Problem 1: Compute the first and second derivative of this likelihood w.r.t. θ. Then compute first and second derivative of the log-likelihood $\log \theta^t(1-\theta)^h$.

$$f = \theta^t \cdot (1-\theta)^h$$

$$f' = t\theta^{t-1} \cdot (1-\theta)^h - h(1-\theta)^{h-1} \cdot \theta^t$$

$$f'' = t(t-1)\theta^{t-2} \cdot (1-\theta)^h - t\theta^{t-1} \cdot h(1-\theta)^{h-1} + h(h-1)(1-\theta)^{h-2} \cdot \theta^t - t\theta^{t-1} \cdot h(1-\theta)^{h-1}$$

$$g = \log\left(\theta^t(1-\theta)^h\right) = t\log\theta + h\log(1-\theta)$$

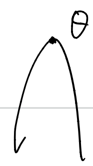
$$g' = t \cdot \frac{1}{\theta} + h\frac{-1}{1-\theta} = \frac{t}{\theta} - \frac{h}{1-\theta}$$ (fall in love with log now ...)

$$g'' = \frac{t}{-\theta^2} - h\left(+\frac{1}{(1-\theta)^2} \cdot (+1)\right) = \frac{-t}{\theta^2} - \frac{h}{(1-\theta)^2} = \frac{-t}{\theta^2} - \frac{h}{(\theta-1)^2}$$

Problem 2: Show that every local maximum of $\log f(\theta)$ is also a local maximum of the differentiable, positive function $f(\theta)$. Considering this and the previous exercise, what is your conclusion?

let $g(\theta) = \log f(\theta)$

For an arbitrary local maximum of $g(\theta)$:



the followings must hold:

$$\begin{cases} ① & g'(\theta) = 0 \\ ② & g''(\theta) < 0 \end{cases} \Rightarrow \begin{cases} ①: & \frac{f'(\theta)}{f(\theta)} = 0 \quad \text{with } f(\theta) > 0 \\ & \Rightarrow f'(\theta) = 0 \quad \cdots A \\ ② & \frac{f''(\theta) \cdot f(\theta) - f'(\theta)^2}{f(\theta)^2} < 0 \quad \text{with } f(\theta)^2 > 0 \end{cases}$$

$\Rightarrow$ $f''(\theta) \cdot f(\theta) - f'(\theta)^2 < 0$

from ① we know $f'(\theta) = 0$

i.e. $f''(\theta) \cdot f(\theta) - 0 < 0$

$\Rightarrow$ $f''(\theta) \cdot f(\theta) < 0$ with $f(\theta) > 0$

$\Rightarrow$ $f''(\theta) < 0$ $\cdots B$

From A, B $f(\theta)$ is also a local maximum.

conclusion: logarithm of a function preserves the monotonicity of that function, hence can sometimes be used to calculate the argmax/min, which reduces the complexity a lot.

Properties of MLE and MAP

Problem 3: You model a coin flip f as a Bernoulli distribution with a parameter θ

$$p(f \mid \theta) = \text{Bern}(f \mid \theta) = \theta^{\mathbb{I}[f=T]}(1-\theta)^{\mathbb{I}[f=H]}.$$

That is, the probability of landing tails (T) is θ, and probability of heads (H) is $(1-\theta)$ respectively.

Your prior on θ is a Beta(6, 4) distribution

$$p(\theta) = \text{Beta}(\theta \mid 6, 4).$$

You observe $(M+N)$ coin flips, out of which M are tails and N are heads. After you do maximum a posteriori estimation of θ, you obtain the result $\theta_{MAP} = 0.75$.

Name any possible values of M and N that can lead to such result. Show your work.

Likelihood : $P(F \mid \theta) = \theta^M \cdot (1-\theta)^N$

prior : $p(\theta) = \text{Beta}(\theta \mid 6, 4)$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1}$$

$$= \frac{\Gamma(10)}{\Gamma(6)\Gamma(4)} \cdot \theta^5 \cdot (1-\theta)^3$$

$$= \frac{9!}{5! \cdot 3!} \cdot \theta^5 \cdot (1-\theta)^3$$

$$= \frac{\cancel{6} \times 7 \times 8 \times 9}{1 \times \cancel{2} \times \cancel{3}} \cdot \theta^5 \cdot (1-\theta)^3$$

$$= 504 \cdot \theta^5 \cdot (1-\theta)^3$$

posterior $\propto$ likelihood $\cdot$ prior

log Posterior $\propto$ log (likelihood) + log (prior)

let $g(\theta) = M \log \theta + N \log(1-\theta) + \log 504 + 5 \log \theta + 3 \log(1-\theta)$

$$g'(\theta) = \frac{M}{\theta} + \frac{N}{\theta - 1} + \frac{5}{\theta} + \frac{3}{\theta - 1}$$

$$= \frac{M+5}{\theta} + \frac{N+3}{\theta - 1} = 0 \quad \text{with } \theta = \frac{3}{4}$$

$$\Rightarrow \frac{M+5}{\frac{3}{4}} - \frac{N+3}{\frac{1}{4}} = 0 \iff M+5 - 3(N+3) = 0$$

$$\iff M - 3N - 4 = 0 \text{ with } M, N \in \mathbb{Z}^+$$

so e.g. $\begin{cases} M = 7 \\ N = 1 \end{cases}$

Problem 4: Consider a Bernoulli random variable X and suppose we have observed m occurrences of $X = 1$ and l occurrences of $X = 0$ in a sequence of $N = m + l$ Bernoulli experiments. We are only interested in the number of occurrences of $X = 1$—we will model this with a Binomial distribution with parameter θ. A prior distribution for θ is given by the Beta distribution with parameters a, b. Show that the posterior *mean* value $\mathbb{E}[\theta \mid \mathcal{D}]$ (not the MAP estimate) of θ lies between the prior mean of θ and the maximum likelihood estimate for θ.

To do this, show that the posterior mean can be written as λ times the prior mean plus $(1 - \lambda)$ times the maximum likelihood estimate, with $0 \leq \lambda \leq 1$. This illustrates the concept of the posterior mean being a compromise between the prior distribution and the maximum likelihood solution.

The probability mass function of the Binomial distribution for some $m \in \{0, 1, \dots, N\}$ is

$$p(x = m \mid N, \theta) = \binom{N}{m} \theta^m (1 - \theta)^{N-m}.$$

Hint: Identify the posterior distribution. You may then look up the mean rather than computing it.

to prove: $E[\theta | D] = \lambda \cdot$ prior mean $+ (1-\lambda) \cdot$ MLE

Since I'm just in 3rd. semester of bacholer informatik, I've not learned the probabity theory yet, so I can't even understand this problem, sry about that!