Reinforcement Learning

强化学习是一类算法, 是让计算机实现从一开始什么都不懂, 脑袋里没有一点想法, 通过不断地尝试, 从错误中学习, 最后找到规律, 学会了达到目的的方法. 这就是一个完整的强化学习过程.

Reinforcement Learning is a branch of Machine Learning, also called Online Learning. It is used to solve interacting problems where the data observed up to time t is considered to decide which action to take at time t + 1.

Desired outcomes provide the AI with reward, undesired with punishment. Machines learn through trial and error.

Multi-Armed Bandit (多臂老虎机)

给定 K 个老虎机，每台老虎机的赢钱概率不同。在不确定哪台老虎机赢钱概率最高的情况下，每玩一台老虎机，在损失一次机会的同时我们以一定的概率收到报酬。假定第 $i = 1, 2, 3, \ldots K$ 个老虎机给我们报酬 $r \in R$ 的概率是 $P_i(r)$ ，且报酬的均值为 h_i 。那么决策便是，给定有限的机会次数 T ，如何玩这些老虎机才能使得期望累积收益最大化。

医生如果只给病人开出目前疗效最好的药，那么也许疗效更好的新药永远也得不到测试。但如果不停地试新药，新药也许效果不好，病人会怨声载道。如何trade-off “探索”(exploration)新鲜事和“利用”(exploitation)已有知识 就是多臂老虎机问题的核心。

探索（Exploration）vs 利用（Exploitation）

先考虑一种极端的方式，那就是均匀的玩每台老虎机，这样可以保证对每台老虎机的收益情况我们都能足够了解。然而，这往往会浪费不必要的资源在太差的老虎机上，就像无头苍蝇一样。这种策略被称之为**探索（exploration）**，很显然如果信息太过于匮乏的话，这种策略不失为一种好方法。

再考虑另一个极端的方式，那就是只玩当前给我们收益报酬最高的那台老虎机，显然这一策略可以在初期较快的获得更高的回报，然而却因为过度贪婪捡了芝麻丢了西瓜，以至于长远错过真正好的老虎机，就像辛勤的蜜蜂一样。这种策略被称之为 **利用 （exploitation）**，很显然如果信息足够充分的话，这种策略不失为一种好方法。

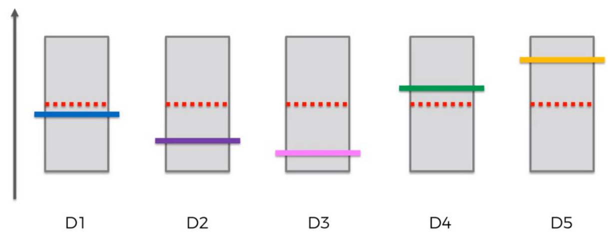
最好的策略显然不是这二者之一，中和这两种截然矛盾的资源分配策略可以给我们更好的思路。比如说，前期信息匮乏，我们采用更多的探索；而后期，信息了解差不多后，我们转向利用，诸如高中生以及博士生的关系。再比如说，再边利用的同时也进行探索，诸如Google的传统部门以及Google X之间的关系。**但这里值得一提的是，如果问题规模过大，比如说资源的数量不足以支撑探索尽量多的信息，那么反而利用是一种更『现实』的策略。**

- We have d arms. For example, arms are ads that we display to users each time they connect to a web page.
- Each time a user connects to this web page, that makes a round.
- At each round n, we choose one ad to display to the user.
- At each round n, ad_i gives reward $r_i(n) \in \{0, 1\}$: $r_i(n) = 1$ if the user clicked on the ad_i, 0 if the user didn't.
- Our goal is to maximize the total reward we get over many rounds.

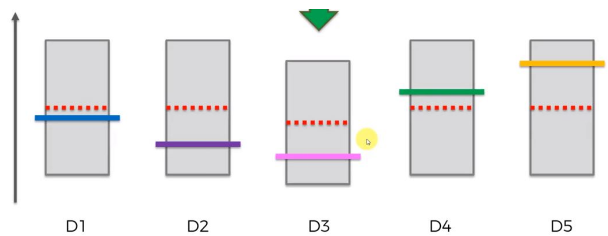
Upper Confidence Bound (UCB)

在开始阶段，每个对象都给定相同的 Expected Value(红虚线)以及 Confidence Bound(灰框)。 随机选择对象，如果该对象成功(赢钱，或者 user clicked)，则 EV 和 CB 上移，且 CB 范围缩减(只要执行该对象，不论成功失败，CB都会缩减)。如果对象失败，则 EV 和 CB 下移，且 CB 缩减。

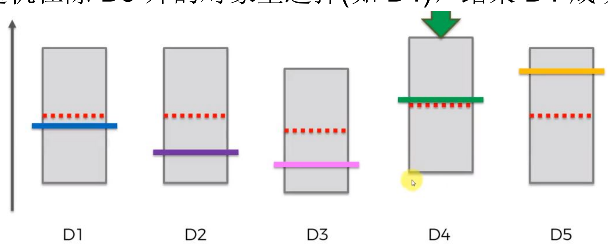
初始设定(彩色实线是测量后的 distribution 参考线)



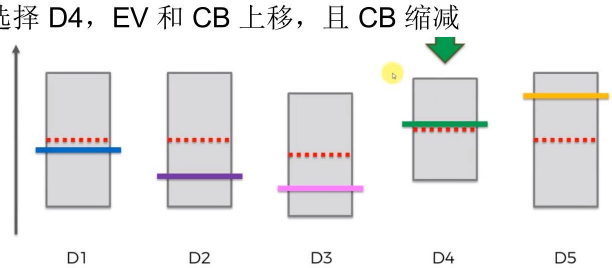
Round 1: 随机选择其中一个对象(如 D3)，结果 D3 失败(输钱或者 user 没有 click)。EV 和 CB 下移，并且 CB 缩减



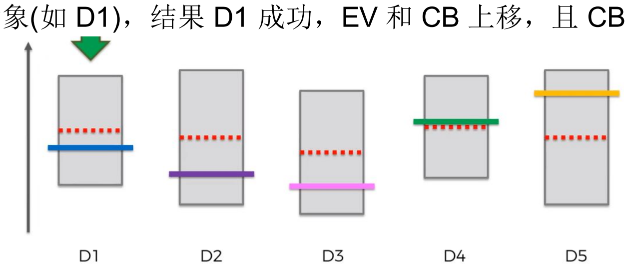
Round 2: 因为 D3 失败，所以随机在除 D3 外的对象里选择(如 D4)，结果 D4 成功。EV 和 CB 上移，且 CB 缩减



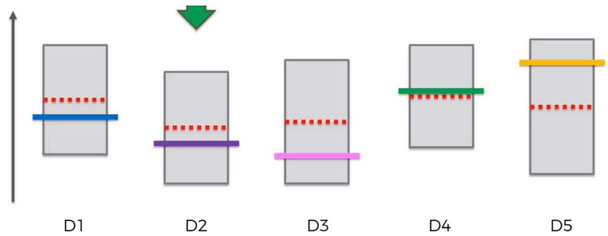
Round 3: 因为 D4 成功，继续选择 D4，EV 和 CB 上移，且 CB 缩减



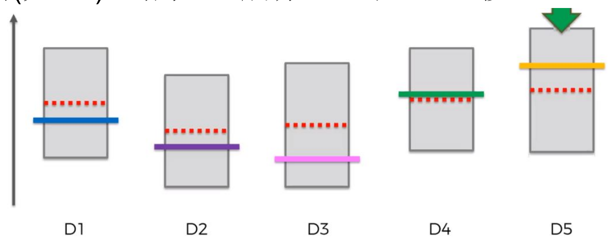
Round 4: 随机选其他剩下的对象(如 D1)，结果 D1 成功，EV 和 CB 上移，且 CB 缩减



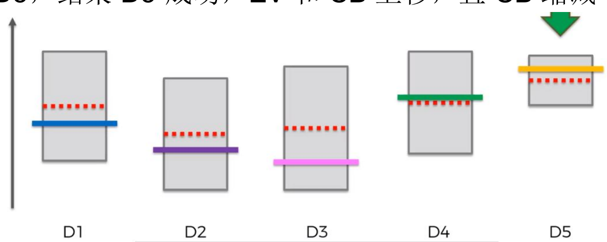
Round 5: 再随机选择(如 D2)，结果 D2 失败，EV 和 CB 下移，且 CB 缩减



Round 6: 随机选择或者选剩下的(如 D5)，结果 D5 成功，EV 和 CB 上移，且 CB 缩减



Round 7: 继续选择，或者利用 D5，结果 D5 成功，EV 和 CB 上移，且 CB 缩减



循环几轮后，可以当 CB 缩减到一定程度后，选择当前最优进行利用(Exploitation)

STEP 1: At each round n, we consider two numbers for each ad_i:

- $N_i(n)$ - the number of times the ad_i was selected up to round n.
- $R_i(n)$ - the sum of rewards of the ad_i up to round n.

STEP 2: From these two numbers we compute:

- the average reward of ad_i up to round n
  $\bar{r}_i(n) = \frac{R_i(n)}{N_i(n)}$
- the confidence interval [$\bar{r}_i(n) - \Delta_i(n), \bar{r}_i(n) + \Delta_i(n)$] at round n with
  $\Delta_i(n) = \sqrt{\frac{3}{2}\frac{log(n)}{N_i(n)}}$

STEP 3: We select the ad_i that has the maximum UCB: $\bar{r}_i(n) + \Delta_i(n)$

$$\frac{R_i(n)}{N_i(n)} + \sqrt{\frac{3}{2}\frac{log(n)}{N_i(n)}}$$

i 表示当前的臂，n 表示目前的尝试次数，$N_i(n)$ 表示臂 i 被选中的次数。公式加号左边表示臂 i 当前的平均收益，右边表示该收益的 Bonus ，本质上是均值的标准差，反应了候选臂效果的不确定性，就是置信区间的上边界。

使用 UCB 算法的流程如下：

1. 对所有臂先尝试一次
2. 按照公式计算每个臂的最终得分

3. 选择得分最高的臂作为本次结果

直观理解下 UCB 算法为什么有效？

- 当一个臂的平均收益较大时，也就是公式左边较大，在每次选择时占有优势
- 当一个臂被选中的次数较少时，即 $N_i(n)$ 较小，那么它的 Bonus 较大，在每次选择时占有优势

所以 UCB 算法倾向选择被选中次数较少以及平均收益较大的臂。

UCB in Python

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

The dataset is a simulation data

In [2]:
```
dataset = pd.read_csv('Ads_CTR_Optimisation.csv')
dataset.head()
```

Out[2]:

	Ad 1	Ad 2	Ad 3	Ad 4	Ad 5	Ad 6	Ad 7	Ad 8	Ad 9	Ad 10
0	1	0	0	0	1	0	0	0	1	0
1	0	0	0	0	0	0	0	0	1	0
2	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0	0	0	0

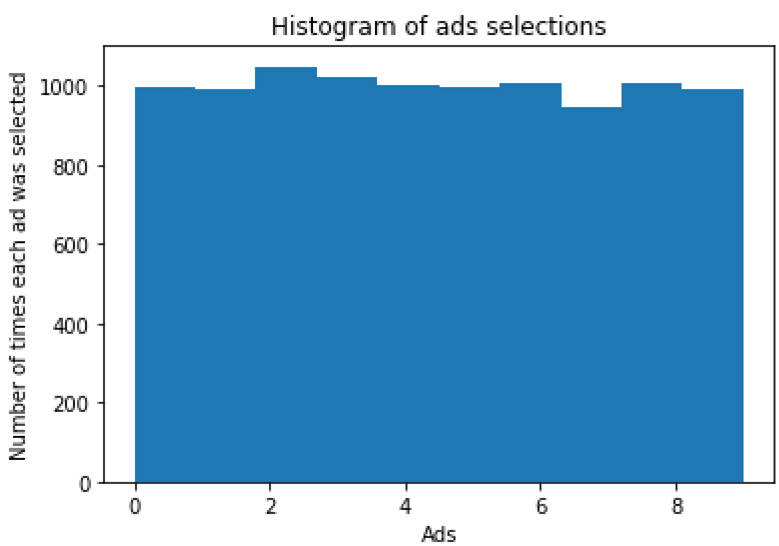
战略1：随机展示 (Random Selection)

有10个版本的广告(ads)。每次用户登入账号就会随机展示其中一个广告，如果用户点击，则记'1'，不点，则记'0'。不考虑探索和利用。

In [3]:
```
import random
N = 10000
d = 10
ads_selected = []
total_reward = 0
for n in range(0,N):
    ad = random.randrange(d)  # 从10个广告中随机抽取。
    ads_selected.append(ad)
    reward = dataset.values[n,ad]  # 根据 dataset 来判断用户是否点击了当前广
告。
    total_reward = total_reward + reward  # 最终总共有多少点击
print(total_reward)
```

```
1264
```

In [4]:
```
# Visualising the Results - Histogram
plt.hist(ads_selected)
plt.title('Histogram of ads selections')
plt.xlabel('Ads')
plt.ylabel('Number of times each ad was selected')
plt.show()
```



战略2：UCB

STEP 1: At each round n, we consider two numbers for each ad_i:

- $N_i(n)$ - the number of times the ad_i was selected up to round n.
- $R_i(n)$ - the sum of rewards of the ad_i up to round n.

In [5]:
```python
import math

# A vector contains each of those numbers of selections of ad_i
d = 10 # d 为 10 个广告数
numbers_of_selections = [0] * d
# The different sums of rewards of each 10 ads at each round
sums_of_rewards = [0] * d
# this vector will give us the list of all the different ad version
# that are selected at each round
ads_selected = []
# reward总数
total_reward = 0
```

STEP 2: From these two numbers we compute:

- the average reward of ad_i up to round n
  $\bar{r}_i(n) = \frac{R_i(n)}{N_i(n)}$
- the confidence interval $[\bar{r}_i(n) - \Delta_i(n), \bar{r}_i(n) + \Delta_i(n)]$ at round n with
  $\Delta_i(n) = \sqrt{\frac{3}{2}\frac{log(n)}{N_i(n)}}$

STEP 3: We select the ad_i that has the maximum UCB: $\bar{r}_i(n) + \Delta_i(n)$

In [6]:
```
for n in range(0,dataset.shape[0]):
    ad = 0
    max_upper_bound = 0
    for i in range(0,d): # Go through 10 ads
        if (numbers_of_selections[i] > 0):
        # Since we don't have any information at beginning,
        # we will select ad1 on round 1, ad2 on round 2... up to round 10
        # we'll use the following UCB stratige as long as we have informati
on
        # (after first 10 rounds)
            # STEP 2
            average_reward = sums_of_rewards[i]/numbers_of_selections[i]
            # in Python, index starts from 0, so n+1 in log function
            delta_i = math.sqrt(3/2*math.log(n+1)/numbers_of_selections[i])
            # STEP 3
            upper_bound = average_reward + delta_i
        else: # for the 1st 10 rounds
            upper_bound = 1e400
        if upper_bound > max_upper_bound:
            max_upper_bound = upper_bound
            ad = i
    # this vector contains all the different ad selected at each round
    ads_selected.append(ad)
    # After updating ads selection, then we need update numbers_of_selectio
n
    numbers_of_selections[ad] = numbers_of_selections[ad] + 1
    # Since we only have simulation data, then we go through it and give th
e
    # rewards for which ad has been selected in the simulation dataset
    # 根据 dataset 来判断用户是否点击了当前广告。
    # (0 if non-select, 1 if select)
    reward = dataset.values[n,ad]
    # Update sums_of_rewards after ad selected at each round
    sums_of_rewards[ad] = sums_of_rewards[ad] + reward
    # Performance Evaluating
    total_reward = total_reward + reward
```

In [7]:
```
print(total_reward)
```

```
2178
```

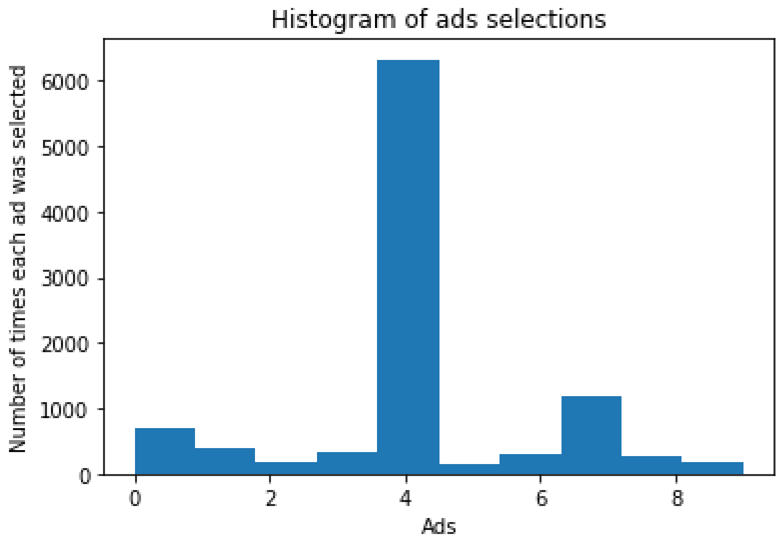
查看最佳广告：根据 EE (探索和利用) 原因，UCB最后停留在“利用”上，即最佳广告会在最后被一直点击。

最佳广告为 4。

In [8]:
```
ads_selected[-10:]
```

Out[8]:
```
[4, 4, 4, 4, 4, 4, 4, 4, 4, 4]
```

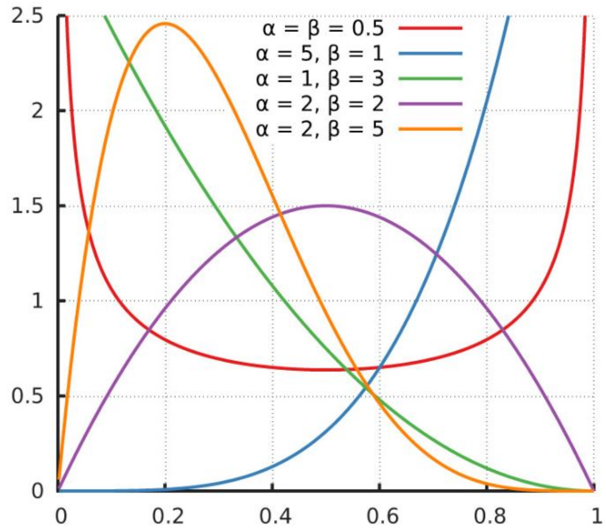
In [9]:
```
# Visualising the Results - Histogram
plt.hist(ads_selected)
plt.title('Histogram of ads selections')
plt.xlabel('Ads')
plt.ylabel('Number of times each ad was selected')
plt.show()
```



Thompson Sampling

介绍汤普森采样之前，可以先来介绍一个分布：$beta$ 分布。$beta$ 分布可以看作一个概率的概率分布，当你不知道一个东西的具体概率是多少时，它可以给出了所有概率出现的可能性大小。

$beta$ 分布有两个控制参数：α 和 β 。先来看下几个 $beta$ 分布的概率密度函数的图形：



$beta$ 分布图形中的 x 轴取值范围是 (0,1)，可以看成是概率值，参数 α 和 β 可以控制图形的形状和位置：

- α + β 的值越大，分布曲线越窄，也就是越集中。
- $\frac{\alpha}{\alpha+\beta}$ 的值是 $beta$ 分布的均值（期望值），它的值越大， $beta$ 分布的中心越靠近 1，否则越靠近 0 。

注意：当参数 α 和 β 确定后，使用 $beta$ 分布生成的随机数有可能不一样，所以汤普森采样法是不确定算法。

$beta$ 分布和 Bandit 算法有什么关联呢？实际上，每个臂是否产生收益的概率 p 的背后都对应一个 $beta$ 分布。**我们将 $beta$ 分布的 α 参数看成是推荐后用户的点击次数， β 参数看成是推荐后用户未点击的次数。**

来看下使用汤普森算法的流程：

1. 每个臂都维护一个 $beta$ 分布的参数，获取每个臂对应的参数 α 和 β，然后使用 $beta$ 分布生成随机数。
2. 选取生成随机数最大的那个臂作为本次结果
3. 观察用户反馈，如果用户点击则将对应臂的 α 加 1，否则 β 加 1

在实际的推荐系统中，需要为每个用户保存一套参数，假设有 m 个用户， n 个臂（选项，可以是物品，可以是策略）， 每个臂包含 α 和 β 两个参数，所以最后保存的参数的总个数是 $2mn$。

可以直观的理解下为什么汤普森采样算法有效：

- 当尝试的次数较多时，即每个臂的 α + β 的值都很大，这时候每个臂对应的 $beta$ 分布都会很窄，也就是说，生成的随机数都非常接近中心位置，每个臂的收益基本确定了。
- 当尝试的次数较少时，即每个臂的 α + β 的值都很小，这时候每个臂对应的 $beta$ 分布都会很宽，生成的随机数有可能会比较大，增加被选中的机会。
- 当一个臂的 α + β 的值很大，并且 $\frac{\alpha}{\alpha+\beta}$ 的值也很大，那么这个臂对应的 $beta$ 分布会很窄，并且中心位置接近 1 ，那么这个臂每次选择时都很占优势。

Thompson Sampling Algorithm

Step 1. At each round n, we consider two numbers for each ad i:

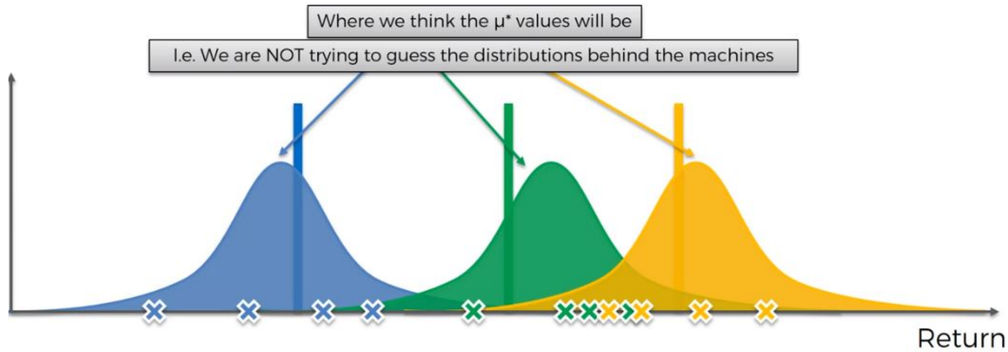
- $N_i^1(n)$ - the number of times the ad i got reward 1 up to round n,
- $N_i^0(n)$ - the number of times the ad i got reward 0 up to round n.

Step 2. For each ad i, we take a random draw from the distribution below:

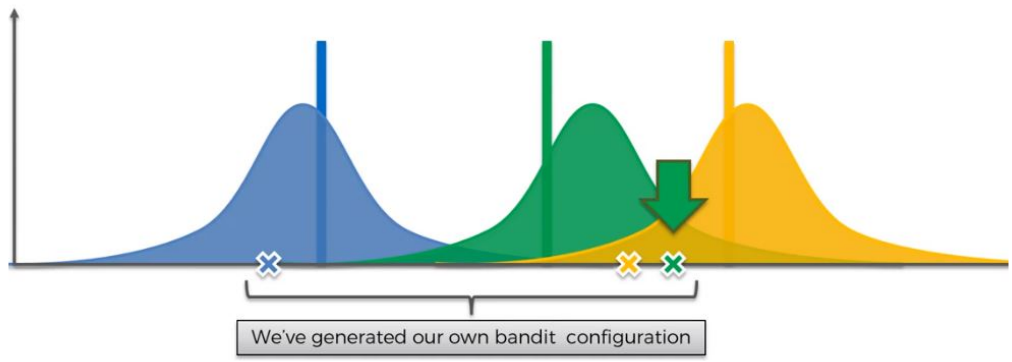
$$\theta_i(n) = \beta(N_i^1(n) + 1, N_i^0(n) + 1)$$

Step 3. We select the ad that has the highest $\theta_i(n)$.

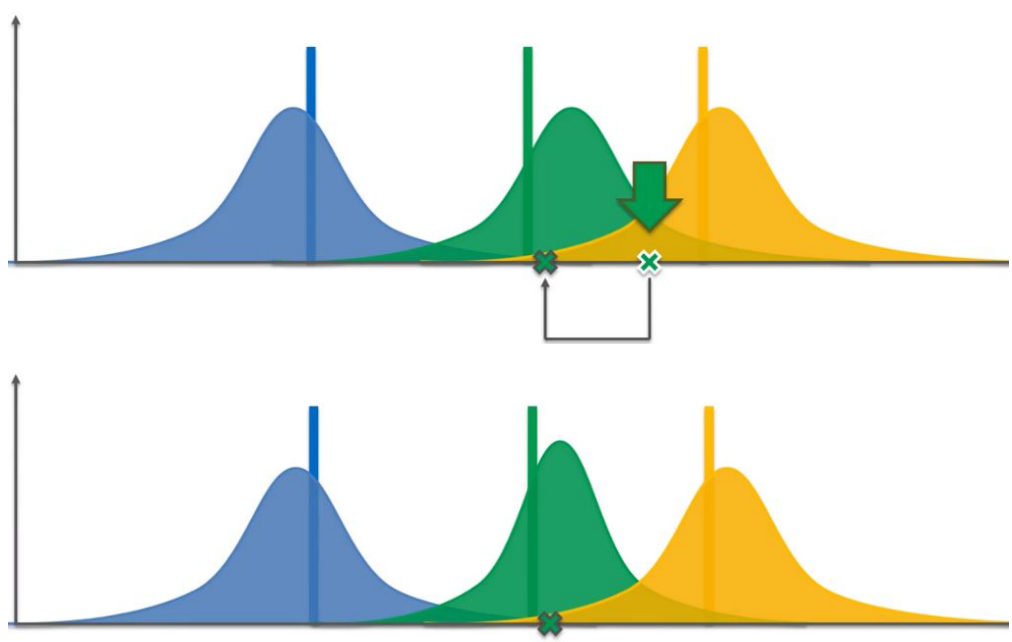
开始阶段，通过在各个对象上的尝试数次并记录结果数据。通过这些数据建立各自的 distribution 来预测各个 EV 的位置(竖直彩色实线为最终真实 Expected Value，在图中为参考线，但是仍是当前未知状态。且当前仅仅预测真实 EV 应该在 distribution 范围内，而非预测精确位置)。



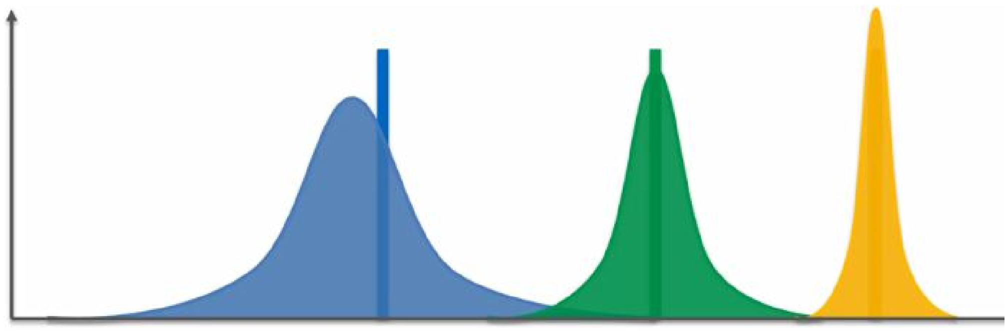
在新回合里，先从各个 distribution 里**随机**提取结果数值将其视为新回合将会获得的 EV。数值最高的对象，即为新回合将要执行的对象。(如图，绿 distribution 中选出的值最高，所以新回合将执行绿色对象)



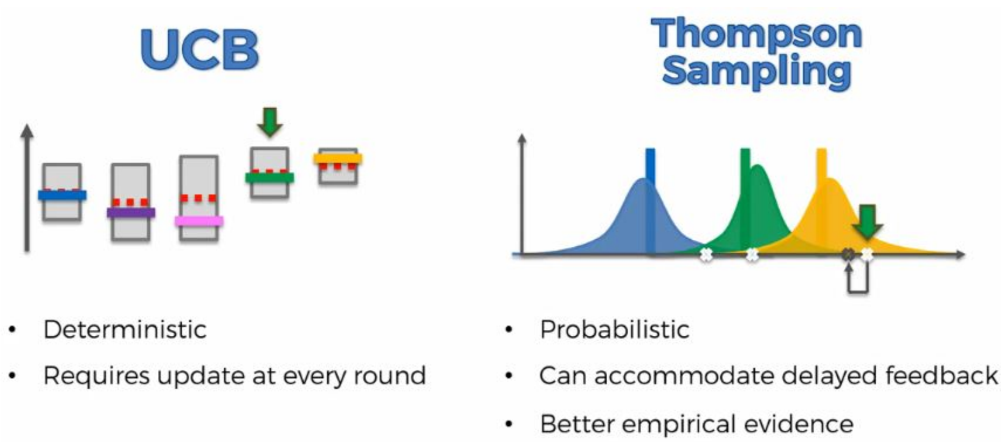
执行绿色对象后，会得到一个绿色对象的真实反馈数据，利用此数据对绿色的 distribution 进行更新。



每个回合，根据各个 distribution 提取的数值后，执行相应的对象。随后再根据产生的真实新数据来更新相对应的 distribution。不断重复此步骤，直至最佳对象的 distribution 范围收敛。



UCB vs. Thompson Sampling



Thompson Sampling in Python

战略3：Thompson Sampling

In [26]:
```
N = dataset.shape[0]
# A vector contains each of those numbers of selections of ad_i
d = 10 # d 为 10 个广告数
# this vector will give us the list of all the different ad version
# that are selected at each round
ads_selected = []
# reward总数
total_reward = 0
```

STEP 1: At each round n, we consider two numbers for each ad_i:

- $N_i^1(n)$ - the number of times the ad_i got reward 1 up to round n.
- $N_i^0(n)$ - the number of times the ad_i got reward 0 up to round n.

In [27]:
```
numbers_of_rewards_1 = [0] * d
numbers_of_rewards_0 = [0] * d
```

STEP 2: For each ad_i, we take a random draw from the $beta$ distribution below:
$\theta_i(n) = \beta(N_i^1(n) + 1, N_i^0(n) + 1)$

Bayesian Inference

- Ad i gets rewards $\mathbf{y}$ from Bernoulli distribution $p(\mathbf{y}|\theta_i) \sim \mathcal{B}(\theta_i)$.
- θ_i is unknown but we set its uncertainty by assuming it has a uniform distribution $p(\theta_i) \sim \mathcal{U}([0,1])$, which is the prior distribution.
- Bayes Rule: we approach θ_i by the posterior distribution

$$\underbrace{p(\theta_i|\mathbf{y})}_{\text{posterior distribution}} = \frac{p(\mathbf{y}|\theta_i)p(\theta_i)}{\int p(\mathbf{y}|\theta_i)p(\theta_i)d\theta_i} \propto \underbrace{p(\mathbf{y}|\theta_i)}_{\text{likelihood function}} \times \underbrace{p(\theta_i)}_{\text{prior distribution}}$$

- We get $p(\theta_i|\mathbf{y}) \sim \beta(\text{number of successes} + 1, \text{number of failures} + 1)$
- At each round n we take a random draw $\theta_i(n)$ from this posterior distribution $p(\theta_i|\mathbf{y})$, for each ad i.
- At each round n we select the ad i that has the highest $\theta_i(n)$.

θ_i can be interpreted as the numbers of successes divided by the total number of times we selected the ad_i (probability of success)

STEP 3: We select the ad that has the highest $\theta_i(n)$

In [28]:
```
# Need random library for random draw from beta distribution
import random

for n in range(0,dataset.shape[0]):
    ad = 0
    max_random = 0 # highest random draw for Thompson Sampling
    for i in range(0,d): # Go through 10 ads
        # STEP 2 :
        # Random draws from Beta distribtuion of parameters that we choose
        random_beta = random.betavariate(numbers_of_rewards_1[i] + 1,
                                         numbers_of_rewards_0[i] + 1)
        # STEP 3 : We still need take the highest random draw
        if random_beta > max_random:
            max_random = random_beta
            ad = i
    # this vector contains all the different ad selected at each round
    ads_selected.append(ad)

    # Since we only have simulation data, then we go through it and give th
e
    # rewards for which ad has been selected in the simulation dataset
    # 根据 dataset 来判断用户是否点击了当前广告。
    # (0 if non-select, 1 if select)
    reward = dataset.values[n,ad]

    # Need increment variables in STEP 1 when we got the reward or not
    if reward == 1:
        # Need increment the reward variable only for the ad which is the h
ighest random draw
        # This ad is also the ad we selected in the real world
        numbers_of_rewards_1[ad] = numbers_of_rewards_1[ad] + 1
    else:
        # Need increment if no reward
        numbers_of_rewards_0[ad] = numbers_of_rewards_0[ad] + 1
    # Performance Evaluating
    total_reward = total_reward + reward
```

Since Thompson Sampling Algo has some random factors, so the total_reward will be varied little bit.
In the case, it's averaging at 2600.

It beats UCB !!!!!

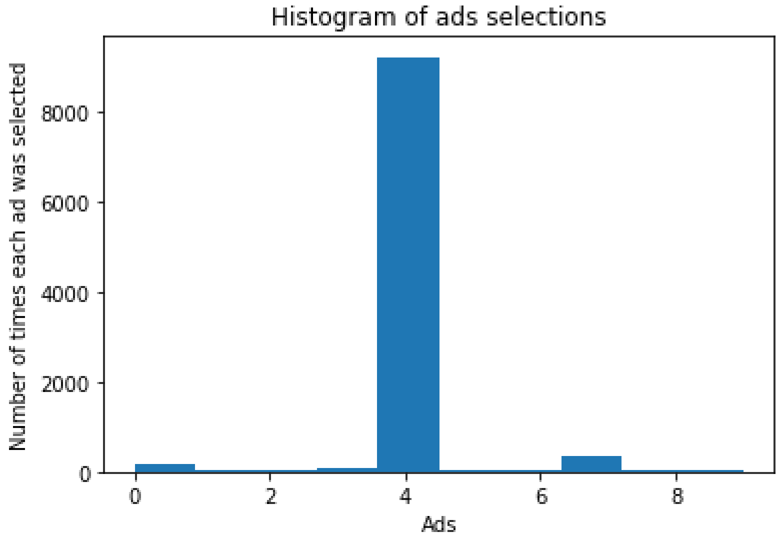
In [29]:
```
print(total_reward)
```

```
2605
```

In [30]:
```
ads_selected[-10:]
```

Out[30]:
```
[4, 0, 4, 4, 4, 4, 4, 4, 4, 4]
```

In [31]:
```
# Visualising the Results - Histogram
plt.hist(ads_selected)
plt.title('Histogram of ads selections')
plt.xlabel('Ads')
plt.ylabel('Number of times each ad was selected')
plt.show()
```



The # 5 Ad (best Ad) has been more selected than UCB.

Thompson Sampling can quickly figure out which Ad is the best to select