Problem Set 10

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① Use Laplace transform to verify that step response of system $\dot{y}+y=x$ is $y(t)=(1-e^{-t})u(t)$.

$$\dot{y}(t) \overset{L}{\Longleftrightarrow} sY(s)$$
$$y(t) \overset{L}{\Longleftrightarrow} Y(s)$$
$$x(t) \overset{L}{\Longleftrightarrow} X(s)$$

In $\dot{y}+y=x$, the Laplace transform is:

$$sY(s)+Y(s)=X(s)$$
$$Y(s)(1+s)=X(s)$$
$$\frac{Y(s)}{X(s)}=\frac{1}{1+s}$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{1+s}, \quad \text{ROC: } Re\{s\}>-1$$

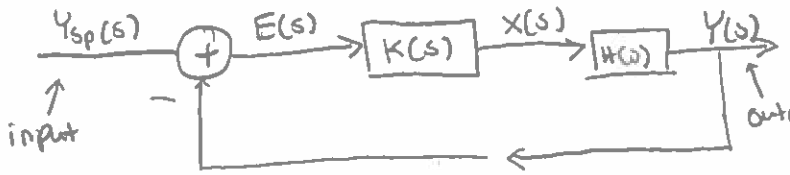
Now, we need to find $Y(s)$ to find step response:

$$Y(s)=H(s)X(s)$$
$$=\frac{X(s)}{1+s}$$
$$=\frac{1}{s}\left(\frac{1}{1+s}\right)$$
$$=\frac{1}{s(1+s)}$$
$$=\frac{s+1-s}{s(1+s)}$$
$$=\frac{\cancel{s+1}}{s(\cancel{1+s})}-\frac{s}{s(1+s)}$$
$$Y(s)=\frac{1}{s}-\frac{1}{s+1}, \quad \text{ROC: } Re\{s\}>0$$

$x(t)$ is unit step, $\cancel{x}(t) \overset{L}{\Longleftrightarrow} \frac{1}{s}$ ← $X(s)$

(2)

A) Find DC gain of system $Y(s)/Y_{sp}(s)$ when you use an integral controller $K(s) = K_I/s$ for any $H(s)$.



Overall transfer function:

$$K(\underbrace{Y_{sp} - HX}_{error}) = X$$

$$\frac{X}{Y_{sp}} = \frac{K}{1+KH}$$

$$\frac{Y}{Y_{sp}} = \frac{KH}{1+KH} \leftarrow \text{Black's Formula}$$

Using an integral controller: $K(s) = K_I/s$

$$\frac{Y}{Y_{sp}} = \frac{\frac{K_I}{s}(H)}{1+\frac{K_I}{s}(H)} = \frac{\frac{K_I(H)}{s}}{1+\frac{K_I H}{s}} = \frac{\frac{1}{s}(K_I H)}{\frac{1}{s}(s+K_I H)} = s \cdot \frac{1}{s}\left(\frac{K_I H}{s+K_I H}\right)$$

$$\text{DC gain} = \lim_{s\to 0} \cancel{s} \cdot \frac{1}{\cancel{s}}\left(\frac{K_I H}{s+K_I H}\right)$$

$$= \lim_{s\to 0}\left(\frac{K_I H}{K_I H + s}\right)$$

$$= 1$$

The DC gain, using integral controller is 1, which means that the system approaches 1 when the time approaches 0. The DC gain does not depend on K_I, because they cancel each other out when s is really small.

B) $H(s) = \frac{1/\tau}{s+1/\tau}$. Find $Y(s)/Y_{sp}(s)$.

$$\frac{Y}{Y_{sp}} = \frac{K_I\left(\frac{1/\tau}{s+1/\tau}\right)}{K_I\left(\frac{1/\tau}{s+1/\tau}\right)+s} = \frac{\frac{\frac{K_I}{\tau}}{\cancel{s+1/\tau}}}{\frac{K_I/\tau + s^2 + s/\tau}{\cancel{s+1/\tau}}} = \frac{\frac{K_I}{\tau}}{\frac{K_I}{\tau}+s^2+\frac{s}{\tau}}$$

In numerator, when it approaches 0, ~~the~~ s is a zero. There are no places where this is applicable

Poles: denom → 0

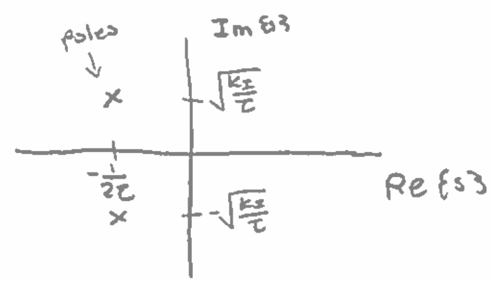
$$\frac{K_I}{\tau} + s^2 + \frac{s}{\tau} = 0$$

Quadratic formula: $\frac{-1/\tau \pm \sqrt{\frac{1}{\tau^2} - \frac{4K_I}{\tau}}}{2}$ $\qquad \frac{1}{\tau^2}$ will get so small, it is negligible

$$= \frac{-1/\tau \pm \sqrt{-\frac{4K_I}{\tau}}}{2}$$

$$= -\frac{1}{2\tau} \pm \frac{2j\sqrt{\frac{K_I}{\tau}}}{2}$$

$$= -\frac{1}{2\tau} \pm j\sqrt{\frac{K_I}{\tau}}$$



Number 3

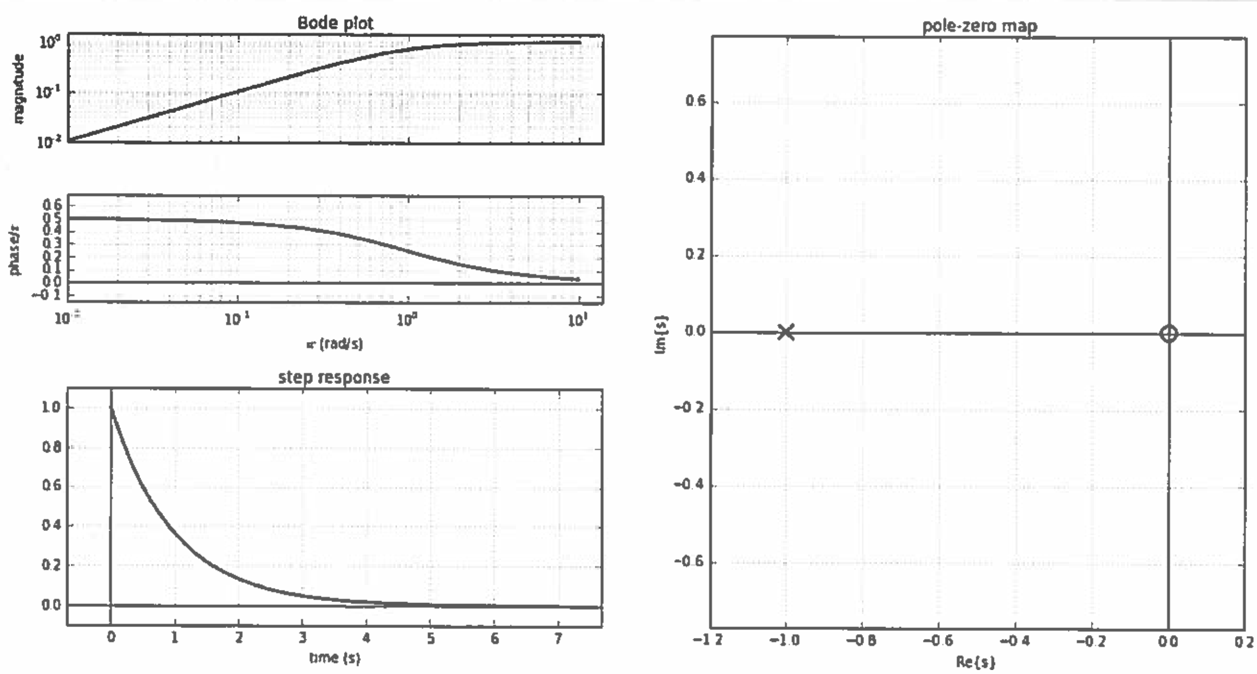
In [1]:
```
%matplotlib inline
%run convenience.ipynb
from __future__ import print_function
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal

np.set_printoptions(precision=2,suppress=True) # numpy output options

pi=np.pi
j=1j
```

Part A

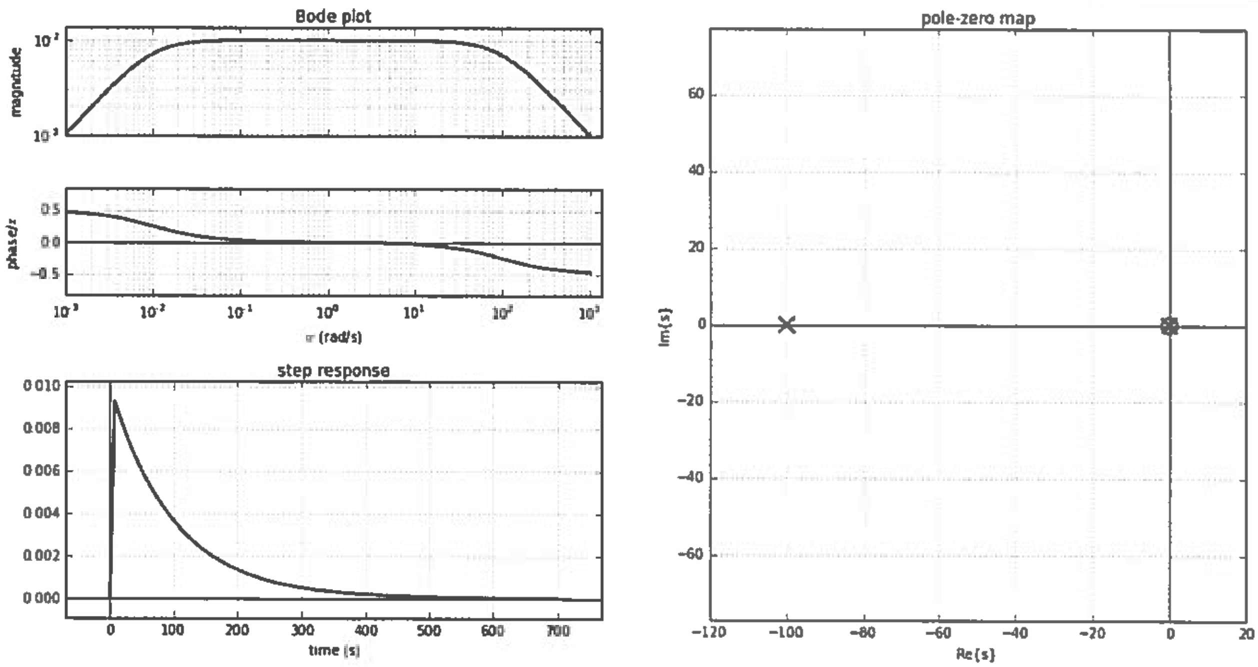
In [21]:
```
combinedplot(signal.lti([1,0],[1,1]))
```



This system is a first-order system, and has the characteristics of a high-pass filter. The system approaches 1 as w approaches infinity, and the system approaches 0 as w approaches 0. In addition, the step response approaches 0, as time approaches 0, and the step response approaches 1 when the w approaches 0, which shows decay as time goes by. In addition, the bode plot shows that lower freqencies are cut-off. There is one pole around -1, and one zero around 0. Lastly, there are no oscillations, and thus, is stable.

Part B

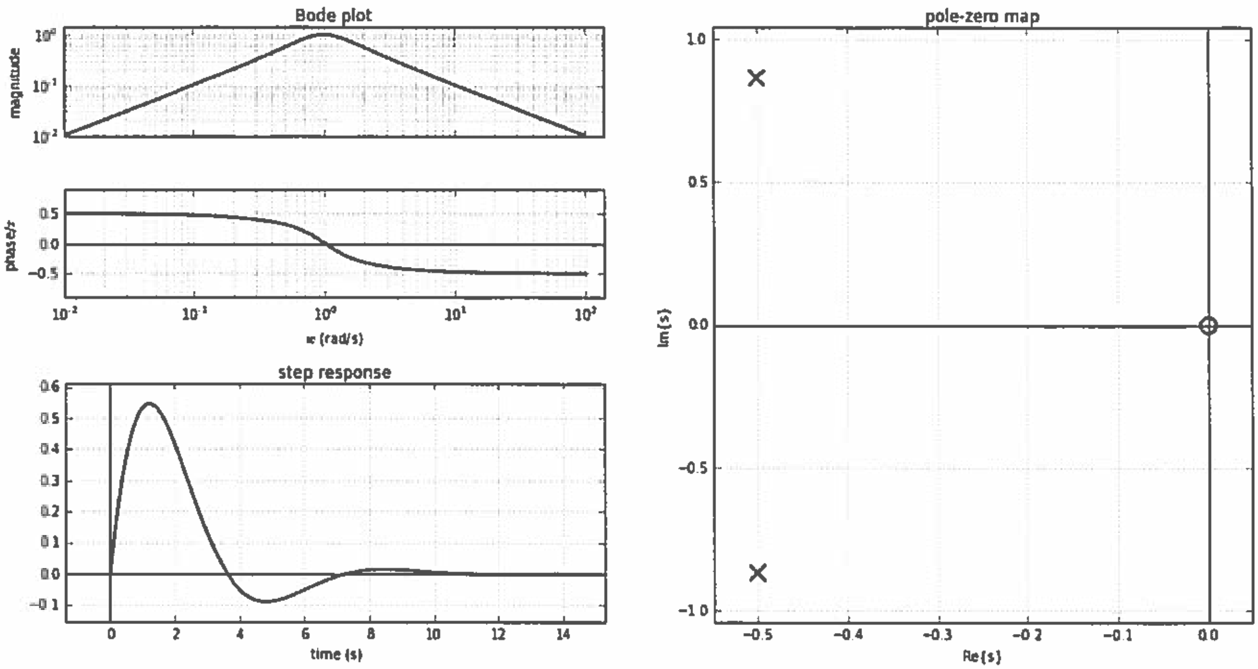
```
In [37]: combinedplot(signal.lti([1,0],[1,100,1]))
```



This system is a second-order system and has the characteristics of a band-pass filter, in that all frequencies after 1/100 and before -1/100 are attenuated (approaching 0 magnitude). There are no oscillations present, and there is over-dampening happening on this pretty stable system. There are two poles, located around -100 and 0, and one zero located around 0.

Part C

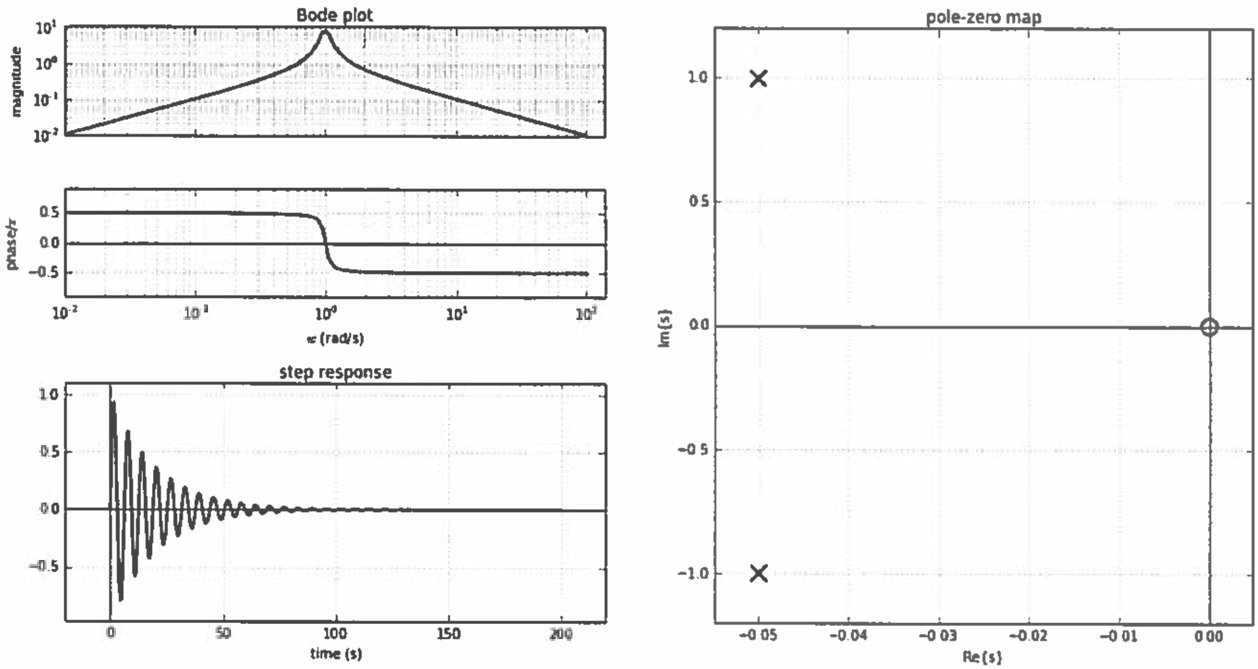
In [26]:
```
combinedplot(signal.lti([1,0],[1,1,1]),T=np.arange(0,14,.1),w=np.logsp
ace(-2,2,100),axis='tight')
```



This system is a second-order system and has the characteristics of a band-pass filter that has a small oscillation that indicates underdampening that is not too intense but means that the system is slightly unstable, but mostly stable. There is one complex pole located around -0.5-0.75j, and a zero located at 0. The zero location makes sense in that the numerator of the step response is only zero when s is zero. The band-pass filter can be characterized by the fact that there is a decay a little after and a little before, which demonstrates that lower and higher frequencies than that region are attenuated. Since the real number for poles are negative, the system is stable.

Part D

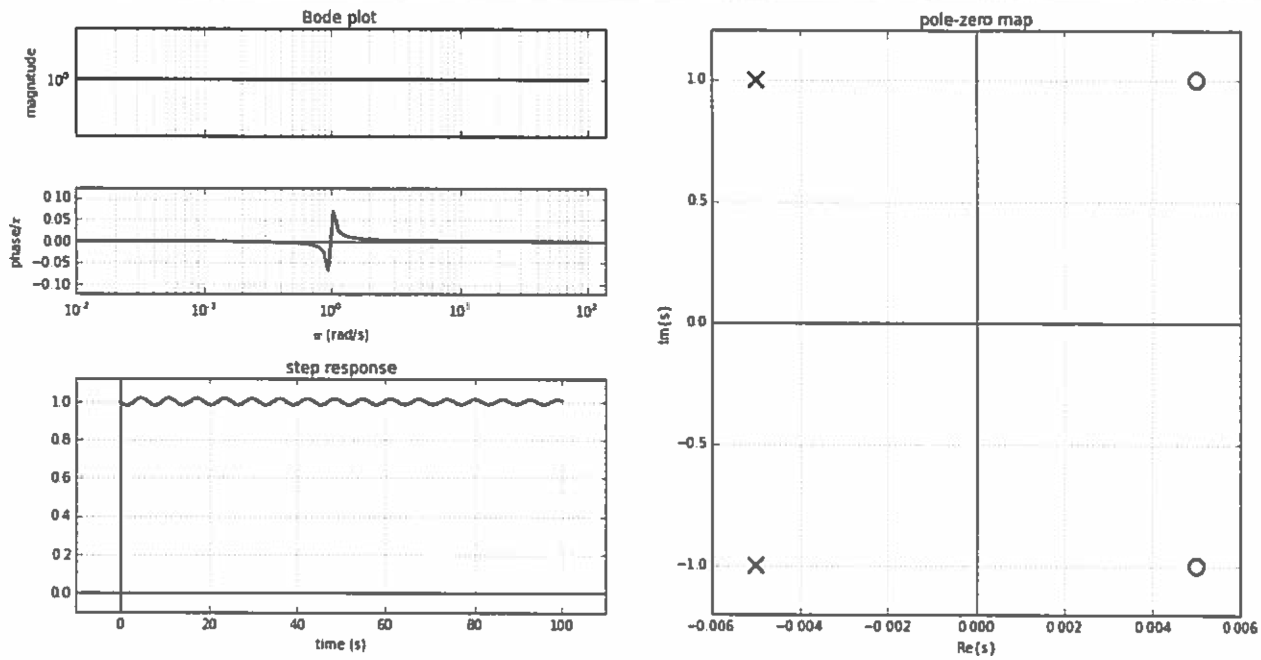
In [22]:
```
combinedplot(signal.lti([1,0],[1,.1,1]),T=np.arange(0,200,.1),w=np.log
space(-2,2,100),axis='tight')
```



This system is a second-order system, and has the characteristic of a band-pass filter that is unstable at first and has a lot of oscillations, but the oscillations stabilize eventually. The frequencies almost immediately before and after 1 are attenuated, and the decay in magnitude happens very quickly. There are two complex poles, located at -0.05+1j and -0.05-1j, and one zero located at 0. The real numbers of the poles are negative, which indicates stability

Part E

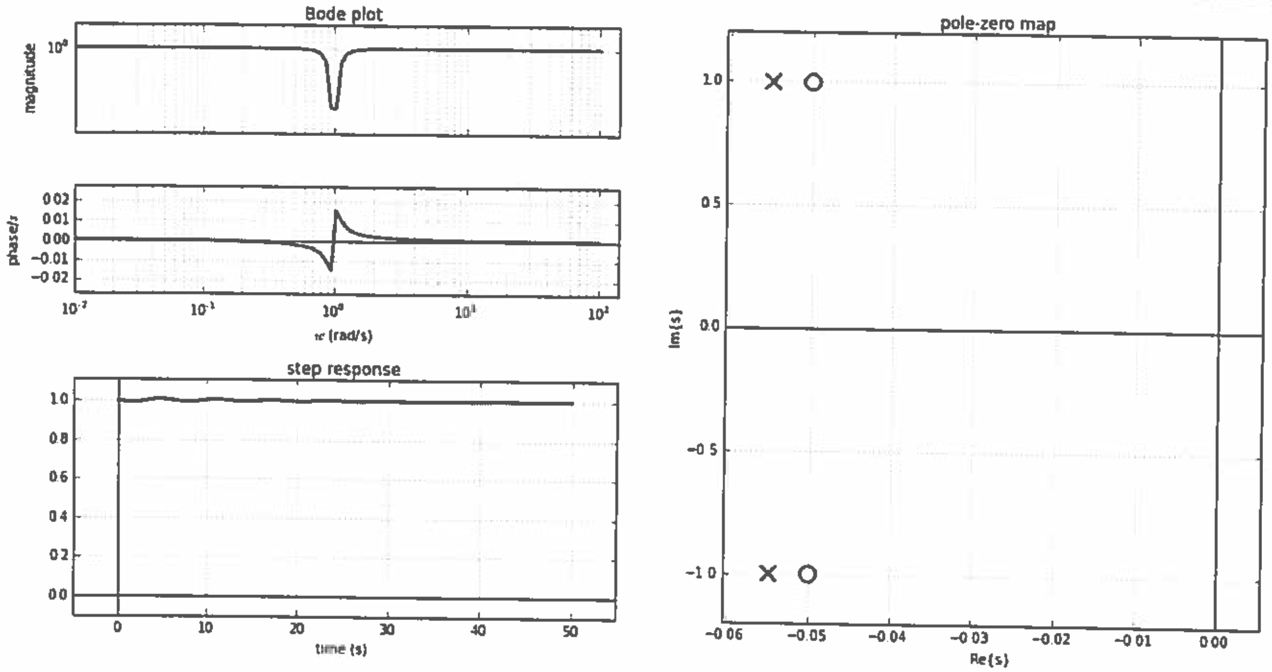
In [41]:
```
combinedplot(signal.lti([1,-0.01,1],[1,.01,1]),T=np.arange(0,10
0,.1),w=np.logspace(-2,2,100),axis='tight')
```



This system is a second-order system that doesn't quite fit any filter characteristic (band-pass, high and low), but preserves all of the frequencies. The step response remains very much at 1 at all times, and the magnitude remains at 1 at all w's. There are two complex poles, located at -0.005+1j and -0.005-1j. There are also two zeros, located at 0.005+1j and -0.005-1j. The step response and magnitude remaining about zero makes sense, in that the numerator and denominator are very close, and at all s's, the system is about 0. Neg-real poles = stability

Part F

In [46]:
```
combinedplot(signal.lti([1,0.1,1],[1,.11,1]),T=np.arange(0,50,.1),w=np.logspace(-2,2,100),axis='tight')
```



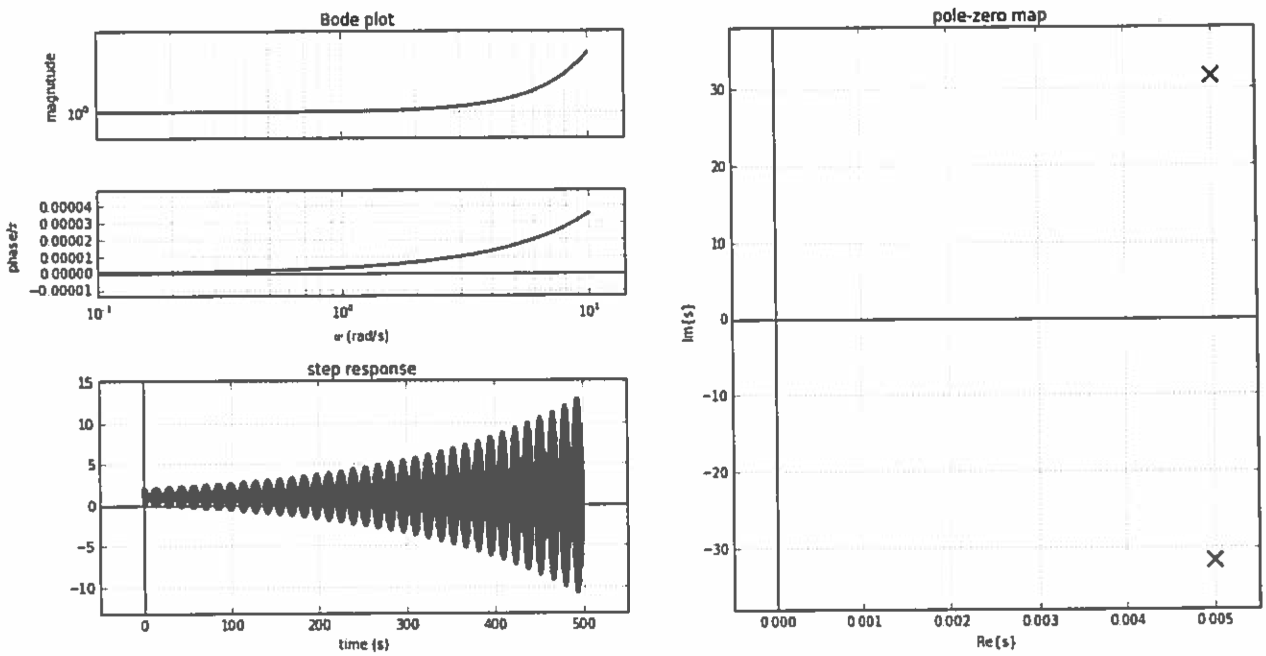
This system is a second-order system that has the characteristic of a notch filter, in that only the frequencies a little after and a little before an w of 1 is passed through, and everything in between is attenuated. The step response stays even closer to zero, and there are two complex poles and two complex zeros. The complex poles are located around -0.055+1j and -0.055-1j, and the complex zeros are located around -0.05+1j and -0.05-1j. Negative poles = stability.

(4) Try to stabilize this system:

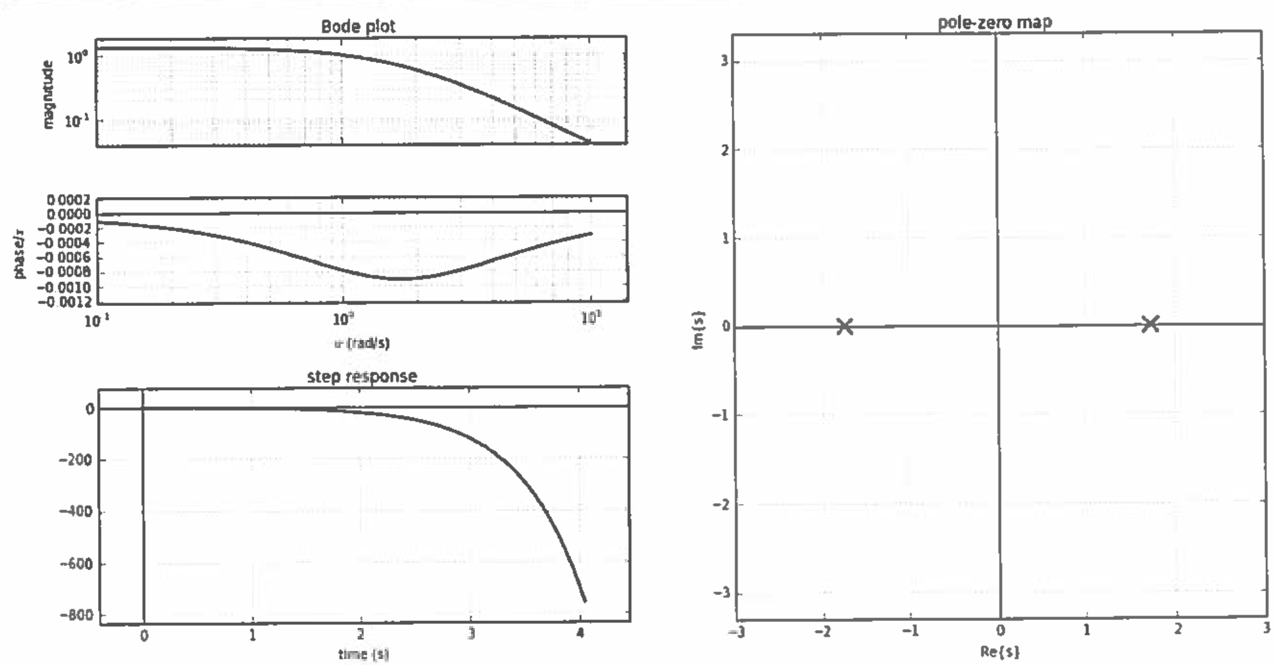
$$H(s) = \frac{1}{s^2 - 0{,}01s + 1}$$

A) Plot step response and pole-zero map:
At a very big value of k, and very small value of k:

In [86]:
```
k=1000
combinedplot(signal.lti([k],[1,-0.01,1+k]),T=np.arange(0,500,.1),w=n
p.logspace(-1,1,100),axis='tight')
```



In [74]:
```
k=-4.000001
combinedplot(signal.lti([k],[1,-0.01,1+k]))
```



B) The effect of the use of proportional control on the system is as follows:

No matter how high or small the feedback gain is, the step response is always unstable.

Calculation of the

$$\frac{Y}{Y_{sp}} = \frac{KH}{1+KH} = \frac{K\left(\frac{1}{s^2-0.01s+1}\right)}{1+K\left(\frac{1}{s^2-0.01s+1}\right)} = \frac{\frac{K}{\cancel{s^2-0.01s+1}}}{\frac{s^2-0.01s+1+K}{\cancel{s^2-0.01s+1}}} = \frac{K}{s^2-0.01s+1+K}$$

no zeros, because no s that makes numerator 0.

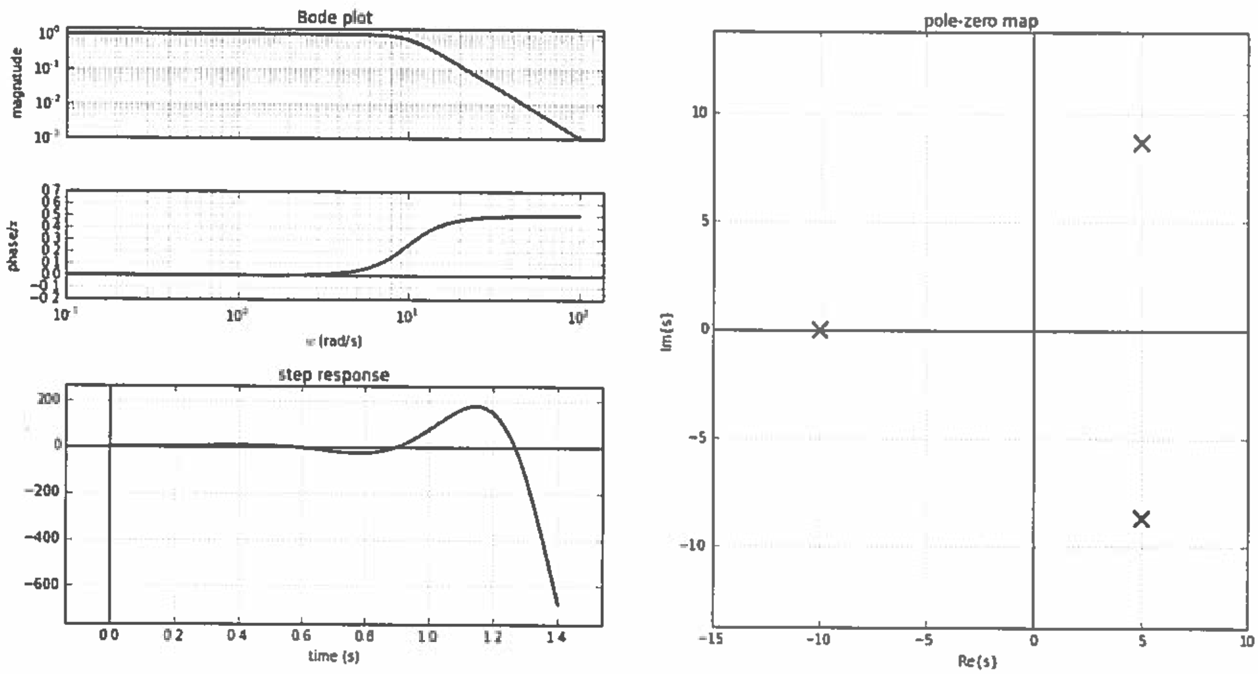
poles: $s^2 - 0.01s + (1+K) = 0$

$$s = \frac{0.01 \pm \sqrt{(0.01)^2 - 4 - K}}{2} = \frac{0.01 \pm \sqrt{0.0001 - 4 - K}}{2} = \frac{0.01 \pm \sqrt{-3.9999 - K}}{2}$$

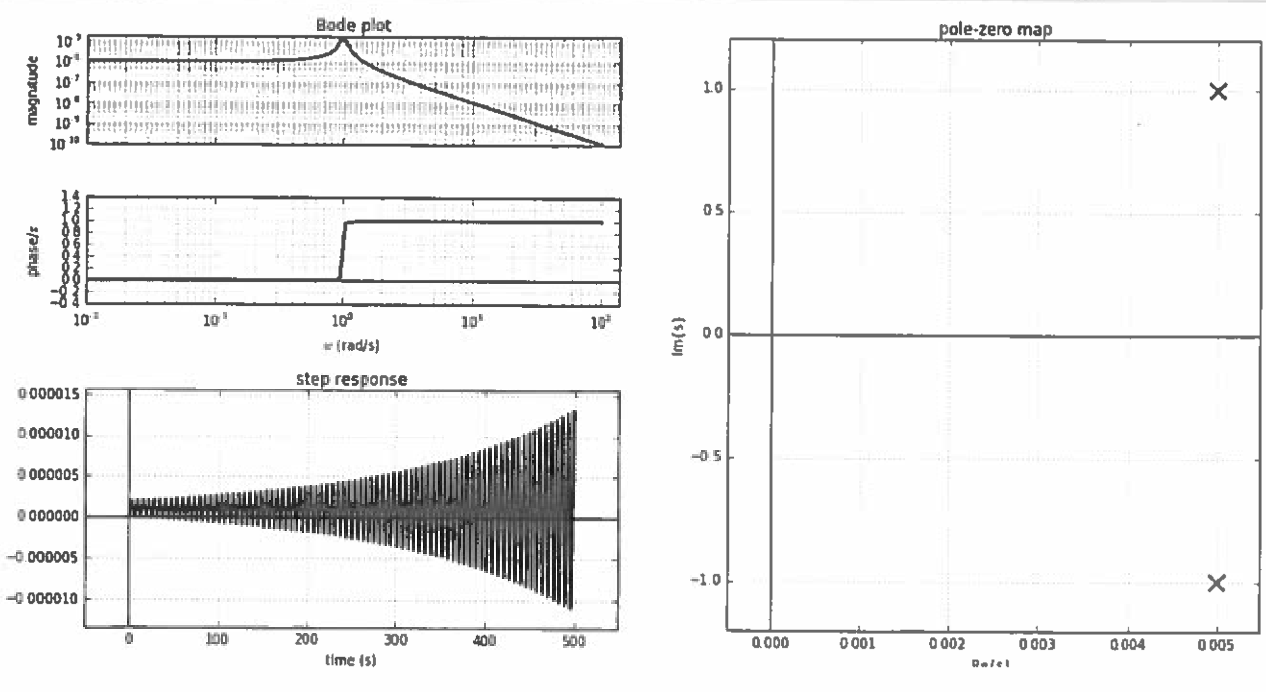
No matter what value of K, the poles will never be real on the left side of the pole-zero map, so there are always at least one positive pole numbers which means that stability and ideal system is never obtained

c) Step response and pole-zero map with integral control applied:

In [75]:
```
k=1000
combinedplot(signal.lti([k],[1,-0.01,1,k]))
```



In [87]:
```
k=0.000001
combinedplot(signal.lti([k],[1,-0.01,1+k]),T=np.arange(0,500,.1),w=n
p.logspace(-2,2,100),axis='tight')
```



Using Black's formula to find the transfer function:

$$\frac{Y}{Y_{sp}} = \frac{\frac{k}{s}H}{1+\frac{k}{s}H} = \frac{\frac{k}{s}\left(\frac{1}{s^2-0.01s+1}\right)}{1+\frac{k}{s}\left(\frac{1}{s^2-0.01s+1}\right)}$$

$$= \frac{\frac{k}{s}\left(\frac{1}{s^2-0.01s+1}\right)}{\frac{s^3-0.01s^2+s+k}{s(s^2-0.01s+1)}}$$

$$= \frac{K}{s^3-0.01s^2+s+K}$$

Zeros: none

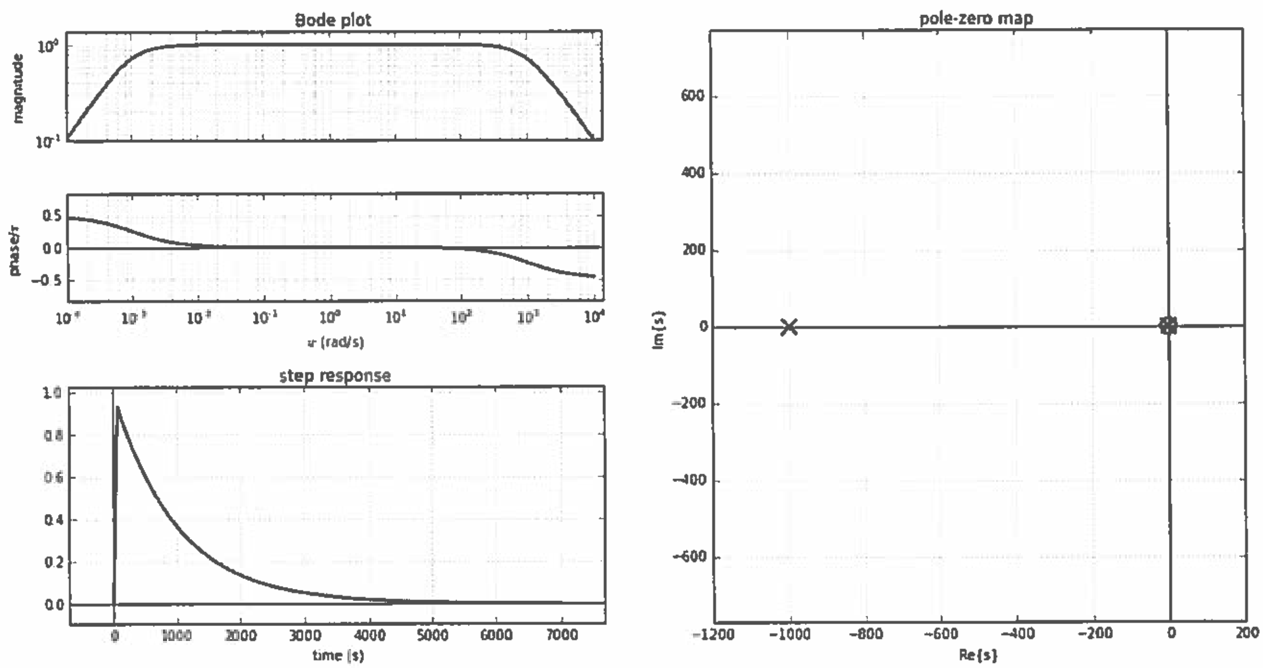
poles: $$s \approx \frac{(49.9983 \pm 86.5997i)}{\sqrt[3]{-13500000k+5196.15\sqrt{6750000k^2+44999k+499975}-4449}} - \frac{(0.00166667 \pm 0.00288675i)}{\sqrt[3]{-13500000k+5196.15\sqrt{6750000k^2+44999k+49975}-4449}}$$

$$s \approx 0.00333333 + 0.00333333\sqrt[3]{5196.15\sqrt{6750000k^2+44999k+499975}-13500000k-44999} - \frac{99.9967}{\sqrt[3]{5196.15\sqrt{6750000k^2+44999k+499975}-13500000k-44999}}$$

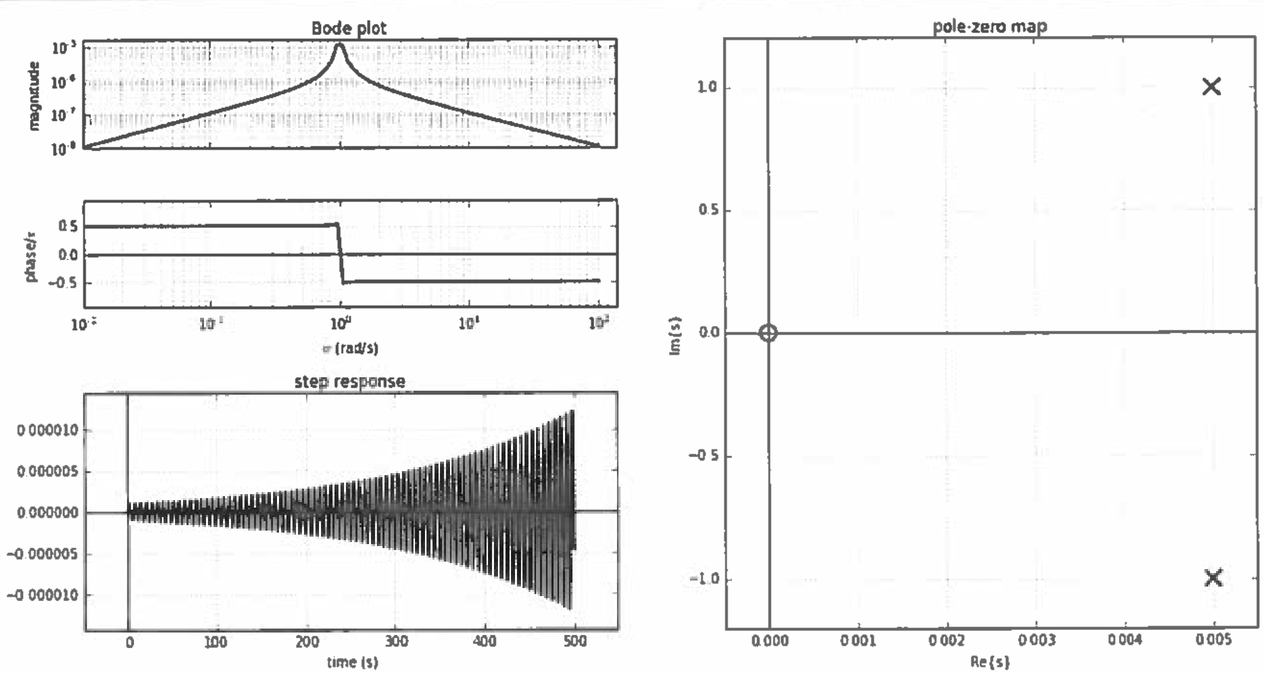
This system can never stabilize, no matter how large ~~k~~ or small K is, because the step response will always approach negative infinity and the poles of the system will never be all ~~the~~ the negative real numbers with region of convergence to the right.

d) The pole-zero map and step response with differential control applied:

In [61]:
```
k=1000
combinedplot(signal.lti([k,0],[1,-0.01+k,1]))
```



In [88]:
```
k=0.000001
combinedplot(signal.lti([k,0],[1,-0.01+k,1]),T=np.arange(0,500,.1),w=n
p.logspace(-2,2,100),axis='tight')
```



Using Black's formula to find overall transfer function:

$$\frac{Y}{Y_{sp}} = \frac{skH}{1+skH} = \frac{sk\left(\frac{1}{s^2-0.01s+1}\right)}{1+sk\left(\frac{1}{s^2-0.01s+1}\right)}$$

$$= \frac{\frac{sk}{\cancel{s^2-0.01s+1}}}{\frac{s^2-0.01s+1+sk}{\cancel{s^2-0.01s+1}}}$$

$$= \frac{sk}{s^2-0.01s+1+sk}$$

zeros: $s = 0$

poles: $s^2+(-0.01+k)s+1=0$

$$s = \frac{0.01-k \pm \sqrt{(-0.01+k)^2-4}}{2}$$

The higher the value of k, the more stable the system becomes, because your poles are more negative in real number, but has a region of convergence going to the right.