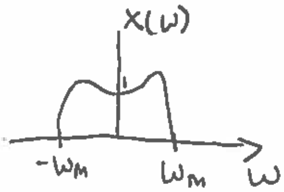
PS08

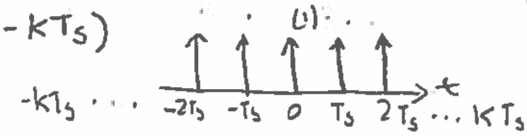
Cynthia Chen

1) $x(t)$:



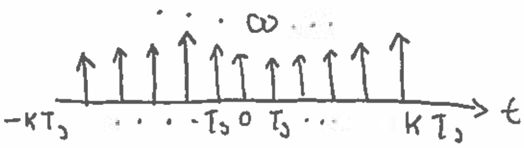
Representative band-limited signal

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

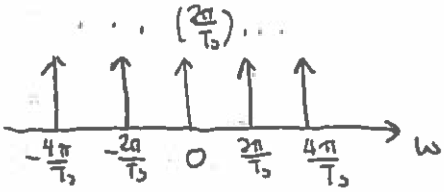


$$x_p(t) = x(t)\,p(t)$$

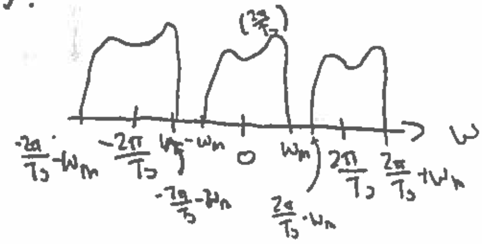
ⓐ $x_p(t)$:



ⓑ $P(\omega)$:



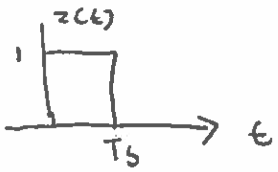
ⓒ $X_p(\omega)$:

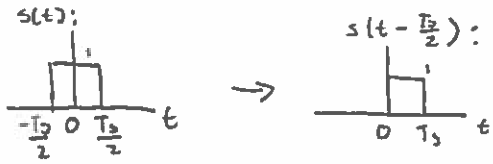


ⓓ The relationship between T_s and ω_m that ensures that $X_p(\omega)$ contains all info present in $X(\omega)$ is $\frac{2\pi}{T_s} \geq 2\omega_m$. Sampling rate is at least 2 times the maximum frequency.

ⓔ You can recover $x(t)$ from $x_p(t)$ by convolving with a sinc function, or apply a filter with a cutoff frequency.

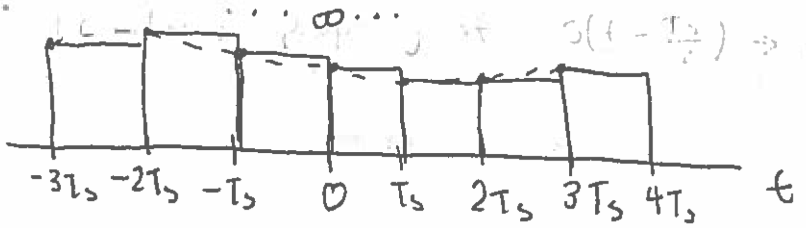
(f) $z(t)$:





$z(t)$ is a rectangle (centered at 0) shifted to the right by $T_s/2$.

(g) $x_2(t) = x_p * z(t)$ sketch:



(h) $X_2(\omega)$ sketch: $X_p(\omega) Z(\omega)$

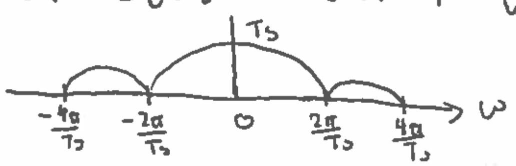
$Z(\omega)$: Fourier Transform property of $s(t-\frac{T_s}{2}) \rightarrow e^{-j\omega t_0} S(\omega)$

$= e^{-j\omega \frac{T_s}{2}} S(\omega)$

$s(t)$ is a rectangular box. In the notes $S(\omega)$ of a rectangular box is $\frac{2\sin(\omega \frac{T_s}{2})}{\omega}$, so

$$Z(\omega) = \frac{2\sin(\omega \frac{T_s}{2})}{\omega} e^{-j\omega \frac{T_s}{2}}$$

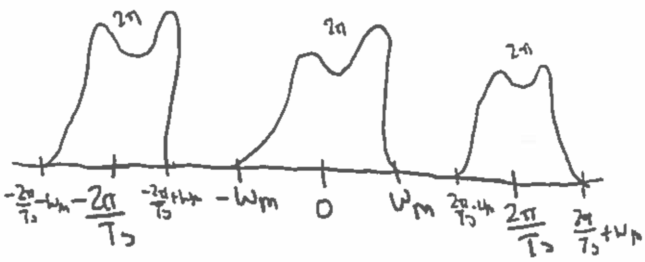
Sketch of $Z(\omega)$: $|Z(\omega)| = \left| \frac{2\sin(\omega \frac{T_s}{2})}{\omega} \right|$ abs



at 0: l'hopital rule

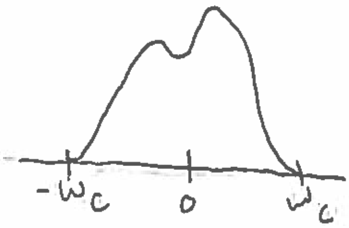
$$\lim_{\omega \to 0} \frac{T_s \cos(\omega \frac{T_s}{2})}{1} = T_s$$

$X_2(\omega)$:

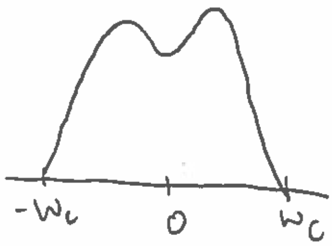


(i) $\bar{x}(\omega) = X_z(\omega) H(\omega)$ $\quad \hat{x}(\omega) = X_p(\omega) H(\omega)$ where $H(\omega)$ with $\omega_c = \frac{\pi}{T_s}$ is:

$\bar{x}(\omega)$:



$\hat{x}(\omega)$:



(j) $\bar{x}(\omega)$ and $\hat{x}(\omega)$ are different by the fact that $\bar{x}(\omega)$ is tapered and $\hat{x}(\omega)$ is not.

(k) ratio of $\bar{x}(\omega_m)$ to $\hat{x}(\omega_m)$:

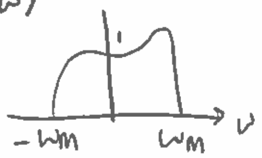
$$\frac{\bar{x}(\omega_m)}{\hat{x}(\omega_m)} = \frac{X_z(\omega_m)\cancel{H(\omega_m)}}{X_p(\omega_m)\cancel{H(\omega_m)}} = \frac{\cancel{X_p(\omega_m)}\,Z(\omega_m)}{\cancel{X_p(\omega_m)}} = Z(\omega_m$$

when $\omega_m = \frac{\pi}{T_s}$

$$Z\left(\frac{\pi}{T_s}\right) = \frac{2\sin\left(\frac{\pi}{T_s}\left(\frac{T_s}{2}\right)\right)}{\left(\frac{\pi}{T_s}\right)} = \frac{2}{\frac{\pi}{T_s}} = \boxed{\frac{2T_s}{\pi}}$$

2) $y(t) = x_1(t)\cos(w_1 t) + x_2(t)\cos(w_2 t)$

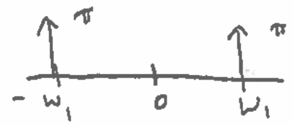
$X_1(w)$ $X_2(w)$



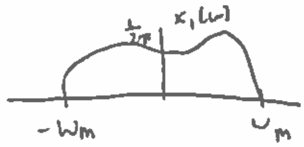
(a) $Y(w) = \frac{1}{2\pi} X_1 * (\pi\delta(w-w_1) + \pi\delta(w+w_1))$
$+ \frac{1}{2\pi} X_2 * (\pi\delta(w-w_2) + \pi\delta(w+w_2))$

$x_1(t)\cos(w_1 t) \rightarrow \frac{1}{2\pi} X_1 * (\pi\delta(w-w_1) + \pi\delta(w+w_1))$

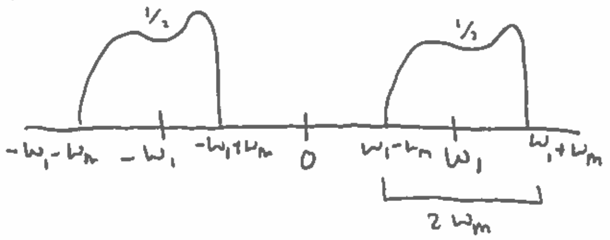
$a = \pi\delta(w-w_1) + \pi\delta(w+w_1)$:



$b = \frac{1}{2\pi} X_1$:

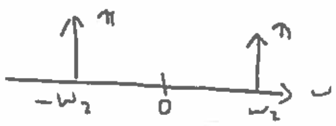


$b * a$:



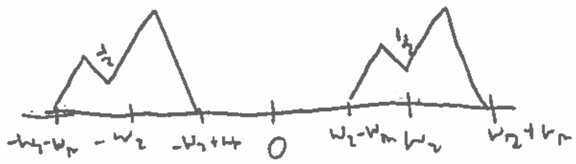
$x_2(t)\cos(w_2 t) \rightarrow \frac{1}{2\pi} X_2 * (\pi\delta(w-w_2) + \pi\delta(w+w_2))$

$a_2 = \pi\delta(w-w_2) + \pi\delta(w+w_2)$:

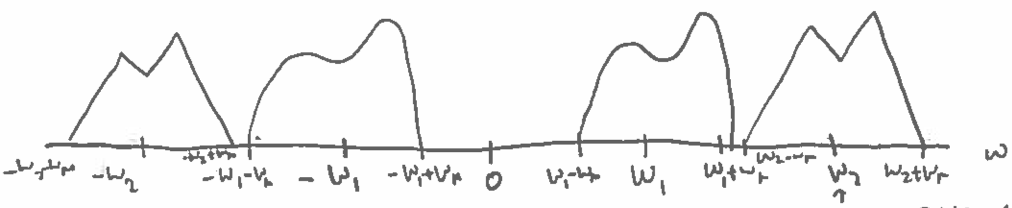


$b_2 = \frac{1}{2\pi} X_2$:

$a_2 * b_2$:



$w_2 > w_1 + 2w_m$

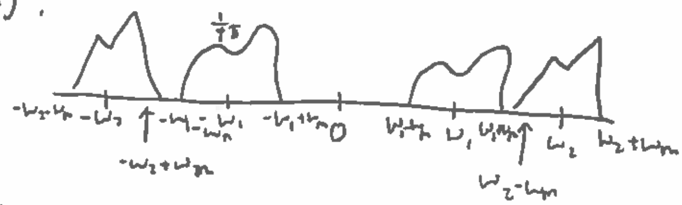


$w_2 > w_1 + 2w_m$ ← never overlaps

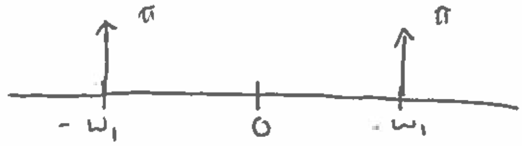
(b) Fourier transforms of $y(t)\cos(\omega_1 t)$ and $y(t)\cos(\omega_2 t)$

$$FT\{y(t)\cos(\omega_1 t)\} = \frac{1}{2\pi} Y(\omega) * (\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1))$$

$a_3 = \frac{1}{2\pi} Y(\omega)$:



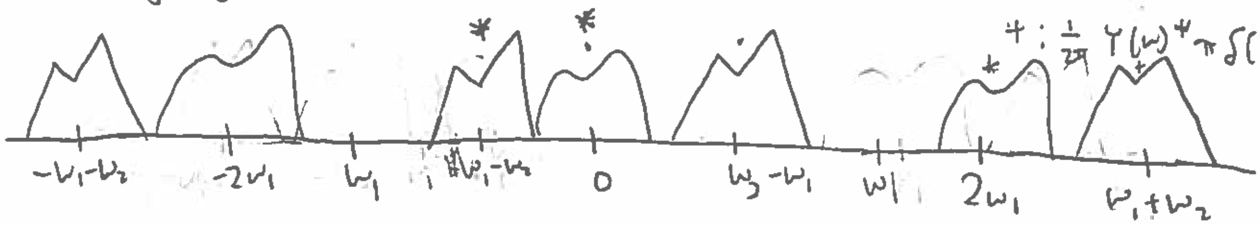
$b_3 = \pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)$:



$a_3 * b_3$:

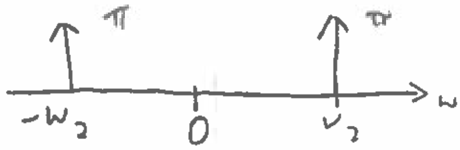
$\bullet$: $\frac{1}{2\pi} Y(\omega) * \pi\delta(\omega+\omega$

$+$: $\frac{1}{2\pi} Y(\omega) * \pi\delta(\omega-\omega_1$



$$FT\{y(t)\cos(\omega_2 t)\} = \frac{1}{2\pi} Y(\omega) * (\pi\delta(\omega-\omega_2) + \pi(\omega+\omega_2))$$

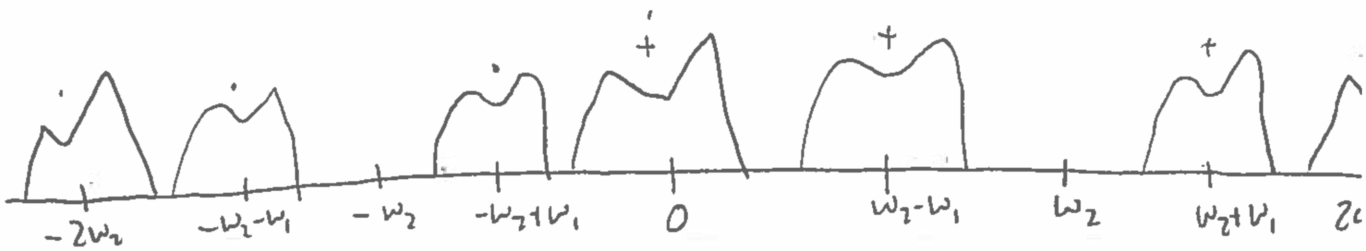
$b_4 = \pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)$;



$\bullet$: $\frac{1}{2\pi} Y(\omega) * \pi\delta(\omega+\omega$

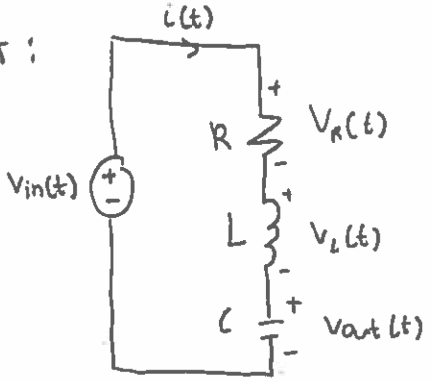
$+$: $\frac{1}{2\pi} Y(\omega) * \pi\delta(\omega-\omega_2$

$a_3 * b_4$:



(c) You can recover $x_1(t)$ and $x_2(t)$ from $y(t)$ by multiplying $y(t)$ with $\cos(\omega_1 t)$, and then applying a low-pass filter to recover $x_1(t)$, and multiplying $y(t)$ with $\cos(\omega_2 t)$, and then applying low-pass filter to recovery $x_2(t)$. This shows 2 different AM transmitters can output the different frequency compo

3) RCL circuit:



$$i(t) = C\frac{d}{dt}V_{out}(t)$$
$$V_L(t) = L\frac{d}{dt}i(t)$$
$$V_R(t) = R\,i(t)$$

ⓐ Diff. equation relating $V_{out}(t)$ and $V_{in}(t)$

$$V_{in}(t) = V_R(t) + V_L(t) + V_{out}(t)$$
$$= Ri(t) + L\frac{d}{dt}i(t) + V_{out}(t)$$
$$\boxed{V_{in}(t) = RC\frac{d}{dt}V_{out}(t) + LC\frac{d^2}{dt^2}V_{out}(t) + V_{out}(t)}$$

ⓑ Find $H(\omega)$:

$$FT\{V_{in}(t)\} = RC\left(FT\{\tfrac{d}{dt}V_{out}(t)\}\right) + LC\left(FT\{\tfrac{d^2}{dt^2}V_{out}(t)\}\right) + FT\{V_o$$
$$V_{in}(\omega) = RCj\omega V_{out}(\omega) + LCj^2\omega^2 V_{out}(\omega) + V_{out}(\omega)$$
$$= V_{out}(\omega)\left(RCj\omega + LCj^2\omega^2 + 1\right)$$
$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{RCj\omega + LCj^2\omega^2 + 1} = \boxed{\frac{1}{RCj\omega - LC\omega^2 + 1}}$$

ⓒ Expression for magnitude of $H(\omega)$:

$$|H(\omega)| = \frac{1}{|RCj\omega - LC\omega^2 + 1|} = \frac{1}{\sqrt{(RC)^2\omega^2 + (1-LC\omega^2)^2}}$$
$$\boxed{|H(\omega)| = \frac{1}{\sqrt{R^2C^2\omega^2 + 1 + L^2C^2\omega^4 - 2LC\omega^2}}}$$

(d) Find ω that maximizes $|H(\omega)|$:

$$|H(\omega)| = \frac{1}{\sqrt{R^2C^2\omega^2+1+L^2C^2\omega^4-2LC\omega^2}}$$

When the 1st deriv. of $|H(\omega)|$ is 0, maximum

$$|H(\omega)|' = -\frac{1}{2}\left(R^2C^2\omega^2+1+L^2C^2\omega^4-2LC\omega^2\right)^{-3/2}\left(2R^2C^2\omega+4L^2C^2\omega^3-4LC\omega\right)$$

When numerator is 0, $|H(\omega)|'$ will be 0:

$$2R^2C^2\omega+4L^2C^2\omega^3-4LC\omega=0$$

$$\cancel{\omega}\left(2R^2C^2+4L^2C^2\omega^2-4LC\right)=0$$

$$4L^2C^2\omega^2 = 4LC-2R^2C^2$$

$$\omega^2 = \frac{4LC-2R^2C^2}{4L^2C^2}$$

$$\boxed{\omega = \pm\sqrt{\frac{1}{LC}-\frac{R^2}{2L^2}}}$$

(e) plots on the next page:

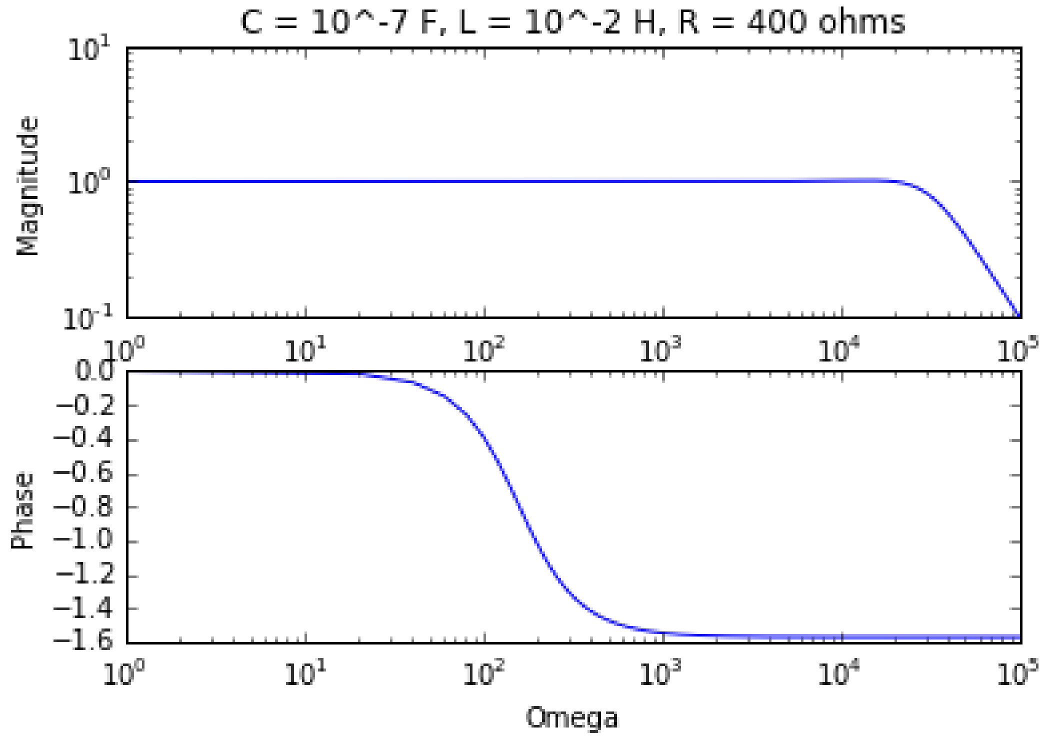
$$\text{magnitude} = \frac{1}{\sqrt{R^2C^2\omega^2+1+L^2C^2\omega^4-2LC\omega^2}}$$

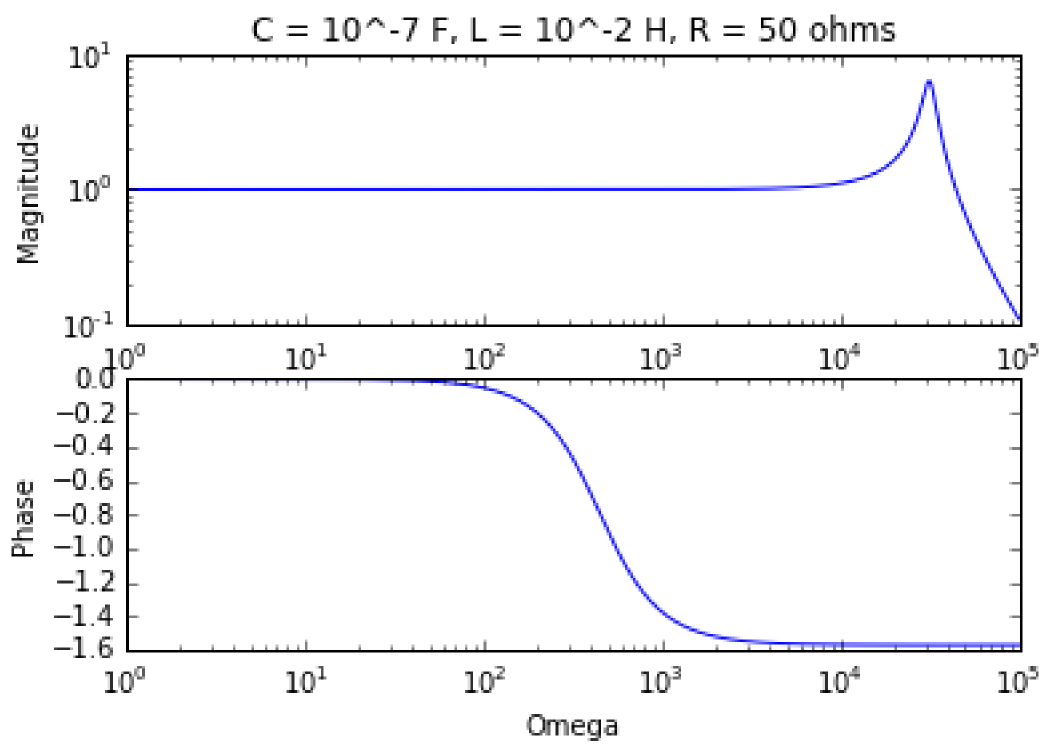
phase: $H(\omega) = \dfrac{1}{1-LC\omega^2+RCj\omega}$ we want $a+bi$ $\quad a = 1-LC\omega^2$, $b = RC\omega j$

cmath.ph finds ph of complex numbe

$$\frac{1}{(1-LC\omega^2+RCj\omega)}\left(\frac{1-LC\omega^2-RCj\omega}{1-LC\omega^2-RCj\omega}\right) = \frac{1-LC\omega^2-RCj\omega}{(1-LC\omega^2-\cancel{RCj\omega}-LC\omega^2+L^2C^2\omega^4+\cancel{RLC^2j\omega^3}+\cancel{RCj\omega}-\cancel{RLC^2j\omega^3}-R^2C^2j^2\omega^2)}$$

$$= \frac{1-LC\omega^2-RCj\omega}{1-2LC\omega^2+L^2C^2\omega^4+R^2C^2\omega^2} = \frac{(1-LC\omega^2)}{1-2LC\omega^2+L^2C^2\omega^4+R^2C^2\omega^2} - \frac{RCj\omega}{1-2LC\omega^2+L^2C^2\omega^4+R^2C^2\omega^2}$$


C = 10^-7 F, L = 10^-2 H, R = 400 ohms
Magnitude
Phase
Omega


C = 10^-7 F, L = 10^-2 H, R = 50 ohms
Magnitude
Phase
Omega