AE 4132 HW2 - Yunqing Jia

Problem 1

In[21]:=
```
Remove["Global`*"] (* clear workspace *)
```

1.1. Corresponding expression for the elastic potential

The elastic potential is defined as: $\Pi = W - E$.

We assume that the bar has constant cross-sectional area A, the material is elastic isotropic, and the stress is uniaxial (i.e. only $\sigma_x \neq 0$, all other stresses = 0). We get the expression

$$W = \int \frac{1}{2} \sigma_x \epsilon_x \, dV$$

Express σ_x in terms of strain using Hooke's law

In[22]:= $\sigma_x = \lambda (\epsilon_x + \epsilon_y + \epsilon_z) + 2 \mu \epsilon_x;$

In[23]:= $\sigma_y = \lambda (\epsilon_x + \epsilon_y + \epsilon_z) + 2 \mu \epsilon_y;$

In[24]:= $\sigma_z = \lambda (\epsilon_x + \epsilon_y + \epsilon_z) + 2 \mu \epsilon_z;$

Apply the uniaxial stress assumption and solve for σ_y and σ_z as functions of σ_x

In[25]:= soln = Solve[{σ_y == 0, σ_z == 0}, {ϵ_y, ϵ_z}] // Simplify

Out[25]= $\left\{\left\{\epsilon_y \to -\frac{\lambda \epsilon_x}{2 (\lambda + \mu)}, \epsilon_z \to -\frac{\lambda \epsilon_x}{2 (\lambda + \mu)}\right\}\right\}$

Solve for σ_x as strictly a function of (ϵ_x, μ, λ)

In[26]:= σ_x /. soln // Simplify

Out[26]= $\left\{\frac{\mu (3 \lambda + 2 \mu) \epsilon_x}{\lambda + \mu}\right\}$

Express coefficient relating strain to stress as E, Young's modulus

In[27]:= $\frac{\sigma_x}{\epsilon_x}$ /. soln // Simplify

Out[27]= $\left\{\frac{\mu (3 \lambda + 2 \mu)}{\lambda + \mu}\right\}$

Recall that $\epsilon_x = \frac{\delta u}{\delta x}$, and that displacement u does not vary w.r.t. δy or δz. We can replace ∂V with $A\, \delta x$.

The strain energy can then be defined as:

$$W = \int_\Omega \frac{1}{2} E \epsilon_x^2 \, dV = \frac{E A}{2} \int_0^L \left(\frac{\delta u}{\delta x}\right)^2 dx$$

In[28]:=
$$W[x] = \frac{E A}{2} \int_0^L (\partial_x u[x])^2 \, dx$$

Out[28]=
$$\frac{1}{2} A E \int_0^L u'[x]^2 \, dx$$

The work done by distributed and applied forces can be expressed as:

In[29]:=
$$E_0[x] = \int_0^L q[x] \times u[x] \, dx + (P u[x] \text{ /. } x \to L)$$

Out[29]=
$$\int_0^L q[x] \times u[x] \, dx + P u[L]$$

The elastic potential is:

In[30]:=
$$\Pi[x] == W[x] - E_0[x];$$

In[31]:= % // Framed

Out[31]=
$$\boxed{\Pi[x] == -\int_0^L q[x] \times u[x] \, dx + \frac{1}{2} A E \int_0^L u'[x]^2 \, dx - P u[L]}$$

The solution is obtained by minimizing Π for all possible displacement fields that satisfy the B.C.s. ($\delta\Pi$ = 0)

$$\delta\Pi = \frac{E A}{2} \delta \left(\int_0^L \left(\frac{\delta u}{\delta x} \right)^2 dx \right) - \delta \left(\int_0^L q[x] \, u \, dx \right) - \delta \, (P u[L])$$

$$= \frac{E A}{2} \int_0^L \delta \left(\frac{\delta u}{\delta x} \right)^2 dx - \int_0^L \delta \, (q[x] \, u) \, dx - P \, \delta u[L]$$

$$= \frac{E A}{2} \int_0^L 2 \frac{\delta u}{\delta x} \delta \left(\frac{\delta u}{\delta x} \right) dx - \int_0^L q[x] \, \delta u \, dx - P \, \delta u[L]$$

$$\delta\Pi = 0$$

Apply integration by part, the first integral becomes:

$$r = \frac{\delta u}{\delta x}, \quad ds = \frac{\delta \, \delta u}{\delta x} dx, \quad dr = \frac{\delta^2 u}{\delta x^2}, \quad s = \delta u$$

$$\int_0^L 2 \frac{\delta u}{\delta x} \delta \left(\frac{\delta u}{\delta x} \right) dx = \frac{\delta u}{\delta x} \delta u[L] - \int_0^L \delta u \frac{\delta^2 u}{\delta x^2} dx$$

$$\delta\Pi = E A \frac{\delta u}{\delta x} \delta u[L] - E A \int_0^L \delta u \frac{\delta^2 u}{\delta x^2} dx - \int_0^L q[x] \, \delta u \, dx - P \, \delta u[L] == 0$$

The final expression for the partial of Π w.r.t. *x* can be expressed as the governing equation + the traction boundary condition equation:

$$\delta\Pi = -\int_0^L \left(E A \frac{\delta^2 u}{\delta x^2} dx - q[x] \right) \delta u \, dx + \left(E A \frac{\delta u}{\delta x} - P \right) \delta u[L] == 0$$

1.2. Rayleigh-Ritz Method

(a) $\hat{u} = a\,x + b$

(b) $\hat{u} = a\,x^2 + b\,x + c$

(c) $\hat{u} = a\,x^3 + b\,x^2 + c\,x + d$

(d) $\hat{u} = a\,x + b$ for $0 < x < \frac{L}{2}$ && $c\,x + d$ for $\frac{L}{2} < x < L$

Boundary condition for the uni-axially loaded bar is $\hat{u}(0) = 0$

(a) $\hat{u}(x) = a\,x + b$

By applying the B.C., we know that $\hat{u}(x) = a\,(0) + b$, therefore $b = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a\,x, \quad \frac{\delta\hat{u}}{\delta x} = a, \quad \hat{u}(L) = a\,L$

Substitute these expressions back into the equation for elastic potential along with the assumption $q(x) = \text{const.} = q$

In[32]:= $\hat{u}_1[x\_] = a\,x;$

In[33]:= $\hat{\Pi}_1 = \frac{E\,A}{2}\int_0^L (\partial_x \hat{u}_1[x])^2\, dx - \int_0^L q\,(\hat{u}_1[x])\, dx - P\,\hat{u}_1[x];$

Now minimize $\hat{\Pi}$ w.r.t. a

In[34]:= $\partial_a \hat{\Pi}_1$

Out[34]= $-\frac{L^2\,q}{2} - P\,x + a\,A\,L\,E$

In[35]:= soln1 = Solve$[\partial_a \hat{\Pi}_1 == 0,\ a]$

Out[35]= $\left\{\left\{a \to \frac{L^2\,q + 2\,P\,x}{2\,A\,L\,E}\right\}\right\}$

Substitute the solved expression of a back into the equation for $\hat{u}$:

In[36]:= $\hat{u}_1[x] = \hat{u}_1[x]$ /. soln1[[1]];

In[37]:= % // Framed

Out[37]= $\boxed{\frac{x\,(L^2\,q + 2\,P\,x)}{2\,A\,L\,E}}$

(b) $\hat{u} = a\,x^2 + b\,x + c$

By applying the B.C., we know that $\hat{u}(x) = a\,(0)^2 + b(0) + c$, therefore $c = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a\,x^2 + b\,x, \quad \frac{\delta\hat{u}}{\delta x} = 2\,a\,x + b, \quad \hat{u}(L) = a\,L^2 + b\,L$

In[38]:= $\hat{u}_2[x\_] = a\,x^2 + b\,x;$

In[39]:= $\hat{\Pi}_2 = \frac{E\,A}{2}\int_0^L (\partial_x \hat{u}_2[x])^2\, dx - \int_0^L q\,(\hat{u}_2[x])\, dx - P\,\hat{u}_2[L];$

Now minimize $\hat{\Pi}_2$ w.r.t. both *a* and *b*

In[40]:= $\partial_a \hat{\Pi}_2$

Out[40]= $-L^2 P - \frac{L^3 q}{3} + \frac{1}{2} A \left(2 b L^2 + \frac{8 a L^3}{3}\right) \mathbb{E}$

In[41]:= $\partial_b \hat{\Pi}_2$

Out[41]= $-L P - \frac{L^2 q}{2} + \frac{1}{2} A \left(2 b L + 2 a L^2\right) \mathbb{E}$

In[42]:= `soln2 = Solve[{∂a Π̂2 == 0, ∂b Π̂2 == 0}, {a, b}]`

Out[42]= $\left\{\left\{a \to -\frac{q}{2 A \mathbb{E}},\ b \to -\frac{-P - L q}{A \mathbb{E}}\right\}\right\}$

In[43]:= `û2[x] = û2[x] /. soln2[[1]] // Simplify;`

In[44]:= `% // Framed`

Out[44]= $\boxed{\frac{x \left(2 P + 2 L q - q x\right)}{2 A \mathbb{E}}}$

(c) $\hat{u} = a x^3 + b x^2 + c x + d$

By applying the B.C., we know that $\hat{u}(0) = a\,(0)^3 + b(0)^2 + c(0) + d$, therefore $d = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a x^3 + b x^2 + c x$, $\frac{\delta \hat{u}}{\delta x} = 3 a x^2 + 2 b x + c$, $\hat{u}(L) = a L^3 + b L^2 + c L$

In[45]:= $\hat{u}_3[x\_] = a x^3 + b x^2 + c x;$

In[46]:= $\hat{\Pi}_3 = \frac{\mathbb{E} A}{2} \int_0^L \left(\partial_x \hat{u}_3[x]\right)^2 dx - \int_0^L q \left(\hat{u}_3[x]\right) dx - P \hat{u}_3[L];$

In[47]:= $\partial_a \hat{\Pi}_3$

Out[47]= $-L^3 P - \frac{L^4 q}{4} + \frac{1}{2} A \left(2 c L^3 + 3 b L^4 + \frac{18 a L^5}{5}\right) \mathbb{E}$

In[48]:= $\partial_b \hat{\Pi}_3$

Out[48]= $-L^2 P - \frac{L^3 q}{3} + \frac{1}{2} A \left(2 c L^2 + \frac{8 b L^3}{3} + 3 a L^4\right) \mathbb{E}$

In[49]:= $\partial_c \hat{\Pi}_3$

Out[49]= $-L P - \frac{L^2 q}{2} + \frac{1}{2} A \left(2 c L + 2 b L^2 + 2 a L^3\right) \mathbb{E}$

Now solve all three equations simultaneously (thanks Mathematica)

In[50]:= `soln3 = Solve[{∂a Π̂3 == 0, ∂b Π̂3 == 0, ∂c Π̂3 == 0}, {a, b, c}]`

Out[50]= $\left\{\left\{a \to 0,\ b \to -\frac{q}{2 A \mathbb{E}},\ c \to -\frac{-P - L q}{A \mathbb{E}}\right\}\right\}$

In[51]:=
```
û3[x] = û3[x] /. soln3[[1]] // Simplify;
```

In[52]:=
```
% // Framed
```

Out[52]= $\boxed{\dfrac{x\,(2P + 2Lq - qx)}{2AE}}$

Note: the 3rd-order polynomial approximation yields the exact same solution as the 2nd-order polynomial. The 2nd-order solution is already the exact solution.

In[53]:=
```
û2[x] == û3[x]
```

Out[53]= True

(d) $\hat{u} = a\,x + b$ for $0 < x < \frac{L}{2}$ && $c\,x + d$ for $\frac{L}{2} < x < L$

Let the left half of the bar be denoted $\hat{u}_{4\,\text{left}}$ and the right half $\hat{u}_{4\,\text{right}}$.

By applying the B.C. at the left, fixed end, we know that $\hat{u}_{4\,\text{left}}(0) = a\,(0) + b$, therefore $b = 0$. The expressions for $\hat{u}_{4\,\text{left}}$ then become $\hat{u}_{4\,\text{left}}(x) = a\,x$, $\frac{\delta \hat{u}_{4\,\text{left}}}{\delta x} = a$

In[54]:=
```
û4left[x_] = a x;  û4right[x_] = c x + d;
```

In[55]:= $d = d\ /.\ \text{Solve}\left[\hat{u}_{4\,\text{left}}\left[\frac{L}{2}\right] == \hat{u}_{4\,\text{right}}\left[\frac{L}{2}\right],\ d\right][[1]]$

Out[55]= $\frac{1}{2}\,(a\,L - c\,L)$

In[56]:= $\hat{\Pi}_4 = \frac{E\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x\,\hat{u}_{4\,\text{left}}[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x\,\hat{u}_{4\,\text{right}}[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\left(\hat{u}_{4\,\text{left}}[x]\right) dx + \int_{\frac{L}{2}}^{L} q\,\left(\hat{u}_{4\,\text{right}}[x]\right) dx\right) - P\,\hat{u}_{4\,\text{right}}[L]$

Out[56]= $-\left(c\,L + \frac{1}{2}\,(a\,L - c\,L)\right)P - \frac{3}{8}\,a\,L^2\,q - \frac{1}{8}\,c\,L^2\,q + \frac{1}{2}\,A\left(\frac{a^2\,L}{2} + \frac{c^2\,L}{2}\right)E$

In[57]:= $\partial_a\,\hat{\Pi}_4$

Out[57]= $-\frac{L\,P}{2} - \frac{3\,L^2\,q}{8} + \frac{1}{2}\,a\,A\,L\,E$

In[58]:= $\partial_c\,\hat{\Pi}_4$

Out[58]= $-\frac{L\,P}{2} - \frac{L^2\,q}{8} + \frac{1}{2}\,A\,c\,L\,E$

Apply the continuity B.C. at the center of the bar $\left(x = \frac{L}{2}\right)$: $\hat{u}_{4\,\text{left}}\left(\frac{L}{2}\right) = \hat{u}_{4\,\text{right}}\left(\frac{L}{2}\right)$.

In[59]:= $\text{soln4} = \text{Solve}\left[\left\{\partial_a\,\hat{\Pi}_4 == 0,\ \partial_c\,\hat{\Pi}_4 == 0\right\},\ \{a,\ c\}\right]$

Out[59]= $\left\{\left\{a \to \frac{4\,P + 3\,L\,q}{4\,A\,E},\ c \to \frac{4\,P + L\,q}{4\,A\,E}\right\}\right\}$

In[60]:=
```
d /. soln4 // Simplify
```

Out[60]= $\left\{\frac{L^2 q}{4 A \mathbb{E}}\right\}$

In[61]:=
```
û4left[x] = û4left[x] /. soln4[[1]] // Simplify;
```

In[62]:=
```
% // Framed
```

Out[62]= $\boxed{\frac{(4 P + 3 L q) x}{4 A \mathbb{E}}}$

In[63]:=
```
û4right[x] = û4right[x] /. soln4[[1]] // Simplify;
```

In[64]:=
```
% // Framed
```

Out[64]= $\boxed{\frac{L^2 q + 4 P x + L q x}{4 A \mathbb{E}}}$

1.3. N-segment piece-wise linear approximation

*Derived by hand. But recovered the exact expression (integral form) for elastic potential as the $N \to \infty$ (*i.e.* $\Delta x \to 0$) in the piecewise approximation.

1.4. Plot the specific case in Python

-> Use matrix format

In[65]:=
```
Remove["Global`*"]
```

N=4

In[66]:=
```
û41[x_] = a x;  û42[x_] = c x + d;  û43[x_] = e x + f;  û44[x_] = g x + h;
```

In[67]:=
```
d = d /. Solve[(û41[x] /. x → L/4) == (û42[x] /. x → L/4), d][[1]];
```

In[68]:=
```
f = f /. Solve[û42[2 L/4] == û43[2 L/4], f][[1]];
```

In[69]:=
```
h = h /. Solve[û43[3 L/4] == û44[3 L/4], h][[1]];
```

In[70]:= $\hat{\Pi}_4 =$

$$((\mathbb{E}\,A)/2)\left(\int_0^{\frac{L}{4}}(\partial_x\,\hat{u}_{41}[x])^2\,dx+\int_{\frac{L}{4}}^{\frac{2L}{4}}(\partial_x\,\hat{u}_{42}[x])^2\,dx+\int_{\frac{2L}{4}}^{\frac{3L}{4}}(\partial_x\,\hat{u}_{43}[x])^2\,dx+\int_{\frac{3L}{4}}^{\frac{4L}{4}}(\partial_x\,\hat{u}_{44}[x])^2\,dx\right)-$$

$$\left(\int_0^{\frac{L}{4}}q\,(\hat{u}_{41}[x])\,dx+\int_{\frac{L}{4}}^{\frac{2L}{4}}q\,(\hat{u}_{42}[x])\,dx+\int_{\frac{2L}{4}}^{\frac{3L}{4}}q\,(\hat{u}_{43}[x])\,dx+\int_{\frac{3L}{4}}^{\frac{4L}{4}}q\,(\hat{u}_{44}[x])\,dx\right)-P\,\hat{u}_{44}[L]$$

Out[70]=
$$-\left(g\,L+\frac{1}{4}\,(a\,L+c\,L+e\,L-3\,g\,L)\right)P-\frac{7}{32}\,a\,L^2\,q-\frac{5}{32}\,c\,L^2\,q-\frac{3}{32}\,e\,L^2\,q-\frac{1}{32}\,g\,L^2\,q+\frac{1}{2}\,A\left(\frac{a^2\,L}{4}+\frac{c^2\,L}{4}+\frac{e^2\,L}{4}+\frac{g^2\,L}{4}\right)\mathbb{E}$$

In[71]:= $\partial_a\,\hat{\Pi}_4$

Out[71]=
$$-\frac{L\,P}{4}-\frac{7\,L^2\,q}{32}+\frac{1}{4}\,a\,A\,L\,\mathbb{E}$$

In[72]:= $\partial_c\,\hat{\Pi}_4$

Out[72]=
$$-\frac{L\,P}{4}-\frac{5\,L^2\,q}{32}+\frac{1}{4}\,A\,c\,L\,\mathbb{E}$$

In[73]:= $\partial_e\,\hat{\Pi}_4$

Out[73]=
$$-\frac{L\,P}{4}-\frac{3\,L^2\,q}{32}+\frac{1}{4}\,A\,e\,L\,\mathbb{E}$$

In[74]:= $\partial_g\,\hat{\Pi}_4$

Out[74]=
$$-\frac{L\,P}{4}-\frac{L^2\,q}{32}+\frac{1}{4}\,A\,g\,L\,\mathbb{E}$$

Apply the continuity B.C. at the center of the bar $\left(x=\frac{L}{2}\right)$: $\hat{u}_{4\,\text{left}}\left(\frac{L}{2}\right)=\hat{u}_{4\,\text{right}}\left(\frac{L}{2}\right)$.

In[75]:= soln5 = Solve[{$\partial_a\,\hat{\Pi}_4$ == 0, $\partial_c\,\hat{\Pi}_4$ == 0, $\partial_e\,\hat{\Pi}_4$ == 0, $\partial_g\,\hat{\Pi}_4$ == 0}, {a, c, e, g}]

Out[75]=
$$\left\{\left\{a\to\frac{8\,P+7\,L\,q}{8\,A\,\mathbb{E}},\ c\to\frac{8\,P+5\,L\,q}{8\,A\,\mathbb{E}},\ e\to\frac{8\,P+3\,L\,q}{8\,A\,\mathbb{E}},\ g\to\frac{8\,P+L\,q}{8\,A\,\mathbb{E}}\right\}\right\}$$

In[76]:= $\hat{u}_{41}$[x] = $\hat{u}_{41}$[x] /. soln5[[1]]

Out[76]=
$$\frac{(8\,P+7\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[77]:= $\hat{u}_{42}$[x] = $\hat{u}_{42}$[x] /. soln5[[1]]

Out[77]=
$$\frac{1}{4}\left(-\frac{L\,(8\,P+5\,L\,q)}{8\,A\,\mathbb{E}}+\frac{L\,(8\,P+7\,L\,q)}{8\,A\,\mathbb{E}}\right)+\frac{(8\,P+5\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[78]:= $\hat{u}_{43}$[x] = $\hat{u}_{43}$[x] /. soln5[[1]]

Out[78]=
$$\frac{1}{4}\left(-\frac{L\,(8\,P+3\,L\,q)}{4\,A\,\mathbb{E}}+\frac{L\,(8\,P+5\,L\,q)}{8\,A\,\mathbb{E}}+\frac{L\,(8\,P+7\,L\,q)}{8\,A\,\mathbb{E}}\right)+\frac{(8\,P+3\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[79]:=
```
û44[x] = û44[x] /. soln5[[1]]
```
Out[79]= $\frac{1}{4}\left(-\frac{3L(8P+Lq)}{8A\mathbb{E}}+\frac{L(8P+3Lq)}{8A\mathbb{E}}+\frac{L(8P+5Lq)}{8A\mathbb{E}}+\frac{L(8P+7Lq)}{8A\mathbb{E}}\right)+\frac{(8P+Lq)x}{8A\mathbb{E}}$

In[80]:=
```
d
```
Out[80]= $\frac{1}{4}(aL-cL)$

In[81]:=
```
f
```
Out[81]= $\frac{1}{4}(aL+cL-2eL)$

In[82]:=
```
h
```
Out[82]= $\frac{1}{4}(aL+cL+eL-3gL)$

Problem 2

2.1. Case 1

In[83]:=
```
Remove["Global`*"]
```

Approximate with quadratic functions based on solution from HW#1

In[84]:=
```
û1[x_] = a x^2 + b x + c; (* 0 ≤ x ≤ L/2 *)
```

In[85]:=
```
û2[x_] = d x^2 + e x + f; (* L/2 ≤ x ≤ L *)
```

Apply B.C. at the left end of the beam $u_1[x=0]=0$

In[86]:=
```
c = c /. Solve[û1[0] == 0, c][[1]]
```
Out[86]= 0

Apply continuity in the middle of the beam $u_1\left[x=\frac{L}{2}\right]=u_1\left[x=\frac{L}{2}\right]$ and $\frac{\delta u_1}{\delta x}\left[x=\frac{L}{2}\right]=\frac{\delta u_2}{\delta x}\left[x=\frac{L}{2}\right]$

In[87]:=
```
f = f /. Solve[û1[L/2] == û2[L/2], f][[1]]
```
Out[87]= $\frac{1}{4}(2bL-2eL+aL^2-dL^2)$

In[88]:=
```
e = e /. Solve[(∂x û1[x] /. x → L/2) == (∂x û2[x] /. x → L/2), e][[1]]
```
Out[88]= $b+aL-dL$

Now we can express $\hat{\Pi}$ as a function of variables $a,\ b,\ d$. Also note that $q(x)=+q$ for $0\le x\le\frac{L}{2}$ and $q(x)=-q$ for $\frac{L}{2}\le x\le L$

In[89]:= $$\hat{\Pi} = \frac{\mathbb{E}\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right) - \left(P\,\hat{u}_2[x]\ /.\ x \to L\right)$$

Out[89]= $$-\left(d L^2 + L\left(b + a L - d L\right) + \frac{1}{4}\left(2 b L + a L^2 - d L^2 - 2 L\left(b + a L - d L\right)\right)\right) P + \frac{1}{4} b L^2 q + \frac{5}{24} a L^3 q + \frac{1}{24} d L^3 q + \frac{1}{2} A\left(\frac{b^2 L}{2} + \frac{1}{2} a b L^2 + \frac{a^2 L^3}{6} + \frac{-\left(b + a L\right)^3 + \left(b + \left(a + d\right) L\right)^3}{6 d}\right)\mathbb{E}$$

In[90]:= $\partial_a \hat{\Pi}$

Out[90]= $$-\frac{3 L^2 P}{4} + \frac{5 L^3 q}{24} + \frac{1}{2} A\left(\frac{b L^2}{2} + \frac{a L^3}{3} + \frac{-3 L\left(b + a L\right)^2 + 3 L\left(b + \left(a + d\right) L\right)^2}{6 d}\right)\mathbb{E}$$

In[91]:= $\partial_b \hat{\Pi}$

Out[91]= $$-L P + \frac{L^2 q}{4} + \frac{1}{2} A\left(b L + \frac{a L^2}{2} + \frac{-3\left(b + a L\right)^2 + 3\left(b + \left(a + d\right) L\right)^2}{6 d}\right)\mathbb{E}$$

In[92]:= $\partial_d \hat{\Pi}$

Out[92]= $$-\frac{L^2 P}{4} + \frac{L^3 q}{24} + \frac{1}{2} A\left(\frac{L\left(b + \left(a + d\right) L\right)^2}{2 d} - \frac{-\left(b + a L\right)^3 + \left(b + \left(a + d\right) L\right)^3}{6 d^2}\right)\mathbb{E}$$

In[93]:= $\text{soln1} = \text{Solve}\left[\left\{\partial_a \hat{\Pi} == 0,\ \partial_b \hat{\Pi} == 0,\ \partial_d \hat{\Pi} == 0\right\},\ \{a,\ b,\ d\}\right]$

Out[93]= $$\left\{\left\{a \to -\frac{q}{2 A \mathbb{E}},\ b \to \frac{P}{A \mathbb{E}},\ d \to \frac{q}{2 A \mathbb{E}}\right\}\right\}$$

In[94]:= $\hat{u}_1[x] = \hat{u}_1[x]\ /.\ \text{soln1}[[1]]\ //\ \text{Simplify}$

Out[94]= $$-\frac{x\left(-2 P + q x\right)}{2 A \mathbb{E}}$$

In[95]:= $\hat{u}_2[x] = \hat{u}_2[x]\ /.\ \text{soln1}[[1]]\ //\ \text{Simplify}$

Out[95]= $$\frac{L^2 q + 4 P x - 4 L q x + 2 q x^2}{4 A \mathbb{E}}$$

Compare these to the exact solutions

In[96]:= $$u_{1\,\text{exact}} = \frac{1}{\mathbb{E}\,A}\left(\frac{-q}{2} x^2 + P x\right);$$

In[97]:= $\hat{u}_1[x] == u_{1\,\text{exact}}\ //\ \text{Simplify}$

Out[97]= True

In[98]:= $$u_{2\,\text{exact}} = \frac{1}{\mathbb{E}\,A}\left(\frac{q}{2} x^2 + P x - q L x + \frac{q}{4} L^2\right);$$

In[99]:=
```
û₂[x] == u2 exact // Simplify
```
Out[99]= True

We recovered the exact solution from the piecewise function with quadratic approximations. Yay! Now compute $N(x)$ by taking the derivative.

In[100]:=
```
N₁[x_] = EA∂x û₁[x] // Simplify
```
Out[100]= $P - q\,x$

In[101]:=
```
N₂[x_] = EA∂x û₂[x] // Simplify
```
Out[101]= $P + q\,(-L + x)$

With numerical values:

In[102]:=
```
N₁[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[102]= $90 - x$

In[103]:=
```
N₂[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[103]= $-6 + x$

2.2. Case 2

In[104]:=
```
Remove["Global`*"]
```

Use same setup as case 1 except for different B.C. at $\hat{u}_2[x = L] = 0$ because it's a fixed joint. $\hat{u}_1[x = 0] = 0$ still applies.

In[105]:=
```
û₁[x_] = a x² + b x ; (* 0 ≤ x ≤ L/2 *)
```

In[106]:=
```
û₂[x_] = d x² + e x + f; (* L/2 ≤ x ≤ L *)
```

Apply B.C. at the right end of the beam: $\hat{u}_2[x = L] = 0$.

In[107]:=
```
f = f /. Solve[û₂[L] == 0, f][[1]]
```
Out[107]= $-e\,L - d\,L^2$

Apply continuity in the middle of the beam $u_1\left[x = \frac{L}{2}\right] = u_1\left[x = \frac{L}{2}\right]$ and $\frac{\delta u_1}{\delta x}\left[x = \frac{L}{2}\right] = \frac{\delta u_2}{\delta x}\left[x = \frac{L}{2}\right]$

In[108]:=
```
e = e /. Solve[û₁[L/2] == û₂[L/2], e][[1]]
```
Out[108]= $\frac{1}{2}\left(-2\,b - a\,L - 3\,d\,L\right)$

Now we can express $\hat{\Pi}$ as a function of variables $a,\ b,\ d$. Also note that $q(x) = +q$ for $0 \le x \le \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \le x \le L$

In[109]:= $\hat{\Pi} = \frac{\mathbb{E}\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right)$

Out[109]= $\frac{1}{48} a L^3 q - \frac{1}{48} d L^3 q + \frac{1}{2} A \left(b^2 L + a b L^2 + \frac{7 a^2 L^3}{24} + \frac{d^2 L^3}{24}\right) \mathbb{E}$

In[110]:= $\partial_a \hat{\Pi}$

Out[110]= $\frac{L^3 q}{48} + \frac{1}{2} A \left(b L^2 + \frac{7 a L^3}{12}\right) \mathbb{E}$

In[111]:= $\partial_b \hat{\Pi}$

Out[111]= $\frac{1}{2} A \left(2 b L + a L^2\right) \mathbb{E}$

In[112]:= $\partial_d \hat{\Pi}$

Out[112]= $-\frac{L^3 q}{48} + \frac{1}{24} A d L^3 \mathbb{E}$

In[113]:= soln2 = Solve$\left[\left\{\partial_a \hat{\Pi} == 0,\ \partial_b \hat{\Pi} == 0,\ \partial_d \hat{\Pi} == 0\right\},\ \{a, b, d\}\right]$

Out[113]= $\left\{\left\{a \to -\frac{q}{2 A \mathbb{E}},\ b \to \frac{L q}{4 A \mathbb{E}},\ d \to \frac{q}{2 A \mathbb{E}}\right\}\right\}$

In[114]:= $\hat{u}_1[x] = \hat{u}_1[x]$ /. soln2[[1]] // Simplify

Out[114]= $\frac{q\,(L - 2 x)\,x}{4 A \mathbb{E}}$

In[115]:= $\hat{u}_2[x] = \hat{u}_2[x]$ /. soln2[[1]] // Simplify

Out[115]= $\frac{q\,(L^2 - 3 L x + 2 x^2)}{4 A \mathbb{E}}$

Compare these to the exact solutions

In[116]:= $u_{1\,exact} = \frac{q}{4\,\mathbb{E}\,A}\left(-2 x^2 + L x\right)$;

In[117]:= $\hat{u}_1[x] == u_{1\,exact}$ // Simplify

Out[117]= True

In[118]:= $u_{2\,exact} = \frac{q}{4\,\mathbb{E}\,A}\left(2 x^2 - 3 L x + L^2\right)$;

In[119]:= $\hat{u}_2[x] == u_{2\,exact}$ // Simplify

Out[119]= True

Once again recovered the exact solution! Now compute $N(x)$ by taking the derivative.

In[120]:= $N_1[x\_] = \mathbb{E}\,A\,\partial_x \hat{u}_1[x]$ // Simplify

Out[120]= $\frac{1}{4} q\,(L - 4 x)$

In[121]:=
```
N₂[x_] = E A ∂x û₂[x] // Simplify
```
Out[121]= $-\frac{3 L q}{4} + q x$

With numerical values:

In[122]:=
```
N₁[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[122]= $24 - x$

In[123]:=
```
N₂[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[123]= $-72 + x$

2.3. Case 1 (linear approximation)

In[124]:=
```
Remove["Global`*"]
```

Approximate with linear functions to see how it affects $N(x)$ results

In[125]:=
```
û₁[x_] = a x + b; (* 0 ≤ x ≤ L/2 *)
```

In[126]:=
```
û₂[x_] = c x + d; (* L/2 ≤ x ≤ L *)
```

Apply B.C. at the left end of the beam $u_1[x = 0] = 0$

In[127]:=
```
b = b /. Solve[û₁[0] == 0, b][[1]]
```
Out[127]= 0

Apply continuity in the middle of the beam $u_1\left[x = \frac{L}{2}\right] = u_1\left[x = \frac{L}{2}\right]$

In[128]:=
```
d = d /. Solve[û₁[L/2] == û₂[L/2], d][[1]]
```
Out[128]= $\frac{1}{2} (a L - c L)$

Now we can express $\hat{\Pi}$ as a function of variables a, c. Also note that $q(x) = +q$ for $0 \le x \le \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \le x \le L$

In[129]:= $\hat{\Pi} = \frac{E A}{2} \left(\int_0^{\frac{L}{2}} (\partial_x \hat{u}_1[x])^2 \, dx + \int_{\frac{L}{2}}^{L} (\partial_x \hat{u}_2[x])^2 \, dx \right) - \left(\int_0^{\frac{L}{2}} q \, \hat{u}_1[x] \, dx + \int_{\frac{L}{2}}^{L} -q \, \hat{u}_2[x] \, dx \right) - (P \, \hat{u}_2[x] \text{ /. } x \to L)$

Out[129]= $-\left(c L + \frac{1}{2} (a L - c L)\right) P + \frac{1}{8} a L^2 q + \frac{1}{8} c L^2 q + \frac{1}{2} A \left(\frac{a^2 L}{2} + \frac{c^2 L}{2}\right) E$

In[130]:= $\partial_a \hat{\Pi}$

Out[130]= $-\frac{L P}{2} + \frac{L^2 q}{8} + \frac{1}{2} a A L E$

In[131]:= $\partial_c \hat{\Pi}$

Out[131]= $-\frac{L P}{2}+\frac{L^2 q}{8}+\frac{1}{2} A c L \mathbb{E}$

In[132]:= soln1 = Solve[{$\partial_a \hat{\Pi}$ == 0, $\partial_c \hat{\Pi}$ == 0}, {a, c}]

Out[132]= $\left\{\left\{a \to \frac{4 P - L q}{4 A \mathbb{E}}, c \to \frac{4 P - L q}{4 A \mathbb{E}}\right\}\right\}$

In[133]:= $\hat{u}_1$[x] = $\hat{u}_1$[x] /. soln1[[1]] // Simplify

Out[133]= $\frac{(4 P - L q)\, x}{4 A \mathbb{E}}$

In[134]:= $\hat{u}_2$[x] = $\hat{u}_2$[x] /. soln1[[1]] // Simplify

Out[134]= $\frac{(4 P - L q)\, x}{4 A \mathbb{E}}$

Compare these to the exact solutions

In[135]:= $u_{1\,\text{exact}} = \frac{1}{\mathbb{E} A}\left(\frac{-q}{2} x^2 + P x\right);$

In[136]:= $\hat{u}_1$[x] == $u_{1\,\text{exact}}$ // Simplify

Out[136]= $\frac{q\,(L - 2 x)\, x}{A \mathbb{E}} == 0$

In[137]:= $u_{2\,\text{exact}} = \frac{1}{\mathbb{E} A}\left(\frac{q}{2} x^2 + P x - q L x + \frac{q}{4} L^2\right);$

In[138]:= $\hat{u}_2$[x] == $u_{2\,\text{exact}}$ // Simplify

Out[138]= $\frac{q\,(L^2 - 3 L x + 2 x^2)}{A \mathbb{E}} == 0$

The linear approximation did not yield the exact solution. Now compute the $N(x)$ expressions

In[139]:= N_1[x_] = $\mathbb{E}$ A ∂_x $\hat{u}_1$[x] // Simplify

Out[139]= $P - \frac{L q}{4}$

In[140]:= N_2[x_] = $\mathbb{E}$ A ∂_x $\hat{u}_2$[x] // Simplify

Out[140]= $P - \frac{L q}{4}$

With numerical values:

In[141]:= N_1[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[141]= 66

In[142]:= N_2[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[142]= 66

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2.4. Case 2 (linear approximation)

In[143]:=
```
Remove["Global`*"]
```

Use same setup as case 1 except for different B.C. at $\hat{u}_2[x = L] = 0$ because it's a fixed joint.
$\hat{u}_1[x = 0] = 0$ still applies.

In[144]:= $\hat{u}_1[x\_] = a\,x;$ (* $0 \le x \le \frac{L}{2}$ *)

In[145]:= $\hat{u}_2[x\_] = c\,x + d;$ (* $\frac{L}{2} \le x \le L$ *)

Apply B.C. at the right end of the beam: $\hat{u}_2[x = L] = 0$.

In[146]:= $d = d$ /. Solve$\left[\hat{u}_2[L] == 0,\ d\right]$[[1]]

Out[146]= $-c\,L$

Now we can express $\hat{\Pi}$ as a function of variable a. Also note that $q(x) = +q$ for $0 \le x \le \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \le x \le L$

In[147]:= $\hat{\Pi} = \frac{E\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right)$

Out[147]= $-\frac{1}{8} a\,L^2 q - \frac{1}{8} c\,L^2 q + \frac{1}{2} A\left(\frac{a^2 L}{2} + \frac{c^2 L}{2}\right) E$

In[148]:= $\partial_a \hat{\Pi}$

Out[148]= $-\frac{L^2 q}{8} + \frac{1}{2} a\,A\,L\,E$

In[149]:= $\partial_c \hat{\Pi}$

Out[149]= $-\frac{L^2 q}{8} + \frac{1}{2} A\,c\,L\,E$

In[150]:= soln2 = Solve$\left[\left\{\partial_a \hat{\Pi} == 0,\ \partial_c \hat{\Pi} == 0\right\},\ \{a,\ c\}\right]$

Out[150]= $\left\{\left\{a \to \frac{L\,q}{4\,A\,E},\ c \to \frac{L\,q}{4\,A\,E}\right\}\right\}$

In[151]:= $\hat{u}_1[x] = \hat{u}_1[x]$ /. soln2[[1]] // Simplify

Out[151]= $\frac{L\,q\,x}{4\,A\,E}$

In[152]:= $\hat{u}_2[x] = \hat{u}_2[x]$ /. soln2[[1]] // Simplify

Out[152]= $\frac{L\,q\,(-L + x)}{4\,A\,E}$

Compare these to the exact solutions

In[153]:= $u_{1\,exact} = \frac{q}{4\,E\,A}\left(-2\,x^2 + L\,x\right);$

In[154]:=
```
û₁[x] == u1 exact // Simplify
```
Out[154]= $\frac{q\,x}{A\,E} == 0$

In[155]:= $u_{2\,exact} = \frac{q}{4\,E\,A}\left(2\,x^2 - 3\,L\,x + L^2\right);$

In[156]:=
```
û₂[x] == u2 exact // Simplify
```
Out[156]= $\frac{q\,(L - x)}{A\,E} == 0$

Once again recovered the exact solution! Now compute $N(x)$ by taking the derivative.

In[157]:= $N_1[x\_] = E\,A\,\partial_x\,\hat{u}_1[x]$ // Simplify

Out[157]= $\frac{L\,q}{4}$

In[158]:= $N_2[x\_] = E\,A\,\partial_x\,\hat{u}_2[x]$ // Simplify

Out[158]= $\frac{L\,q}{4}$

With numerical values:

In[159]:=
```
N₁[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[159]= 24

In[160]:=
```
N₂[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[160]= 24

Problem 3

In[161]:=
```
Remove["Global`*"]
```

3.1. Corresponding expression for the elastic potential

*Derived by hand. Let $v(x)$ be the vertical displacement as a function of x

$$\Pi = W - E = \frac{E\,I}{2}\int_0^L \left(\frac{\delta^2 v}{\delta x^2}\right)^2 dx - \int_0^L q\,v\,dx - P\,v[x = L]$$

3.2. Rayleigh-Ritz Method

(a) $\hat{v} = a\,x^2 + b\,x + c$

(b) $\hat{v} = a\cos(b\,x) + c$

Boundary condition for the uni-axially loaded bar is $\hat{v}(0) = 0$

a) $\hat{v} = a\,x^2 + b\,x + c$

In[162]:= $\hat{v}_1 = a\,x^2 + b\,x + c;$

In[163]:= $\hat{v}_1 = \hat{v}_1$ /. Solve$\left[\left\{\left(\hat{v}_1 \text{ /. } x \to 0\right) == 0,\ \left(\partial_x \hat{v}_1 \text{ /. } x \to 0\right) == 0\right\},\ \{b,\ c\}\right][[1]]$

Out[163]= $a\,x^2$

In[164]:= $\hat{\Pi}_1 = \frac{E\,I}{2}\int_0^L \left(\partial_{x,x}\hat{v}_1\right)^2 dx - \int_0^L q\,\hat{v}_1\,dx - \left(P\,\hat{v}_1 \text{ /. } x \to L\right)$

Out[164]= $-a\,L^2\,P - \frac{1}{3}\,a\,L^3\,q + 2\,a^2\,L\,E\,I$

In[165]:= $\partial_a \hat{\Pi}_1$

Out[165]= $-L^2\,P - \frac{L^3\,q}{3} + 4\,a\,L\,E\,I$

In[166]:= soln1 = Solve$\left[\partial_a \hat{\Pi}_1 == 0,\ a\right]$

Out[166]= $\left\{\left\{a \to \frac{3\,L\,P + L^2\,q}{12\,E\,I}\right\}\right\}$

In[167]:= $\hat{v}_1 = \hat{v}_1$ /. soln1[[1]];

In[168]:= % // Framed

Out[168]= $\boxed{\frac{\left(3\,L\,P + L^2\,q\right)\,x^2}{12\,E\,I}}$

b) $\hat{v} = a\cos(b\,x) + c$

In[169]:= $\hat{v}_2 = a\,\mathrm{Cos}[b\,x] + c;$

In[170]:= $\hat{v}_2 = \hat{v}_2$ /. Solve$\left[\left(\hat{v}_2 \text{ /. } x \to 0\right) == 0,\ c\right][[1]]$

Out[170]= $-a + a\,\mathrm{Cos}[b\,x]$

Apply B.C. at $x = L$: since there is no applied load at the tip, we know that

$\frac{\delta^2\,\hat{v}_2}{\delta\,x^2}[x = L] = a\,\mathrm{Cos}[b\,L] = 0$

For $a,\ b \neq 0$, $\mathrm{Cos}[b\,L] = 0$. Let $b\,L = \frac{\Pi}{2}$, we get $b = \frac{\Pi}{2L}$.

In[171]:= $b = \frac{\pi}{2\,L}$

Out[171]= $\frac{\pi}{2\,L}$

In[172]:= $\hat{v}_2$

Out[172]= $-a + a\,\text{Cos}\left[\frac{\pi x}{2 L}\right]$

In[173]:= $\hat{\Pi}_2 = \frac{E I}{2}\int_0^L \left(\partial_{x,x}\hat{v}_2\right)^2 dx - \int_0^L q\,\hat{v}_2\,dx - \left(P\,\hat{v}_2 \text{ /. } x \to L\right)$

Out[173]= $a P + \frac{a L (-2+\pi) q}{\pi} + \frac{a^2 \pi^4 E I}{64 L^3}$

In[174]:= `soln2 = Solve[{`$\partial_a \hat{\Pi}_2$` == 0}, {a}]`

Out[174]= $\left\{\left\{a \to -\frac{32 L^3 (P \pi - 2 L q + L \pi q)}{\pi^5 E I}\right\}\right\}$

In[175]:= `v̂2 = v̂2 /. soln2[[1]] // Simplify`

Out[175]= $\frac{64 L^3 \left(P\pi + L(-2+\pi) q\right) \text{Sin}\left[\frac{\pi x}{4 L}\right]^2}{\pi^5 E I}$

In[176]:= `% // Framed`

Out[176]= $\boxed{\frac{64 L^3 \left(P\pi + L(-2+\pi) q\right) \text{Sin}\left[\frac{\pi x}{4 L}\right]^2}{\pi^5 E I}}$

3.3. Total potential energy and plot Π vs. β

In[177]:= `Π̂1 /. soln1[[1]] /. P → β q L /. {q → 40, L → 1, E → 120 × 10^9, I → 8 × 10^-9} // Simplify;`

In[178]:= `% // Framed`

Out[178]= $\boxed{-\frac{5}{216}(1+3\beta)^2}$

In[179]:= `Π̂2 /. soln2[[1]] /. P → β q L /. {q → 40, L → 1, E → 120 × 10^9, I → 8 × 10^-9} // Simplify;`

In[180]:= `% // Framed`

Out[180]= $\boxed{-\frac{80(-2+\pi+\pi\beta)^2}{3\pi^6}}$

3.4. Which is better? Why?

Quadratic is better because the elastic potential is closer to 0 (the exact solution is always at the minimum).