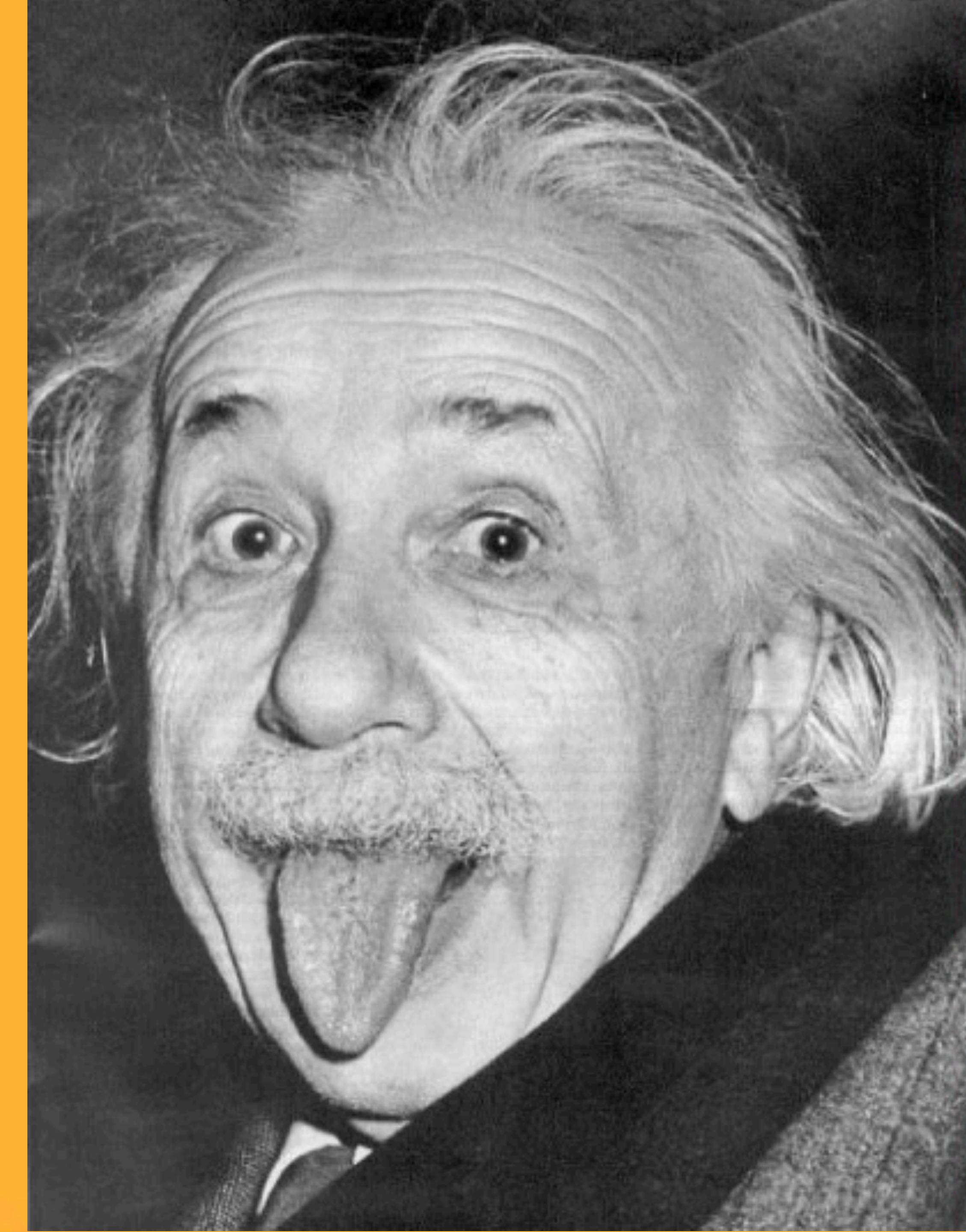


Continuous Distributions



“Not everything that can be counted, counts; not everything that counts can be counted.”

- Albert Einstein

Discrete to Continuous

Discrete distributions: Countable # values (finite or countably-infinite)

Continuous distributions: Uncountable # values, intervals

Why Continuous

Anything physics

Time

flight

delivery

disease

life

Space

height

storm area

Mass

pet

cookie

Temperature

air

body

Nearly continuous variables

Cost

stock

house

pork bellies

Rates

interest

exchange

unemployment

Probability Density Function

Replaces the discrete pmf

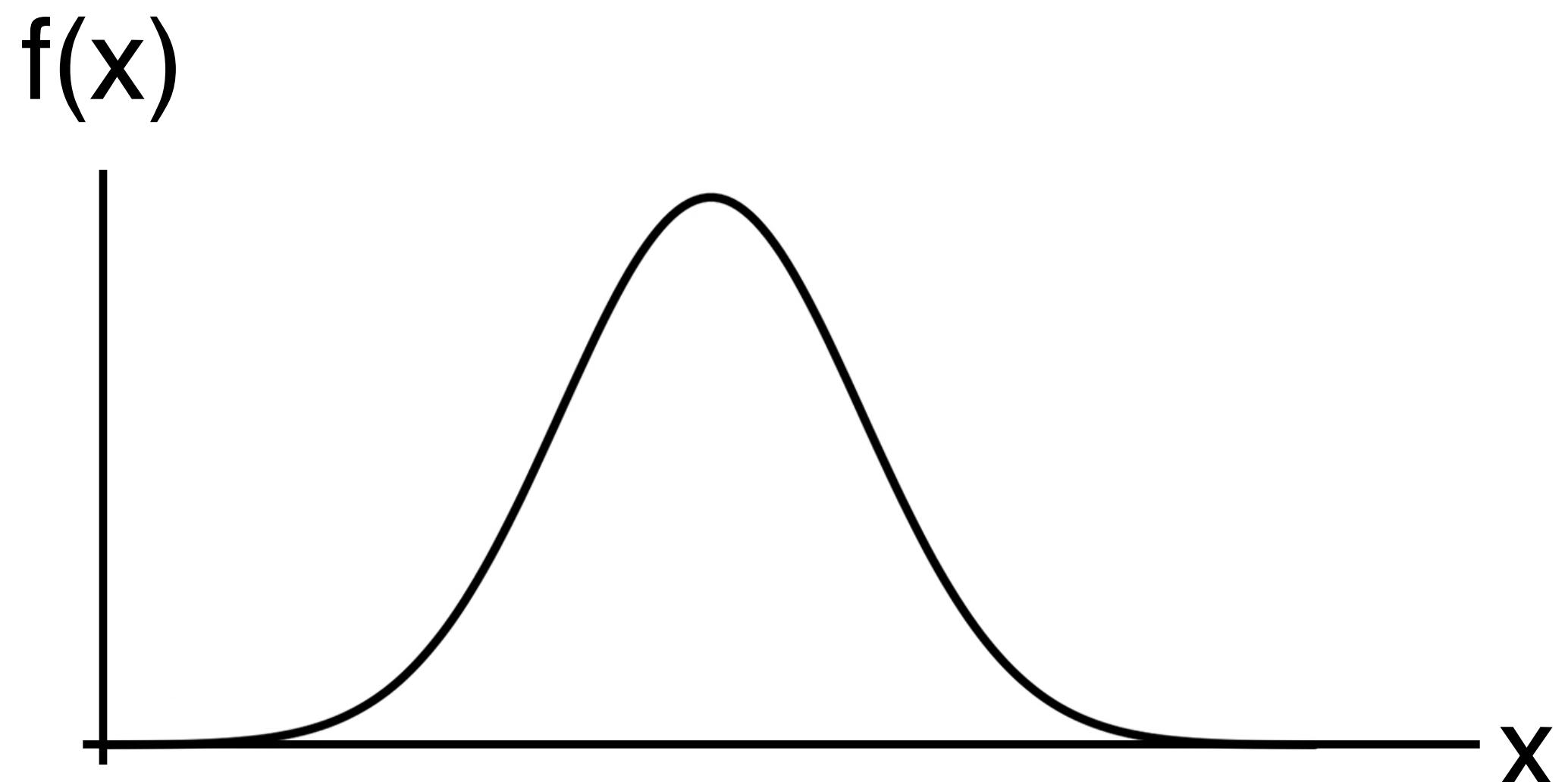
$$f(x) \geq 0$$

relative likelihood of x

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

area under curve

(area)

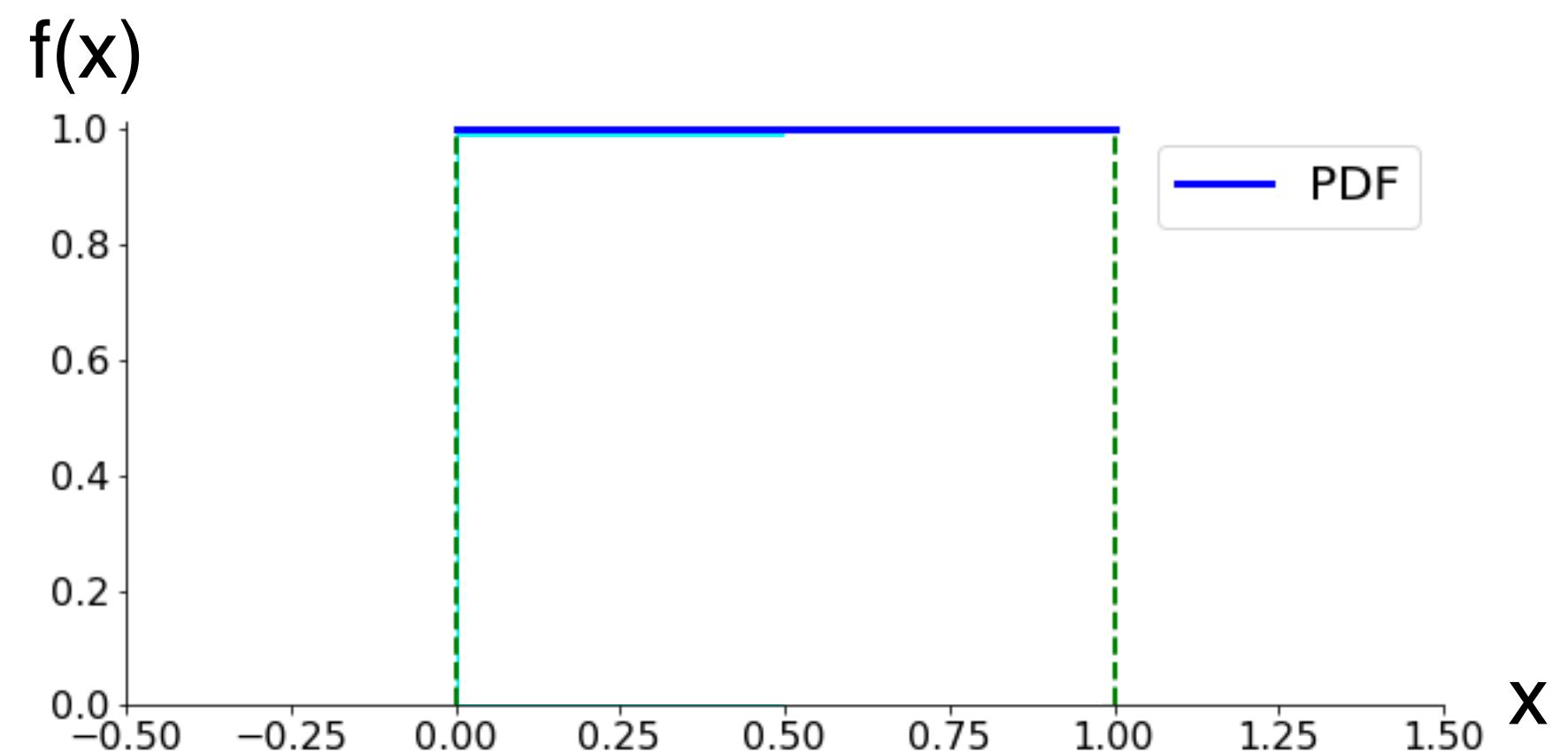


Comparison to Discrete

	Discrete	Continuous
Probability function	mass (pmf)	density (pdf)
≥ 0	$p(x) \geq 0$	$f(x) \geq 0$
$\sum = 1$	$\sum_x p(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1$

Uniform

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Will it \int ? Area = $1 \cdot 1 = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = x \Big|_0^1 = 1$$

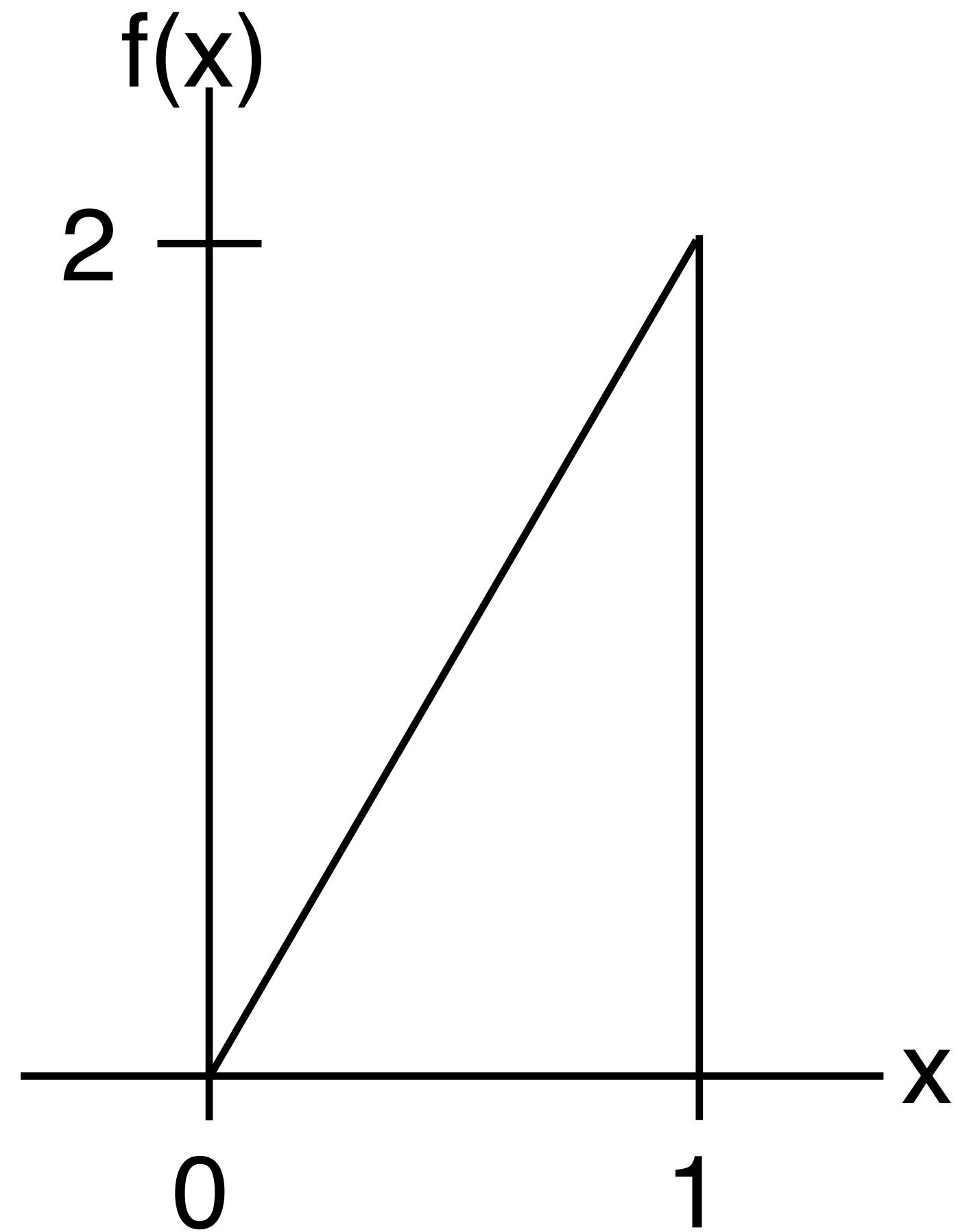
Triangle

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Will it \int ?

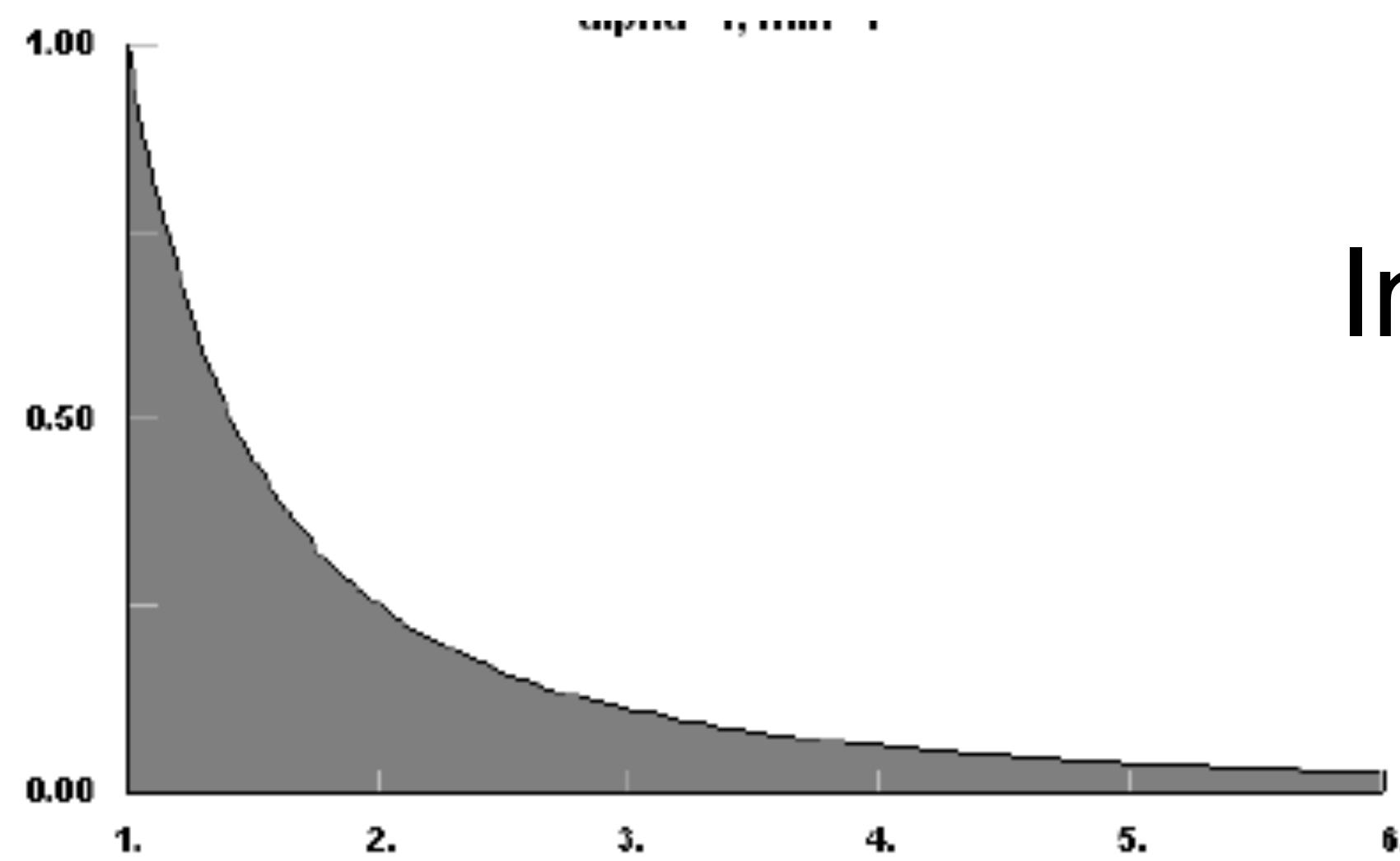
$$\text{Area} = 2 \cdot 1 \cdot \frac{1}{2} = 1$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 2xdx = x^2 \Big|_0^1 = 1 - 0 = 1$$



Power Law

$$f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$$



Infinite support

Will it Σ ?

$$\int_{-\infty}^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{u^2} du = \left. \frac{-1}{u} \right|_1^{\infty} = 1$$

Event Probability

	Discrete	Continuous
P(A)	$\sum_{x \in A} p(x)$	$\int_{x \in A} f(x)dx$

Typically interested in interval probability $P(a \leq X \leq b)$

Area between a and b

$P(X \leq b) - P(X \leq a)$

Cumulative distribution function

Cumulative Distribution Function (CDF)

$$F(x) \triangleq P(X \leq x)$$

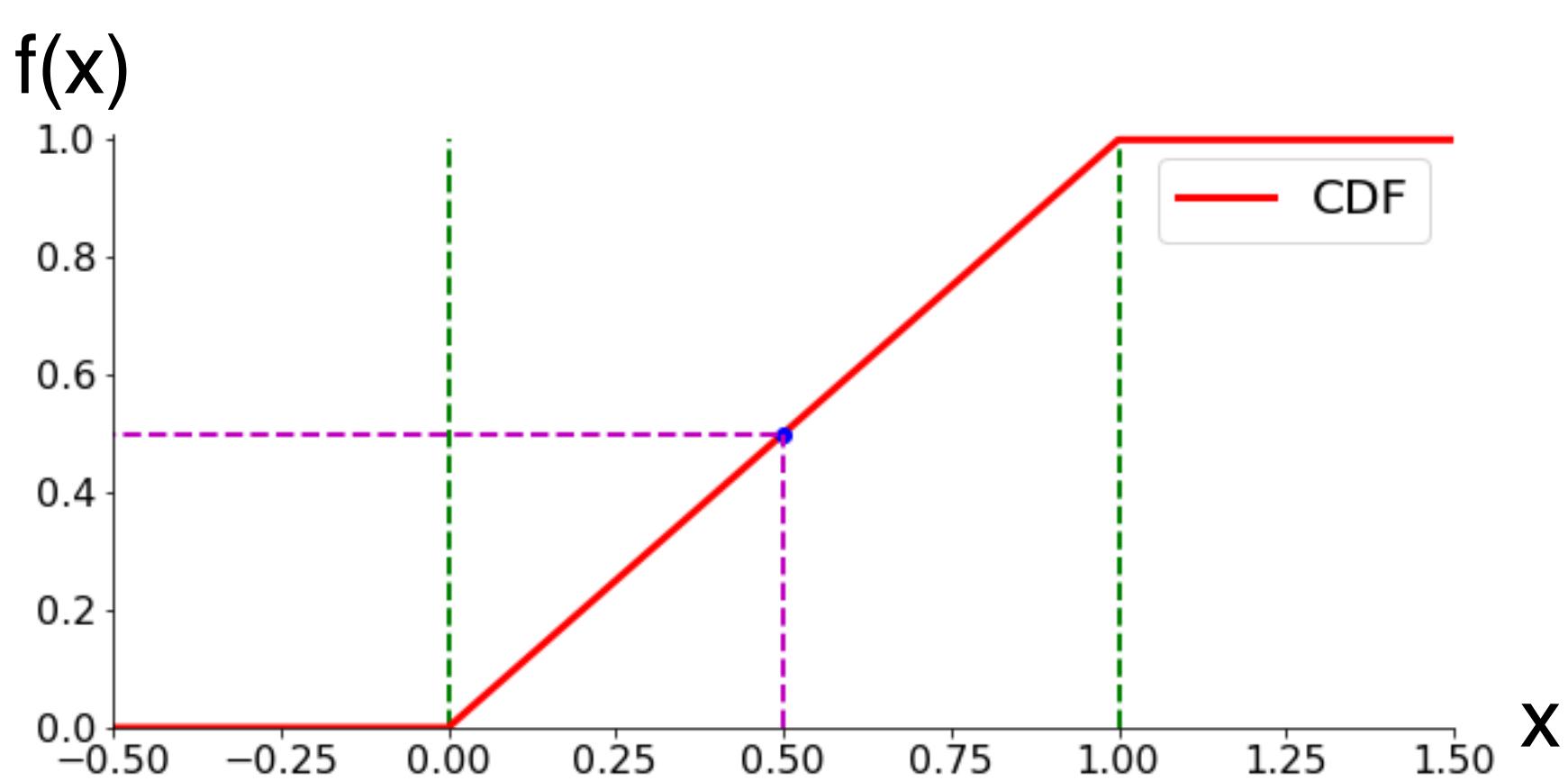
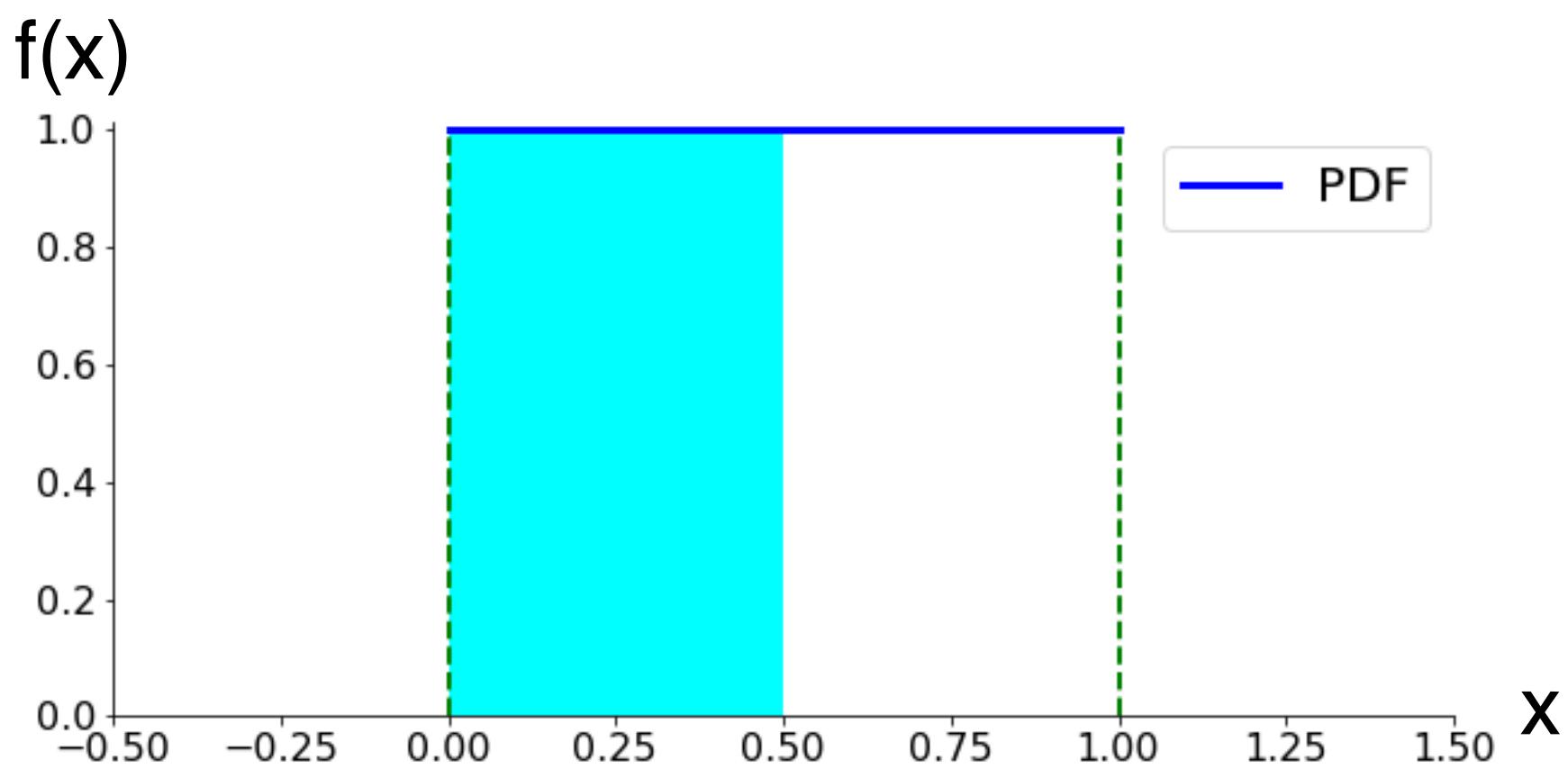
	Discrete	Continuous
PF → CDF	$\sum_{u \leq x} p(u)$	$\int_{-\infty}^x f(u) du$
CDF → PF	$p(x) = F(x) - F(x^*)$	$f(x) = F'(x)$

x^* - element preceding x

Uniform

$$F(x) = \int_{-\infty}^x f(u)du = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1 du = u \Big|_0^x = x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

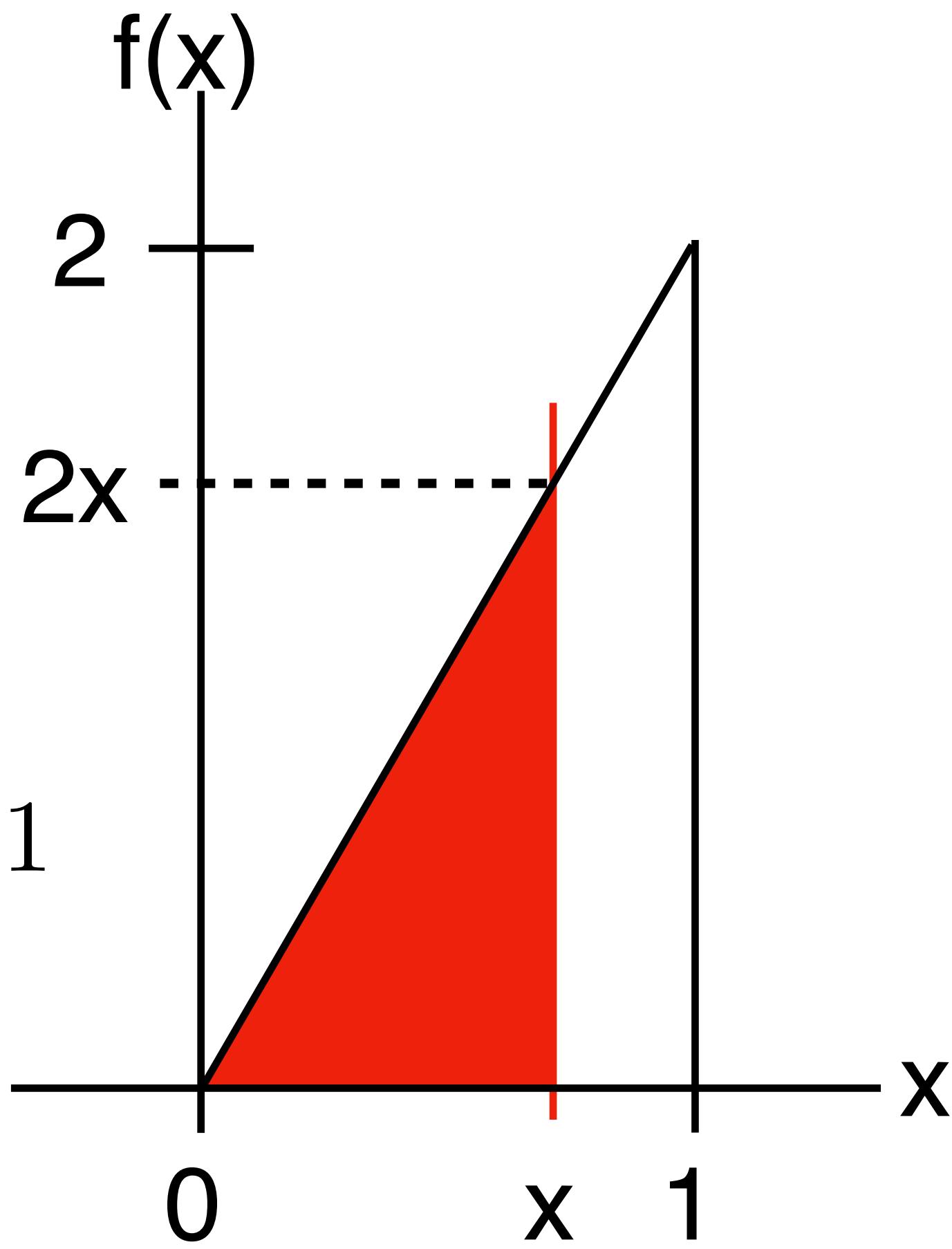
$$F'(x) = \begin{cases} (0)' = 0 & x \leq 0 \\ (x)' = 1 & 0 < x < 1 \\ (1)' = 0 & 1 < x \end{cases}$$



Triangle

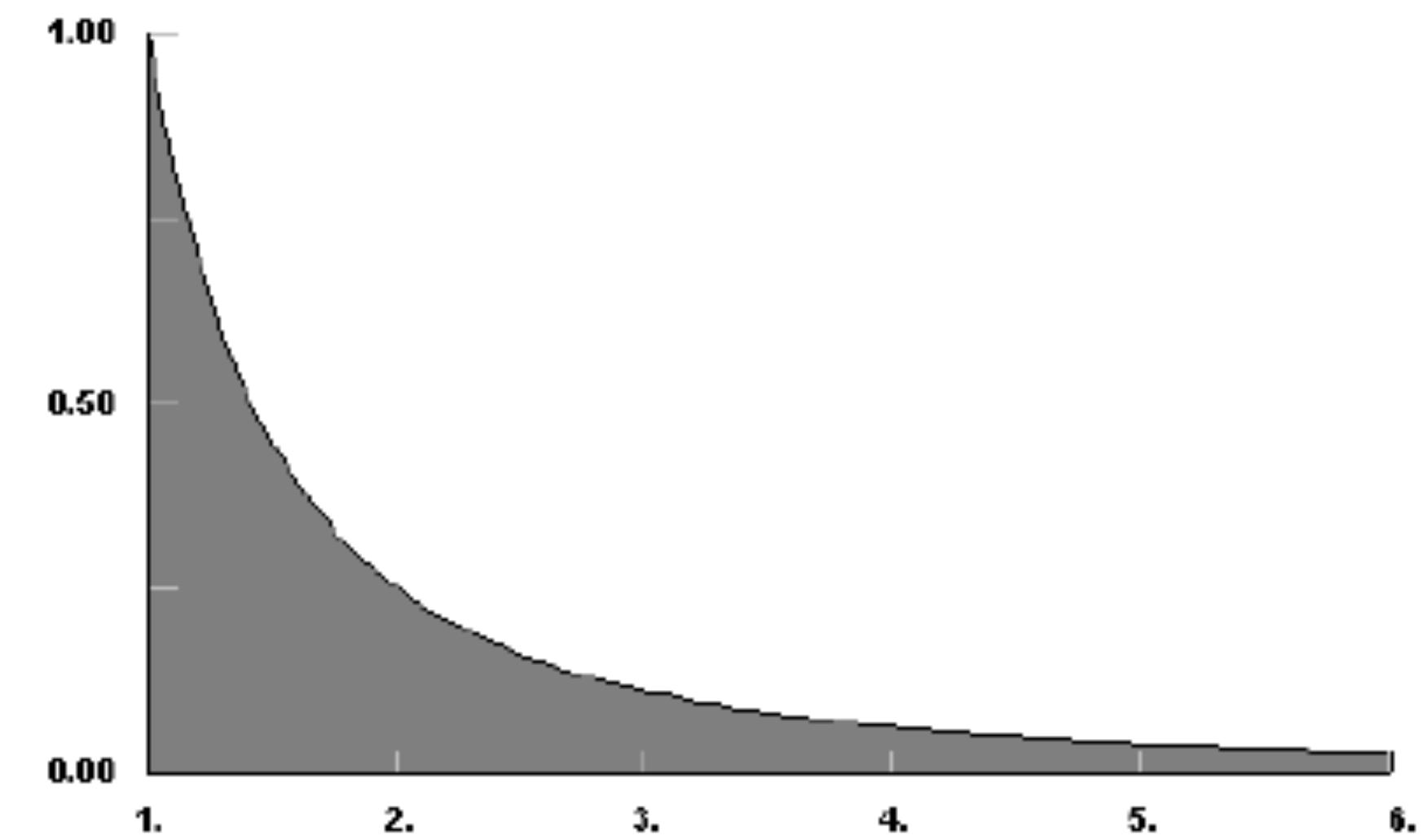
$$F(x) = \int_{-\infty}^x f(u)du = \begin{cases} 0 & x \leq 0 \\ \int_0^x 2udu = u^2 \Big|_0^x = x^2 & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

$$F'(x) = \begin{cases} (0)' = 0 & x < 0 \\ (x^2)' = 2x & 0 < x < 1 \\ (1)' = 0 & 1 < x \end{cases}$$



Power Law

$$F(x) = \begin{cases} 0 & x \leq 1 \\ \int_1^x \frac{1}{u^2} du = \frac{-1}{u} \Big|_1^x = 1 - \frac{1}{x} & x \geq 1 \end{cases}$$



$$F'(x) = \begin{cases} (0)' = 0 & x < 1 \\ \left(1 - \frac{1}{x}\right)' = \frac{1}{x^2} & x > 1 \end{cases}$$

Properties of the CDF

$F(x) = \text{integral}$

Nondecreasing

$F(-\infty)=0$

$F(\infty)=1$

Continuous

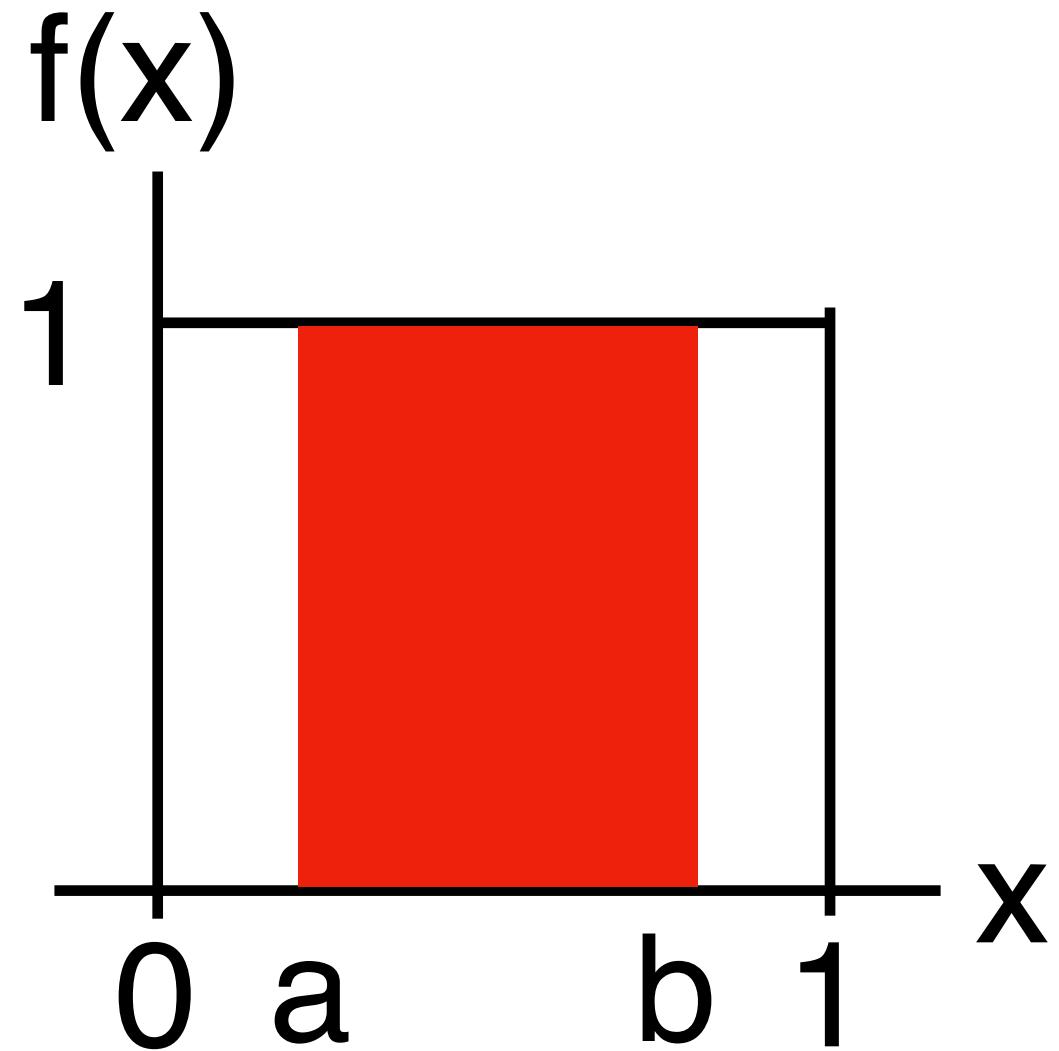
Interval Probabilities

$$P(a,b) = P[a,b] = P(a,b] = F(b)-F(a)$$

Examples

Uniform

$$0 \leq a \leq b \leq 1$$
$$P(a \leq X \leq b) = \begin{cases} \text{Area} = (b - a) \cdot 1 = b - a \\ \int_a^b f(x)dx = \int_a^b 1 dx = x \Big|_a^b = b - a \\ F(b) - F(a) = b - a \end{cases}$$



$$\begin{aligned} P(0.6 \leq X \leq 1.3) &= P(0.6 \leq X \leq 1) = 0.4 \\ &= F(1.3) - F(0.6) = 1 - 0.6 = 0.4 \end{aligned}$$

Power law

$$1 \leq a \leq b \quad P(a \leq X \leq b) = F(b) - F(a) = \left(1 - \frac{1}{b}\right) - \left(1 - \frac{1}{a}\right) = \frac{1}{a} - \frac{1}{b}$$

Differences

Discrete	Continuous
$p(x) \leq 1$	$f(x)$ can be > 1
Generally $p(x) \neq 0$	$p(x) = 0$
Generally $P(X \leq a) \neq P(X < a)$	$P(X \leq a) = P(X < a) = F(a)$
	$P(X \geq a) = P(X > a) = 1 - F(a)$
	$P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a)$

Expectation

	Discrete	Continuous
EX	$\sum x \cdot p(x)$	$\int_{-\infty}^{\infty} xf(x)dx$

As discrete:

Average of many samples

Properties

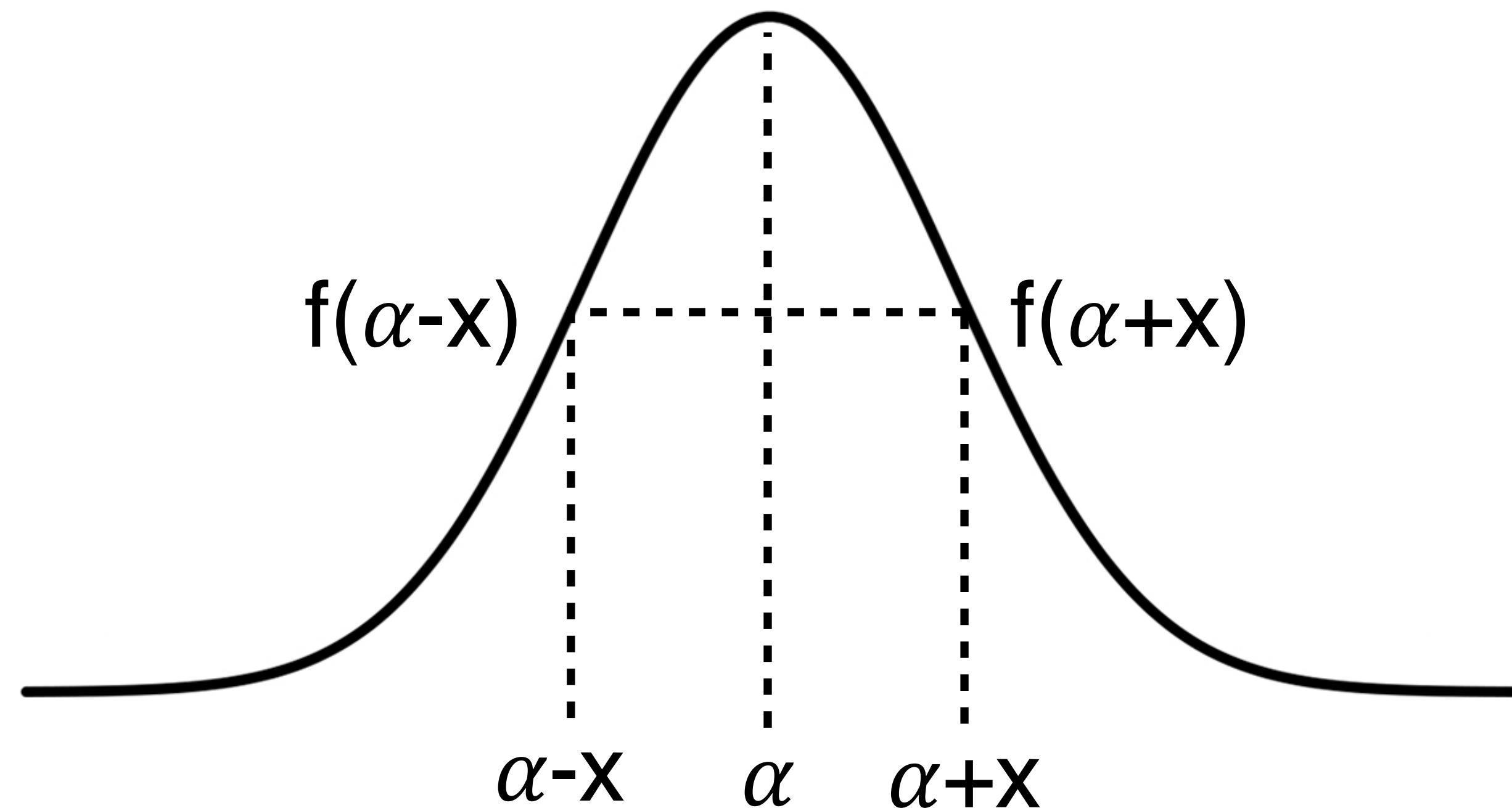
Support set = $[a, b]$

$$a \leq EX \leq b$$

Symmetry

If for some α , $f(\alpha+x) = f(\alpha-x)$ for all x

then $EX = \alpha$



Examples

Uniform

$$EX = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot 1 \cdot dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$

Triangle

$$EX = \int_0^1 x \cdot 2x \cdot dx = \frac{2x^3}{3} \Big|_0^1 = \frac{2}{3}$$

Power law

$$EX = \int_1^{\infty} x \cdot \frac{1}{x^2} \cdot dx = \int_1^{\infty} \frac{1}{x} \cdot dx = \ln x \Big|_1^{\infty} = \infty$$

Later: power laws with finite expectation

Variance

	Discrete	Continuous
$V(X) \triangleq E(X - \mu)^2$	$\sum_x p(x)(x - \mu)^2$	$\int_{-\infty}^{\infty} f(x)(x - \mu)^2 dx$
		$E(X^2) - \mu^2$

As for discrete

$$\begin{aligned} E(X - \mu)^2 &= \int (x - \mu)^2 f(x) dx \\ &= \int (x^2 - 2x\mu + \mu^2) f(x) dx \\ &= \int x^2 f(x) dx - 2\mu \int x f(x) dx + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - \mu^2 \end{aligned}$$

$$\sigma = \sqrt{V(X)}$$

Examples

Uniform

$$EX = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2 \cdot 1 \, dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

$$V(X) = E(X^2) - (EX)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\sigma = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$$

Triangle

$$EX = \frac{2}{3}$$

$$EX^2 = \int_0^1 x^2 \cdot 2x \, dx = \frac{2}{4}x^4 \Big|_0^1 = \frac{1}{2}$$

$$V(X) = E(X^2) - (EX)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{9-8}{18} = \frac{1}{18}$$

$$\sigma = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$$

Discrete vs. Continuous

	Discrete	Continuous
Prob. Fun.	pmf - p	pdf - f
≥ 0	$p(x) \geq 0$	$f(x) \geq 0$
$\sum = 1$	$\sum p(x) = 1$	$\int f(x)dx = 1$
P(A)	$\sum_{x \in A} p(x)$	$\int_{x \in A} f(x)dx$
F(X)	$\sum_{u \leq x} p(u)$	$\int_{-\infty}^x f(u)du$
$\mu = E(X)$	$\sum xp(x)$	$\int xf(x)dx$
V(X)	$\sum (x - \mu)^2 p(x)$	$\int (x - \mu)^2 f(x)dx$

$p(x) \leq 1$ f can be larger

$$f(x) = F'(x)$$

$$P(a \leq X \leq b) = P(a < X < b) = F(b) - F(a)$$

A cartoon illustration of a young boy with brown hair and freckles. He is wearing a blue and white striped shirt under an orange vest and a green t-shirt. He is looking through a large blue telescope with a red eyepiece. He has a wide-eyed, excited expression on his face.

Functions of Continuous Random Variables