Homework 1

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1 Problem 1

$X \sim \mathcal{N}(5,4) \Rightarrow f_X(x) = \frac{1}{4\sqrt{2\pi}}exp[\frac{-(x-5)^2}{16}]$

i. $P[X \leq 6] = \int_{-\infty}^{6} f_X(x)dx$
$\approx$

ii. $P[X > 4] = \int_{4}^{\infty} f_X(x)dx$
$\approx$

iii. $P[|X-5| > 1] = \int_{|x-5|>1} f_X(x)dx = \int_{x-5>1} f_X(x)dx + \int_{x-5<-1} f_X(x)dx$
$= \int_{x>6} f_X(x)dx + \int_{x<4} f_X(x)dx$
$= \int_{6}^{\infty} f_X(x)dx + \int_{-\infty}^{4} f_X(x)dx$
$= (1 - P[X < 6]) \ + \ (1 - P[X > 4])$
$\approx$

2 Problem 2

$n \ = \ 10, \ p = q = \frac{1}{2}$

i. P[beat market all 10 years] $= p^n = (\frac{1}{2})^{10} = \frac{1}{1024} \approx 0.977\%$

ii. P[at least 1 beats market all 10 years] $= 1 - P$[no one beat market all 10 years]
$= 1 - ((\binom{4170}{0})(0.977\%)^0(1-0.977\%)^{4170})$
$\approx 1 - 0 = 1$

Out of 4170 funds it is virtually guaranteed that at least one will beat the market all 10 years.

iii. P[at least 5 funds beats market all 10 years] $= \sum_{i=5}^{4170}\binom{4170}{i}(0.977\%)^i(1-0.977\%)^{4170-i}$
$\approx$

3 Problem 3

Y_i are i.i.d, $E[Y_i] = \mu$, $Var[Y_i] = \sigma^2$ and $\overline{Y} = \frac{1}{4}\sum_{i=1}^{4} Y_i$

i. Calculate $E[\overline{Y}]$ and $Var[\overline{Y}]$
$E[\overline{Y}] = E[\frac{1}{4}\sum_{i=1}^{4} Y_i] = \frac{1}{4}\sum_{i=1}^{4} E[Y_i] = \frac{1}{4}(4\mu)$
$= \mu$
$Var[\overline{Y}] = Var[\frac{1}{4}\sum_{i=1}^{4} Y_i] = (\frac{1}{4})^2 \sum_{i=1}^{4} Var[Y_i]$ $\quad\because$ Y_i are i.i.d
$= \frac{1}{16}(4\sigma^2) = \frac{1}{4}\sigma^2$

ii. $W = \frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{4}Y_3 + \frac{1}{2}Y_4$ calculate $E[W]$ and $Var[W]$
$E[W] = E[\frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{4}Y_3 + \frac{1}{2}Y_4] = \frac{1}{8}E[Y_1] + \frac{1}{8}E[Y_2] + \frac{1}{4}E[Y_3] + \frac{1}{2}E[Y_4]$
$= \frac{1}{8}\mu + \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{2}\mu$
$= \mu \quad \Rightarrow$ W is an unbiased estimator of μ $Var[W] = Var[\frac{1}{8}Y_1 + \frac{1}{8}Y_2 + \frac{1}{4}Y_3 + \frac{1}{2}Y_4]$
$= (\frac{1}{8})^2 Var[Y_1] + (\frac{1}{8})^2 Var[Y_2] + (\frac{1}{4})^2 Var[Y_3] + (\frac{1}{2})^2 Var[Y_4]$ $\quad\because$ Y_i are i.i.d
$= (\frac{1}{8})^2\sigma^2 + (\frac{1}{8})^2\sigma^2 + (\frac{1}{4})^2\sigma^2 + (\frac{1}{2})^2\sigma^2$
$= \frac{11}{32}\sigma^2 > \frac{1}{4}\sigma^2$

iii. $\overline{Y}$ would be a better estimator of μ since its variance is less than that of W

4 Problem 4

Consider Y_i, $1 \le i \le n$, and $E[Y_i] = \mu$, $Var[Y_i] = \sigma^2$, and $Cov[Y_i, Y_j] = 0$ for $i \ne j$

i. Define $W_a = \sum_{i=1}^{n} a_i Y_i$
$\Rightarrow E[W] = E[\sum_{i=1}^{n} a_i Y_i] = \sum_{i=1}^{n} E[a_i Y_i] = \sum_{i=1}^{n} a_i E[Y_i]$
$= \sum_{i=1}^{n} a_i \mu = \mu \sum_{i=1}^{n} a_i$
W is an unbiased estimator of $\mu \iff E[W] = \mu \Rightarrow \mu \sum_{i=1}^{n} a_i = \mu$
$\Rightarrow \sum_{i=1}^{n} a_i = 1$ and $\mu \ne 0$

ii. Find $Var[W]$
$Var[W] = Var[\sum_{i=1}^{n} a_i Y_i] = \sum_{i=1}^{n} Var[a_i Y_i]$ $\quad\because Cov[Y_i, Y_j] = 0$ for $i \ne j$
$= \sum_{i=1}^{n} a_i^2 Var[Y_i] = \sum_{i=1}^{n} a_i^2 \sigma^2$
$= \sigma^2 \sum_{i=1}^{n} a_i^2$

iii. Give $\frac{1}{n}(\sum_i a_i)^2 \le \sum_i a_i^2$ show $\forall a \; s.t \; E[W] = \mu$, $Var[W_a] \ge Var[\overline{Y}]$
$Var[W_a] = \sigma^2 \sum_{i=1}^{n} a_i^2 \ge \frac{(\sum_{i=1}^{n} a_i)^2}{n}\sigma^2 = \frac{1}{n}\sigma^2$
$Var[\overline{Y}] = Var[\frac{1}{n}\sum_{i=1}^{n} Y_i] = \frac{1}{n^2}\sum_{i=1}^{n} Var[Y_i] = \frac{1}{n^2} n\sigma^2 = \frac{1}{n}\sigma^2$
$\Rightarrow \; Var[W_a] \ge Var[\overline{Y}]$

5 Problem 5

Let $\overline{Y} = \frac{1}{n}\sum_i X_i$ where $E[X_i] = \mu$ and $Var[X_i] = \sigma^2$
Consider:
$W_1 = [\frac{n-1}{n}]\overline{Y}$
$W_2 = \frac{\overline{Y}}{2}$

i. Show W_1 and W_2 are both biased estimators, and take limit $n \to \infty$
$E[W_1] = E[[\frac{n-1}{n}]\overline{Y}] = [\frac{n-1}{n}]E[\overline{Y}]$
$= [\frac{n-1}{n}]\mu$
$\Rightarrow \mu - E[W_1] = \frac{1}{n}\mu$
$\Rightarrow \lim_{n\to\infty}(\mu - E[W_1]) = 0$
W_1 approaches an unbiased estimator of μ as the sample size becomes large
$E[W_2] = E[\frac{\overline{Y}}{2}] = \frac{E[\overline{Y}]}{2}$
$= \frac{\mu}{2}$
$\Rightarrow \mu - E[W_2] = \frac{1}{2}\mu$
$\Rightarrow \lim_{n\to\infty}(\mu - E[W_2]) = \frac{\mu}{2}$
As sample size gets large, W_2 remains a biased estimator of μ with constant bias

ii. Find probability limits of W_1 and W_2
$plim_{n\to\infty} W_1 = \lim_{n\to\infty} E[W_1] = \lim_{n\to\infty}[\frac{n-1}{n}]\mu$
$= \mu$
$plim_{n\to\infty} W_2 = \lim_{n\to\infty} E[W_2] = \lim_{n\to\infty} \frac{\mu}{2}$
$= \frac{\mu}{2}$
$\Rightarrow W_1$ is a consistent estimator of μ

iii. Find $Var[W_1]$ and $Var[W_2]$
$Var[W_1] = Var[[\frac{n-1}{n}]\overline{Y}] = [\frac{n-1}{n}]^2 Var[\overline{Y}]$
$= [\frac{n-1}{n}]^2 \frac{\sigma^2}{n}$
$Var[W_2] = Var[\frac{\overline{Y}}{2}]$
$= \frac{1}{4}\frac{\sigma^2}{n}$

iv. Consider when μ is close to 0
$\lim_{\mu\to 0} E[W_1] = 0 = \lim_{\mu\to 0} E[\overline{Y}]$
$Var[W_1] = [\frac{n-1}{n}]^2 \frac{\sigma^2}{n} = [\frac{n-1}{n}]^2 Var[\overline{Y}] < Var[\overline{Y}]$
$\Rightarrow$ For finite number of samples, as μ approaches 0, the expected values of W_1 and $\overline{Y}$ both approach 0, but the variance of W_1 is bounded above by variance of $\overline{Y}$. Therefore W_1 is a better estimator of μ than $\overline{Y}$

6 Problem 6

$X, Y > 0$ and $E[Y|X] = \theta X$

i. Define $Z = \frac{Y}{X}$, show $E[Z] = \theta$

Let $a(X) = \frac{1}{X}$ and $b(X) = 0$, by property (1) $\Rightarrow E[Z|X] = E[\frac{Y}{X}|X] = \frac{1}{X}E[Y|X]$

$= \frac{1}{X}\theta X = \theta$

By property (2) $\Rightarrow E[Z] = E[E[Z|X]] = E[\theta] = \theta$

ii. Define $W_1 = \frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{X_i}$, show W_1 is unbiased for θ

$E[W_1] = E[\frac{1}{n}\sum_{i=1}^{n}\frac{Y_i}{X_i}] = \frac{1}{n}\sum_{i=1}^{n}E[\frac{Y_i}{X_i}]$

$= \frac{1}{n}\sum_{i=1}^{n}\theta \qquad \because$ (i)

$= \theta \quad \Rightarrow \quad W_1$ is an unbiased estimator of θ

iii.Define $W_2 = \frac{\overline{Y}}{\overline{X}}$, show $W_2 \neq W_1$ but W_2 is an unbiased estimator of θ

$W_2 = \frac{\frac{1}{n}\sum_i Y_i}{\frac{1}{n}\sum_i X_i} = \frac{\sum_i Y_i}{\sum_i X_i} \neq W_1$

7 Problem 7

Consider Y, a Bernoulli random variable $0 < \theta < 1$, let $\gamma = \frac{\theta}{1-\theta}$ For $\{Y_i | 1 \leq i \leq n\}$, define $G = \frac{\overline{Y}}{1-\overline{Y}}$

i. Why is G not an unbiased estimator of γ

$E[\gamma - G] = E[\frac{\theta}{1-\theta} - \frac{\overline{Y}}{1-\overline{Y}}] = E[\frac{\theta(1-\overline{Y})-\overline{Y}(1-\theta)}{(1-\theta)(1-\overline{Y})}]$

$= E[\frac{\theta-\overline{Y}}{(1-\theta)(1-\overline{Y})}] \qquad$ (a)

$= E[\frac{\theta}{(1-\theta)(1-\overline{Y})}] - E[\frac{\overline{Y}}{(1-\theta)(1-\overline{Y})}] \qquad$ (b)

$\Rightarrow \lim_{\theta\to 0+} E[\gamma - G] = E[\frac{\overline{Y}}{1-\overline{Y}}] > 0$

and $\quad \lim_{\theta\to 1-} E[\gamma - G] = \infty > 0$

$\Rightarrow G$ is not an unbiased estimator of γ

ii. Show that G is a consistent estimator of γ

Let $X_n = \overline{Y}_n = \frac{1}{n}\sum_i Y_i \quad \Rightarrow plim(X_n) = \theta$

Let $Z_n = 1 - \overline{Y}_n = 1 - \frac{1}{n}\sum_i Y_i \quad \Rightarrow plim(Z_n) = 1 - \theta$

$\because plim(X_n) = \alpha$ and $plim(Z_n) = \beta \quad \Rightarrow \quad plim(\frac{X_n}{Z_n}) = \frac{\alpha}{\beta}$

$\Rightarrow plim(G) = \frac{\theta}{1-\theta}$

$\Rightarrow$ G is a consistent estimator of γ

8 Problem 8

Consider the survey as a Bernoulli trial with $p = 0.65,\ q = 0.35,\ n = 200$

i. Find $E[X]$
$E[X] = np = 130$

ii. Find $\sigma[X]$
$\sigma[X] = \sqrt{npq} = 6.75$

iii. Only 115 people of the sample voted yes
$\frac{130-115}{6.75} = 2.22$
The number of people who voted in the sampled population is about 2.22 standard deviations below expected$\Rightarrow P[X \leq 115] = \int_{-\infty}^{-2.22} N(0,1)dx$