PROBLEM 1.

1.1. $\Pi = W - E$

$W = \int_\Omega \frac{1}{2} \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} \, dV$

assume uniaxial stress : only $\sigma_x \neq 0$

$W = \int_\Omega \frac{1}{2} \sigma_x \varepsilon_x \, dV$

Hooke's Law : $\sigma(\varepsilon)$

$$\left.\begin{aligned}
\sigma_x &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_x \\
\sigma_y &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_y = 0 \\
\sigma_z &= \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2\mu\varepsilon_z = 0
\end{aligned}\right\} \text{solve for } \varepsilon_y, \varepsilon_z$$

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{-2\mu}{\lambda}\varepsilon_y = \frac{-2\mu}{\lambda}\varepsilon_z \Rightarrow \varepsilon_y = \varepsilon_z$$

$$\varepsilon_x + 2\varepsilon_y = \frac{-2\mu}{\lambda}\varepsilon_y \Rightarrow \left(\frac{2\lambda}{\lambda} + \frac{2\mu}{\lambda}\right)\varepsilon_y = -\varepsilon_x \Rightarrow \varepsilon_y = \varepsilon_z = -\frac{\lambda}{2(\lambda+\mu)}\varepsilon_x$$

$$\sigma_x = \left(\lambda\left(1 - \frac{\lambda}{\lambda+\mu}\right) + 2\mu\right)\varepsilon_x = \frac{\cancel{\lambda^2} + \lambda\mu - \cancel{\lambda^2} + 2\lambda\mu + \mu^2}{\lambda+\mu}\varepsilon_x = \underbrace{\frac{\mu(3\lambda+\mu)}{\lambda+\mu}}_{\text{Young's modulus } (E)}\varepsilon_x$$

$$W = \int_\Omega \frac{1}{2} E \varepsilon_x^2 \, dV = \frac{EA}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$E = \int_0^L q(x) u \, dx + Pu\Big|_{x=L}$$

$$\boxed{\Pi = W - E = \frac{EA}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L q(x) u \, dx - Pu\Big|_{x=L}}$$

* not needed for this part

$$\begin{aligned}
\delta\Pi &= \frac{EA}{2}\delta\left(\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right) - \delta\left(\int_0^L q(x) u \, dx\right) - \delta Pu\Big|_{x=L} \\
&= \frac{EA}{2}\int_0^L \delta\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L \delta\left(q(x)u\right) dx - P\delta u\Big|_{x=L} \\
&= \frac{EA}{2}\int_0^L 2\frac{\partial u}{\partial x} \cdot \delta\left(\frac{\partial u}{\partial x}\right) dx - \int_0^L q(x)\,\delta u \, dx - P\delta u\Big|_{x=L}
\end{aligned}$$

1.2. a) $\hat{u}(x) = ax + b$

* plugged $\hat{u}(x)$ into the equation for elastic potential from 1.1.

* B.C. : $\hat{u}(0) = 0 \Rightarrow b = 0$

* Solve : $\frac{\partial \hat{\Pi}}{\partial a} = 0$ in Mathematica (see script attached)

$$\boxed{\hat{u}(x) = \frac{P}{EAL}x^2 + \frac{qL}{2EA}x}$$

b) $\hat{u}(x) = ax^2 + bx + c$

* B.C. : $\hat{u}(0) = 0 \Rightarrow c = 0$

* odes : $\left(\frac{\partial \hat{\Pi}}{\partial a} = 0\right)$ & $\left(\frac{\partial \hat{\Pi}}{\partial b} = 0\right)$

$$\boxed{\hat{u}(x) = -\frac{q}{2EA}x^2 + \frac{P+Lq}{EA}x}$$

c) $\hat{u}(x) = ax^3 + bx^2 + cx + d$

* B.C. : $\hat{u}(0) = 0 \Rightarrow d = 0$

* odes : $\left(\frac{\partial \hat{\Pi}}{\partial a} = 0\right)$ & $\left(\frac{\partial \hat{\Pi}}{\partial b} = 0\right)$ & $\left(\frac{\partial \hat{\Pi}}{\partial c} = 0\right)$

$$\boxed{\hat{u}(x) = -\frac{q}{2EA}x^2 + \frac{P+Lq}{EA}x}$$

⇐ note : same as quadratic approximation (also the exact soln.)

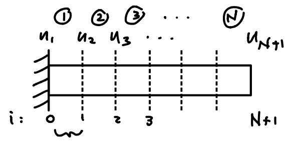
d) $\begin{cases} \hat{u}_{left} = ax + b & \text{for } 0 \le x \le \frac{L}{2} \\ \hat{u}_{right} = cx + d & \text{for } \frac{L}{2} \le x \le L \end{cases}$

* B.C. : $\hat{u}_{left}(0) = 0$; $\hat{u}_{left}\left(\frac{L}{2}\right) = \hat{u}_{right}\left(\frac{L}{2}\right) \Rightarrow$ $b = 0$, solve for d as func. of a and c

* odes : $\left(\frac{\partial \hat{\Pi}}{\partial a} = 0\right)$ & $\left(\frac{\partial \hat{\Pi}}{\partial c} = 0\right)$

$$\boxed{\hat{u}(x) = \begin{cases} \frac{4P+3Lq}{4EA}x & \text{for } 0 \le x \le \frac{L}{2} \\ \frac{4P+Lq}{4EA}x + \frac{L^2 q}{4EA} & \text{for } \frac{L}{2} \le x \le L \end{cases}}$$

1.3.



$\Delta x = \frac{L}{N}$ ← const.

use linear approximation ; but also remember that each consecutive element needs to have continuous B.C.s

$$\hat{u}_1(1\cdot\Delta x) = \hat{u}_2(1\cdot\Delta x)$$
$$\hat{u}_2(2\cdot\Delta x) = \hat{u}_3(2\cdot\Delta x)$$

$$\hat{u} = \begin{cases} u_1 + \frac{u_2-u_1}{L/N}(x-0) \\ u_2 + \frac{u_3-u_2}{L/N}\left(x-\frac{L}{N}\right) \end{cases}$$

generalize : $\hat{u}_i(i\Delta x) = \hat{u}_{i+1}(i\Delta x)$

$$\hat{u} = \begin{cases} u_i + \frac{u_{i+1}-u_i}{\Delta x}\left(x-(i-1)\Delta x\right) \\ u_{i+1} + \frac{u_{i+1}-u_i}{\Delta x}\left(x-i\Delta x\right) \\ \vdots \\ u_{N-1} + \frac{u_N-u_{N-1}}{\Delta x}\left(x-(N-1)\Delta x\right) \end{cases}$$

$$\hat{u}_i = u_i + \frac{u_{i+1}-u_i}{\Delta x}\left(x-(i-1)\Delta x\right) \Rightarrow \hat{u} = \sum_{i=1}^{n}\hat{u}_i$$

$$\frac{\partial \hat{u}_i}{\partial x} = \frac{u_{i+1}-u_i}{\Delta x} \Rightarrow \frac{\partial \hat{u}}{\partial x} = \sum_{i=1}^{n}\hat{u}_i$$

assume $q(x) = q$ = const. ; move outside of the integral

$$\hat{\Pi} = \frac{EA}{2}\sum_{i=1}^{N}\underbrace{\int_{(i-1)\Delta x}^{i\Delta x}\left(\frac{\partial \hat{u}_i}{\partial x}\right)^2 dx}_{\mathrm{I}} - \sum_{i=1}^{N}\underbrace{\int_{(i-1)\Delta x}^{i\Delta x} q\hat{u}_i\, dx}_{\mathrm{II}} - P\hat{u}_{N+1}\big|_{x=L}$$

$\cancel{i^2\Delta x^2} - (\cancel{i^2}-2i+1)\Delta x^2$
$(2i-1)\Delta x^2$
$2i-\cancel{i}-i+\cancel{1} = i$

I : $\displaystyle\int_{(i-1)\Delta x}^{i\Delta x}\frac{(u_{i+1}-u_i)^2}{(\Delta x)^2}dx$

$$= \left[\frac{(u_{i+1}-u_i)^2}{(\Delta x)^2}x\right]_{(i-1)\Delta x}^{i\Delta x}$$

$$= \frac{(u_{i+1}-u_i)^2}{(\Delta x)^2}\Delta x$$

$$= \frac{(u_{i+1}-u_i)^2}{\Delta x}$$

II : $\displaystyle q\int_{(i-1)\Delta x}^{i\Delta x} u_i + \frac{u_{i+1}-u_i}{\Delta x}\left(x-(i-1)\Delta x\right)dx$

$$= q\left[u_i x + \frac{u_{i+1}-u_i}{2\Delta x}\left(x^2-(i-1)\Delta x\cdot x\right)\right]_{(i-1)\Delta x}^{i\Delta x}$$

$$= q\left[u_i\Delta x + \frac{u_{i+1}-u_i}{2\cancel{\Delta x}}\left((2i-1)(\Delta x)^{\cancel{2}} - (i-1)(\Delta x)^{\cancel{2}}\right)\right]$$

$$= q\left(u_i\Delta x + \frac{1}{2}(u_{i+1}-u_i)\cdot i\cdot\Delta x\right)$$

$$\hat{\Pi} = \frac{EA}{2}\sum_{i=1}^{N}\frac{(u_{i+1}-u_i)^2}{\Delta x} - \sum_{i=1}^{N} q\left(u_i\Delta x + \frac{1}{2}(u_{i+1}-u_i)\,i\cdot\Delta x\right) - P\hat{u}_{N+1}\big|_{x=L}$$

$$\lim_{N\to\infty}\Delta x = \frac{L}{N} = 0 \Rightarrow \boxed{\lim_{\Delta x\to 0}\hat{\Pi} = \frac{EA}{2}\int_0^L\left(\frac{\partial \hat{u}}{\partial x}\right)^2 dx - \int_0^L q\hat{u}\,dx - P\hat{u}_{N+1}\big|_{x=L}}$$

* as each element becomes diminishingly small ($N\to\infty$ $\therefore$ $\Delta x\to 0$), the summation can be replaced with integration w.r.t. dx, allowing us to recover the exact solution (discrete → continuous)

1.4. N=50 Piecewise R-R

element @ $i=1$: $\frac{\partial\Pi}{\partial u_1} = EA\int_{x_1}^{x_2}\frac{u_2-\cancel{u_1}}{l}\cdot\frac{-1}{l}\,dx = k(\cancel{u_1}-u_2)$

$i=2$: $\frac{\partial\Pi}{\partial u_2} = k(u_2-\cancel{u_1}-u_3+u_2)$

$i=3$: $\frac{\partial\Pi}{\partial u_3} = k(u_3-u_2-u_4+u_3)$

$\vdots$

$i=N$: $\frac{\partial\Pi}{\partial u_4} = k(u_N-u_{N-1})$

* note
let $\frac{EA}{l} = k$
$u_1 = 0$ from B.C. @ $x=0$
⇒ first row & first column → 0

$$\begin{bmatrix}\frac{\partial\Pi}{\partial u_1}\\ \frac{\partial\Pi}{\partial u_2}\\ \frac{\partial\Pi}{\partial u_3}\\ \vdots\\ \frac{\partial\hat{\Pi}}{\partial u_N}\end{bmatrix} = k\cdot\begin{bmatrix}0&0&0&0&\cdots&0&0\\0&2&-1&0&\cdots&0&0\\0&-1&2&-1&\cdots&0&0\\&&&\cdots&&&\\0&0&0&0&\cdots&-1&1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\\vdots\\u_N\end{Bmatrix} = \vec{0}$$

solve for all the coefficients u_i by computing the eigenvalue of the matrix (done in python)

PROBLEM 2

We know from HW # 1 that the exact solution is quadratic for both cases.

Approximate using a quadratic assumption for $\hat{u}$

recall:

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L q(x)\,u\,dx - Pu|_{x=L}$$

Case 1.

$$\hat{u}_1 = \begin{cases} \hat{u}_{11}(x) = ax^2 + bx + c & \text{for } 0 \le x \le \frac{L}{2} \quad q(x) = +q \\ \hat{u}_{12}(x) = dx^2 + ex + f & \text{for } \frac{L}{2} \le x \le L \quad q(x) = -q \end{cases}$$

B.C. : $\hat{u}_{11}(0) = 0 \Rightarrow c = 0$

Continuity : $\hat{u}_{11}(\frac{L}{2}) = \hat{u}_{12}(\frac{L}{2}) \Rightarrow \frac{1}{4}aL^2 + \frac{1}{2}bL = \frac{1}{4}dL^2 + \frac{1}{2}eL + f \Rightarrow f = \frac{L^2}{4}(a-d) + \frac{L}{2}(b-e)$

$$\hat{\Pi} = \frac{EA}{2}\left(\int_0^{L/2}\left(\frac{\partial \hat{u}_{11}}{\partial x}\right)^2 dx + \int_{L/2}^{L}\left(\frac{\partial \hat{u}_{12}}{\partial x}\right)^2 dx\right) - \left(\int_0^{L/2} q\,\hat{u}_{11}\,dx + \int_{L/2}^{L} -q\,\hat{u}_{12}\,dx\right) - Pu|_{x=L} \quad \text{s.t.} \quad \frac{\partial \hat{\Pi}}{\partial a} = \frac{\partial \hat{\Pi}}{\partial b} = \frac{\partial \hat{\Pi}}{\partial d} = 0$$

Case 2.

$$\hat{u}_2 = \begin{cases} \hat{u}_{21}(x) = ax^2 + bx + c & \text{for } 0 \le x \le \frac{L}{2} \quad q(x) = +q \\ \hat{u}_{22}(x) = dx^2 + ex + f & \text{for } \frac{L}{2} \le x \le L \quad q(x) = -q \end{cases}$$

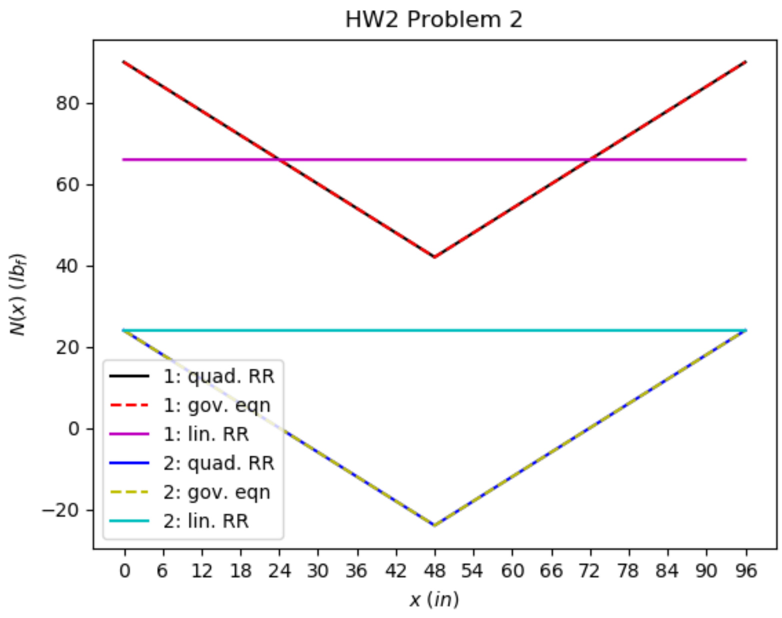
B.C. : $\hat{u}_{21}(0) = 0 \Rightarrow c = 0$

$\hat{u}_{22}(L) = 0 \Rightarrow f = -dL^2 - eL$

$\hat{u}_{21}(\frac{L}{2}) = \hat{u}_{22}(\frac{L}{2}) \Rightarrow a(\frac{L}{2})^2 + b(\frac{L}{2}) = d(\frac{L}{2})^2 + e(\frac{L}{2}) + (-dL^2 - eL)$

$$\hat{\Pi} = \frac{EA}{2}\left(\int_0^{L/2}\left(\frac{\partial \hat{u}_{11}}{\partial x}\right)^2 dx + \int_{L/2}^{L}\left(\frac{\partial \hat{u}_{12}}{\partial x}\right)^2 dx\right) - \left(\int_0^{L/2} q\,\hat{u}_{11}\,dx + \int_{L/2}^{L} -q\,\hat{u}_{12}\,dx\right) - Pu|_{x=L} \quad \text{s.t.} \quad \frac{\partial \hat{\Pi}}{\partial a} = \frac{\partial \hat{\Pi}}{\partial b} = \frac{\partial \hat{\Pi}}{\partial d} = 0$$

* algebra in Mathematica $\Rightarrow$ quadratic approximation allows us to recover the exact soln.
* plotted in python $\Rightarrow$ also plotted linear approximation as comparison



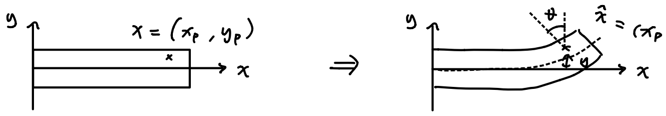
PROBLEM 3

3.1. $\Pi = W - E$

$W = \int_\Omega \frac{1}{2} \underline{\underline{\sigma}} \, \underline{\underline{\varepsilon}} \, dV$

$= \int_\Omega \frac{1}{2} \sigma_x \, \varepsilon_x \, dV$

(assuming uniaxial stress)



$\hat{x} = (x_p - y_p \sin\theta \, , \; y + y_p \cos\theta)$

$y = y(x)$

$\theta = y'$

assume : small deformation angle , i.e. $\theta << 1$, $\sin\theta \approx \theta$, $\cos\theta \approx 1$

displacement : $u = \hat{x} - x = \begin{pmatrix} -y_p \sin\theta \\ y + (\cos\theta - 1)\, y_p \end{pmatrix} \approx \begin{pmatrix} -y_p \theta \\ y \end{pmatrix}$ $\quad$ * let $V(x)$ denote the vertical displacement

$\varepsilon_x = \frac{\partial u_x}{\partial x} = -y \frac{\partial^2 V}{\partial x^2}$; $\varepsilon_y = \frac{\partial V}{\partial y} = 0$; $\varepsilon_{xy} = 0$

$\sigma_x = E \varepsilon_x = -E y \frac{\partial^2 V}{\partial x^2}$ $\quad \Big\} \Rightarrow$ plug into strain energy

$W = \int_\Omega \frac{1}{2} E \varepsilon_x \varepsilon_x \, dV$

$= \int_\Omega \frac{1}{2} E y^2 \left(\frac{\partial^2 V}{\partial x^2}\right)^2 dA \, dx$

$= \int_0^L \frac{1}{2} E \left(\frac{\partial^2 V}{\partial x^2}\right)^2 \Big(\underbrace{\iint y^2 dy \, dz}_{I} \Big) dx$

$W = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 V}{\partial x^2}\right)^2 dx$

Work done by body forces :

$E = \int_0^L q v \, dx + P v|_{x=L}$

elastic potential

$\boxed{\Pi = W - E = \frac{EI}{2} \int_0^L \left(\frac{\partial^2 V}{\partial x^2}\right)^2 dx - \int_0^L q v \, dx - P v|_{x=L}}$

3.2. a) $\hat{V}_1 = ax^2 + bx + c$

B.C. : $\hat{V}_1(0) = \frac{\partial \hat{V}_1}{\partial x}\Big|_{x=0} = 0 \Rightarrow b = c = 0 \Rightarrow \hat{V}_1 = ax^2$

solve $\frac{\partial \Pi}{\partial a}$ in Mathematica

$\boxed{\hat{V}_1(x) = \frac{3LP + L^2 q}{12 EI} x^2}$

b) $\hat{V}_2 = a \cos(bx) + c$

B.C. : $\hat{V}_2(0) = 0 \Rightarrow c = 0$

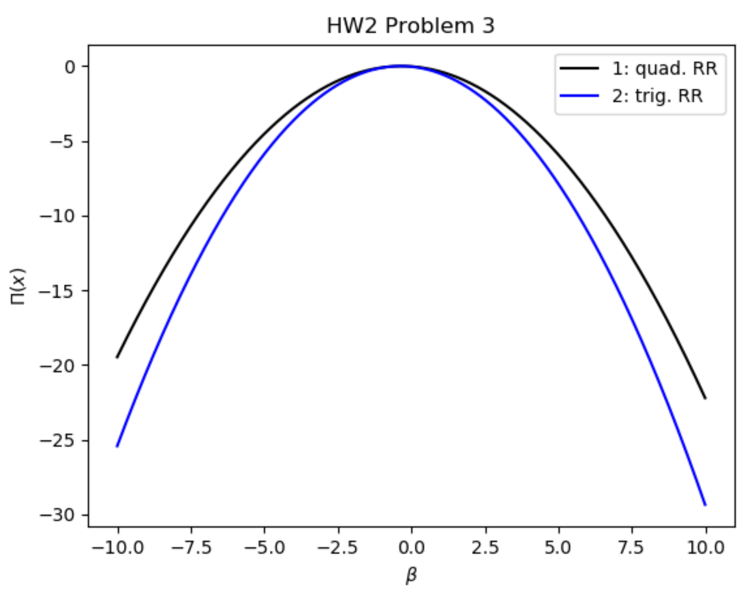
$\frac{\partial^2 \hat{V}_2}{\partial x^2}\Big|_{x=L} = a \cos(bL) = 0$ b/c no applied load at tip $\quad$ let $bL = \frac{\pi}{2}$

for $a, b \neq 0$, $\cos(bL) = 0$ $\quad$ $b = \frac{\pi}{2L}$

$\boxed{\hat{V}_2(x) = \frac{32 L^3 (P\pi - 2Lq + L\pi q)}{\pi^5 EI} \left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)}$

3.3. plugging in all #'s in Mathematica & express potential in terms of β

$\boxed{\hat{\Pi}_1 = -\frac{5}{216}(1 + 3\beta)^2}$ $\qquad$ $\boxed{\hat{\Pi}_2 = -\frac{80(-2 + \pi + \pi\beta)^2}{3\pi^6}}$



3.4.

$\boxed{\text{quadratic}}$ is better b/c the elastic potential is closer to 0 for all β

AE 4132 HW2 - Yunqing Jia

Problem 1

In[21]:=
```
Remove["Global`*"] (* clear workspace *)
```

1.1. Corresponding expression for the elastic potential

The elastic potential is defined as: $\Pi = W - E$.
We assume that the bar has constant cross-sectional area A, the material is elastic isotropic, and the stress is uniaxial (i.e. only $\sigma_x \neq 0$, all other stresses = 0). We get the expression

$$W = \int \frac{1}{2}\, \sigma_x\, \epsilon_x\, dV$$

Express σ_x in terms of strain using Hooke's law

In[22]:= $\sigma_x = \lambda\,(\epsilon_x + \epsilon_y + \epsilon_z) + 2\,\mu\,\epsilon_x;$

In[23]:= $\sigma_y = \lambda\,(\epsilon_x + \epsilon_y + \epsilon_z) + 2\,\mu\,\epsilon_y;$

In[24]:= $\sigma_z = \lambda\,(\epsilon_x + \epsilon_y + \epsilon_z) + 2\,\mu\,\epsilon_z;$

Apply the uniaxial stress assumption and solve for σ_y and σ_z as functions of σ_x

In[25]:= soln = Solve[{σ_y == 0, σ_z == 0}, {ϵ_y, ϵ_z}] // Simplify

Out[25]= $\left\{\left\{\epsilon_y \to -\frac{\lambda\,\epsilon_x}{2\,(\lambda+\mu)},\ \epsilon_z \to -\frac{\lambda\,\epsilon_x}{2\,(\lambda+\mu)}\right\}\right\}$

Solve for σ_x as strictly a function of $(\epsilon_x, \mu, \lambda)$

In[26]:= σ_x /. soln // Simplify

Out[26]= $\left\{\frac{\mu\,(3\,\lambda + 2\,\mu)\,\epsilon_x}{\lambda+\mu}\right\}$

Express coefficient relating strain to stress as E, Young's modulus

In[27]:= $\frac{\sigma_x}{\epsilon_x}$ /. soln // Simplify

Out[27]= $\left\{\frac{\mu\,(3\,\lambda + 2\,\mu)}{\lambda+\mu}\right\}$

Recall that $\epsilon_x = \frac{\delta u}{\delta x}$, and that displacement u does not vary w.r.t. δy or δz. We can replace ∂V with $A\,\delta$x.
The strain energy can then be defined as:

$$W = \int_\Omega \frac{1}{2}\, E\, {\epsilon_x}^2\, dV = \frac{E\,A}{2}\int_0^L \left(\frac{\delta u}{\delta x}\right)^2 dx$$

In[28]:= $W[x] = \frac{E A}{2} \int_0^L (\partial_x u[x])^2 \, dx$

Out[28]= $\frac{1}{2} A E \int_0^L u'[x]^2 \, dx$

The work done by distributed and applied forces can be expressed as:

In[29]:= $E_0[x] = \int_0^L q[x] \times u[x] \, dx + (P\, u[x] \text{ /. } x \to L)$

Out[29]= $\int_0^L q[x] \times u[x] \, dx + P\, u[L]$

The elastic potential is:

In[30]:= $\Pi[x] == W[x] - E_0[x];$

In[31]:= `% // Framed`

Out[31]= $\boxed{\Pi[x] == -\int_0^L q[x] \times u[x] \, dx + \frac{1}{2} A E \int_0^L u'[x]^2 \, dx - P\, u[L]}$

The solution is obtained by minimizing Π for all possible displacement fields that satisfy the B.C.s. ($\delta\Pi$ = 0)

$$\delta\Pi = \frac{E A}{2} \delta \left(\int_0^L \left(\frac{\delta u}{\delta x} \right)^2 dx \right) - \delta \left(\int_0^L q[x] \, u \, dx \right) - \delta \, (P\, u[L])$$

$$= \frac{E A}{2} \int_0^L \delta \left(\frac{\delta u}{\delta x} \right)^2 dx - \int_0^L \delta \, (q[x] \, u) \, dx - P \, \delta u[L]$$

$$= \frac{E A}{2} \int_0^L 2 \, \frac{\delta u}{\delta x} \, \delta \left(\frac{\delta u}{\delta x} \right) dx - \int_0^L q[x] \, \delta u \, dx - P \, \delta u[L]$$

$$\delta\Pi = 0$$

Apply integration by part, the first integral becomes:

$$r = \frac{\delta u}{\delta x}, \quad ds = \frac{\delta \, \delta u}{\delta x} dx, \quad dr = \frac{\delta^2 u}{\delta x^2}, \quad s = \delta u$$

$$\int_0^L 2 \, \frac{\delta u}{\delta x} \, \delta \left(\frac{\delta u}{\delta x} \right) dx = \frac{\delta u}{\delta x} \, \delta u[L] - \int_0^L \delta u \, \frac{\delta^2 u}{\delta x^2} \, dx$$

$$\delta\Pi = E A \, \frac{\delta u}{\delta x} \, \delta u[L] - E A \int_0^L \delta u \, \frac{\delta^2 u}{\delta x^2} \, dx - \int_0^L q[x] \, \delta u \, dx - P \, \delta u[L] == 0$$

The final expression for the partial of Π w.r.t. *x* can be expressed as the governing equation + the traction boundary condition equation:

$$\delta\Pi = -\int_0^L \left(E A \, \frac{\delta^2 u}{\delta x^2} \, dx - q[x] \right) \delta u \, dx + \left(E A \, \frac{\delta u}{\delta x} - P \right) \delta u[L] == 0$$

1.2. Rayleigh-Ritz Method

(a) $\hat{u} = a\,x + b$
(b) $\hat{u} = a\,x^2 + b\,x + c$
(c) $\hat{u} = a\,x^3 + b\,x^2 + c\,x + d$
(d) $\hat{u} = a\,x + b$ for $0 < x < \frac{L}{2}$ && $c\,x + d$ for $\frac{L}{2} < x < L$

Boundary condition for the uni-axially loaded bar is $\hat{u}(0) = 0$

(a) $\hat{u}(x) = a\,x + b$

By applying the B.C., we know that $\hat{u}(x) = a\,(0) + b$, therefore $b = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a\,x$, $\frac{\delta \hat{u}}{\delta x} = a$, $\hat{u}(L) = a\,L$

Substitute these expressions back into the equation for elastic potential along with the assumption $q(x) = \text{const.} = q$

In[32]:= $\hat{u}_1[x\_] = a\,x;$

In[33]:= $\hat{\Pi}_1 = \frac{E\,A}{2} \int_0^L (\partial_x \hat{u}_1[x])^2 \, dx - \int_0^L q\,(\hat{u}_1[x]) \, dx - P\,\hat{u}_1[x];$

Now minimize $\hat{\Pi}$ w.r.t. a

In[34]:= $\partial_a \hat{\Pi}_1$

Out[34]= $-\frac{L^2 q}{2} - P\,x + a\,A\,L\,E$

In[35]:= $\text{soln1} = \text{Solve}[\partial_a \hat{\Pi}_1 == 0,\ a]$

Out[35]= $\left\{\left\{a \to \frac{L^2 q + 2\,P\,x}{2\,A\,L\,E}\right\}\right\}$

Substitute the solved expression of a back into the equation for $\hat{u}$:

In[36]:= $\hat{u}_1[x] = \hat{u}_1[x]$ /. soln1[[1]];

In[37]:= % // Framed

Out[37]= $\boxed{\frac{x\,(L^2 q + 2\,P\,x)}{2\,A\,L\,E}}$

(b) $\hat{u} = a\,x^2 + b\,x + c$

By applying the B.C., we know that $\hat{u}(x) = a\,(0)^2 + b(0) + c$, therefore $c = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a\,x^2 + b\,x$, $\frac{\delta \hat{u}}{\delta x} = 2\,a\,x + b$, $\hat{u}(L) = a\,L^2 + b\,L$

In[38]:= $\hat{u}_2[x\_] = a\,x^2 + b\,x;$

In[39]:= $\hat{\Pi}_2 = \frac{E\,A}{2} \int_0^L (\partial_x \hat{u}_2[x])^2 \, dx - \int_0^L q\,(\hat{u}_2[x]) \, dx - P\,\hat{u}_2[L];$

Now minimize $\hat{\Pi}_2$ w.r.t. both a and b

In[40]:= $\partial_a \hat{\Pi}_2$

Out[40]= $-L^2 P - \frac{L^3 q}{3} + \frac{1}{2} A \left(2 b L^2 + \frac{8 a L^3}{3}\right) \mathbb{E}$

In[41]:= $\partial_b \hat{\Pi}_2$

Out[41]= $-L P - \frac{L^2 q}{2} + \frac{1}{2} A \left(2 b L + 2 a L^2\right) \mathbb{E}$

In[42]:= **soln2 = Solve**$\left[\left\{\partial_a \hat{\Pi}_2 == 0, \partial_b \hat{\Pi}_2 == 0\right\}, \{a, b\}\right]$

Out[42]= $\left\{\left\{a \to -\frac{q}{2 A \mathbb{E}}, b \to -\frac{-P - L q}{A \mathbb{E}}\right\}\right\}$

In[43]:= **û₂[x] = û₂[x] /. soln2[[1]] // Simplify;**

In[44]:= **% // Framed**

Out[44]= $\boxed{\frac{x \left(2 P + 2 L q - q x\right)}{2 A \mathbb{E}}}$

(c) $\hat{u} = a x^3 + b x^2 + c x + d$

By applying the B.C., we know that $\hat{u}(0) = a\,(0)^3 + b(0)^2 + c(0) + d$, therefore $d = 0$. The expressions for $\hat{u}$ then become $\hat{u}(x) = a x^3 + b x^2 + c x$, $\frac{\delta \hat{u}}{\delta x} = 3 a x^2 + 2 b x + c$, $\hat{u}(L) = a L^3 + b L^2 + c L$

In[45]:= **û₃[x_] = a x³ + b x² + c x;**

In[46]:= $\hat{\Pi}_3 = \frac{\mathbb{E} A}{2} \int_0^L \left(\partial_x \hat{u}_3[x]\right)^2 dx - \int_0^L q \left(\hat{u}_3[x]\right) dx - P \hat{u}_3[L];$

In[47]:= $\partial_a \hat{\Pi}_3$

Out[47]= $-L^3 P - \frac{L^4 q}{4} + \frac{1}{2} A \left(2 c L^3 + 3 b L^4 + \frac{18 a L^5}{5}\right) \mathbb{E}$

In[48]:= $\partial_b \hat{\Pi}_3$

Out[48]= $-L^2 P - \frac{L^3 q}{3} + \frac{1}{2} A \left(2 c L^2 + \frac{8 b L^3}{3} + 3 a L^4\right) \mathbb{E}$

In[49]:= $\partial_c \hat{\Pi}_3$

Out[49]= $-L P - \frac{L^2 q}{2} + \frac{1}{2} A \left(2 c L + 2 b L^2 + 2 a L^3\right) \mathbb{E}$

Now solve all three equations simultaneously (thanks Mathematica)

In[50]:= **soln3 = Solve**$\left[\left\{\partial_a \hat{\Pi}_3 == 0, \partial_b \hat{\Pi}_3 == 0, \partial_c \hat{\Pi}_3 == 0\right\}, \{a, b, c\}\right]$

Out[50]= $\left\{\left\{a \to 0, b \to -\frac{q}{2 A \mathbb{E}}, c \to -\frac{-P - L q}{A \mathbb{E}}\right\}\right\}$

In[51]:=
```
û3[x] = û3[x] /. soln3[[1]] // Simplify;
```

In[52]:=
```
% // Framed
```

Out[52]= $\boxed{\dfrac{x\,(2\,P + 2\,L\,q - q\,x)}{2\,A\,E}}$

Note: the 3rd-order polynomial approximation yields the exact same solution as the 2nd-order polynomial. The 2nd-order solution is already the exact solution.

In[53]:=
```
û2[x] == û3[x]
```

Out[53]= True

(d) $\hat{u} = a\,x + b$ for $0 < x < \frac{L}{2}$ && $c\,x + d$ for $\frac{L}{2} < x < L$

Let the left half of the bar be denoted $\hat{u}_{4\,\text{left}}$ and the right half $\hat{u}_{4\,\text{right}}$.

By applying the B.C. at the left, fixed end, we know that $\hat{u}_{4\,\text{left}}(0) = a\,(0) + b$, therefore $b = 0$. The expressions for $\hat{u}_{4\,\text{left}}$ then become $\hat{u}_{4\,\text{left}}(x) = a\,x$, $\frac{\delta\,\hat{u}_{4\,\text{left}}}{\delta x} = a$

In[54]:=
```
û4left[x_] = a x;  û4right[x_] = c x + d;
```

In[55]:=
```
d = d /. Solve[û4left[L/2] == û4right[L/2], d][[1]]
```

Out[55]= $\frac{1}{2}\,(a\,L - c\,L)$

In[56]:= $\hat{\Pi}_4 = \frac{E\,A}{2}\left(\int_0^{\frac{L}{2}} \left(\partial_x\,\hat{u}_{4\,\text{left}}[x]\right)^2 dx + \int_{\frac{L}{2}}^{L} \left(\partial_x\,\hat{u}_{4\,\text{right}}[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\left(\hat{u}_{4\,\text{left}}[x]\right) dx + \int_{\frac{L}{2}}^{L} q\,\left(\hat{u}_{4\,\text{right}}[x]\right) dx\right) - P\,\hat{u}_{4\,\text{right}}[L]$

Out[56]= $-\left(c\,L + \frac{1}{2}\,(a\,L - c\,L)\right)P - \frac{3}{8}\,a\,L^2\,q - \frac{1}{8}\,c\,L^2\,q + \frac{1}{2}\,A\left(\frac{a^2\,L}{2} + \frac{c^2\,L}{2}\right)E$

In[57]:= $\partial_a\,\hat{\Pi}_4$

Out[57]= $-\frac{L\,P}{2} - \frac{3\,L^2\,q}{8} + \frac{1}{2}\,a\,A\,L\,E$

In[58]:= $\partial_c\,\hat{\Pi}_4$

Out[58]= $-\frac{L\,P}{2} - \frac{L^2\,q}{8} + \frac{1}{2}\,A\,c\,L\,E$

Apply the continuity B.C. at the center of the bar $\left(x = \frac{L}{2}\right)$: $\hat{u}_{4\,\text{left}}\left(\frac{L}{2}\right) = \hat{u}_{4\,\text{right}}\left(\frac{L}{2}\right)$.

In[59]:=
```
soln4 = Solve[{∂a Π̂4 == 0, ∂c Π̂4 == 0}, {a, c}]
```

Out[59]= $\left\{\left\{a \to \frac{4\,P + 3\,L\,q}{4\,A\,E},\ c \to \frac{4\,P + L\,q}{4\,A\,E}\right\}\right\}$

In[60]:=
```
d /. soln4 // Simplify
```

Out[60]= $\left\{\frac{L^2 q}{4 A \mathbb{E}}\right\}$

In[61]:=
```
û4left[x] = û4left[x] /. soln4[[1]] // Simplify;
```

In[62]:=
```
% // Framed
```

Out[62]= $\boxed{\frac{(4 P + 3 L q) x}{4 A \mathbb{E}}}$

In[63]:=
```
û4right[x] = û4right[x] /. soln4[[1]] // Simplify;
```

In[64]:=
```
% // Framed
```

Out[64]= $\boxed{\frac{L^2 q + 4 P x + L q x}{4 A \mathbb{E}}}$

1.3. N-segment piece-wise linear approximation

*Derived by hand. But recovered the exact expression (integral form) for elastic potential as the $N \to \infty$ (*i.e.* $\Delta x \to 0$) in the piecewise approximation.

1.4. Plot the specific case in Python

-> Use matrix format

In[65]:=
```
Remove["Global`*"]
```

N=4

In[66]:=
```
û41[x_] = a x;  û42[x_] = c x + d;  û43[x_] = e x + f;  û44[x_] = g x + h;
```

In[67]:=
```
d = d /. Solve[(û41[x] /. x → L/4) == (û42[x] /. x → L/4), d][[1]];
```

In[68]:=
```
f = f /. Solve[û42[2 L/4] == û43[2 L/4], f][[1]];
```

In[69]:=
```
h = h /. Solve[û43[3 L/4] == û44[3 L/4], h][[1]];
```

In[70]:=
$$\hat{\Pi}_4 = ((\mathbb{E}\,A)/2)\left(\int_0^{\frac{L}{4}}(\partial_x\,\hat{u}_{41}[x])^2\,dx + \int_{\frac{L}{4}}^{\frac{2L}{4}}(\partial_x\,\hat{u}_{42}[x])^2\,dx + \int_{\frac{2L}{4}}^{\frac{3L}{4}}(\partial_x\,\hat{u}_{43}[x])^2\,dx + \int_{\frac{3L}{4}}^{\frac{4L}{4}}(\partial_x\,\hat{u}_{44}[x])^2\,dx\right) - \left(\int_0^{\frac{L}{4}} q\,(\hat{u}_{41}[x])\,dx + \int_{\frac{L}{4}}^{\frac{2L}{4}} q\,(\hat{u}_{42}[x])\,dx + \int_{\frac{2L}{4}}^{\frac{3L}{4}} q\,(\hat{u}_{43}[x])\,dx + \int_{\frac{3L}{4}}^{\frac{4L}{4}} q\,(\hat{u}_{44}[x])\,dx\right) - P\,\hat{u}_{44}[L]$$

Out[70]=
$$-\left(g\,L + \frac{1}{4}(a\,L + c\,L + e\,L - 3\,g\,L)\right)P - \frac{7}{32}a\,L^2\,q - \frac{5}{32}c\,L^2\,q - \frac{3}{32}e\,L^2\,q - \frac{1}{32}g\,L^2\,q + \frac{1}{2}A\left(\frac{a^2\,L}{4} + \frac{c^2\,L}{4} + \frac{e^2\,L}{4} + \frac{g^2\,L}{4}\right)\mathbb{E}$$

In[71]:= $\partial_a\,\hat{\Pi}_4$

Out[71]=
$$-\frac{L\,P}{4} - \frac{7\,L^2\,q}{32} + \frac{1}{4}a\,A\,L\,\mathbb{E}$$

In[72]:= $\partial_c\,\hat{\Pi}_4$

Out[72]=
$$-\frac{L\,P}{4} - \frac{5\,L^2\,q}{32} + \frac{1}{4}A\,c\,L\,\mathbb{E}$$

In[73]:= $\partial_e\,\hat{\Pi}_4$

Out[73]=
$$-\frac{L\,P}{4} - \frac{3\,L^2\,q}{32} + \frac{1}{4}A\,e\,L\,\mathbb{E}$$

In[74]:= $\partial_g\,\hat{\Pi}_4$

Out[74]=
$$-\frac{L\,P}{4} - \frac{L^2\,q}{32} + \frac{1}{4}A\,g\,L\,\mathbb{E}$$

Apply the continuity B.C. at the center of the bar $\left(x = \frac{L}{2}\right)$: $\hat{u}_{4\,\text{left}}\left(\frac{L}{2}\right) = \hat{u}_{4\,\text{right}}\left(\frac{L}{2}\right)$.

In[75]:= soln5 = Solve[$\{\partial_a\,\hat{\Pi}_4 == 0,\ \partial_c\,\hat{\Pi}_4 == 0,\ \partial_e\,\hat{\Pi}_4 == 0,\ \partial_g\,\hat{\Pi}_4 == 0\}$, {a, c, e, g}]

Out[75]=
$$\left\{\left\{a \to \frac{8\,P + 7\,L\,q}{8\,A\,\mathbb{E}},\ c \to \frac{8\,P + 5\,L\,q}{8\,A\,\mathbb{E}},\ e \to \frac{8\,P + 3\,L\,q}{8\,A\,\mathbb{E}},\ g \to \frac{8\,P + L\,q}{8\,A\,\mathbb{E}}\right\}\right\}$$

In[76]:= $\hat{u}_{41}$[x] = $\hat{u}_{41}$[x] /. soln5[[1]]

Out[76]=
$$\frac{(8\,P + 7\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[77]:= $\hat{u}_{42}$[x] = $\hat{u}_{42}$[x] /. soln5[[1]]

Out[77]=
$$\frac{1}{4}\left(-\frac{L\,(8\,P + 5\,L\,q)}{8\,A\,\mathbb{E}} + \frac{L\,(8\,P + 7\,L\,q)}{8\,A\,\mathbb{E}}\right) + \frac{(8\,P + 5\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[78]:= $\hat{u}_{43}$[x] = $\hat{u}_{43}$[x] /. soln5[[1]]

Out[78]=
$$\frac{1}{4}\left(-\frac{L\,(8\,P + 3\,L\,q)}{4\,A\,\mathbb{E}} + \frac{L\,(8\,P + 5\,L\,q)}{8\,A\,\mathbb{E}} + \frac{L\,(8\,P + 7\,L\,q)}{8\,A\,\mathbb{E}}\right) + \frac{(8\,P + 3\,L\,q)\,x}{8\,A\,\mathbb{E}}$$

In[79]:=
```
û44[x] = û44[x] /. soln5[[1]]
```

Out[79]= $\frac{1}{4}\left(-\frac{3 L (8 P+L q)}{8 A \mathbb{E}}+\frac{L (8 P+3 L q)}{8 A \mathbb{E}}+\frac{L (8 P+5 L q)}{8 A \mathbb{E}}+\frac{L (8 P+7 L q)}{8 A \mathbb{E}}\right)+\frac{(8 P+L q) x}{8 A \mathbb{E}}$

In[80]:=
```
d
```

Out[80]= $\frac{1}{4}(a L-c L)$

In[81]:=
```
f
```

Out[81]= $\frac{1}{4}(a L+c L-2 e L)$

In[82]:=
```
h
```

Out[82]= $\frac{1}{4}(a L+c L+e L-3 g L)$

Problem 2

2.1. Case 1

In[83]:=
```
Remove["Global`*"]
```

Approximate with quadratic functions based on solution from HW#1

In[84]:=
```
û1[x_] = a x^2 + b x + c; (* 0 ≤ x ≤ L/2 *)
```

In[85]:=
```
û2[x_] = d x^2 + e x + f; (* L/2 ≤ x ≤ L *)
```

Apply B.C. at the left end of the beam $u_1[x=0]=0$

In[86]:=
```
c = c /. Solve[û1[0] == 0, c][[1]]
```

Out[86]= 0

Apply continuity in the middle of the beam $u_1\left[x=\frac{L}{2}\right]=u_1\left[x=\frac{L}{2}\right]$ and $\frac{\delta u_1}{\delta x}\left[x=\frac{L}{2}\right]=\frac{\delta u_2}{\delta x}\left[x=\frac{L}{2}\right]$

In[87]:=
```
f = f /. Solve[û1[L/2] == û2[L/2], f][[1]]
```

Out[87]= $\frac{1}{4}\left(2 b L-2 e L+a L^2-d L^2\right)$

In[88]:=
```
e = e /. Solve[(∂x û1[x] /. x → L/2) == (∂x û2[x] /. x → L/2), e][[1]]
```

Out[88]= $b+a L-d L$

Now we can express $\hat{\Pi}$ as a function of variables $a,\ b,\ d$. Also note that $q(x)=+q$ for $0\le x\le \frac{L}{2}$ and $q(x)=-q$ for $\frac{L}{2}\le x\le L$

In[89]:= $$\hat{\Pi} = \frac{\mathbb{E}\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right) - \left(P\,\hat{u}_2[x]\ /.\ x \to L\right)$$

Out[89]= $$-\left(d L^2 + L\left(b + a L - d L\right) + \frac{1}{4}\left(2 b L + a L^2 - d L^2 - 2 L\left(b + a L - d L\right)\right)\right) P + \frac{1}{4} b L^2 q + \frac{5}{24} a L^3 q + \frac{1}{24} d L^3 q + \frac{1}{2} A\left(\frac{b^2 L}{2} + \frac{1}{2} a b L^2 + \frac{a^2 L^3}{6} + \frac{-\left(b + a L\right)^3 + \left(b + \left(a + d\right) L\right)^3}{6 d}\right)\mathbb{E}$$

In[90]:= $\partial_a \hat{\Pi}$

Out[90]= $$-\frac{3 L^2 P}{4} + \frac{5 L^3 q}{24} + \frac{1}{2} A\left(\frac{b L^2}{2} + \frac{a L^3}{3} + \frac{-3 L\left(b + a L\right)^2 + 3 L\left(b + \left(a + d\right) L\right)^2}{6 d}\right)\mathbb{E}$$

In[91]:= $\partial_b \hat{\Pi}$

Out[91]= $$-L P + \frac{L^2 q}{4} + \frac{1}{2} A\left(b L + \frac{a L^2}{2} + \frac{-3\left(b + a L\right)^2 + 3\left(b + \left(a + d\right) L\right)^2}{6 d}\right)\mathbb{E}$$

In[92]:= $\partial_d \hat{\Pi}$

Out[92]= $$-\frac{L^2 P}{4} + \frac{L^3 q}{24} + \frac{1}{2} A\left(\frac{L\left(b + \left(a + d\right) L\right)^2}{2 d} - \frac{-\left(b + a L\right)^3 + \left(b + \left(a + d\right) L\right)^3}{6 d^2}\right)\mathbb{E}$$

In[93]:= $\text{soln1} = \text{Solve}\left[\left\{\partial_a \hat{\Pi} == 0,\ \partial_b \hat{\Pi} == 0,\ \partial_d \hat{\Pi} == 0\right\},\ \{a,\ b,\ d\}\right]$

Out[93]= $$\left\{\left\{a \to -\frac{q}{2 A \mathbb{E}},\ b \to \frac{P}{A \mathbb{E}},\ d \to \frac{q}{2 A \mathbb{E}}\right\}\right\}$$

In[94]:= $\hat{u}_1[x] = \hat{u}_1[x]\ /.\ \text{soln1}[[1]]\ //\ \text{Simplify}$

Out[94]= $$-\frac{x\left(-2 P + q x\right)}{2 A \mathbb{E}}$$

In[95]:= $\hat{u}_2[x] = \hat{u}_2[x]\ /.\ \text{soln1}[[1]]\ //\ \text{Simplify}$

Out[95]= $$\frac{L^2 q + 4 P x - 4 L q x + 2 q x^2}{4 A \mathbb{E}}$$

Compare these to the exact solutions

In[96]:= $$u_{1\,\text{exact}} = \frac{1}{\mathbb{E}\,A}\left(\frac{-q}{2} x^2 + P x\right);$$

In[97]:= $\hat{u}_1[x] == u_{1\,\text{exact}}\ //\ \text{Simplify}$

Out[97]= True

In[98]:= $$u_{2\,\text{exact}} = \frac{1}{\mathbb{E}\,A}\left(\frac{q}{2} x^2 + P x - q L x + \frac{q}{4} L^2\right);$$

In[99]:=
```
û₂[x] == u₂ exact // Simplify
```
Out[99]= True

We recovered the exact solution from the piecewise function with quadratic approximations. Yay! Now compute $N(x)$ by taking the derivative.

In[100]:=
```
N₁[x_] = E A ∂x û₁[x] // Simplify
```
Out[100]= $P - q\, x$

In[101]:=
```
N₂[x_] = E A ∂x û₂[x] // Simplify
```
Out[101]= $P + q\, (-L + x)$

With numerical values:

In[102]:=
```
N₁[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[102]= $90 - x$

In[103]:=
```
N₂[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify
```
Out[103]= $-6 + x$

2.2. Case 2

In[104]:=
```
Remove["Global`*"]
```

Use same setup as case 1 except for different B.C. at $\hat{u}_2[x = L] = 0$ because it's a fixed joint.
$\hat{u}_1[x = 0] = 0$ still applies.

In[105]:=
```
û₁[x_] = a x^2 + b x ; (* 0 ≤ x ≤ L/2 *)
```

In[106]:=
```
û₂[x_] = d x^2 + e x + f; (* L/2 ≤ x ≤ L *)
```

Apply B.C. at the right end of the beam: $\hat{u}_2[x = L] = 0$.

In[107]:=
```
f = f /. Solve[û₂[L] == 0, f][[1]]
```
Out[107]= $-e\, L - d\, L^2$

Apply continuity in the middle of the beam $u_1\left[x = \frac{L}{2}\right] = u_1\left[x = \frac{L}{2}\right]$ and $\frac{\delta u_1}{\delta x}\left[x = \frac{L}{2}\right] = \frac{\delta u_2}{\delta x}\left[x = \frac{L}{2}\right]$

In[108]:=
```
e = e /. Solve[û₁[L/2] == û₂[L/2], e][[1]]
```
Out[108]= $\frac{1}{2}\left(-2\, b - a\, L - 3\, d\, L\right)$

Now we can express $\hat{\Pi}$ as a function of variables $a,\ b,\ d$. Also note that $q(x) = +q$ for $0 \leq x \leq \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \leq x \leq L$

In[109]:= $\hat{\Pi} = \frac{E A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\, \hat{u}_1[x]\, dx + \int_{\frac{L}{2}}^{L} -q\, \hat{u}_2[x]\, dx\right)$

Out[109]= $\frac{1}{48} a L^3 q - \frac{1}{48} d L^3 q + \frac{1}{2} A \left(b^2 L + a b L^2 + \frac{7 a^2 L^3}{24} + \frac{d^2 L^3}{24}\right) E$

In[110]:= $\partial_a \hat{\Pi}$

Out[110]= $\frac{L^3 q}{48} + \frac{1}{2} A \left(b L^2 + \frac{7 a L^3}{12}\right) E$

In[111]:= $\partial_b \hat{\Pi}$

Out[111]= $\frac{1}{2} A \left(2 b L + a L^2\right) E$

In[112]:= $\partial_d \hat{\Pi}$

Out[112]= $-\frac{L^3 q}{48} + \frac{1}{24} A d L^3 E$

In[113]:= soln2 = Solve$\left[\left\{\partial_a \hat{\Pi} == 0, \partial_b \hat{\Pi} == 0, \partial_d \hat{\Pi} == 0\right\}, \{a, b, d\}\right]$

Out[113]= $\left\{\left\{a \to -\frac{q}{2 A E}, b \to \frac{L q}{4 A E}, d \to \frac{q}{2 A E}\right\}\right\}$

In[114]:= $\hat{u}_1[x] = \hat{u}_1[x]$ /. soln2[[1]] // Simplify

Out[114]= $\frac{q (L - 2 x) x}{4 A E}$

In[115]:= $\hat{u}_2[x] = \hat{u}_2[x]$ /. soln2[[1]] // Simplify

Out[115]= $\frac{q \left(L^2 - 3 L x + 2 x^2\right)}{4 A E}$

Compare these to the exact solutions

In[116]:= $u_{1\,exact} = \frac{q}{4 E A}\left(-2 x^2 + L x\right);$

In[117]:= $\hat{u}_1[x] == u_{1\,exact}$ // Simplify

Out[117]= True

In[118]:= $u_{2\,exact} = \frac{q}{4 E A}\left(2 x^2 - 3 L x + L^2\right);$

In[119]:= $\hat{u}_2[x] == u_{2\,exact}$ // Simplify

Out[119]= True

Once again recovered the exact solution! Now compute $N(x)$ by taking the derivative.

In[120]:= $N_1[x\_] = E A\, \partial_x \hat{u}_1[x]$ // Simplify

Out[120]= $\frac{1}{4} q (L - 4 x)$

In[121]:=
$$N_2[x\_] = \mathbb{E}\,A\,\partial_x \hat{u}_2[x] \text{ // Simplify}$$

Out[121]=
$$-\frac{3\,L\,q}{4} + q\,x$$

With numerical values:

In[122]:=
$$N_1[x] \text{ /. } \{P \to 90,\ q \to 1,\ L \to 8 * 12\} \text{ // Simplify}$$

Out[122]=
$$24 - x$$

In[123]:=
$$N_2[x] \text{ /. } \{P \to 90,\ q \to 1,\ L \to 8 * 12\} \text{ // Simplify}$$

Out[123]=
$$-72 + x$$

2.3. Case 1 (linear approximation)

In[124]:=
```
Remove["Global`*"]
```

Approximate with linear functions to see how it affects $N(x)$ results

In[125]:=
$$\hat{u}_1[x\_] = a\,x + b;\ (*\ 0 \le x \le \tfrac{L}{2}\ *)$$

In[126]:=
$$\hat{u}_2[x\_] = c\,x + d;\ (*\ \tfrac{L}{2} \le x \le L\ *)$$

Apply B.C. at the left end of the beam $u_1[x = 0] = 0$

In[127]:=
$$b = b \text{ /. Solve}\left[\hat{u}_1[0] == 0,\ b\right][[1]]$$

Out[127]=
$$0$$

Apply continuity in the middle of the beam $u_1\left[x = \frac{L}{2}\right] = u_1\left[x = \frac{L}{2}\right]$

In[128]:=
$$d = d \text{ /. Solve}\left[\hat{u}_1\left[\frac{L}{2}\right] == \hat{u}_2\left[\frac{L}{2}\right],\ d\right][[1]]$$

Out[128]=
$$\frac{1}{2}\,(a\,L - c\,L)$$

Now we can express $\hat{\Pi}$ as a function of variables $a,\ c$. Also note that $q(x) = +q$ for $0 \le x \le \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \le x \le L$

In[129]:=
$$\hat{\Pi} = \frac{\mathbb{E}\,A}{2}\left(\int_0^{\frac{L}{2}} \left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L} \left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right) - \left(P\,\hat{u}_2[x] \text{ /. } x \to L\right)$$

Out[129]=
$$-\left(c\,L + \frac{1}{2}\,(a\,L - c\,L)\right) P + \frac{1}{8}\,a\,L^2\,q + \frac{1}{8}\,c\,L^2\,q + \frac{1}{2}\,A\left(\frac{a^2\,L}{2} + \frac{c^2\,L}{2}\right)\mathbb{E}$$

In[130]:=
$$\partial_a \hat{\Pi}$$

Out[130]=
$$-\frac{L\,P}{2} + \frac{L^2\,q}{8} + \frac{1}{2}\,a\,A\,L\,\mathbb{E}$$

In[131]:= $\partial_c \hat{\Pi}$

Out[131]= $-\frac{L P}{2}+\frac{L^2 q}{8}+\frac{1}{2} A c L \mathbb{E}$

In[132]:= soln1 = Solve[{$\partial_a \hat{\Pi}$ == 0, $\partial_c \hat{\Pi}$ == 0}, {a, c}]

Out[132]= $\left\{\left\{a \to \frac{4 P - L q}{4 A \mathbb{E}}, c \to \frac{4 P - L q}{4 A \mathbb{E}}\right\}\right\}$

In[133]:= $\hat{u}_1$[x] = $\hat{u}_1$[x] /. soln1[[1]] // Simplify

Out[133]= $\frac{(4 P - L q) x}{4 A \mathbb{E}}$

In[134]:= $\hat{u}_2$[x] = $\hat{u}_2$[x] /. soln1[[1]] // Simplify

Out[134]= $\frac{(4 P - L q) x}{4 A \mathbb{E}}$

Compare these to the exact solutions

In[135]:= $u_{1\,\text{exact}} = \frac{1}{\mathbb{E} A}\left(\frac{-q}{2} x^2 + P x\right)$;

In[136]:= $\hat{u}_1$[x] == $u_{1\,\text{exact}}$ // Simplify

Out[136]= $\frac{q (L - 2 x) x}{A \mathbb{E}} == 0$

In[137]:= $u_{2\,\text{exact}} = \frac{1}{\mathbb{E} A}\left(\frac{q}{2} x^2 + P x - q L x + \frac{q}{4} L^2\right)$;

In[138]:= $\hat{u}_2$[x] == $u_{2\,\text{exact}}$ // Simplify

Out[138]= $\frac{q \left(L^2 - 3 L x + 2 x^2\right)}{A \mathbb{E}} == 0$

The linear approximation did not yield the exact solution. Now compute the $N(x)$ expressions

In[139]:= N_1[x_] = $\mathbb{E}$ A ∂_x $\hat{u}_1$[x] // Simplify

Out[139]= $P - \frac{L q}{4}$

In[140]:= N_2[x_] = $\mathbb{E}$ A ∂_x $\hat{u}_2$[x] // Simplify

Out[140]= $P - \frac{L q}{4}$

With numerical values:

In[141]:= N_1[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[141]= 66

In[142]:= N_2[x] /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[142]= 66

2.4. Case 2 (linear approximation)

In[143]:=
```
Remove["Global`*"]
```

Use same setup as case 1 except for different B.C. at $\hat{u}_2[x = L] = 0$ because it's a fixed joint. $\hat{u}_1[x = 0] = 0$ still applies.

In[144]:= $\hat{u}_1[x\_] = a\,x;\ (*\ 0 \le x \le \frac{L}{2}\ *)$

In[145]:= $\hat{u}_2[x\_] = c\,x + d;\ (*\ \frac{L}{2} \le x \le L\ *)$

Apply B.C. at the right end of the beam: $\hat{u}_2[x = L] = 0$.

In[146]:= $d = d\ /.\ \text{Solve}\left[\hat{u}_2[L] == 0,\ d\right][[1]]$

Out[146]= $-c\,L$

Now we can express $\hat{\Pi}$ as a function of variable a. Also note that $q(x) = +q$ for $0 \le x \le \frac{L}{2}$ and $q(x) = -q$ for $\frac{L}{2} \le x \le L$

In[147]:= $$\hat{\Pi} = \frac{E\,A}{2}\left(\int_0^{\frac{L}{2}}\left(\partial_x \hat{u}_1[x]\right)^2 dx + \int_{\frac{L}{2}}^{L}\left(\partial_x \hat{u}_2[x]\right)^2 dx\right) - \left(\int_0^{\frac{L}{2}} q\,\hat{u}_1[x]\,dx + \int_{\frac{L}{2}}^{L} -q\,\hat{u}_2[x]\,dx\right)$$

Out[147]= $$-\frac{1}{8}a\,L^2\,q - \frac{1}{8}c\,L^2\,q + \frac{1}{2}A\left(\frac{a^2\,L}{2} + \frac{c^2\,L}{2}\right)E$$

In[148]:= $\partial_a \hat{\Pi}$

Out[148]= $$-\frac{L^2\,q}{8} + \frac{1}{2}a\,A\,L\,E$$

In[149]:= $\partial_c \hat{\Pi}$

Out[149]= $$-\frac{L^2\,q}{8} + \frac{1}{2}A\,c\,L\,E$$

In[150]:= $\text{soln2} = \text{Solve}\left[\left\{\partial_a \hat{\Pi} == 0,\ \partial_c \hat{\Pi} == 0\right\},\ \{a,\ c\}\right]$

Out[150]= $$\left\{\left\{a \to \frac{L\,q}{4\,A\,E},\ c \to \frac{L\,q}{4\,A\,E}\right\}\right\}$$

In[151]:= $\hat{u}_1[x] = \hat{u}_1[x]\ /.\ \text{soln2}[[1]]\ //\ \text{Simplify}$

Out[151]= $$\frac{L\,q\,x}{4\,A\,E}$$

In[152]:= $\hat{u}_2[x] = \hat{u}_2[x]\ /.\ \text{soln2}[[1]]\ //\ \text{Simplify}$

Out[152]= $$\frac{L\,q\,(-L + x)}{4\,A\,E}$$

Compare these to the exact solutions

In[153]:= $$u_{1\,\text{exact}} = \frac{q}{4\,E\,A}\left(-2\,x^2 + L\,x\right);$$

In[154]:= $\hat{u}_1[x] == u_{1\,exact}$ // Simplify

Out[154]= $\frac{q\,x}{A\,E} == 0$

In[155]:= $u_{2\,exact} = \frac{q}{4\,E\,A}\left(2\,x^2 - 3\,L\,x + L^2\right)$;

In[156]:= $\hat{u}_2[x] == u_{2\,exact}$ // Simplify

Out[156]= $\frac{q\,(L - x)}{A\,E} == 0$

Once again recovered the exact solution! Now compute $N(x)$ by taking the derivative.

In[157]:= $N_1[x\_] = E\,A\,\partial_x\,\hat{u}_1[x]$ // Simplify

Out[157]= $\frac{L\,q}{4}$

In[158]:= $N_2[x\_] = E\,A\,\partial_x\,\hat{u}_2[x]$ // Simplify

Out[158]= $\frac{L\,q}{4}$

With numerical values:

In[159]:= $N_1[x]$ /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[159]= 24

In[160]:= $N_2[x]$ /. {P → 90, q → 1, L → 8 * 12} // Simplify

Out[160]= 24

Problem 3

In[161]:= Remove["Global`*"]

3.1. Corresponding expression for the elastic potential

*Derived by hand. Let $v(x)$ be the vertical displacement as a function of x

$$\Pi = W - E = \frac{E\,I}{2}\int_0^L \left(\frac{\delta^2 v}{\delta x^2}\right)^2 dx - \int_0^L q\,v\,dx - P\,v[x = L]$$

3.2. Rayleigh-Ritz Method

(a) $\hat{v} = a\,x^2 + b\,x + c$

(b) $\hat{v} = a\cos(b\,x) + c$

Boundary condition for the uni-axially loaded bar is $\hat{v}(0) = 0$

a) $\hat{v} = a x^2 + b x + c$

In[162]:= $\hat{v}_1 = a x^2 + b x + c;$

In[163]:= $\hat{v}_1 = \hat{v}_1$ /. Solve[{($\hat{v}_1$ /. x → 0) == 0, ($\partial_x \hat{v}_1$ /. x → 0) == 0}, {b, c}][[1]]

Out[163]= $a x^2$

In[164]:= $\hat{\Pi}_1 = \frac{EI}{2} \int_0^L (\partial_{x,x} \hat{v}_1)^2 \, dx - \int_0^L q \hat{v}_1 \, dx - (P \hat{v}_1$ /. x → L)

Out[164]= $-a L^2 P - \frac{1}{3} a L^3 q + 2 a^2 L EI$

In[165]:= $\partial_a \hat{\Pi}_1$

Out[165]= $-L^2 P - \frac{L^3 q}{3} + 4 a L EI$

In[166]:= soln1 = Solve[$\partial_a \hat{\Pi}_1$ == 0, a]

Out[166]= $\left\{\left\{a \to \frac{3 L P + L^2 q}{12 EI}\right\}\right\}$

In[167]:= $\hat{v}_1 = \hat{v}_1$ /. soln1[[1]];

In[168]:= % // Framed

Out[168]= $\boxed{\frac{(3 L P + L^2 q) x^2}{12 EI}}$

b) $\hat{v} = a \cos(b x) + c$

In[169]:= $\hat{v}_2$ = a Cos[b x] + c;

In[170]:= $\hat{v}_2 = \hat{v}_2$ /. Solve[($\hat{v}_2$ /. x → 0) == 0, c][[1]]

Out[170]= - a + a Cos[b x]

Apply B.C. at $x = L$: since there is no applied load at the tip, we know that

$\frac{\delta^2 \hat{v}_2}{\delta x^2}$[x = L] = a Cos[b L] = 0

For a, $b \neq 0$, Cos[$b L$] = 0. Let $b L = \frac{\Pi}{2}$, we get $b = \frac{\Pi}{2L}$.

In[171]:= $b = \frac{\pi}{2 L}$

Out[171]= $\frac{\pi}{2 L}$

In[172]:= $\hat{v}_2$

Out[172]= $-a + a\,\mathrm{Cos}\left[\frac{\pi x}{2 L}\right]$

In[173]:= $\hat{\Pi}_2 = \frac{E I}{2}\int_0^L \left(\partial_{x,x}\hat{v}_2\right)^2 dx - \int_0^L q\,\hat{v}_2\,dx - \left(P\,\hat{v}_2 \,/.\, x \to L\right)$

Out[173]= $a P + \frac{a L (-2+\pi) q}{\pi} + \frac{a^2 \pi^4 E I}{64 L^3}$

In[174]:= `soln2 = Solve[{`$\partial_a \hat{\Pi}_2$` == 0}, {a}]`

Out[174]= $\left\{\left\{a \to -\frac{32 L^3 (P\pi - 2 L q + L \pi q)}{\pi^5 E I}\right\}\right\}$

In[175]:= $\hat{v}_2$ `=` $\hat{v}_2$ `/. soln2[[1]] // Simplify`

Out[175]= $\frac{64 L^3 \left(P\pi + L(-2+\pi) q\right) \mathrm{Sin}\left[\frac{\pi x}{4 L}\right]^2}{\pi^5 E I}$

In[176]:= `% // Framed`

Out[176]= $\boxed{\frac{64 L^3 \left(P\pi + L(-2+\pi) q\right) \mathrm{Sin}\left[\frac{\pi x}{4 L}\right]^2}{\pi^5 E I}}$

3.3. Total potential energy and plot Π vs. β

In[177]:= $\hat{\Pi}_1$ `/. soln1[[1]] /. P →` β `q L /. {q → 40, L → 1, E → 120 × 10^9, I → 8 × 10^-9} // Simplify;`

In[178]:= `% // Framed`

Out[178]= $\boxed{-\frac{5}{216}(1+3\beta)^2}$

In[179]:= $\hat{\Pi}_2$ `/. soln2[[1]] /. P →` β `q L /. {q → 40, L → 1, E → 120 × 10^9, I → 8 × 10^-9} // Simplify;`

In[180]:= `% // Framed`

Out[180]= $\boxed{-\frac{80(-2+\pi+\pi\beta)^2}{3\pi^6}}$

3.4. Which is better? Why?

Quadratic is better because the elastic potential is closer to 0 (the exact solution is always at the minimum).