PS07

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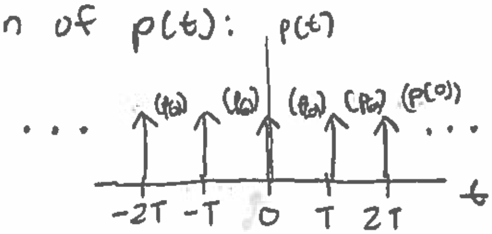
1) Consider a train of unit impulses separated by T time units, given by the following expression

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

ⓐ Representation of $p(t)$:



ⓑ Fourier series representation of $p(t)$ within an infinite number of terms:

$$C_K = \frac{1}{T}\int_{-T/2}^{T/2} p(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$$

substitute $p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

$$C_K = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t-kT) e^{-j\frac{2\pi}{T}kt}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} \underbrace{\delta(t-kT)}\underbrace{e^{-j\frac{2\pi}{T}kt}}\, dt$$

picking property: $\int_{-\infty}^{\infty} x(t)\delta(t-t_o)\,dt = x(t_o)$

$$x(t) = e^{-j\frac{2\pi}{T}kt}$$

$$C_K = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{-j\frac{2\pi}{T}kt_o} = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{0} = \sum_{k=-\infty}^{\infty} \frac{1}{T}$$

$$\boxed{\text{fourier series rep: } \hat{p}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}kt}}$$

ⓒ $x(t) = \sum_{k=-\infty}^{\infty} C_K e^{j\frac{2\pi}{T}kt}$ Find $X(\omega)$, in terms of C_K

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_K e^{j\frac{2\pi}{T}kt} e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} C_K e^{jt(\frac{2\pi}{T}k-\omega)}\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_K \int_{-\infty}^{\infty} e^{jt(\frac{2\pi}{T}k-\omega)}\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_K\, 2\pi\, \delta(\omega - \tfrac{2\pi}{T}k + \omega)\, dt$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} C_K\, 2\pi\, \delta(2\omega - \tfrac{2\pi k}{T})}$$

$x(t) = e^{jt\omega_o}$

$\omega_o = \frac{2\pi}{T}k - \omega$

$x(t) = e^{jt(\frac{2\pi}{T}k-\omega)}$

$X(\omega) = 2\pi\,\delta(\omega-\omega_o)$

(d) Find $P(\omega)$:

fourier series: $p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}kt}$

$C_K = \frac{1}{T}$

$$X(\omega) = \sum_{k=-\infty}^{\infty} C_K \, 2\pi \, \delta\left(2\omega - \frac{2\pi k}{T}\right)$$

$$P(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi \, \delta\left(2\omega - \frac{2\pi k}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(2\omega - \frac{2\pi k}{T}\right) \qquad \omega_0 = \frac{2\pi}{T}$$

$$= \omega_0 \sum_{k=-\infty}^{\infty} \delta(2\omega - \omega_0 k)$$

$$\boxed{P(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(2\omega - \omega_0 k)}$$

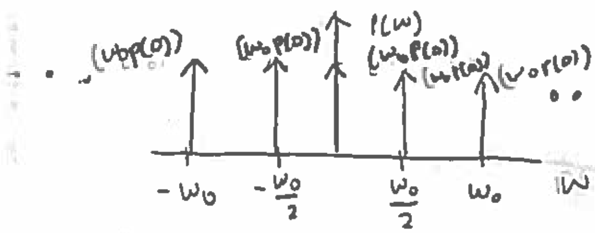
(e) Sketch $P(\omega)$. Effect of changing T to $p(t)$ and $P(\omega)$?

$P(\omega)$: $P(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(2\omega - \omega_0 k)$

$2\omega - \omega_0 k = 0$

$\omega = \frac{\omega_0 k}{2}$

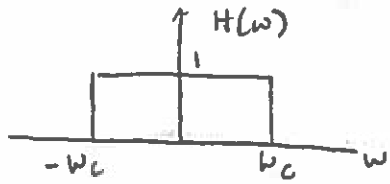
↑ ω when the impulse repeats itself every k



Since the fundamental frequency T is equal to $\frac{2\pi}{\omega_0}$, and thus $\omega_0 = \frac{2\pi}{T}$, increasing T would make ω_0 smaller and smaller, which would make the spread of impulses closer and closer in $P(\omega)$, but the train of unit impulses in $p(t)$ would be more spread out.

Lower T, means less spread out in time domain, but more spread out in frequency domain.

2) LTI system with impulse response $h(t)$, input signal $x(t)$ and output $y(t)$. $H(\omega)$ is the following:



From $-\omega_c$ to ω_c:

$H(\omega) = 1$

ⓐ Find $h(t)$:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{jt}\left(e^{j\omega t}\right)\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi j t}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

$$= \frac{1}{\pi t}\left(\frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)\right) \quad \theta = \omega_c t$$

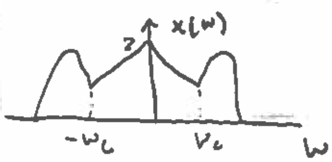
sine property:

$$\sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta}$$

$$= \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\boxed{h(t) = \frac{\sin(\omega_c t)}{\pi t}}$$

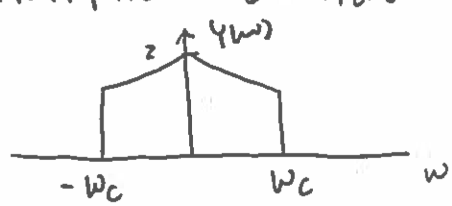
ⓑ $X(\omega)$:



Please sketch $Y(\omega)$:

$$Y(\omega) = H(\omega)X(\omega)$$

Element-wise multiplication of $H(\omega)$ with $X(\omega)$:

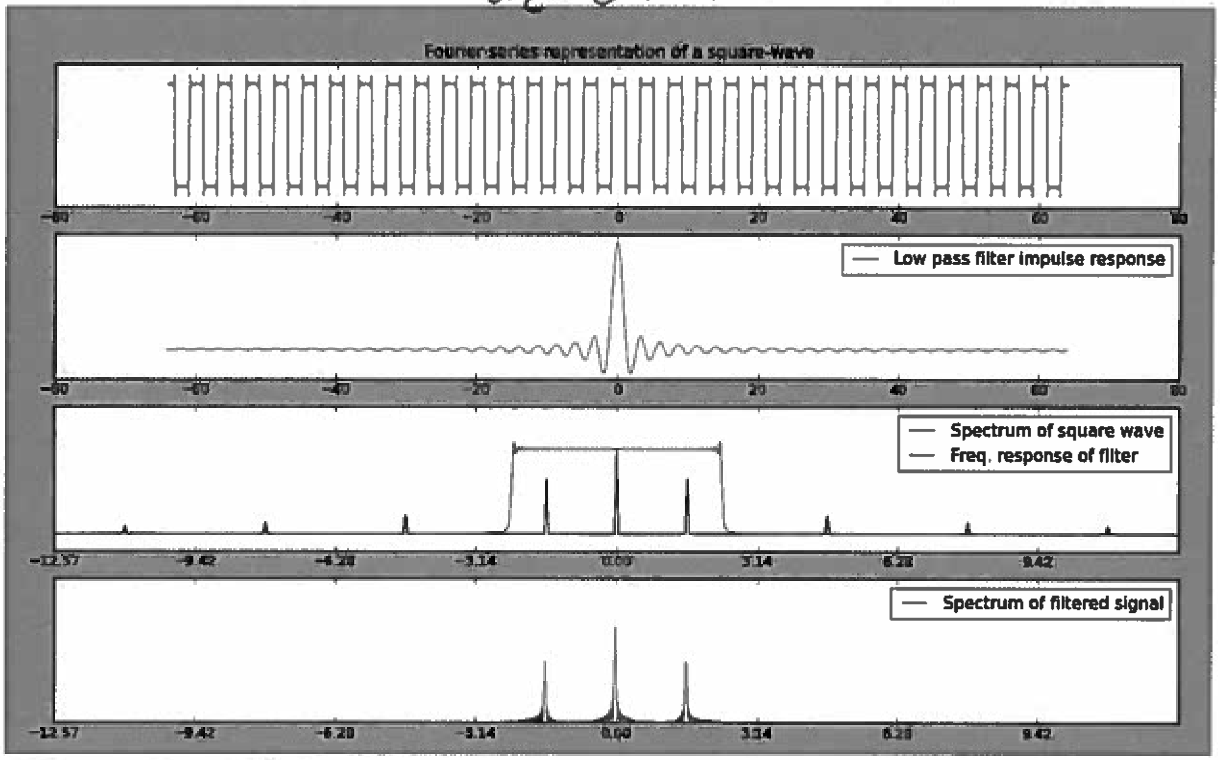


ⓒ Why is the LTI system known as an ideal low-pass filter with cutoff frequency ω_0?

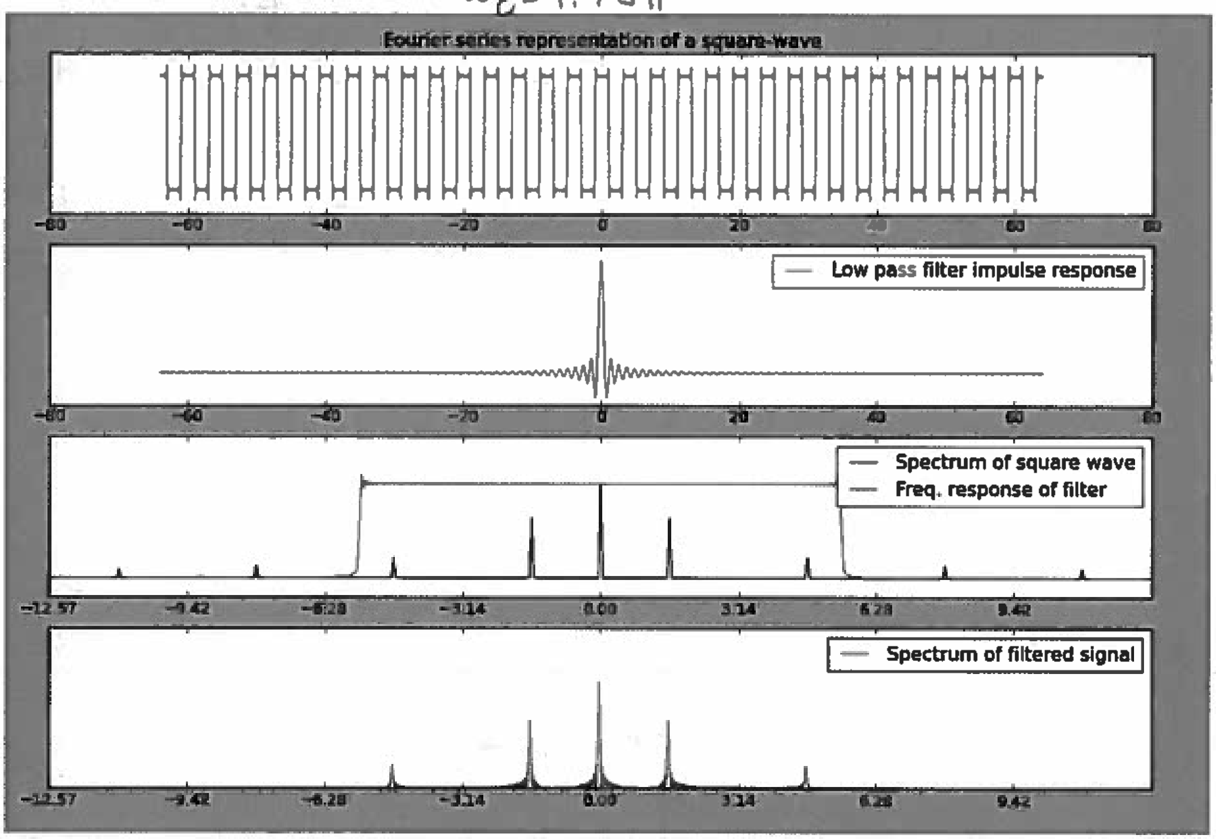
The LTI system $H(\omega)$ is an ideal low-pass filter with cut off frequency, because it attenuates components above a given cutoff frequency ω_c, once the convolution of $h(t)$ with $x(t)$ occurs, which is also the element-wise multiplication of $H(\omega)$ and $X(\omega)$. Before $-\omega_c$ and after ω_c, $H(\omega)$ is zero, and when multiplied with anything, outputs zero also, which eliminates those frequencies

(d)

$w_c = 0.75\pi$

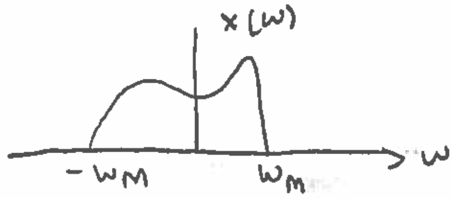


$w_c = 1.75\pi$



The graphs show that, with a larger frequency, the time speed gets smaller, but making the spectrum of square wave applied larger.

3) $x(t)$ is band-limited to range $[-\omega_M, \omega_M]$. $X(\omega) = 0$ for $\omega < -\omega_M$ and $\omega > \omega_M$. Let $y(t) = x(t)\cos(\omega_c t)$, where $\omega_c >> \omega_M$.



Please sketch $Y(\omega)$:

$$y(t) = x(t)\underbrace{\cos(\omega_c t)}$$

Fourier Transform prop.: $y(t) = x(t)\,h(t) \rightarrow Y(\omega) = \frac{1}{2\pi} X * H(\omega)$

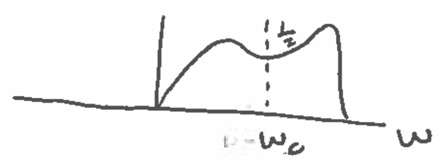
$$h(t) = \cos(\omega_c t) \rightarrow H(\omega) = \pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)$$

$$Y(\omega) = \frac{1}{2\pi} X * (\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c))$$

$$= \frac{1}{2\pi} X * \pi\delta(\omega-\omega_c)$$

$$+ \frac{1}{2\pi} X * \pi\delta(\omega+\omega_c)$$

① $\frac{1}{2\pi} X * \pi\delta(\omega-\omega_c)$:



X(m) shifted right

② $\frac{1}{2\pi} X * \pi\delta(\omega+\omega_c)$:

X(m) be shifted left

$Y(\omega) = ① + ②$:

Multiplying rel. low bandwidth signal $(X(\omega))$ with a high frequency cosine wave is basis of amplitude modulation. At different assigned frequencies (ω_c), the same propagation medium but negligible interference is obtained, because the whole impulse/area of $X(\omega)$ will be obtained at all possible ω_c's, but the split around $\omega = 0$ would be different.