

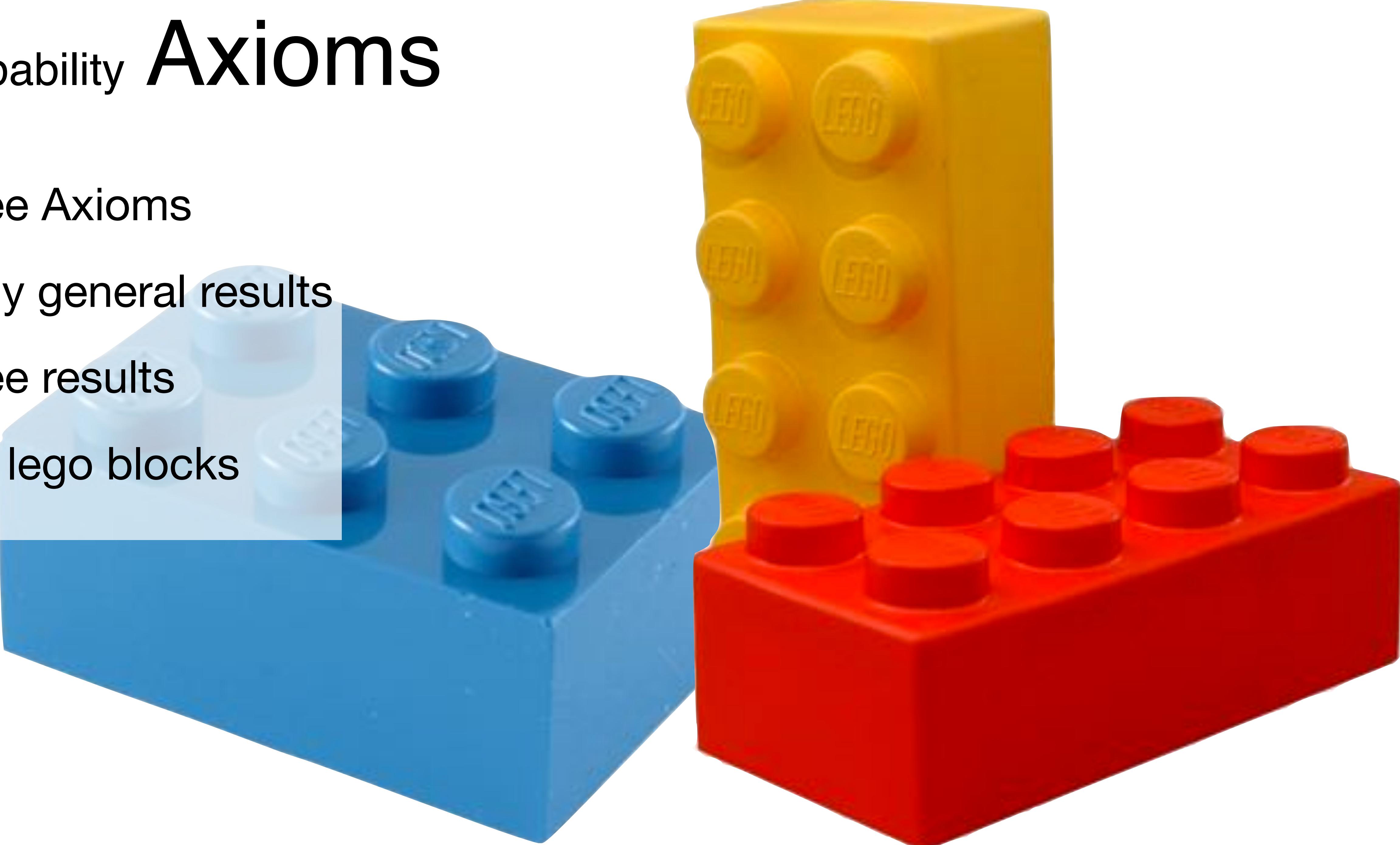
Probability Axioms

Three Axioms

Imply general results

Three results

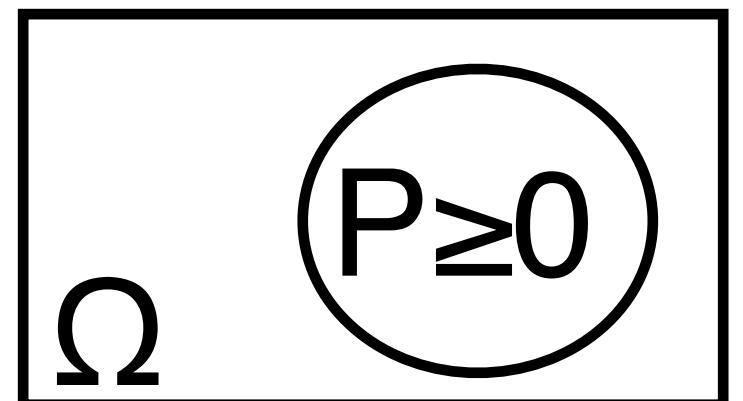
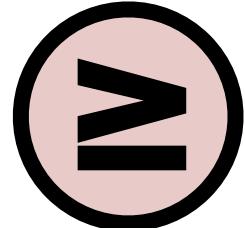
Use lego blocks



Three Axioms

Non-negativity

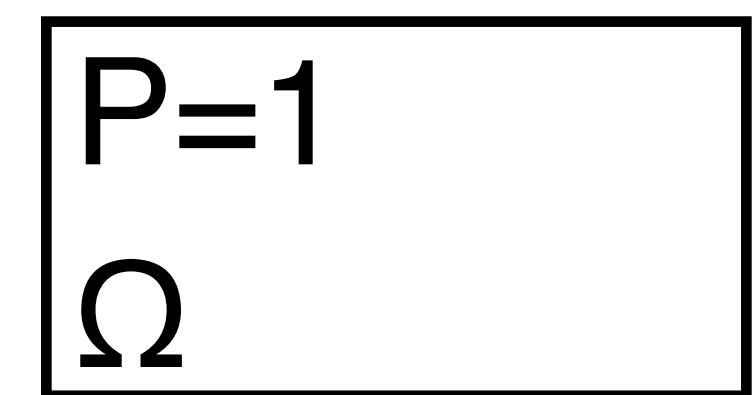
$$P(A) \geq 0$$



Short for $\forall A \ P(A) \geq 0$

Unitarity

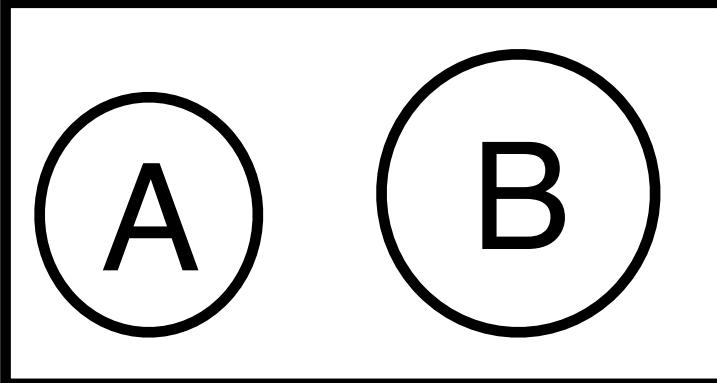
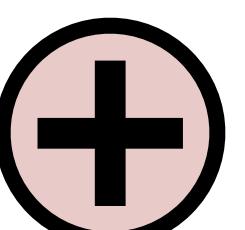
$$P(\Omega) = 1$$



Addition rule

A, B disjoint \rightarrow

$$P(A \cup B) = P(A) + P(B)$$



A_1, A_2, \dots disjoint

$$\rightarrow P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$

Countable unions only

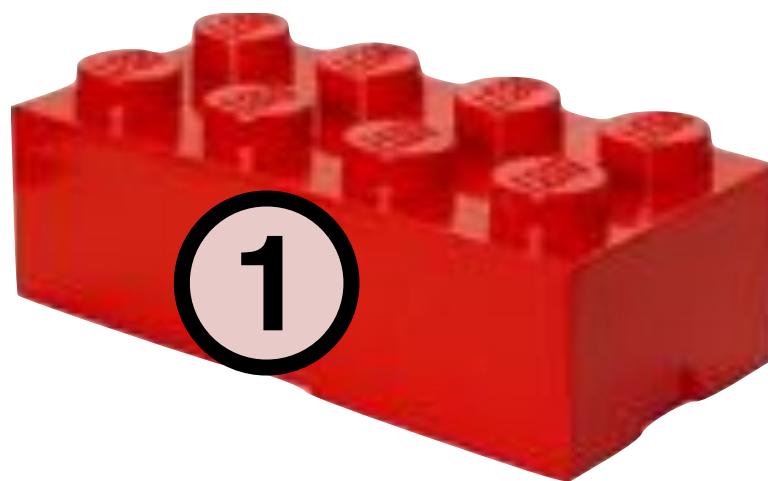
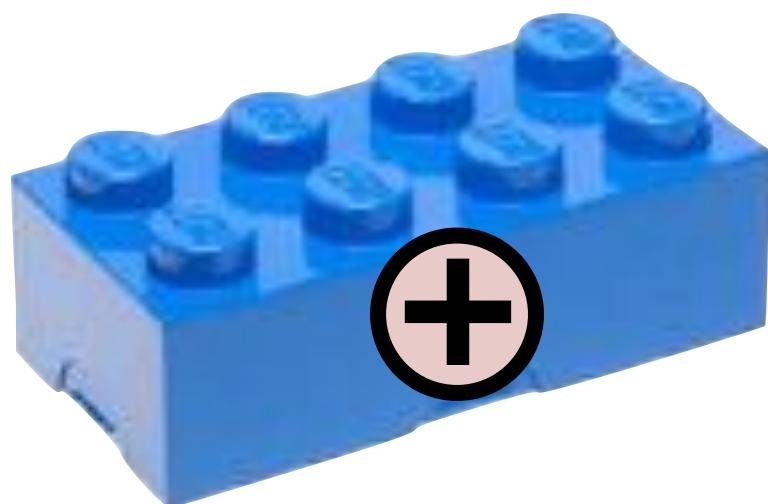
Uncountable later

“Building” Results

Use lego blocks

construct three simple results

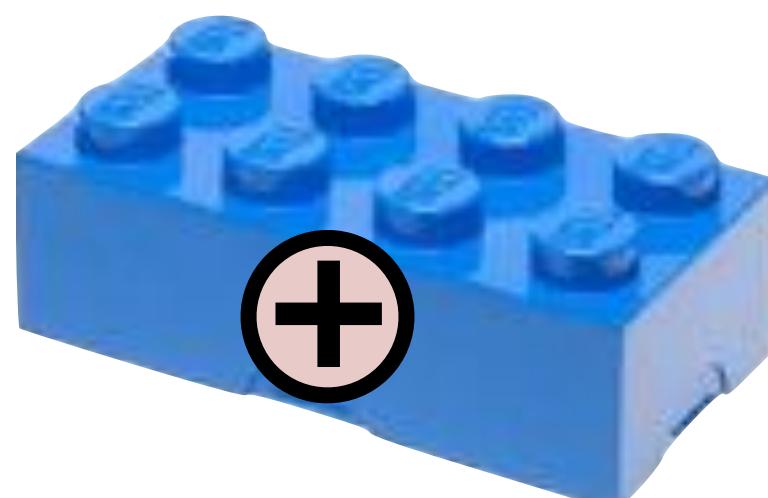
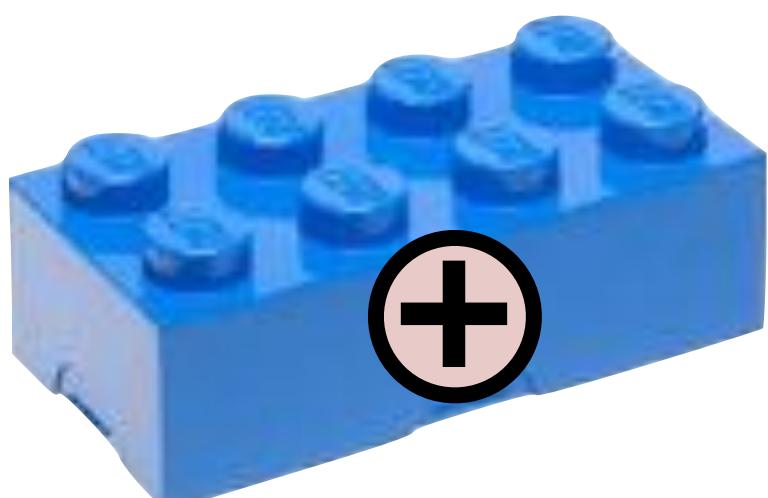
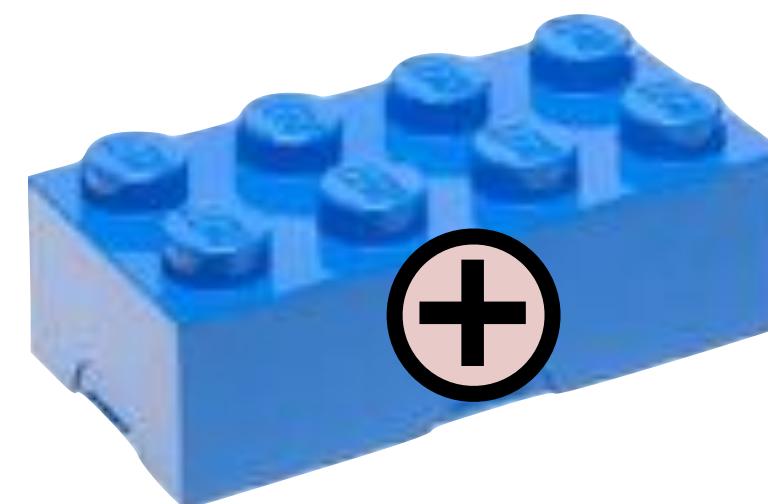
Complement rule



Subtraction

nested sets

general sets



Inclusion-exclusion

Complement Rule

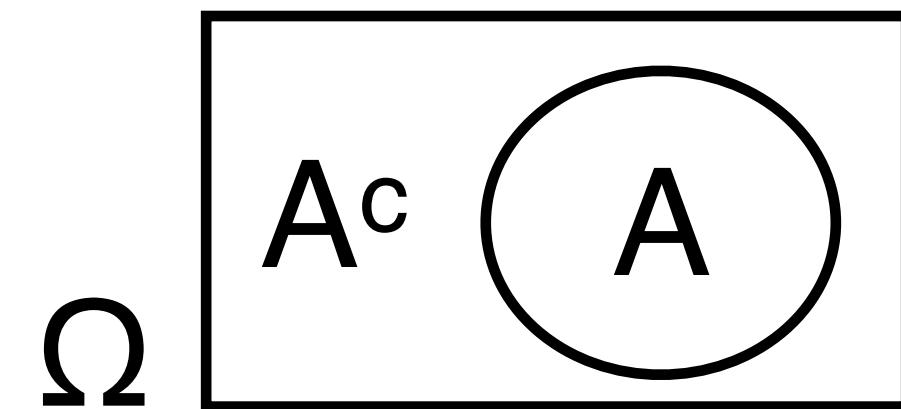
Complement rule for counting

$$|A^c| = |\Omega| - |A|$$

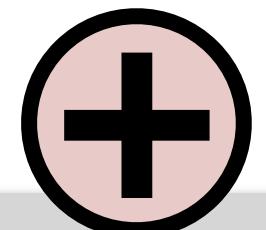
Same for



$$A \cup A^c = \Omega$$

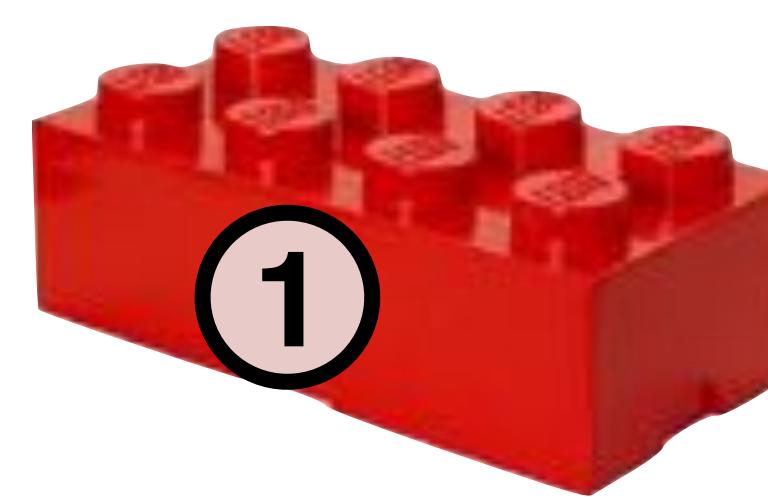
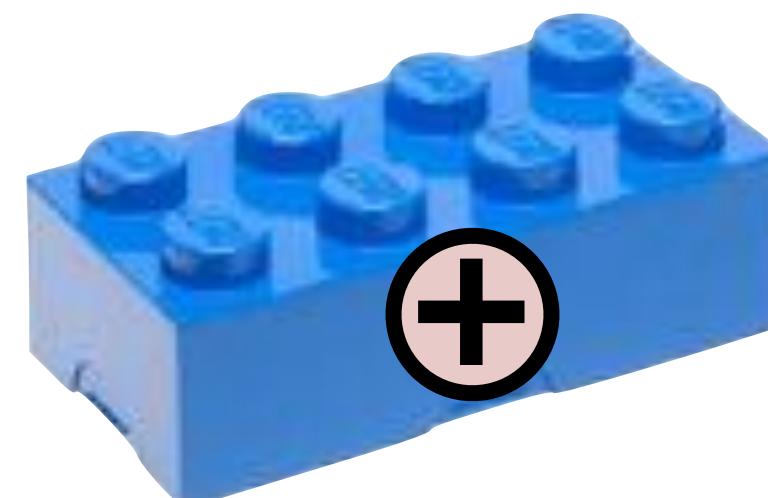
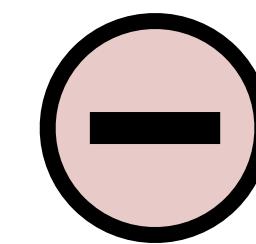


$$P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1$$



$$P(A^c) = 1 - P(A)$$

Complement rule for probability



Subtraction Rule - Nested Sets

Complement rule

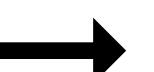
$$A \subseteq \Omega$$

$$P(A^c) = 1 - P(A)$$

$$P(\Omega - A) = P(\Omega) - P(A)$$

Generalize

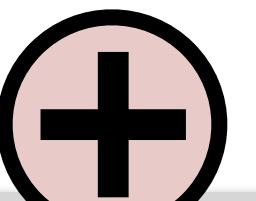
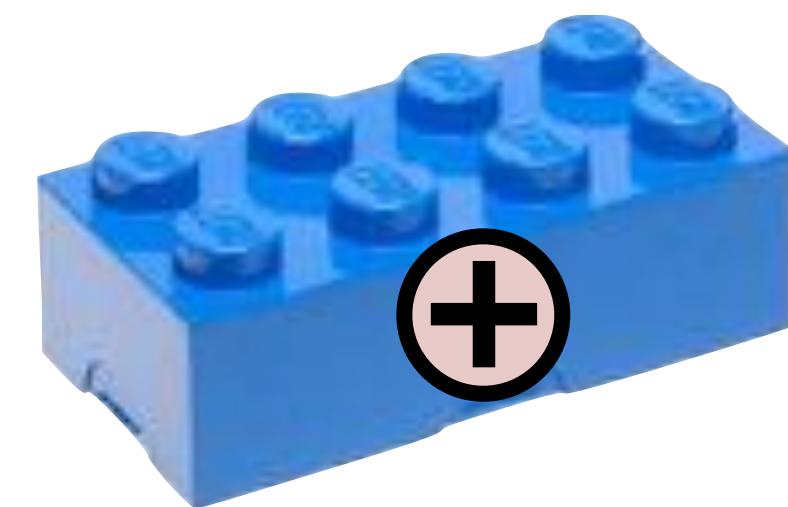
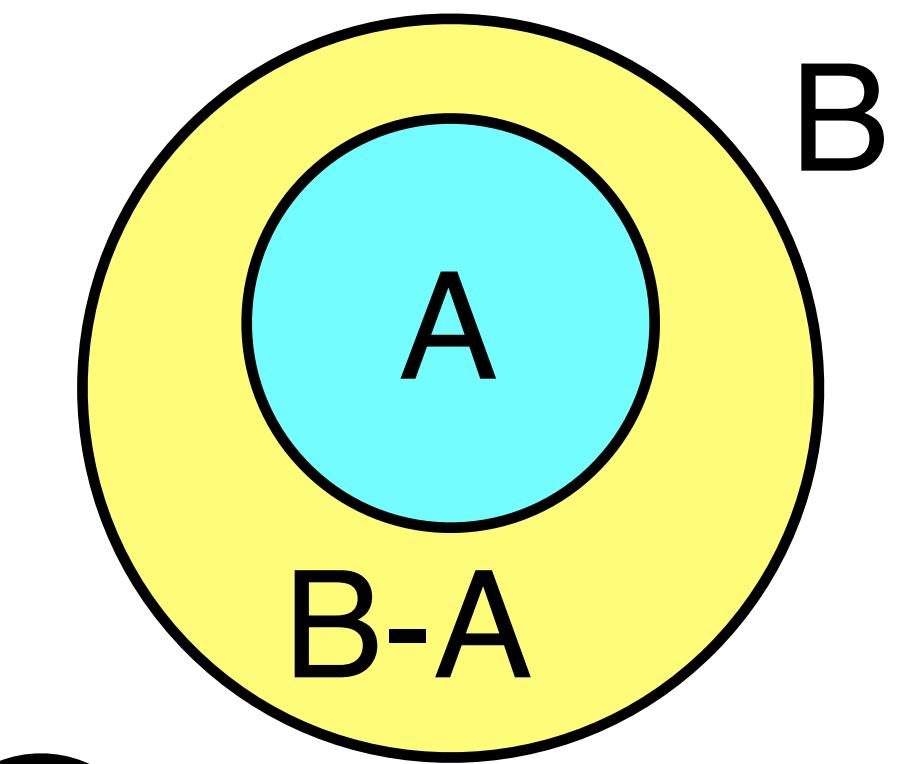
$$A \subseteq B$$



$$P(B - A) = P(B) - P(A)$$

$$A \subseteq B \rightarrow$$

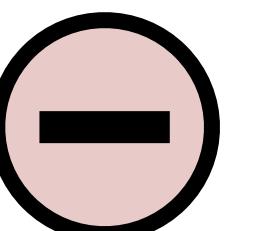
$$B = A \cup (B - A)$$



$$P(B) = P(A \cup (B - A)) = P(A) + P(B - A)$$

$$P(B - A) = P(B) - P(A)$$

Subtraction rule for nested sets



Subtraction Rule - General Sets

Nested

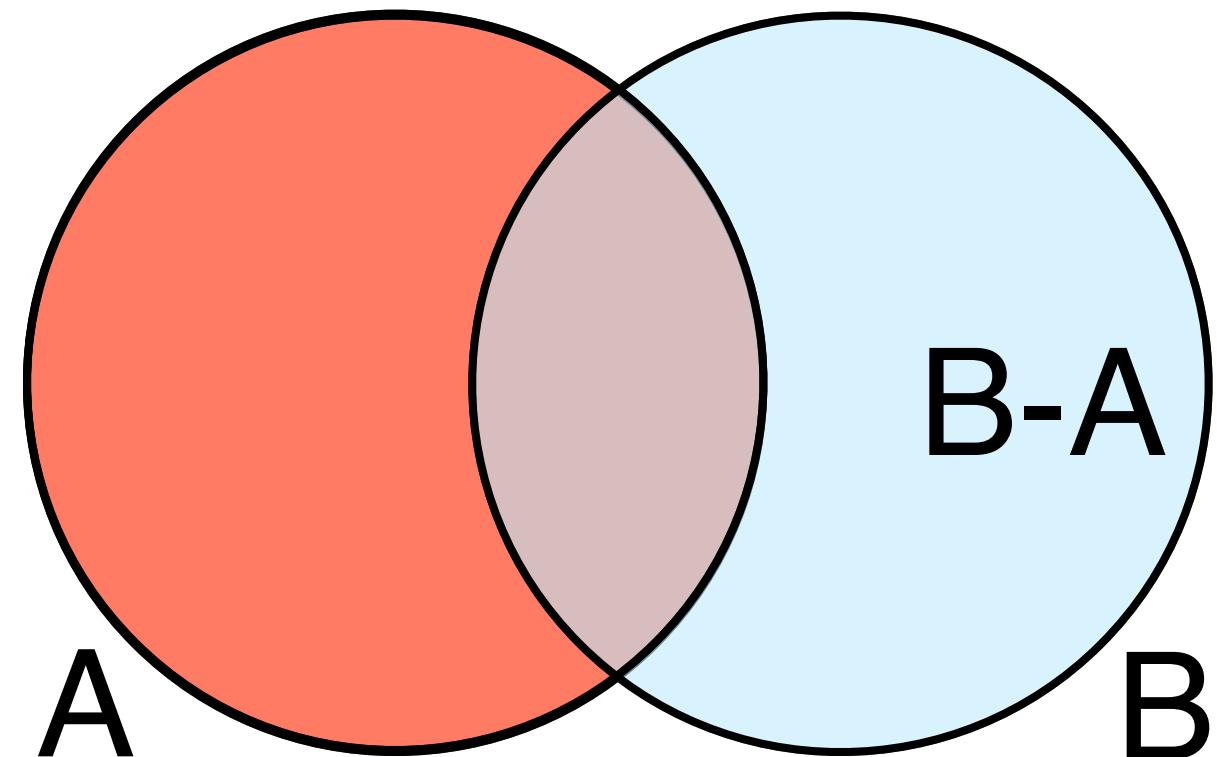
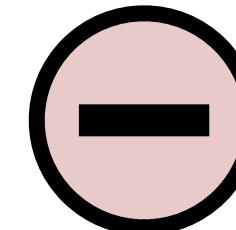
$$A \subseteq B$$

$$P(B-A) = P(B) - P(A)$$

General

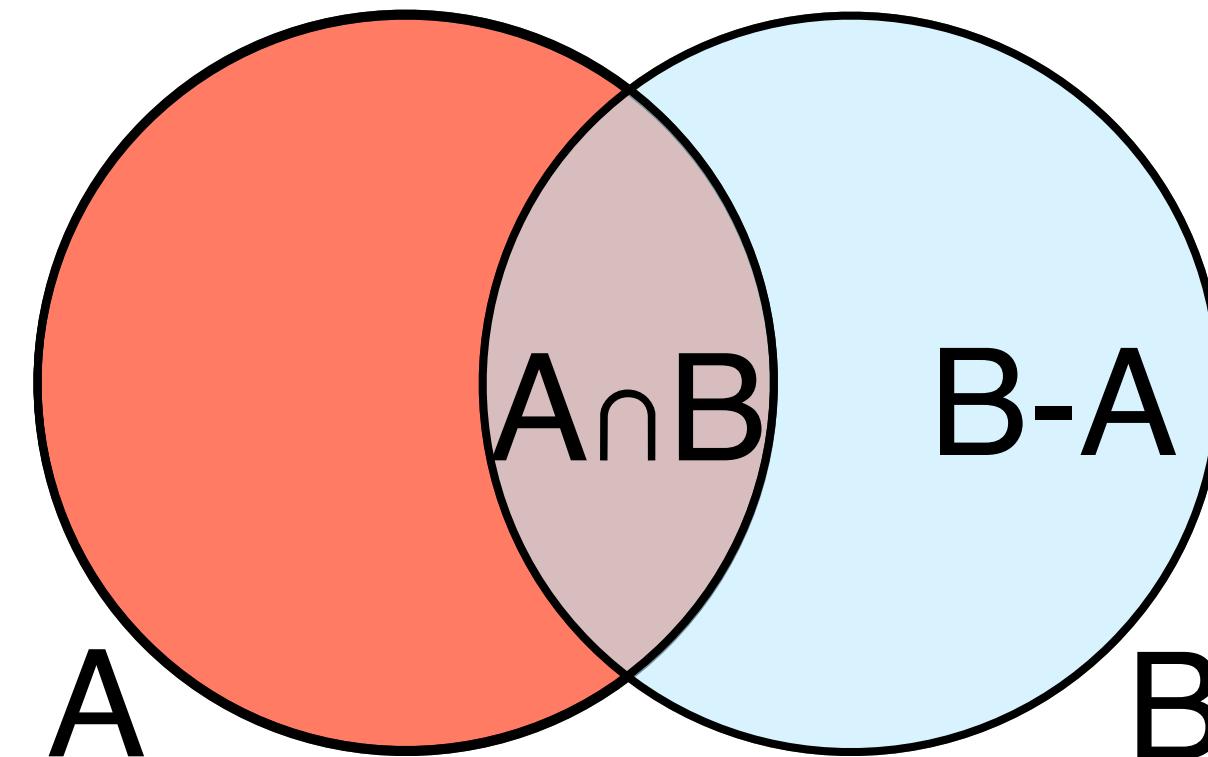
$$\forall A, B$$

$$P(B-A) = P(B) - P(A \cap B)$$

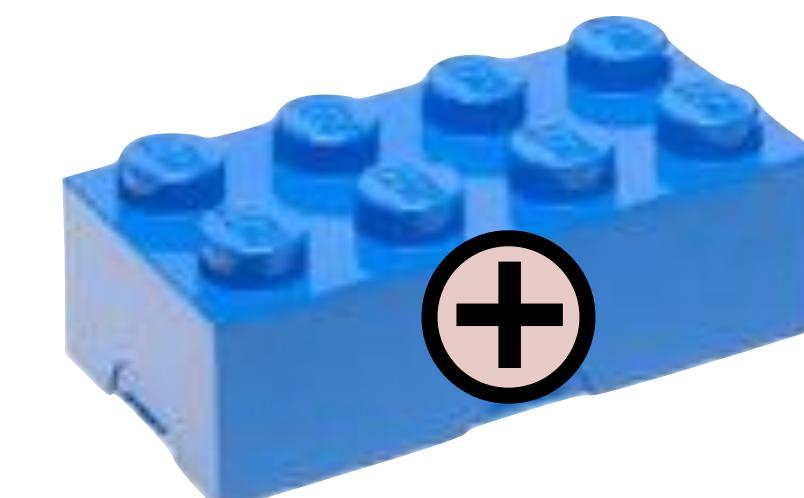


$$B - A = B - (A \cap B)$$

$$A \cap B \subseteq B$$



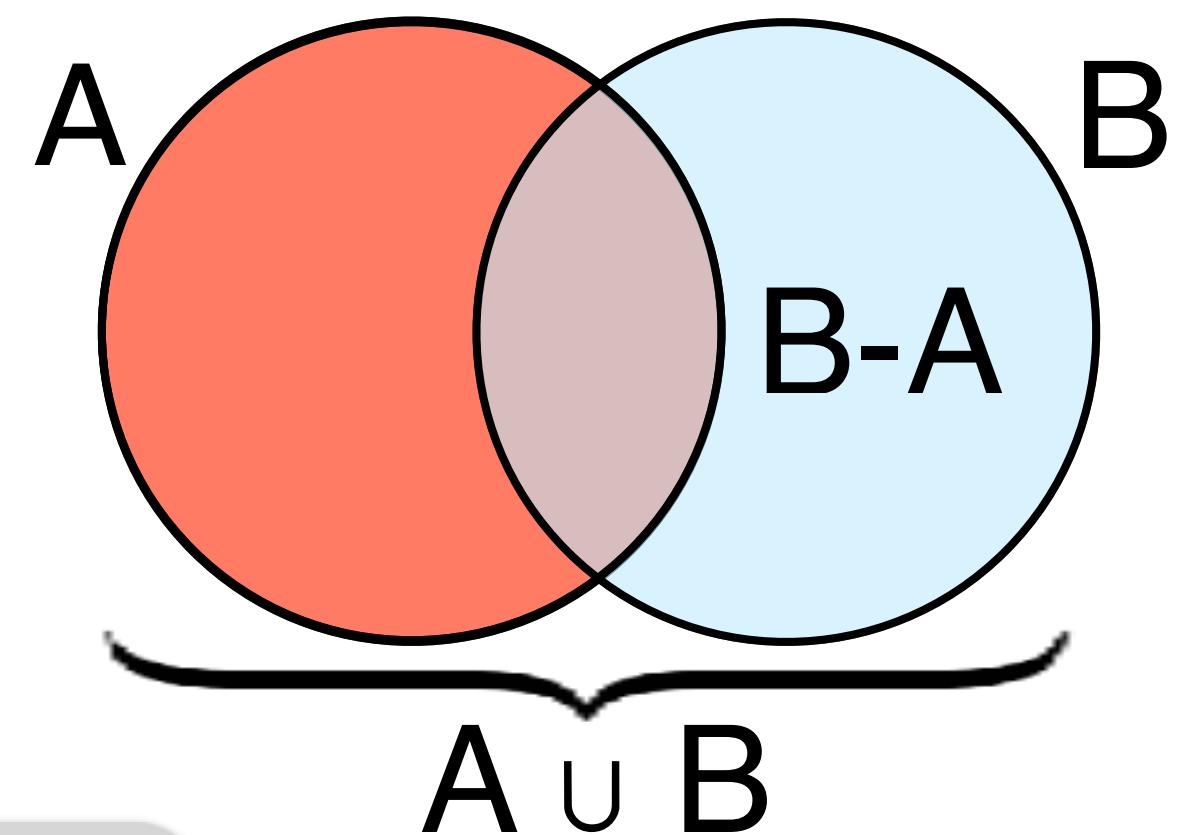
$$P(B - A) = P(B - (A \cap B)) = P(B) - P(A \cap B)$$



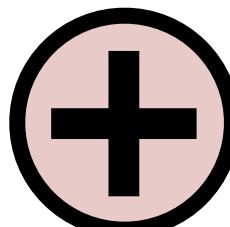
Inclusion Exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$A \cup B = A \cup B - A$$



$$P(A \cup B) = P(A \cup B - A)$$

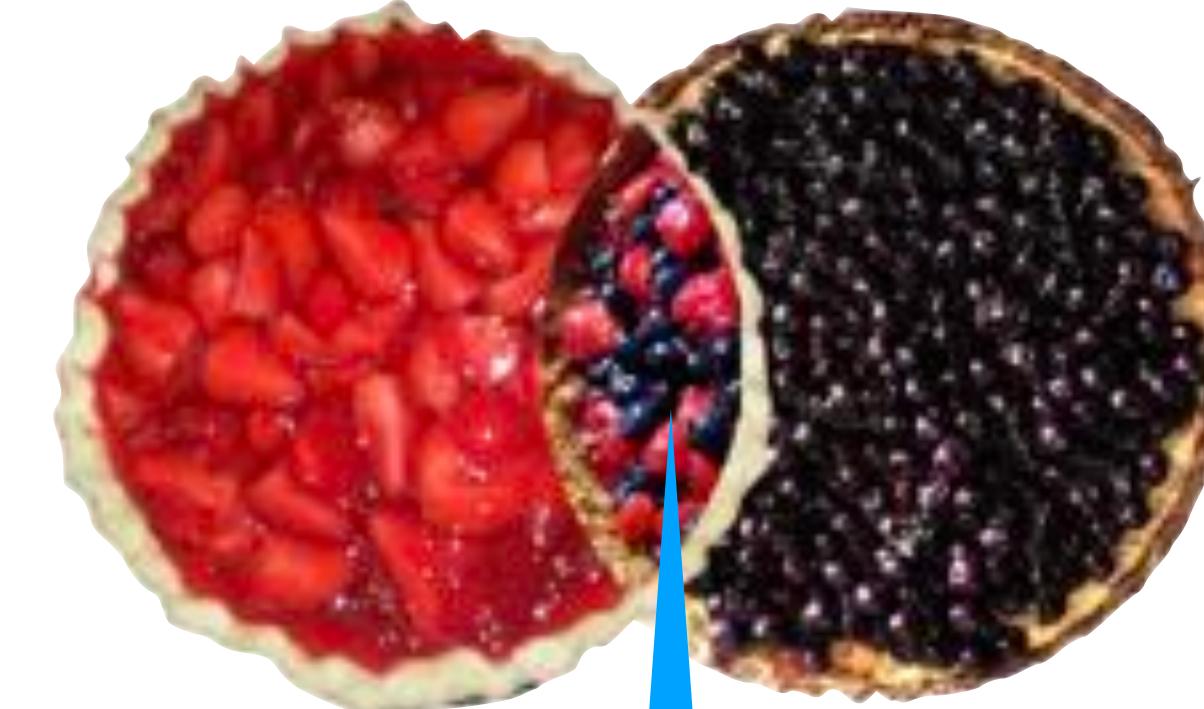


$$= P(A) + P(B - A)$$

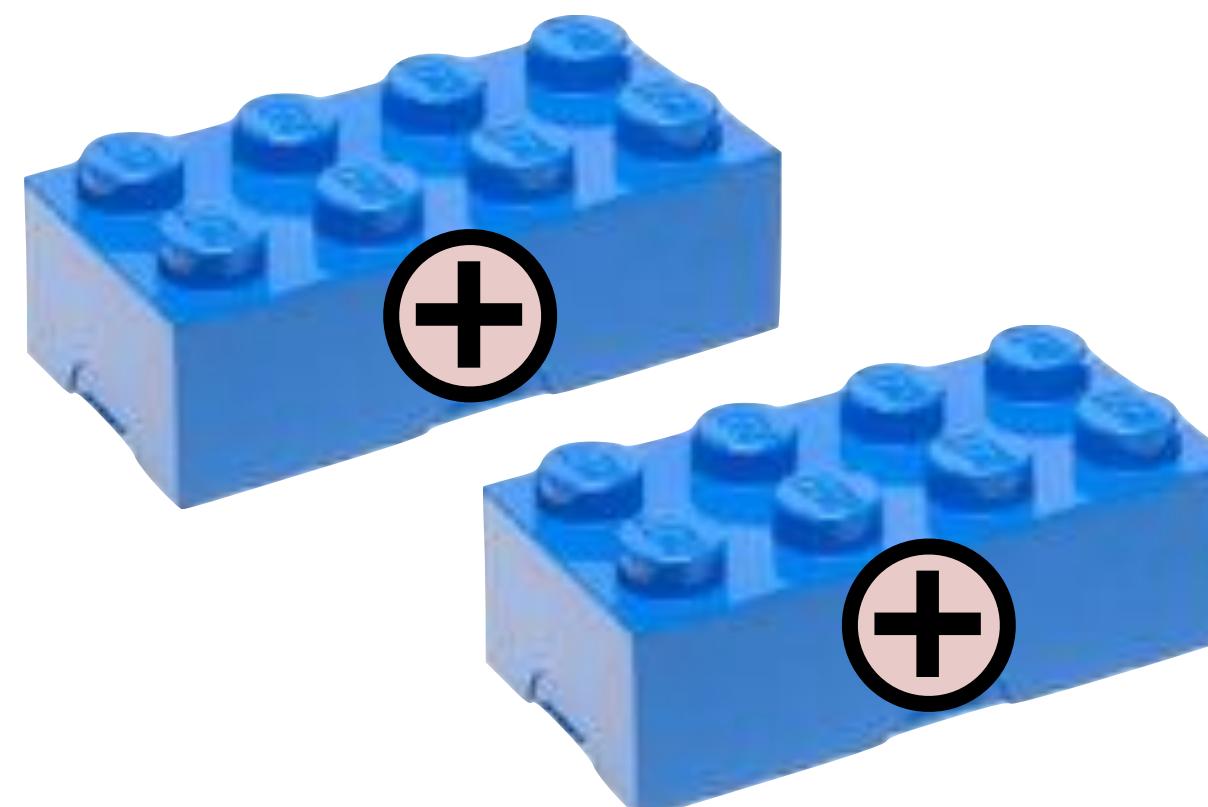
$$= P(A) + P(B) - P(A \cap B)$$

General subtraction

$$P(B - A) = P(B) - P(A \cap B)$$



X 2



More Sets

Two sets

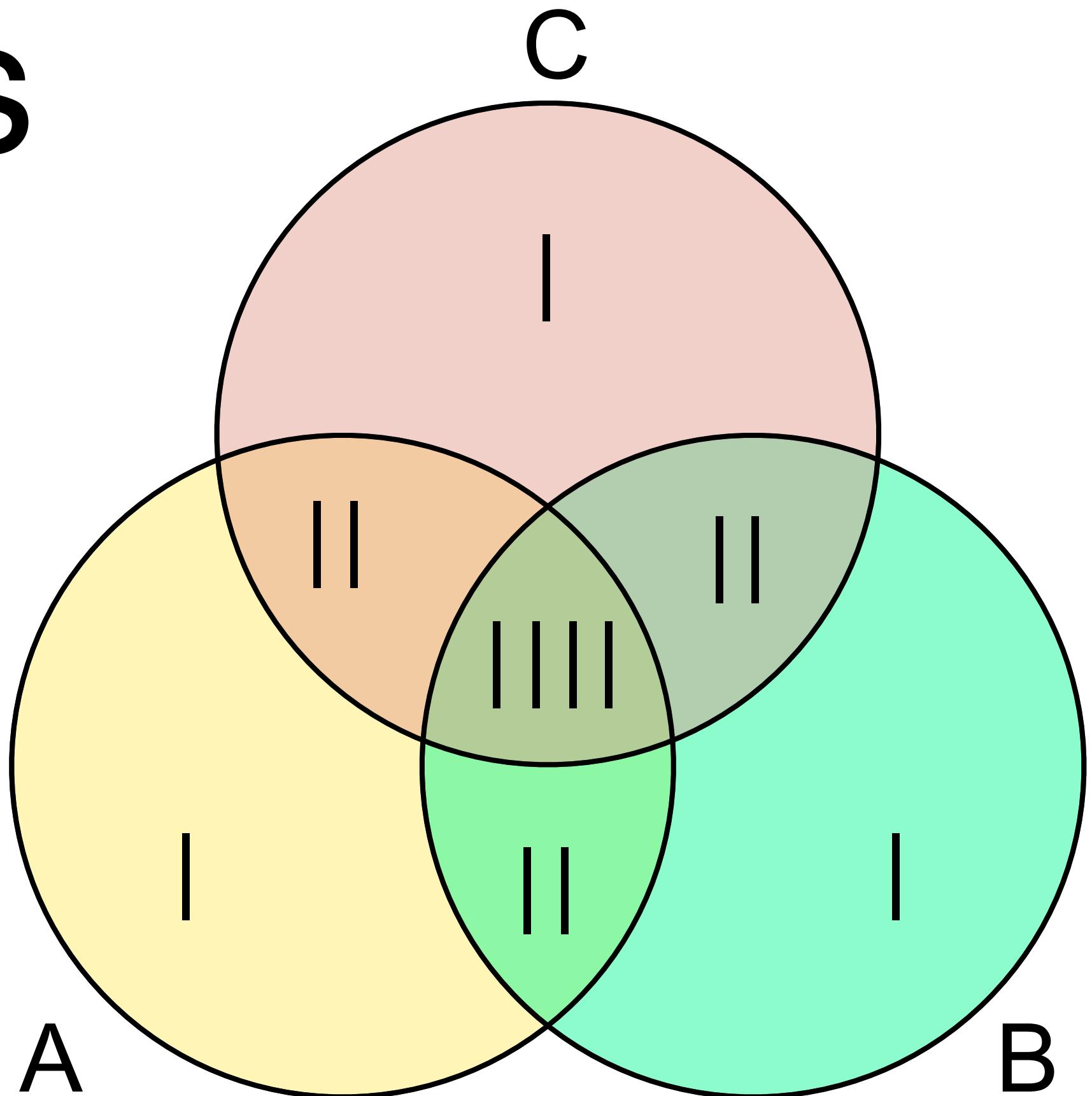
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Three sets

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

n sets

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{1 \leq i \leq n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \dots + (-1)^{n-1} P\left(\bigcap_{i=1}^n A_i\right)$$



Probability Inequalities

Three Axioms

Inequalities

Union bound

\$1.4M question

The

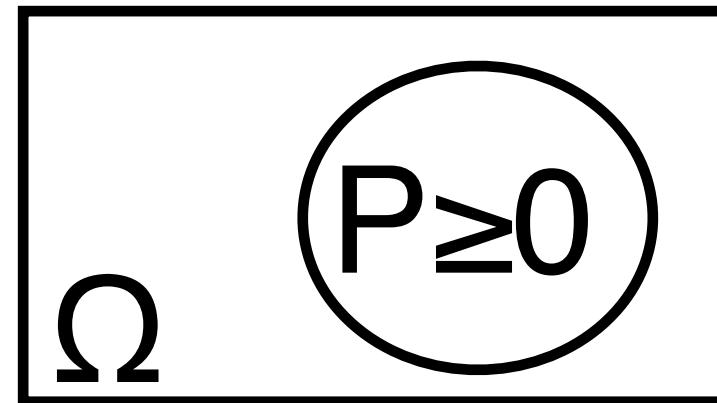
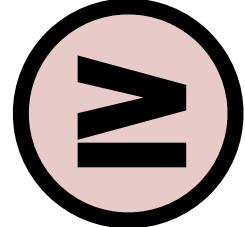
\$64,000

Question

Three Axioms

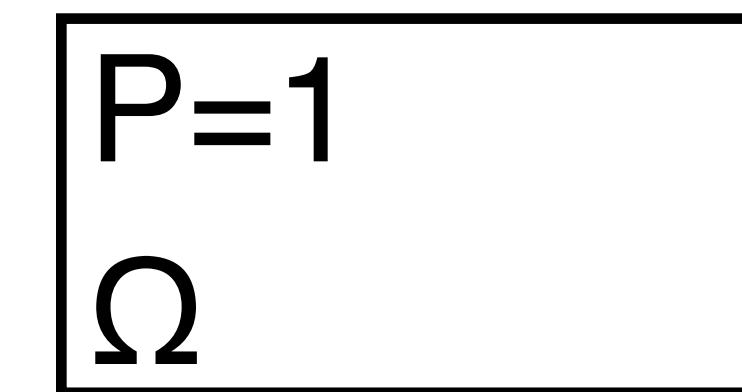
Non-negativity

$P(A) \geq 0$



Unitarity

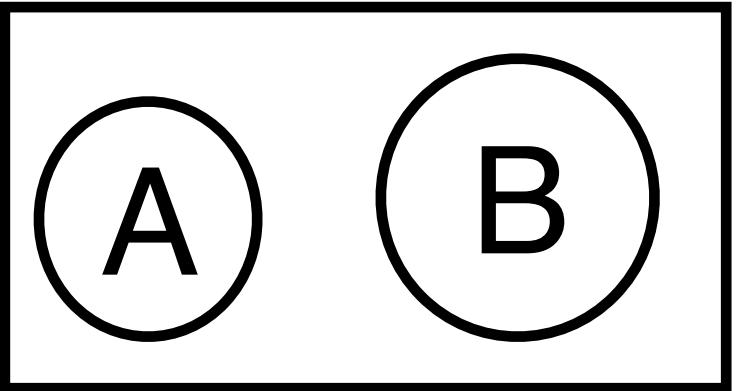
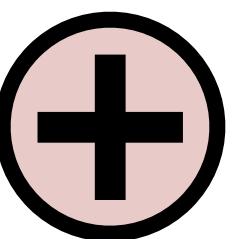
$P(\Omega) = 1$



Addition rule

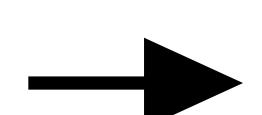
A, B disjoint \rightarrow

$P(A \cup B) = P(A) + P(B)$



A_1, A_2, \dots disjoint $\rightarrow P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$

Equalities



Inequalities

Probability of Null Event

1

$$P(\Omega) = 1$$

What about $P(\emptyset)$?

$$\Omega = \emptyset \cup \Omega$$



Show

$$P(\emptyset) = 0$$

~~$P(\Omega) = P(\emptyset \cup \Omega) = P(\emptyset) + P(\Omega)$~~

$$P(\emptyset) = 0$$

The null event has nil probability

0

$$0 \leq P(A) \leq 1$$

N $P(A) \geq 0$

Show $P(A) \leq 1$

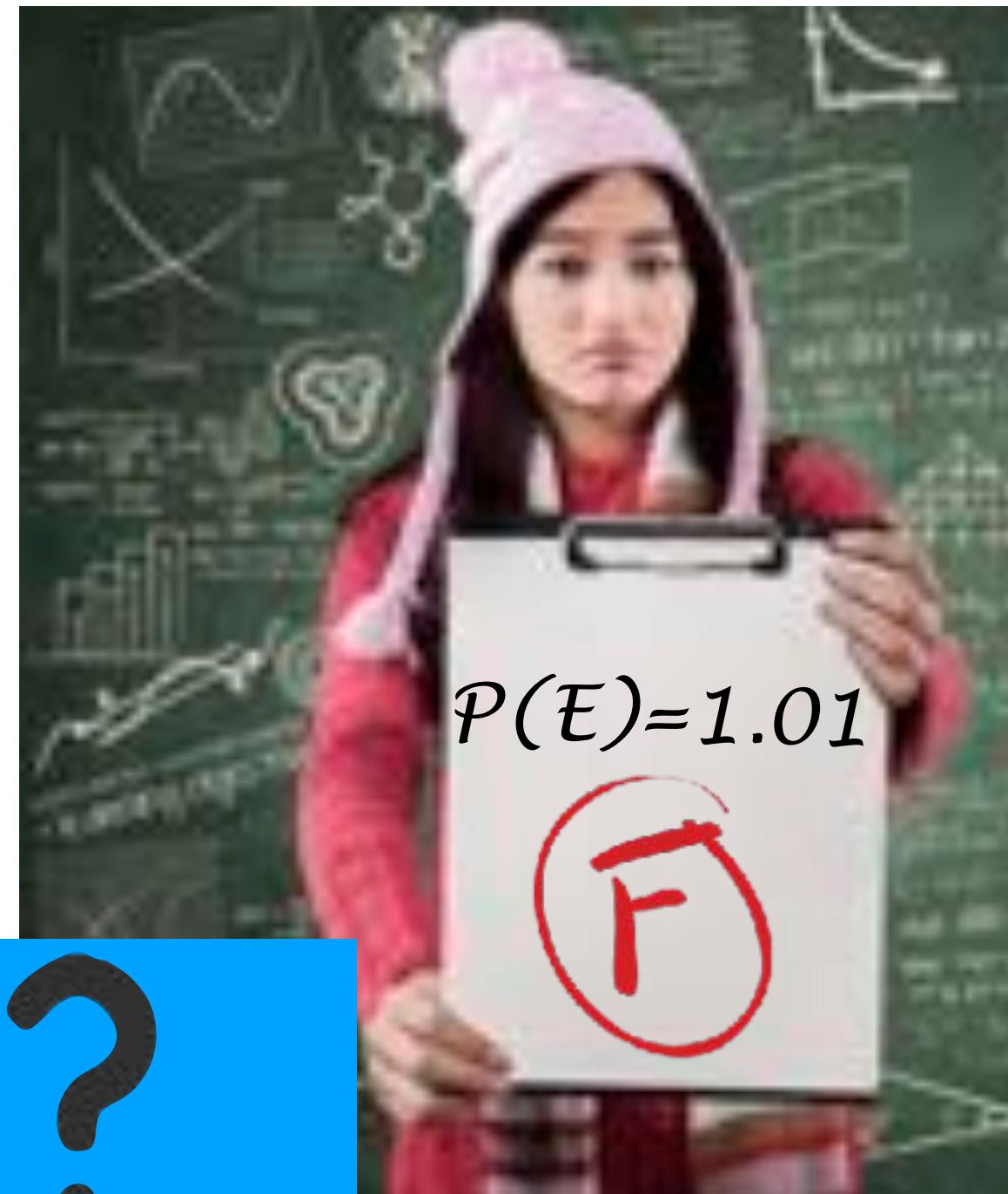
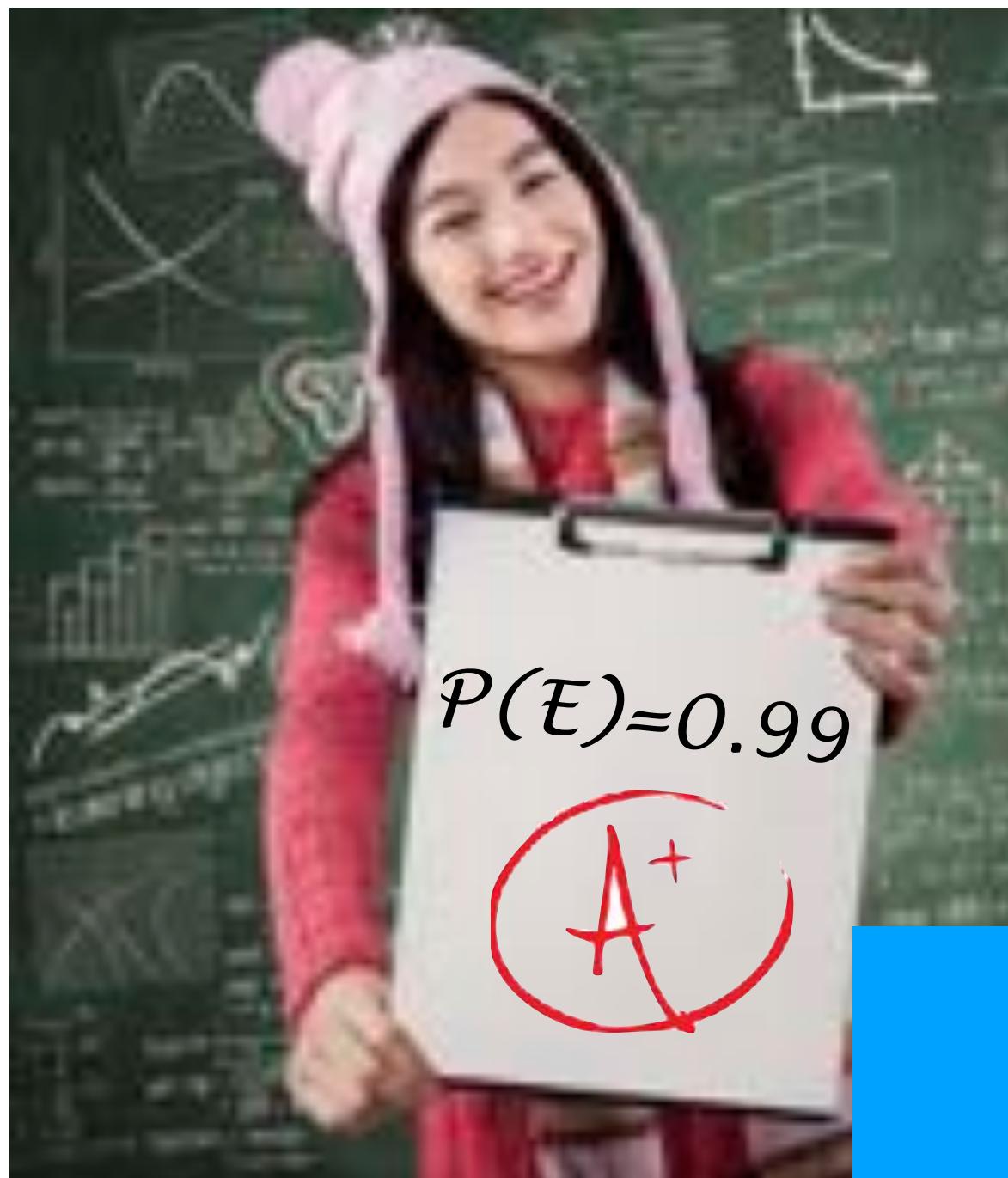
$$A \cup A^c = \Omega$$

N $P(A) \leq P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1$



$P(A) \leq 1$

Probability always between 0 and 1



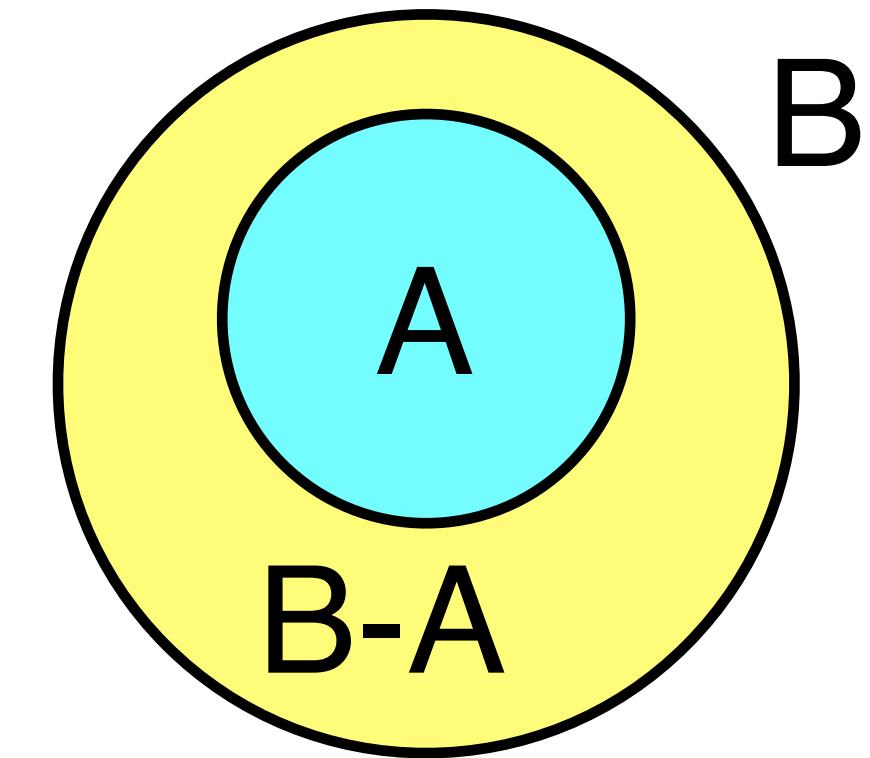
Subset

$$A \subseteq B$$

$$P(A) \leq P(B)$$

$$A \subseteq B$$

$$B = A \cup (B-A)$$



$$P(B) = P(A \cup (B-A)) = P(A) + P(B-A) \geq P(A)$$

Obvious?



Union

$$\max(P(A), P(B)) \leq P(A \cup B) \leq P(A) + P(B)$$

Left \leq

$A, B \subseteq A \cup B$

$P(A), P(B) \leq P(A \cup B)$

Right \leq

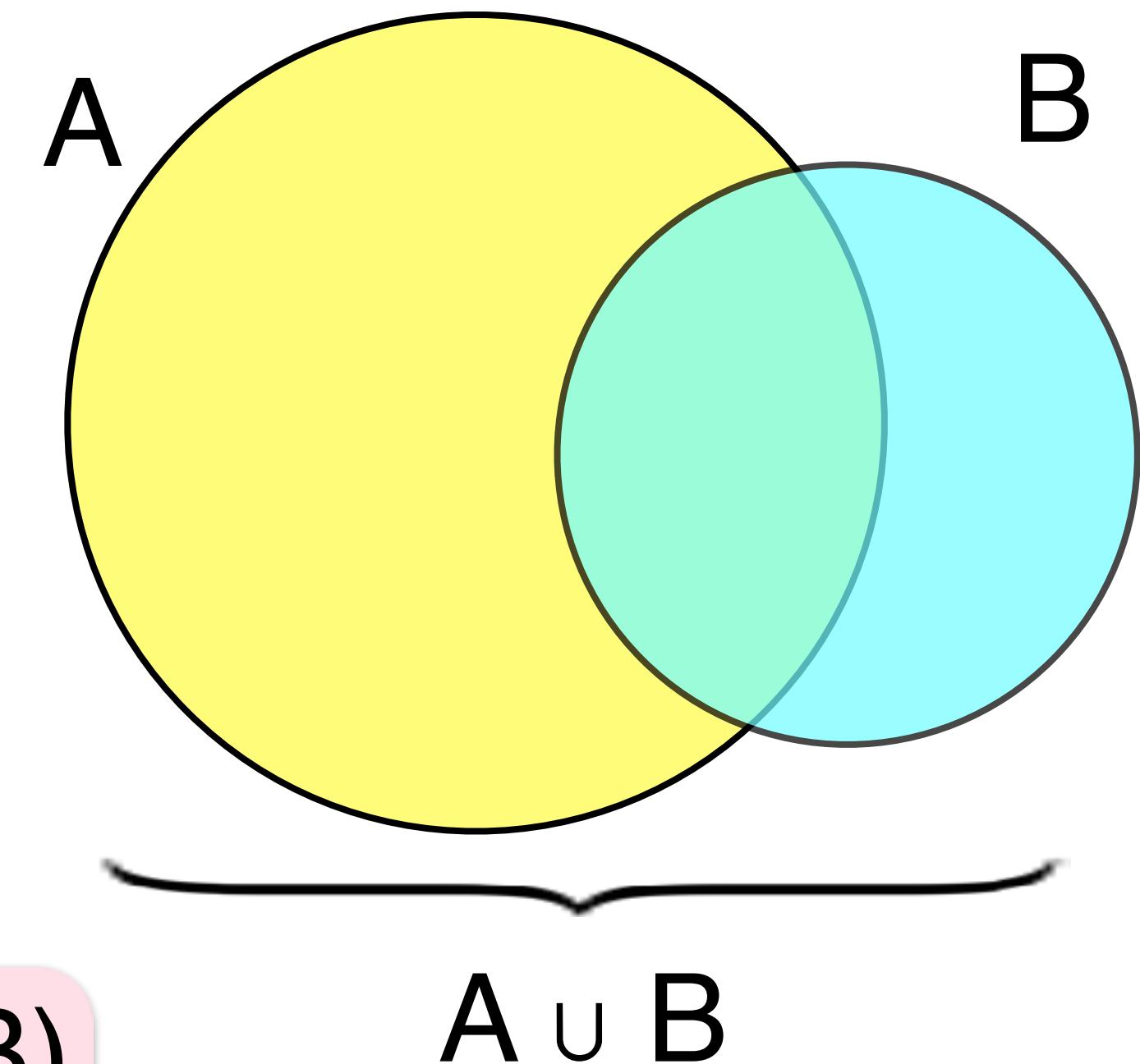


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B)$$

Union bound

Very useful



\$1,400,000 Question

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

- 1 Linda is a bank teller
- 2 Linda is a bank teller and is active in the feminist movement

Please answer in the poll below



The Linda Problem



88 UBC Students

85% Bank teller & activist more likely

B - Bank teller

A - Active in feminist movement

$P(B)$ vs. $P(B \cap A)$

$B \supseteq B \cap A$

$P(B) \geq P(B \cap A)$

Irrational concept of probability

Related Questions

Several conjunction fallacy problems

Bjorn Borg

Preeminent tennis player of late '70's

6 French opens Wimbledon '76-'80

1980

Suppose Borg reaches the '81 Wimbledon Finals.

More likely?

A Borg will lose first set

B Borg will lose first set but win match

72% chose B

Again $B \subseteq A$

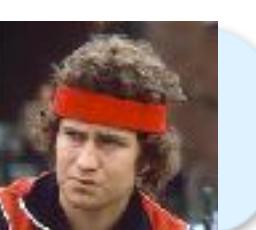
$P(B) \leq P(A)$

1981

Reached final

Won first set

Lost match to



1983

Retired

Age 26



The Ultimate Reward

T & K

Many probability-perception studies

1996

Tversky passed away

2002

Kahneman



best known

“for integrating insights from psychological research into economic science, especially regarding human judgment and decision-making under uncertainty”

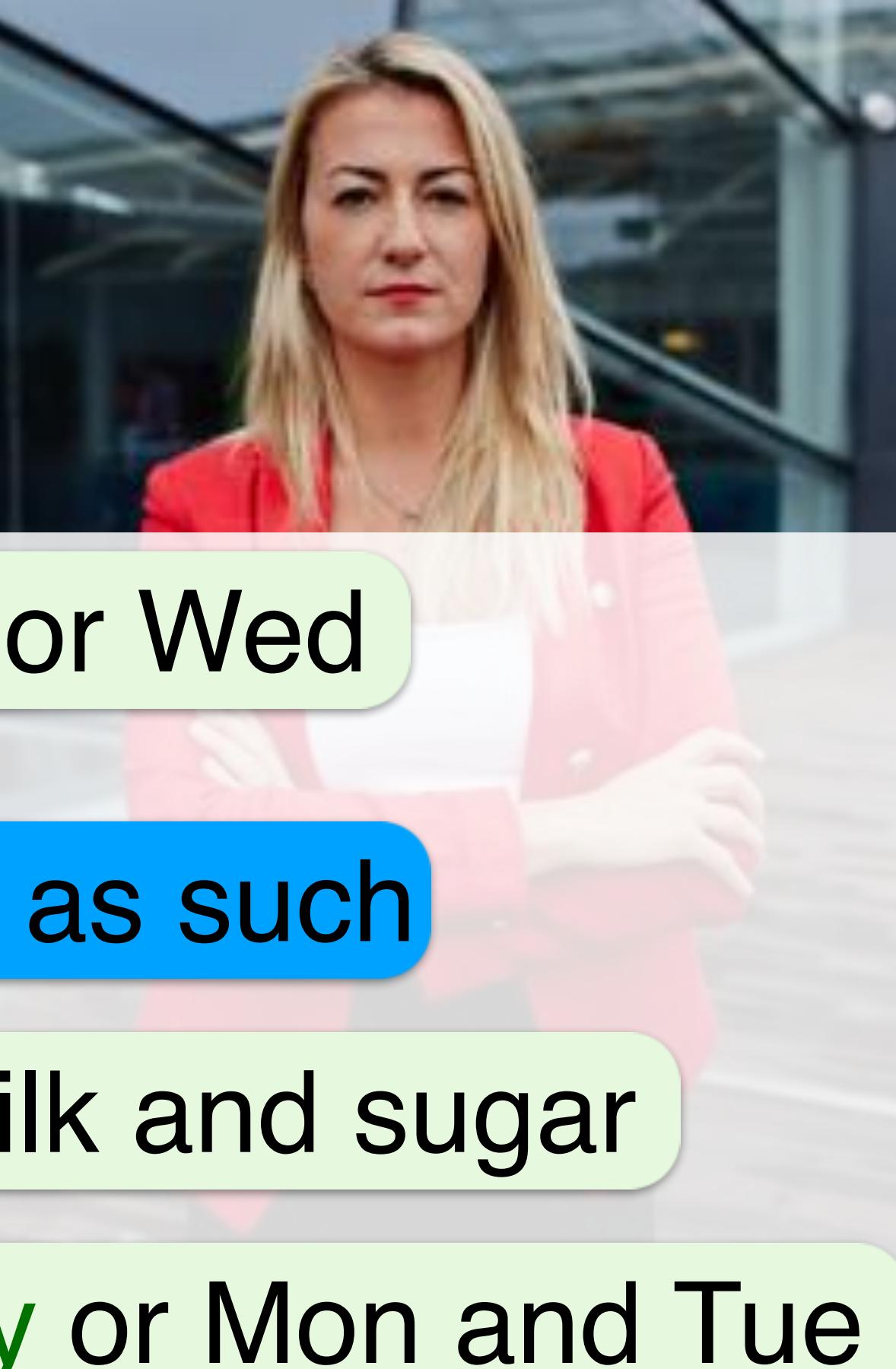


Economics
Nobel Prize



Criticism

Multiple choice and ranking questions often disjoint



Coffee or Tea

Tea hot or cold

Rank Mon, Tue, or Wed

Often, when not explicitly disjoint, we still interpret them as such

With milk or with milk and sugar → Milk only or Milk and sugar

Meet Monday or Monday and Tuesday → Mon only or Mon and Tue

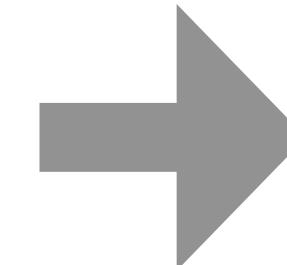
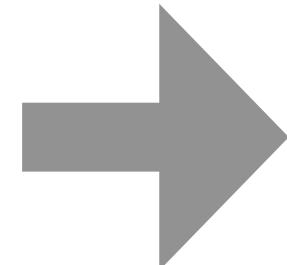
Students may have similarly interpreted the Linda question

Teller or Teller and activist → Teller only or Teller and activist

Simply thought Linda more likely active in movement than not

Bottom Line

Humans only moderately good at estimating probability



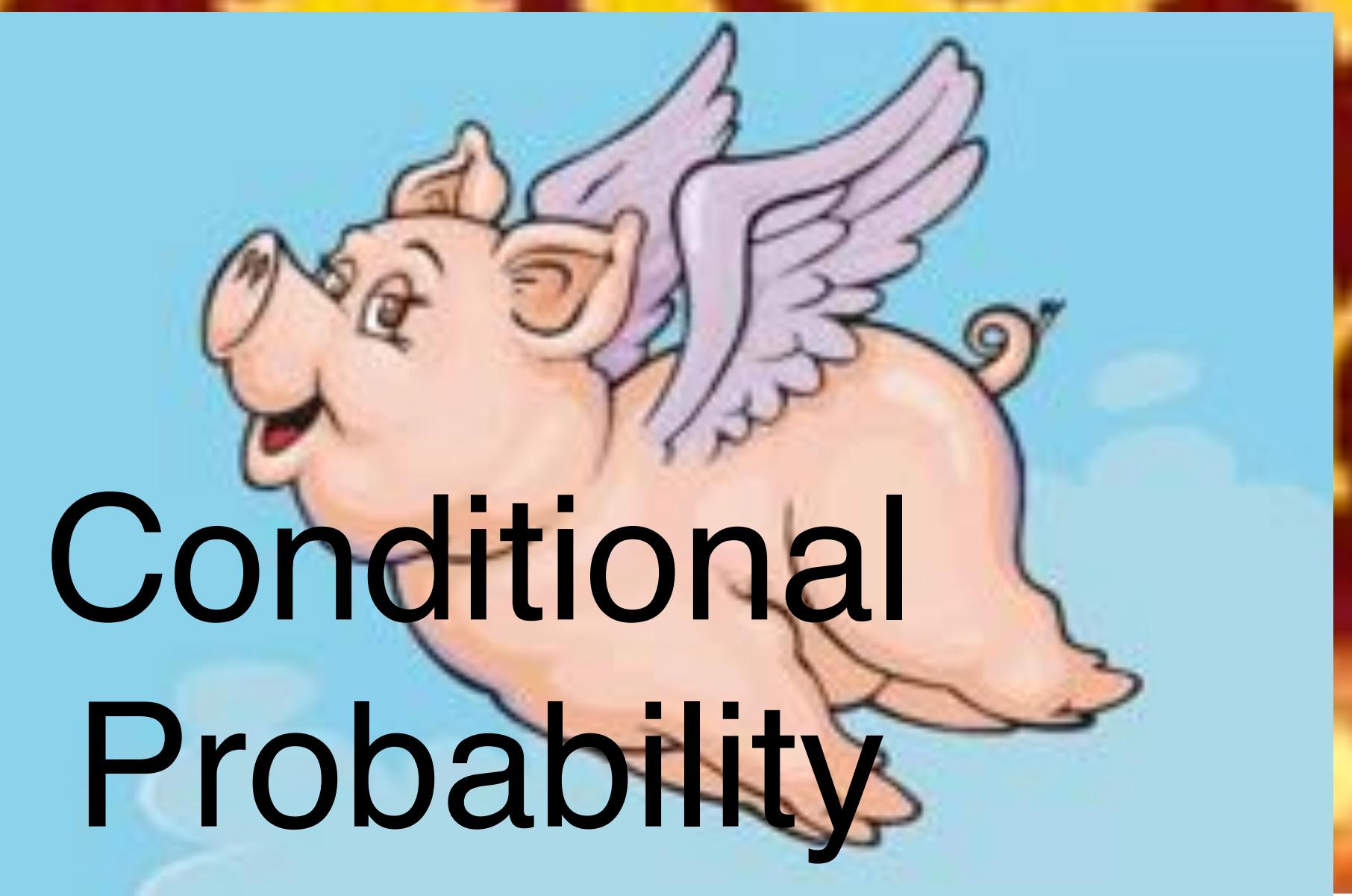
Probability Inequalities

Three Axioms

Inequalities

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\$1.4M question



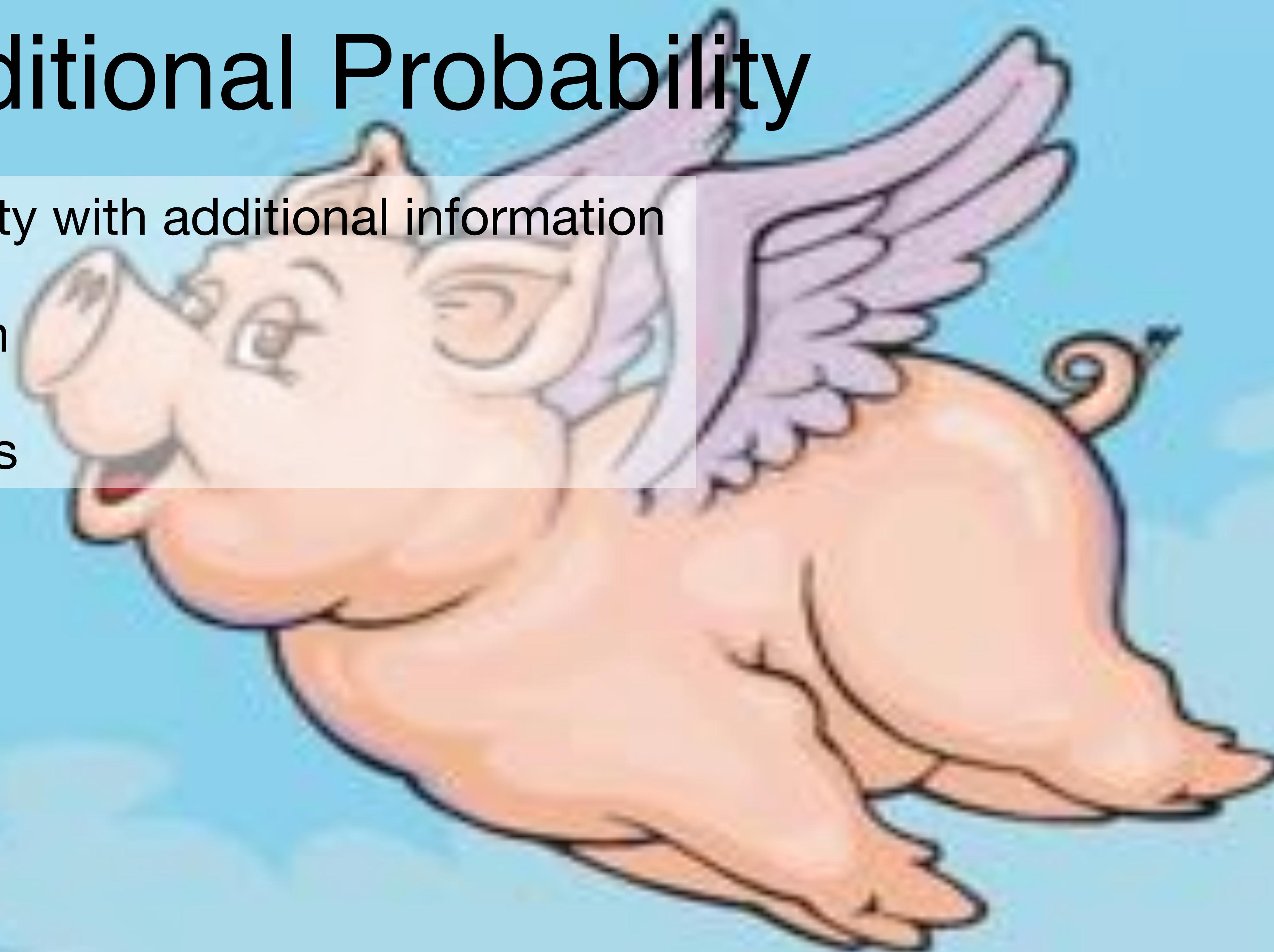
Conditional
Probability

Conditional Probability

Probability with additional information

Definition

Examples



Why Condition

Often have partial information about the world

Modifies event probabilities

Unemployment numbers - stock prices

LeBron James injured - Cavaliers game result

Sunny weekend - beach traffic

Can help

Improve estimates

Determine original unconditional probabilities

Back to Basics

Empirical frequency interpretation of probability

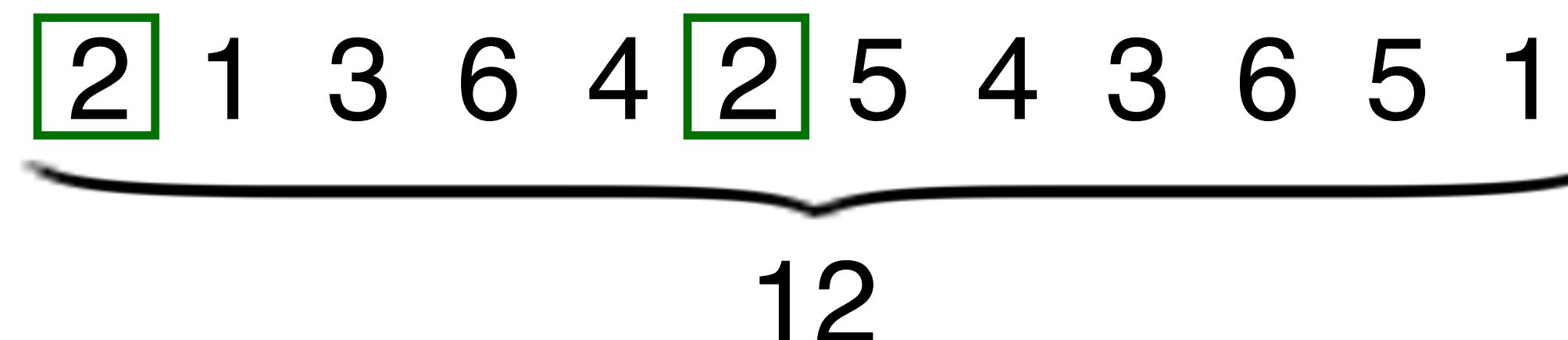
The probability $P(E)$ of an event E is the fraction of experiments where E occurs when the number of experiments grows to infinity

To estimate $P(E)$ repeat the experiment many times, calculate the fraction of experiments where E occurs

Fair Die

$$P(2) = \frac{2}{12} = \frac{1}{6}$$

Estimate



Conditional Probability

Let E and F be events. The conditional probability $P(F | E)$ of F given E is the fraction of times F occurs in experiments where E occurs

To estimate $P(F | E)$ take many samples, consider only experiments where E occurs, and calculate the fraction therein where F occurs too

$$P(2 | \text{Even}) = \frac{2}{6} = \frac{1}{3}$$

Even = {2, 4, 6}

2 1 3 6 4 2 5 4 3 6 5 1

Die

$$P(\{2\}) = P(2) = \frac{1}{6}$$



$$P(2) = ?$$

$$P(2 \mid \text{Odd}) = P(2 \mid \{1,3,5\}) = \frac{0}{6} = 0$$

2 **1** **3** 6 4 2 **5** 4 **3** 6 **5** **1**

$$P(\leq 2) = P(\{1,2\}) = \frac{1}{3}$$



$$P(\leq 2) = ?$$

$$P(\leq 2 \mid \geq 2) = P(\{1,2\} \mid \{2,3,4,5,6\}) = \frac{2}{10} = \frac{1}{5}$$

2 1 **3** **6** **4** **2** 5 4 3 6 5 1

General Events - Uniform Spaces

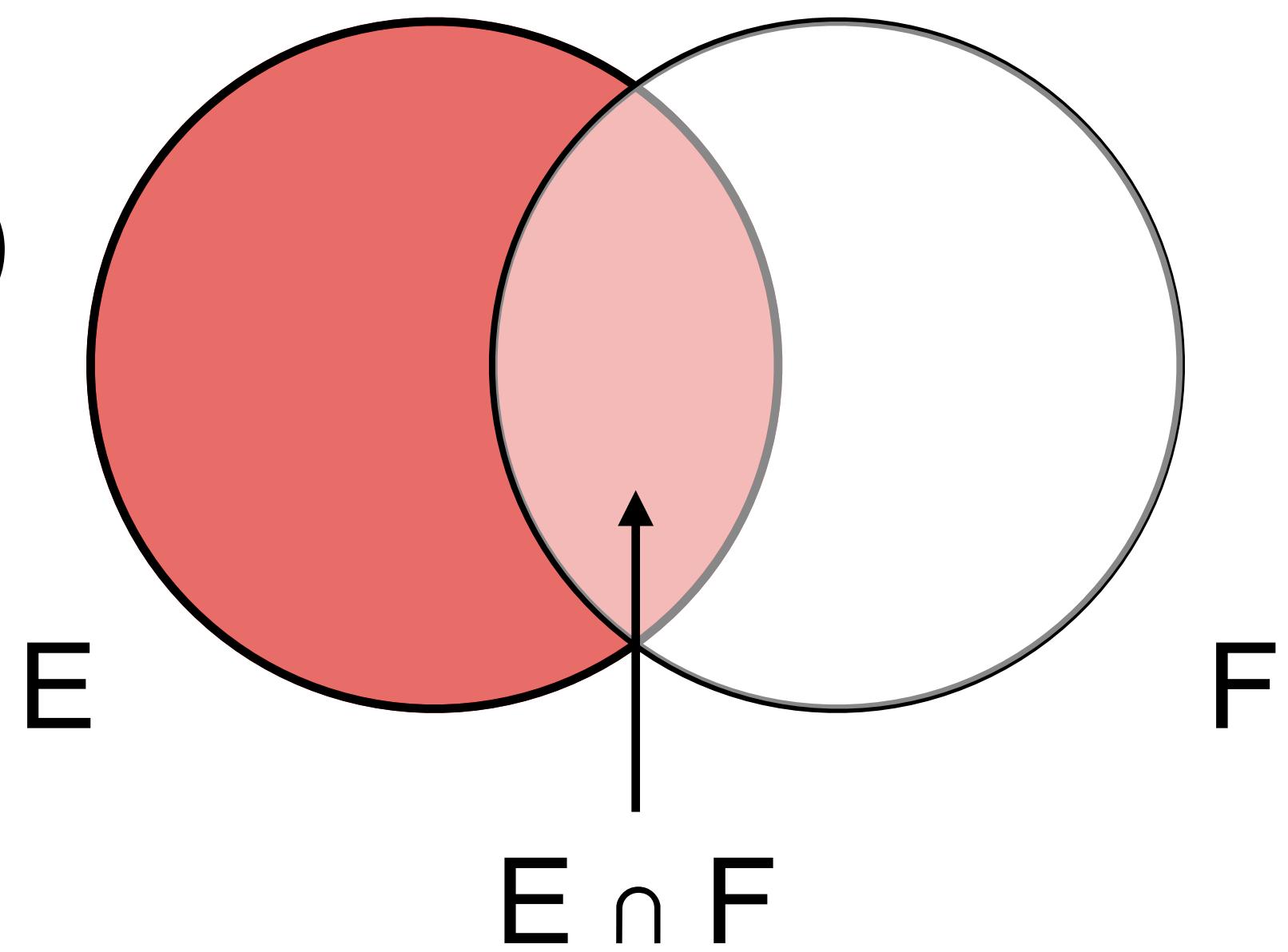
$$P(F | E) = P(X \in F | X \in E)$$

$$= P(X \in E \text{ and } X \in F | X \in E)$$

$$= P(X \in E \cap F | X \in E)$$

$$= P(E \cap F | E)$$

$$= \frac{|E \cap F|}{|E|}$$



Fair Die Again

$$P(\text{Prime} \mid \text{Odd}) = P(\{2,3,5\} \mid \{1,3,5\})$$

$$= \frac{|\{2,3,5\} \cap \{1,3,5\}|}{|\{1,3,5\}|} = \frac{|\{3,5\}|}{|\{1,3,5\}|} = \frac{2}{3}$$

$$P(\{4\} \mid \text{Prime}) = P(\{4\} \mid \{2,3,5\})$$

$$= \frac{|\{4\} \cap \{2,3,5\}|}{|\{2,3,5\}|} = \frac{|\emptyset|}{|\{2,3,5\}|} = 0$$

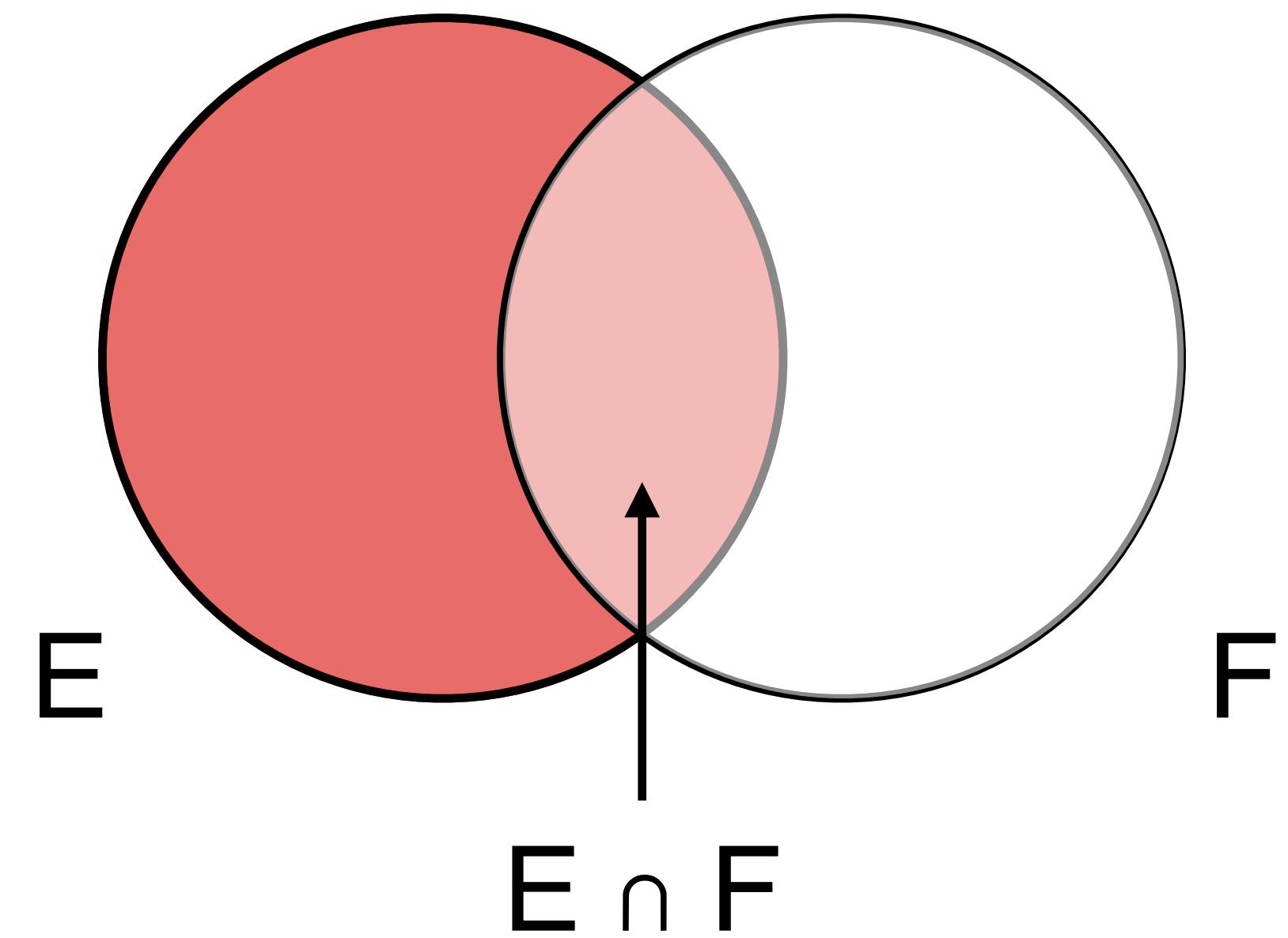
General Spaces

$$P(F | E) = P(X \in F | X \in E)$$

$$= P[X \in E \cap X \in F | X \in E]$$

$$= P[X \in E \cap F | X \in E]$$

$$= \frac{P(E \cap F)}{P(E)}$$



4-Sided Die

$$P(\geq 2 \mid \leq 3)$$

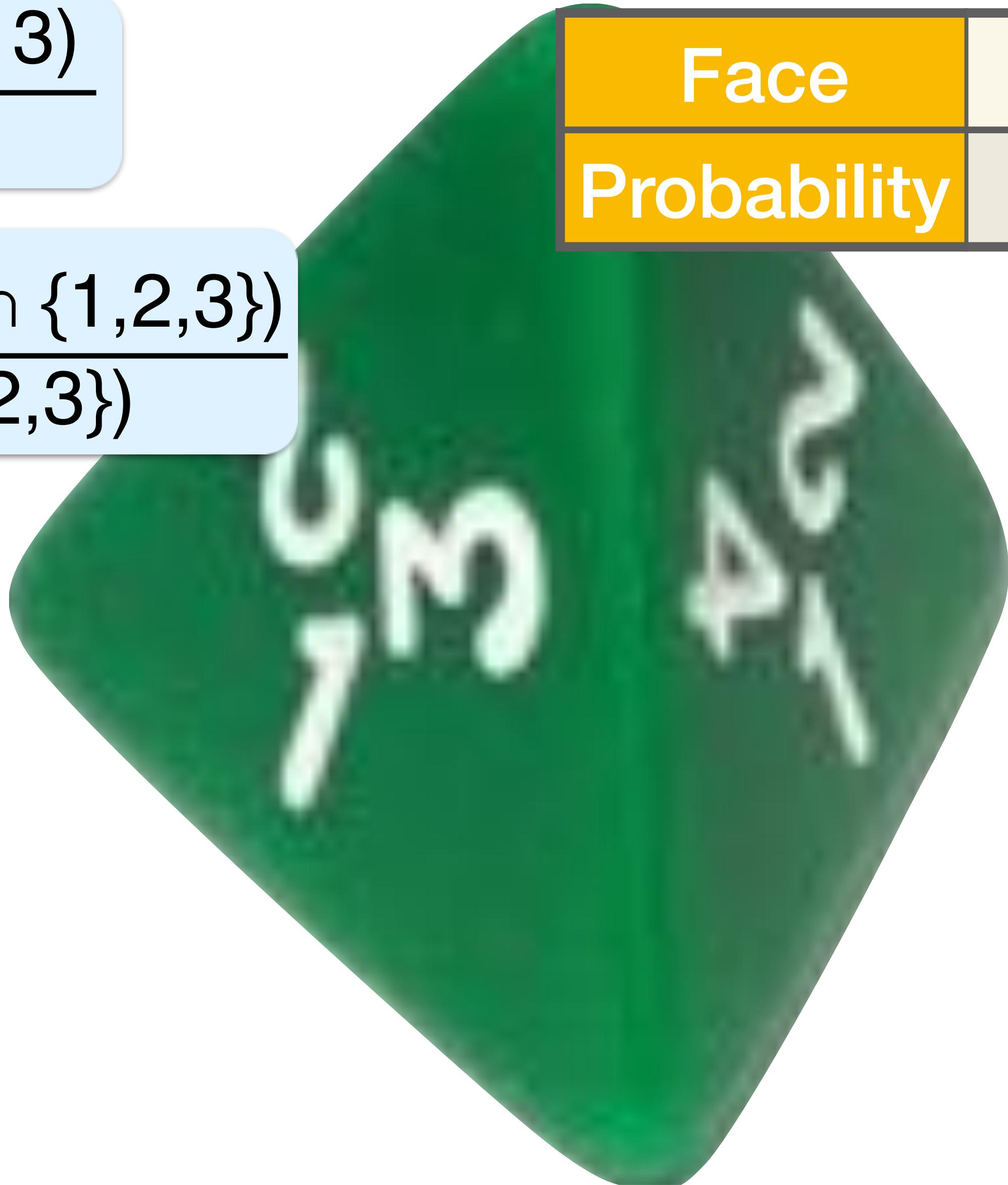
$$= \frac{P(\geq 2 \cap \leq 3)}{P(\leq 3)}$$

$$= \frac{P(\{2,3,4\} \cap \{1,2,3\})}{P(\{1,2,3\})}$$

$$= \frac{P(\{2,3\})}{P(\{1,2,3\})}$$

$$= \frac{.5}{.6} = \frac{5}{6}$$

Face	1	2	3	4
Probability	.1	.2	.3	.4



KIDS
ARE PEOPLE TOO



Conditionals are Probabilities Too

$$P(x | a) \geq 0$$

$$\sum_{x \in \Omega} P(x | a) = 1$$

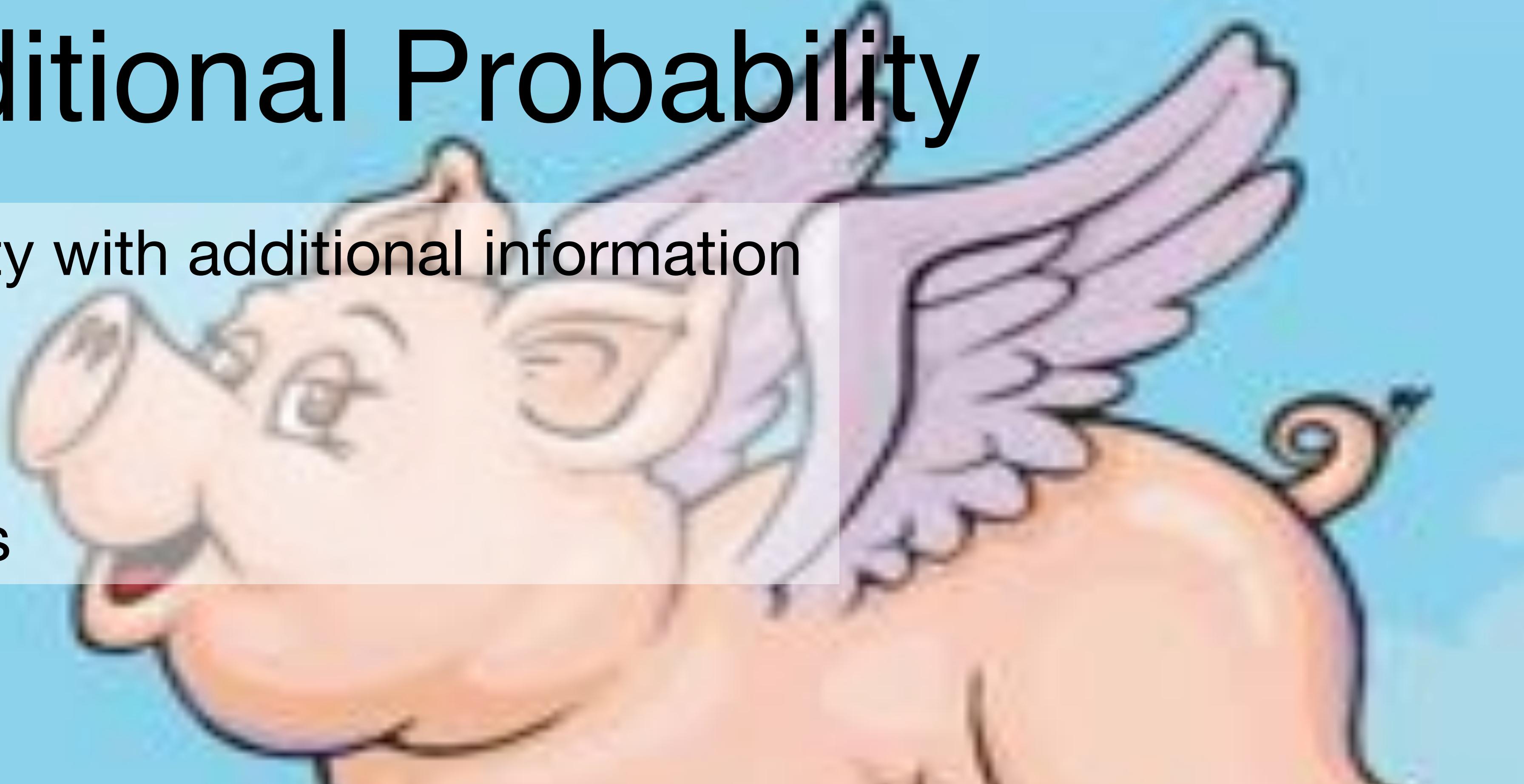
$$P(x | a) \geq 0$$

Conditional Probability

Probability with additional information

Definition

Examples



Independence





Independence

Independence

Informal and formal definition

Examples



Motivation

$$P(F | E) > P(F)$$

$$P(F | E) < P(F)$$

$$P(2 | \text{Even}) = \frac{1}{3} > \frac{1}{6} = P(2)$$

$$P(2 | \text{Odd}) = 0 < \frac{1}{6} = P(2)$$

$E \nearrow$ probability of F

$E \searrow$ probability of F

$$P(F | E) = P(F)$$

$$P(\text{Even} | l \leq 4) = \frac{1}{2} = P(\text{Even})$$

E neither \nearrow nor \searrow probability of F

Whether or not E occurs, does not change P(F)

motivation \rightarrow intuitive definition \rightarrow formal

Independence - Intuitive

Events E and F are **independent**, denoted $E \perp\!\!\!\perp F$, if the occurrence of one does not affect the other's probability

$$P(F | E) = P(F)$$

Visually

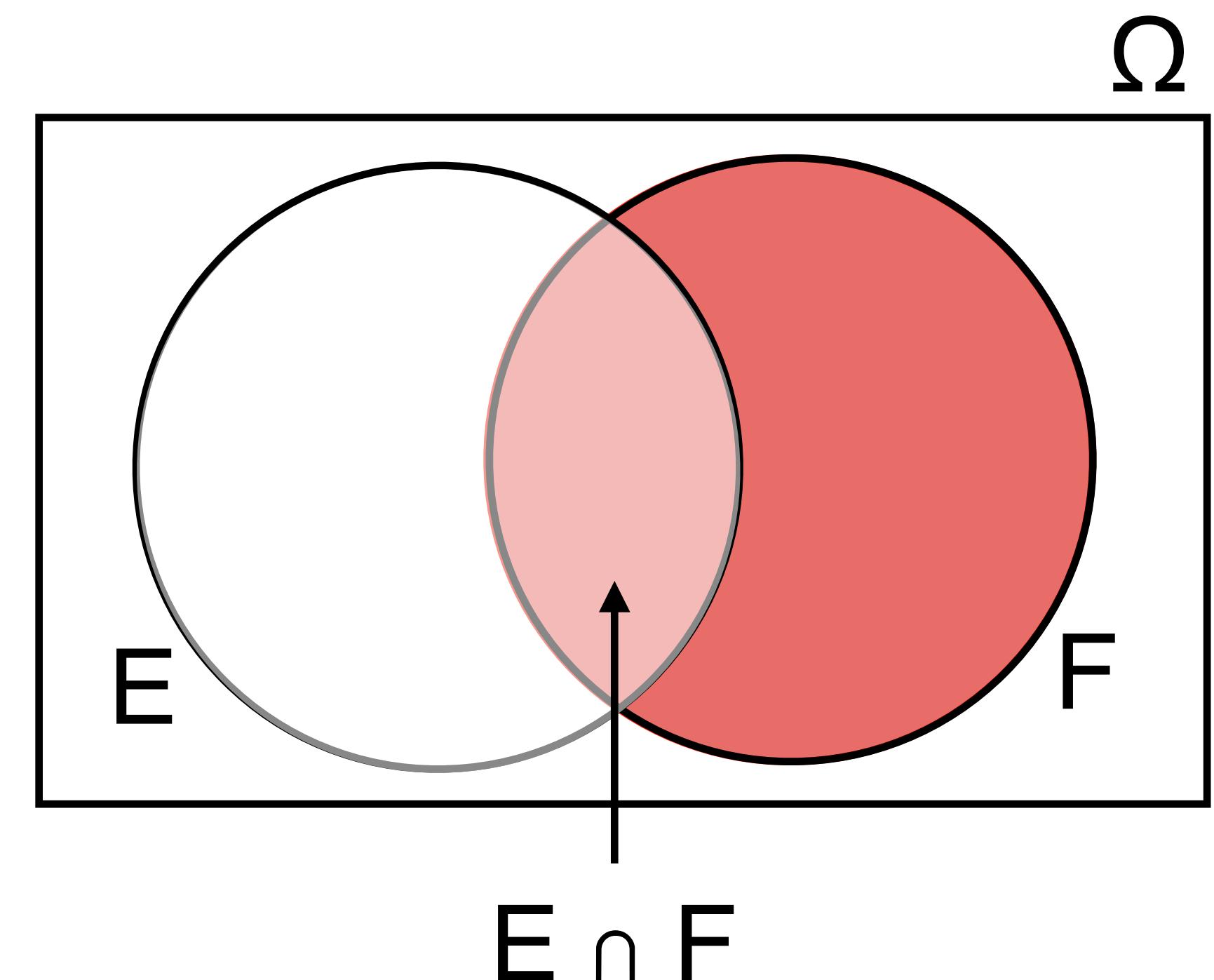
$$P(F) = \frac{P(F)}{P(\Omega)}$$

$$P(F | E) \triangleq \frac{P(E \cap F)}{P(E)}$$

F as a fraction of Ω

=

$E \cap F$ as a fraction of E



Two issues

Asymmetric

Undefined if $P(E)=0$

Independence - Formal

Informally

$$P(F) = P(F | E) \triangleq \frac{P(E \cap F)}{P(E)}$$

Asymmetric

Undefined if $P(E)=0$

Formally

E and F are **independent** if $P(E \cap F) = P(E) \cdot P(F)$

Otherwise, **dependent**

Symmetric



Applies when $P = 0$



Implies

intuitive def. $P(F|E) = P(F)$

$P(E|F) = P(E)$

$$P(F | \bar{E}) = P(F)$$

$$P(E | \bar{F}) = P(E)$$

Non-Surprising Independence

Two coins

H_1

First coin heads

$P(H_1) = \frac{1}{2}$



H_2

Second coin heads

$P(H_2) = \frac{1}{2}$



$H_1 \cap H_2$

Both coins heads

$P(H_1 \cap H_2) = \frac{1}{4}$

$$P(H_1 \cap H_2) = \frac{1}{4} = P(H_1) \cdot P(H_2)$$

$H_1 \perp\!\!\!\perp H_2$

Not surprising as two separate coins

Can have $\perp\!\!\!\perp$ even for one experiment

Single Die

Three events

Event	Set	Probability
Prime	{ 2, 3, 5 }	$\frac{1}{2}$
Odd	{ 1, 3, 5 }	$\frac{1}{2}$
Square	{ 1, 4 }	$\frac{1}{3}$

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	Product	=?	Independence
Prime \cap Odd	{ 3, 5 }	$\frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	\neq	dependent
Prime \cap Square	\emptyset	0	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$	\neq	dependent
Odd \cap Square	{ 1 }	$\frac{1}{6}$	$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$	$=$	independent

Three Coins

Three events

Event	Description	Set	Probability
H_1	first coin heads	{h***}	$\frac{1}{2}$
H_2	second coin heads	{*h*}	$\frac{1}{2}$
HH	exactly 2 heads in a row	{hht, thh}	$\frac{1}{4}$

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	=?	Product	Independence
$H_1 \cap H_2$	{hh*}	$\frac{1}{4}$	=	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	independent
$H_2 \cap HH$	{hht, thh}	$\frac{1}{4}$	\neq	$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$	dependent
$H_1 \cap HH$	{hht}	$\frac{1}{8}$	=	$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$	independent

Independence of Ω and \emptyset

$\forall A$

$$P(\Omega \cap A) = P(A) = P(\Omega) \cdot P(A)$$

$\Omega \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of Ω

$\forall A$

$$P(\emptyset \cap A) = P(\emptyset) = P(\emptyset) \cdot P(A)$$

$\emptyset \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of \emptyset

Independence

Informal and formal definitions

Examples



Total Probability

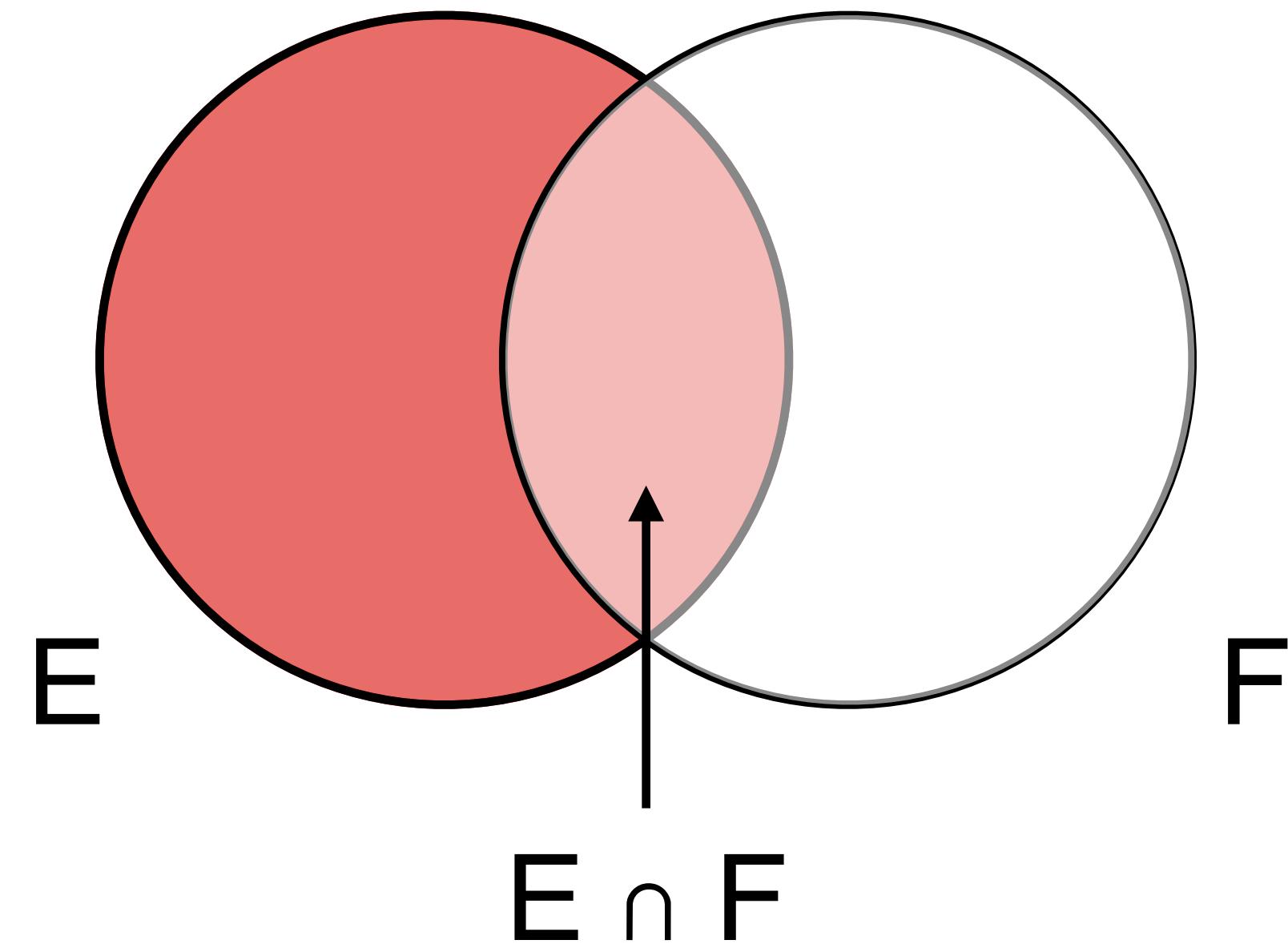
Sequential Probability

Product Rule

Recall conditional probability

$$P(F | E) = \frac{P(E \cap F)}{P(E)}$$

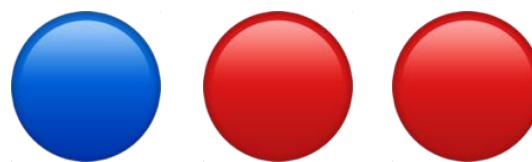
$$P(E \cap F) = P(E) \cdot P(F | E)$$



Helps calculate regular (not conditional) probabilities

Sequential Selection

1 blue, 2 red balls



Draw 2 balls without replacement

$P(\text{both red}) = ?$

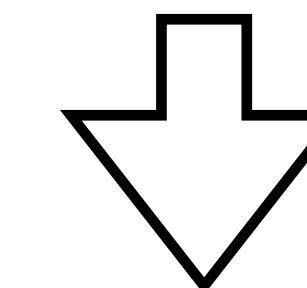
R_i - i'th ball is red

$$P(\text{both red}) = P(R_1, R_2)$$

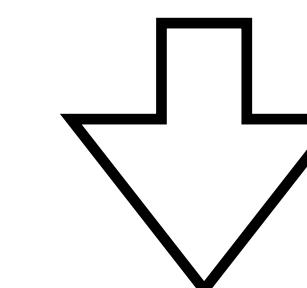
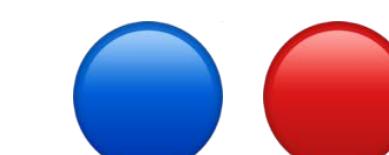
$$= P(R_1) \cdot P(R_2 | R_1)$$

$$= \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{1}{3}$$



$$P(R_1) = \frac{2}{3}$$



$$P(R_2 | R_1) = \frac{1}{2}$$



General Product Rule

For 3 events

$$P(E \cap F \cap G) = P((E \cap F) \cap G)$$

$$= P(E \cap F) \cdot P(G | E \cap F)$$

$$= P(E) \cdot P(F | E) \cdot P(G | E \cap F)$$

Similarly for more events

Odd Ball

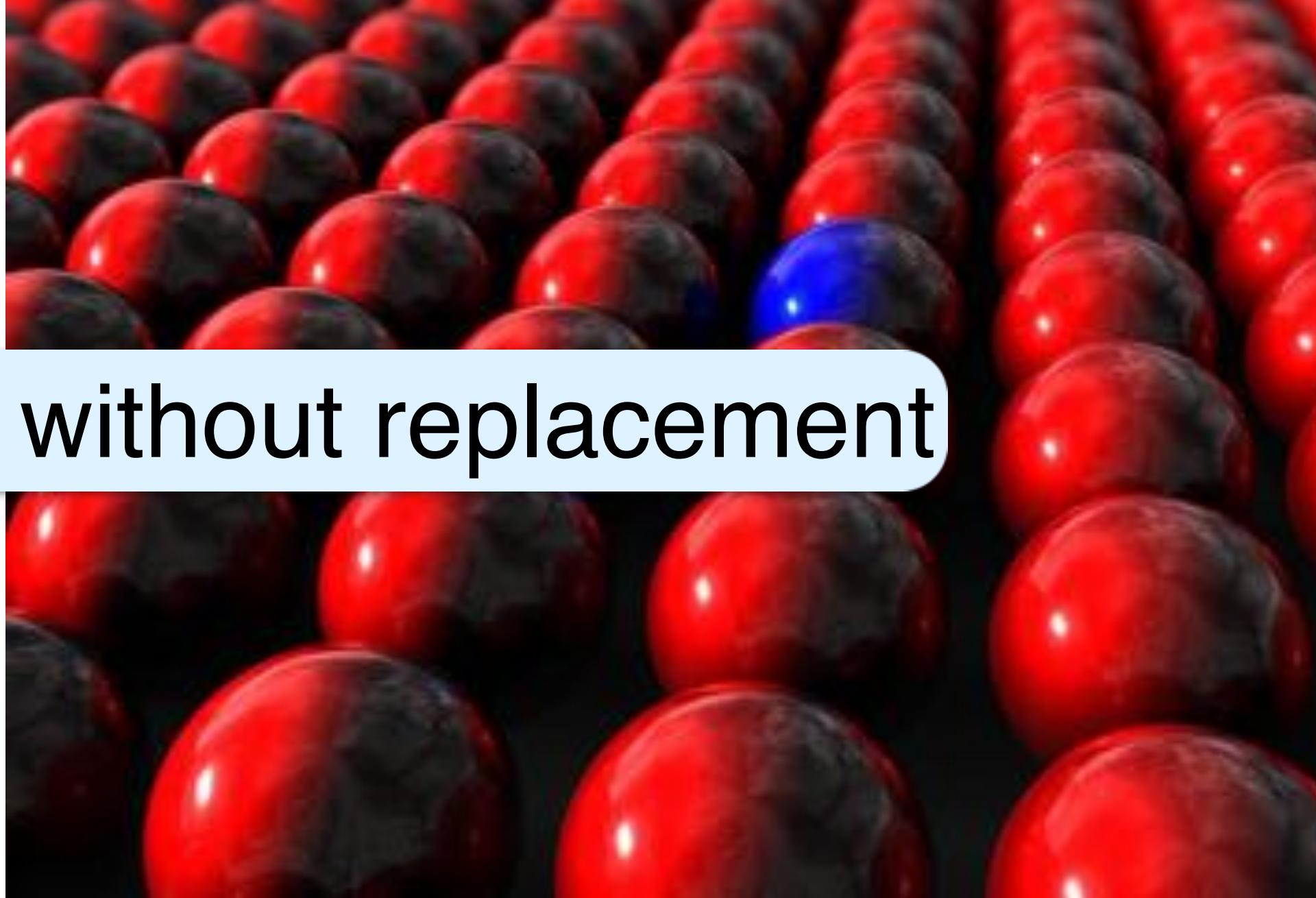
n-1 red balls and one blue ball

Pick n balls without replacement

$P(\text{last ball is blue}) = ?$

R_i - ith ball is red

$R^i - R_1, R_2, \dots, R_i$



$P(\text{last ball blue}) = P(R_1)P(R_2|R_1)P(R_3|R_2) \dots P(R_{n-1}|R^{n-2})$

$$= \frac{n-1}{n} \frac{n-2}{n-1} \frac{n-3}{n-2} \cdots \frac{2}{3} \frac{1}{2} = \frac{1}{n}$$

Or.. Arrange in row, probability last ball is blue = 1/n

The Birthday Paradox

How many people does it take so that two will likely share a birthday?

Assume that every year has 365 days

Everyone is equally likely to be born on any given day

Probabilistically

Choose n random integers, each $\in \{1, \dots, 365\}$, with replacement

$B(n)$ - probability that two (or more) are the same

For which n does $B(n)$ exceed, say, $1/2$?

Some first think it $n \approx 365$, but in fact much smaller

First Attempt

Consider the n people in order, say alphabetically

List their birthdays

2, 10, 365, 180, 10, ...

Selection with replacement

Set of all possible birthdays sequences

$\Omega = \{1, 2, \dots, 365\}^n$

$|\Omega| = 365^n$

Individual birthday uniform



Ω uniform

B_n

{sequences with repetition}

$P(\text{repetition}) = |B_n| / |\Omega|$

Evaluating $|B_n|$ involved

Complement

B_n

n people have birthday repetition

B_n^c

n people, no two share a birthday

Evaluate sequentially

Person i different b/day from all previous



Calculation

$$1 - x \leq e^{-x}$$

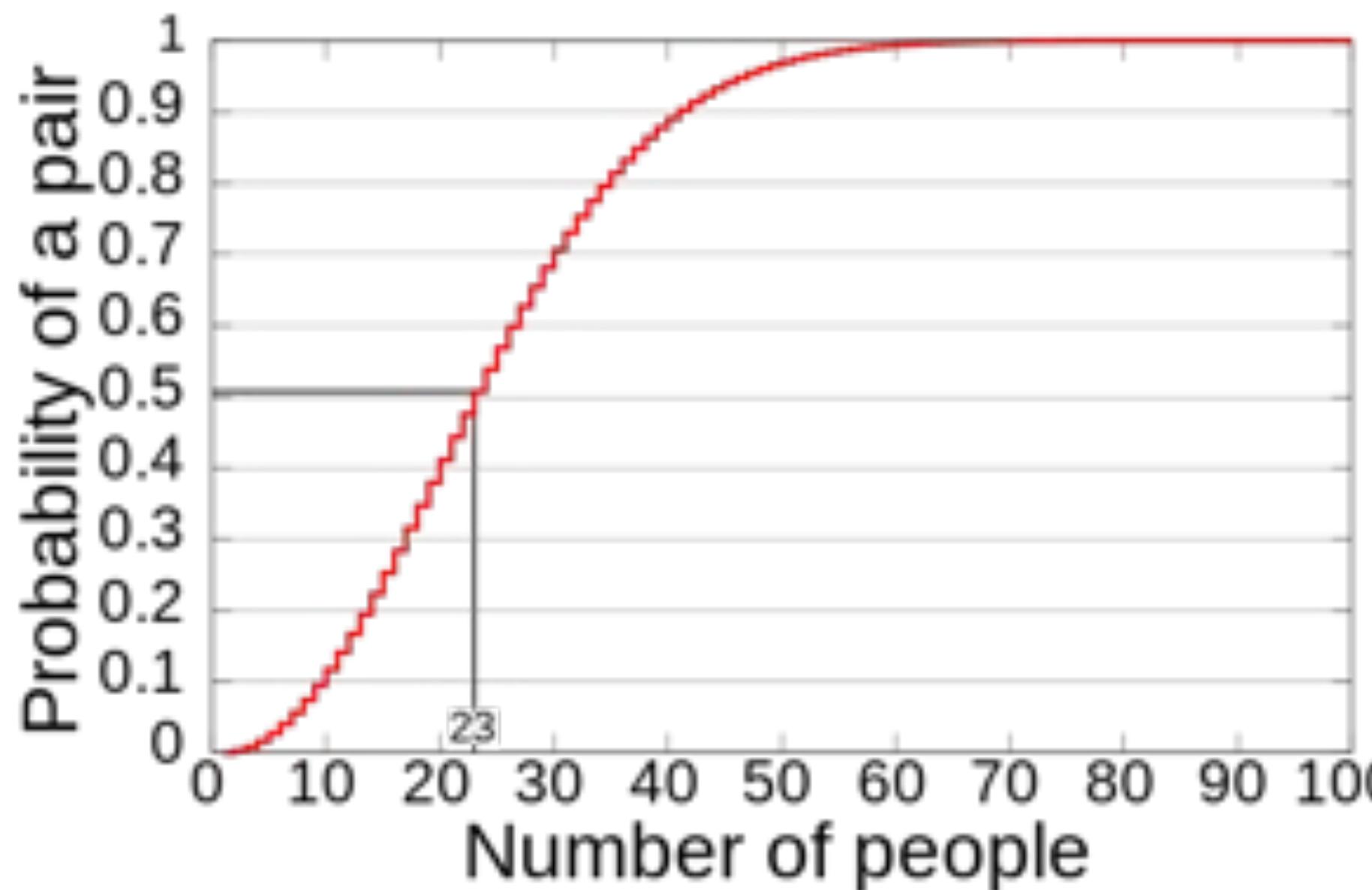
Among n people

P(no two people share a birthday)

When the probability is 0.5

$$-\frac{n^2}{2 \cdot 365} = \ln 0.5 = -\ln 2$$

$$n \approx \sqrt{-2 \cdot 365 \cdot \ln 0.5} = 22.494$$



$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365-n+1}{365}$$

$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$$\leq \prod_{i=1}^n e^{-\frac{i}{365}}$$

$$= \exp \left(-\frac{1}{365} \cdot \sum_{i=1}^{n-1} i \right)$$

$$= \exp \left(-\frac{n(n-1)}{2 \cdot 365} \right)$$

$$\approx \exp \left(-\frac{n^2}{2 \cdot 365} \right) = 0.5$$

This lecture: Sequential Probability

Next: Total Probability

Total Probability



Divide and Conquer

When evaluating probability of an event

Sometimes easier to split event into different parts

Calculate probability of each part

Add probabilities

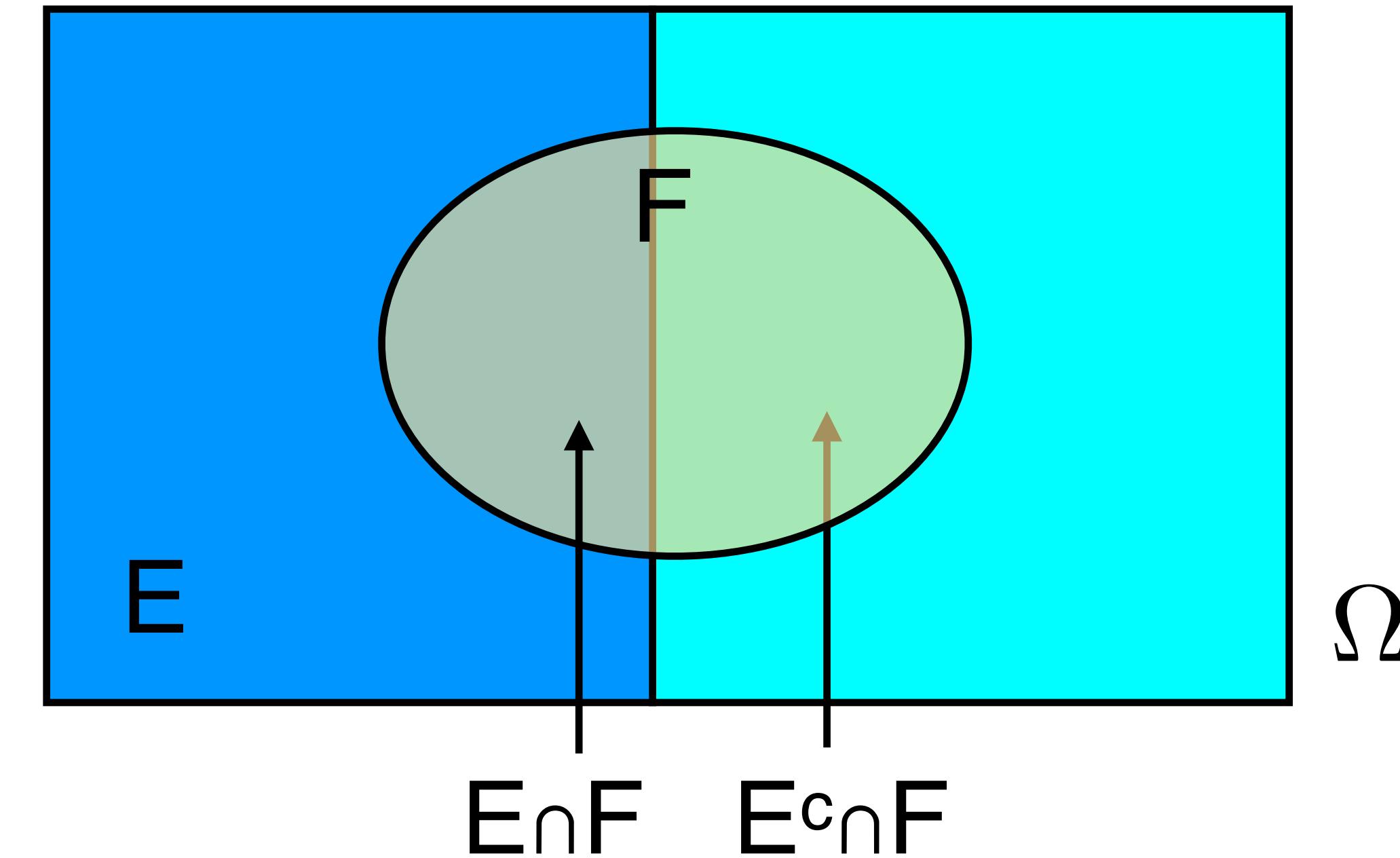
Law of Total Probability

E,F events, $P(F)=?$

$$F = E \cap F \cup E^c \cap F$$

$$P(F) = P(E \cap F) + P(E^c \cap F)$$

$$= P(E) \cdot P(F | E) + P(E^c) \cdot P(F | E^c)$$



Product rule

2 Fair Coins

H_i - coin i is h

$\exists H$ - at least one h

$P(\exists H)$?

$$P(\exists H) = P(H_1 \cap \exists H) + P(H_1^C \cap \exists H)$$

$$= P(H_1) \cdot P(\exists H | H_1) + P(H_1^C) \cdot P(\exists H | H_1^C)$$

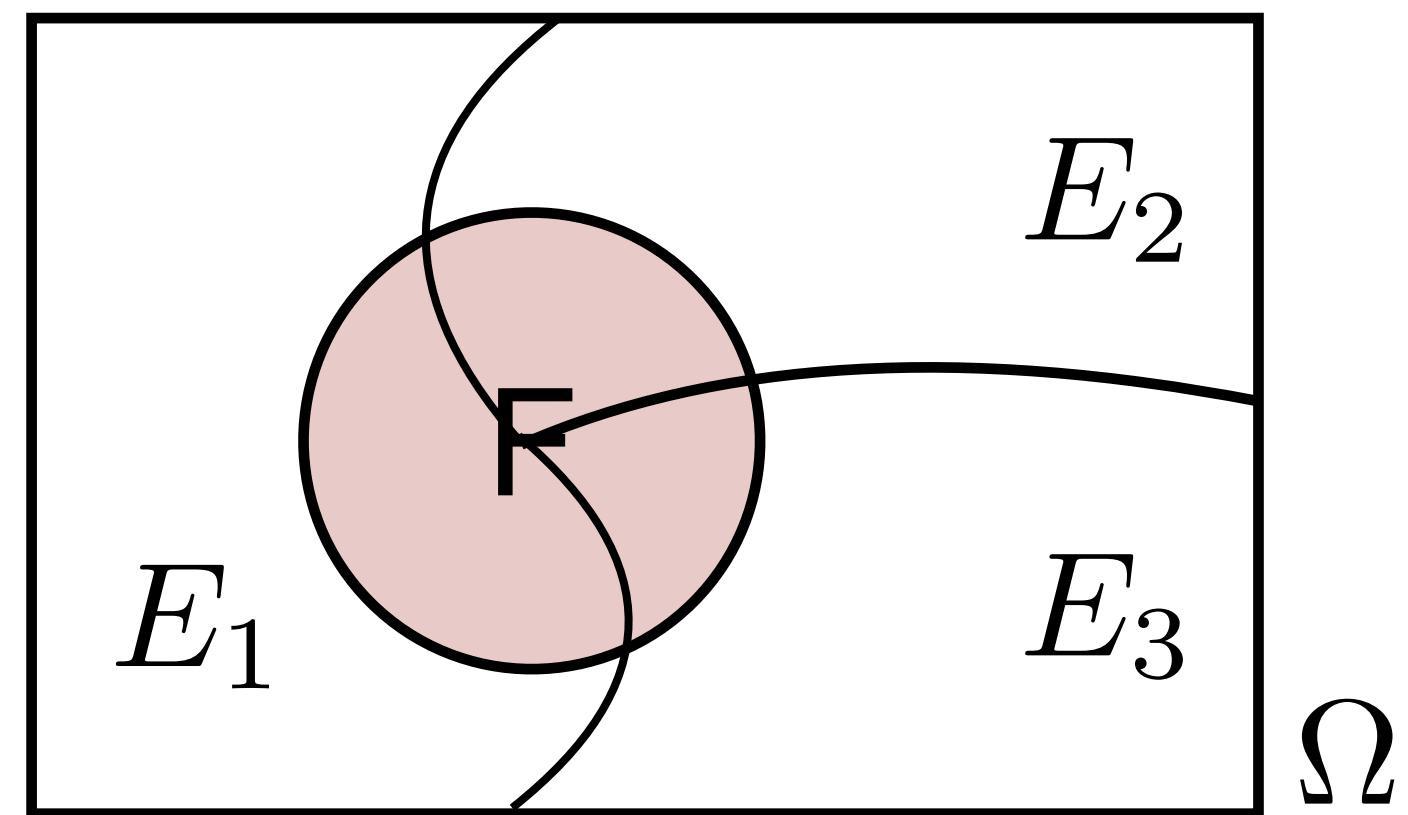
$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

h h
h t
t h
t t

Total Probability - n Conditions

Let E_1, E_2, \dots, E_n partition Ω

$$F = \bigcup_{i=1}^n (E_i \cap F)$$



$$P(F) = \sum_{i=1}^n P(E_i \cap F) = \sum_{i=1}^n P(E_i) \cdot P(F|E_i)$$

2 Dice

D_i - outcome of die i

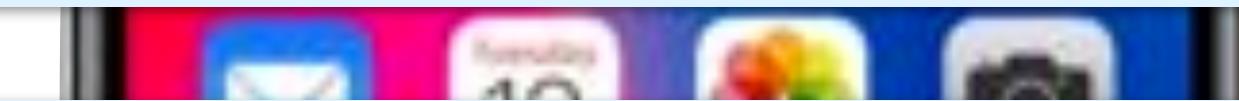
$S = D_1 + D_2$ sum of 2 dice

$$P(S = 5) = ?$$

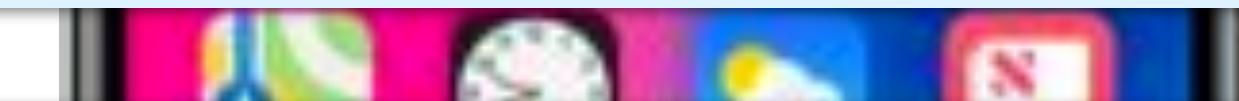
$$\begin{aligned} P(S = 5) &= \sum_{i=1}^4 P(D_1 = i) \cdot P(D_2 = 5 - i \mid D_1 = i) \\ &= \sum_{i=1}^4 P(D_1 = i) \cdot P(D_2 = 5 - i) \\ &= 4 \cdot \frac{1}{36} = \frac{1}{9} \end{aligned}$$

iPhone X

Three factories produce 50%, 30%, and 20% of iPhones



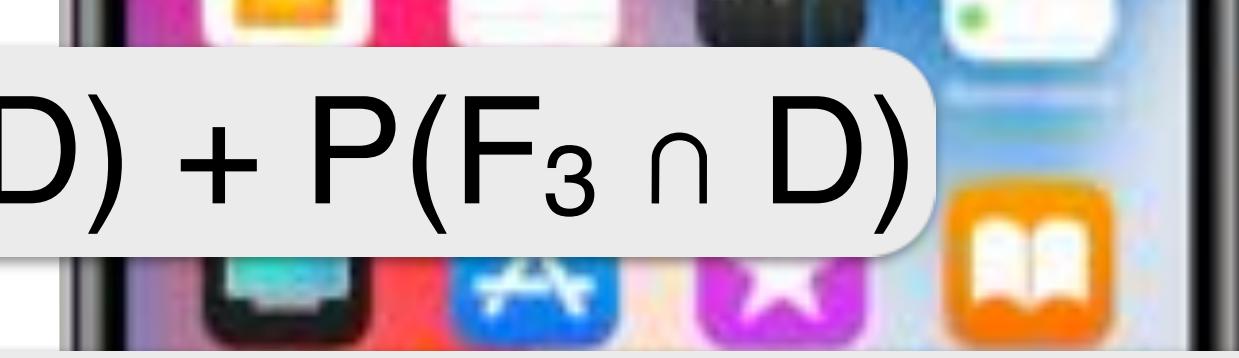
Their defective rates are 4%, 10%, and 5% respectively



What is the overall fraction of defective iPhones?



$$P(D) = P(F_1 \cap D) + P(F_2 \cap D) + P(F_3 \cap D)$$



$$= P(F_1)P(D | F_1) + P(F_2)P(D | F_2) + P(F_3)P(D | F_3)$$



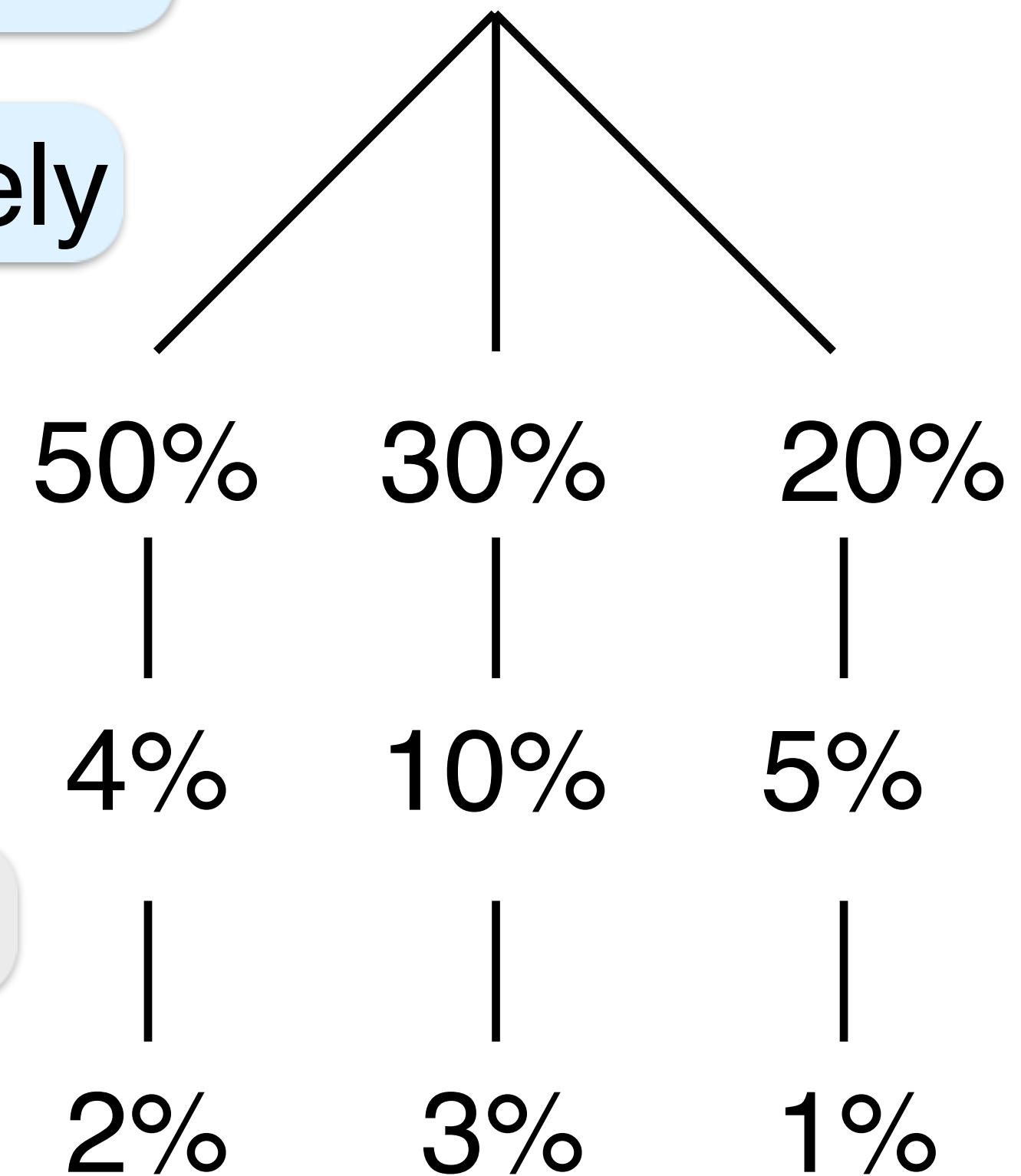
$$= .5 \times .04 + .3 \times .1 + .2 \times .05$$

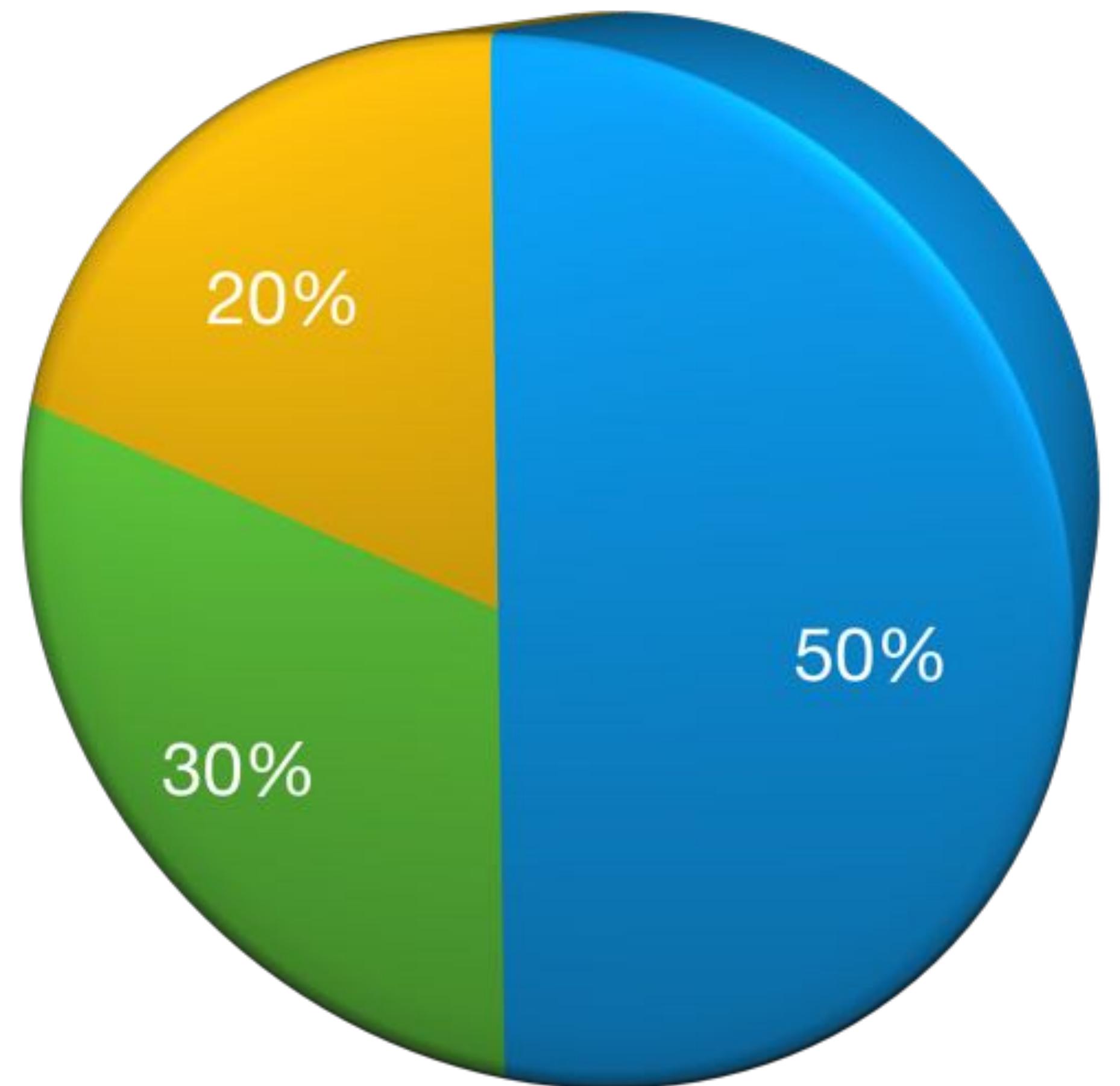


$$= .02 + .03 + .01$$



$$= .06$$





This lecture: Total Probability

Next: Bayes' Rule

Bayes' Rule



Asymmetry

Bought Google at IPO → Wealthy ← ?

Alive today → Born after 1800 ← ?

Forward - Backward

At times

$P(F | E)$ - easy

$P(E | F)$ - hard

2 coins

H_i - coin i is h

$\exists H$ - at least one h

$P(\exists H | H_1) = 1$

$P(H_1 | \exists H)?$

2 dice

D_i - face of die i

$S = D_1 + D_2$ sum of 2 faces

$P(S=5 | D_1=2) = P(D_2=3) = \frac{1}{6}$

$P(D_1=2 | S=5)?$

Bayes' Rule

Method for converting $P(F | E)$ to $P(E | F)$

Bayes' Rule

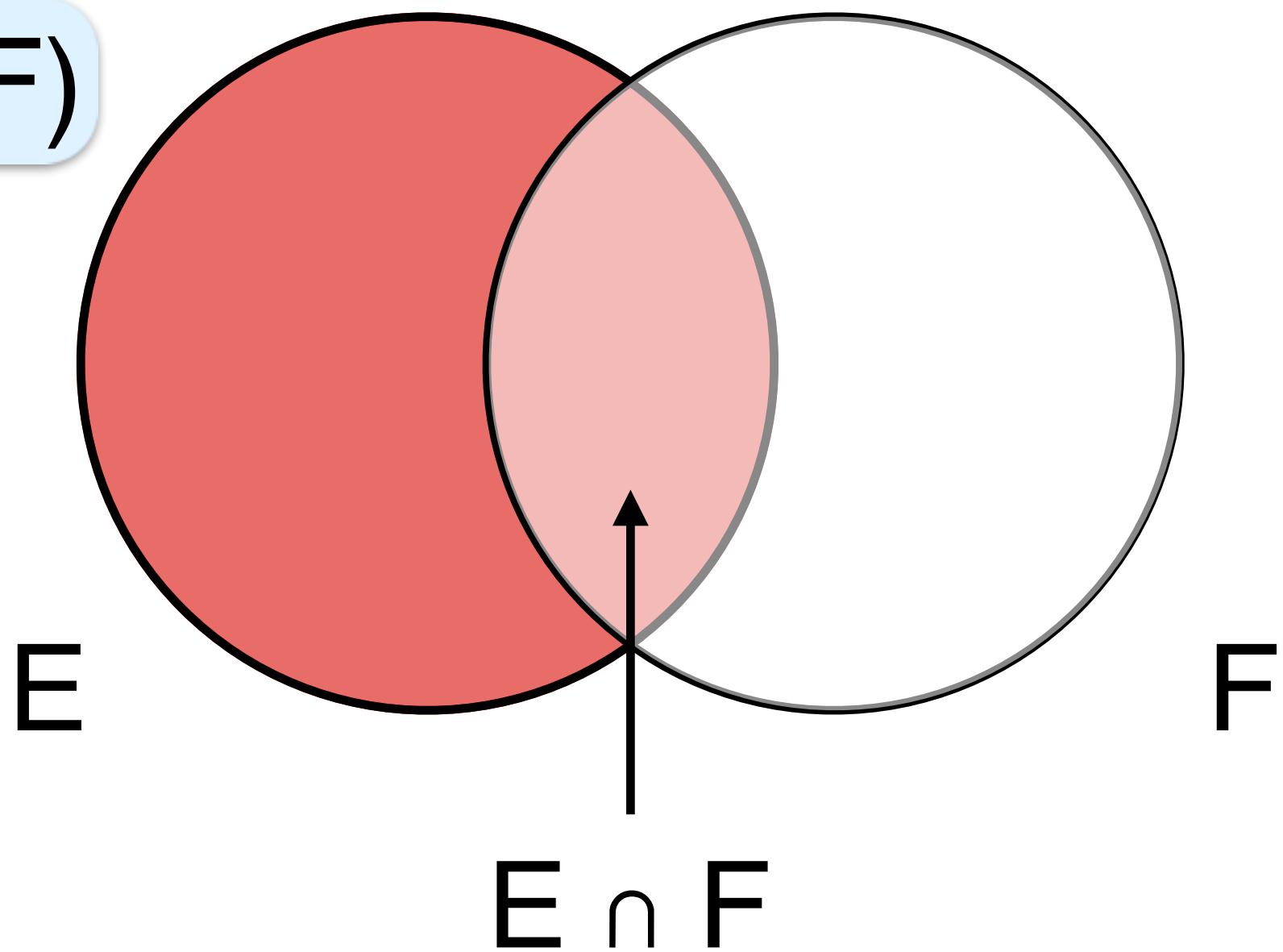
P(E), P(F)

Given $P(F | E)$ (and a bit more) determine $P(E | F)$

$$P(E | F) = \frac{P(E) \cdot P(F | E)}{P(F)}$$

μ -proof

$$P(E | F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \cdot P(F | E)}{P(F)}$$



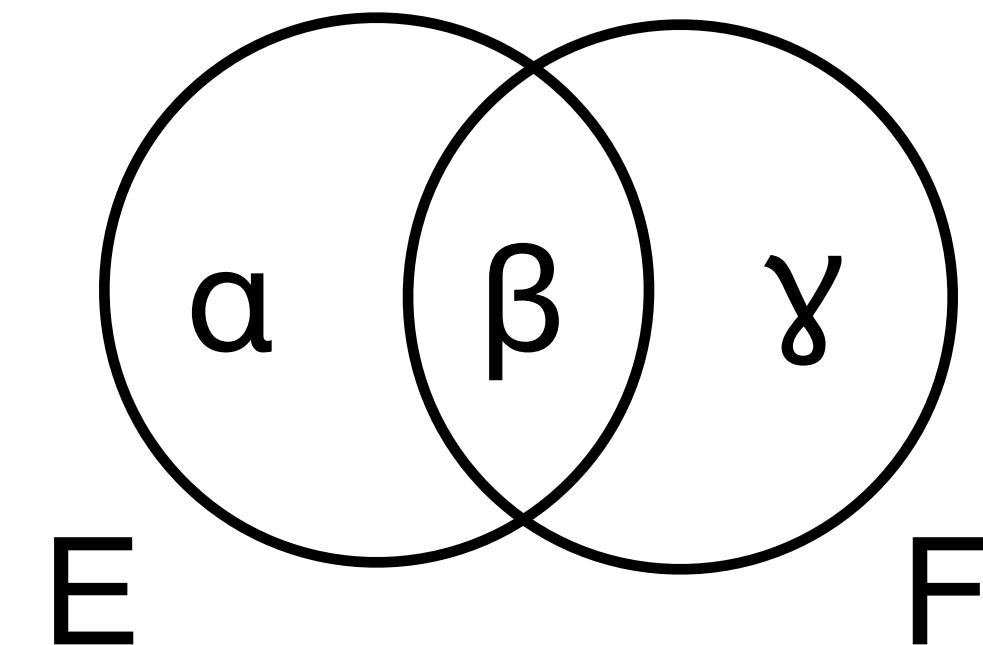
Two More Views

$$P(E|F) = \frac{P(E) \cdot P(F|E)}{P(F)}$$

$$P(F) \cdot P(E|F) = P(E \cap F) = P(E) \cdot P(F|E)$$

$$P(F|E) = \frac{\beta}{\alpha + \beta}$$

$$P(E|F) = \frac{\beta}{\beta + \gamma}$$



$$P(E|F) = \frac{\beta}{\beta + \gamma} = \frac{\beta}{\alpha + \beta} \cdot \frac{\alpha + \beta}{\beta + \gamma} = \frac{P(F|E) \cdot P(E)}{P(F)}$$

Two Fair Coins

H_i - coin i is h

$\exists H$ - at least one h

$$P(H_1 | \exists H) = ?$$

$$P(H_1 | \exists H) = P(\exists H | H_1) \cdot \frac{P(H_1)}{P(\exists H)} = 1 \cdot \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

$$P(\exists H | H_1) = 1$$

$$P(H_1) = \frac{1}{2}$$

$$P(\exists H) = \frac{3}{4}$$

Last video

$$P(H_1 | \exists H) = \frac{|H_1 \cap \exists H|}{|\exists H|}$$

$$\exists H \left\{ \begin{array}{c} h \ h \\ h \ t \\ t \ h \\ t \ t \end{array} \right\} H_1$$

Two Fair Dice

$S = D_1 + D_2$ Sum of 2 dice

D_i - outcome of die i

$$P(D_1 = 2 \mid S = 5) ?$$

$$P(D_1 = 2 \mid S = 5) = \frac{P(S = 5 \mid D_1 = 2) \cdot P(D_1 = 2)}{P(S = 5)} = \frac{\frac{1}{6} \cdot \frac{1}{6}}{\frac{1}{9}} = \frac{1}{4}$$

$$P(S = 5 \mid D_1 = 2) = P(D_2 = 3 \mid D_1 = 2) = P(D_2 = 3) = \frac{1}{6}$$

$$P(D_1 = 2) = \frac{1}{6}$$

$$P(S = 5) = \frac{1}{9}$$

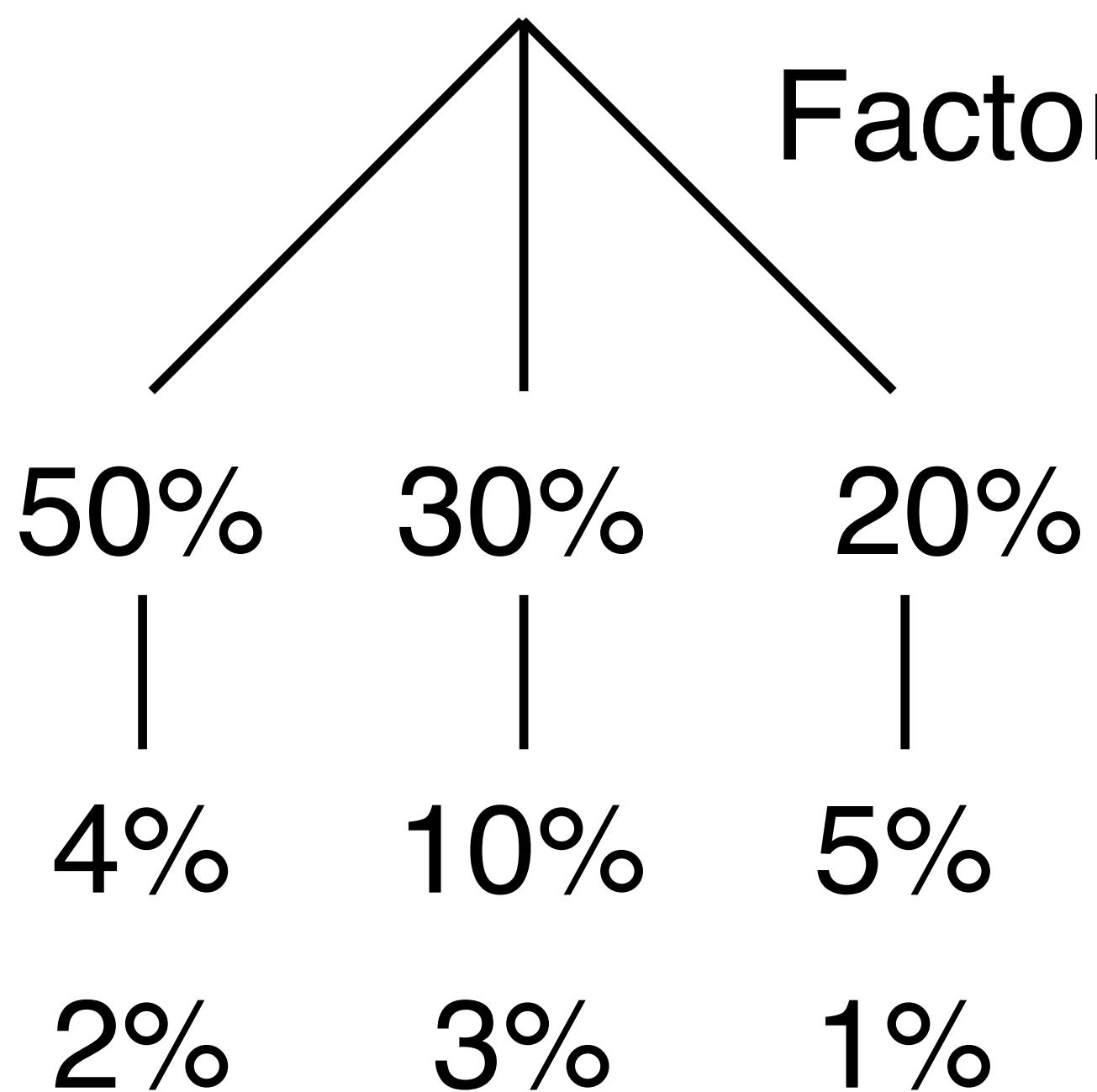
Last video

$$P(D_1 = 2 \mid S = 5) = \frac{|D_1 = 2 \cap S = 5|}{|S = 5|}$$

$$S = 5 \begin{cases} 1 & 4 \\ 2 & 3 \end{cases} \quad D_1 = 2 \\ \begin{cases} 3 & 2 \\ 4 & 1 \end{cases}$$

Foxconn

Foxconn has 3 factories producing 50%, 30%, and 20% of its iPhones



Factory defective fractions 4%, 10%, and 5% respectively

Overall fraction of defective phones?

$$\begin{aligned} P(D) &= P(D \cap F_1) + P(D \cap F_2) + P(D \cap F_3) \\ &= P(F_1)P(D | F_1) + P(F_2)P(D | F_2) + P(F_3)P(D | F_3) \\ &= .5 \times .04 + .3 \times .1 + .2 \times .05 \\ &= .02 + .03 + .01 \\ &= .06 \end{aligned}$$

Culprit?

$$P(F_1 | D) = \frac{P(D | F_1) \cdot P(F_1)}{P(D)} = \frac{.04 \cdot .5}{.06} = \frac{.02}{.06} = \frac{1}{3}$$

$$P(D | F_1) = .04$$

$$P(F_1) = .5$$

$$P(D) = .06$$

$$P(F_2 | D) = \frac{.1 \cdot .3}{.06} = \frac{.03}{.06} = \frac{1}{2}$$

$$P(F_3 | D) = \frac{.05 \cdot .2}{.06} = \frac{.01}{.06} = \frac{1}{6}$$

Conditional probabilities add to 1

Conditional order determined by both $P(F_i)$ and $P(D | F_i)$

Taxi



This Lecture: Bayes' Rule

Next: Random Variables