Q.1 $y \cdot z = [3 \ \ 1]\begin{bmatrix}3\\2\end{bmatrix} = 9+2 = 11$

$Xy = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$, $z^T X = [3 \ \ 2]\begin{bmatrix}1&2\\2&1\end{bmatrix} = [7 \ \ 8]$

$X^{-1} = \frac{1}{det(X)}\begin{pmatrix}1&-2\\-2&1\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}1&-2\\-2&1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$

$X = \begin{pmatrix}1&2\\2&1\end{pmatrix} \to \begin{bmatrix}1&2\\0&-1\end{bmatrix} \to \begin{bmatrix}1&2\\0&1\end{bmatrix} \to \begin{bmatrix}1&0\\0&1\end{bmatrix}$, $rank(X) = 2$

$det(X) = \begin{vmatrix}1&2\\2&1\end{vmatrix} = 1-4 = -3$

$det(\lambda I - X) = 0$

$\left|\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} - \begin{bmatrix}1&2\\2&1\end{bmatrix}\right| = 0 \Rightarrow \begin{vmatrix}\lambda-1&-2\\-2&\lambda-1\end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 - 4 = 0$

$\lambda = \pm 2 + 1$

$\lambda_1 = 3$ $\quad \Lambda = \begin{bmatrix}3&\\&-1\end{bmatrix}$

$\lambda_2 = -1$

$XV = \Lambda V$

$X - 3I = \begin{bmatrix}1&2\\2&1\end{bmatrix} - \begin{bmatrix}3&0\\0&3\end{bmatrix}$

$= \begin{bmatrix}-2&2\\2&-2\end{bmatrix} \to \left[\begin{array}{cc|c}-2&2&0\\2&-2&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-1&0\\2&-2&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]$

$x_1 - x_2 = 0$ s.t. $x_1^2 + x_2^2 = 1 \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}$

$X + I = \begin{bmatrix}1&2\\2&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$= \begin{bmatrix}2&2\\2&2\end{bmatrix}$

$\begin{bmatrix}2&2&0\\2&2&0\end{bmatrix}$

$\to \begin{bmatrix}1&1&0\\0&0&0\end{bmatrix} \to x_1 + x_2 = 0$

s.t. $x_1^2 + x_2^2 = 1$

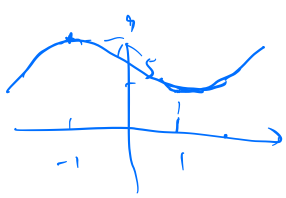
$\Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}$

$\Rightarrow X = \begin{bmatrix}\frac{\sqrt{2}}{2}&-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{bmatrix}\begin{bmatrix}3&0\\0&-1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\end{bmatrix}$

Q.2 $\frac{df}{dx} = 3x^2 - 3$, $\quad f(x) = x^3 - 3x + 7$.

$\frac{df}{dx} = 0 \Rightarrow 3x^2 = 3 \Rightarrow \begin{cases}x = 1\\x = -1\end{cases}$ $\quad f(1) = 1 - 3 + 7 = 5$

$f(-1) = -1 + 3 + 7 = 9$



$f'(0) = -3 < 0$

$f'(-2) = 9 > 0$

$f'(2) = 9 > 0$

$f(0) = 7$, $f(2) = 8 - 6 + 7 = 9$

then maximum in $[0,2]$ is 9

minimum is $[0,2]$ is 5

$f(-2) = -8+6+7 = 5$

then maximum in $[-2,0]$ is 9

minimum is 5

$f(x,y) = x^2+y^2+xy$

$\frac{\partial f}{\partial x} = 2x+y$ $\quad \frac{\partial^2 f}{\partial x^2} = 2$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 1$

$\frac{\partial f}{\partial y} = 2y+x$ $\quad \frac{\partial^2 f}{\partial y^2} = 2$ $\quad \frac{\partial^2 f}{\partial y \partial x} = 1$

Hessian $= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

let $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} 2x+y=0 \\ 2y+x=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$ $\quad f(0,0)=0$

$f(x,y,z) = (x+1)^2 + (y+2)^2 + (z-2)^2$

s.t. $g(x,y,z) = x^2+y^2+z^2-36=0$

$L = (x+1)^2+(y+2)^2+(z-2)^2 + \lambda(x^2+y^2+z^2-36) = 0$

$\frac{\partial L}{\partial x} = 2(x+1) + 2\lambda x = 0$

$\frac{\partial L}{\partial y} = 2(y+2) + 2\lambda y = 0$

$\frac{\partial L}{\partial z} = 2(z-2) + 2\lambda z = 0$

$2x+2+2\lambda x = 0$

$(\lambda+1)x+1=0$

$x = -\frac{1}{\lambda+1}$

$2y+4+2\lambda y = 0$

$y+2+\lambda y = 0$

$(\lambda+1)y = -2$

$y = \frac{-2}{\lambda+1}$

$z-2+\lambda z = 0$

$z(1+\lambda) = 2$

$z = \frac{2}{1+\lambda}$

$\frac{1}{(\lambda+1)^2} + \frac{4}{(\lambda+1)^2} + \frac{4}{(\lambda+1)^2} = 36$

$\frac{1}{4} = (\lambda+1)^2$

$\lambda+1 = \pm\frac{1}{2}$

$\lambda = -1 \pm \frac{1}{2} \Rightarrow \lambda = \begin{cases} \frac{1}{2} \\ \frac{3}{2} \end{cases}$

$\Rightarrow \begin{cases} x = -\frac{2}{3} \\ y = \frac{4}{3} \\ z = -\frac{4}{3} \end{cases}$ or $\begin{cases} x = -\frac{2}{5} \\ y = -\frac{4}{5} \\ z = \frac{4}{5} \end{cases}$

$f(x) = x^4$

$\frac{df}{dx} = 4x^3 = 0 \Rightarrow x = 0$

Q.3

$p(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

first four moment $= E[(X-E(X))^4]$

$= E[(X-\frac{1}{2})^4]$

$= \int_{-\infty}^{+\infty} (x-\frac{1}{2})^4 x \, dx$

$= \int_{-\infty}^{+\infty} [(C_4^0 x^4 + C_4^1 x^3 \frac{1}{2} + C_4^2 x^2 \frac{1}{4} + C_4^3 x \frac{1}{8} + C_4^4 x \cdot \frac{1}{16})x \, dx$

$= \int_{-\infty}^{+\infty} (x^5 + 2x^4 + \frac{3}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{16}x^2) \, dx$