Basic Mathematical Tools for Imaging and Visualization

winter term 2019/20

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Inverse Problems in Tomography
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Program today

Linear spaces

Linear space (over $\mathbb{R}$)

Let V be a nonempty set with the two operations $a + b \in V$ and $\lambda a \in V$ for every $a, b \in V$ and $\lambda \in \mathbb{R}$.
V is called a **linear space** if the following rules are fulfilled:

(V1) $a + b = b + a$ for all $a, b \in V$ (commutativity)

(V2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in V$ (associativity)

(V3) there exists a **zero element** $0 \in V$ such that $a + 0 = a$ for all $a \in V$

(V4) for every $a \in V$ there exists an **inverse element** $-a \in V$ such that $a + (-a) = 0$

(V5) $1a = a$ for all $a \in V$ ($1 \in \mathbb{R}$ real number)

(V6) $\lambda(\mu a) = (\lambda\mu)a$ for all $\lambda, \mu \in \mathbb{R}$, $a \in V$

(V7) $\lambda(a + b) = \lambda a + \lambda b$ for all $\lambda \in \mathbb{R}$, $a, b \in V$

(V8) $(\lambda + \mu)a = \lambda a + \mu a$ for all $\lambda, \mu \in \mathbb{R}$, $a \in V$

Program today

1. Linear Algebra
   - Linear spaces
   - Subspaces and linear spans
   - Coordinate systems and bases

Parts of linear spaces

Subspace of a linear space

Let V be a linear space. A nonempty set $U \subset V$ is called subspace of V if

- $x, y \in U \implies x - y \in U$,
- $x \in U$, $\lambda \in \mathbb{R} \implies \lambda x \in U$.

Remarks:

- a subspace U of V is a linear space itself.
- short notation: $U \leq V$.

Subspace properties

Let V be a linear space.

- Let U_i, $i \in I$, be subspaces of V, then $\bigcap_{i \in I} U_i$ is a subspace of V.
- Let $M \subset V$, then there exists a smallest subspace $\langle M \rangle$ of V that contains M. $\langle M \rangle$ is called the (linear) span of M in V. We have

$$\langle M \rangle = \bigcap_{M \subset U \leq V} U$$

(intersection of all subspaces U containing M) and we have

$$\langle M \rangle = \left\{ \sum_{x \in M} \lambda_x x : \ \lambda_x \in \mathbb{R}, \text{ almost all } \lambda_x = 0 \right\},$$

("almost all" means all except finitely many λ_x are 0), i.e. $\langle M \rangle$ is the set of finite linear combinations of elements of M.

Generating linear spaces

Generating set

Let V be a linear space. A set of vectors $(b_i)_{i \in I}$ in V is called generating set of V if

$$\left\langle (b_i)_{i \in I} \right\rangle = V.$$

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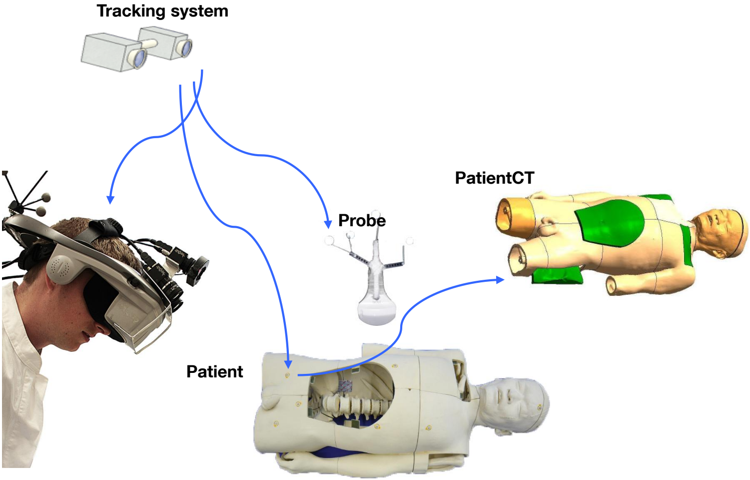
Augmented Reality



Independence

Linear independence

Let V be a linear space. A finite set of vectors $a_1, \ldots, a_n \in V$ is called linearly independent if

$$\lambda_1 a_1 + \ldots + \lambda_n a_n = 0 \qquad \Longrightarrow \qquad \lambda_1 = \ldots = \lambda_n = 0.$$

A set of vectors $(a_i)_{i \in I}$ in V is called linearly independent, if all finite subsets of $(a_i)_{i \in I}$ are linearly independent.

Coordinate systems generalized

Basis

Let V be a linear space. A set of vectors $(b_i)_{i \in I}$ in V is called basis of V, if it is a generating set of V that is linearly independent.

Basis properties

- Every linear space V has a basis.
- Let $b_1, \ldots, b_n$ be a basis of the linear space V. Then every $a \in V$ can be written as a unique linear combination

$$a = \sum_{i=1}^{n} \lambda_i b_i \quad \text{with } \lambda_i \in \mathbb{R}.$$

- Let V be a linear space. Then $b_1, \ldots, b_n \in V$ basis
  $\iff$ $b_1, \ldots, b_n \in V$ minimal generating set of V
  $\iff$ $b_1, \ldots, b_n \in V$ maximal linearly independent set in V.
- Let V be a linear space with basis $b_1, \ldots, b_n$. Then every basis has exactly n elements.
  This invariant is called dimension of V, $\dim V := n$.

Images

Summary