Understanding Scattering Amplitudes through Spinor Helicity Formalism

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—包昀嘉庚子年五月于珀特兰

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Abstract

Scattering amplitude is a fundamental physical quantity used in perturbative quantum mechanics and quantum field theory to identify the probability of some physical process. Frequently in quantum field theory, one calculates the scattering amplitude via Feynman rules. Although Feynman rules are closely related to the Lagrangian density of the field, this calculation technique usually turns into an evaluation of many Feynman diagrams, which then have nontrivial cancellations when summed over. Spinor helicity formalism is a technique discovered when evaluating challenging Feynman diagrams in gauge theories. It seems that with this technique, many calculations simplify. This review is adapted from the author's undergraduate thesis with an aim to provide background from the fundamentals of spinor helicity formalism to research frontiers in scattering amplitudes. No knowledge in quantum field theory is assumed, and minimal undergraduate-level knowledge in quantum mechanics may be helpful. In this document, we investigate the connection between spinor helicity formalism and scattering amplitudes. Specifically, we restrict our attention to scattering amplitudes with massless external particles at tree level and one-loop level. We noticed that, with spinor helicity, we could expose many interesting properties of scattering amplitudes and simplify their evaluations. Our investigation shows that spinor helicity formalism can provide us a new perspective on calculating and understanding amplitudes, a perspective distinct from the traditional Feynman-rule approach.

Dedication

To a great man and the stars he has guarded
To a great family and the members it connects
To a lovely harbor and the anchor where I rested
To the Nature, I sail towards its starlit dreams

致一位伟大的人和他守护的那片星空
致一个伟大的家庭和它紧紧联系的一家人
致一个可爱的海港和我歇息过的那个浅湾
致自然，向那片星辉，扬帆远航

"We have this one life to appreciate the grand design of the Universe, and for that, I am extremely grateful." — S. W. Hawking

"我们有此生得以欣赏宇宙的伟大设计，为此我心怀感激。" — 斯蒂芬·霍金

Introduction

In the quest to understand the physical world, physicists have obtained finer and finer descriptions of the fundamental constituents and their interactions. These agents are called **elementary particles**, and their interactions are described by **quantum field theories**. These particles are localized excitations of a sheet of underlying background spreading over the spacetime, known as the quantum field. The state of the particle, such as their spin, charge, momentum, helicity, is encoded in a **wavefunction**.[1] Quite triumphantly, physicists have come up with the **Standard Model** (see fig. 1), an almost comprehensive description of the fundamental interactions to high accuracy. These interactions include: electromagnetism enabling us to see and touch the world around us, strong interaction gluing nucleons together, and weak interaction generating the mysterious neutrinos. Roughly, the Standard Model describes the Universe in which the material formers, *i.e.* fermions, interact with each other through the force carriers, *i.e.* bosons.

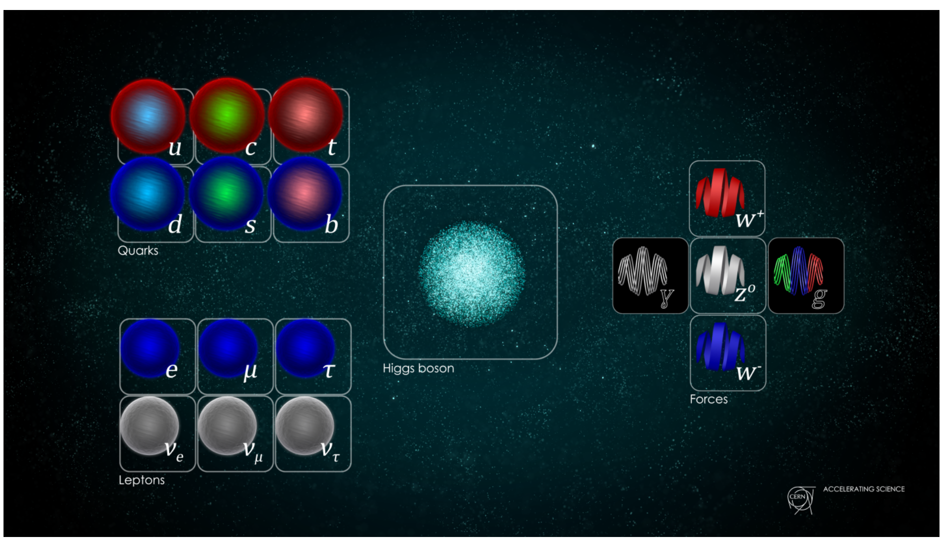


Figure 1: An illustration of the standard model particles [credit: Daniel Dominguez, CERN]

To gauge the robustness of the theory, physicists conduct various kinds of experiments with these particles, one of which is a collision experiment. In a collision experiment, physi-

[1]The notion of "a" wavefunction, albeit pictorial and intuitive, can be misinterpreted as there is no such thing as *a* wavefunction of the electron or *a* wavefunction of the photon. Instead, the wavefunction completely describes the world, *i.e.* there is only one wavefunction for both the field and particles. And particles are excitations of the field.

cists accelerate particles to adequately high energy and collide them to a target or another bunch of particles. The collision produces various particles, and the resultants are studied to test theoretical predictions and probe potentially new interactions or particles. The particles before a collision are referred to as **initial-state particles**, and the resulting particles from a collision are **final-state particles**. Some of these final-state particles can be captured by various detectors in the experiment, and one or several physical quantities are then measured by these detectors to provide descriptions of the particles. This class of physical quantities is named **observables**, and as its name suggests, these are the quantities that the experimentalists can see and probe.

0.1 Scattering Amplitudes

In particle collision experiments, particles interact through scattering processes in which experimentalists have some control over the initial-state particles' types and kinematics, such as their energies and momenta. Due to the quantum mechanical nature of scattering, particles can turn into different kinds of particles with a certain probability. This probability is described by the overlapping wavefunction or the **amplitude** of the scattering process. Then, with this probability as a function of particles' momenta and other labels, physicists can predict the probability of acquiring certain final-state configurations with certain energies, momenta, charges, *etc.* One can also calculate other observables, such as the jet pull [1], to make predictions about how effective this new observable differentiates one scattering process from another [2]. Calculating an amplitude thus is a crucial step for comparing theoretical predictions with experimental data.

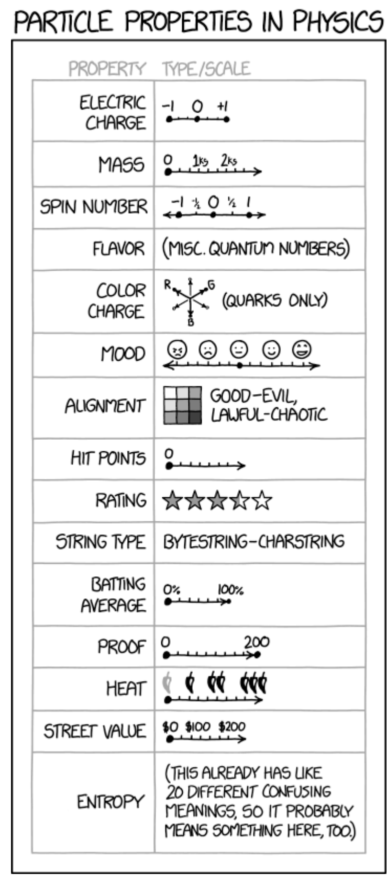
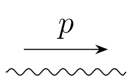
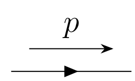


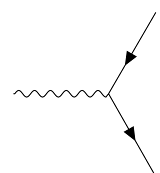
Figure 2: Examples of true (and potentially true) particle properties: they can be packaged into the wavefunction [credit: xkcd]

The canonical picture for calculating scattering amplitudes comes from the **perturbation theory** in quantum mechanics in which a perturbation is a process of having another interaction "perturbing" the (often known) evolution of a given physical system. This interaction is controlled by the small perturbation parameter or the **coupling constant**. As one usually does in quantum mechanics, one may attempt to find the new wavefunction which completely describes the evolution of the perturbed system, but this process can be quite arduous. Instead, we often use another way of understanding and solving a perturbative problem by rephrasing the question of interest. We ask how likely it is for one initial state to evolve into a final state, similar to a finite-wall scattering problem (or a scattering problem in optics) in which we ask what the probability (or the intensity of the light) is to observe an incoming wave to be transmitted. Then, the scattering process will depend on the initial state, the final state, and the intermediate interactions (analogous to the width of the wall or the opacity of the surface through which the light travels). The **scattering amplitude**, sometimes also known as the matrix element of the **S-matrix** due to historical reasons, is to encode these pieces of information in a scattering process and can be found by calculating the matrix element $\langle f \,|\, S \,|\, i \rangle$ in which $|f\rangle$ ($|i\rangle$) denotes the final (initial) state's wavefunction, and S is the unitary[2] scattering operator governed by the Lagrangian.

Due to the appearance of various phase space integrals and constraints, even setting up the mathematical expression for a scattering amplitude can be quite involved. Fortunately, due to Feynman's insight [3] and later rigorous developments from several physicists [4, 5], each perturbative expression can be represented pictorially with **Feynman diagrams**. Improving formal accuracy in perturbative corrections to the scattering amplitude can be straightforwardly carried out by adding more and more complicated diagrams. A Feynman diagram consists of **lines** representing the momentum of a particle and **vertices** representing interactions of particles in the process. Instead of performing the full perturbation theory, physicists have collected a large set of rules, *viz.* **Feynman rules**, for utilizing this diagrammatic tool to represent some interactions. In this document, we will restrict ourselves mostly to Feynman rules for **quantum electrodynamics (QED)**. These rules include



$$= \frac{-ig_{\mu\nu}}{p^2 + i\varepsilon}, \text{ (Feynman gauge)}$$


$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon},$$

$$= -ie\gamma^\mu. \tag{0.1.1}$$

The canonical approach with these rules is to first evaluate the tensorial part of the diagram with Feynman rules, and then, one can input the "helicity information", *i.e.* whether the external particles are left- or right-handed polarized. It seems that this approach is reasonable and relatively simple. However, as we will discuss in section 0.2 and chapter 1, this approach may contain some redundancies, troubling our calculation.

0.2 Massless Particle and Spinor Helicity Formalism

What do we mean by a particle? Frequently, we ignored this simple question and assumed a point-like entity that carries certain properties and travels with a certain momentum. Although this idea is on the right track, the very idea of a point-like entity seems to contradict the fundamentals of the study of a field continuum. In fact, we care less about the point-like structure than the fact that a particle can be seen as a collection of properties over a localized region. Specifically, we should be able to attribute a momentum to this entity. In light of this idea, Wigner and Bargmann [6, 7, 8] developed a precise definition for a particle state in a quantum field. Given a momentum p of a particle, we can find all the coordinate transformations that leave each component of p^μ unchanged. For example, over $\mathbb{R}^3$, a momentum $\mathbf{p}$ remains unchanged if we perform any rotation about $\hat{\mathbf{p}}$, *e.g.* in some

[2] Recall that the unitary condition on an operator S is $S^\dagger S = SS^\dagger = \mathbb{1}$ in which $\dagger$ denotes complex conjugate transpose, also known as the Hermitian conjugate or Hermitian adjoint.

coordinate with $\mathbf{p} = p\hat{\mathbf{z}}$

$$R(\theta)\mathbf{p} = \mathbb{R}_\theta \mathbf{p} = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} = \mathbf{p}. \tag{0.2.1}$$

These rotational symmetries form a group called $SO(2)$. Now, to understand what a momentum vector of a particle is, we can refer to this symmetry and argue that a particle is formed since the momentum of the particle remains unchanged under this symmetry. Although this is a rather circumventing approach to define a particle, it seems that there is a good connection between the symmetry group and the fact that we can attribute the momentum to a particle.

The same idea applies to a relativistic particle with a four-momentum p^μ. The set of transformations that leave p^μ invariant is defined as the **little group** of p^μ. A massive particle can be at rest in some inertial frame with $p^\mu = (m, \mathbf{0})^\mu$, and we can straightforwardly see that its little group is $SO(3)$, the set of $3D$ rotations. However, according to special relativity, a massless particle, such as photons or gluons, cannot be at rest. Instead, its momentum can only be parameterized as $p^\mu = (E, 0, 0, E)^\mu$ with some energy E. It is obvious that the little group for a massless particle should at least contain a copy of $SO(2)$ since it contains a spatial component of the form $\mathbf{p} = (0, 0, E)$. However, it is perhaps not so obvious that there are two more symmetries. The rough idea is that one can boost along $\hat{\mathbf{x}}$ direction and rotate about $\hat{\mathbf{y}}$ direction appropriately so that the rotation mixes the x- and z-component of the momentum while the boost mixes the x- and t-component in the right amount to cancel each others' contribution, leaving p^μ invariant. A similar thing happens to a boost along $-\hat{\mathbf{y}}$ and rotation about $\hat{\mathbf{x}}$. As it turns out, this new group of transformation can be identified as an isometry of a 2D Euclidean plane with a rotation and two translations, *i.e.* $ISO(2) = SO(2) \ltimes T(2)$. Therefore, a massless particle with a certain momentum is chosen to be the "simplest state" under these transformations, mathematically known as the **irreducible representations** or **irreps** of the little group.

Now, let's imagine observing a left-handed photon with some momentum $p^\mu = (E, 0, 0, E)^\mu$ along the z-direction. We may designate a state vector $|p, -\rangle$ in the Hilbert space to this particle since we know that a left-handed photon is indeed a simplest state. We also know, from the experiment if you wish, that with the same momentum, there is another right-handed state which we call $|p, +\rangle$, and $|-\rangle$ and $|+\rangle$ suffice to describe any photon with momentum p. Let Λ be a rotation about the z-direction. If photon were a particle without "internal structure" such as spin,[3] we could naturally imagine that $\Lambda |p\rangle = |\Lambda p\rangle$. However, for a photon with certain polarization, as illustrated in fig. 3, the polarization vector is actually changed. If we parameterize the polarization vector as $\epsilon^\mu = (0, 1, i, 0)^\mu$, then a rotation Λ will change it by

$$\begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} = e^{-i\theta}\epsilon, \tag{0.2.2}$$

[3] In other words, the particle is governed by a scalar field.

which means that the polarization vector picks up a phase. This is to say that any rotation Λ in the photon's little group will actually change the quantum state of the photon by $\Lambda |p,-\rangle \to e^{-i\theta} |\Lambda p,-\rangle$. This phase shift is hinting at a much bigger reorganization of our calculation techniques for scattering amplitudes.

Reviewing our Feynman rules shown in eq'n (0.1.1), we realize that the calculation from Feynman rules is treated as a process of constructing Lorentz invariants so that under any Lorentz transformation, the scattering amplitude is just a scalar. But if we perform a little group transformation of one of the particles, then, at least, a phase factor should naturally appear in the scattering amplitude, demonstrating that the amplitude cannot simply be a scalar as we first assumed. As one expects, the probability remains a Lorentz scalar since the overall phase factor has magnitude $\left|e^{i\theta}\right|^2 = 1$; however, we have packaged this extra phase factor dependency in Feynman rules by introducing redundancies into them. These redundancies are causing certain difficulties in calculating scattering amplitudes. As we will discuss in chapter 1, the spinor helicity formalism makes the little group transformation of a massless particle manifest in calculations while leaving a formal structure, called the spinor product, Lorentz invariant. Therefore, we can both ensure the Lorentz invariance of the probability and identify the appropriate phase factor from the little group.

0.3 Conventions

In this document, we use the high-energy physics (HEP) spacetime signature $(+,-,-,-)$ and the natural units in HEP $c = \hbar = 1$. In particular, this means $p^2 = m^2$ in which p means (four-)momentum and m denotes the mass of a particle. In case we are interested in discussing the spatial components of a four-vector, we will bold-face the express as $p^\mu = (p^0, \mathbf{p})^\mu$. In most cases, Greek indices (such as μ, ν, ρ, $\dots$) run over the spacetime index; Latin indices starting from a (such as a, b, $\dots$) run over the spinor index; Latin indices starting from the middle, (such as i, j, $\dots$) will be free indices left for other uses. All momenta of the external particles are to be understood as outgoing to avoid any confusion. One can re-interpret them as initial- or final-state particles by crossing.

In natural units, each physical quantity can have only a mass dimension; for instance, we call the energy of "having mass dimension 1". A few dimensions of some useful quantities include

$$[E] = [p] = 1, \quad [x] = [\mathrm{d}x] = -1, \quad [\mathcal{S}] = \left[\int \mathrm{d}^4x\ \mathcal{L}\right] = 0, \quad [\phi] = [A^\mu] = 1, \quad [\chi] = \frac{3}{2}, \tag{0.3.1}$$

in which E denotes energy, p momentum, x position, $\mathcal{S}$ action, $\mathcal{L}$ Lagrangian density, ϕ scalar field, A vector field, and χ fermion field.

0.4 Organization of the Document

The document is organized as follows: in chapter 1, we provides an analysis about the transformation property of scattering amplitudes, and how spinor helicity suits the transformation properties of amplitudes with massless external particles. Then, we discuss how spinor

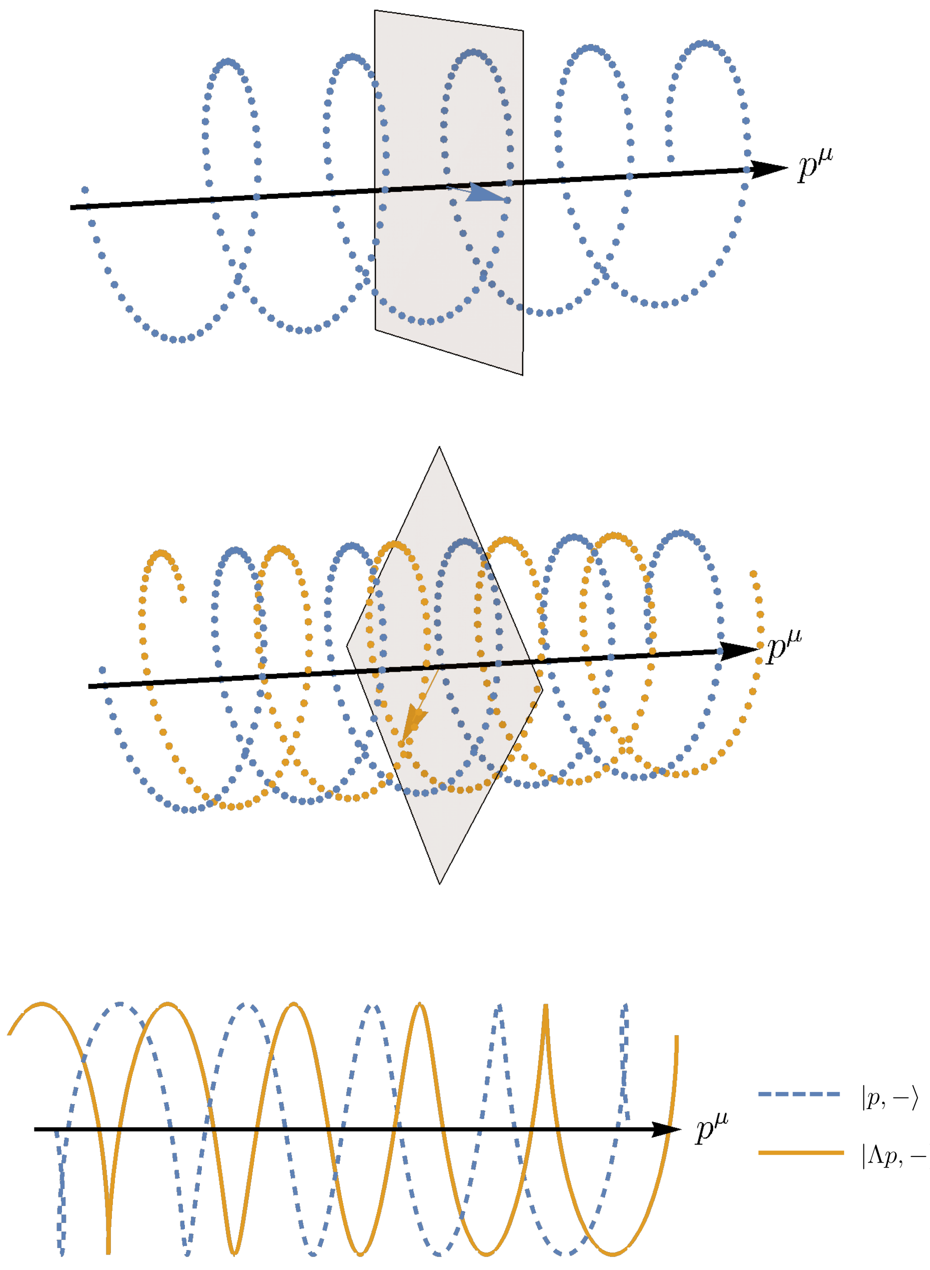


Figure 3: A little group transformation of a photon generating an extra $e^{-i\theta}$ phase factor.

emerges from Lorentz transformations, what spinor products are, and how these spinor products give rise to spinor helicity formalism. In Chapter 2, we dive into investigations about three-point and four-point tree-level amplitudes. In particular, there is an intriguing connection between lower-point tree-level amplitudes to higher-point tree-level amplitudes through an on-shell recursion technique. We demonstrate the insight one gains when treating three-point amplitudes as building blocks for constructing higher-point amplitudes. Then, we investigate the constraint that four-point tree-level amplitudes put on the field theory. In particular, the factorization property of four-point amplitudes provides certain locality constraints demanded by field theory. We present two arguments with the spinor helicity formalism: one is that the only consistent massless spin-1 self-interacting field theory is the Yang-Mills theory, and the other is that there is no self-interacting massless spin-> 2 field theory. In chapter 3, we discuss several techniques to evaluate one-loop scattering amplitudes, including Feynman shift parameterization, simplification of numerator algebra, and dimensional regularization. These techniques are demonstrated through an example in which we evaluate the vacuum polarization amplitude. Then, we discuss how spinor helicity formalism fits into the evaluation of one-loop amplitudes. Specifically, we use light-by-light scattering as the second example in chapter 3, and present techniques to leverage the little group properties of spinor variables and to simplify calculations with tree-level amplitudes. These techniques include: gauge fixing with spinor variables, scalar loop reduction, and unitarity cuts. We also provides several potential extensions to our calculations which warrant further explorations.

Chapter 1

Spinor Helicity Formalism

Spinor helicity formalism is a powerful and concise calculation tool that provides a new picture about scattering amplitudes, different from the traditional Feynman diagram approach in quantum field theory calculations. It re-expresses the kinematic and helicity information found in a scattering amplitude, removes or repackages some redundancies in the calculation, and works well with the transformation properties of scattering amplitudes. In this chapter, we will mainly provide a contemporary view about scattering amplitudes and their close relation to spinor helicity formalism. Along the way, we will provide details about manipulations in this new formalism, connections with the traditional Feynman diagram approach, some classic examples of this technique, and more. However, before we directly jump into spinor helicity formalisms, it would be nice to provide some historical backgrounds on this formalism.

Spinor helicity formalism was first motivated by calculations of Feynman diagrams with respect to the chiral basis [9, 10, 11]. With a new 2-dimensional kinematic notation that decomposes the massless particles' momenta, many previous inconspicuous cancellations in Feynman-rule calculations were made evident. These decompositions were related to a particular choice of a set of **gamma matrices** in the Dirac equation, known as the Weyl basis or the chiral basis [12]. It seemed that QED or, more generally, Yang-Mills theory has certain interesting preferences for using a particular basis to simplify Feynman diagram calculations for some unclear reason. On the other hand, theoretical calculations in Yang-Mills theory were cornered to face a factorially large number of diagrams to be summed over with numerous terms to ensure gauge symmetry [13]. However, people realized that for particular choices of the helicity of external particles, the scattering amplitude squared simplifies dramatically into a simple combination of Lorentz invariants [14]. This particular choice of helicity configuration leads to a surprising possibility for calculating similar higher-point amplitude squared recursively [15], which differs from the traditional Feynman diagram approach.

At this stage, physicists realized that a new framework for calculating the scattering amplitude was emerging in the form of shortcuts and rules that were not previously understood. This new framework seems to have connections with the spinor variables and chiral basis. This new calculation technique was, therefore, named **spinor helicity formalism**. Quickly, people gathered various rules and arguments in this new formalism. Some nice reviews are presented in [16, 17, 18]. Although spinor helicity greatly simplifies for massless particles, several recent developments have extended the spinor helicity formalism to even massive case [19, 20], demonstrating its potential for efficient calculations in more complicated theories.

1.1 Transformation Properties of Scattering Amplitudes

Continuing the train of logic of our discussion on massless particles in section 0.2, we define a particle as a state which remains unchanged under a little group transformation. To obtain a slightly more concrete picture, let's consider a set of momentum eigenstates $\{|p,\sigma\rangle\}$ in which p denotes the momentum, and σ denotes some other labels of the particle's properties. Then, the scattering amplitude is calculated by $\langle p_f, \sigma_f \,|\, S \,|\, p_i, \sigma_i\rangle$; however, just like in quantum mechanics, an overall phase factor $e^{i\phi}$ will not matter. More generally, if we are given any **unitary operator**[1] Λ describing a Lorentz transformation, *i.e.* $\Lambda\,|p\rangle = |\Lambda p\rangle$, then we must demand that the amplitude $\mathcal{A}$

$$\mathcal{A} = \langle p_f \,|\, S \,|\, p_i\rangle = \left\langle p_f \,\middle|\, \Lambda^\dagger S \Lambda \,\middle|\, p_i\right\rangle = \langle \Lambda p_f \,|\, S \,|\, \Lambda p_i\rangle \,, \tag{1.1.1}$$

which follows from the Lorentz covariance[2] of the field theory. This motivates the idea that we must write a scattering amplitude in terms of Lorentz invariant quantities through Feynman rules. So far, we have laid out some crucial reasons for performing efficient calculations with Feynman rules, and it seems that the train of logic follows nicely. However, notice that we have assumed that $\Lambda\,|p\rangle \to |\Lambda p\rangle$ without referring to the label σ. This argument is on the right track but is missing a significant piece regarding the label σ. We will see in the following discussion that $\mathcal{M}$ should not, in general, be Lorentz invariant.

1.1.1 Little Group Representation: A Missing Piece in the Puzzle

To be more careful, let's find two Lorentz transformations, one general Λ and one $L_{p\to\Lambda p}$. We demand that

$$L_{p\to\Lambda p}\,|p,\sigma\rangle \triangleq |\Lambda p,\sigma\rangle\,, \quad L_{\Lambda p\to p} = L^{-1}_{p\to\Lambda p}, \tag{1.1.2}$$

which can be viewed as a definition for the operator $L_{p\to\Lambda p}$. (See, for example, Section 2.5 of [8] and Section 2 of [19].) Notice that $L_{p\to\Lambda p}$ is not unique due to the existence of little group transformations. Also, there is a subtle difference between L and Λ. The critical distinction between L and Λ is that while Λ is completely general, L is a particular Lorentz transformation such that the σ label remains unchanged as the momentum eigenstate is boosted to Λp. Let's now observe the transformation property of $|p,\sigma\rangle$ under a general boost. Of course, we are looking for $\Lambda\,|p,\sigma\rangle \overset{?}{\to} |\Lambda p,\sigma\rangle$. After all, this is what we mean by Λ representing a Lorentz transformation. However, this is not allowed since only L can transform $|p,\sigma\rangle \to |\Lambda p,\sigma\rangle$ but not any general Λ. Therefore, our second best choice is to insert some identities to the expression

$$\Lambda\,|p,\sigma\rangle = \mathbb{1}\Lambda\,|p,\sigma\rangle = L_{p\to\Lambda p}\underbrace{L_{\Lambda p\to p}\Lambda}_{\triangleq W}|p,\sigma\rangle\,, \tag{1.1.3}$$

[1] Since a Lorentz transformation on a momentum eigenstate should not affect the probability of observing the state, we will assume that any Lorentz transformation in the Hilbert space is a unitary operator.

[2] In general, we also would demand Poincarè covariance. To achieve that is to add in translational invariance through a momenta-conserving delta function to the amplitude.

in which we defined a new operator W. Let's consider the effect of W on any momentum eigenstate $|p\rangle$. To probe this, we use the momentum operator p on $W|p\rangle$

$$pW|p\rangle = p^\dagger L^\dagger_{p\to\Lambda p}\Lambda|p\rangle = (L_{p\to\Lambda p}p)^\dagger\Lambda|p\rangle = (\Lambda p)^\dagger\Lambda|p\rangle = p\Lambda^{-1}\Lambda|p\rangle = p|p\rangle\,. \tag{1.1.4}$$

Therefore, W is just a unitary operator in the little group. However, we have no guarantee that $W|p,\sigma\rangle = |p,\sigma\rangle$ since the little group may not leave the σ labels invariant, and in fact, they don't in general. Nonetheless, we are allowed to pick a basis of σ such that $\mathrm{Span}\{|p,\sigma\rangle \mid \text{all } \sigma\}$ is the entire eigenspace for $|p\rangle$, and $\{|p,\sigma\rangle \mid \text{all } \sigma\}$ is an orthonormal basis. Then, we can represent the little group operator W as

$$W|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}|p,\sigma'\rangle\,, \tag{1.1.5}$$

in which $D_{\sigma\sigma'}$ is just a number. In other words,

$$D_{\sigma\sigma'} = \langle\sigma'\,|\,W\,|\,\sigma\rangle\,, \tag{1.1.6}$$

is a **unitary representation** of the little group action over the label basis. Thus,

$$\Lambda|p,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}L_{p\to\Lambda p}|p,\sigma'\rangle = \sum_{\sigma'} D_{\sigma\sigma'}|\Lambda p,\sigma'\rangle\,, \tag{1.1.7}$$

or that under a general Lorentz transformation Λ, the particle states pick up a little group representation by

$$|p,\sigma\rangle \to D_{\sigma\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle = W(\Lambda,p)\,|\Lambda p\rangle\,. \tag{1.1.8}$$

Plugging this result into $\mathcal{A}$, we notice that now[3]

$$\mathcal{A} \to \mathcal{A}^\Lambda = \left[\prod_{\text{particle } r} W(\Lambda,p_r)\right]\mathcal{A}. \tag{1.1.9}$$

This is saying that $\mathcal{A}$ in general should not be a Lorentz invariant quantity. Instead, under a general Lorentz transformation Λ, $\mathcal{A}$ should reflect its little group transformation, which is completely hidden in our current approach with Feynman rules. Why then does our field theoretic approach yield a valid quantity that transforms erroneously?

1.1.2 Gauge Symmetry and Its Redundancy

This little group transformation has actually been successfully packaged into an **equivalence class of gauge**, and this is especially obvious for massless particles. For simplicity, we will consider the case for photons. Recall from QED that external photons should possess some positive- or negative-polarization, and this **polarization vector** ϵ is a lightlike "four-vector" perpendicular to the momentum p. In terms of mathematical language,

$$\epsilon\cdot\epsilon = 0, \quad p\cdot\epsilon = 0. \tag{1.1.10}$$

[3] The product over particles comes from the fact that multiple-particle states are in Fock states over a Fock space rather than a Hilbert state, but effectively we only need to convert an initial or a final state into tensor products of single-particle states.

And we usually input this helicity information about a photon into a Feynman diagram by dotting this polarization vector to appropriate tensorial index. The goal of a polarization vector is to introduce the little group's representation to the tensorial "amplitude". For instance, given a tensorial "amplitude" M^μ found with Feynman rules and a Lorentz transformation Λ, the amplitude looks like $\mathcal{A} = \epsilon_\mu M^\mu$, and after performing the Lorentz transformation Λ, $M^\mu \to \Lambda^\mu{}_\nu M^\nu$ and $\epsilon^\mu \to \tilde{\epsilon}^\mu$. If ϵ were a four-vector, then $\epsilon^\mu \to \Lambda^\mu{}_\nu \epsilon^\nu$, and $\mathcal{A}$ would be a scalar, which has the wrong transformation property.

Fortunately, ϵ^μ is *not* a four-vector. Consider an infinitesimal Lorentz transformation

$$\mathfrak{L} = \eta \begin{pmatrix} & 1 & & \\ 1 & & & -1 \\ & & & \\ & 1 & & \end{pmatrix}, \tag{1.1.11}$$

acting on a massless momentum $p^\mu = (1,0,0,1)^\mu$ by the usual matrix multiplication. Evidently, this infinitesimal Lorentz transformation leaves the first-order perturbation on p^μ unchanged, *i.e.* $\exp(\mathfrak{L})$ lives in the little group of p^μ. The full Lorentz transformation for this algebra is

$$\exp(\mathfrak{L}) = \begin{pmatrix} \eta^2/2+1 & \eta & & -\eta^2/2 \\ \eta & 1 & & -\eta \\ & & 1 & \\ \eta^2/2 & \eta & & 1-\eta^2/2 \end{pmatrix}. \tag{1.1.12}$$

Again, we can confirm that $\exp(\mathfrak{L})p = p$, *i.e.* $\exp(\mathfrak{L})$ is a little group transformation for p.

With the same idea, if the polarization ϵ is a four-vector transforming just like p, we would expect $\exp(\mathfrak{L})\epsilon \stackrel{?}{=} \epsilon$. This is, however, not the case,

$$\exp(\mathfrak{L})\epsilon = \begin{pmatrix} \eta^2/2+1 & \eta & & -\eta^2/2 \\ \eta & 1 & & -\eta \\ & & 1 & \\ \eta^2/2 & \eta & & 1-\eta^2/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ 1 \\ i \\ \eta \end{pmatrix} = \epsilon + \eta p. \tag{1.1.13}$$

This means that $\epsilon \to \epsilon + \eta p$ under some Lorentz transformation, and ϵ is surely *not* a four-vector. In general, a little group transformation of a massless p will transform

$$\epsilon^\mu \to e^{i\theta}\epsilon^\mu + (\eta + i\xi)p^\mu, \quad \theta, \eta, \xi \in \mathbb{R}. \tag{1.1.14}$$

However, it is naturally allowed since we say that $\epsilon^\mu \to \epsilon^\mu + \lambda p^\mu$ is a **gauge symmetry** which we may freely choose and has no physical consequence. This gauge symmetry is certainly consistent with eq'n (1.1.10) as long as $p^2 = 0$. By choosing another gauge, we realize that the $\epsilon \to e^{i\theta}\epsilon$ so that $\epsilon_\mu M^\mu \to e^{i\theta}\epsilon_\mu M^\mu = W(\Lambda)\mathcal{A}$ has the correct transformation property.

By using a polarization "vector", we have introduced "infinite degrees of freedom" into our description by introducing the equivalence class of $\epsilon \equiv \epsilon + \lambda p$, just to insert back the $W(\Lambda)$ for each particle state. This formulation, albeit smoothly connected particles to fields, also introduced some significant difficulties to evaluating the Feynman diagram, not only because **gauge fixing** in general is an artistic choice rather than a systematic process, but also since gauge fixing has no physical consequence. To simplify this process, we must, therefore, look for a new way of representing a Lorentz invariant quantity with sufficiently nice manifestation for the little group and, potentially, permit the identification of a gauge to be obvious or simply eliminate the gauge dependence.

1.2 Spin and Spinors: Lorentz Transformations from a New Perspective

To have a firm understanding of the spinor helicity formalism, we first need to learn about the emergence of spin and spinors in our framework. In many classic textbooks for quantum mechanics and quantum field theory[21, 22, 23, 24], the introduction of spinor is either through the introduction of Pauli matrices as a calculation tool or through the derivation for Dirac equation and gamma matrices. These approaches make the emergence of spinor as a consequence of quantum mechanics. However, as we will see shortly, the introduction of spin and spinors can be entirely classical and follows from the Lorentz group for special relativity.

In this section, we follow the idea presented in Chapter 10 of [25] and present the relation between spinor and Lorentz transformations. The collection of all Lorentz transformations form a Lorentz group SO(3,1). This group is a **Lie group** and consists of only rotations and boosts which can be generated by elements in its **Lie algebra** $\mathfrak{so}(3,1)$. The elements of the Lie algebra are sometimes referred to as **infinitesimal transformations** or **generators**. These elements can be expressed in terms of J_i (infinitesimal rotations about x_i-axis) and K_i (infinitesimal boosts along x_i-axis). In the four-dimensional representation, these generators are

$$(J_i)_{jk} = -i\epsilon_{ijk}, \quad (K_i)_{\mu\nu} = -i(\delta_{0\mu}\delta_{i\nu} + \delta_{i\mu}\delta_{0\nu}). \tag{1.2.1}$$

Any element in the Lorentz group can be found by an exponential map,[4] which "accumulates" infinitesimal transformations to a finite one

$$\Lambda = \exp(i\theta_j J_j + i\eta_k K_k). \tag{1.2.2}$$

Conveniently, this exponential map over the four-dimensional representation can be treated just like a matrix exponential. As a demonstrative calculation, consider a boost along the z-direction with rapidity η_z. In a four-dimensional representation, the boost generator looks like

$$K_z = \begin{pmatrix} 0 & & & -i \\ & 0 & & \\ & & 0 & \\ -i & & & 0 \end{pmatrix} = -i\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & 1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 1 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & & & 1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 1 \end{pmatrix}\right]. \tag{1.2.3}$$

[4] The conventional exponential map used in physics is frequently a complex exponential with an extra factor of i from the convention used in mathematical literatures. This convention aims to exhibit the unitarity of the chosen representation.

Therefore, the element in SO(3,1) generated by K_z is

$$\Lambda_{tz} = \exp \begin{pmatrix} 0 & & & \eta_z \\ & 0 & & \\ & & 0 & \\ \eta_z & & & 0 \end{pmatrix} = \begin{pmatrix} \cosh \eta_z & & & \sinh \eta_z \\ & 0 & & \\ & & 0 & \\ \sinh \eta_z & & & \cosh \eta_z \end{pmatrix}, \tag{1.2.4}$$

which is precisely what we expect from a relativistic boost on a four-vector. These generators form a complete basis of $\mathfrak{so}(3,1)$ so that any element of the Lie algebra is a linear combination of six generators, and each element in the Lie algebra can be mapped to some Lorentz transformation.[5]

In general, the study of all Lorentz transformations can sometimes be challenging; however, we can gain a clear picture of the Lorentz group from its corresponding Lie algebra. An algebra demands both a vector addition over its generators and a bilinear multiplication. For a Lie algebra, the *defining* multiplication is the **commutation relations**, *i.e.* the **Lie brackets**. In particular, $\mathfrak{so}(3,1)$ has the following commutations relations.

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{1.2.5}$$

We will show these commutation relations with our four-vector representation shown in eq'n (1.2.1).

$$\begin{aligned}
([J_i, J_j])_{mn} =& - (\epsilon_{imk}\epsilon_{jkn} - \epsilon_{jmk}\epsilon_{ikn}) = -(\delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn} - \delta_{jn}\delta_{im} + \delta_{ij}\delta_{mn}) \\
=& \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = \epsilon_{ijk}\epsilon_{mnk} = i\epsilon_{ijk}(J_k)_{mn}. \\
([K_i, K_j])_{\mu\nu} =& - (\delta_{0\mu}\delta_{i\rho} + \delta_{i\mu}\delta_{0\rho})(\delta_{0\rho}\delta_{j\nu} + \delta_{j\rho}\delta_{0\nu}) + (\delta_{0\mu}\delta_{j\rho} + \delta_{j\mu}\delta_{0\rho})(\delta_{0\rho}\delta_{i\nu} + \delta_{i\rho}\delta_{0\nu}) \\
=& 0\delta_{0\mu}\delta_{0\nu} - \delta_{i\mu}\delta_{j\nu} + \delta_{j\mu}\delta_{i\nu} \\
=& \epsilon_{ijm}\epsilon_{\nu\mu m} \quad (\mu, \nu \neq 0) \\
=& - i\epsilon_{ijk}(J_k)_{\mu\nu} \quad (\mu, \nu \neq 0) \\
([J_i, K_j])_{\mu\nu} =& - \epsilon_{i\mu\rho}(\delta_{0\rho}\delta_{j\nu} + \delta_{j\rho}\delta_{0\nu}) + (\delta_{0\mu}\delta_{j\rho} + \delta_{j\mu}\delta_{0\rho})\epsilon_{i\rho\nu} = -\epsilon_{i\mu j}\delta_{0\nu} + \delta_{0\mu}\epsilon_{ij\nu} \\
=& \epsilon_{ijk}(\delta_{0\mu}\delta_{k\nu} + \delta_{0\nu}\delta_{k\mu}) = i\epsilon_{ijk}(K_k)_{\mu\nu}.
\end{aligned}$$

Therefore, the **commutators** shown in eq'n (1.2.5) are correct. In this calculation, we picked the four-dimensional representation for our convenience. However, since the defining feature of a Lie algebra is these commutators, the commutator is unique regardless of the representation. From now on, we will refer to these generators abstractly instead of picking a particular representation if possible.

1.2.1 Direct Sum Decomposition: Taking the Square-root of $\mathfrak{so}(3,1)$

Now, we can incorporate the properties of the Lorentz group generated by these exponential maps of $\mathfrak{so}(3,1)$ to our calculation. Analyzing these elements of $\mathfrak{so}(3,1)$ can help us understand the flat spacetime on which we put our field theory. However, there is, in fact, a more

[5] This follows from Lie's third theorem that every Lie algebra over $\mathbb{R}$ is isomorphic to a Lie algebra of a connected Lie group (See, for example, the discussion in Appendix B of [26]).

convenient way to understand $\mathfrak{so}(3,1)$ by a "change of basis". In a slightly more mathematical term, we can decompose $\mathfrak{so}(3,1) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which can be easier to consider separately.

Let's define the new generators as

$$J_i^+ \triangleq \frac{1}{2}(J_i + iK_i), \quad J_i^- \triangleq \frac{1}{2}(J_i - iK_i). \tag{1.2.6}$$

Note that we can easily express any old J and K generators as a $\mathbb{C}$-linear combination of J^+ and J^-. Then, the new commutators read

$$\left[J_i^\pm, J_i^\pm\right] = i\epsilon_{ijk}J_k^\pm, \quad \left[J_i^\pm, J_j^\mp\right] = 0. \tag{1.2.7}$$

Some simple algebra can demonstrate the validity of these commutators shown below.

$$\begin{aligned}
\left[J_i^+, J_j^+\right] =& \frac{1}{2}([J_i, J_j] - [K_i, K_j] + i[J_i, K_j] + i[K_i, J_j]) \\
=& \frac{1}{2}(i\epsilon_{ijk}J_k + i\epsilon_{ijk}J_k - \epsilon_{ijk}K_k - i[J_j, K_i]) \\
=& i\epsilon_{ijk}J_k - \epsilon_{ijk}K_k = i\epsilon_{ijk}(J_k + iK_k) = i\epsilon_{ijk}J_k^+, \\
\left[J_i^-, J_j^-\right] =& \frac{1}{2}([J_i, J_j] - [K_i, K_j] - i[J_i, K_j] - i[K_i, J_j]) = i\epsilon_{ijk}J_k + \epsilon_{ijk}K_k = i\epsilon_{ijk}J_k^-, \\
\left[J_i^+, J_j^-\right] =& \frac{1}{2}([J_i, J_j] + [K_i, K_j] - i[J_i, K_i] + i[K_i, J_j]) = 0.
\end{aligned}$$

Note that the last equation is significant for our decomposition. It states that the algebra contains two subalgebras which commute with each other, enabling the direct sum decomposition. Even more fortunately, this commutation relation is very familiar. Recall that the angular momentum operators from quantum mechanics satisfy the following commutators $[L_i, L_j] = i(\hbar)\epsilon_{ijk}L_k$. This means that we have decomposed $\mathfrak{so}(3,1)$ into two copies of the symmetry groups of the angular momentum $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Seemingly abstract, this decomposition actually has an intuitive interpretation. Each J_i^+ generates both a boost along x_i-axis and a rotation about x_i-axis. A rotation about x_i may be ambiguous, but we can pick it to be a rotation in a **left-handed** fashion with respect to $+x_i$. Similarly, a J_i^- generates a boost and a **right-handed** rotation along x_i. Then, $\mathfrak{so}(3,1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is simply stating the intuitive fact that any general infinitesimal Lorentz transformation can be decomposed into a linear combination of six boost-rotation generators $J_{1,2,3}^\pm$. Therefore, any Lorentz transformation can be expressed as

$$\Lambda = \exp\left(\sum_{r=1}^{3} i\theta_r J_r^\pm\right) = \exp\left(\sum_r \frac{i}{2}\theta_r J_r \mp \frac{1}{2}\theta_r K_r\right). \tag{1.2.8}$$

Because of the factor of i in the exponential map, we obtain a relative sign change between the rotation generators J and the boost generators K. This explains why J^+ denotes generators for the left-handed boost-rotation and J^- for the right-handed one.

Before this decomposition, we are right to consider the four-vectors, a four-dimensional object, to be the fundamental object transforming over a flat spacetime manifold equipped

$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$(0,0)$	$(1/2,0)$	$(0,1/2)$	$(1/2,1/2)$	$(1,0)$	$(1,1/2)$	$(1,1)$
$\mathfrak{so}(3) \leq \mathfrak{so}(3,1)$	0	$1/2$	$1/2$	$1 \oplus 0$	1	$1/2 \oplus 3/2$	$2 \oplus 1 \oplus 0$

Table 1.1: Decomposition of irreps of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into irreps of $\mathfrak{so}(3)$ subalgebra describing spin (Table 10.1 of [25], some details can be found in section 5.6 of [8])

with $(+,-,-,-)$ metric. However, this decomposition shows that we can understand these objects as two objects transforming individually under a copy of $\mathfrak{su}(2)$, similar to angular momentum operators and angular momentum eigenstates in quantum mechanics. This object is named a **spinor**. In physics, we restrict the definition of **spinor** to the two-dimensional object transforming under the spin[6]-1/2 **irrep** of $\mathfrak{su}(2)$. The dimension of a representation[7] of the angular momentum group can be found by $2l+1$ in which l denotes the spin. For the spin-1/2, we obtain that this is a 2-dimensional space. This is expected because a spin-1/2 particle can have only two eigenstates, *viz.* $\{|\uparrow\rangle, |\downarrow\rangle\}$. Since there are two copies of $\mathfrak{su}(2)$, we label their spin individually by (l_+, l_-). There is a correspondence between the particular choice of spin in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and the resulting spin in $\mathfrak{so}(3) \leq \mathfrak{so}(3,1)$ shown in table 1.1. From table 1.1, we noticed that the $(1/2, 1/2)$ representation has an interesting transformation property that it contains a spin-0 part, *i.e.* a scalar, and a spin-1 part, *i.e.* a vector. It thus may be reasonable to expect to form the Lorentz invariants (scalars) and the four-vectors in terms of the $(1/2, 1/2)$ irrep. Investigating the $(1/2, 1/2)$ irrep will be our next goal towards building the spinor helicity formalism. But before that, we first need some calculation tools for the spin-1/2 irrep.

1.2.2 Finding a Representation: Usage of Pauli Matrices

As we focus on the spin-1/2 representation, we know that the object which we will manipulate are 2-dimensional objects. To provide concrete calculation tools, we can view them as some 2-dimensional vectors, and transformations on them are generated by some two dimensional matrices. Since we are working with spinors which look similar to angular momentum eigenstates, it is natural to investigate **Pauli's spin matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2.9}$$

which have been used to represent angular momentum eigenstates in quantum mechanics. We seek again the defining feature of a Lie algebra, its **commutators**. Fortunately, the commutation relation of Pauli matrices are simple

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{1.2.10}$$

A proof of this claim is done by direct computation in appendix A. This is precisely the commutation relation for $\mathfrak{su}(2)$, and 2×2 Pauli matrices indeed can operate on 2-dimensional

[6] In mathematics, the spin l is identified with the weight of the representation $m = 2l$, and here the spin-1/2 representation of $\mathfrak{su}(2)$ corresponds to picking its fundamental representation.

[7] Technically, we mean the dimension of the **representation space** here, but we usually identify the representation space, the element in a representation space, and the representation itself together.

objects; therefore, Pauli matrices are the generators of $\mathfrak{su}(2)$ in a spin-1/2 irrep. However, they also have another interesting property in terms of their **anticommutators**

$$\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\delta_{ij}}{2}\mathbb{1}. \tag{1.2.11}$$

Later, when using spinor helicity formalism, we will notice that these properties provide a convenient linkage between spinor helicity formalism and the conventional Feynman diagram approach.

With some handle about how to manipulate a spin-1/2 irrep of $\mathfrak{su}(2)$, let's redirect our attention back to the $(1/2, 1/2)$ irrep of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Recall the remark from the previous section, a right-handed spinor is generated by J_- while a left-handed is by J^+. Thus, the sign difference implies that they may have different Lorentz generators in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In particular, the right-handed generator J_- does not operate on a left-handed spinor, and *vice versa*. So $J_+ = 0$ for a right-handed spinor, and $J_- = 0$ for a left-handed spinor. By eq'n (1.2.6), for a right-handed spinor, its rotation and boost can be represented by

$$J_i = J_i^+ + J_i^- = 0 + J_i^- = \frac{\sigma_i}{2}, \quad K_i = i\big(J_i^- - J_i^+\big) = i\Big(\frac{\sigma_i}{2} - 0\Big) = \frac{i\sigma_i}{2}, \tag{1.2.12}$$

whereas for a left-handed spinor,

$$J_i = \frac{\sigma_i}{2}, \quad K_i = -\frac{i\sigma_i}{2}. \tag{1.2.13}$$

Therefore, an infinitesimal Lorentz transformation in a spin-1/2 representation will take the form

$$\Lambda = \exp[i(\theta_j J_j + \eta_k K_k)] \approx \mathbb{1} + (i\theta_j \mp \eta_j)\frac{\sigma_j}{2}. \tag{1.2.14}$$

in which the left-handed generator takes the negative sign whereas the right-handed generator takes the positive sign. This is precisely what we concluded in eq'n (1.2.8) but with a more concrete representation of the algebra. Notice that the presence of a factor of 1/2 is the manifestation that we are manipulating a spin-1/2 object under Lorentz transformations.

Now, we are ready to investigate the great convenience introduced by spinors. Sometimes, it is convenient to write out a 2-dimensional vector signaling the spinor; sometimes, it is convenient to call out one of its two components with an index notation; sometimes, it is useful to regard a spinor as a single entity. Here, we will establish some conventions which will assist our discussion in the next section. By convention, a **left-handed spinor** is called "transforming like $(1/2, 0)$" with its components denoted as λ_a in which $a \in \{1, 2\}$ is the index. Similarly, a **right-handed spinor** is called "transforming like $(0, 1/2)$" with its components denoted as $\tilde{\lambda}_{\dot{a}}$ in which $\dot{a} \in \{\dot{1}, \dot{2}\}$. Notice that we insisted on separating the index for the left-handed spinor from that for the right-handed one. This is in light of the argument that a left-handed spinor is never influenced by a right-handed generator, thus cannot be contracted with a right-handed index. These two types of spinors together are called **spinor variables**.[8]

[8]Traditionally, these spinors are discovered by choosing a specific basis (representation) of the gamma matrices in Dirac equation (see for example [12]). In the Weyl (or chiral) basis, the Dirac equation decouples into two copies of Weyl equations with these simple 2-dimensional spinors, instead of the traditional 4-dimension bispinors. Therefore, these spinors are sometimes called Weyl spinors in old literature.

At this point, we are on the verge of discovering a deep motivation for the spinor helicity formalism. Recall that we noticed a copy of spin-0 irrep in the $\mathfrak{so}(3)$ subalgebra given by a $(1/2, 1/2)$ irrep of the Lorentz group. This is hinting at a possible Lorentz invariant forming out of spinors. How is this invariant formed? We will investigate this in the next section.

1.2.3 Rediscover Lorentz Invariants

To form an invariant out of 2-dimensional (complex) vectors, one immediately thinks about the familiar inner product in Hilbert space formed by dotting a vector with its Hermitian conjugate. Therefore, we may try to form a Lorentz invariant with a $(1/2, 0)$ spinor by $\lambda_b^\dagger \lambda_a$. Here we will temporarily suspend the usage of index summation convention and treat $\lambda_{a(b)}$ as the name for the 2-dimensional spinor. This is because we have not yet identified an appropriate dot product through which we contract, raise and lower the indices. Now, we may ask what happens if we perform a Lorentz transformation on the quantity. Specifically, we are interested whether or not $\lambda_b^\dagger \lambda_a \xrightarrow{?} \lambda_b^\dagger \lambda_a$ just like a scalar. As we discussed in eq'n (1.2.14), we can always transform a $(1/2, 0)$ spinor infinitesimally by

$$\delta(\lambda_a) \approx (i\theta_j - \eta_j)\frac{\sigma_j}{2}\lambda_a, \quad \delta(\lambda_b^\dagger) \approx -(i\theta_j + \eta_j)\lambda_b^\dagger \frac{\sigma_j^\dagger}{2}, \tag{1.2.15}$$

in which $\dagger$ denotes Hermitian conjugate. However, sadly, the usual trick of forming dot products by Hermitian conjugate is no longer useful for spinors because

$$\delta(\lambda_b^\dagger \lambda_a) = \delta(\lambda_b^\dagger)\lambda_a + \lambda_b^\dagger \delta(\lambda_a) = [-(i\theta_j + \eta_j) + (i\theta_j - \eta_j)]\left(\lambda_b^\dagger \frac{\sigma_j}{2}\lambda_a\right) = -\eta_j \lambda_b^\dagger \sigma_j \lambda_a \neq 0. \tag{1.2.16}$$

This means that $\lambda_b^\dagger \lambda_a$ is not a Lorentz invariant. Nonetheless, a Lorentz invariant can be formed with spinors by using the 2D Levi-Civita symbol ϵ. In particular, we will introduce the **spinor product** $\lambda_b^T \epsilon \lambda_a$, in which T denotes transpose. Observe that

$$\delta\lambda_b^T \approx (i\theta_j - \eta_j)\lambda_b^T \frac{\sigma_j^T}{2} = \lambda_b^T \left[(i\theta_1 - \eta_1)\frac{\sigma_1}{2} + (i\theta_3 - \eta_3)\frac{\sigma_3}{2} - (i\theta_2 - \eta_2)\frac{\sigma_2}{2}\right] \tag{1.2.17}$$

Notice that the 2D Levi-Civita $\epsilon = i\sigma_2$, and quite interestingly, by eq'n (1.2.11), Pauli matrices satisfy

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1, \quad \sigma_3\sigma_2 = -\sigma_2\sigma_3. \tag{1.2.18}$$

Therefore, we can notice that an infinitesimal Lorentz transformation yields

$$\delta(\lambda_b^T \epsilon \lambda_a) = i\delta(\lambda_b^T \sigma_2 \lambda_a) = i\big[\delta(\lambda_b^T)\sigma_2\lambda_a + \lambda_b^T \sigma_2 \delta(\lambda_a)\big] = 0, \tag{1.2.19}$$

which is precisely what we were seeking. We have successfully found a Lorentz invariant quantity in the spinor space, $\lambda_b^T \epsilon \lambda_a$. A similar story holds for taking the norm of a right-handed spinor with $\tilde{\lambda}_{\dot{b}}^T \epsilon \tilde{\lambda}_{\dot{a}}$. But, as we discussed before, left-handed and right-handed spinors transform under different sets of generators, so their spinor product, *e.g.* $\tilde{\lambda}_{\dot{b}}^T \epsilon \lambda_a$, is not Lorentz invariant,[9] and, in general, we will simply say that these expressions are illegal. At

[9] To see this more concretely, one can notice that the spinors transform differently and the $(i\theta \mp \eta)$ will not cancel in the $\delta(\tilde{\lambda}_{\dot{b}}^T \epsilon \lambda_a)$.

this stage, we have found an appropriate inner product (more precisely, a bilinear form) that bridges Lorentz invariants to spinor variables and should establish a convention for contracting, raising, and lowering the indices. We will elaborate them in the following section

1.2.4 Inner Product in Details

So far, we have consistently use the index, such as $a, \dot{a}, b, \dot{b}$, as a name for the spinor. With a well-defined spinor product, we can now define a meaningful way to manipulate indices. Let's consider again that

$$\lambda_L = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad \tilde{\lambda}_R = \begin{pmatrix} \tilde{\lambda}_{\dot{1}} \\ \tilde{\lambda}_{\dot{2}} \end{pmatrix} \tag{1.2.20}$$

Also, we define the (Levi-Civita) **epsilon symbol** in its index notation as[10]

$$\epsilon^{ab} \triangleq -\epsilon_{ab} \triangleq \left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right]_{ab}, \quad \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}}. \tag{1.2.21}$$

so that what we discussed in the previous section is

$$\text{“}\chi_b^T \epsilon \psi_a\text{”} = \epsilon^{ab} \chi_a \psi_b. \tag{1.2.22}$$

Conveniently,

$$\epsilon^{ac}\epsilon_{cb} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \delta^a_b. \tag{1.2.23}$$

We then have almost every gadget we need to construct a "dot product". But noticeably, this is not a "dot product" in the usual sense because it is antisymmetric, *i.e.*

$$\text{“}\chi_b^T \epsilon \psi_a\text{”} = \epsilon^{ab} \chi_a \psi_b = -\epsilon^{ba} \chi_a \psi_b = -\text{“}\psi_a^T \epsilon \chi_b\text{”}. \tag{1.2.24}$$

So we have to be very mindful and avoid identifying "$\chi \cdot \psi$" = "$\psi \cdot \chi$" as we frequently do for other dot products.[11] Nevertheless, the epsilon symbol provides a Lorentz invariant "metric" over the spinor space, and raising and lowering its index behaves just like what one would expect.

Not only does the epsilon symbol function as a "metric" over the spinor space, but also it interplays well with the Pauli matrices. Then, Pauli matrices will connect the 2-dimensional spinors to the 4-dimensional bispinors used in Dirac equation. We can also obtain some fun identities. If we define the following four-vector-like Pauli matrices

$$\sigma^\mu_{a\dot{a}} \triangleq (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)^\mu_{a\dot{a}}, \quad \bar{\sigma}^{\mu\dot{a}a} \triangleq \sigma_{\mu a\dot{a}} = (\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3)^{\mu\dot{a}a}, \tag{1.2.25}$$

[10] Unfortunately, just like we have a group of $(+,-,-,-)$ believers and a group of $(-,+,+,+)$ believers in physics, there are also two different conventions for this poor epsilon symbol. Some physicists use $\epsilon^{ab} = \epsilon_{ab}$ so that raising and lowering are done in a consistent way. We introduced an additional minus so that the conversion between the spinor helicity formalism and the usual Dirac notation will be more intuitive and straightforward. There is no difference in the two conventions, but one must keep track of many additional minus signs if we were to use the the other convention.

[11] Although if these products are over spinor fields rather than spinors, this spinor product becomes symmetric due to additional antisymmetry of the field. For more details, consult p. 180 of [25].

then we can obtain the following identities

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}} = \bar{\sigma}^{\mu\dot{a}a}, \quad g_{\mu\nu}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}} = 2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}. \tag{1.2.26}$$

The first identity guarantees the meaningful raising and lowering indices with epsilon symbols, and the second identity is known as the **Fierz identity**. Let's prove them with direct matrix multiplications. For the first identity, we can try

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^{\mu}_{b\dot{b}} = -\epsilon^{ab}\sigma^{\mu}_{b\dot{b}}\epsilon^{\dot{b}\dot{a}}; \tag{1.2.27}$$

however, in matrix notation, this means

$$-\epsilon\mathbb{M}\epsilon = -\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{1.2.28}$$

Substituting Pauli matrices into $\mathbb{M}$, we find that, indeed, the first identity holds. For the Fierz identity, it may be convenient to utilize the tensor product or the Kronecker product shown as follows.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{12}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_{21}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} & a_{22}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix}. \tag{1.2.29}$$

Then, computationally, we are interested in

$$\begin{aligned} & g_{\mu\nu}\sigma^{\mu}_{a\dot{a}}\sigma^{\nu}_{b\dot{b}} \\ =& \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ =& \left[\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} - \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} - \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}\right] \\ =& \begin{pmatrix} 0 & & & 0 \\ & 2 & -2 & \\ & -2 & 2 & \\ 0 & & & 0 \end{pmatrix} = \begin{cases} 2, & \left(a, \dot{a}, b, \dot{b}\right) = (1,1,2,2) \quad \text{or} \quad (2,2,1,1) \\ -2, & \left(a, \dot{a}, b, \dot{b}\right) = (1,2,2,1) \quad \text{or} \quad (2,1,1,2) \\ 0, & \text{else} \end{cases} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \end{aligned}$$

These identities provide foundations for our attempt to use spinor variables describing 4-dimensional objects such as momentum, polarization, and bispinors. A quick remark on the two relations: we can combine the two identities and use the anticommutators of Pauli matrices and find that

$$\sigma^{\mu}_{a\dot{a}}\sigma^{\dot{b}b}_{\mu} = \epsilon^{\dot{b}\dot{c}}\epsilon^{bc}\sigma^{\mu}_{a\dot{a}}\sigma_{\mu\dot{c}c} = 2\epsilon^{bc}\epsilon^{\dot{b}\dot{c}}\epsilon_{ac}\epsilon_{\dot{a}\dot{c}} = 2\delta^{b}_{a}\delta^{\dot{b}}_{\dot{a}}, \tag{1.2.30}$$

$$g^{\mu\nu}\frac{\delta^{a}_{b}}{2} = \left\{\frac{\sigma^{\mu}}{2}, \frac{\bar{\sigma}^{\nu}}{2}\right\}^{a}_{b} \tag{1.2.31}$$

This equality will help us identify some interesting relation between traces of the "spinor matrices" and Lorentz dot products discussed in section 1.3.2.

1.3 Spinor Helicity Basics

We have witnessed that the **spinor product** defined in the previous section is a Lorentz invariant. This motivates the use of spinor variables to describe kinematics and substitute four-vectors. In this section, we present some basics about **spinor helicity formalism**. We will follow the convention used by Dixon's article[27] and Schwartz's textbook[25]. Given some Weyl spinors $\left\{\lambda_1, \lambda_2, \tilde{\lambda}_1, \tilde{\lambda}_2\right\}$, one can define the following quantities.

$$\langle\lambda_1\,\lambda_2\rangle \triangleq \epsilon^{ab}(\lambda_1)_a(\lambda_2)_b, \quad \left[\tilde{\lambda}_1\,\tilde{\lambda}_2\right] \triangleq \epsilon^{\dot{a}\dot{b}}\left(\tilde{\lambda}_1\right)_{\dot{a}}\left(\tilde{\lambda}_2\right)_{\dot{b}}, \tag{1.3.1}$$

which are, as discussed above, conveniently Lorentz invariant. We can also use a shorthand

$$\langle 1\,2\rangle \triangleq \langle\lambda_1\,\lambda_2\rangle\,, \quad [1\,2] \triangleq \left[\tilde{\lambda}_1\,\tilde{\lambda}_2\right] \tag{1.3.2}$$

as long as the implication of the index is clear. Also, by eq'n (1.2.24), we know that

$$\langle 1\,2\rangle = -\langle 2\,1\rangle\,, \quad [1\,2] = -[2\,1]\,, \quad \langle 1\,1\rangle = [1\,1] = 0. \tag{1.3.3}$$

Also, it may be notationally concise to just define

$$|1\rangle = (\lambda_1)^a, \quad |1] = \left(\tilde{\lambda}_1\right)_{\dot{a}}, \quad \langle 1| = (\lambda_1)_a, \quad [1| = \left(\tilde{\lambda}_1\right)^{\dot{a}}, \tag{1.3.4}$$

in which we raise and lower the indices with the epsilon symbol defined in eq'n (1.2.21).

1.3.1 Schouten Identity: Spinor Version of the Jacobi Identity

The epsilon symbol provides an interesting way for us to contract spinors in an antisymmetric fashion. This antisymmetry has an implication that will be frequently used in spinor manipulations. In this section, we aim to show the following identity, called **Schouten identity**

$$0 = |1\rangle\langle 2\,3\rangle + |2\rangle\langle 3\,1\rangle + |3\rangle\langle 1\,2\rangle = |i\rangle\langle j\,k\rangle + (\text{cyclic in } i, j, k). \tag{1.3.5}$$

Let's consider the projection of $|1\rangle$ into some other spinors $|2\rangle$ and $|3\rangle$. Note that these spinors are just in $\mathbb{C}^2$, we can always project $|1\rangle$ onto another two spinors by spinor products as long as $|2\rangle$ and $|3\rangle$ are linearly independent. This is straightforward because

$$|1\rangle = \frac{\langle 3\,1\rangle}{\langle 3\,2\rangle}|2\rangle + \frac{\langle 2\,1\rangle}{\langle 2\,3\rangle}|3\rangle\,. \tag{1.3.6}$$

One can check that both left-hand side and the right-hand side of the equation yields the correct result when contracting with $\langle 2|$ and $\langle 3|$. Since $|2\rangle$ and $|3\rangle$ are linearly independent, we know that $\langle 2\,3\rangle \neq 0$.

$$|1\rangle = \frac{\langle 3\,1\rangle}{\langle 3\,2\rangle}|2\rangle + \frac{\langle 2\,1\rangle}{\langle 2\,3\rangle}|3\rangle \implies |1\rangle\langle 2\,3\rangle + |2\rangle\langle 3\,1\rangle + |3\rangle\langle 1\,2\rangle = 0, \tag{1.3.7}$$

which is the Schouten identity. But even if $|2\rangle = |3\rangle$, $\langle i\,1\rangle \langle 2\,3\rangle = 0$, and $\langle i\,2\rangle \langle 3\,1\rangle = -\langle i\,2\rangle \langle 1\,3\rangle = -\langle i\,3\rangle \langle 1\,2\rangle$. Thus, the equality still holds. This provides us some Jacobi-like identity. Similarly, the same identity holds for squary brackets

$$0 = |1]\,[2\,3] + |2]\,[3\,1] + |3]\,[1\,2]\,. \tag{1.3.8}$$

Since raising, lowering, or contracting the free index will not affect the result, we can also claim that

$$0 = \langle 1\,2\rangle \langle 3| + \langle 2\,3\rangle \langle 1| + \langle 3\,1\rangle \langle 2| = [1\,2]\,[3| + [2\,3]\,[1| + [3\,1]\,[2|\,. \tag{1.3.9}$$

1.3.2 Expressing and Manipulating Four-Momenta

As shown in table 1.1, we choose to use the $(1/2, 1/2)$ irrep so that we can express both a scalar and a vector. The spinor product is formed to provide a scalar under Lorentz transformations. In this section, we would like to investigate how to express a four-momentum, and, in general, any four-vector. As discussed in section 1.2.4, Pauli matrices can bridge the spinor index and the Lorentz index with nice properties in either space. So we may use Pauli matrices to express four-vectors. For a four-momentum p^μ, we may use

$$p^{a\dot{a}} \triangleq p^\mu \sigma_\mu^{a\dot{a}} = \begin{pmatrix} p^t - p^z & -p^x + ip^y \\ -p^x - ip^y & p^t + p^z \end{pmatrix} \triangleq \not{p}, \tag{1.3.10}$$

in which σ^μ is defined as in eq'n (1.2.25). Due to eq'n (1.2.26), we also can infer that the original four-momentum can be expressed as

$$p^\mu = \frac{1}{2}\sigma^{\mu a\dot{a}} p_{\dot{a}a}. \tag{1.3.11}$$

Performing an infinitesimal Lorentz transformation on $p^\mu \to \Lambda^\mu{}_\nu p^\nu$, we find that

$$\delta(\Lambda^\mu{}_\nu p^\nu) = \frac{1}{2}\Lambda^\mu{}_\nu \sigma^\nu_{a\dot{a}} p^{a\dot{a}}, \tag{1.3.12}$$

which is how we defined the transformation laws for spinors shown in eq'n (1.2.14).

The spinor version of a four-momentum also has an interesting property. Observe that

$$\det\left(p^{a\dot{a}}\right) = \left(p^t\right)^2 - (p^z)^2 - (p^x)^2 - (p^y)^2 = p^2. \tag{1.3.13}$$

In particular, for a massless particle ($p^2 = 0$), this determinant is zero. This means that we can express the matrix $p^{a\dot{a}}$ as an outer product of two 2-dimensional objects written suggestively as

$$p^{a\dot{a}} = \lambda_p^a \tilde{\lambda}_p^{\dot{a}}, \tag{1.3.14}$$

but using eq'n (1.3.12), we know that each 2-dimensional object transform just like spinors, so they are in fact $|p\rangle$ and $[p|$ we identified and defined in eq'n (1.3.9). In many cases, we don't care much about the specific placement of the spinor index, so, in general, we write in spinor helicity

$$p = |p\rangle\,[p| = |p]\,\langle p|\,. \tag{1.3.15}$$

This equation also assigns each spinor variables a mass dimension $[\lambda] = [\tilde{\lambda}] = 1/2$. So far, λ_p and $\tilde{\lambda}_p$ are completely general, but if we enforce the fact that $p^\mu \in \mathbb{R}^4$, then

$$p^{a\dot{a}} = \left(p^{a\dot{a}}\right)^\dagger \implies |p\rangle = [p|^*, \quad \langle p| = |p]^*, \tag{1.3.16}$$

in which $*$ denotes complex conjugate. Eq'n (1.3.16) is called the **reality condition**. Admittedly, physical momentum should always be real. But there is no need to enforce this fact until it is absolutely necessary. It is because the fact that we can decompose a massless p^μ into an outer product of some general λ^a and $\tilde{\lambda}^{\dot{a}}$ gives us a freedom to analytically continue the kinematic variables and manage singularities of the phase space more systematically. So when using spinor helicity formalism, we will treat left-handed and right-handed spinors as independent objects and enforce the **reality condition** by the end.

Knowing how to represent a four-vector as a spinor, how can we express the usual Lorentz dot products of k^μ and p^μ? We will use eq'n (1.3.11) and eq'n (1.2.30) and find that

$$p \cdot k = \left(\frac{1}{2}\sigma^\mu_{a\dot{a}} p^{a\dot{a}}\right)\left(\frac{1}{2}\sigma^{b\dot{b}}_\mu k_{b\dot{b}}\right) = \frac{1}{2} p^{a\dot{a}} k_{a\dot{a}} = \frac{1}{2}\operatorname{Tr}(|p\rangle\,[p|\,|k]\,\langle k|) = \frac{1}{2}\langle k\,p\rangle\,[p\,k]\,. \tag{1.3.17}$$

One may notice that according to eq'n (1.3.11), we can also express p^μ as

$$p^\mu = \frac{1}{2}\,[p|\,\sigma^\mu\,|p\rangle\,, \tag{1.3.18}$$

then the dot product becomes very notational

$$p^\mu k_\mu = \frac{1}{2}\,[p|\,\sigma^\mu\,|p\rangle\,\frac{1}{2}\,\langle k|\,\sigma_\mu\,|k] = [p|\,\frac{\sigma^\mu}{2}\,|p\rangle\,\langle k|\,\frac{\sigma_\mu}{2}\,|k] = \frac{1}{2}\,\langle k\,p\rangle\,[p\,k]\,, \tag{1.3.19}$$

in which we invoked the Fierz identity. However, these notations have a severe drawback, ambiguities. By the notation "$[p|\,\sigma^\mu\,|p\rangle$", we can either mean "$[p|\,\sigma^\mu_{a\dot{a}}\,|p\rangle$" $= \tilde{\lambda}^{\dot{a}}_p \sigma^\mu_{a\dot{a}} \lambda^a_p = \lambda^a_p \sigma^\mu_{a\dot{a}} \tilde{\lambda}^{\dot{a}}_p = |p\rangle\,\sigma^\mu_{a\dot{a}}\,[p|$ or $[p|\,\sigma^{\mu a\dot{a}}\,|p\rangle = [p|\,\bar{\sigma}_{\mu\dot{a}a}\,|p\rangle$. This ambiguity can be quite confusing as we are unsure which way we should interpret the Lorentz index. To ensure that we obtain the information correctly, we can use a symmetrized version of the definition

$$p^\mu = \frac{1}{2}\tilde{\lambda}^{\dot{a}}_p \bar{\sigma}^\mu_{\dot{a}a}\lambda^a_p + \frac{1}{2}\lambda_{pb}\sigma^{\mu b\dot{b}}\tilde{\lambda}_{p\dot{b}} = \frac{1}{2}\,\langle p|_b \left(\bar{\sigma}^\mu_{\dot{a}a} + \sigma^{\mu b\dot{b}}\right)|p]_{\dot{b}} = \frac{1}{2}\,[p|^{\dot{a}}\left(\bar{\sigma}^\mu_{\dot{a}a} + \sigma^{\mu b\dot{b}}\right)|p\rangle^a\,, \tag{1.3.20}$$

in which the unpaired index is not summed over and contributes 0 to the final result since the expression does not parse. This new 4-indexed object is shorthanded as the **gamma matrices** $(\gamma^\mu)^{b\dot{b}}_{\dot{a}a}$ so that

$$p^\mu = \frac{1}{2}\,\langle p|\,\gamma^\mu\,|p] = \frac{1}{2}\,[p|\,\gamma^\mu\,|p\rangle\,. \tag{1.3.21}$$

Notice that along with eq'n (1.2.31), we realize

$$(\gamma^\mu)^{a\dot{c}}_{\dot{a}c}(\gamma^\nu)^{c\dot{b}}_{\dot{c}b} = (\bar{\sigma}^\mu\sigma^\nu)^{\dot{a}}_{\dot{b}} + (\sigma^\mu\bar{\sigma}^\nu)^a_b \implies (\gamma^\mu)^{a\dot{c}}_{\dot{a}c}(\gamma^\nu)^{c\dot{b}}_{\dot{c}b} + (\gamma^\nu)^{a\dot{c}}_{\dot{a}c}(\gamma^\mu)^{c\dot{b}}_{\dot{c}b} = 2g^{\mu\nu}\left(\delta^a_b + \delta^{\dot{a}}_{\dot{b}}\right). \tag{1.3.22}$$

Both the loose index in defining $[p|\,\gamma^\mu\,|p\rangle$ and the double contraction of c and $\dot{c}$ in taking an "anticommutator" seems strange. To remedy these, we use a concrete matrix notation with off-diagonal block matrices to visualize the gamma matrices

$$(\gamma^\mu)^{b\dot{b}}_{\dot{a}a} = \begin{pmatrix} & \sigma^\mu_{\dot{a}a} \\ \bar{\sigma}^{\mu b\dot{b}} & \end{pmatrix}. \tag{1.3.23}$$

Now, the strange c and $\dot{c}$ double contraction is just a matrix multiplication over a 4×4 matrix, and the anticommutator-like statement becomes an actual anticommutator

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 2g^{\mu\nu} \mathbb{1}_4. \tag{1.3.24}$$

The loose index can be explained if we regard the spinor variables as some four-dimensional objects

$$\lambda_a = \begin{pmatrix} 0 & \langle\lambda|_a \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = \begin{pmatrix} [\lambda|_{\dot{a}} & 0 \end{pmatrix}, \quad \lambda^a = \begin{pmatrix} 0 \\ |\lambda\rangle_a \end{pmatrix}, \quad \tilde{\lambda}^{\dot{a}} = \begin{pmatrix} |\lambda]^{\dot{a}} \\ 0 \end{pmatrix}. \tag{1.3.25}$$

Of course, these are for our notational convenience rather than some actual four-vectors, but, historically, these objects were first identified as spinors. These objects look like four-vectors but do not transform like four-vectors or spinors as we know from angular momentum eigenstates. This is because they are fundamentally **bispinors** carrying two sets of spinor indices, and they can be constructed purely from decomposing the Lie algebra of $\mathfrak{so}(3,1) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, each of which is the familiar spinor group for an angular momentum operator.

1.4 Little Group, Gauge, and Polarization

Taking some time decomposing the Lorentz group, we have concluded that there is a new way of representing any four-vector and Lorentz invariants with the $(1/2, 1/2)$ irrep. Four-vectors are formed by outer products of one left-handed spinor and one right-handed spinor, and Lorentz invariants are formed by spinor products. Although refreshed by this new type of quantity, we have not identified how spinor helicity is useful for the study of scattering amplitudes. Returning to eq'n (1.1.9), we claimed that a Lorentz transformation over an amplitude should exhibit the unitary representation of the little group of each particle

$$\mathcal{A} \to \left[\prod_{\text{particle r}} W(\Lambda, p_r) \right] \mathcal{A}. \tag{1.4.1}$$

For a massless particle, its little group is $SO(2) \cong U(1)$.[12][13] Therefore, the unitary representation of the little group is in the form of a complex phase factor $e^{i\theta}$ with $\theta \in \mathbb{R}$ for each initial- and final-state particle. According to eq'n (1.3.15), we can decompose the momentum of a massless particle, and the little group transformation leaves the momentum invariant.

[12] *Caveat lector*: we are very sloppy about the notation $\cong$ here. By $\cong$, we mean that the two groups have the same Lie algebra. In general, we are interested in the Lie algebra instead of the Lie group since infinitesimal transformations generates finite transformations. The same warning applies to the following footnote.

[13] One may be also interested in the little group transformation for massive particles. Since the little group for a massive particle is $SO(3) \cong SU(2)$, we can also infer that the appropriate spinor variables should be some 2×2 objects, one index contributing to the "outer product" and one index contributing to the little group transformations.

So in spinor helicity formalism, the little group transformation looks like $|p\rangle \to \lambda\,|p\rangle$ and $[p| \to \lambda^{-1}\,[p|$ for some arbitrary $\lambda \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$. This is called the **little group scaling** of spinor variables. Enforcing the reality condition shown in eq'n (1.3.16), we know that $\lambda = e^{i\theta/2}$ for some $\theta \in \mathbb{R}$, and the factor of $1/2$ comes from the fact that spinors variables is only spin-1/2. In spinor helicity formalism, we can factor out the helicity information $W(\Lambda, p)$ from the amplitude $\mathcal{A}$ which is the quantity of interest. So spinor variables are natural variables to describe the transformation of scattering amplitudes for massless particles. To count the helicity of a particle i, we can either count the $1/2\Big(-\text{exponent of } \lambda_i + \text{ exponent of } \tilde{\lambda}_i\Big)$, or simply use the following **helicity operator** on the amplitude to extract its eigenvalue

$$h_i = \frac{1}{2}\bigg(-\lambda_i \frac{\partial}{\partial \lambda_i} + \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}\bigg). \tag{1.4.2}$$

Having understood the relation between spinor helicity formalism and scattering amplitude, we can continue to investigate the manifestation of gauge redundancy described in section 1.1.2 in terms of spinor variables. Specifically, we would like to find the **polarization vector** ϵ for a given momentum p in spinor helicity formalism. Recall that polarization vectors satisfy $\epsilon \cdot \epsilon = 0$ and $\epsilon \cdot p = 0$. The first condition hints expressing the polarization vector as an outer product of two spinors, and the second condition shows that one of its spinors should contain a factor of p so that the spinor product of ϵ and p vanishes. Following the argument presented in [28], we can express polarization vectors as

$$\epsilon_+ = \sqrt{2}\frac{|r\rangle\,[p|}{\langle r\,p\rangle} = \sqrt{2}\frac{|p]\,\langle r|}{\langle r\,p\rangle}, \quad \epsilon_- = \sqrt{2}\frac{|p\rangle\,[r|}{[p\,r]} = \sqrt{2}\frac{|r]\,\langle p|}{[p\,r]}, \tag{1.4.3}$$

in which $\lambda_r \neq \lambda_p$ and $\tilde{\lambda}_r \neq \tilde{\lambda}_p$ denote some arbitrary reference vector *not* along p direction. These polarization vectors are exactly the eigenvectors of the helicity operator because

$$h\epsilon_\pm = \frac{1}{2}[-(\mp 1) + (\pm 1)]\epsilon_\pm = \pm\epsilon_\pm. \tag{1.4.4}$$

Similar to eq'n (1.3.21), we can also use the gamma matrices to rewrite the polarization vector as

$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}\frac{[p|\,\gamma^\mu\,|r\rangle}{\langle r\,p\rangle} = \frac{1}{\sqrt{2}}\frac{\langle r|\,\gamma^\mu\,|p]}{\langle r\,p\rangle}, \quad \epsilon_-^\mu = \frac{1}{\sqrt{2}}\frac{[r|\,\gamma^\mu\,|p\rangle}{[p\,r]} = \frac{1}{\sqrt{2}}\frac{\langle p|\,\gamma^\mu\,|r]}{[p\,r]}. \tag{1.4.5}$$

Let's digress a little to discuss another important conservation law. Here we have encountered a new kind of outer product that is formed by a spinor related to a particle's momentum and a spinor of another vector. This is legal since the Lorentz invariant can still be formed in the spinor space with the usual epsilon symbol as we did with $\epsilon\cdot\epsilon$ or $\epsilon\cdot p$. But this freedom yields an important result: the manifestation of momentum conservation in spinor space. In terms of four-vectors, we know that $0^\mu = \sum_{\text{particle } i} p_i^\mu$ for a scattering process, and its spinor form is $0 = \sum_{\text{particle } i} |i\rangle\,[i| = \sum_{\text{particle } i} |i]\,\langle i|$. Then, we can form spinor products by sandwiching them in between another pair of left-handed and right-handed spinors, which can depend on different momenta. Then, we realize that there will be vanishing terms such as $\langle j\, j \rangle$ or $[k\, k]$. Therefore, the **conservation of momentum** in spinor helicity formalism is of the form

$$\sum_{i \neq j,k} \langle j\, i \rangle \, [i\, k] = 0. \tag{1.4.6}$$

For example, in a four-point (four-external-particle) configuration, we can deduce that

$$\langle 1\, 2 \rangle \, [2\, 4] = - \langle 1\, 3 \rangle \, [3\, 4] \, . \tag{1.4.7}$$

When there is no ambiguity, we will also write these kind of expressions as

$$\langle 1 | \, 2 \, | 4] = - \langle 1 | \, 3 \, | 4] \tag{1.4.8}$$

with the "jam" of the sandwich understood to be a four-vector (which may or may not be massless) expressed in terms of gamma matrices (general four-vector) or outer products of spinor variables (massless vector).

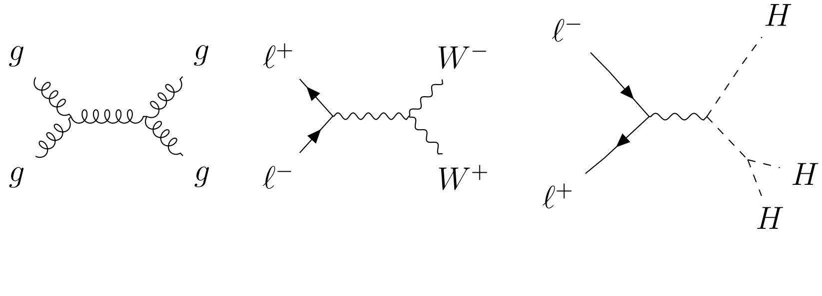
Knowing how the helicity information of the external particle manifests in amplitudes through spinor helicity formalism, we can twist the original question about solving for the scattering amplitude into another interesting question: if there are only massless external particles with dimensionless coupling constants, then what kind of scattering amplitudes are we permitted to write down? This question yields some surprising constraints on possible amplitudes and sometimes constraints on possible theories, and we will explore this aspect in the following chapter.

Chapter 2

Tree-Level Amplitudes

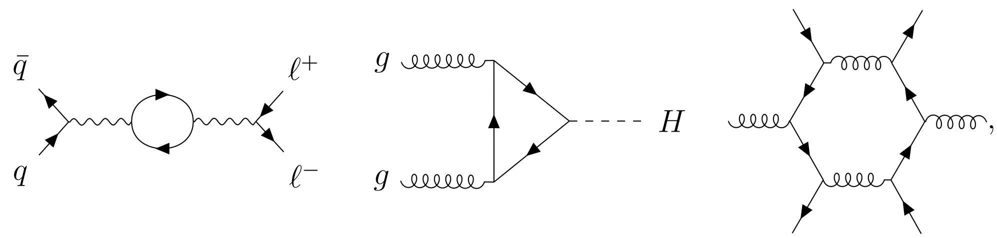
In the previous chapter, we thoroughly examined the transformation properties of scattering amplitudes and recognized that spinor helicity formalism is the suitable formulation to present a scattering amplitude. By the end of the last chapter, we also identified how "helicity information" is represented in spinor helicity formalism, and by applying helicity operator, we can retrieve this information.

Quite intriguingly, the transformation property also shows that the helicity information of each external particle must factorize into unitary representations of their individual little groups. This means that we would expect the amplitude written in spinor helicity formalism to exhibit a similar factorization of the little group. This allows us to determine the helicity part of a scattering amplitude. For example, if $\mathcal{A}$ is an amplitude with an external spin-1 positively-polarized particle labeled as 1, then $h_1\mathcal{A} = \mathcal{A}$; if $\mathcal{A}$ also describes an external spin-1/2 negatively-polarized particle labeled as 2, then $h_2\mathcal{A} = -\mathcal{A}/2$. Therefore, the helicity part of an amplitude will severely constrain our ability to write down an amplitude for any process. In fact, in some simple cases, we are forced to acknowledge certain restrictions on the process. In this chapter, we will leverage this knowledge about helicity and demonstrate the physical significance of spinor helicity formalism over **tree-level amplitudes**, *i.e.* amplitudes with Feynman diagrams without any cycles (between vertices) in them or amplitudes in which each pair of external particles is connected via a unique path of propagators. To draw a few tree-level amplitudes, they may include:



, (2.0.1)

but we must exclude processes like



(2.0.2)

in which (at least) a cycle (*viz.* a loop) appears. The main motivation for investigating tree-level amplitudes is that according to Feynman rules of the field theory, we know that tree-level amplitudes must be rational functions of Lorentz invariants. In contrast, for an amplitude with a loop, since the internal loop momentum is not observed, we must integrate over all possible loop momenta which leads to functions with potentially more complicated analytic structures than those of rational functions.

2.1 Three-Point Amplitudes

First, we will investigate the simplest amplitudes, three-point amplitudes. Interestingly, although three-point amplitudes should be the fundamental type of interaction described by field theory,[1] it seems that we do not encounter this type of amplitude in the final result too frequently. The reason appears straightforwardly with spinor helicity formalism. For simplicity, we will assume from now on that all momenta for an amplitude are either all outgoing or all incoming. Note that we have implicitly assumed so when deriving the momentum conservation condition in eq'n (1.4.6).

Let's start arguing for the possible form for a 3-point amplitude. In general, the amplitude should be Poincaré invariant, *i.e.* invariant under Lorentz transformations and spatial and temporal translations. But we can simply restrict our attention to the Lorentz invariant part of the amplitude, *viz.* **stripped amplitude**. This is because an amplitude usually takes the form

$$\mathcal{A} = \delta^4\left(\sum_{\text{particle } i} p_i\right)A, \tag{2.1.1}$$

in which A denotes the stripped amplitude. The delta function over the momentum space enforces the energy-momentum conservation, *i.e.* the Noether current generated by translations. Since Lorentz transformation has determinant 1, any Lorentz transformation will not change the argument of the delta function; thus, $\delta(\sum p) \to \delta(\sum p)$ under any Lorentz transformation. But $\mathcal{A} \to [\prod W]\mathcal{A}$, so $A \to [\prod W]A$, *i.e.* **little group scaling** of an amplitude is entirely captured by the its stripped amplitude. From now on, we will use the word **amplitude** as a shorthand for a **stripped amplitude**.

We know that the amplitude describes the interaction between particles, and particles $|p, \sigma\rangle$ are described by their momenta and little group label. For massless particles, this little group label is the helicity of the particle; therefore, the amplitude must only be a function of the external particles' momenta and helicity, $A = A(p_i, h_i)$. As shown in the

[1]For example, recall that the only interaction vertex in QED is a three-point $\gamma e^- e^+$ vertex.

previous chapter, Lorentz invariance is preserved with spinor products, so a general three-point amplitude takes the form.

$$A(1,2,3) = C \langle 1\,2\rangle^{p_{\langle 1\,2\rangle}} \langle 2\,3\rangle^{p_{\langle 2\,3\rangle}} \langle 3\,1\rangle^{p_{\langle 3\,1\rangle}} [1\,2]^{p_{[1\,2]}} [2\,3]^{p_{[2\,3]}} [3\,1]^{p_{[3\,1]}} . \tag{2.1.2}$$

This is already an amazingly stringent constraint on the possible form of a three-point amplitude. But the kinematics and the antisymmetry of the spinor product provides a stronger constraint. Momentum conservation (both in four-momentum and in spinor helicity) demands that

$$\langle 1\,2\rangle [2\,1] = \langle 1\,2\rangle [2\,3] = \langle 2\,1\rangle [1\,3] = 0. \tag{2.1.3}$$

This means many combinations of spinor products vanish in this three-point configuration. Now, assume that $\langle 1\,2\rangle \neq 0$. Then, we are forced to conclude that $[1\,2] = [2\,3] = [3\,1] = 0$. A similar argument holds if we assume that $[1\,2] \neq 0$, then all $\langle i\,j\rangle$ vanishes. This is called **three-point special kinematics** and is frequently recognized as

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3] . \tag{2.1.4}$$

Sometimes, these two scenarios are named λ-parallel or $\tilde{\lambda}$-parallel case since the proportionality statement implies that the three spinors are parallel in the spinor space $\mathbb{C}^2$. This observation illustrates why vertices in Feynman rules are not regarded as amplitudes because three-point special kinematics only shows up with complex momenta. If we enforce the reality condition $|p\rangle = [p|^*$, then we are forced to conclude that all spinor products vanish. This is understandable. After all, massless three-particle scattering happens only if three particles are collinear.

Now, let's continue to assume that $\langle 1\,2\rangle \neq 0$. To write down a 3-point amplitude, we can only use $\langle i\,j\rangle$s, *i.e.*

$$A(1,2,3) = C \langle 1\,2\rangle^{p_{\langle 1\,2\rangle}} \langle 2\,3\rangle^{p_{\langle 2\,3\rangle}} \langle 3\,1\rangle^{p_{\langle 3\,1\rangle}} . \tag{2.1.5}$$

We can yet impose an even stronger but frequently neglected constraint, mass dimension. An n-point amplitude A must have a **mass dimension** $[A] = 4 - n$. To see this, recall that the scattering cross section can be found via Fermi's Golden rule as

$$\frac{1}{\left|\epsilon_{\mu x y \nu} p_{i_1}^{\mu} p_{i_2}^{\nu}\right|} \underbrace{\int \left[\mathrm{d}^4\Pi\ \delta(p^2)\right]^{n-2} |A|^2 \delta^4\Big(\sum p\Big)}_{\text{dimensionless}} \implies \left[A^2\right] + (n-2)[\text{on-shell LIPS}] - 4 = 0, \tag{2.1.6}$$

in which the on-shell Lorentz-invariant phase space (LIPS) is denoted as $\mathrm{d}^4\Pi\delta(p^2)$. This means that

$$p_{\langle 1\,2\rangle} + p_{\langle 2\,3\rangle} + p_{\langle 3\,1\rangle} = 1 \tag{2.1.7}$$

with the assumption that C is dimensionless. Along with the helicity information about the external particles, we are forced to reach the following over-constrained system of linear equations of the form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} p_{\langle 1\,2\rangle} \\ p_{\langle 2\,3\rangle} \\ p_{\langle 3\,1\rangle} \end{pmatrix} = \begin{pmatrix} 1 \\ -2h_1 \\ -2h_2 \\ -2h_3 \end{pmatrix}, \tag{2.1.8}$$

Particles	(AAA)	$(A\chi\chi)$	$(A\phi\phi)$	$(\phi\chi\chi)$	(χAA)
Helicity	$(-1,-1,+1)$	$\left(-1,+\frac{1}{2},-\frac{1}{2}\right)$	$(-1,0,0)$	$\left(0,-\frac{1}{2},-\frac{1}{2}\right)$	—
A	$\frac{\langle 1\,2\rangle^3}{\langle 1\,3\rangle\,\langle 3\,2\rangle}$	$\frac{\langle 1\,3\rangle^2}{\langle 2\,3\rangle}$	$\frac{\langle 1\,2\rangle\,\langle 3\,1\rangle}{\langle 2\,3\rangle}$	$\langle 1\,2\rangle\,\langle 3\,1\rangle$	forbidden
Particles	$(\chi\chi\chi)$	$(\phi\phi\phi)$	$(\chi\phi\phi)$	(ϕAA)	$(A\chi\phi)$
A	all forbidden				

Table 2.1: Possible three-point massless amplitude with dimensionless coupling: A denotes gauge boson, χ a spin-1/2 fermion, ϕ a scalar; a dimensionless coupling C is understood in very amplitude.

in which $h_i = \pm s_i$ denotes the helicity of particle i with spin s_i. Let's consider the following possible species of massless external particles: scalar with spin 0, fermion with spin 1/2, gauge boson with spin 1. With these external particles, we can check all 10 possible combinations of these particles shown in table 2.1. Many processes can be interpreted as standard model vertices such as cubic-g vertex in QCD or $\gamma e^+ e^-$ vertex in QED. Notably, many three-point amplitudes are excluded on the ground of dimensionless coupling. Of course, if we replace $\langle i\,j\rangle \to [i\,j]$,[2] the amplitudes are still valid but with a negated helicity $h_i \to -h_i$. These eight three-point amplitudes can also be regarded as a replacement for interaction vertices, so we have found several building blocks for our investigations about tree-level amplitudes.

On the other hand, if we allow the coupling to be dimensionful, we can conclude that the three-point amplitude for particle $\{1,2,3\}$ with helicity (weighted by their spins as usual) $\{h_1,h_2,h_3\}$ will have the form

$$\begin{aligned} A(h_1,h_2,h_3) &= C\,[1\,2]^{h_1+h_2-h_3}\,[2\,3]^{h_2+h_3-h_1}\,[3\,1]^{h_3+h_1-h_2} \\ &= C\,\langle 1\,2\rangle^{h_3-h_1-h_2}\,\langle 2\,3\rangle^{h_1-h_2-h_3}\,\langle 3\,1\rangle^{h_2-h_3-h_1}\,, \end{aligned} \tag{2.1.9}$$

with a coupling constant C of mass dimension $[C] = -(h_1+h_2+h_3+1)$. Here, the ambiguity is whether the left-handed spinors or right-handed spinors should be chosen. However, as we discussed previously, all three-point amplitude vanishes at real external momenta; therefore, we must enforce that the amplitude remains finite (and vanish) when taken all $\langle i\,j\rangle$ or $[i\,j] \to 0$. This means that there will be higher powers of either left-handed or right-handed spinors in the numerator than those in the denominator. As a quick demonstration, we prefer $\langle 1\,2\rangle^3\,\langle 1\,3\rangle^{-1}\,\langle 3\,2\rangle^{-1} \sim \mathcal{O}(\langle\cdot\,\cdot\rangle)$ over $[1\,2]^{-3}\,[1\,3]^1\,[3\,2]^1 \sim \mathcal{O}\big([\cdot\,\cdot]^{-1}\big)$.

2.2 Four-Point Amplitudes

In the last section, we examined how spinor helicity formalism, kinematics, and dimensional analysis can jointly constrain a unique form of amplitudes. Sadly (and fortunately), the four-point kinematics is complicated enough to relax the constraint. Instead of using our previous argument about restricting a unique form of the amplitude, we will try to leverage

[2]This is simply to switch from the $\tilde{\lambda}$-parallel case to the λ-parallel case.

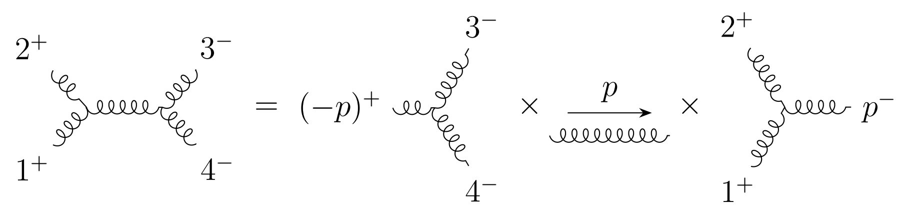
our knowledge about these three-point building blocks to construct higher-point amplitudes. First, we will present some pragmatic methods to find four-point amplitudes from three-point ones. These methods are a combination of reasonable guesses and conjectures. Then, we will demonstrate the reason behind this approach, *viz.* Britto-Cachazo-Feng-Witten on-shell recursion. This recursion method can be easily generalized to find a $n+1$-point amplitude from known $< n$-point amplitudes.

2.2.1 Some Pragmatic Ansätze

In physics, one of our favorite methods to solve a problem is proposing an Ansatz to test whether the problem is solved. If so, we can tweak our Ansatz to see its physical significance and its relation to the problem; if not, we can modify our Ansatz to reattempt solving the problem. There are many known theoretical data, such as scattering amplitudes or probability density, which we may use to guess higher-point amplitudes from lower-point ones. Here we will find the four-gauge-boson (Parke-Taylor) amplitude and Compton scattering amplitude with some guesses and compare them with known results.

Four-point Parke-Taylor Amplitude

Let's first consider the four-gauge-boson[3] amplitude with helicity $(++--)$. Note that our helicity is with respect to an incoming or outgoing momentum convention; thus, to interpret the diagram as a two-to-two scattering involves taking appropriate crossing on external particles. One can attempt to construct four-point amplitude following propagator and Feynman rules but with vertices replaced by three-point amplitudes. For example, we can write

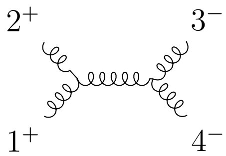


$$= \frac{\langle 3\,4\rangle^3}{\langle 3\,(-\hat{p})\rangle\,\langle(-\hat{p})\,4\rangle}\frac{1}{p^2}\frac{[1\,2]^3}{[1\,\hat{p}]\,[\hat{p}\,2]}, \tag{2.2.1}$$

in which we guessed that the propagator takes the form $1/p^2$ from the equation of motion for a gauge boson, $\Box A^\mu = 0$. Note that it is crucial to have p^+ and p^- in each part of the diagram so that the internal propagator's helicity cancel out, *i.e.* the helicity information about the internal propagator is hidden from the little group scaling. But due to momentum conservation, $p = p_1 + p_2$. So in spinor helicity formalism, $p^2 = (p_1+p_2)^2 = 2p_1 \cdot p_2 =$

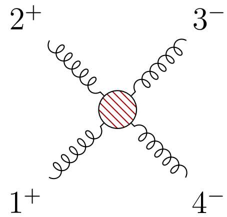
[3]Here, we will use the familiar gluon to represent a massless gauge boson in the diagram.

$\langle 1\,2\rangle\,[2\,1]$. Therefore, we can write $\not{p} = |1\rangle\,[1| + |2\rangle\,[2|$. In this case, the amplitude reduces to



$$
\begin{aligned}
&= \frac{\langle 3\,4\rangle^3}{\langle 3\,\hat{p}\rangle\,\langle \hat{p}\,4\rangle}\frac{1}{\cancel{[1\,2]}\,\langle 2\,1\rangle}\frac{[1\,2]^{\cancel{3}}}{[1\,\hat{p}]\,[\hat{p}\,2]} = \frac{\langle 3\,4\rangle^3\,[1\,2]^2}{[1\,\hat{p}]\,\langle \hat{p}\,3\rangle\,\langle 4\,\hat{p}\rangle\,[\hat{p}\,2]\,\langle 2\,1\rangle} \\
&= \frac{\langle 3\,4\rangle^3\,\cancel{[1\,2]}^2}{\cancel{[1\,2]}\,\langle 2\,3\rangle\,\langle 4\,1\rangle\,\cancel{[1\,2]}\,\langle 2\,1\rangle} = -\frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}.
\end{aligned}
\tag{2.2.2}
$$

Readers may be curious about the notation $\hat{p}$ here, but we will temporarily ignore this subtlety and treat it as a decoration. This gives $(1^+2^+3^-4^-)$ scattering amplitude[4]



$$
= -\frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}. \tag{2.2.3}
$$

One can also use a similar method to find the $(+\,+\,++)$ amplitudes for four external gauge bosons. But one will encounter that the internal propagator will have the same helicity in each vertex, resulting in a manifestation of addition helicity information, which is unphysical. With a similar method for $(-+++)$ helicity, one will find that the conservation of momentum demands this diagram to vanish too.[5] Thus, it is impossible to have $(++++)$ and $(-+++)$ scattering. Interesting, this is exactly the conclusion reached by Parke and Taylor[13, 14] via direct computation with Feynman rules.[6] Thus, the amplitude shown in eq'n (2.2.3) is also known as the four-point **Parke-Taylor amplitude** or the four-point **maximally helicity violating (MHV) amplitude**. With confirmation from a previous calculation, we are more certain about the soundness of spinor helicity formalism. Also, we notice that these calculations involve only simple algebraic manipulations that can be done with pen and paper while Parke and Taylor's Feynman diagram approach involve some careful choice of gauge, non-trivial cancellations across many terms, and arduous evaluations of tensorial expressions. This hints that our previous approach with Feynman rules may have introduced redundancies to the problem.

Compton Scattering Amplitude

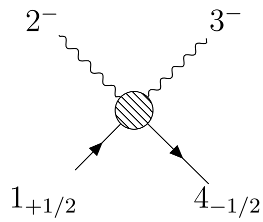
Let's now consider a more familiar Feynman diagram appearing as **Compton scattering** in QED. In QED, the only interaction is mediated by the $\gamma e^+ e^-$-vertex corresponding to the $(A\chi\chi)$ diagram we had. There may be several possible helicity choices for the external photon and electrons in this process. But note that regardless of the photon's helicity, the $(A\chi\chi)$ diagram will always have one plus-helicity and one minus-helicity fermion. To glue

[4] In fact, this argument finds only the **color-ordered amplitude** which I use a red blob to denote.

[5] We will see this in the calculation for Compton scattering amplitudes.

[6] Parke and Taylor's conclusion was about the squared amplitude, but one can square the amplitude straightforwardly by imposing the reality condition on spinor momenta.

the two three-point function up, we must have external fermions of opposite helicity. Since flipping all helicities of the external particles together simply requires to flip the chiral spinors by $|i\rangle \leftrightarrow |i]$, we are left with only two diagrams, one with all external photon with the same helicity and one with all external photon with an opposite helicity. Let's first calculate the same-helicity configuration.



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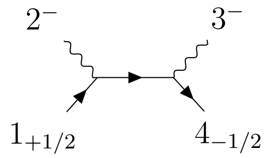

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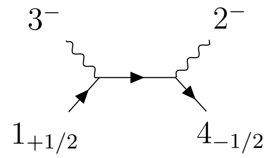
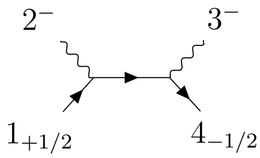


(2.2.4)

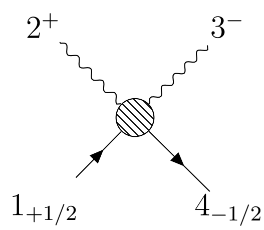
But each subamplitude vanishes due to the conservation of momentum across the propagator. To see that, we can pick one amplitude and find



$$= \frac{\langle 3\,4\rangle^2}{\langle \hat{p}\,4\rangle} \frac{1}{p^2} \frac{\langle 2\,\hat{p}\rangle^2}{\langle 1\,\hat{p}\rangle} = \frac{\langle 3\,4\rangle^2 \langle 2\,\hat{p}\rangle^2 [4\,\hat{p}]\,[\hat{p}\,1]}{4p^2(\hat{p}\cdot p_1)(\hat{p}\cdot p_4)} \propto \langle 2|\,\hat{p}\,|1] = 0, \tag{2.2.5}$$

So the same-helicity-photon configuration has zero amplitude. Another way to see this is that three-point amplitudes which we glued up implies that $|1] \propto |2] \propto |p] \propto |3] \propto |4]$, *i.e.* the amplitude must be entirely a function of products of left-handed spinor $\langle i\,j\rangle$. By momentum conservation in four-point kinematics, we can eliminate one spinor product, say $\langle 3\,4\rangle$, so that $A = \langle 1\,2\rangle^{p_{\langle 1\,2\rangle}} \langle 1\,3\rangle^{p_{\langle 1\,3\rangle}} \langle 1\,4\rangle^{p_{\langle 1\,4\rangle}} \langle 2\,3\rangle^{p_{\langle 2\,3\rangle}} \langle 2\,4\rangle^{p_{\langle 2\,4\rangle}}$. But there is no such amplitude that has the correct little group scaling and dimensionality. Thus, this amplitude vanishes.

Now, for the opposite-helicity case, we again notice that



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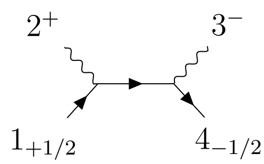

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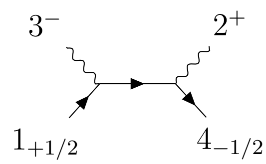
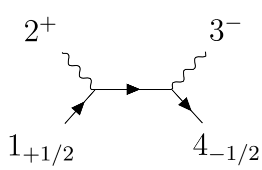


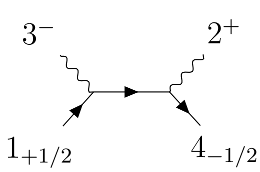
. (2.2.6)

Therefore, we may perform a similar calculation for each diagram.



$$= \frac{\langle 3\,4\rangle^2}{\langle \hat{p}\,4\rangle} \frac{1}{p^2} \frac{[2\,1]^2}{[\hat{p}\,1]} = -\frac{\langle 3\,4\rangle^2\,[1\,2]^2}{[1\,\hat{p}]\,\langle \hat{p}\,4\rangle} \frac{1}{[1\,2]\,\langle 2\,1\rangle} = \frac{\langle 3\,4\rangle^2}{\langle 1\,2\rangle\,\langle 2\,4\rangle}. \tag{2.2.7}$$

One can perform a similar calculation to find that the other contributing diagram evaluates to zero, again, by momentum conservation



$$= \frac{\langle 3\,\hat{p}\rangle^2}{\langle 1\,\hat{p}\rangle} \frac{1}{p^2} \frac{[2\,\hat{p}]^2}{[4\,\hat{p}]} = \frac{\langle 3\,\hat{p}\rangle^2\,[\hat{p}\,1]\,[2\,\hat{p}]^2\,\langle \hat{p}\,4\rangle}{p^2(\hat{p}+p_1)^2(\hat{p}+p_4)^2} \propto \langle 3|\,\hat{p}\,|1] = 0. \tag{2.2.8}$$

So $A(f_1^{+1/2}\gamma_2^+\gamma_3^- f_4^{-1/2}) = \langle 3\,4\rangle^2 \langle 1\,2\rangle^{-1} \langle 2\,4\rangle^{-1}$. One can also prescribe the three-point amplitude as its vertex counterpart in the QED's Feynman rules as carrying a factor of $-\sqrt{2}ie$ for each vertex.[7] Then, $A(f_1^{+1/2}\gamma_2^+\gamma_3^- f_4^{-1/2}) = -2e^2 \langle 3\,4\rangle^2 \langle 1\,2\rangle^{-1} \langle 2\,4\rangle^{-1}$. A similar calculation holds for fermions with the opposite helicities, *i.e.* $A(f_1^{-1/2}\gamma_2^+\gamma_3^- f_4^{+1/2}) = -2e^2 \langle 1\,3\rangle^2 \langle 1\,2\rangle^{-1}$ $\langle 2\,4\rangle^{-1}$; therefore, the probability density for Compton scattering is

$$|A|^2 = \frac{1}{2^2}\left(\left|-\frac{2e^2\langle 3\,4\rangle^2}{\langle 1\,2\rangle\langle 2\,4\rangle}\right|^2 + \left|-\frac{2e^2\langle 1\,3\rangle^2}{\langle 1\,2\rangle\langle 2\,4\rangle}\right|^2\right) = 2e^4\left(\frac{p_1\cdot p_2}{p_1\cdot p_3} + \frac{p_1\cdot p_3}{p_1\cdot p_2}\right), \tag{2.2.9}$$

which agrees with the calculation from the field theoretic approach.[8] Interestingly, Compton scattering amplitude has a "dual" version in terms of the right-handed spinors. To see this, notice that

$$\begin{aligned} A(f_1^{+1/2}\gamma_2^+\gamma_3^- f_4^{-1/2}) &= \frac{\langle 3\,4\rangle^2}{\langle 1\,2\rangle\langle 2\,4\rangle} = \frac{\langle 3\,4\rangle^2[1\,2]^2}{\langle 1\,2\rangle[1\,2]^2\langle 2\,4\rangle} = -\frac{\langle 3\,4\rangle^2[1\,2]^2}{2(p_1\cdot p_2)[1\,2]\langle 2\,4\rangle} \\ =&\frac{\langle 3\,4\rangle^2[1\,2]^2}{(p_1+p_2)^2[1\,3]\langle 3\,4\rangle} = \frac{\langle 3\,4\rangle[1\,2]^2}{(-p_3+p_4)^2[1\,3]} = \frac{\langle 3\,4\rangle[1\,2]^2}{\langle 3\,4\rangle[4\,3][1\,3]} = -\frac{[1\,2]^2}{[1\,3][3\,4]}. \end{aligned} \tag{2.2.10}$$

It is straightforward to check that each particle has correct helicity, and when the amplitude is squared, both expressions yield the same probability density.

One can notice again that our approach involved nothing but simple identities in spinor variables, and it yields a result that usually requires demanding algebraic manipulations. Another interesting feature that both calculation exhibits is the use of three-point amplitude as building blocks. Curiously, this idea is distinct from the Feynman rule approach since Feynman rules involve vertices and exchange of off-shell virtual particles. Combinations of off-shell propagators and off-shell vertices eventually yield on-shell diagrams. Our approach, albeit containing off-shell propagators, involves combination of on-shell three-point diagrams. Why on-shell diagrams can replace off-shell vertices and propagators will be a theme of our discussion in the next section.

2.2.2 Britto-Cachazo-Feng-Witten (BCFW) On-shell Recursion

In the previous section, we introduced some fanciful (but erudite) Ansatz that four-point amplitudes consist of propagators p^{-2} and on-shell three-point amplitudes. The fluidity of our calculations and arguments hint on some exciting hidden method for finding scattering amplitude. In this section, we will follow arguments presented in [29, 30] and explore this new method, **Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion**.

To obtain an on-shell diagram, we need a method to shift the momentum of an internal particle while observing its behavior until the momentum goes on-shell, *i.e.* analytically continue the momentum. Let's consider adding external momenta in spinor helicity formalism.

$$p_i + p_j = |i\rangle[i| + |j\rangle[j| = |i\rangle[i| + |j\rangle[j| + z|j\rangle[i| - z|j\rangle[i| = (|i\rangle + z|j\rangle)[i| + |j\rangle([j| - z[i|). \tag{2.2.11}$$

[7] The factor of $\sqrt{2}$ corresponds to evaluating the polarization vector in a Feynman-rule approach.
[8] See, for example, eq'n (5.87) of [23].

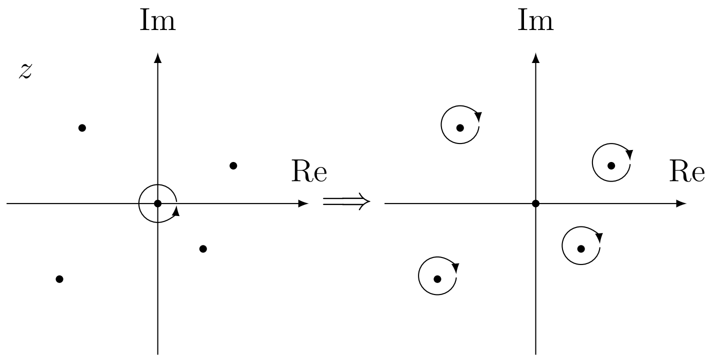


Figure 2.1: Choice of integration contour and application of residue theorem in BCFW on-shell recursion

This means that whenever we have a statement about adding external momenta, there is a symmetry $|i\rangle \to |i\rangle + z\,|j\rangle$ and $[j| \to [j| - z\,[i|$ so that the statement about momentum addition remains unchanged. However, if we consider some other particle $k \neq i, j$, then

$$\begin{gathered} p_i + p_k \to \hat{p}_i + p_k = (|i\rangle + z\,|j\rangle)\,[i| + |k\rangle\,[k| = p_i + p_k + z\,|j\rangle\,[i| \\ (p_i + p_k)^2 \to (p_i + p_k)^2 + z\,\langle k\,j\rangle\,[i\,k]\,, \end{gathered} \tag{2.2.12}$$

in which the $\hat{p}$ denotes a **shifted momentum**. Note that a few spinor products remain unchanged under this shift such as

$$\langle j|\,\hat{p}_i + p_k\,|l] = \langle j|\,(p_i + p_k + z\,|j\rangle\,[i|)\,|l] = \langle j|\,p_i + p_k\,|l]\,, \tag{2.2.13}$$

$$\langle l|\,\hat{p}_i + p_k\,|i] = \langle l|\,(p_i + p_k + z\,|j\rangle\,[i|)\,|i] = \langle l|\,k\,|i]\,, \tag{2.2.14}$$

in which l denotes an arbitrary spinor. When $z = r_{ik} \triangleq -\langle k\,i\rangle / \langle k\,j\rangle$, we find that $(p_i + p_k)^2 + z\,\langle k\,j\rangle\,[i\,k] = 0$. To find the original amplitude, we would like to find $A(z = 0)$ when introducing this new parameter z. According to Cauchy's theorem, we can integrating over a small complex contour around $z = 0$ shown in fig. 2.1 and find that

$$A(z=0) = \frac{1}{2\pi i}\oint_0 \mathrm{d}z\,\frac{A(z)}{z} = \frac{1}{2\pi i}\oint_\infty \mathrm{d}z\,\frac{A(z)}{z} - \sum_{\text{residue } i} \mathrm{Res}\left[\frac{A(z)}{z}, r_i\right], \tag{2.2.15}$$

in which the integral of $A(z)/z$ at $r \to \infty$ is also called the **boundary term**. The residue turns out to be relatively straightforward to evaluate. Since the only term in a tree-level amplitude contributing to a pole is the $(\sum p)^{-2}$ propagators, we learn from eq'n (2.2.12) that the amplitude can only have simple poles at $z = r_{ik}$. The scattering amplitude near the pole should factorize into two separate on-shell amplitudes with an on-shell propagator,[9] *i.e.*

$$\lim_{z\to r_{ik}} A(z) = \frac{A_L(r_{ik})A_R(r_{ik})}{(p_i + p_k)^2 + z\,\langle k\,j\rangle\,[i\,k]} = \frac{A_L(r_{ik})A_R(r_{ik})}{(z - r_{ij})\,\langle k\,j\rangle\,[i\,k]} = -\frac{1}{(z - r_{ij})}\,\frac{A_L(r_{ik})A_R(r_{ik})}{r_{ij}^{-1}(p_i + p_k)^2} \tag{2.2.16}$$

[9]The physical picture of this factorization is clear. If a virtual particle goes on-shell, it becomes a real particle propagating indefinitely. Thus, the processes on the left and right of the propagator must individually go on-shell.

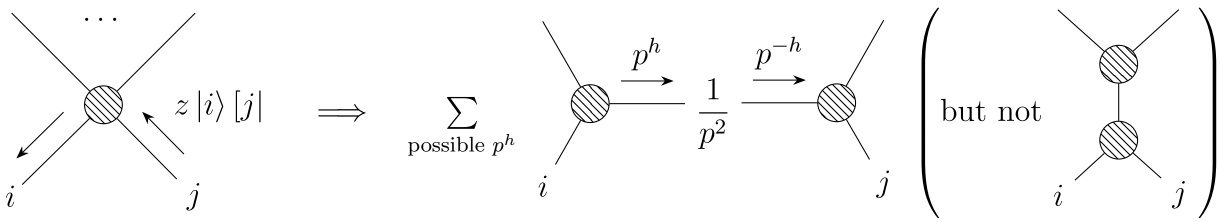


Figure 2.2: Illustration for the interpretation of the BCFW on-shell recursion

Then, under the assumption that the integral of $A(z)/z$ at infinity evaluates to zero, the residue of interest becomes

$$A = \sum_{\text{residue } r} -\operatorname{Res}\left[\frac{A(z)}{z}, r\right] = \sum_{r_{ik},\,\mu} \frac{A_{L\mu}(r_{ik})A_{R\mu}(r_{ik})}{(p_i+p_k)^2}, \tag{2.2.17}$$

in which we denote all possible on-shell propagator states as μ. A diagrammatic interpretation of this conclusion is shown in fig. 2.2. This justifies our Ansätze proposed in the previous section. It may be surprising that the deformed amplitude has no **boundary term**. After all, if A_L and A_R are linear functions of z, then the $A(z)/z$ is $\mathcal{O}(z)$ for a large z. But it has been shown that for Yang-Mills theory and perturbative gravity that there is always an appropriate choice of deformation parameter z such that $A(z) \to 0$, thus yielding a zero integral at infinity [31]. The general rule of thumb for deformation as discussed in [31] is to avoid shifting $|i^-\rangle \to |i^-\rangle + z\,|j^+\rangle$ and $[j^+| \to [j^+| - z\,[i^-|$ between particle i^- with a negative helicity and particle j^+ with a positive helicity, called a $[+,-\rangle$ shift. Other shifts, such as $[+,+\rangle$, $[-,-\rangle$, $[-,+\rangle$, are guaranteed to have no boundary contributions from infinity. There are other theories in which BCFW on-shell recursion should invoke its boundary contribution [32, 33].

How BCFW on-shell recursion was performed in the previous section?

In the previous section, it seemed that we did not invoke BCFW on-shell recursion but simply glued up propagators and three-point amplitudes with mysterious magics. After all, we performed no analysis about the residue r_{ik} whatsoever. This is because we chose the shift appropriately so that all shifted momenta looks the same as the propagator. Specifically, this is because we invoked eq'ns (2.2.13) and (2.2.14) appropriately. In the calculation for both the four-point Parke-Taylor amplitude and the Compton scattering amplitude, we decorated the shifted momenta as $\hat{p}$ while denoting the unshifted $(p_i+p_k)^{-2}$ propagator as p.

For the Parke-Taylor amplitude, we performed a $[4,1\rangle$ shift, and the only pole[10] is when g_1g_2 are on the same side while g_3g_4 on the other. Then, $r_{ik} = r_{12} - \langle 1\,2\rangle\,/\,\langle 4\,2\rangle$. But we noticed that in the denominator, we have

$$\langle 3(l)|\,\hat{p}\,|1(i)] = \langle 3|\,2\,|1]\,, \quad \langle 4(j)|\,\hat{p}\,|2(k)] = \langle 4|\,1\,|2] \tag{2.2.18}$$

according to eq'ns (2.2.13) and (2.2.14). So there is no need to perform a shift to r_{12} in the first place, but to glue up the expression as if $\hat{p}$ were the same as p. It is important, however, to stress that $\hat{p} \neq p$ in general.

[10]Again, this is because the Parke-Taylor amplitude is a **color-ordered amplitude**.

For the Compton scattering amplitude, we pulled out the same trick by performing a $[4,1\rangle$ shift. This case, we have two poles: one pole put $f_1\gamma_2$ on-shell, and the other put $f_1\gamma_3$ on-shell. Just like what we discussed, $r_{12} = -\langle 1\,2\rangle\,/\,\langle 4\,2\rangle$ and $r_{13} = -\langle 1\,3\rangle\,/\,\langle 4\,3\rangle$. But the shifted amplitude at $z = r_{12}$ contains only expressions of the form $\langle 4|\,\hat{p}\,|1] = \langle 4|\,2\,|1]$, and the shifted amplitude at $z = r_{13}$ contains only $\langle 3|\,\hat{p}\,|1] = 0$. So it, again, appears as if $\hat{p}$ were the same as p. But as shown in eq'ns (2.2.13) and (2.2.14), the z dependence of the shifted momenta in these spinor products drop out due to on-shell conditions of p_i or p_j. In a complete calculation, the safest option is to identify r_{ik} first and simplify each spinor product accordingly instead of guessing the result of the spinor product.

What is the advantage of BCFW?

In general, this BCFW recursion method is used to construct higher-point tree-level scattering amplitudes from lower-point ones. Unlike the Feynman-rule approach in which higher-point amplitudes are broken into more complicated off-shell vertices with delicate gauge choices, BCFW recursion leverages all the on-shell lower-point amplitudes and turns the evaluation of higher-point amplitude into a combinatorial game, which frequently yields much fewer terms to be simplified.

2.3 Implication of Locality

In the previous section, we discussed the on-shell factorization of tree-level amplitudes into a sum of pairs of left and right subamplitudes. This idea provides us an efficient method to evaluate tree-level amplitudes without using Feynman rules and fixing gauge. But this factorization can be, in turn, used to examine the consistency of a field theory, *i.e.* tree-level interactions manifest through local propagators. We will demonstrate this by considering two related arguments: the emergence of structure constants of the underlying Lie algebra describing symmetries of particles and the impossibility of field-theoretic interactions of high-spin massless particles.

2.3.1 Jacobi Identity Emerges Naturally: Yang-Mills is the Only Spin-1 Theory

First, we will demonstrate that the structure constants of a particle's Lie algebra emerge naturally. Consider a theory involving several massless particles of identical spin σ.[11] We will also relax the condition that the coupling must be massless so that any three-point interaction, according to eq'n (2.1.9), can be described as

$$A(1^-2^-3^+) = \left(\frac{\langle 1\,2\rangle^3}{\langle 1\,3\rangle\,\langle 3\,2\rangle}\right)^{\sigma}, \quad A(1^+2^+3^-) = \left(\frac{[1\,2]^3}{[1\,3]\,[3\,2]}\right)^{\sigma}. \tag{2.3.1}$$

[11] Clarification of nomenclature: We choose to denote the spin as σ rather than s as we did in the previous section mainly because we will refer to the Mandelstam variables of four-point kinematics in our following analysis. It may be confusing to talk about both s channel and spin s.

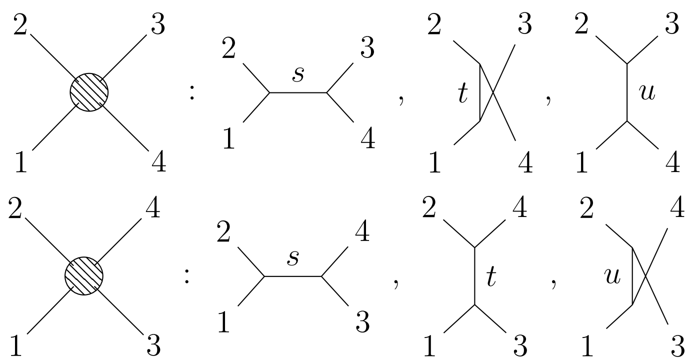


Figure 2.3: Two conventions for labeling Mandelstam variables: the top one is the convention adopted by this paper, and the bottom one is the traditional convention for 2-to-2 scattering. Note that all interactions are in tree level, so there is no interaction when lines of external particles cross each other.

Now, let's consider a four-point process involving some external spin-σ particles of distinct flavors or colors labeled as a, b, c, d, e. Starting from a three-point amplitude, we may give each particle a different weight of coupling by introducing a constant f^{abc} dependent on the particle flavor a, b, c so that the previous equation becomes

$$A(1_a^- 2_b^- 3_c^+) = f^{abc}\left(\frac{\langle 1\,2\rangle^3}{\langle 1\,3\rangle\,\langle 3\,2\rangle}\right)^\sigma, \quad A(1_a^+ 2_b^+ 3_c^-) = f^{cab}\left(\frac{[1\,2]^3}{[1\,3]\,[3\,2]}\right)^\sigma. \tag{2.3.2}$$

In the four-point kinematics, we can obtain a simple set of four-momentum invariants named as **Mandelstam variables**. These invariants are defined as

$$s \triangleq (p_1+p_2)^2, \quad t \triangleq (p_1+p_3)^2, \quad u \triangleq (p_1+p_4)^2. \tag{2.3.3}$$

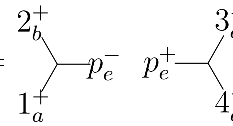
Some important consequences of the outgoing massless four-point kinematics on Mandelstam variables are $s+t+u=0$ and $s = \langle 1\,2\rangle\,[2\,1] = \langle 3\,4\rangle\,[4\,3]$ (and similar relation holds for t and u). One may refer to appendix B for a more detailed discussion. The process which has a propagator of the form $1/s$ is called an s-**channel process** and similarly for $1/t$ and $1/u$ propagators.

Nota bene: traditionally, Mandelstam variables are used in 2-to-2 scattering in which $p_1+p_2-p_3-p_4=0$ with external particles labeled as $1, 2, 4, 3$ clockwise on a diagram as shown in fig. 2.3. In this way, particle 1 has neighbors 2 and 3 so that s- and <u>t-channel</u> have either a horizontal or a vertical propagator. However, it is our convention to use $1, 2, 3, 4$ clockwise labeling with $p_1+p_2+p_3+p_4=0$. In this case, particle 1 has neighbors 2 and 4 so that s- and <u>u-channel</u> have either a horizontal or a vertical propagator. We match up Mandelstam variables in its algebraic form, but some authors studying scattering amplitudes also use the convention $t \triangleq (p_1+p_4)^2$, $u \triangleq (p_1+p_3)^2$ to match up the diagrammatic form. Either convention is perfectly fine as long as we stick to one of them throughout.

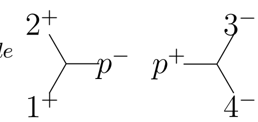
Returning to our argument about factorization in different channels, we now know that any four-point kinematic can be described by these Mandelstam variables. As we discussed in BCFW on-shell recursion, when we observe the residue in an s-channel process (so that

$s \to 0$ goes on-shell), the four-point amplitude factorizes into two three-point amplitudes. The residue in s channel should be a product of two on-shell three-point amplitudes. More importantly, since the internal propagator enforces locality, *i.e.* interaction of particles over spacetime is always "at a point", we know that this residue is solely contributed by a simple pole in s. This provides a severe constraint on the form of the tree-level amplitude. With this general picture in mind, we can express the residue in s channel as

$$\mathrm{Res}\big[A(1_a^+ 2_b^+ 3_c^- 4_d^-), s\big] =$$



$$= f^{abe} f^{cde}$$



$$\begin{aligned}
&= f^{abe} f^{cde} \left(\frac{[1\,2]^3 \langle 3\,4\rangle^3}{[1\,p]\,[p\,2]\,\langle 3\,p\rangle\,\langle p\,4\rangle} \right)^\sigma \\
&= f^{abe} f^{cde} \left(\frac{[1\,2]^3 \langle 3\,4\rangle^3}{[1\,2]\,[3\,2]\,\langle 3\,2\rangle\,\langle 3\,4\rangle} \right)^\sigma \\
&= -\, f^{abe} f^{cde} \left(\frac{[1\,2]^2 \langle 3\,4\rangle^2}{u} \right)^\sigma ,
\end{aligned} \tag{2.3.4}$$

in which a summation over e is understood. One can play a similar game in other channels. In general, we find that

$$\mathrm{Res}\big[A(1_a^+ 2_b^+ 3_c^- 4_d^-), s\big] = -f^{abe} f^{ecd} \left(\frac{[1\,2]^2 \langle 3\,4\rangle^2}{u} \right)^\sigma , \tag{2.3.5}$$

$$\mathrm{Res}\big[A(1_a^+ 2_b^+ 3_c^- 4_d^-), t\big] = -f^{ace} f^{edb} \left(\frac{[1\,2]^2 \langle 3\,4\rangle^2}{s} \right)^\sigma , \tag{2.3.6}$$

$$\mathrm{Res}\big[A(1_a^+ 2_b^+ 3_c^- 4_d^-), u\big] = -f^{dae} f^{ebc} \left(\frac{[1\,2]^2 \langle 3\,4\rangle^2}{t} \right)^\sigma . \tag{2.3.7}$$

Some more detailed discussions can be found in [34].

Let's consider the case when $\sigma = 1$. Since tree-level amplitude is a rational function in Lorentz invariants with simple poles in each variable, we can observe the previous equations about residues in all channels and propose that[12]

$$A(1_a^+ 2_b^+ 3_c^- 4_d^-) = -\,[1\,2]^2 \langle 3\,4\rangle^2 \left(\frac{A^{abcd}}{st} + \frac{B^{abcd}}{su} + \frac{C^{abcd}}{tu} \right), \tag{2.3.8}$$

in which A, B, C are constants with respect to a given set of external particle states $\{|p_1, +, a\rangle\,,\ |p_2, +, b\rangle\,, |p_3, -, c\rangle\,, |p_4, -, d\rangle\}$. When $s \to 0$, $s + t + u = 0 \implies t \to -u$ and similarly for $t, u \to 0$. Therefore, our Ansatz demands that

$$(B - A)^{abcd} = f^{abe} f^{ecd}, \quad (A - C)^{abcd} = f^{ace} f^{edb}, \quad (C - B)^{abcd} = f^{dae} f^{ebc}. \tag{2.3.9}$$

[12] More rigorously, one should include non-residue part of a rational function, *i.e.* Polynomial(s, t, u). But in high-energy limit, *e.g.* $s \gg t, u$, we want the amplitude to be finite; therefore, the polynomial term drops out.

Adding three equations up, we find that the left-hand side is zero, and the right-hand side is some constraints on fs which reads

$$0^{abcd} = 0 \stackrel{!}{=} f^{abe} f^{ecd} + f^{ace} f^{edb} + f^{dae} f^{ebc}. \tag{2.3.10}$$

This is precisely the **Jacobi identity** of the structure constants of the underlying Lie algebra. Now, noticing that for eq'n (2.3.2) to describe a valid amplitude for $\sigma = 1$, we must demand that f^{abc} is antisymmetric in its flavor index since the spinor product is antisymmetric in particle labels. The antisymmetry of flavor index and Jacobi identity provides sufficient restrictions about the flavor symmetry. The restriction is that a self-interacting massless spin-1 particle must have a flavor symmetry described by a (compact[13]) Lie algebra. This means that *the* field theory for self-interacting spin-1 particles is **Yang-Mills theory** [35], a theory describing interactions of spin-1 gauge bosons under the symmetry of various compact Lie algebras [36]. Note that the kinematic variables describing the scattering amplitudes stringently constrain the relation of the flavor symmetry. This correspondence between kinematics and flavor is called **color-kinematics duality**.[14] More remarkably, if there is no distinct flavor, *i.e.* if we drop all a, b, c, d, e labels and set all $f^{abc} = f$, then similar to eq'n (2.3.10), we will conclude that $0 = 3f^2$. This means that there is no field theory of self-interacting massless spin-1 particles without distinct flavors. An immediate extension of this conclusion is that in QED (in which photon has a unique flavor), there is no photon self-interaction at tree level.

2.3.2 Spin->2 Particles Do Not Interact Locally

To show that spin->2 massless particles cannot have local interaction, we will first investigate the critical case when particles in our theory have exactly spin $\sigma = 2$. According to eq'ns (2.3.5) to (2.3.7), we can find that

$$\mathrm{Res}[A_{++--,\sigma=2}, s] \propto \frac{1}{u^2}, \quad \mathrm{Res}[A_{++--,\sigma=2}, t] \propto \frac{1}{s^2}, \quad \mathrm{Res}[A_{++--,\sigma=2}, u] \propto \frac{1}{t^2}. \tag{2.3.11}$$

To ensure that we have simple poles in each variable while still obtaining the desired u^{-2}, s^{-2}, t^{-2} dependence in each channel, we realize that the only possible rational function is

$$A(1^+, 2^+, 3^-, 4^-)_{\sigma=2} = [1\,2]^4 \langle 3\,4\rangle^4 \frac{C^{abcd}}{stu}. \tag{2.3.12}$$

Then, when $s \to 0$, the residue is proportional to $1/(tu) = -u^{-2}$. We also conclude that $f^{abe} f^{ecd} = f^{ace} f^{edb} = f^{dae} f^{ebc}$. A trivial way to satisfy this condition is to turn off the flavor distinctions so that $C = f^2$. In this case, the scattering amplitude will have the form

$$A(1^+, 2^+, 3^-, 4^-)_{\sigma=2} = C \frac{[1\,2]^4 \langle 3\,4\rangle^4}{stu}, \tag{2.3.13}$$

[13] See for example Chapter 4 of [26].

[14] Sometimes, the striking similarity between graviton scattering amplitude shown in eq'n (2.3.13) and gluon scattering amplitude shown in eq'n (2.3.8) is also included in the color-kinematic duality under the slogan "(gravity) = (gauge)^2". It is another fascinating topic beyond the scope of this document.

in which the mass dimension of $[C] = -2$ and is the same as that of the gravitational constant G. This is in fact the four-point scattering amplitude for gravitons (perturbative metric over flat spacetime) which can be obtained through bruteforce Feynman rule approach as shown in [37]. Therefore, there can be a field theory describing self-interacting spin-2 particles according to the locality condition.

However, for $\sigma > 2$ case, one can straightforwardly observe the stringency of the locality condition. For example, let $\sigma = 3$. Then, the residue in s channel is proportional to u^{-3}, and there will be no rational function with simple poles in s, t, and u to satisfy this residue condition. Therefore, there is no local self-interaction for particles with $\sigma > 2$ as promised.

2.3.3 Concluding Remarks

In this section, we have seen the power of turning the on-shell factorization into a locality condition and using spinor helicity formalism to simplify three-point amplitudes. Investigating the residue and pole structure of the amplitude allows us to eliminate non-local propagators and keep those propagators that have simple poles in Mandelstam variables. The impossibility to write down an amplitude with local propagators shows that a process cannot be governed by field-theoretic interactions. We focused on studying the manifestation of locality in self-interacting massless bosons and find that

1. There is no theory for a self-interacting spin-1 particle without flavor/color symmetry. Hence, no photon self-interaction is permitted at tree level in QED.

2. The only theory for self-interacting spin-1 particles with flavors/colors is Yang-Mills theory due to color-kinematic duality.

3. There is no theory for self-interacting spin->2 particles.

Chapter 3

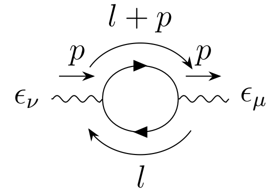
One-Loop Amplitudes

In the previous section, we investigated some interesting tree-level amplitudes with spinor helicity formalism. We noticed that this formalism exposes the helicity information and factorization properties of an amplitude, which leads to on-shell recursion methods to generate higher-point amplitudes from lower-point ones. The three-point kinematic is degenerate enough to provide unique expressions for three-point amplitudes of different-spin particles. The four-point amplitudes, then, can be generated through the BCFW on-shell recursion. Checking factorization properties of these four-point amplitudes then ensures locality is still preserved, and locality condition can further restrict the symmetry of the particle's flavor.

Now, we move onto **loop-level amplitudes** with a specialization in one-loop scattering amplitudes. Tree-level amplitudes are defined as amplitudes in which each pair of external particles is connected via a unique path of propagators. A loop-level amplitude, on the other hand, has more than one path of propagators from some external particle to some other external particle. All diagrams shown in eq'n (2.0.2) in the previous chapter are valid diagrams representing one-loop amplitudes. Conceptually, loop amplitudes frequently occur with a higher power of coupling constant (or perturbation parameter) and can be regarded as a correction to its tree-level counterpart, so **loop-level amplitudes** are also called **radiative corrections**. Therefore, loop-level amplitudes are a starting point for high-precision calculations and predictions of quantum field theories and play a central role in recent developments in high-energy physics. In this chapter, we will start with some textbook calculations about one-loop amplitudes with Feynman rules. Gradually, we will return to our investigation with spinor helicity and study how this formalism interplays with our known approach.

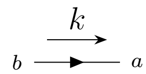
3.1 Vacuum Polarization

In our first calculation, we will consider the following diagram



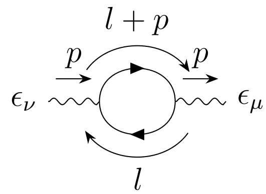
. (3.1.1)

This diagram is called **vacuum polarization** or **photon self-energy**. Physically, this describes the effect that when a photon mediates the electric field at a high enough energy, it can create an e^-e^+ pair from a vacuum that looks like a dipole. Under the influence of a current, this dipole will respond just like a polarized material would to a charge source, whence the name. Recall that a fermion propagator[1] takes the form



$$= \frac{i(\not{k}+m)_{ab}}{k^2-m^2+i\varepsilon}, \tag{3.1.2}$$

in which $\not{k} \triangleq k^\mu\gamma_\mu$ denotes gamma slash product, m mass of the fermion, and $\{a,b,c,d\}$ are indices of the gamma matrices. Here the notation of $(\not{k}+m)_{ab}$ is a shorthand for $k_\mu\gamma^\mu_{ab}+m\mathbb{1}_{ab}$. Therefore, the amplitude for this diagram is



$$= -\epsilon^*_\mu\epsilon_\nu \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,(-ie)\gamma^\mu_{ab}\frac{i(\not{p}+\not{l}+m)_{bc}}{(p+l)^2-m^2}(-ie)\gamma^\nu_{cd}\frac{i(\not{l}+m)_{da}}{l^2-m^2}, \tag{3.1.3}$$

in which the integral runs over 4-dimensional loop momentum space, m denotes the mass of the electron, ϵ polarization vector. One can recognize that the numerator takes the form of a trace over gamma matrices $A_{ab}B_{bc}C_{cd}D_{da} = \mathrm{Tr}(ABCD)$. Therefore, we can re-express the amplitude as $\epsilon^*_\mu\epsilon_\nu M^{\mu\nu}$ in which

$$M^{\mu\nu} \triangleq -\frac{e^2}{(2\pi)^4}\int \mathrm{d}^4 l\,\frac{\mathrm{Tr}[\gamma^\mu(\not{p}+\not{l}+m)\gamma^\nu(\not{l}+m)]}{(l^2-m^2)\left[(p+l)^2-m^2\right]}. \tag{3.1.4}$$

The integral over l can be tricky, but we can simplify it a little if we can make the integrand a spherically symmetric[2] function in l so that $\mathrm{d}^4 l = 4\pi l^2 \mathrm{d}l\mathrm{d}l_0$. To render the numerator of the integrand into a spherically symmetric function just requires us to find its even part, whereas rendering the denominator into a spherically symmetric function is slightly tricky.

3.1.1 Feynman Shift Parameterization

To put the denominator into a spherically symmetric form, we would like it to look like a polynomial in l^2 (or some other variables). Observe that the denominator takes the form $1/(AB)$ which can be massaged into

$$\frac{1}{AB} = \frac{1}{B-A}\left(\frac{1}{A}-\frac{1}{B}\right) = \frac{1}{B-A}\int_A^B \frac{\mathrm{d}z}{z^2} \overset{z\triangleq(B-A)x+A}{=\!=\!=\!=} \int_0^1 \frac{\mathrm{d}x}{[xB+(1-x)A]^2}. \tag{3.1.5}$$

More generally,

$$\frac{1}{A_1A_2\ldots A_n} = (n-1)!\int_0^1 \mathrm{d}x_1\ldots\int_0^1 \mathrm{d}x_n\left[\sum_{i=1}^n x_iA_i\right]^{-n}\delta\!\left(\sum_{i=1}^n x_i - 1\right). \tag{3.1.6}$$

[1] One can check this by taking the Dirac equation $(i\not{\partial}-m)\psi = 0$ into the momentum space.

[2] Technically, we mean "hyperbolically symmetric" since l^2 is a hyperbolic length. But we will see that this corresponds to a spherically symmetric expression after Wick rotation as shown in section 3.1.3.

This trick is known as **Feynman shift**. By introducing additional parameters x_i, we trade a degree of freedom for manipulating the denominator into any form we please. Here, we will rewrite it into a spherically symmetric form

$$\begin{aligned}
&\left[x\left[(p+l)^2-m^2\right]+(1-x)\left(l^2-m^2\right)\right]^{-2} \\
=&\left[l^2-m^2+x\left(p^2+2p\cdot l\right)\right]^{-2} \\
=&\left[l^2+2xp\cdot l+x^2p^2-x^2p^2+xp^2-m^2\right]^{-2} \\
=&\left[(l+xp)^2+x(1-x)p^2-m^2\right]^{-2},
\end{aligned} \tag{3.1.7}$$

Therefore, we can define a new integration variable

$$k \triangleq (l+xp), \quad \Delta \triangleq m^2-x(1-x)p^2 \tag{3.1.8}$$

so that

$$M^{\mu\nu}=-\frac{e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \int \mathrm{d}^4k\, \frac{\mathrm{Tr}[\gamma^\mu(\not{k}+(1-x)\not{p}+m)\gamma^\nu(\not{k}-x\not{p}+m)]}{\left(k^2-\Delta\right)^2}, \tag{3.1.9}$$

and now the integral is in a more symmetric form in k. Our next step is to simplify the numerator algebra and potentially eliminate some terms that is odd in k since any even integral over an odd function is zero.

3.1.2 Numerator Algebra

Before we dive into simplifying the numerator of the integral of interest, we may start with understanding traces of gamma matrices. To spice things up a little in a "helicity flavor", we can utilize the relation between gamma matrices and Pauli matrices in terms of spinor helicity formalism to make some progress. Recall from eq'n (1.3.23) that gamma matrices can be expressed as[3]

$$\gamma^\mu=\begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \triangleq \begin{pmatrix} & |\mu\rangle\,[\mu| \\ |\mu]\,\langle\mu| & \end{pmatrix} = |\mu\rangle\,[\mu| \oplus |\mu]\,\langle\mu|\,, \tag{3.1.10}$$

with a new operation $\oplus$ denoting

$$A\oplus B \triangleq \begin{pmatrix} & A \\ B & \end{pmatrix}. \tag{3.1.11}$$

Then, the matrix multiplication between $\oplus$ matrices takes the form

$$(A\oplus B)(C\oplus D)=\begin{pmatrix} AD & \\ & BC \end{pmatrix}, \quad (A\oplus B)(C\oplus D)(E\oplus F)=ADE\oplus BCF. \tag{3.1.12}$$

[3]Here, we are using $|\mu\rangle\,[\mu|$ to denote the four index of the gamma matrices. It seems illegal, but one can imagine that instead of denoting the index, we are taking the dot product $\not{\mu}=\gamma^\nu\mu_\nu$ on a four-vector named μ. In this notation, $\langle\mu\,\nu\rangle\,[\nu\,\mu]=2\mu\cdot\nu=2g^{\mu\nu}\mu_\mu\nu_\nu\equiv 2g^{\mu\nu}$.

Therefore,

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}\left[\begin{pmatrix} |\mu\rangle\,[\mu|\,|\nu]\,\langle\nu| & \\ & |\mu]\,\langle\mu|\,|\nu\rangle\,[\nu| \end{pmatrix}\right] \\ &= [\mu\,\nu]\,\langle\nu\,\mu\rangle + \langle\mu\,\nu\rangle\,[\nu\,\mu] = 2\,\langle\mu\,\nu\rangle\,[\nu\,\mu] = 4g^{\mu\nu},\end{aligned} \tag{3.1.13}$$

and

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0. \tag{3.1.14}$$

In fact, it is straightforward to see that Tr[odd number of γ] = 0. Now, we can perform a similar manipulation on the trace for four gamma matrices

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= \langle\sigma\,\mu\rangle\,[\mu\,\nu]\,\langle\nu\,\rho\rangle\,[\rho\,\sigma] + [\sigma\,\mu]\,\langle\mu\,\nu\rangle\,[\nu\,\rho]\,\langle\rho\,\sigma\rangle \\ =&(\langle\sigma\,\nu\rangle\,\langle\mu\,\rho\rangle + \langle\sigma\,\rho\rangle\,\langle\nu\,\mu\rangle)\,[\mu\,\nu]\,[\rho\,\sigma] + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&\,\langle\sigma\,\nu\rangle\,\langle\mu\,\rho\rangle\,[\mu\,\nu]\,[\rho\,\sigma] + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&\,\langle\sigma\,\nu\rangle\,\langle\mu\,\rho\rangle\,([\mu\,\sigma]\,[\rho\,\nu] + [\mu\,\rho]\,[\nu\,\sigma]) + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&\,\langle\sigma\,\nu\rangle\,\langle\mu\,\rho\rangle\,[\mu\,\sigma]\,[\rho\,\nu] - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&(\langle\sigma\,\rho\rangle\,\langle\mu\,\nu\rangle + \langle\sigma\,\mu\rangle\,\langle\nu\,\rho\rangle)\,[\mu\,\sigma]\,[\rho\,\nu] - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&\,\langle\sigma\,\rho\rangle\,\langle\mu\,\nu\rangle\,[\mu\,\sigma]\,[\rho\,\nu] + 4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =& - [\sigma\,\mu]\,\langle\mu\,\nu\rangle\,[\nu\,\rho]\,\langle\rho\,\sigma\rangle + 4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma} + [\sigma|\,\mu\nu\rho\,|\sigma\rangle \\ =&4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma}.\end{aligned} \tag{3.1.15}$$

These trace identities can also be derived with anticommutators of gamma matrices shown in appendix B.4.

For the numerator of the integral of interest, we can leave out terms containing odd number of gamma matrices by

$$\mathrm{Tr}[\gamma^\mu(\not{k} + (1-x)\not{p} + m)\gamma^\nu(\not{k} - x\not{p} + m)] = \mathrm{Tr}[\gamma^\mu(\not{k} + (1-x)\not{p})\gamma^\nu(\not{k} - x\not{p})] + m^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu]. \tag{3.1.16}$$

Now, we may factor out the four-vectors from the slash notation and invoke the trace identities of gamma matrices. In the end, we find that the numerator reads

$$\begin{aligned}\mathrm{Tr}[\ldots] = 4\,[&(k^\mu + (1-x)p^\mu)(k^\nu - xp^\nu) + (k^\nu + (1-x)p^\nu)(k^\mu - xp^\mu) \\ &+ \left[m^2 - (k - xp)\cdot(k + (1-x)p)\right]g^{\mu\nu}\right].\end{aligned} \tag{3.1.17}$$

This is still a long tensorial express to be evaluated in the integral. However, since our integral runs over the entire $\mathbb{R}^4$ phase space, we know that only "even" terms will contribute. For instance, if $\mu \neq \nu$, then $k^\mu k^\nu$ is an odd function in either k^μ or k^ν. After integrating over $\mathrm{d}^4k \supset \mathrm{d}k^\mu \mathrm{d}k^\nu$, this part must be zero. Since $M^{\mu\nu}$ is a Lorentz tensor that is nonvanishing if $\mu = \nu$, we know that under the integral $k^\mu k^\nu \equiv Ck^2 g^{\mu\nu}$ in which C is a dimensionless constant. Contracting $g_{\mu\nu}$ to $k^\mu k^\nu$, we can infer that $k^\mu k^\nu \equiv k^2 g^{\mu\nu}/4$ since $g^{\mu\nu}g_{\mu\nu} = 4$ (or D for any D-dimensional metric tensor). Note that the equivalence sign is used instead of an equal sign. This prescription is only valid if the region of integration is symmetric. Hence, the numerator, after removing odd part and asymmetric part, looks like

$$\begin{aligned}\mathrm{Tr}[\ldots] \equiv& 4\left[2k^\mu k^\nu - 2x(1-x)p^\mu p^\nu + \left[m^2 - \left(k^2 - x(1-x)p^2\right)\right]g^{\mu\nu}\right] \\ \equiv& 4\left[\left(-\frac{k^2}{2} + \Delta + 2x(1-x)p^2\right)g^{\mu\nu} - 2x(1-x)p^\mu p^\nu\right].\end{aligned} \tag{3.1.18}$$

It seems that the integrand is in its best shape, but the main reason for us to use a spherically symmetric function in the integrand is so that our integral is just over a radial direction $k = \sqrt{k^2}$. However, over a hyperbolic space with $(+ - --)$ metric, it may be difficult to prescribe a radial direction. Therefore, we introduce a complexified momentum by $\tilde{k}^i = k^i$ but $\tilde{k}^0 = ik^0$ so that $\tilde{k}^\mu \tilde{k}^\nu \delta_{\mu\nu} = -k^2$. This technique is called **Wick rotation** which rotates the temporal region of integration from the real axis to the imaginary axis. Then, $\mathrm{d}^4 k = i\mathrm{d}^4\tilde{k}$. So our integral becomes

$$M^{\mu\nu} = -\frac{ie^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \int \mathrm{d}\Omega_4 \int_0^\infty \tilde{k}^3 \mathrm{d}\tilde{k}\, \frac{4\left[\left(\Delta + 2x(1-x)p^2 + \tilde{k}^2/2\right)g^{\mu\nu} - 2x(1-x)p^\mu p^\nu\right]}{\left(\tilde{k}^2 + \Delta\right)^2}, \tag{3.1.19}$$

in which $\mathrm{d}\Omega_4$ denotes the solid angle in four dimensions, *i.e.* the unit 3-sphere lying on the boundary of a 4-ball.

At this point, we are almost ready to perform the $\tilde{k}$ integration. However, one immediately notices that it may be illegal to carry out this integral. Observe that the integrand at large $\tilde{k}$ limit behaves like $\mathcal{O}\left(\tilde{k}\right)$, implying that the integral for large $\tilde{k}$ diverges like $\mathcal{O}\left(\tilde{k}^2\right)$. It seems that we are at an impasse; however, we may isolate the divergence systematically and extract the relevant finite part.

3.1.3 Dimensional Regularization

To isolate the divergence, we will perform the integral over a $D = (4-2\epsilon)$-dimensional phase space[4] instead of in 4D. In this approach, we render the original quadratic divergence in the integral to a logarithmic divergence so that it is possible to analytically continue the dimension back to 4-dimensional phase space while subtracting off unwanted divergences in terms of ϵ. In this approach, $\mathrm{d}^4\tilde{k} \to \mathrm{d}^D\tilde{k} = \mathrm{d}\Omega_D \tilde{k}^{3-2\epsilon}\mathrm{d}\tilde{k}$. If so, $M^{\mu\nu}$ will no longer have the correct dimension; thus, we also introduce an artificial mass parameter $\mu^{2\epsilon}$ to match up to dimension. Hence, $M^{\mu\nu}$ in $(4-2\epsilon)$-dimensional regularization is

$$\begin{aligned} M^{\mu\nu} = &-\frac{i(\mu^\epsilon e)^2}{(2\pi)^D}\int_0^1 \mathrm{d}x \int \mathrm{d}\Omega_D \int_0^\infty \tilde{k}^{3-2\epsilon}\mathrm{d}\tilde{k} \\ &\times \frac{4\left[\left(\Delta + 2x(1-x)p^2 + (1-2/D)\tilde{k}^2\right)g^{\mu\nu} - 2x(1-x)p^\mu p^\nu\right]}{\left(\tilde{k}^2+\Delta\right)^2}, \end{aligned} \tag{3.1.20}$$

in which we rewrite $k^\mu k^\nu \equiv k^2 g^{\mu\nu}/D$. To understand the solid angle (in a spherically symmetric integral) in D dimension, we may use a D-dimensional Gauss integral and find

[4] Some author also use $(4-\epsilon)$ dimension, but $(4-2\epsilon)$, in general, yields simpler algebraic results since it captures the correct dimension of transverse space away from the light cone.

that

$$\pi^{D/2} = \left(\int \mathrm{d}x\ e^{-x^2}\right)^D = \int \mathrm{d}\Omega_D \int_0^\infty r^{D-1}\mathrm{d}r\ e^{-r^2} = \frac{1}{2}\Gamma\left(\frac{D}{2}\right)\int \mathrm{d}\Omega_D,$$
$$\implies \int \mathrm{d}\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{3.1.21}$$

Now, we notice that the $\tilde{k}$ integral only has two type of integrands, one has $\tilde{k}^2/\left(\tilde{k}^2+\Delta\right)^2$ and the other has $1/\left(\tilde{k}^2+\Delta\right)^2$. And these can be done with a general integral

$$\int_0^\infty \mathrm{d}\tilde{k}\ \frac{\tilde{k}^p}{\left(\tilde{k}^2+\Delta\right)^q} = \frac{\Gamma[(p+1)/2]\Gamma[(2q-p-1)/2]}{2\Delta^{(2q-p-1)/2}\Gamma(q)}. \tag{3.1.22}$$

Thus,

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}, \tag{3.1.23}$$

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{k^2}{(k^2-\Delta)^2} = -\frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(3-\epsilon)}{\Gamma(2-\epsilon)}\frac{\Gamma(\epsilon-1)}{\Delta^{\epsilon-1}} = -\frac{i}{(4\pi)^{2-\epsilon}}\frac{D}{2}\frac{\Gamma(\epsilon-1)}{\Delta^{\epsilon-1}} + \mathcal{O}\left(\epsilon^3\right). \tag{3.1.24}$$

With these identities, we can find that

$$M^{\mu\nu} = -\frac{4i(\mu^\epsilon e)^2}{(4\pi)^{2-\epsilon}}\int_0^1 \mathrm{d}x\ \frac{(1-\epsilon)\Gamma(\epsilon-1)}{\Delta^{\epsilon-1}}g^{\mu\nu} + \left[\left(\Delta+2x(1-x)p^2\right)g^{\mu\nu} - 2x(1-x)p^\mu p^\nu\right]\frac{\Gamma(\epsilon)}{\Delta^\epsilon}. \tag{3.1.25}$$

We trade the nitty-gritty of performing the D-dimensional integral with an integral in parameter x, but completing this step of the integration can be challenging due to the "conservation of difficulty" in problem solving. Gamma function behaves just like a continuous factorial, so $(\epsilon-1)\Gamma(\epsilon-1) = \Gamma(\epsilon)$. This gives us

$$M^{\mu\nu} = -\frac{8i(\mu^\epsilon e)^2\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\int_0^1 \mathrm{d}x\ \frac{x(1-x)}{\Delta^\epsilon} = \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\Pi_2(p), \tag{3.1.26}$$

in which

$$\Pi_2(p) \triangleq -\frac{8i(\mu^\epsilon e)^2\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}}\int_0^1 \mathrm{d}x\ \frac{x(1-x)}{\Delta^\epsilon}. \tag{3.1.27}$$

At this point, we may take the limit of $\Pi_2(p)$ when $D\to 4$ or $\epsilon\to 0$ and assign the value when $D\to 4$ to be the value for $M^{\mu\nu}$ at $D=4$. Note that $\Gamma(\epsilon)$ has a singularity of order $\mathcal{O}(\epsilon^{-1})$ while other terms are well-defined at $\epsilon=0$. The integral near $\epsilon\to 0$ has the form

$$\Pi_2(p) = -\frac{ie^2}{2\pi^2}\int_0^1 \mathrm{d}x\ x(1-x)\left[\frac{1}{\epsilon} + \ln\left(\frac{4\pi\mu^2}{e^{\gamma_E}\Delta}\right)\right], \tag{3.1.28}$$

in which γ_E denotes the Euler gamma constant from the expansion of $\Gamma(\epsilon)$. Since μ is an artificial mass scale, we may free to define a new scale $\tilde{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ so that we may drop the constant term contained in the logarithm. This scheme of removing $\ln(4\pi)$ and γ_E from the expansion is called **modified minimal subtraction ($\overline{\text{MS}}$) scheme**. This subtraction scheme can be seen as a redefinition of the coupling constant in dimensional regularization $\alpha_e = e^2/4\pi \to (\mu^\epsilon e)/4\pi \to (\tilde{\mu}^\epsilon e)/4\pi$, which should return to the usual dimensionless coupling α_e as we take D back to 4. In $\overline{\text{MS}}$ scheme,

$$\Pi_2(p) = -\frac{ie^2}{2\pi^2}\int_0^1 \mathrm{d}x\; x(1-x)\left[\frac{1}{\epsilon} + \ln\left(\frac{\tilde{\mu}^2}{\Delta}\right)\right], \tag{3.1.29}$$

Ignoring the $\mathcal{O}(1/\epsilon)$ divergence part, the physical quantity in which we are interested is

$$M^{\mu\nu} = \big(p^2 g^{\mu\nu} - p^\mu p^\nu\big)\tilde{\Pi}_2 \tag{3.1.30}$$

in which

$$\tilde{\Pi}_2(p) \triangleq \Pi_2(p) - \Pi_2(0) = -\frac{ie^2}{2\pi^2}\int_0^1 \mathrm{d}x\; x(1-x)\ln\left(\frac{m^2}{\Delta}\right). \tag{3.1.31}$$

Here we subtract off the divergence by choosing a reference scale $p = 0$. In this case, the photon has an infinite wavelength, thus giving us the long-distance observed behavior. We argue that what we calculated may have a divergence at infinity (just like that electric potential may divergence at infinity), but as long as we fix what we mean by "infinitely far away", the physical quantity has no divergence.

3.1.4 Remarks and Discussions

As we see from the previous calculation, loop diagrams are frequently higher-order corrections to a tree-level process. If an n-point tree-level process in QED is proportional to e^n, then its one-loop correction will be of $\mathcal{O}(e^{n+2})$ since one needs two additional vertices to form a loop. The traditional Feynman-rule approach usually involves several main steps: numerator simplification, denominator parameterization, dimensional regularization, and integration over parameters. But one can realize that the challenge snowballs as the loop gets more and more complicated. The main challenge comes from the integration over parameters. In our example, we have one Feynman parameter x. However, even in this case, we never integrate over this parameter unless necessary. In $M^{\mu\nu}$, terms, such as

$$\int_0^1 \mathrm{d}x\; x(1-x)\ln\Delta = \int_0^1 \mathrm{d}x\; x(1-x)\ln[m^2 - p^2 x(1-x)], \tag{3.1.32}$$

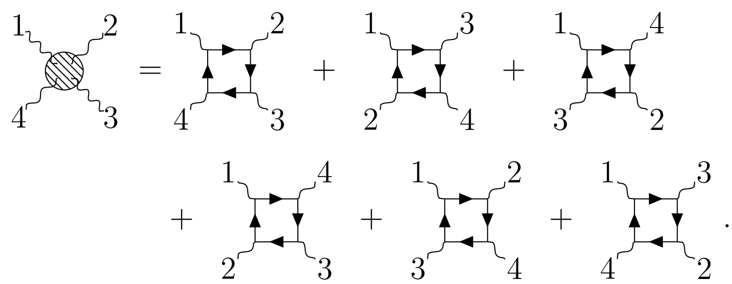
can be extremely challenging to evaluate. Frequently, the one-loop correction involves not only two propagators but more. In that case, we must introduce more Feynman parameters, *e.g.* for a four-propagator diagram (also called a box diagram), one should introduce three Feynman parameters x, y, z. One can imagine that the integral should take roughly the form

$$\int_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}z\; f(x,y,z)\ln(g(x,y,z)), \tag{3.1.33}$$

in which f and g are rational functions in x, y, z. But this integral becomes extremely difficult to evaluate in general due to the logarithmic term in the integrand. Therefore, we need a more creative approach to solve a loop-level amplitude, and some of these approaches are presented in the next calculation.

3.2 Light-by-light Scattering: A Four-photon Process

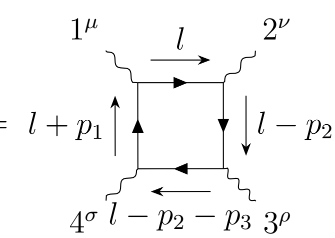
In this section, we are interested in a process that can be reinterpreted as $\gamma\gamma \to \gamma\gamma$ scattering as known as the **light-by-light scattering**. As discussed in the previous chapter, there cannot be a vertex with two photons in QED due to the fact that photon has a unique flavor. This process can only be realized at loop level, and its one-loop contribution are represented by diagrams with a fermion loop, *i.e.*



(3.2.1)

Evaluating these diagrams with Feynman rules can be tricky. However, since each diagram has the same topology, we may evaluate one diagram and permute the particle labels to obtain other diagrams. For the simplicity of our calculation and the introduction of spinor helicity variables, let's assume that the fermion loop has zero mass. In this case, we may have one diagram being

$M^{\mu\nu\rho\sigma} =$



$$= \int \mathrm{d}\Pi_l e^4 \,\mathrm{Tr}\left[\gamma^\mu \frac{\not{l} + \not{p}_1}{(l+p_1)^2}\gamma^\sigma \frac{\not{l} - \not{p}_2 - \not{p}_3}{(l-p_2-p_3)^2}\gamma^\rho \frac{\not{l} - \not{p}_2}{(l-p_2)^2}\gamma^\nu \frac{\not{l}}{l^2}\right]. \tag{3.2.2}$$

Following our previous approach, we parameterize the denominator by the general Feynman shift formula shown in eq'n (3.1.6).

$$\begin{aligned}
&\left[l^2(l+p_1)^2(l-p_2-p_3)^2(l-p_2)^2\right]^{-1} \\
&= \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\; \Gamma(n)\left[xl^2 + y(l+p_1)^2 + z(l-p_2-p_3)^2 + (1-x-y-z)(l-p_2)^2\right]^{-4} \\
&= \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\; 6\begin{bmatrix} (l+yp_1+mp_2-zp_3)^2 - \\ \quad \left(-yp_1^2 - z(p_2+p_3)^2 + (m+z)p_2^2 + (yp_1+mp_2-zp_3)^2\right)\end{bmatrix}^{-4} \\
&= \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\; 6\left[(l+yp_1+mp_2-zp_3)^2 - \Delta\right]^{-4},
\end{aligned} \tag{3.2.3}$$

in which $m \triangleq (x+y-1)$. Thus, we can define the following shifted momentum

$$l = k - yp_1 - mp_2 + zp_3 \triangleq k - s, \tag{3.2.4}$$

and focus on manipulating the numerator.

3.2.1 Absence of UV Divergence

As discussed in section 3.1.3, the k-integral may contain various divergences. In this section, we focus on the **ultraviolet (UV) divergence**, which is the divergence when $k \to \infty$. For the vacuum polarization, we know that it has a UV divergence of $\mathcal{O}(k^2)$ which we regularize as a logarithmic divergence in $\tilde{\mu}$. In the 4γ amplitude, we argue that it has no UV divergence.

Since the trace is a linear function in its argument and l is at most $\mathcal{O}(k)$, we can infer that as $k \to \infty$ the integral grows roughly like

$$\frac{\mathrm{d}^4 k \ \mathcal{O}(k^4)}{(k^2-\Delta)^4} \sim \mathcal{O}(\ln k), \tag{3.2.5}$$

and any lower-order term in the numerator cannot contribute the the UV divergence. Due to the tensor structure, the $\mathcal{O}(k^4)$ term in the numerator is of the form

$$\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\sigma \not{k} \gamma^\rho \not{k} \gamma^\nu \not{k}] = A g^{\mu\nu} g^{\rho\sigma} + B g^{\mu\rho} g^{\sigma\nu} + C g^{\mu\sigma} g^{\nu\rho}, \tag{3.2.6}$$

in which A, B, and C are functions of k. Summing over six diagrams, we find that

$$\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\sigma \not{k} \gamma^\rho \not{k} \gamma^\nu \not{k}] \propto g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\sigma\nu} + g^{\mu\sigma} g^{\nu\rho}. \tag{3.2.7}$$

Note that to find the amplitude, we must dot the polarizations of external particles. Thus, the UV divergence part of the amplitude is proportional to

$$\mathrm{Div}_{\mathrm{UV}}(A) \propto (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot \epsilon_4) + (\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_2) + (\epsilon_1 \cdot \epsilon_4)(\epsilon_2 \cdot \epsilon_3). \tag{3.2.8}$$

Our goal is to show that this divergent part evaluates to zero under some particular gauge choice. Usually, fixing gauge is a challenging task. A good gauge simplifies the calculation by eliminating unnecessary gauge redundancies while another gauge may obscure the physical dependence amid a swamp of nontrivial cancellation of unphysical terms. However, when we use spinor helicity formalism, the gauge choice is entirely packed into the reference vector of each polarization. We may utilize a good (sometimes obvious) choice of reference vectors to eliminate as many spinor products as possible, resulting in a good gauge. It suffice to consider when the four particles are of helicity $(++++)$, $(-+++)$, and $(--++)$.

$(++++)$ Helicity

For all plus helicity configuration, recall from section 1.4 that the polarization vector can be expressed as $\epsilon_i^+ = \sqrt{2}\,|r_i\rangle\,[i|\,/\,\langle r_i\, i\rangle$. The dot product between any two polarization is of the form

$$\epsilon_i^+ \cdot \epsilon_j^+ = \frac{2\,[i\,j]\,\langle r_j\, r_i\rangle}{\langle r_i\, i\rangle\,\langle r_j\, j\rangle}. \tag{3.2.9}$$

Therefore, there is a gauge choice $|r\rangle = |r_1\rangle = |r_2\rangle = |r_3\rangle = |r_4\rangle$ such that $\langle r_i\, r_j\rangle = 0 \implies \epsilon_i \cdot \epsilon_j = 0 \implies \mathrm{Div}_{\mathrm{UV}}(A^{++++}) = 0$.

$(-+++)$ Helicity

For three-plus one-minus helicity configuration, each term will still contain a dot product of polarization vectors of two positive-helicity photons. Hence, there exists a gauge $|r] = |r_2] = |r_3] = |r_4]$ such that $\epsilon^+ \cdot \epsilon^+ = 0 \implies \mathrm{Div}_{\mathrm{UV}}(A^{-+++}) = 0$.

$(--++)$ Helicity

For two-plus two-minus helicity configuration,

$$\mathrm{Div}_{\mathrm{UV}}(A^{--++}) \propto \frac{\langle 1\, 2\rangle\, [r_2\, r_1]\, [3\, 4]\, \langle r_4\, r_3\rangle + \langle 1\, r_3\rangle\, [3\, r_1]\, [r_2\, 4]\, \langle r_4\, 2\rangle + \langle 1\, r_4\rangle\, [4\, r_1]\, [r_2\, 3]\, \langle r_3\, 2\rangle}{[r_1\, 1]\, [r_2\, 2]\, \langle r_3\, 3\rangle\, \langle r_4\, 4\rangle}. \tag{3.2.10}$$

Now, we notice that there exists a gauge choice $|r_1] = |r_2] = |3]$ and $|r_3\rangle = |r_4\rangle = |1\rangle$ such that every term vanishes. Thus, $\mathrm{Div}_{\mathrm{UV}}(A^{--++}) = 0$.

Other helicity configurations will be a matter of relabeling particle names or switching left-handed and right-handed spinors. So as promised, all helicity amplitudes for a 4γ process at one-loop level has no UV divergence. Although this argument is plausible, the process may be a little too slick to convince some readers. After all, the gauge fixing seems a little artificial. So we also supply an argument without explicit gauge fixing.

An Argument Without Gauge Fixing

We will directly simplify the numerator algebra shown in eq'n (3.2.6). Sometimes, we will use some identities presented in appendix B.4.

$$\begin{aligned}
&\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\sigma \not{k} \gamma^\rho \not{k} \gamma^\nu \not{k}] \\
=&k^\alpha k^\beta k^\delta k^\lambda\, \mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\sigma \gamma^\beta \gamma^\rho \gamma^\delta \gamma^\nu \gamma^\lambda\big] \\
=&\frac{k^4}{D(D+2)} \big(g^{\alpha\beta} g^{\delta\lambda} + g^{\alpha\delta} g^{\beta\lambda} + g^{\alpha\lambda} g^{\beta\delta}\big)\, \mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\sigma \gamma^\beta \gamma^\rho \gamma^\delta \gamma^\nu \gamma^\lambda\big] \\
=&\frac{k^4}{D(D+2)} \big(\mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\sigma \gamma_\alpha \gamma^\rho \gamma^\delta \gamma^\nu \gamma_\delta\big] + \mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\sigma \gamma^\beta \gamma^\rho \gamma_\alpha \gamma^\nu \gamma_\beta\big] + \mathrm{Tr}\big[\gamma^\mu \gamma^\alpha \gamma^\sigma \gamma^\beta \gamma^\rho \gamma_\beta \gamma^\nu \gamma_\alpha\big]\big) \\
=&\frac{k^4}{D(D+2)} \big(2(D-2)^2\, \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\rho \gamma^\nu] + 16(4-D) g^{\mu\sigma} g^{\rho\nu} - (4-D)^2\, \mathrm{Tr}[\gamma^\mu \gamma^\sigma \gamma^\rho \gamma^\nu] \\
&-32 g^{\sigma\nu} g^{\nu\rho} + 2(4-D)\, \mathrm{Tr}[\gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\nu]\big) \\
=&\frac{4k^4}{D(D+2)} [D(D-2)(g^{\mu\sigma} g^{\rho\nu} + g^{\mu\nu} g^{\sigma\rho}) + (D+4)(D-2) g^{\mu\rho} g^{\sigma\nu}] \\
=&4k^4 \frac{1-\epsilon}{3-\epsilon} \left[g^{\mu\sigma} g^{\rho\nu} + g^{\mu\nu} g^{\sigma\rho} - \frac{(4-\epsilon)}{(2-\epsilon)} g^{\mu\rho} g^{\sigma\nu} \right] \approx \frac{4k^4}{3} (g^{\mu\sigma} g^{\rho\nu} + g^{\mu\nu} g^{\sigma\rho} - 2 g^{\mu\rho} g^{\sigma\nu}),
\end{aligned} \tag{3.2.11}$$

in which we fixed $D = 4 - 2\epsilon$. After calculating eq'n (3.2.6), one notice that $B = -2A = -2C$ so that when summed over all six diagrams by permuting Lorentz indices $\{\mu, \nu, \rho, \sigma\}$, these metric tensors identically vanish, regardless of the gauge choice. But as we argued, if the divergent part vanishes in one particular choice of gauge, then the divergent part vanishes.

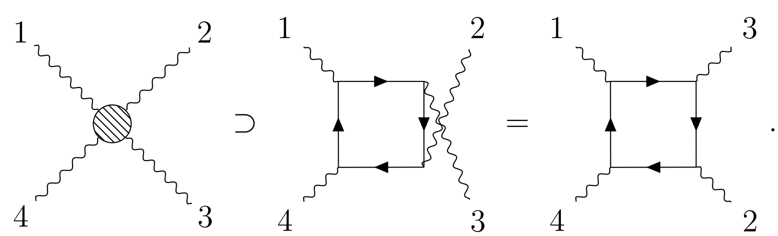
With some understanding of the divergence structure of the amplitude, we would like to proceed to calculate the loop with the help of spinor helicity as much as possible. But as we recognized in the previous section, the calculation with Feynman shifts poses a challenge for us once we finish the k integral and start to integrate over Feynman parameters. Integrating over one Feynman parameter already leads to daunting manipulations with logarithmic integrals, if we integrate over three parameters as shown in eq'n (3.2.3), we probably will expect a more puzzling expression. Therefore, instead of using Feynman shift, we will try another approach.

3.2.2 One-loop Scalar Loop Decomposition

With a traditional Feynman rule approach, we know that n-point one-loop amplitudes will take the form

$$A = \sum_{\text{diagram } i} f_i \,\text{Diagram}_i = \sum_{\text{diagram } i} f_i \int \mathrm{d}^D\Pi_l \, \frac{\mathcal{N}(l)}{\prod_j^{m\leq n} P_j^2(l)}, \tag{3.2.12}$$

in which f_i is some prefactor of diagram i, $\mathrm{d}^D\Pi_l = \mathrm{d}^D l/(2\pi)^D$ is the D-dimensional phase space for the loop momentum l, $\mathcal{N}(l)$ is the numerator, and P_j^2 are loop propagators. In particular, we may organize diagrams by permutations up to cyclic permutations. For example, in our case of 4γ amplitude, we sum over six different diagrams that effectively permutes label of $\{1,2,3,4\}$ up to cyclic symmetry. This is because while we may label external photons in some fashion when observing a scattering event, this label is artificial and does not correspond to the underlying process. Taking the (1324) diagram as an example, we include this diagram in the amplitude because there is a possible scenario when the underlying physical process is described by amplitude (1324) but when observing the momentum of 3 and the momentum of 2 points in the "opposite direction", *i.e.*



(3.2.13)

Each underlying diagram now has only cyclic symmetry in its external particles. These diagrams are sometimes called **ordered amplitudes**.[5] With these ordered amplitudes, we are permitted to rewrite the n-point one-loop amplitude shown in eq'n (3.2.12) as

$$A = \sum_{\text{ordered amplitudes } i} f_i \tilde{A}_i = \sum_{\sigma_i \in S_n/\mathbb{Z}_n} f_i \int \mathrm{d}^D\Pi_l \, \frac{\mathcal{N}(l)}{\prod_j^{m\leq n} P_{\sigma_i(j)}^2(l)}, \tag{3.2.14}$$

[5]The reason for calling these diagrams *amplitudes* is that in a non-Abelian gauge theory such as QCD, one may assign a flavor index to each external particles as discussed in section 2.3.1, then these ordered amplitudes become individually a physical process, hence an actual amplitude.

in which the propagators are

$$\prod_{j}^{m\leq n} P^2_{\sigma_i(j)}(l) = l^2\big(l - p_{\sigma_i(k_1)}\big)^2\big(l - p_{\sigma_i(k_1)} - p_{\sigma_i(k_2)}\big)^2 \ldots \big(l - p_{\sigma_i(k_1)} - p_{\sigma_i(k_2)} - \ldots - p_{\sigma_i(k_m)}\big)^2, \tag{3.2.15}$$

in which $1 \leq k_1 < k_2 < \ldots < k_m \leq n$ and p_j denote some external momenta so that these propagator labels respect the ordering of external momenta. Here, we also assumed that the loop propagators are massless, but a massive case can be extended straightforwardly.

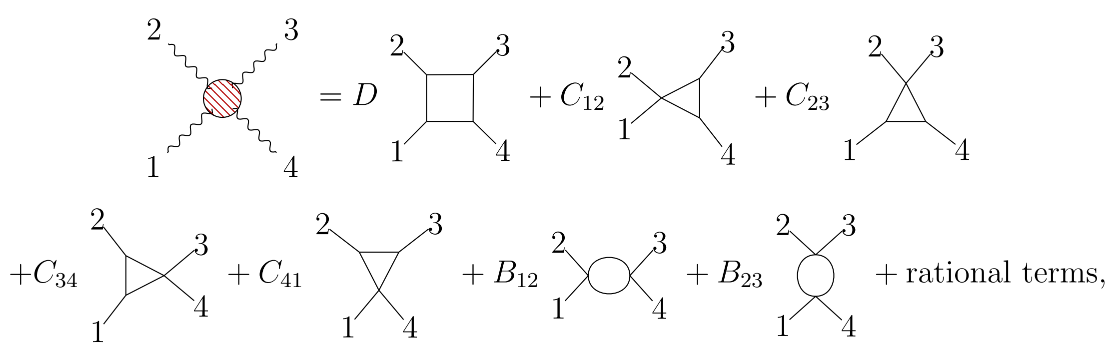
For any one-loop **ordered amplitude**, the loop integral can be reduced to linear combinations of 4, 3, and 2-point scalar loops

$$\begin{aligned}\tilde{A} = &\sum_{1\leq i<j<k<l\leq n} D_{ijkl} \int \mathrm{d}^D\Pi_l \frac{1}{P_i^2 P_j^2 P_k^2 P_l^2} + \sum_{1\leq i<j<k\leq n} C_{ijk} \int \mathrm{d}^D\Pi_l \frac{1}{P_i^2 P_j^2 P_k^2} \\ &+ \sum_{1\leq i<j<k<l\leq n} B_{ij} \int \mathrm{d}^D\Pi_l \frac{1}{P_i^2 P_j^2} + \text{rational terms},\end{aligned} \tag{3.2.16}$$

in which

$$P_i^2 = (l - K_i)^2, \quad K_i \triangleq \sum_{r=1}^{i} p_r. \tag{3.2.17}$$

Fondly, we call 4-point scalar functions box integrals, 3-point scalar functions triangle integrals, and 2-point scalar functions bubble integrals. With this general conclusion, we can infer that for each ordered amplitude in our calculation, the amplitude reads



(3.2.18)

and instead of evaluating the complicated integral, we break it up into scalar integrals and look for coefficients in front of each scalar integral. Our goal of this section is to provide a sketch that for our 4γ amplitude, this Ansatz makes sense. The following paragraph only provides a gist of the general proof [38, 39, 40, 41] for the one-loop decomposition. However, there are many technical details omitted in this sketch and beyond the scope of this document. Some of the argument is adopted from Chapter 3.4 of [17].

Let's know turn our attention to the numerator $\mathcal{N}(l)$. As we saw in the previous calculation, when dealing with amplitudes, these numerators are simply dot products of the loop momentum l, external momenta p_i, and perhaps polarization vector ϵ_i. Note that if there are parts of the numerator that only depends on external data, such as terms of the form $p_1 \cdot \epsilon_2$ or $p_3 \cdot p_4$, we can factor out that part into f_i. So the numerator $\mathcal{N}$ is just a polynomial of $l \cdot l$ and $l \cdot k$ in which k denotes some external data. But in 4-dimension, there are only 4 independent vectors. Hence, we can choose a set of variables $\{l \cdot K_1, l \cdot K_2, l \cdot K_3, l \cdot r_4, l^2\}$, in which $r_4 \cdot K_i = 0$ for any K_i. Any other dot products between l and external data k can be expressed as

$$l \cdot k = (l \cdot r_4)(r_4 \cdot k) + \sum_{r=1}^{3} (l \cdot K_r)(K_r \cdot k). \tag{3.2.19}$$

Note that $K_r \cdot k$ depends only on external data and can be just treated as a constant. According to eq'n (3.2.2), we know that the numerator is an expression at most of $\mathcal{O}(l^4)$. Thus, $\mathcal{N}(l)$ is just polynomial in these variables of at most degree 4. But since $P_i^2 = l^2 + K_i^2 - 2(l \cdot K_i)$, we may reorganize the numerator as polynomials in P_i^2s. Now, we may perform cancellations and partial fractions in P_i^2 to reduce the number of propagators in the denominator which results in these lower-point integrals. In the meantime, we may drop terms that have odd power in l such as terms proportional to $l^2(l \cdot K_1)$ as these will vanish after taking the integral. One may think that 1-point integral (also known as the tadpole integral)

$$I_1 = \int \mathrm{d}^D \Pi_l \, \frac{1}{l^2} \tag{3.2.20}$$

may also contribute to the amplitude. However, since the sum of external momenta is 0 and the fermion propagator is massless, it is an integral without a mass scale. Since the tadpole integral itself is manifestly Lorentz invariant, we know that it must be a function of Lorentz invariant quantities. In this case, there is no such a mass scale, so when setting $D \to 4 - 2\epsilon$, this integral at most contributes some constant offsetting the amplitude and is a part of the rational terms.

3.2.3 Unitarity Cut Method

Recall that the S matrix defined as $S = \mathbb{1} + iT$, in which T describes the interactions from one state to another. Now, the unitary constraints on the S matrix yields that

$$\mathbb{1} \stackrel{!}{=} S^\dagger S = \big(\mathbb{1} - iT^\dagger\big)\big(\mathbb{1} + iT\big) = \mathbb{1} + i\big(T - T^\dagger\big) + T^\dagger T, \tag{3.2.21}$$

or simply

$$T^\dagger T = i\big(T^\dagger - T\big). \tag{3.2.22}$$

For the interaction part of S, we usually expand it in terms of the coupling[6] α and recognize that $T \propto \alpha T_{\text{tree}} + \alpha^2 T_{\text{1-loop}} + \alpha^3 T_{\text{2-loop}} + \mathcal{O}(\alpha^4)$. Therefore, by perturbative expansion in α, we conclude that the unitarity of S implies

$$\alpha i \Big(T_{\text{tree}}^\dagger - T_{\text{tree}}\Big) = 0, \tag{3.2.23}$$

$$\alpha^2 i \Big(T_{\text{1-loop}}^\dagger - T_{\text{1-loop}}\Big) = \alpha^2 T_{\text{tree}}^\dagger T_{\text{tree}}. \tag{3.2.24}$$

The first statement is that the tree-level interactions are Hermitian while the second statement describes the imaginary part of a one-loop interaction is connected to the total cross

[6]In QED, the coupling constant is proportional to e^2.

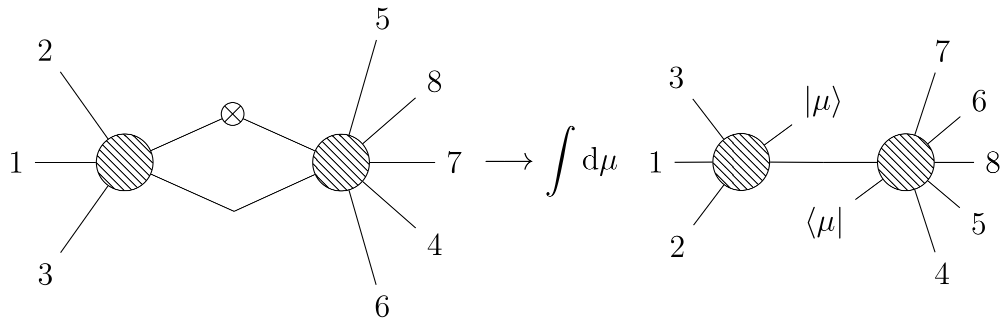


Figure 3.1: Illustration for cutting a one-loop diagram and producing two tree-level diagram

section of its tree-level counterpart. Consider the evolution of some initial state $|i\rangle$ to some final state $|f\rangle$. The second statement says

$$i(\langle i\,|\,T_{\text{1-loop}}\,|\,f\rangle^* - \langle f\,|\,T_{\text{1-loop}}\,|\,i\rangle) = 2\,\mathrm{Im}(\langle f\,|\,T_{\text{1-loop}}\,|\,i\rangle) = \int \mathrm{d}\mu\ \langle f|T^\dagger_{\text{tree}}|\mu\rangle\,\langle\mu\,|\,T_{\text{tree}}\,|\,i\rangle\,, \tag{3.2.25}$$

in which $|\mu\rangle$ are intermediate on-shell states. It seems mysterious whence the imaginary part comes. After all, the phase space integral over loops are real with real-valued numerators. However, as we defined the propagator $1/p^2$, we have made a choice such that the causality (time-ordering) is enforced (See for example [25, 23]), *i.e.*

$$\frac{1}{p^2} \cong \frac{1}{p^2 + i\varepsilon} = \mathrm{P.V.}\left\{\frac{1}{p^2}\right\} - i\pi\delta(p^2). \tag{3.2.26}$$

This last term follows from the fact that for any analytic function $f(z)$ has the following property

$$\oint \mathrm{d}z\ \frac{f(z)}{z - i\varepsilon} = \mathrm{P.V.}\left\{\int \mathrm{d}z\ \frac{f(z)}{z}\right\} - \pi i \int \mathrm{d}z\ \delta(z - i\varepsilon) f(z). \tag{3.2.27}$$

Then, the imaginary part of the loop digram comes from these $i\pi\delta(p^2)$ terms in the propagator which puts the internal particles on-shell. Therefore, in a more familiar term, eq'n (3.2.25) states that whenever we put a loop propagator on-shell, we obtain a product of two tree-level amplitudes. Pictorially, this is sensible if we interpret the on-shell condition as cutting the loop propagator to produce two on-shell states of the same momentum (one incoming and one outgoing), then the rest of the diagram is a tree-level diagram as illustrated in fig. 3.1. Thus, putting a loop momentum on-shell is also called performing a **unitarity cut**.

This pictorially intuitive but paradigmatically profound consequence leads to a new idea on reconstructing loop, especially one-loop, integral. The first idea was proposed by Cutkosky in [42]. Cutkosky noticed that these delta functions denotes simple poles at the integrand level, and when integrating over simple poles, these delta functions and only these delta functions will produce branch cuts appearing in the logarithmic terms of the loop integral. Hence, when cutting two loop momenta, the left-hand side of eq'n (3.2.25) captures the **discontinuity** across the branch cut by replacing loop propagator $l_i^2 \to -i\pi\delta(l_i^2)$ whereas the right-hand side of eq'n (3.2.25) produces two separated on-shell diagram, *i.e.* with the replacement

$$\begin{aligned} \mathrm{d}\mu =&\mathrm{d}^4 l_1 \mathrm{d}^4 l_2 \delta(l_1^2)\delta(l_2^2)\delta^4\left(l_1 + l_2 + \sum_{i \in \text{left amplitude}} p_i\right) \\ =&\mathrm{d}^4 l_1 \mathrm{d}^4 l_2 \delta(l_1^2)\delta(l_2^2)\delta^4\left(l_1 + l_2 - \sum_{j \in \text{right amplitude}} p_j\right), \end{aligned} \tag{3.2.28}$$

we must have

$$\mathrm{Disc}\Big|_{l_1+l_2} (A_{\text{one-loop}}) = \int \mathrm{d}\mu \; A_{\text{left tree}} A_{\text{right tree}}. \tag{3.2.29}$$

By knowing the discontinuity across various pairs of loop momenta, we may be able to reconstruct the original loop integral without performing Feynman shifts. For instance, with the scalar loop decomposition shown in eq'n (3.2.18), we can simply "glue" up the left and right subamplitudes on the right-hand side of eq'n (3.2.29) and identify the coefficients of different scalar integrals

$$\int \mathrm{d}\mu \; AA = \mathrm{Disc}\Big|_{l_1+l_2} (D\,(\text{box}) + C\,(\text{triangles}) + B\,(\text{bubbles})). \tag{3.2.30}$$

This is known as a **double cut** or the **Cutkosky rules**. This new idea opens up a possibility of finding one-loop amplitudes without using Feynman rules.[7] But we still face a challenge of fixing factors in front of each scalar integral.

To further pin down the appropriate coefficients, we cut all four propagators following the description of [43, 44, 45]. This approach is called the **quadruple cut** method. Once we fix some ordered amplitude, we may cut each loop momentum between external momenta. When cutting four loop propagators, we realize that it is necessary that the scalar loop integrand is a box as other scalar integrands cannot provide sufficient poles. In this way, we isolated the discontinuity of the box and discarded any other discontinuity, *i.e.*

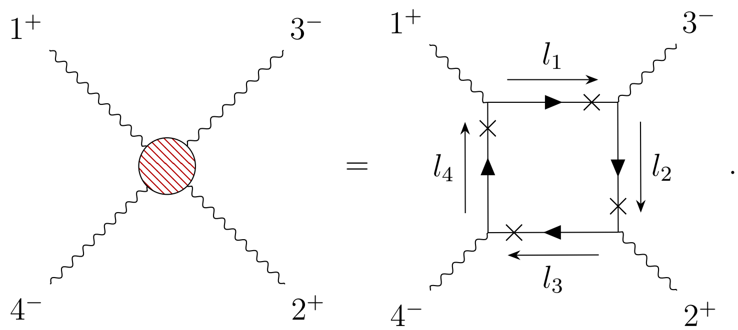
$$\int \mathrm{d}\mu \; A_1 A_2 A_3 A_4 = \mathrm{Disc}\Big|_{1|2|3|4|} (D\,(\text{box})). \tag{3.2.31}$$

Traditionally, this task is challenging as subamplitude may contain three-point amplitudes which vanish when the on-shell states have real momentum. But fortunately, with spinor helicity formalism, we can analytically continue external (massless) momenta to complex numbers so that we have a non-vanishing expression for each subamplitude.

[7] Recall from the previous chapter that we may "glue" up tree-level amplitudes with the BCFW on-shell recursion if the contour integral has no boundary terms. Therefore, knowing the Feynman rule is similar to enumerating all permitted three-point amplitudes of a theory and pin down their appropriate coupling constants.

3.2.4 Quadruple Cuts in Action for $(++--)$

Let's focus on the helicity choice $(1^+2^+3^-4^-)$ and start with cutting the ordered amplitude $(1^+3^-2^+4^-)$.



(3.2.32)

Each corner is just a three-point $e^+e^-\gamma$ amplitude which we found in the previous chapter. To sum over $\mathrm{d}\mu$, we must enforce momentum conservation between each momentum propagator l_i and add up two possible helicity choices at each propagator, *i.e.*

$$2\int \mathrm{d}\mu \prod_{i=1}^{4} A_i = A(1^+, l_1^+, -l_4^-)\cdots A(4^-, l_4^+, -l_3^-) + A(1^+, -l_4^+, l_1^-)\cdots A(4^-, -l_3^+, l_4^-). \tag{3.2.33}$$

To ensure that $-l_i$ is represented correctly in spinor variables, we pick $|(-l_i)\rangle = |l_i\rangle$ and $|(-l_i)] = -|l_i]$ so that $|-l_i\rangle[-l_i| = -|l_i\rangle[l_i|$. There are a few other prescriptions such as $|(-l_i)\rangle = -|l_i\rangle, |(-l_i)] = |l_i]$ or $|(-l_i)\rangle = i|l_i\rangle, |(-l_i)] = i|l_i]$. There is no difference between these prescriptions as long as one stick to one consistently when evaluating an amplitude. We will mainly use the first one, but when other prescription is manifestly easier, *e.g.* when the amplitudes are exclusively in terms of $|i]$, we may switch to another prescription. Putting four 3-point amplitudes down and summing over two fermion helicities, we find the quadruple cut yields

$$\begin{aligned} 2\int \mathrm{d}\mu\ A_{1|3|2|4|} &= \frac{\langle 3|\,l_1\,|1]^2\,\langle 4|\,l_3\,|2]^2 + \langle 3|\,l_2\,|1]^2\,\langle 4|\,l_4\,|2]^2}{\langle l_1|\,l_2l_3l_4\,|l_1]} \\ &= \frac{\langle 3|\,l_4\,|1]^2\,\langle 4|\,l_2\,|2]^2 + \langle 3|\,l_2\,|1]^2\,\langle 4|\,l_4\,|2]^2}{\langle l_1|\,l_2l_3l_4\,|l_1]}, \end{aligned} \tag{3.2.34}$$

in which we used momentum conservation $l_1 = l_4 - p_1$ and $l_3 = l_2 - p_2$. Due to **three-point special kinematics**, we know that

$$|1\rangle \propto |l_4\rangle \propto |l_1\rangle\,, \quad |2\rangle \propto |l_2\rangle \propto |l_3\rangle\,, \quad |3] \propto |l_2] \propto |l_1]\,, \quad |4] \propto |l_4] \propto |l_3]\,. \tag{3.2.35}$$

But we can always perform a **little group scaling** on l_2 and l_4 so that

$$|1\rangle = |l_4\rangle = C_{|l_1\rangle}\,|l_1\rangle\,, \quad |2\rangle = |l_2\rangle = C_{|l_3\rangle}\,|l_3\rangle\,, \quad |3] = |l_2] = C_{|l_1]}\,|l_1]\,, \quad |4] = |l_4] = C_{|l_3]}\,|l_3]\,. \tag{3.2.36}$$

This means that we may set

$$|l_2\rangle\,[l_2| = |2\rangle\,[3|\,, \quad |l_4\rangle\,[l_4| = |1\rangle\,[4|\,, \tag{3.2.37}$$

and find the amplitude becomes

$$2\int \mathrm{d}\mu\ A_{1|3|2|4|} = \frac{\langle 1|\,4\,|1]^2\,\langle 2|\,3\,|2]^2 + \langle 2|\,413\,|2]^2}{\langle 1|\,l_3\,|3]\,\langle 2|\,l_1\,|4]} = \frac{\langle 1|\,4\,|1]^2\,\langle 2|\,3\,|2]^2 + \langle 2|\,413\,|2]^2}{C_{|l_1\rangle}C_{|l_1]}C_{|l_3\rangle}C_{|l_3]}\,\langle 1\,2\rangle^2\,[3\,4]^2}. \quad (3.2.38)$$

It seems that this scaling argument is rather arbitrary, but there is a nontrivial consistency check. Notice that the numerator carries no helicity weight in any external particle, and the helicity weight is entirely captured by the denominator. Our consistency check, hence, will be to examine whether $C_{|l_1\rangle}C_{|l_1]}C_{|l_3\rangle}C_{|l_3]}$ carries no helicity weight.

To find these constants, observe the following relation from momentum conservation

$$\langle 3|\,l_1\,|1] = \langle 3|\,l_4\,|1] \implies C_{|l_1\rangle}C_{|l_1]}\,\langle 3\,1\rangle\,[3\,1] = \langle 3\,1\rangle\,[4\,1] \implies C_{|l_1\rangle}C_{|l_1]} = \frac{[4\,1]}{[3\,1]}, \quad (3.2.39)$$

$$\langle 4|\,l_3\,|2] = \langle 4|\,l_2\,|2] \implies C_{|l_3\rangle}C_{|l_3]} = \frac{[3\,2]}{[4\,2]}. \quad (3.2.40)$$

Indeed, $C_{|l_1\rangle}C_{|l_1]}C_{|l_3\rangle}C_{|l_3]}$ carries no helicity weight as we expected. Substituting these factors into eq'n (3.2.38), we find that

$$2\int \mathrm{d}\mu\ A_{1|3|2|4|} = \frac{4(p_1\cdot p_4)(p_2\cdot p_3)\,\langle 2|\,314\,|2] + 4(p_1\cdot p_3)(p_2\cdot p_4)\,\langle 2|\,413\,|2]}{\langle 1\,2\rangle^2\,[3\,4]^2}. \quad (3.2.41)$$

With Mandelstam invariants and simplification of spinor variables, we can find that

$$2\int \mathrm{d}\mu\ A_{1|3|2|4|} = \frac{(u^2+t^2)(u^2+t^2-s^2)/2}{\langle 1\,2\rangle^2\,[3\,4]^2} = \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]}\frac{-tu(t^2+u^2)}{s^2}. \quad (3.2.42)$$

Now, we may turn to the discontinuity of the box diagram, but there is not much to perform in this case. Since the discontinuity of the branch cuts comes from integrating over $l_i^2 \to -i\pi\delta(l_i^2)$, we may simply compare the left-hand side and the right-hand side of eq'n (3.2.31). The left-hand side is a statement about on-shellness and momentum conservation

$$\begin{aligned}\int \mathrm{d}\mu &= \int \left[\prod_{r=1}^{4} \mathrm{d}^4 l_r\,\delta(l_r^2)\right]\delta^4(l_1+p_1-l_4)\delta^4(l_2+p_3-l_1)\delta^4(l_3+p_2-l_2)\\ &= \int \mathrm{d}^4 l_1\,\delta\big(l_1^2\big)\delta\big((l_1-p_3)^2\big)\delta\big((l_1-p_3-p_2)^2\big)\delta\big((l_1-p_3-p_2-p_4)^2\big),\\ &= \int \mathrm{d}^4 l_1\,\delta\big(l_1^2\big)\delta\big(l_2^2\big)\delta\big(l_3^2\big)\delta\big(l_4^2\big),\end{aligned} \quad (3.2.43)$$

but this is the discontinuity of a box if we substitute $l_i^2 \to -i\pi\delta(l_i^2)$, *i.e.*

$$\int \mathrm{d}\mu\ A_{1|3|2|4|} = D\ \mathrm{Disc}\Big|_{1|3|2|4|}(\text{box}). \quad (3.2.44)$$

Thus, the part in $\int \mathrm{d}\mu\ A$ which is independent of the loop momenta can be moved outside the integral and identified as the coefficient D. In this scenario, we happen to find that $\int \mathrm{d}\mu\ A_{1|3|2|4|}$ is entirely independent of any loop momenta. Hence,

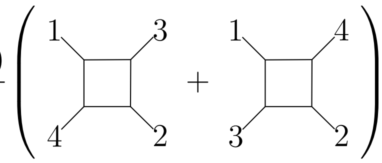
$$D_{1324} = \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]}\frac{-tu(t^2+u^2)}{2s^2}. \quad (3.2.45)$$

As we are agnostic about the direction of the loop momentum, we may simply replace $l_i \to -l_i$ and find the same box coefficient. But this coefficient now corresponds to the ordered amplitude (1423); thus,

$$D_{1423} = D_{1324} = \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \frac{-tu(t^2+u^2)}{2s^2}. \tag{3.2.46}$$

Therefore, we know that the one-loop 4γ amplitude in QED

$$A_{\text{1-loop}}(1^+2^+3^-4^-) \supset \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \frac{-tu(t^2+u^2)}{s^2}$$



. (3.2.47)

As discussed in chapter 2, Compton scattering has vanishing amplitude if the two external photons have the same helicity. Therefore, when we cut any other ordered amplitudes, such as $(1^+2^+3^-4^-)$ or $(1^+4^-3^-2^+)$, there will always be two of the four unitarity cuts that partition the amplitude into products with a vanishing Compton scattering subamplitude. For instance, for ordered amplitude $(1^+4^-3^-2^+)$ or equivalently amplitude $(4^-3^-2^+1^+)$, a cut between 1 and 4 and a cut between 2 and 3 will produce two vanishing subamplitudes. Then, we cannot further cut the amplitude. Hence, we can conclude that the only box contribution of $A_{\text{1-loop}}$ comes from what we identified from the previous cuts.

Now, we may substitute in the known result for the box [46] and find that to $\mathcal{O}(\epsilon)$ the box integral evaluates to

$$\begin{aligned} A_{\text{1-loop}}(1^+2^+3^-4^-) \approx \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \frac{-tu(t^2+u^2)}{2s^2} & \left\{ \frac{\mu^{2\epsilon}}{tu} \left[\frac{2}{\epsilon^2}\left((-t)^{-\epsilon} + (-u)^{-\epsilon}\right) \right.\right. \\ & \left.\left. - \ln^2\left(\frac{-t}{-u}\right) - \pi^2 \right] + (t \leftrightarrow u) \right\}. \end{aligned} \tag{3.2.48}$$

This loop contains both an infrared (IR) divergent piece (in terms of $1/\epsilon$ and μ) and a finite piece. One may expanding in orders of ϵ to find that the scalar loop evaluates to

$$\begin{aligned} A_{\text{1-loop}}(1^+2^+3^-4^-) = & - \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \frac{2(t^2+u^2)}{s^2} \left[\ln\left(-\frac{s}{t}\right) \ln\left(-\frac{s}{u}\right) - \frac{2\pi^2}{3} \right] \\ & - \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]} \frac{2(t^2+u^2)}{s^2} \left\{ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left[\ln\left(-\frac{s}{t}\right) + \ln\left(-\frac{s}{u}\right) + 2\ln\left(\frac{\mu^2}{s}\right) \right] \right. \\ & \left. + \ln\left(\frac{\mu^2}{s}\right) \left[\ln\left(-\frac{s}{t}\right) + \ln\left(-\frac{s}{u}\right) + \ln\left(\frac{\mu^2}{s}\right) \right] \right\}. \end{aligned} \tag{3.2.49}$$

After restoring the factor of $\sqrt{2}e$ at each three-point amplitude, we noticed that the finite piece agree very well with the **leading singularity** contribution[8] from a known result calculated in supersymmetric QED [47]. With ease, we performed a calculation that starts not from Feynman rules but tree-level amplitudes. This permits us to leverage as much as the known theoretical data, such as tree amplitudes, tree-level cross sections, to construct finer radiative corrections to a physical process.

[8]The term **leading singularity** refers to the maximal unitarity cuts applied to an n-loop diagram[45]. At one-loop level, the maximal cut is a quadruple cut.

3.2.5 Quadruple Cuts for $(++++)$

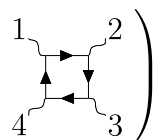
In this section, we will go through the calculation for the $(++++)$ helicity amplitude for the light-by-light scattering. Unlike the approach starting with four-point amplitudes shown in the previous section, we will use spinor helicity formalism as a guide to fix a good gauge so that finding the helicity amplitude becomes relative easy. As we established in section 3.1.2, there are relations between the trace of gamma matrice and spinor products. Therefore, we may rewrite the numerator of eq'n (3.2.2) as spinor products. Similar to the analysis on the UV divergence of the amplitude, we will pick good reference vectors for the $(++++)$ amplitude so that it factors into a helicity part and a part that can be treated just with Lorentz invariant. Taking the (1234) ordered amplitude's numerator as an example, we can express it as

$$\begin{aligned}
&\mathrm{Tr}[\not{\epsilon}_2 \not{l} \not{\epsilon}_1 (\not{l}+\not{p}_1) \not{\epsilon}_4 (\not{l}-\not{p}_2-\not{p}_3) \not{\epsilon}_3 (\not{l}-\not{p}_2)] \\
&= \frac{[2|\, l\, |r_1\rangle\, [1|\, l+1\, |r_4\rangle\, [4|\, l-2-3\, |r_3\rangle\, [3|\, l-2\, |r_2\rangle + \text{c.c.}}{\langle 1\, r_1\rangle \langle 2\, r_2\rangle \langle 3\, r_3\rangle \langle 4\, r_4\rangle},
\end{aligned} \tag{3.2.50}$$

in which c.c. denotes complex conjugate. For other ordered amplitudes, the spinor products in the numerator will be different; however, the denominator will always be $\langle 1\, r_1\rangle \langle 2\, r_2\rangle \langle 3\, r_3\rangle$ $\langle 4\, r_4\rangle$. Therefore, we may pick a gauge to gather all the helicity weights into the denominator so that the numerator can only be Lorentz invariants without helicity information. One of such gauge is to let $|r_1\rangle = |2\rangle$, $|r_2\rangle = |1\rangle$, $|r_3\rangle = |4\rangle$, and $|r_4\rangle = |3\rangle$. Then, the denominator becomes $\langle 1\, 2\rangle^2 \langle 3\, 4\rangle^2$ which has the correct helicity information for every photon. The numerator in this case becomes

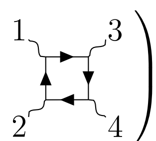
$$\begin{aligned}
&[2|\, l\, |2\rangle\, [4|\, l+1\, |4\rangle\, [1|\, l3(l-2)\, |1\rangle + \text{c.c.} \\
=&4(p_2\cdot l)[p_4\cdot(l+p_1)]([1|\, l3(l-2)\, |1\rangle + \langle 1|\, l3(l-2)\, |1]) \\
=&4(p_2\cdot l)[p_4\cdot(l+p_1)]\,\mathrm{Tr}[\not{p}_1 \not{l} \not{p}_3 (\not{l}-\not{p}_2)].
\end{aligned} \tag{3.2.51}$$

With similar techniques, one may find that the numerator of the six diagrams corresponds to the following expressions

$\mathcal{N}\Bigg($ 

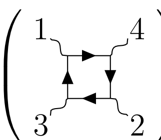
$\Bigg)$ $= [2|\, l\, |2\rangle\, [4|\, l+1\, |4\rangle\, [1|\, l3(l-2)\, |1\rangle + \text{c.c.}$

$$= 4(p_2\cdot l)[p_4\cdot(l+p_1)]\,\mathrm{Tr}[p_1 l p_3 (l-p_2)], \tag{3.2.52}$$

$\mathcal{N}\Bigg($ 

$\Bigg)$ $= [1|\, l\, |1\rangle\, [4|\, l-3\, |4\rangle\, [3|\, l2(l+1)\, |3\rangle + \text{c.c.}$

$$= 4(p_1\cdot l)[p_4\cdot(l-p_3)]\,\mathrm{Tr}[p_3 l (l+p_1)], \tag{3.2.53}$$

$\mathcal{N}\Bigg($ 

$\Bigg)$ $= [4|\, l2(l-4)3(l+1)1l\, |4\rangle + \text{c.c.} = \mathrm{Tr}[p_4 l p_2 (l-p_4) p_3 (l+p_1) l],$ (3.2.54)

in which a slash notation is understood in each term inside the trace. The other three diagrams can be obtained by "reflecting" the aforementioned three diagrams and relabeling the loop momenta so that the total amplitude is just twice of the contribution from the previous three diagrams.

At this point, keen readers may be curious about whether this amplitude vanishes. After all, if what we argued in the previous section about Compton scattering amplitude is correct, the quadruple-cut amplitude should vanish since all external particles have the same helicity. It is true that this amplitude will have no 4-dimensional scalar box contribution; however, as it turns out, the $(++++)$ amplitude does have a contribution from dimensional regularization[47, 48]. Specifically, there is a contribution to the rational terms from a box loop when we take $D \to 4$. However, a D-dimensional unitarity cut should invoke massive spinor helicity formalism. This is because a 4-dimensional massless momentum in $4-2\epsilon$ will take the form $(|\mathbf{k}|, \mathbf{k}, k_{-2\epsilon})$, and the extra -2ϵ-dimensional piece provides a mass term[9] that is not observed in 4 dimensions.

Performing an analysis with massive spinor helicity is beyond the scope of our discussion; however, since it has been shown that the only contribution comes from the -2ϵ-dimensional piece of a box integral. It is possible to reconstruct the calculation with the help of some symbolic algebraic package relatively straightforwardly. In our calculation, we used the FeynCalc package[49, 50, 51] to perform this routine and further simplified the integral into rational and logarithm functions with FeynHelpers package[52] assisted by Package X[53]. The advantage of using FeynCalc is that it permits both 4-dimensional and D-dimensional manipulations. It also enables prescriptions of 4-dimensional Lorentz dot products to D-dimensional Lorentz dot products, and *vice versa*. This flexibility permits us to isolate the -2ϵ-dimensional piece from the 4-dimensional piece. In a D-dimensional quadratic cut method, all external particles' momenta are in 4-dimension while the loop momenta are in D dimensions [54]. In a naïve calculation, one may put all Lorentz invariants, such as $(p_2 \cdot l)$ and $\mathrm{Tr}[p_1 l p_3 (l-p_2)]$ in 4-dimension, so that the D-dimensional contribution comes from the loop propagators in the denominator. This is almost correct since if all external momenta are in 4 dimensions, then $(l \cdot p_i)_D$ in D dimension will be $(l \cdot p_i)_D = (l \cdot p_i)_4 + (l \cdot p_i)_{-2\epsilon} = (l \cdot p_i)_4$ since p_i has no -2ϵ-dimensional piece. However, when considering the trace of gamma matrices, *e.g.*

$$\mathrm{Tr}[\not{p_1} \not{l} \not{p_3} (\not{l} - \not{p_2})] = 4\{(p_1 \cdot l)[p_3 \cdot (l-p_2)] + (p_3 \cdot l)[p_1 \cdot (l-p_2)] - (p_1 \cdot p_3)[l \cdot (l-p_2)]\}, \tag{3.2.55}$$

there may be terms of the form $(l \cdot l)_D \neq (l \cdot l)_4$. These terms can also contribute to the (-2ϵ)-dimensional piece of the box integral, *i.e.* the difference between l_D^2 and l_4^2 is $l_{-2\epsilon}^2$. Thus, while evaluating the numerator and denominator of the three diagrams, we carefully identified all l^2 terms from gamma traces; then, we performed one calculation in which all such l^2s are in 4 dimension and another calculation in which all such l^2s are in D dimension so that their difference is the correct (-2ϵ)-dimensional piece of the scalar box loop. While FeynCalc outputs the loop integrals in terms of Passarino-Veltman functions which are basis functions of tensorial decomposition of one-loop integrals [55], we utilized Package X to further simplify these functions into special functions. Fortunately, the final result is

[9] The reason is similar to why we must include a renormalization scale μ to match up the dimension.

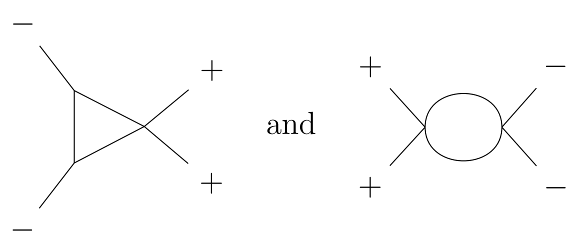
exceptionally simple.

$$A(++++)_{\text{1-loop}} = -\frac{e^4}{\langle 1\,2\rangle^2 \langle 3\,4\rangle^2}\frac{is^2}{2\pi^2} = -\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle \langle 3\,4\rangle}\frac{ie^4}{2\pi^2} \tag{3.2.56}$$

which agrees again with the calculation from SUSY QED [47], and as promised, this is only a rational term. Interestingly, this means that even if we observe the $4\gamma^+$ (or by crossing $4\gamma^-$) interaction, this interaction may be extremely difficult to detect as its amplitude has no dependence on momenta of external particles.

3.2.6 Lesson from Compton Scattering

Our observation about Compton scattering amplitudes helps us to skip evaluating the box contribution of four ordered amplitudes that have the helicity configuration $(++--)$ up to cyclic permutations. But this observation is actually more efficacious than what we just applied in the argument. With the same argument about vanishing subamplitudes, we learned that any scalar integral with two same-helicity photons must have zero coefficient. These 4-dimensional contributions include



(3.2.57)

With this argument, we can eliminate 12 coefficients and scalar loop diagrams.

But there is one more detail we learned while calculating the Compton scattering amplitude through BCFW on-shell recursion: the pole structure. When performing a momentum shift and gluing up three-point amplitudes to form a four-point amplitude, we noticed that one channel yields some amplitude while another channel has zero contribution. Therefore, in terms of the pole structure, when the photon and fermion have the same helicity on each side of the diagram, *i.e.* $f^{+1/2}\gamma^+ \to f^{-1/2}\gamma^-$, then the diagram factorizes into two three-point amplitudes whereas if the photon and fermion on each side have opposite helicity, *i.e.* $f^{+1/2}\gamma^- \to f^{-1/2}\gamma^+$, then there is no way to factorize that diagram further. This can potentially provides insights about how one should perform a **triple cut** [56, 57, 58] or a **double cut**. We will provide more discussions about double cuts in the next section

3.2.7 Double Cut on $(++--)$: Some Insights for Future Work

As we learned in section 3.2.3, the unitarity cut method aims to find the discontinuity of the scalar loops and infer the original scalar integral cut by putting loop momenta on-shell. This is why a replacement rule is used in unitarity cuts $1/l^2 \to \delta(l^2)$. According to fig. 3.2, when we cut two loop propagators l_1 and l_2, the discontinuities for the box, triangle, and

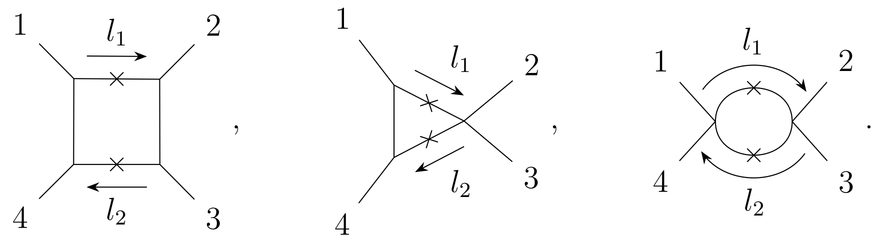


Figure 3.2: Double cuts on scalar box, triangle, and bubble loops

bubble loop drawn are

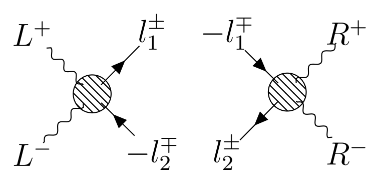
$$\mathrm{Disc}_{l_1+l_2}(\text{box}) = \int \mathrm{d}\mu \, \frac{1}{(l_1 - p_2)^2 (l_2 - p_4)^2} = \int \mathrm{d}\mu \, \frac{1}{(l_1 + p_1)^2 (l_2 + p_3)^2}, \tag{3.2.58}$$

$$\mathrm{Disc}_{l_1+l_2}(\text{triangle}) = \int \mathrm{d}\mu \, \frac{1}{(l_2 - p_4)^2} = \int \mathrm{d}\mu \, \frac{1}{(l_1 + p_1)^2}, \tag{3.2.59}$$

$$\mathrm{Disc}_{l_1+l_2}(\text{bubble}) = \int \mathrm{d}\mu \, 1, \tag{3.2.60}$$

in which $\mathrm{d}\mu$ is the Cutkosky on-shell condition shown in eq'n (3.2.28). Therefore, when we glue up the two on-shell tree-level amplitudes as shown in eq'n (3.2.29), we should observe the distinct pole structure of the remaining loop propagators. These poles will be telltale features that helps us identify the coefficient in front of each scalar loop integral.

As we discussed before, the only non-vanishing amplitude for a Compton scattering is to have $\gamma^+\gamma^- f^+ f^-$. Therefore, any cut on the light-by-light scattering diagram in QED will necessarily involve a left subamplitude with a γ^+ with momentum L^+ and a γ^- with a momentum L^-, and a right subamplitude with a γ^+ with momentum R^+ and a γ^- with a momentum R^-. Similar to the quadruple cut, when we perform a double cut on loop propagators, we must sum over two possible helicity configurations of the cut fermion propagators. The left and right subamplitudes will take the form



(3.2.61)

$$= \frac{\langle L^- \, l_2\rangle^2 \langle R^- \, l_1\rangle^2 + \langle L^- \, l_2\rangle^2 \langle R^- \, l_1\rangle^2}{\langle (-l_2) \, L^+\rangle \langle L^+ \, l_1\rangle \langle (-l_1) \, R^+\rangle \langle R^+ \, l_2\rangle}.$$

Note that the numerator is symmetric under the exchange $l_1 \leftrightarrow l_2$. Therefore, we can focus on simplifying the first term and the other term follows from a relabeling. For each subamplitude (left or right), we may have several propagators of the internal fermion. For the left subamplitude, the two possible propagators are

$$P_1^2 \triangleq \left(l_1 + L^+\right)^2 = \left(l_2 - L^-\right)^2, \quad P_2^2 \triangleq \left(l_1 + L^-\right)^2 = \left(l_2 - L^+\right)^2, \tag{3.2.62}$$

and for the right subamplitude, the two possible propagators are

$$P_3^2 \triangleq \left(l_2 + R^+\right)^2 = \left(l_1 - R^-\right)^2, \quad P_4^2 \triangleq \left(l_2 + R^-\right)^2 = \left(l_1 - R^+\right)^2, \tag{3.2.63}$$

in which we used the momentum conservation

$$\begin{cases} l_1 = l_2 + R^+ + R^-, \\ l_2 = l_1 + L^+ + L^-. \end{cases} \tag{3.2.64}$$

Then, we may "lift" the denominator by the possible propagators, *e.g.*

$$\frac{1}{\langle l_2\, L^+\rangle} = \frac{[l_2\, L^+]}{\langle l_2\, L^+\rangle\, [l_2\, L^+]} = \frac{[l_2\, L^+]}{P_1^2}. \tag{3.2.65}$$

We can introduce a helicity factor that contains the correct helicity information for the external particles

$$1 = \frac{[L^+\, R^+]\, \langle L^-\, R^-\rangle}{\langle L^+\, R^+\rangle\, [L^-\, R^-]} \frac{\langle L^+\, R^+\rangle\, [L^-\, R^-]}{[L^+\, R^+]\, \langle L^-\, R^-\rangle} = \frac{[L^+\, R^+]\, \langle L^-\, R^-\rangle}{\langle L^+\, R^+\rangle\, [L^-\, R^-]} \frac{\langle L^+\, R^+\rangle^2\, [L^-\, R^-]^2}{4(L^+\cdot R^+)(L^-\cdot R^-)}. \tag{3.2.66}$$

Notice that for every ordered amplitude, the helicity factor is always

$$\frac{[L^+\, R^+]\, \langle L^-\, R^-\rangle}{\langle L^+\, R^+\rangle\, [L^-\, R^-]} = \frac{[1\,2]\,\langle 3\,4\rangle}{\langle 1\,2\rangle\,[3\,4]}, \tag{3.2.67}$$

and it is also an easy check by squaring the helicity factor that this factor carries only helicity information but no momenta dependence. By lifting the denominator and factoring out the helicity part, we find that the double-cut amplitude becomes

$$\begin{aligned} &\frac{1}{\prod_i^4 P_i^2} \frac{[L^+\, R^+]\, \langle L^-\, R^-\rangle}{\langle L^+\, R^+\rangle\, [L^-\, R^-]} \times \left[\frac{\langle L^+\, R^+\rangle^2\, [L^-\, R^-]^2}{4(L^+\cdot R^+)(L^-\cdot R^-)} \right. \\ &\quad \left. \times \left[l_1\, L^+\right] \left[l_2\, L^+\right] \left[l_1\, R^+\right] \left[l_2\, R^+\right] \left(\left\langle L^-\, l_2\right\rangle^2 \left\langle R^-\, l_1\right\rangle^2 + \left\langle L^-\, l_2\right\rangle^2 \left\langle R^-\, l_1\right\rangle^2 \right) \right], \end{aligned} \tag{3.2.68}$$

in which P_i^2 denotes the four possible internal propagators of the subamplitude. Our new goal is to simplify the terms in the square bracket and cancel some propagators so that we can recognize each term as (the discontinuity of) box, triangle, or a bubble integrals and identify their respective coefficients.

The numerator in the square bracket can be expressed as

$$\begin{aligned} &-\langle l_2|\, L^- R^- l_1 R^+ L^+ l_1 R^- L^- l_2 R^+ L^+\, |l_2] - \langle l_2|\, R^- L^- l_1 R^+ L^+ l_1 L^- R^- l_2 R^+ L^+\, |l_2] \\ =&[l_1|\, R^+ L^+\, |l_1]\, [l_2|\, R^+ L^+\, |l_2] \left(\langle l_2|\, L^- R^-\, |l_1\rangle^2 + \langle l_2|\, R^- L^-\, |l_1\rangle^2 \right) \\ =&[l_1|\, R^+ L^+\, |l_1]\, [l_2|\, R^+ L^+\, |l_2] \left[\left(\langle l_2|\, L^- R^- + R^- L^-\, |l_1\rangle \right)^2 - 2\, \langle l_2|\, L^- R^-\, |l_1\rangle\, \langle l_2|\, R^- L^-\, |l_1\rangle \right] \\ =&[l_1|\, R^+ L^+\, |l_1]\, [l_2|\, R^+ L^+\, |l_2] \left[4\big(L^-\cdot R^-\big)^2\, \langle l_2\, l_1\rangle^2 - 2\, \langle l_2|\, L^- R^-\, |l_1\rangle\, \langle l_2|\, R^- L^-\, |l_1\rangle \right] \\ =&-4\big(L^-\cdot R^-\big)^2\, [l_1|\, R^+ L^+ l_1 l_2 R^+ L^+ l_2\, |l_1\rangle \\ &\quad -2\, [l_1|\, R^+ L^+\, |l_2]\, [l_1|\, R^+ L^+\, |l_2]\, \langle l_2|\, L^- R^-\, |l_1\rangle\, \langle l_1|\, R^- L^-\, |l_2\rangle \\ =&-4\big(L^-\cdot R^-\big)^2 \left[R^-\right|\, R^+ L^+ \big(-L^-\big) l_2 R^+ L^+ l_2 \left|R^-\right\rangle + 2\big([l_1|\, R^+ L^+\, |l_2]\, \langle l_2|\, L^- R^-\, |l_1\rangle \big)^2 \\ =&4\big(L^-\cdot R^-\big)^2 \left[R^-\right|\, R^+ L^+ L^- l_2 R^+ L^+ l_2 \left|R^-\right\rangle + 2\big([l_2|\, R^+ L^+\, |l_2]\, \langle l_2|\, L^- R^-\, |l_2\rangle \big)^2 \\ =&-8\big(L^-\cdot R^-\big)^2 \big(R^-\cdot R^+\big) \left[R^-\right|\, L^- l_2 R^+ L^+ l_2 \left|R^-\right\rangle + 2\big([l_2|\, R^+ L^+ l_2 R^- L^-\, |l_2\rangle \big)^2 \\ \triangleq&\mathcal{N}_1 + \mathcal{N}_2 \end{aligned} \tag{3.2.69}$$

Then, we can focus on simplifying the expression

$$\begin{aligned}
&[l_2|\, R^+ L^+ l_2 R^- L^- \,|l_2\rangle \\
=&2\big(l_2\cdot R^-\big)\,[l_2|\, R^+ L^+ L^- \,|l_2\rangle - 2\big(l_2\cdot L^-\big)\,[l_2|\, R^+ L^+ R^- \,|l_2\rangle \\
=&P_4^2\,[l_2|\, R^+ L^+ L^- \,|l_2\rangle + P_1^2\,[l_2|\, R^+ L^+ R^- \,|l_2\rangle \\
=&\frac{P_4^2}{2}\,\mathrm{Tr}\Big[\not{l}_2 \not{R}^+ \not{L}^+ \not{L}^-\Big] + \frac{P_1^2}{2}\,\mathrm{Tr}\Big[\not{l}_2 \not{R}^+ \not{L}^+ \not{R}^-\Big] \\
=&\big(L^+\cdot L^-\big)P_3^2P_4^2 + \big(R^+\cdot L^-\big)P_2^2P_4^2 + \big(R^-\cdot L^+\big)P_1^2P_3^2 + \big(R^+\cdot R^-\big)P_1^2P_2^2 \\
=&\big(L^+\cdot L^-\big)\big(P_3^2P_4^2 + P_1^2P_2^2\big) + \big(R^+\cdot L^-\big)\big(P_2^2P_4^2 + P_1^2P_3^2\big)
\end{aligned} \tag{3.2.70}$$

Note that not only is there a set of four-point Mandelstam invariants for the "global" $\{L^+, L^-, R^+, R^-\}$ kinematics, but also each subamplitude possesses a set of Mandelstam invariants. These relations are

$$0 = l_2^2 = \big(l_1 + L^+ + L^-\big)^2 = \big(l_1 + L^+\big)^2 + 2\big(l_1 + L^+\big)\cdot L^- = P_1^2 + P_2^2 + 2\big(L^+\cdot L^-\big), \tag{3.2.71}$$

$$0 = l_1^2 = \big(l_2 + R^+ + R^-\big)^2 = \big(l_2 + R^+\big)^2 + 2\big(l_2 + R^+\big)\cdot R^- = P_3^2 + P_4^2 + 2\big(R^+\cdot R^-\big). \tag{3.2.72}$$

So, we can further simplify the spinor product in $\mathcal{N}_1$ to

$$\begin{aligned}
&[l_2|\, R^+ L^+ l_2 R^- L^- \,|l_2\rangle \\
=&-\frac{1}{2}\Big(\big(P_1^2+P_2^2\big)P_3^2P_4^2 + P_1^2P_2^2\big(P_3^2+P_4^2\big)\Big) + \big(R^+\cdot L^-\big)\big(P_2^2P_4^2 + P_1^2P_3^2\big) \\
=&-\frac{1}{2}\big(P_1^2P_3^2P_4^2 + P_2^2P_3^2P_4^2 + P_1^2P_2^2P_3^2 + P_1^2P_2^2P_4^2\big) + \big(R^+\cdot L^-\big)\big(P_2^2P_4^2 + P_1^2P_3^2\big)
\end{aligned} \tag{3.2.73}$$

Therefore, we can start recognizing propagators by comparing the term

$$\begin{aligned}
\frac{\langle l_2|\ldots|l_2]}{\prod_i^4 P_i^2} =& -8\big(L^-\cdot R^-\big)^2\big(R^-\cdot R^+\big)\,\Bigg[-\frac{1}{2}\bigg(\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}+\frac{1}{P_4^2}\bigg) \\
&+\big(R^+\cdot L^-\big)\bigg(\frac{1}{P_1^2P_3^2}+\frac{1}{P_2^2P_4^2}\bigg)\Bigg] + \frac{\mathcal{N}_2}{\prod_i^4 P_i^2}
\end{aligned} \tag{3.2.74}$$

For $\mathcal{N}_2$, we apply a similar strategy of simplifying the spinor products first and then multiplying by the four possible propagators. Notice that after dividing the loop propagators, one may encounter expressions of the form $(P_2^2P_4^2)/(P_1^2P_3^2)$ which seems nonlocal. But these terms are actually local since we can use both the global four-point kinematics and that of the subamplitudes to find

$$\frac{P_2^2P_4^2}{P_1^2P_3^2} = \frac{[P_1^2 + 2(L^+\cdot L^-)][P_3^2 + 2(R^+\cdot R^-)]}{P_1^2P_3^2} = 1 + 2\big(L^+\cdot L^-\big)\bigg(\frac{1}{P_1^2}+\frac{1}{P_3^2}\bigg) + \frac{4(L^+\cdot L^-)^2}{P_1^2P_3^2}. \tag{3.2.75}$$

On the other hand, there will be terms of the form $(P_3^2P_4^2)/(P_1^2P_2^2)$ and P_2^2/P_4^2 which again appears nonlocal. We believe that there are systematic partial-fraction-like approaches to combine these nonlocal propagators into local poles; however, due to the time constraint and potentially a missing relation of momentum conservation, we have not identified the local poles for these nonlocal expressions. Nonetheless, we will present our current result for $\mathcal{N}_2$ containing a factor of squared spinor product and hope that the result is enlightening for future investigations

$$
\begin{aligned}
&\frac{\mathcal{N}_2}{\prod_i^4 P_i^2} = \frac{2\left([l_2|\, R^+ L^+ l_2 R^- L^- \,|l_2\rangle\right)^2}{\prod_i^4 P_i^2} \\
&= \Bigg\{ \frac{8(L^+\cdot L^-)^2 (L^+\cdot R^-)^2}{P_1^2 P_3^2} + 4\big(L^+\cdot L^-\big)\big(L^+\cdot R^-\big)^2\left[\frac{1}{P_1^2}+\frac{1}{P_3^2}\right] \\
&\quad -\frac{8}{2}\big(L^+\cdot L^-\big)\big(L^+\cdot R^-\big) + \frac{4}{2}\big(L^+\cdot R^-\big)^2 + \big(P_1^2\to P_2^2, P_3^2\to P_4^2\big)\Bigg\} \\
&\quad + \left\{ \frac{P_1^2P_2^2}{4P_3^2} + \frac{P_1^2P_2^2}{4P_4^2} + \frac{P_3^2P_4^2}{4P_1^2} + \frac{P_3^2P_4^2}{4P_2^2} + \big(L^+\cdot R^-\big)\left[\frac{P_4^2}{P_1^2}+\frac{P_1^2}{P_4^2}+\frac{P_2^2}{P_3^2}+\frac{P_3^2}{P_2^2}\right]\right\}.
\end{aligned} \tag{3.2.76}
$$

Here we organized the expression so that the first curly bracket braces all identified local expressions while the second curly bracket braces all nonlocal expressions which remains to be identified. From the symmetric appearance of the nonlocal contribution, it is likely that there is some partial fraction techniques that further reduces these terms into local terms $\propto P_i^{-2}$.

3.2.8 Remarks and Discussions

In this section, we widely applied spinor helicity formalism and tree-level amplitudes presented in the previous section to survey some one-loop techniques. First, we investigated the UV divergence of this amplitude. We learned that the UV divergence of the loop is potentially logarithmic, but with spinor helicity formalism, we may fix a good gauge so that the divergence vanishes. Divergence of an amplitude is gauge-independent; therefore, we learned with spinor helicity that the one-loop light-by-light scattering has no UV divergence. Then, we move onto evaluating the loop. Unlike the vacuum polarization calculation, for the light-by-light scattering, we used an Ansatz that the one-loop amplitude can always be decomposed into some box, triangle, bubble scalar loops, and some rational terms. This reduction scheme is sometimes known as the **Passarino-Veltman reduction** [39], and quite nontrivially, any n-point $4-2\epsilon$-dimensionally regularized one-loop integral can be reduced to these basis functions and a scalar pentagon loop with **van Neerven-Vermaseren basis** [40]. It was then shown that a scalar pentagon can be reduced to a linear combination of boxes [59]. These conclusions from the scalar loop decompositions warrant our Ansatz about the one-loop light-by-light integral.

To use the Ansatz, we must find an approach to isolate coefficients. Thus, we turn our attention to the analytic structure of these scalar loops, and how one can extract these with known results. We leveraged the unitarity condition of the **S-matrix** and found that the imaginary part of the one-loop amplitude corresponds to gluing two tree-level amplitudes with an intermediate on-shell state $|\mu\rangle$. Combining this observation with the form of the propagator, we realize that this means whenever we put a loop momentum on-shell, it forms a product of two tree-level subamplitudes. This is the general idea behind **unitarity cut**. Then, we investigated how one may use **quadruple cut** to find the **leading singularity** of a helicity amplitude. We find that $(--++)$ has a box contribution from (1324) and (1423) ordered amplitude. Then, we also used spinor helicity to fix a gauge so that the $(++++)$ helicity amplitude factorizes into a helicity part and a non-helicity part. With careful prescription and the help of several numerical packages, we find that the $(++++)$ amplitude has only rational contribution without dependence on external momenta. We also provided some observations about the pole structure of the Compton scattering amplitude and the **double cut** approach.

There are several possible extensions to our current results. First, one may seek some useful partial fraction techniques to render all terms in eq'n (3.2.76) local, and isolate the coefficient in front of each box, triangle, and bubble integrals for every ordered amplitude. These will provide useful insights about how one should perform future calculations with the double cut method. Secondly, one may investigate how to leverage the observation about the factorization properties and pole structures of the Compton scattering to isolate loop propagators, hence identifying the coefficient of each scalar loops. This can provide some idea about how one should organize the unitarity cut calculation from quadruple ones down to double ones in a systematic fashion. Thirdly, one may investigate how one can perform a cut in $4-2\epsilon$-dimension with spinor helicity formalism. One of such is called the spinor integration by replacing the Cutkosky rules with spinor variables $\int \mathrm{d}\mu \to \int_0^\infty \mathrm{d}t \int_{l=\tilde{l}} \langle l\, \mathrm{d}l\rangle\, [\mathrm{d}l\, l]$ as proposed in [60, 61]. Another approach with a similar idea is to separate the spinor variables systematically into pieces in different dimensions, called the $\mathcal{Q}$-cut representation, as proposed in [62, 63]. All of these will lead to a much more sophisticated understanding of loop-level amplitudes as well as efficient calculation techniques for them.

Conclusion

Over the course of this document, we get acquainted with spinor helicity formalism and investigated its relation to scattering amplitudes. We learned that while sensible, scattering amplitudes do not transform like Lorentz scalars. Instead, scattering amplitudes will pick up a little group action. This transformation property makes spinor variables suitable for describing scattering amplitudes. At tree level, three-point amplitudes are completely fixed by helicity information of external particles, the special three-point kinematics, and dimensional analysis. There are more constraints on the possible three-point amplitudes if one only accepts massless coupling constants. However, once we enforce that all external particles have real momenta, the three-point amplitudes vanish. At four-point tree level, one may use BCFW on-shell recursion to find the appropriate four-point amplitude by including all possible three-point amplitudes of a given field theory. Once four-point amplitudes are obtained, we may proceed to investigate the factorization properties and the locality condition on these amplitudes. We noticed that the locality condition restricts the possible field theory we may have, showing the fascinating connection between scattering amplitude and field theory. At one-loop level, the standard textbook approach for evaluating loop integrals is to use Feynman shift to render the loop integral symmetric and dimensional regularization to control the divergence of an amplitude. However, we noticed that these techniques could be quite challenging once the amplitude has a large number of external particles. This is especially the case when evaluating the integral over Feynman parameters. Then, we showed how one uses spinor helicity to fix gauge while understanding the divergence of a one-loop amplitude, and how one leverages the unitarity and scalar loop decomposition to calculate the one-loop integral. In the end, we presented some tentative results from a double-cut technique and provided remarks about some future extensions.

4.1 Future Extensions

As mentioned in the previous chapter, at one-loop level, there are quite a few directions which one may develop based on this work. We will direct the readers to the previous chapter for that discussion. It is also true that tree-level amplitudes are not fully investigated. First, although massless external particles can be regarded as high-energy limit behaviors of a massive particle, it is worth an in-depth investigation of how one extends the spinor helicity formalism beyond massless particles. **Massive spinor helicity formalism** is an active field of research with some interesting work done in [19, 20]. In particular, the symmetrized massive spinor helicity formalism presented in [19] has interesting properties in exposing the IR structure of scattering amplitudes and exhibiting "**Higgsing**" as an **IR unification** of massless (UV) scattering amplitudes from this bottom-up on-shell construction. This "Higgsing" IR unification has interesting implications about on-shell formalisms in effective field theories [64, 65, 66, 67] which can be valuable for extending the standard model as an IR effective field theory of some grand unification theory. The particularly "good" leading and subleading soft behavior [68] of the effective field theories makes spinor helicity formalism a great tool for investigating both the UV and the IR regime of many quantum field theories.

Another possible extension is to investigate the implication of conformal symmetries in tree-level amplitudes [69, 70], or even in one-loop amplitudes [71]. As we briefly discussed in chapter 2, spinor helicity can also be used to describe scattering amplitudes in perturbative gravity; thus, it may be interesting to observe how diffeomorphism symmetry of perturbative gravity manifests in spinor helicity. As a potential continuation in this direction, one can also further investigate the **color-kinematics duality** [72] between Yang-Mills theory and gravity, and its underlying **double copy** structure [73].

Last but not least, since spinor variables manifest the helicity information of external particles, they are frequently used in supersymmetric theories. In the process, people observed a dual conformal symmetry [74, 75] called the **Yangian symmetry**. This symmetry leads to a geometric interpretation of scattering amplitude called **Amplituhedron** [76, 77] by the school of amplitudologists or amplitugicians. This geometrization provides an interesting connection between scattering amplitudes, spinor helicity formalism, and combinatorial topics such as **cluster algebras** [78, 79, 80], which opens a new approach to link pure mathematics to theoretical physics.

There are numerous extensions that cannot be exhausted here, and we would like to direct the readers to the last few sections or chapters of the following references [17, 18, 27, 81]. These reviews, notes, or textbooks provide a fascinating amount of outlooks which still lies on the research frontier and beyond our current understanding about scattering amplitudes.

Physicists seek to find an ultimate unified description of the Universe. We wish it is fundamental, simple, and, perhaps, mundane and boring, but even the most fundamental objects like the scattering amplitudes can contain so much that for sure I, and perhaps we, do not understand. It seems that Nature never forgets to reward those who seek simplicity with a treat of simple extravaganza. I hope you, the reader who is reading this concluding remark, has enjoyed this tease of Nature as much as I do and are willing to carry on this curious quest to unveil more about the mystery of scattering amplitudes.

Appendix A

Pauli Matrices

In this appendix, we present some basic calculations about Pauli matrices. Recall that Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.0.1}$$

First, we would like to show by explicit computation that the commutation relation of Pauli matrices satisfies eq'n (1.2.10), *i.e.*

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{A.0.2}$$

To see this, we will check them by matrix multiplications,

$$\begin{aligned}
\left[\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right] &= \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] = \frac{i\sigma_3}{2}, \\
\left[\frac{\sigma_2}{2}, \frac{\sigma_3}{2}\right] &= \frac{1}{4}\left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right] = \frac{i\sigma_1}{2}, \\
\left[\frac{\sigma_3}{2}, \frac{\sigma_1}{2}\right] &= \frac{1}{4}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = \frac{i\sigma_2}{2}.
\end{aligned}$$

Pauli matrices also have interesting anticommutation relation shown in eq'n (1.2.11), *i.e.*

$$\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\delta_{ij}}{2}\mathbb{1}. \tag{A.0.3}$$

This can again be checked by direction computations,

$$\begin{aligned}
\left\{\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right\} &= \frac{1}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] = 0, \qquad &\left\{\frac{\sigma_1}{2}, \frac{\sigma_1}{2}\right\} &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\mathbb{1}}{2}, \\
\left\{\frac{\sigma_2}{2}, \frac{\sigma_3}{2}\right\} &= \frac{1}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right] = 0, \qquad &\left\{\frac{\sigma_2}{2}, \frac{\sigma_2}{2}\right\} &= \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\mathbb{1}}{2}, \\
\left\{\frac{\sigma_3}{2}, \frac{\sigma_1}{2}\right\} &= \frac{1}{4}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] = 0, \qquad &\left\{\frac{\sigma_3}{2}, \frac{\sigma_3}{2}\right\} &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\mathbb{1}}{2}.
\end{aligned}$$

Appendix B

Useful Identities

In this appendix, we collect conventions and useful identities that we used in this paper.

B.1 Conventions

In this paper, we use $(+---)$ signature so that $p^2 = m^2$ for any particle. We also use the natural unit in which $\hbar = c = 1$. For simplicity, all momenta are taken to all outgoing or all incoming so that momentum conservation reads $\sum_i p_i = 0$. For spinor variables, we define raising and lowering of spinor index by the **epsilon symbol**

$$\epsilon^{ab} \triangleq -\epsilon_{ab} \triangleq \left[\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right]_{ab}, \quad \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}}, \tag{B.1.1}$$

and the **spinor product** is simply

$$\langle \lambda_1 \, \lambda_2 \rangle \triangleq \langle 1\, 2 \rangle \triangleq \epsilon^{ab} (\lambda_1)_a (\lambda_2)_b, \quad \left[\tilde{\lambda}_1 \, \tilde{\lambda}_2\right] \triangleq [1\, 2] \triangleq \epsilon^{\dot{a}\dot{b}} \left(\tilde{\lambda}_1\right)_{\dot{a}} \left(\tilde{\lambda}_2\right)_{\dot{b}}, \tag{B.1.2}$$

so that

$$2p^\mu k_\mu = \langle k\, p \rangle \, [p\, k] = [k\, p] \, \langle p\, k \rangle \,, \tag{B.1.3}$$

for massless p and k. **Pauli matrices** in 4D are defined as

$$\sigma^\mu_{a\dot{a}} \triangleq (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)^\mu_{a\dot{a}}, \quad \bar{\sigma}^{\mu\dot{a}a} \triangleq \sigma_{\mu a\dot{a}} = (\mathbb{1}, -\sigma_1, -\sigma_2, -\sigma_3)^{\mu\dot{a}a}. \tag{B.1.4}$$

Sometimes, it is convenient to have **gamma matrices** to convert expressions into four-dimensional bispinor representation. Here gamma matrices are defined as

$$(\gamma^\mu)^{b\dot{b}}_{\dot{a}a} \triangleq \begin{pmatrix} & \sigma^\mu_{\dot{a}a} \\ \bar{\sigma}^{\mu b\dot{b}} & \end{pmatrix} \tag{B.1.5}$$

so that spinor variables can be mapped to bispinor representation via

$$\lambda_a = \begin{pmatrix} 0 & \langle \lambda|_a \end{pmatrix}, \quad \tilde{\lambda}_{\dot{a}} = \begin{pmatrix} [\lambda|_{\dot{a}} & 0 \end{pmatrix}, \quad \lambda^a = \begin{pmatrix} 0 \\ |\lambda\rangle_a \end{pmatrix}, \quad \tilde{\lambda}^{\dot{a}} = \begin{pmatrix} |\lambda]^{\dot{a}} \\ 0 \end{pmatrix}. \tag{B.1.6}$$

B.2 Spinor Helicity Identities

The antisymmetric epsilon symbol in a spinor product tells us that

$$\langle 1\,2\rangle = -\langle 2\,1\rangle\,, \quad [1\,2] = -[2\,1]\,, \quad \langle 1\,1\rangle = [1\,1] = 0. \tag{B.2.1}$$

This also enforces the **Schouten identity**

$$\begin{aligned} 0 &= |1\rangle\,\langle 2\,3\rangle + |2\rangle\,\langle 3\,1\rangle + |3\rangle\,\langle 1\,2\rangle = |i\rangle\,\langle j\,k\rangle + (\text{cyclic in } i,j,k) \\ &= |i]\,[j\,k] + (\text{cyclic in } i,j,k) = \langle i\,j\rangle\,\langle k| + (\text{cyclic in } i,j,k) = [i\,j]\,[k| + (\text{cyclic in } i,j,k). \end{aligned} \tag{B.2.2}$$

One also can obtain the **Fierz identity** in spinor variables

$$\langle 1|\,\gamma^\mu\,|2]\,\langle 3|\,\gamma_\mu\,|4] = 2\,\langle 1\,3\rangle\,[4\,2]\,. \tag{B.2.3}$$

We can define the momentum in spinor variables by

$$p^{a\dot a} = p^\mu \sigma_\mu^{a\dot a} \tag{B.2.4}$$

so that $\det p = p^2 = m^2$. But if the particle is massless, then we may decompose

$$p = |p\rangle\,[p| = |p]\,\langle p|\,. \tag{B.2.5}$$

If $p_i^2 = 0$ for all i and $\sum_i p_i = 0$, then the **conservation of momentum** can be translated into spinor language via

$$\sum_{j\neq i,k} \langle i|\,j\,|k] = 0 = \sum_{j\neq i,k} [i|\,j\,|k\rangle\,, \tag{B.2.6}$$

in which

$$\langle i|\,j\,|k] \triangleq \langle i\,j\rangle\,[j\,k]\,, \quad [i|\,j\,|k\rangle \triangleq [i\,j]\,\langle j\,k\rangle\,. \tag{B.2.7}$$

In general, $|i\rangle$ and $|i]$ can be in $\mathbb{C}^2$. But if we enforce the condition that p^μ is real or that $p^{a\dot a}$ is Hermitian, then we find the **reality condition**

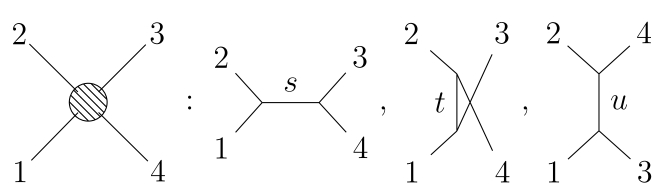
$$|p\rangle = [p|^*\,, \quad \langle p| = |p]^*\,. \tag{B.2.8}$$

B.3 Kinematics

In a three-point process, which is allowed if we complexify the momenta, we derived the **three-point special kinematics**

$$|1\rangle \propto |2\rangle \propto |3\rangle \quad \text{or} \quad |1] \propto |2] \propto |3]\,. \tag{B.3.1}$$

In a four-point process, we label particles by



in which s, t, u are **Mandelstam variables** defined as

$$s \triangleq (p_1+p_2)^2 = (p_3+p_4)^2, \quad t \triangleq (p_1+p_3)^2 + (p_2+p_4)^2, \quad u \triangleq (p_1+p_4)^2 = (p_2+p_3)^2. \tag{B.3.2}$$

Their sum satisfies an interesting relation

$$\begin{aligned} s+t+u =& (p_1+p_2)^2 + (p_1+p_3)^2 + (p_1+p_4)^2 \\ =& 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (p_2+p_3+p_4) \\ =& p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot \cancelto{0}{(p_1+p_2+p_3+p_4)} \\ =& m_1^2 + m_2^2 + m_3^2 + m_4^2, \end{aligned} \tag{B.3.3}$$

and when external particles are all massless, this reduces $s+t+u=0$.

B.4 Numerator Algebras

To generalize the algebraic manipulations of gamma matrices into D dimensions, we demand that $g_\mu{}^\mu = D$. Then, we have the following identities

$$\gamma^\alpha \gamma_\alpha = D, \tag{B.4.1}$$

$$\gamma^\alpha \gamma^\mu \gamma_\alpha = (2-D)\gamma^\mu, \tag{B.4.2}$$

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} - (4-D)\gamma^\mu\gamma^\nu, \tag{B.4.3}$$

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\alpha = (4-D)\gamma^\mu\gamma^\nu\gamma^\rho - 2\gamma^\rho\gamma^\nu\gamma^\mu. \tag{B.4.4}$$

Let's show these identities. Recall that the gamma matrices are defined by the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4, \implies \gamma^\nu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu. \tag{B.4.5}$$

Therefore, for eq'n (B.4.1),

$$\gamma^\alpha\gamma_\alpha = \frac{1}{2}g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = \frac{1}{2}g_{\mu\nu}2g^{\mu\nu} = D;$$

for eq'n (B.4.2),

$$\gamma^\alpha\gamma^\mu\gamma_\alpha = (2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)\gamma_\alpha = (2-D)\gamma^\mu;$$

for eq'n (B.4.3),

$$\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = (2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)(2g_\alpha{}^\nu - \gamma_\alpha\gamma^\nu) = 4g^{\mu\nu} - 4\gamma^\mu\gamma^\nu + D\gamma^\mu\gamma^\nu = 4g^{\mu\nu} - (4-D)\gamma^\mu\gamma^\nu;$$

for eq'n (B.4.4)

$$\begin{aligned} & \gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha \\ =& (2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha)\gamma^\nu\gamma^\rho\gamma_\alpha \\ =& 2\gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\mu(4g^{\nu\rho} - (4-D)\gamma^\nu\gamma^\rho) \\ =& (4-D)\gamma^\mu\gamma^\nu\gamma^\rho - 2(2g^{\nu\rho}\gamma^\mu - \gamma^\nu\gamma^\rho\gamma^\mu) \\ =& (4-D)\gamma^\mu\gamma^\nu\gamma^\rho - 2\gamma^\rho\gamma^\nu\gamma^\mu. \end{aligned}$$

With the anticommutation relation and the cyclic symmetry of trace, we can also find that

$$\operatorname{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}, \tag{B.4.6}$$

$$\operatorname{Tr}(\text{odd number of } \gamma) = 0, \tag{B.4.7}$$

$$\operatorname{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma}. \tag{B.4.8}$$

To show eq'n (B.4.6), we just need to invoke the anticommutation relation once

$$\operatorname{Tr}(\gamma^\mu\gamma^\nu) = \operatorname{Tr}(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu) = 2g^{\mu\nu}\operatorname{Tr}(\mathbb{1}) - \operatorname{Tr}(\gamma^\nu\gamma^\mu) \implies 2\operatorname{Tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu}\cdot 4;$$

for eq'n (B.4.8),

$$\begin{aligned}
&\operatorname{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \\
=&\operatorname{Tr}[(2g^{\mu\nu}\mathbb{1} - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma] \\
=&2g^{\mu\nu}\operatorname{Tr}[\gamma^\rho\gamma^\sigma] - \operatorname{Tr}[\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma] \\
=&2g^{\mu\nu}\operatorname{Tr}[\gamma^\rho\gamma^\sigma] - 2g^{\mu\rho}\operatorname{Tr}[\gamma^\nu\gamma^\sigma] + \operatorname{Tr}[\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma] \\
=&2g^{\mu\nu}\operatorname{Tr}[\gamma^\rho\gamma^\sigma] - 2g^{\mu\rho}\operatorname{Tr}[\gamma^\nu\gamma^\sigma] + 2g^{\mu\sigma}\operatorname{Tr}[\gamma^\nu\gamma^\rho] - \operatorname{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu] \\
=&2g^{\mu\nu}4g^{\rho\sigma} - 2g^{\mu\rho}4g^{\nu\sigma} + 2g^{\mu\sigma}4g^{\nu\rho} - \operatorname{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] \\
\implies& 2\operatorname{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 2(4g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\nu}g^{\rho\sigma}).
\end{aligned}$$

It is straightforward to generalize this proof to trace of arbitrary even number of gamma matrices

$$\operatorname{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_{2n}}] = \sum_{k=2}^{2n}(-1)^k g^{\mu_1\mu_k}\operatorname{Tr}\left[\gamma^{\mu_2}\gamma^{\mu_3}\dots\gamma^{\mu_{k-1}}\cancelto{\mathbb{1}}{\gamma^{\mu_k}}\gamma^{\mu_{k+1}}\dots\gamma^{\mu_{2n}}\right], \tag{B.4.9}$$

in which $\cancelto{\mathbb{1}}{\gamma^{\mu_k}}$ means to replace the k^{th} gamma matrix with an identity matrix or, equivalently, remove the k^{th} gamma matrix from the product. Note that this equation can be quite cumbersome due to its recursive nature. For a trace of $2n$ gamma matrices, it produces $1 \times 3 \times \ldots \times (2n-1) = (2n-1)!!$ terms.

B.5 Loop Integral Techniques

To reparameterize the denominator, one may use **Feynman shift** given by the equality

$$\frac{1}{A_1A_2\dots A_n} = (n-1)!\int_0^1 \mathrm{d}x_1\dots\int_0^1 \mathrm{d}x_n\left[\sum_{i=1}^n x_iA_i\right]^{-n}\delta\left(\sum_{i=1}^n x_i - 1\right) \tag{B.5.1}$$

to manipulate the denominator into the form $(k^2+\Delta)^n$. Then, performing the integral in $D = 4-2\epsilon$ dimension with dimensional regularization, we find

$$\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon)}{\Delta^\epsilon}, \tag{B.5.2}$$

$$\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{k^2}{(k^2-\Delta)^2} = -\frac{i(2-\epsilon)}{(4\pi)^{2-\epsilon}}\frac{\Gamma(\epsilon-1)}{\Delta^{\epsilon-1}} + \mathcal{O}(\epsilon^3). \tag{B.5.3}$$

When integrating over a spherically symmetric region in the k space, it is frequently useful to prescribe tensorial expressions in k with metric tensors by

$$k^\mu k^\nu \equiv \frac{k^2 g^{\mu\nu}}{D}, \quad k^\mu k^\nu k^\rho k^\sigma \equiv \frac{k^4}{D(D+2)}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\sigma\nu} + g^{\mu\sigma}g^{\nu\rho}). \tag{B.5.4}$$

The coefficient in front can be found via dimensional analysis and contraction with metric tensors. As a demonstration, consider the Ansatz $k^\mu k^\nu k^\rho k^\sigma = Ck^4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\sigma\nu} + g^{\mu\sigma}g^{\nu\rho})$ with the correct dimension and tensorial structure. We may find C by multiplying $g_{\mu\nu}g_{\rho\sigma}$ to both sides of the equation

$$k^4 \stackrel{!}{=} Ck^4\big(g^\mu{}_\mu g^\nu{}_\nu + 2g_{\mu\nu}g^{\mu\nu}\big) = Ck^4\big(D^2 + 2D\big) \implies C = [D(D+2)]^{-1}.$$

Appendix C

Mathematical Remarks on Spinor Helicity Formalism

In this appendix, we collect some known formal structures of spinor helicity formalism (for massless particles). We hope this review provides more mathematical backgrounds on how one may develop spinor helicity formalism and how one can extend this formalism into one for massive eigenstates.

In spinor helicity formalism, we map four vectors to a 2×2 matrix by

$$\begin{aligned} \mathbb{R}^{1,3} &\longrightarrow M_2(\mathbb{C}) \\ p^\mu &\longmapsto \not{p} = p^\mu \sigma_\mu^{a\dot{a}}, \end{aligned} \tag{C.0.1}$$

and Lorentz transformations are represented by its Lie algebra $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. However, just like what we observed in chapter 1, we can set the coefficients in front of the Lorentz generators to some complex number for the sake of analytic continuation. This process is known as the **complexification** of a Lie algebra. Formally, taking a coefficient from real to complex is the same as taking a tensor product on the original algebra (which is a vector space). So the **complexification** of a Lie algebra $\mathfrak{g}$ is defined as $\mathfrak{g}_\mathbb{C} \triangleq \mathfrak{g} \otimes \mathbb{C} \cong \mathfrak{g} \oplus i\mathfrak{g}$ so that any generator $c_R J$ with a real coefficient $c_R \in \mathbb{R}$ can be extended to a complex generator as $c_C J = \mathrm{Re}\{c_C\} J + \mathrm{Im}\{c_C\} iJ$ with some complex coefficient $c_C \in \mathbb{C}$. The complexified Lorentz algebra still has the same direct sum decomposition since the commutators remain unchanged. Therefore, $\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. The last equality comes from the identification that $\mathfrak{su}(2)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$, and some enlightening discussion regarding this can be found in [82].

In the correspondence of Lie groups and Lie algebras, one may find the following important claim.

Claim C.1 *If G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ and H is a closed subgroup of $\mathrm{GL}(m, \mathbb{C})$, then the Lie algebra of $G \times H \leq \mathrm{GL}(n+m,\mathbb{C})$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$ in which $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebra of G and H respectively. With some representation of G denoted as $\rho_G : G \to \mathrm{GL}(V_G)$ and some representation of H denoted as $\rho_H : H \to \mathrm{GL}(V_H)$, one can find a **tensor product of Lie groups' representations***

$$\begin{aligned} \rho_G \otimes \rho_H : \quad G \times H \quad &\longrightarrow \quad \mathrm{GL}(V_G \otimes V_H) \\ (A, B) \quad &\longmapsto \quad \rho_G(A) \otimes \rho_H(B). \end{aligned} \tag{C.0.2}$$

Correspondingly, one can find a Lie algebra representation of $\mathfrak{g}$ *(or* $\mathfrak{h}$*) denoted as* $\varrho_{\mathfrak{g}(or\ \mathfrak{h})}$: $\mathfrak{g}(or\ \mathfrak{h}) \to \mathfrak{gl}(V_G(or\ V_H))$*. Then, the* ***tensor product of Lie algebras' representations*** *is found to be*

$$\begin{aligned} \varrho_{\mathfrak{g}} \otimes \varrho_{\mathfrak{h}}: \quad \mathfrak{g} \oplus \mathfrak{h} &\longrightarrow \qquad \mathfrak{gl}(V_G \otimes V_H) \\ (\mathfrak{a}, \mathfrak{b}) &\longmapsto \quad \varrho_{\mathfrak{g}}(\mathfrak{a}) \otimes \mathbb{1} + \mathbb{1} \otimes \varrho_{\mathfrak{h}}(\mathfrak{b}). \end{aligned} \tag{C.0.3}$$

so that we may find the following diagram

$$\begin{tikzcd} G \times H \arrow[r, "\rho_G \otimes \rho_H"] \arrow[d, "\mathrm{d}"] & \mathrm{GL}(V_G \otimes V_H) \\ \mathfrak{g} \oplus \mathfrak{h} \arrow[r, "\varrho_{\mathfrak{g}} \otimes \varrho_{\mathfrak{h}}"] & \mathfrak{gl}(V_G \otimes V_H) \arrow[u, "\exp"'] \end{tikzcd}$$

Since $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we find that $\mathrm{SO}(1,3) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Note that $\mathfrak{so}(1,3)$ has six generators, and correspondingly, $\mathfrak{su}(2)$ has three generators. Consider the action of $\mathrm{SO}(1,3)$ on a four-dimensional vector space $\mathbb{R}^{1,3}$. This is precisely the four-vector representation of the Lorentz group, and we can decompose it into a tensor product of two copies of two-dimensional representations of SU(2), *i.e.* spin-1/2 representation. In this case, V_G can be identified as the space of left-handed spinors, and V_H can be the space of right-handed spinors. With Clebsch-Gordan decomposition, we find that $1/2 \otimes 1/2 \cong 0 \oplus 1$, which warrants us to seek both Lorentz invariants and Lorentz covariants.

However, just like in quantum mechanics, it is easier to analytically continue the spinor space.[1] With this continuation, we find that $\mathrm{SO}(1,3)_{\mathbb{C}} \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, and any complexified Lorentz transformation will act on bispinors individually as

$$\Lambda^{\mu}{}_{\nu} p^{\nu} \longmapsto \rho(\Lambda)_{ba} \not{p}^{a\dot{a}} \tilde{\rho}(\tilde{\Lambda})_{\dot{a}\dot{b}}, \tag{C.0.4}$$

in which $\rho(\Lambda)$ and $\tilde{\rho}(\tilde{\Lambda})$ are two general matrices with determinant 1, *i.e.* they are individually in $\mathrm{SL}(2,\mathbb{C})$. Note that $\mathrm{SO}(1,3)_{\mathbb{C}}$ has six complex dimensions, *i.e.* twelve real dimensions, and correspondingly, $\mathrm{SL}(2,\mathbb{C})$ has three complex dimensions, *i.e.* six real dimensions. Because the action looks like a matrix multiplication of the form "$\mathrm{SL}(2,\mathbb{C})\ \not{p}\ \mathrm{SL}(2,\mathbb{C})$", the only Lorentz invariant quantity formed by $\not{p}$ is $\det(\not{p}) = \det[$"$\mathrm{SL}(2,\mathbb{C})\ \not{p}\ \mathrm{SL}(2,\mathbb{C})$"$]$. On the other hand, the Lorentz invariant quantity in $\mathbb{R}^{1,3}$ is p^2. Therefore, we identify $\det(\not{p}) \propto p^2$. The massless spinor must have determinant 0 in $V_G \otimes V_H$. This means that $\not{p}$ is a rank-1 tensor or that $\not{p}_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ is a simple tensor. When $p^2 \neq 0$, we must promote $\not{p}$ to a rank-2 tensor or that $\not{p}_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}} + \mu_a \tilde{\mu}_{\dot{a}}$. Since $\not{p}$ is a 2×2 matrix, we know that the maximal rank of $\not{p}$ is 2, *i.e.* it suffices to express any momentum with two bispinors.

Taking p^{μ} from $\mathbb{C}^4$ back to $\mathbb{R}^{1,3}$, we realize that $\not{p}$ must be a Hermitian matrix since σ^{μ} are all Hermitian. The Hermiticity of $\not{p}$ implies that $\tilde{\lambda} = \lambda^*$ which is the **reality condition**. The reality condition, in turn, constrains the action of real Lorentz groups, *viz.* $\mathrm{SO}(1,3) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Note that there is a natural embedding from $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$

[1] For example, the spin-up state is denoted as $|\uparrow\rangle = (1,0)^T$, and the spin-down state is denoted as $|\downarrow\rangle = (0,1)^T$. But the eigenstate of the y-projection, represented by the Pauli matrix σ_y, is a $\mathbb{C}$ linear combination of spin-up and spin-down states, *viz.* $|+_y\rangle = (|\uparrow\rangle + i\,|\downarrow\rangle)/\sqrt{2}$.

so that we can identify ρ and $\tilde{\rho}$ in each copy of SU(2). However, $\not{p}$ remains Hermitian in any frame, or explicitly,

$$\rho\not{p}\tilde{\rho} \stackrel{!}{=} (\rho\not{p}\tilde{\rho})^\dagger = \tilde{\rho}^\dagger\not{p}^\dagger\rho^\dagger = \tilde{\rho}^\dagger\not{p}\rho^\dagger, \tag{C.0.5}$$

which is possible only if $\tilde{\rho}^\dagger = \rho$. This gives how the two copies of SU(2) acts on $\not{p}$. Then, the little group of $\not{p}$ is represented by the set of matrices $\{\rho \in \mathrm{SU}(2) \mid \not{p} = \rho\not{p}\rho^\dagger = \rho\not{p}\rho^{-1}\}$. For a massless particle, this is the set $\{e^{i\theta}\mathbb{1} \in \mathrm{GL}(2,\mathbb{C}) \mid \theta \in \mathbb{R}\} \cong \mathrm{U}(1)$. For a massive particle, one can mixes λ and μ by ρ, and sometimes, this mixing is explicitly carried out by a little group index as discussed in [19]. Sometimes, it is useful to have a representation of the operators on the function space of spinor variables for the investigation of scattering amplitudes in spinor helicity formalism [69, 83]. For a massless momentum, these operators are

$$P_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}, \quad J_{ab} = \frac{1}{2}\left(\lambda_a\frac{\partial}{\partial\lambda^b} + \lambda_b\frac{\partial}{\partial\lambda^a}\right), \qquad \tilde{J}_{\dot{a}\dot{b}} = \frac{1}{2}\left(\tilde{\lambda}_{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}^{\dot{b}}} + \tilde{\lambda}_{\dot{b}}\frac{\partial}{\partial\tilde{\lambda}^{\dot{a}}}\right). \tag{C.0.6}$$

so that the Lorentz invariance of the scattering amplitude translates to an eigenproblem

$$J\mathcal{A}(\lambda,\tilde{\lambda}) = 0 = \tilde{J}\mathcal{A}(\lambda,\tilde{\lambda}). \tag{C.0.7}$$

This condition, along with the helicity condition, gives a logarithmic derivative on all the spinor products [34], and this equation restricts three-point amplitudes to take the form shown in eq'n (2.1.9).

In recapitulation, from the isomorphism $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and its complexification $\mathfrak{so}(1,3)_\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, we find that $\mathrm{SO}(1,3) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SO}(1,3)_\mathbb{C} \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. To ensure that we can utilize the exponential map to map from the Lie algebra to the Lie group, we use the tensor product of the representations. Since the tensor product of irreps is frequently not an irrep, we use the Clebsch-Gordan decomposition to find a spin-1 piece (four-vectors) and a spin-0 (Lorentz invariants). The complexification permits us to see that the only Lorentz invariant quantity formed by $\not{p}$ is $\det(\not{p})$. The Hermiticity of $\not{p}$ when $p^\mu \in \mathbb{R}^{1,3}$ imparts us the form of the group action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\not{p}$, and fixing the condition on $\det(\not{p})$ restricts the little group actions on $\not{p}$ to U(1). Finally, identifying the operators on the scattering amplitudes in terms of spinor variables yields a constraint on the form of the amplitude.

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