$$S_n^{OR} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{X_i - \mu}{\sigma}\right)^3 - sk \right]$$

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{X_i - \hat{\mu}}{\hat{\sigma}}\right)^3 - sk \right] \qquad \hat{\mu} = \bar{X} \qquad \hat{\sigma}^2 = \bar{X}^2 - (\bar{X})^2$$

1)

$$S_n^{H_0} = \frac{1}{\sqrt{n}\,\hat{\sigma}^3} \sum_{i=1}^{n} (X_i - \hat{\mu})^3 = \frac{1}{\hat{\sigma}^3 \cdot \sqrt{n}} \sum_{i=1}^{n} (X_i^3 - 3X_i^2\hat{\mu} + 3X_i\hat{\mu}^2 - \hat{\mu}^3)$$

$$= \sqrt{n} \cdot \frac{[\overline{X^3} - 3\bar{X}\,\overline{X^2} + 3\bar{X}(\bar{X})^2 - (\bar{X})^3]}{(\overline{X^2} - (\bar{X})^2)^{\frac{3}{2}}}$$

$$= \sqrt{n}\, g(\overline{X^3}, \overline{X^2}, \bar{X})$$

$$g(a,b,c) = \frac{a - 3bc + 2c^3}{(b-c)^{\frac{3}{2}}}$$

we have $\sqrt{n}\left[\begin{pmatrix} \overline{X^3} \\ \overline{X^2} \\ \bar{X} \end{pmatrix} - \begin{pmatrix} \mathbb{E}(X^3) \\ \mathbb{E}(X^2) \\ \mathbb{E}(X) \end{pmatrix}\right] \xrightarrow{d} N(0, \Sigma)$ CLT.

Delta method:

$$\sqrt{n}\,[g(\overline{X^3}, \overline{X^2}, \bar{X}) - g(\mathbb{E}(X^3), \mathbb{E}(X^2), \mathbb{E}(X))] \xrightarrow{d} \nabla g(\,)^T N(0, \Sigma)$$

S_n is asymptotically normal.

2)

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{X_i - \hat{\mu}}{\hat{\sigma}}\right)^3 - \mathbb{E}\left(\frac{X-\mu}{\sigma}\right)^3 \right]$$

$\mu = \mathbb{E}_{P_X}[X]$

$\mathbb{E}_{\hat{P}_X}[X] = \hat{\mu} = \bar{X}$

$\begin{cases} \text{Replace } P_X \text{ with } \hat{P}_X & ① \\ \text{use principle of substitution \& similarity.} & ② \end{cases}$

$$S_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\underbrace{\left(\frac{X_i^* - \hat{\mu}^*}{\hat{\sigma}^*}\right)^3}_{②} - \underbrace{\overline{\left(\frac{X - \hat{\mu}}{\hat{\sigma}}\right)^3}}_{①} \right]$$

Then, Compute the quantiles of S_n^* $\hat{q}_{\alpha/2}$, $\hat{q}_{1-\alpha/2}$

Reject H_0 iif $S_n^{H_0} \notin [\hat{q}_{\alpha/2}, \hat{q}_{1-\frac{\alpha}{2}}]$

Bootstrap provides no refinements, but it avoids the computation of variance ψ, which is complexe.

3) $S_n^* = \sqrt{n}\left[g\left(\overline{(X^*)^3}, \overline{(X^*)^2}, \overline{X^*}\right) - g\left(\overline{X^3}, \overline{X^2}, \bar{X}\right)\right]$

By Pairwise Bootstrap CLT

$$\sqrt{n}\left[\begin{pmatrix} \overline{(X^*)^3} \\ \overline{(X^*)^2} \\ \overline{X^*} \end{pmatrix} - \begin{pmatrix} \overline{X^3} \\ \overline{X^2} \\ \bar{X} \end{pmatrix}\right] \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

By Bootstrap delta method

$$\sqrt{n}\left[g(\overline{(X^*)^3}, \overline{(X^*)^2}, \overline{X^*}) - g(\overline{X^3}, \overline{X^2}, \overline{X})\right] \xrightarrow{d} \nabla g(\ )^T \mathcal{N}(0, \Sigma)$$

Recall that $S_n \xrightarrow{d} \nabla g(\ )^T \mathcal{N}(0, \Sigma)$

So, $\| \hat{F}_{S_n^*} - F_{S_n} \|_\infty \leq \underbrace{\| \hat{F}_{S_n^*} - \Phi_\varphi \|_\infty}_{o(1)} + \underbrace{\| \Phi_\varphi - F_{S_n} \|_\infty}_{o(1)}$

Then S_n^* estimates consistently the distribution of S_n

Ex2.

$$S_{n,1} = \sqrt{n}\left[(\bar{X})^2 - (\mathbb{E}(X))^2 \right] = \sqrt{n}\left[g(\bar{X}) - g(\mathbb{E}(X)) \right]$$

$$\xrightarrow{d} g'(\mathbb{E}(X)) \cdot \mathcal{N}(0, V(X))$$

CLT + Δ method.

$$S_{n,1}^* = \sqrt{n}\left[g(\overline{X^*}) - g(\bar{X}) \right]$$

$$\xrightarrow{d} g'(\mathbb{E}(X))\, \mathcal{N}(0, V(X)) \quad \text{in probability}$$

$S_{n,1}^*$ consistently the distribution of $S_{n,1}$

3). $S_{n,3}^* = \left[\sqrt{n}\left(\bar{X}^* - \bar{X}\right)\right]^2$

By Bootstrap CLT, $\sqrt{n}\left(\bar{X}^* - \bar{X}\right) \to \mathcal{N}(0, V(X))$

$S_{n,3}^* \xrightarrow{d} \chi^2$

Ex3. with proposition 4.2.1,

we should have: $\mathbb{E}(v) = 1$,

$Var(v) = 1$