Exercice 1.

1) $\pi(\tau^2 \mid y^{(T)}, \sigma^{(T)}, \sigma_0, \beta_0, \beta_1) \propto f(y^{(T)} \mid \sigma^{(T)}, \sigma_0, \beta_0, \beta_1, \tau^2) \cdot \pi(\tau^2)$

The likelihood: $f(y^{(T)} \mid \sigma^{(T)}, \sigma_0, \beta_0, \beta_1, \tau^2)$

$$= \prod_{t=1}^{T} f(y_t \mid \sigma^{(T)}, \sigma_0, \beta_0, \beta_1, \tau^2)$$

$$= \frac{1}{(2\pi)^{\frac{T}{2}} \cdot \exp\left(\frac{1}{2} \cdot \sum_{t=1}^{T} \sigma_t\right)} \exp\left(-\sum_{t=1}^{T} \frac{(y_t)^2}{2 \cdot \exp(\sigma_t)}\right)$$

A posteriori:

$$\pi(\tau^2 \mid y^{(T)}, \sigma^{(T)}, \sigma_0, \beta_0, \beta_1) \propto \frac{1}{(2\pi)^{\frac{T}{2}} \cdot \exp\left(\frac{1}{2} \cdot \sum_{t=1}^{T} \sigma_t\right)} \exp\left(-\sum_{t=1}^{T} \frac{(y_t)^2}{2 \cdot \exp(\sigma_t)}\right) \frac{(d_0)^{c_0}}{\Gamma(c_0)} \cdot \left(\frac{1}{\tau^2}\right)^{c_0+1} \cdot \exp\left(-\frac{d_0}{\tau^2}\right)$$

2)

$$\pi(\beta_0 \mid y^{(T)}, \sigma^{(T)}, \sigma_0, \tau^2, \beta_1) \propto f(y^{(T)} \mid \sigma^{(T)}, \sigma_0, \beta_0, \beta_1, \tau^2) \cdot \pi(\beta_0)$$

$$\propto \frac{1}{(2\pi)^{\frac{T}{2}} \cdot \exp\left(\frac{1}{2} \cdot \sum_{t=1}^{T} \sigma_t\right)} \exp\left(-\sum_{t=1}^{T} \frac{(y_t)^2}{2 \cdot \exp(\sigma_t)}\right) \cdot \frac{1}{\sqrt{2\pi} \cdot \gamma_0} \cdot \exp\left(-\frac{(\beta_0 - \alpha_0)^2}{2\gamma_0^2}\right)$$

3)

$$\pi(\beta_1 \mid y^{(T)}, \sigma^{(T)}, \sigma_0, \tau^2, \beta_0) \propto f(y^{(T)} \mid \sigma^{(T)}, \sigma_0, \beta_0, \beta_1, \tau^2) \cdot \pi(\beta_1)$$

$$\propto \frac{1}{(2\pi)^{\frac{T}{2}} \cdot \exp\left(\frac{1}{2} \cdot \sum_{t=1}^{T} \sigma_t\right)} \exp\left(-\sum_{t=1}^{T} \frac{(y_t)^2}{2 \cdot \exp(\sigma_t)}\right) \cdot \frac{1}{\sqrt{2\pi} \cdot \gamma_1} \cdot \exp\left(-\frac{(\beta_1 - \alpha_1)^2}{2\gamma_1^2}\right) \cdot \mathbb{1}_{(-1,1)}(\beta_1)$$

$\boxed{\text{Exercice 2}}$

1) The likelihood:

$$f(y^T \mid \gamma, \delta, \lambda) = \prod_{t=1}^{\lambda} f(y_t \mid \gamma, \delta, \lambda) \cdot \prod_{t=\lambda+1}^{T} f(y_t \mid \gamma, \delta, \lambda)$$

$$= \prod_{t=1}^{\lambda} \frac{\gamma^{y_t} \cdot e^{-\gamma}}{(y_t)!} \cdot \prod_{t=\lambda+1}^{T} \frac{\delta^{y_t} \cdot e^{-\delta}}{(y_t)!}$$

$$= \frac{\gamma^{\sum_{t=1}^{\lambda} y_t} \cdot e^{-\lambda\gamma} \cdot \delta^{\sum_{t=\lambda+1}^{T} y_t} \cdot e^{-(T-\lambda)\cdot\delta}}{\prod_{t=1}^{T} (y_t)!}$$

$y_t = 0, 1, 2, \cdots \qquad \forall t \in [\![1, T]\!]$

2) Gibbs:

$\theta_0 = (\gamma_0, \delta_0, \lambda_0)$

Sample from:

- $\gamma_1 \sim \pi(\gamma \mid y^T, \delta_0, \lambda_0) \propto f(y^T \mid \lambda_0, \delta_0, \gamma) \cdot \pi(\gamma)$

$$\propto \frac{\gamma^{\sum_{t=1}^{\lambda_0} y_t} \cdot e^{-\lambda_0\gamma} \cdot \delta_0^{\sum_{t=\lambda_0+1}^{T} y_t} \cdot e^{-(T-\lambda_0)\cdot\delta_0}}{\prod_{t=1}^{T} (y_t)!} \cdot \gamma^{a_1} \cdot e^{-\frac{\gamma}{a_2}}$$

$$\propto \gamma^{\sum_{t=1}^{\lambda_0} y_t} \cdot e^{-\lambda_0\gamma} \cdot \gamma^{a_1} \cdot e^{-\frac{\gamma}{a_2}}$$

$$= \gamma^{\sum_{t=1}^{\lambda_0} y_t + a_1} \cdot e^{-\gamma\left(\frac{1}{a_2} + \lambda_0\right)}$$

$$\Gamma\left(\sum_{t=1}^{\lambda_0} y_t + a_1 + 1, \ \frac{1}{a_2} + \lambda_0\right)$$

- $\delta_1 \sim \pi(\delta \mid y^T, \gamma_1, \lambda_0) \propto f(y^T \mid \gamma_1, \lambda_0, \delta) \cdot \pi(\delta)$

$$\propto \delta^{\sum_{t=\lambda_0+1}^{T} y_t} \cdot e^{-(T-\lambda_0)\cdot\delta} \cdot \delta^{d_1} \cdot e^{-\frac{\delta}{d_2}}$$

$$= \delta^{\left(\sum_{t=\lambda_0+1}^{T} y_t + d_1\right)} \cdot e^{-\delta\cdot\left(\frac{1}{d_2} + (T-\lambda_0)\right)}$$

$$\Gamma\left(\sum_{t=\lambda_0+1}^{T} y_t + d_1 + 1,\ \frac{1}{d_2} + T - \lambda_0\right)$$

- $\lambda_1 \sim \pi(\delta \mid y^T, \gamma_1, \delta_1) \propto f(y^T \mid \gamma_1, \delta_1, \lambda) \cdot \pi(\lambda)$

$$\propto \gamma_1^{\sum_{t=1}^{\lambda} y_t} \cdot \delta_1^{\sum_{t=\lambda+1}^{T} y_t} \cdot e^{-(T-\lambda)\delta_1} \cdot \frac{1}{T}$$

$$\propto \gamma_1^{\sum_{t=1}^{\lambda} y_t} \cdot \delta_1^{\sum_{t=\lambda+1}^{T} y_t} \cdot e^{-(T-\lambda)\delta_1}$$

$\lambda_1 \in \{1, 2, \cdots, T-1\}$

In this way, we can obtain the first iterate.

We can repeat iterations as we need.

Statistiques Bayésiens DM1

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November 20, 2022

1 Exercice 2.3

1.1 Code

```
import numpy as np
import matplotlib.pyplot as plt
rng = np.random.default_rng()

with open('acc_usines.txt') as f:
    contents = f.readlines()

y = [] # store the number of accidents
for i in contents:
    y.append(int(i[0]))

T = len(y)
# hyperparameters
a1, a2, d1, d2 = 1, 1, 1, 1
epochs = 100

# initialization
gamma_ = rng.gamma(a1, scale=1/a2)
delta_ = rng.gamma(d1, scale=1/d2)
lambda_ = np.random.choice(range(1, T+1), p=[1/T]*T)

# store the sampling history
l_gamma, l_lambda, l_delta = [gamma_], [lambda_], [delta_]

# Gibbs sampler
for e in range(epochs):
    # define the new parameters for the law of parameter gamma, which follows the law gamma
    shape_gamma_ = sum(y[:lambda_]) + a1 + 1
    rate_gamma_ = 1 / a2 + lambda_
    gamma_ = rng.gamma(shape_gamma_, scale=1/rate_gamma_)
    l_gamma.append(gamma_)

    # define the new parameters for the law of parameter delta, which follows the law gamma
    shape_delta_ = sum(y[lambda_:]) + d1 + 1
    rate_delta_ = 1 / d2 + T - lambda_
    delta_ = rng.gamma(shape_delta_, scale=1/rate_delta_)
    l_delta.append(delta_)

    proba_lambda = [gamma_**(sum(y[:lamb]))*delta_**(sum(y[lamb:]))*np.exp(-(T-lamb)*delta_)
        for lamb in range(1, T)]
    proba_lambda = proba_lambda / sum(proba_lambda)
    lambda_ = np.random.choice(range(1,T), p=proba_lambda)
    l_lambda.append(lambda_)
```

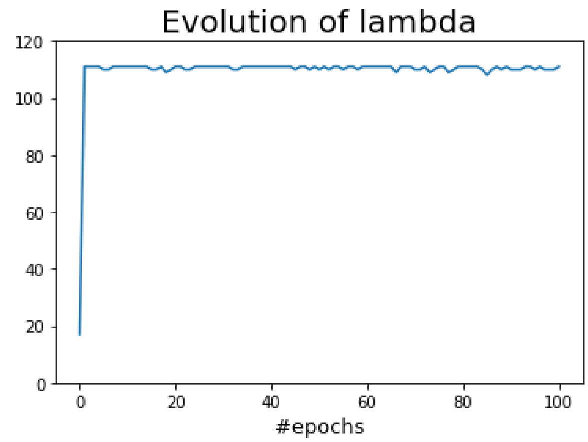

Figure 1: Lambda

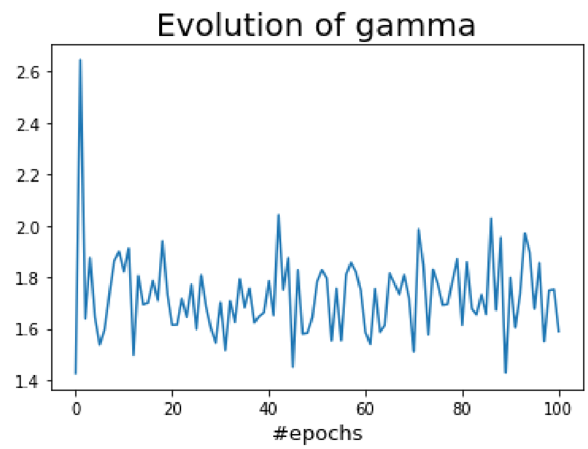

Figure 2: Gamma

```
plt.plot(l_lambda)
plt.xlabel("#epochs", size=13)
plt.ylim((0,120))
plt.title("Evolution of lambda", size=20)

plt.plot(l_gamma)
plt.xlabel("#epochs", size=13)
plt.title("Evolution of gamma", size=20)

plt.plot(l_delta)
plt.xlabel("#epochs", size=13)
plt.title("Evolution of delta", size=20)
```

1.2 Results

There are 100 epochs in Gibbs sampler. Fig. 1, Fig. 2 and Fig. 3 give the evolution of parameters $(\lambda, \gamma, \delta)$ in the model:

After 20 epochs, λ arrives at a stable state. We can then calculate its mean and variance using command **np.mean(l_lambda[20:])** and **np.var(l_lambda[20:])**. We respectively get **110.6 and 0.39** as mean and variance of λ. Therefore, the change point is **around 110 and 111**.

The number of accidents is given by the parameter in law of Poisson. That is to say, before change point, the anticipated number of accident equals to γ, which is around **1.7**; After change point, the

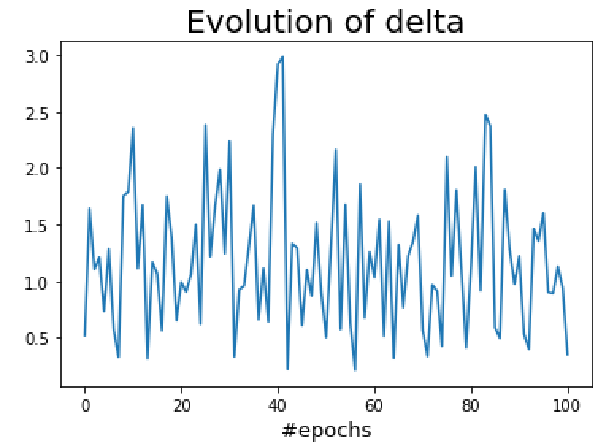

Figure 3: Delta

anticipated number of accident drop to around **1.2** (we should notice that the δ is in fact not as stable as γ and λ).