Analysis and simulation of The Lady in the Lake problem

Winston (Yuntao) Wu

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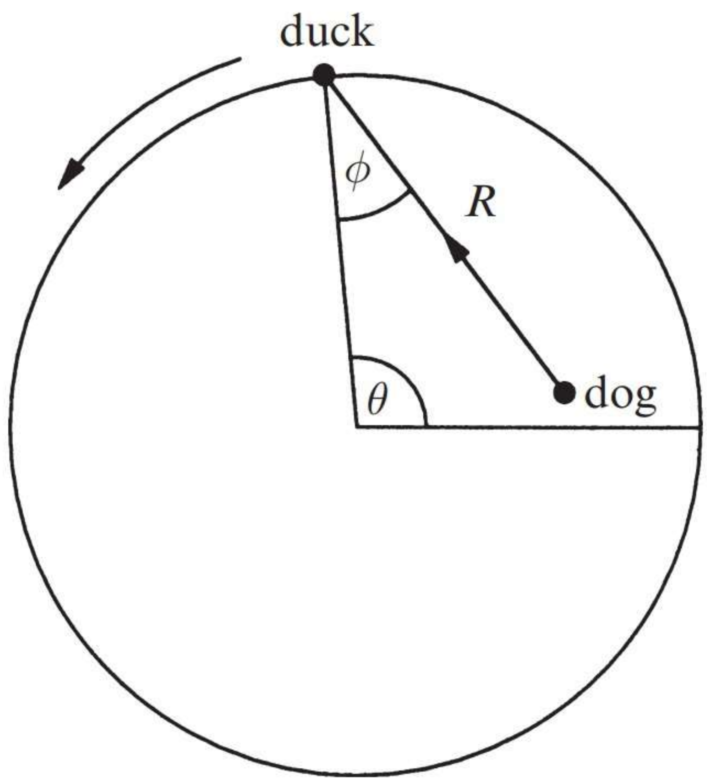
Motivation Problem

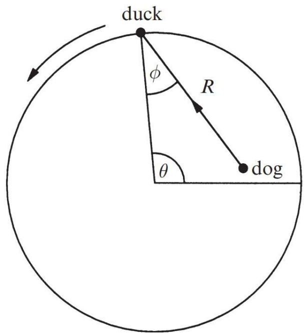
Problem 7.1.9 from (Strogatz), picture taken from the book directly.

Pond with radius $r = 1$
Duck: constant speed $v_{duck} = d\theta$ along the edge of the pond
Dog: constant speed $v_{dog} = kv_{duck}$ from center of the pond ($R = 1$ at the beginning)straight towards the duck

What does the dog do in the long run when $k = 1$ and $k = \frac{1}{2}$?

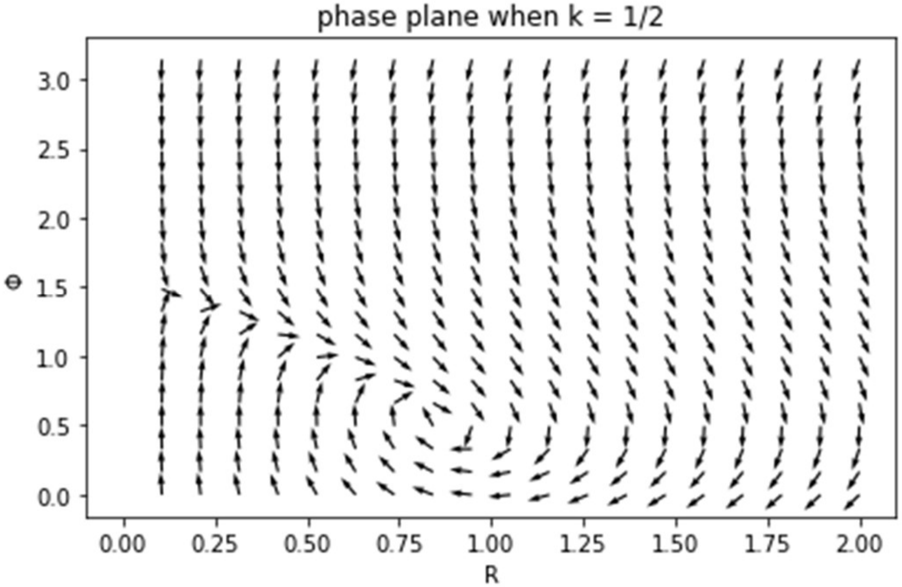


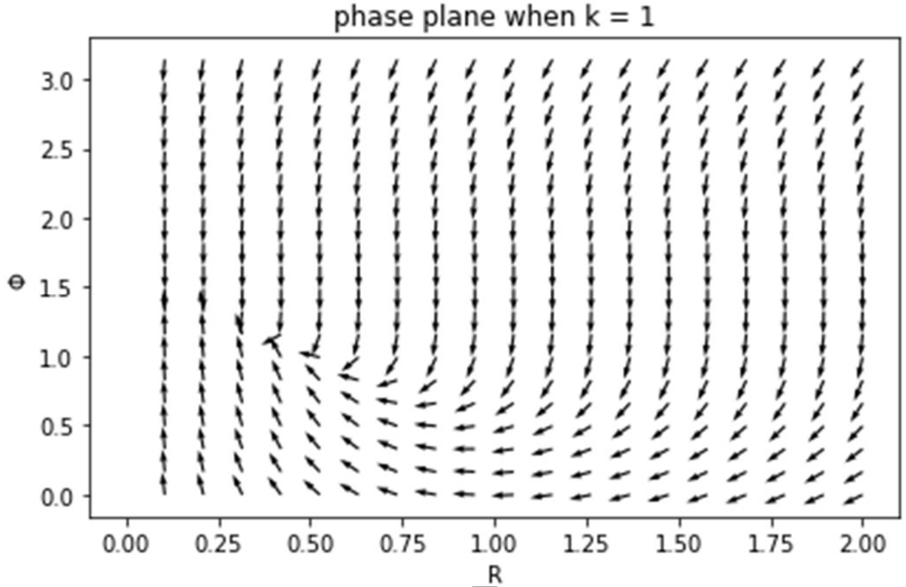
Motivation Problem: Solution



The system can be described by

$$\begin{cases} \dfrac{dR}{d\theta} = \sin\phi - k \\ \dfrac{d\phi}{d\theta} = \dfrac{\cos\phi}{R} - 1 \end{cases} \text{with } R \in [0,2],\ \phi \in \left[0, \frac{\pi}{2}\right]$$





When $k = \frac{1}{2}$, and starts from $R = 1$, attracted to $\phi = \frac{\pi}{6}$, $R = \frac{\sqrt{3}}{2}$

When $k = 1$, and starts from $R = 1$, no clear attraction, but still cannot reach $R = 0$

Motivation

The analysis of phase plane can give us some information about the long term behavior. However, it may not be able to solve the following questions easily.

- For what ks can the dog reach the duck as close as possible? (R cannot be zero in the analysis, so there is no definite number to define "closest".)
- What will happen if the speed of dog and duck are not constant?
- What will happen if the dog doesn't swim straight towards the duck?

Basic Definitions: Dynamical System

The motivation problem is a simple pursuit-evasion game, formulated as a nonlinear continuous-time autonomous dynamic system which can be described by:

$$\dot{\mathbf{x}} = f(\mathbf{x}, a),\ \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{x} \in F^n$, F a field, $\mathbf{x}_0$ the initial condition and a a parameter.
It studies how trajectories and limiting behavior $\lim_{t\to\infty} \mathbf{x}(t)$ changes w.r.t. $\mathbf{x}_0$ and a.

Basic Definitions: Pursuit-Evasion Game

The Lady in the Lake problem is a pursuit-evasion game, formulated as a **two-person deterministic zero-sum differential game** with variable terminal time. The differential system is

$$\dot{\mathbf{x}} = f\big(t, \mathbf{x}, u^1(t), u^2(t)\big),\ \mathbf{x}(0) = \mathbf{x}_0.$$

This is nonautonomous. Parameters become player strategies dependent on time.

Terminal time: $T = \inf\{t \in \mathbb{R}^+, \mathbf{x}(t) \in \Lambda\}$, where Λ is the target set with boundary $\partial\Lambda$ defined by $l(t, x) = 0$

Objective function: $J(u^1, u^2) = \int_0^T g\big(t, \mathbf{x}, u^1(t), u^2(t)\big)dt + q\big(T, x(T)\big)$

The goal is to optimize the objective function.

Basic Definitions: Issacs Equation

Consider the minimax value of the objective function:

$$V(t,\mathbf{x}) = \min_{u^1} \max_{u^2} \left\{ \int_0^T g\big(t,\mathbf{x},u^1(t),u^2(t)\big)dt + q\big(T,\mathbf{x}(T)\big) \right\}$$

If V is C^1 in t and $\mathbf{x}$, then it satisfies the Issacs equation:

$$-\frac{\partial V}{\partial t} = \min_{u^1} \max_{u^2} \left\{ \frac{\partial V}{\partial \mathbf{x}} f\big(t,\mathbf{x},u^1(t),u^2(t)\big) + g\big(t,\mathbf{x},u^1(t),u^2(t)\big) \right\}$$

Let $H = \frac{\partial V}{\partial \mathbf{x}} f\big(t,\mathbf{x},u^1(t),u^2(t)\big) + g\big(t,\mathbf{x},u^1(t),u^2(t)\big)$, the following gives the solution to the costate function $p = \frac{\partial V}{\partial \mathbf{x}}$.

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \mathbf{x}}\right) = -\frac{\partial H}{\partial \mathbf{x}}, \frac{\partial V}{\partial \mathbf{x}}(T) = \frac{\partial}{\partial \mathbf{x}}\Big(q\big(T,\mathbf{x}^*(T)\big)\Big) \text{(solve backwards in time)}$$

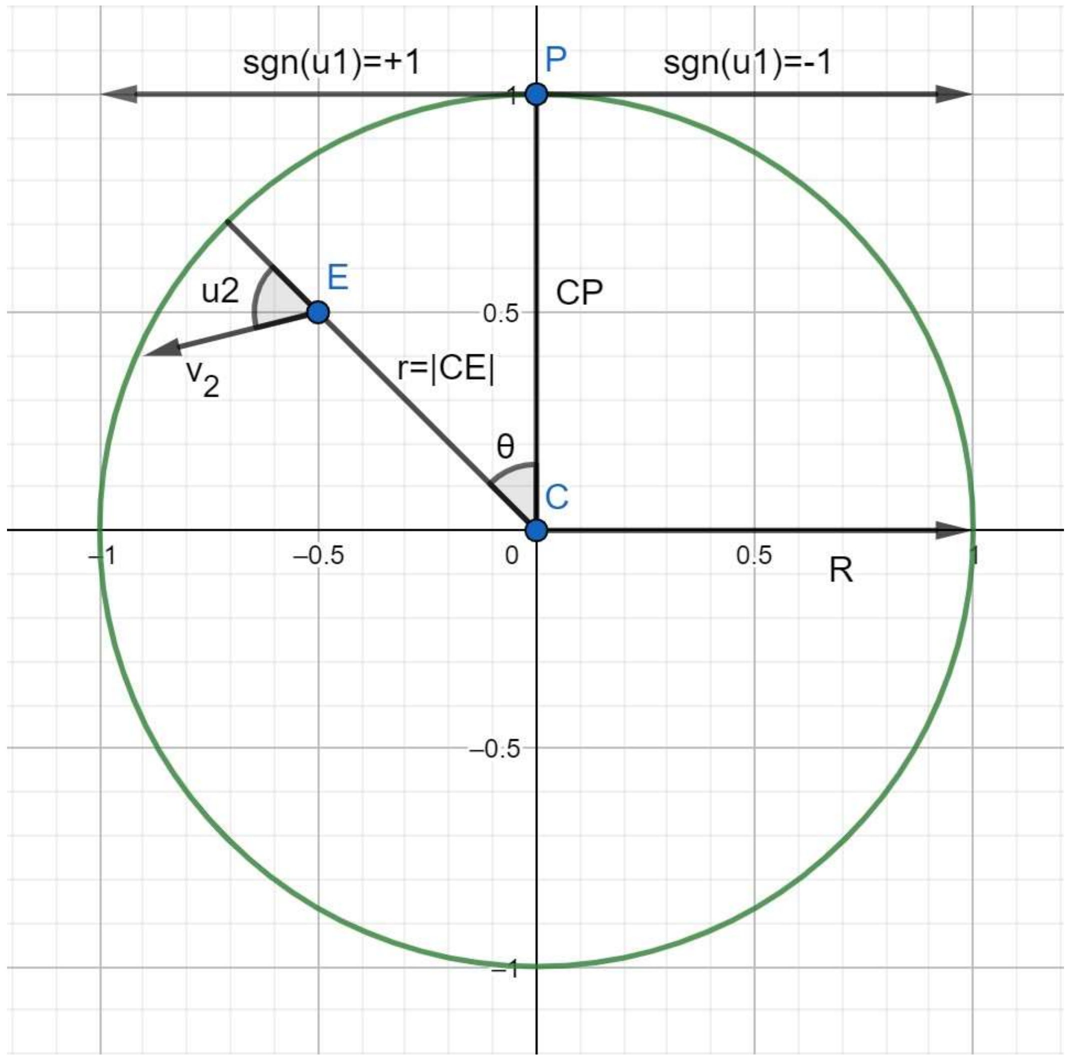
The Lady in the Lake problem

Circular pond with radius R

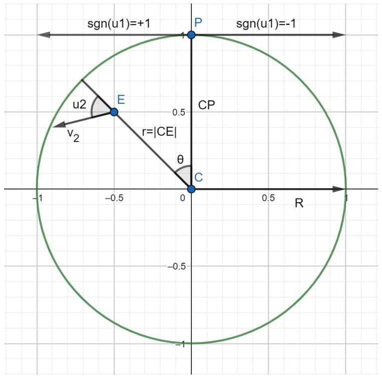
Pursuer $\mathbf{P}$ moves along the perimeter, with strategy $|u^1| \leq 1$, trying to catch the lady (evader) as she gets to the boundary.

Evader $\mathbf{E}$ starts from center C, with constant speed v_2, strategy u^2 representing direction she goes w.r.t. $\overrightarrow{CE}$, trying to reach the edge of the pond without being caught.

Let $\theta = \angle PCE$, $r = |CE|$



The Lady in the Lake problem: Analysis



The differential equations for the system is:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix} = \begin{cases} \dfrac{v_2 \sin u^2}{r} - \dfrac{u^1}{R} \\ v_2 \cos u^2 \end{cases}$$

Objective function $J = |\theta(T)|$ with $T = \inf\{t : r(t) = R\}$, $|\theta(t)| \le \pi$, $\forall t \in [0, T]$

We can see that $g \equiv 0$, $q\big(T, \mathbf{x}(T)\big) = |\theta(T)|$.

So the Issacs equation is:

$$0 = -\frac{\partial V}{\partial t} = \min_{u^1} \max_{u^2} \left\{ \frac{\partial V}{\partial \mathbf{x}} f\big(t, \mathbf{x}, u^1(t), u^2(t)\big) \right\} = \min_{u^1} \max_{u^2} \left\{ \frac{\partial V}{\partial \theta} \left(\frac{v_2 \sin u^2}{r} - \frac{u^1}{R} \right) + \frac{\partial V}{\partial r} \left(v_2 \cos u^2 \right) \right\}$$

It is separable in u^1 and u^2.

$$0 = \min_{u^1} \left\{ -\frac{u^1}{R} \frac{\partial V}{\partial \theta} \right\} + v_2 \max_{u^2} \left\{ \frac{1}{r} \frac{\partial V}{\partial \theta} \sin u^2 + \frac{\partial V}{\partial r} \cos u^2 \right\}$$

The Lady in the Lake problem: Analysis

Let $H = \frac{\partial V}{\partial \theta}\left(\frac{v_2 \sin u^2}{r} - \frac{u^1}{R}\right) + \frac{\partial V}{\partial r}\left(v_2 \cos u^2\right)$

The costate function is given by

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \theta}\right) = -\frac{\partial H}{\partial \theta} = 0, \frac{\partial V}{\partial \theta}(T) = \frac{\partial}{\partial \theta}|\theta(T)| = \operatorname{sgn}(\theta(T))$$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial r}\right) = -\frac{\partial H}{\partial r} = \frac{\partial V}{\partial \theta}\left(\frac{v_2 \sin u^2}{r^2}\right), \frac{\partial V}{\partial r}(T) = 0$$

The θ equation gives $\frac{\partial V}{\partial \theta}(t) = \operatorname{sgn}(\theta(T))$, $\forall t \in [0, T]$.

This will help us find the optimal strategies.

The Lady in the Lake problem: Analysis

$$0 = \min_{u^1}\left\{-\frac{u^1}{R}\frac{\partial V}{\partial \theta}\right\} + v_2 \max_{u^2}\left\{\frac{1}{r}\frac{\partial V}{\partial \theta}\sin u^2 + \frac{\partial V}{\partial r}\cos u^2\right\} (*)$$

$\min_{u^1}\left\{-\frac{u^1}{R}\frac{\partial V}{\partial \theta}\right\} = -\frac{1}{R}\left|\frac{\partial V}{\partial \theta}\right| = -\frac{1}{R}$ is achieved when $u^{1*} = \operatorname{sgn}\left(\frac{\partial V}{\partial \theta}\right) = \operatorname{sgn}\theta(T)$.

$\max_{u^2}\left\{\frac{1}{r}\frac{\partial V}{\partial \theta}\sin u^2 + \frac{\partial V}{\partial r}\cos u^2\right\} = \max_{u^2}\left\{\left(\frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial r}\right) \cdot \left(\sin u^2, \cos u^2\right)\right\}$ is achieved when the two vectors $\left(\frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial r}\right)$ and $\left(\sin u^2, \cos u^2\right)$ are parallel.

Let $\frac{1}{r}\frac{\partial V}{\partial \theta} = k \sin u^{2*}$, $\frac{\partial V}{\partial r} = k\cos u^2$, then $\max_{u^2}\left\{\left(\frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{\partial V}{\partial r}\right) \cdot \left(\sin u^2, \cos u^2\right)\right\} = k$

Substitute u^{1*} and k back into $(*)$, $k = \frac{1}{Rv_2}$.

Then $\sin u^{2*} = \frac{Rv_2}{r}\frac{\partial V}{\partial \theta} = \frac{Rv_2}{r}\operatorname{sgn}\theta(T)$

The Lady in the Lake problem: Analysis

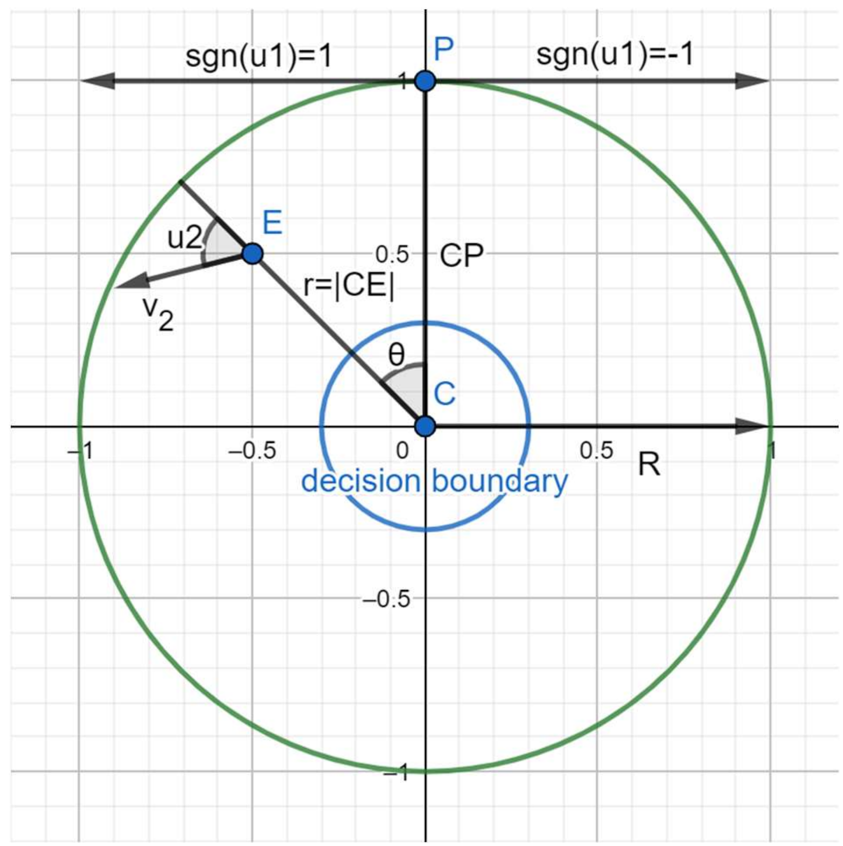
$$u^{1*} = \mathrm{sgn}\big(\theta(T)\big), \sin u^{2*} = \frac{Rv_2}{r}\,\mathrm{sgn}\,\theta(T)$$

What do these strategies mean?

$u^{1*} = \mathrm{sgn}\big(\theta(T)\big)$ means $\mathbf{P}$ should try to move in the direction to minimize θ

$\sin u^{2*} = \frac{Rv_2}{r}\,\mathrm{sgn}\,\theta(T)$ is only valid when $r \geq Rv_2$
$\mathbf{E}$ should try to maximize θ when $\mathbf{E}$ is outside of the decision boundary $\{\mathbf{x} : \|\mathbf{x}\| = Rv_2\}$.

When $r < Rv_2$, $\mathbf{E}$'s maximum angular velocity $\omega_{\mathbf{E}} = \frac{v_2}{r} \geq \frac{1}{R} \geq \omega_{\mathbf{P}}$, so $\mathbf{E}$ can always make θ as large as possible before reaching the decision boundary.



The Lady in the Lake problem: Analysis

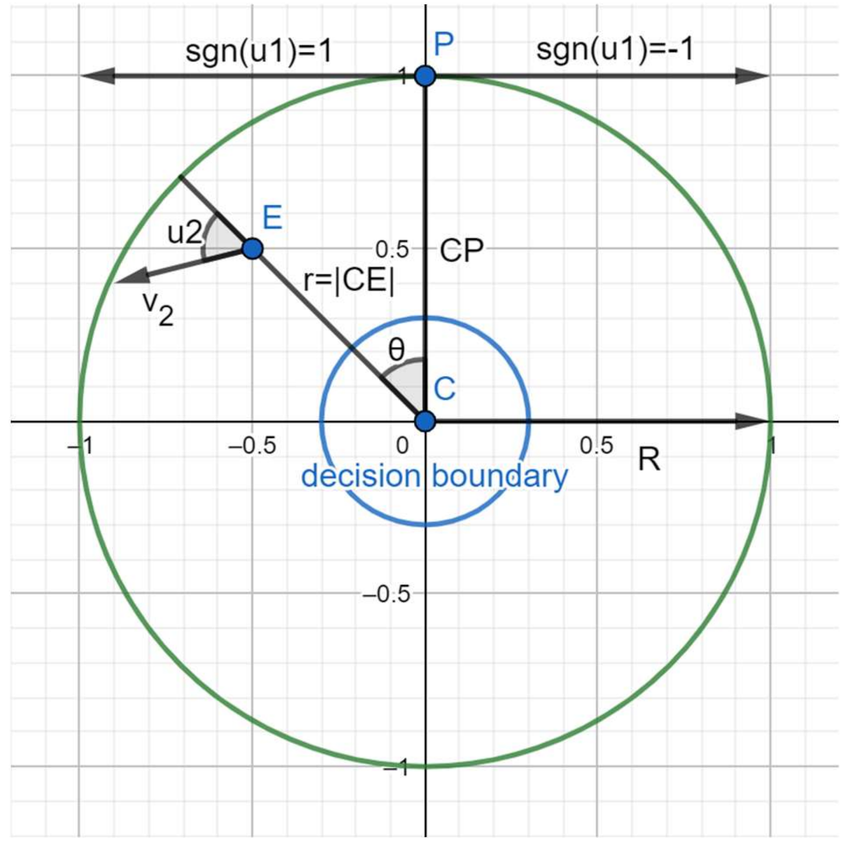
Under what conditions can $\mathbf{E}$ escape?

Ans: $\mathbf{E}$ can escape when $|\theta(T)| > 0$

T. Basar calculated $|\theta(T)| = \pi + \arccos v_2 - \frac{1}{v_2}\sqrt{1 - v_2^2}$.

$|\theta(T)| > 0$ if $v_2 > 0.21723$.

So, $\mathbf{E}$ should swim with speed $v_\mathbf{E} \geq 0.217 v_\mathbf{P}$ to escape.
And from the previous slide, $\mathbf{E}$ should try to maximize θ when reaching the decision boundary, and keep $\sin u^{2*} = \frac{R v_2}{r} \operatorname{sgn} \theta(T)$ at each r.



Inverse problem

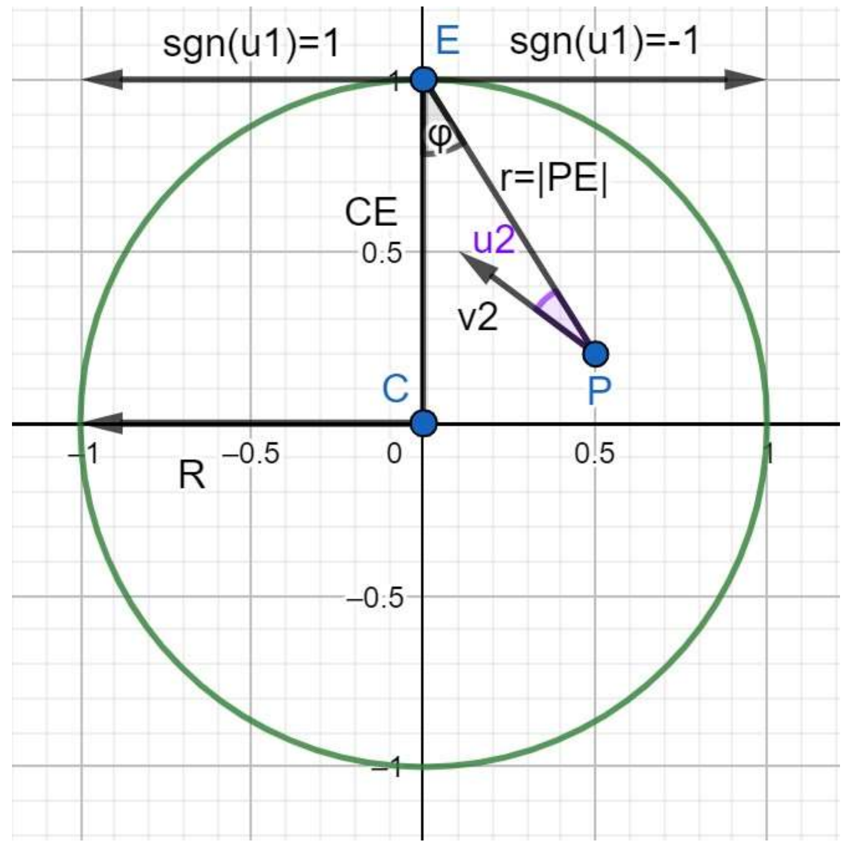
Motivation problem + strategies

Circular pond with radius R

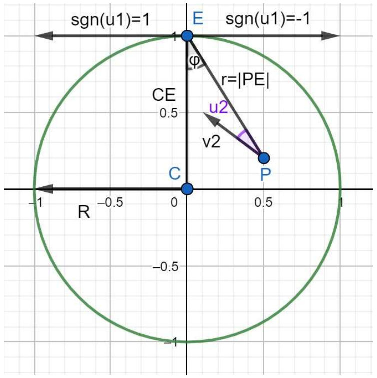
Evader $\mathbf{E}$ moves along the perimeter, with strategy $|u^1| \leq 1$ trying to survive as long as possible without being caught by $\mathbf{P}$.

Pursuer $\mathbf{P}$ starts from center C, with constant speed v_2, strategy u^2 representing direction she goes w.r.t. $\overrightarrow{PE}$, trying to catch $\mathbf{E}$ as fast as possible.

Let $\phi = \angle CEP$, $r = |PE|$



Inverse problem: Analysis



The differential equations for the system is:

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{\phi} \\ \dot{r} \end{pmatrix} = \begin{cases} \left(\dfrac{R\cos\phi}{r} - 1\right)u^1 - \dfrac{v_2 \sin u^2}{r} \\ u^1 \sin\phi - v_2 \cos u^2 \end{cases}$$

Objective function $J = r(T)$ with $T = \min\{\inf\{t : r(t) = 0\}, 15\ sec\}$

The Issacs equation is:

$$0 = \min_{u^2} \max_{u^1} \left\{ \frac{\partial V}{\partial \phi}\left(\left(\frac{R\cos\phi}{r} - 1\right)u^1 - \frac{v_2 \sin u^2}{r}\right) + \frac{\partial V}{\partial r}\left(u^1 \sin\phi - v_2 \cos u^2\right) \right\}$$

It is separable in u^1 and u^2.

$$0 = \max_{u^1}\left\{u^1\left(\frac{\partial V}{\partial \phi}\left(\frac{R\cos\phi}{r} - 1\right) + \frac{\partial V}{\partial r}\sin\phi\right)\right\} + v_2 \min_{u^2}\left\{-\frac{1}{r}\frac{\partial V}{\partial \phi}\sin u^2 - \frac{\partial V}{\partial r}\cos u^2\right\}$$

The optimal strategies are:

$u^{1*} = \operatorname{sgn}\left(\frac{\partial V}{\partial \phi}\left(\frac{R\cos\phi}{r} - 1\right) + \frac{\partial V}{\partial r}\sin\phi\right), (\sin u^{2*}, \cos u^{2*}) \parallel \left(-\frac{1}{r}\frac{\partial V}{\partial \phi}, -\frac{\partial V}{\partial r}\right)$, with reverse direction

Inverse problem: Analysis

Let $H = \frac{\partial V}{\partial \phi}\left(\left(\frac{R\cos}{r} - 1\right)u^1 - \frac{v_2 \sin u^2}{r}\right) + \frac{\partial V}{\partial r}\left(u^1 \sin\phi - v_2 \cos u^2\right)$

The costate function is given by

$$\frac{d}{dt}\left(\frac{\partial V}{\partial \phi}\right) = u^1 \frac{\partial V}{\partial \phi}\frac{R\sin\phi}{r} - u^1\frac{\partial V}{\partial r}\cos\phi\,, \frac{\partial V}{\partial \phi}(T) = \frac{\partial}{\partial \phi} r(T) = 0$$

$$\frac{d}{dt}\left(\frac{\partial V}{\partial r}\right) = \frac{\partial V}{\partial \phi}\left(\frac{R\cos\phi}{r^2}u^1 - \frac{v_2 \sin u^2}{r^2}\right), \frac{\partial V}{\partial r}(T) = 1$$

This time, it doesn't really give any helpful information for solving the system.

Inverse Problem: Analysis

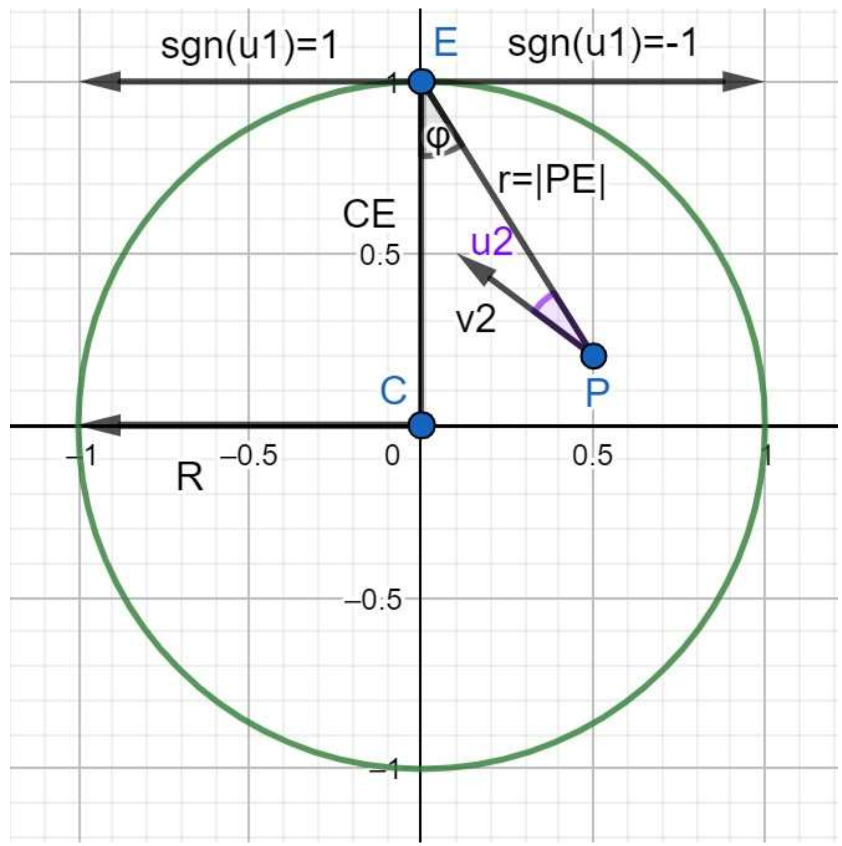
$$u^{1*} = \operatorname{sgn}\left(\frac{\partial V}{\partial \phi}\left(\frac{R\cos}{r} - 1\right) + \frac{\partial V}{\partial r}\sin\phi\right)$$

$$(\sin u^{2*}, \cos u^{2*}) \parallel \left(-\frac{1}{r}\frac{\partial V}{\partial \phi}, -\frac{\partial V}{\partial r}\right)$$

What do these strategies mean?

u^{1*} is the sign of a dot product between $\left(\frac{\partial V}{\partial \phi}, \frac{\partial V}{\partial r}\right)$ and $\mathbf{a} = \left(\left(\frac{R\cos}{r} - 1\right), \sin\phi\right)$.
$\mathbf{a}$ is the $(\dot{\phi}, \dot{r})$ for the simplified problem with speed $v_{\mathbf{P}} = 0$ along $\overrightarrow{PE}$. So $\mathbf{E}$ is trying to make ϕ larger so that $v_2 \cos u^2$ can be as small as possible and it will take longer for $\mathbf{P}$ to minimize r.

However, u^{2*} does not tell much.



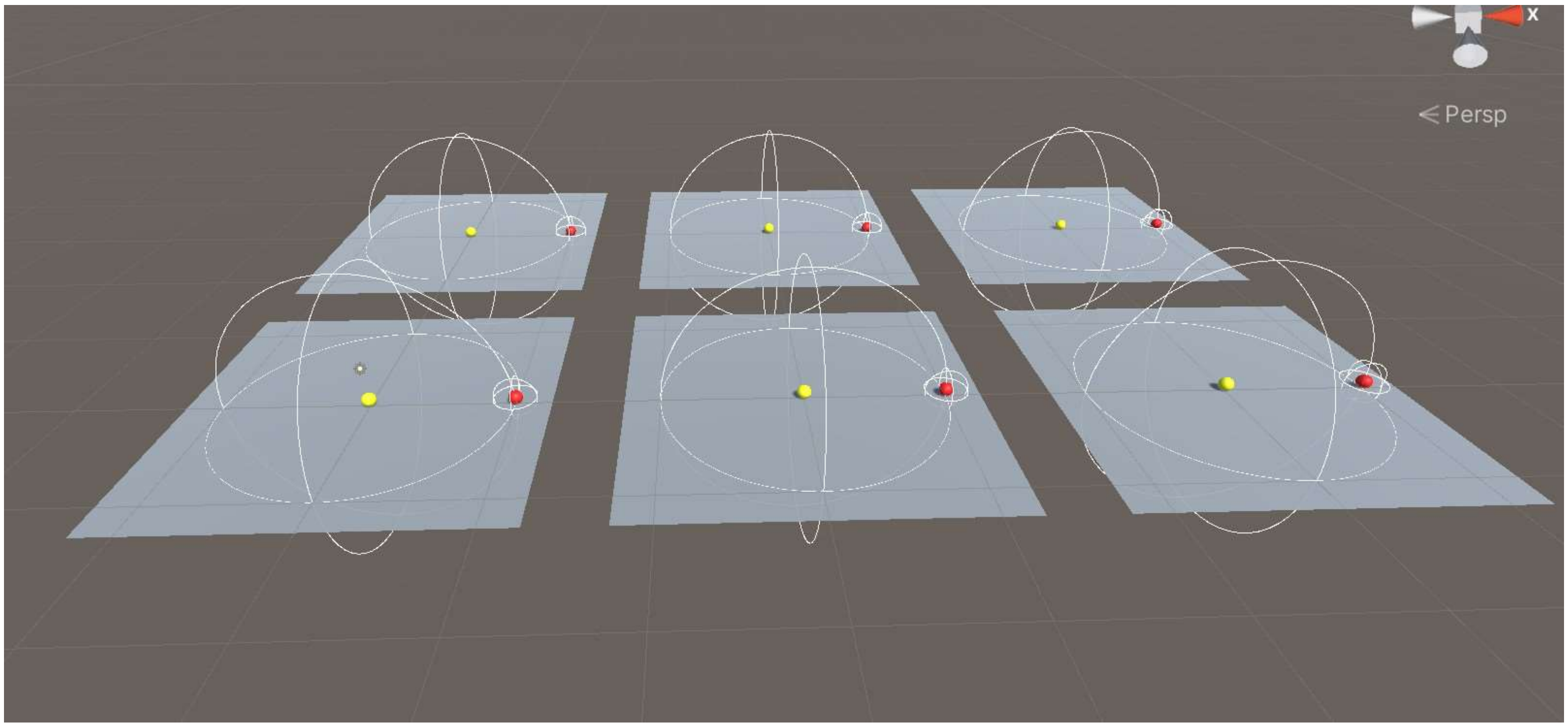
Tools for simulation

Unity with ml-agents is used to simulate the game.

- 6 training regions concurrently.
- Red sphere for $\mathbf{P}$, yellow sphere for $\mathbf{E}$.
- Large gizmos: boundary of the game $|\mathbf{x}| = R = 10$
- Small gizmos around $\mathbf{P}$: criteria for catching $||\mathbf{x_P} - \mathbf{x_E}|| \leq 1.5$ (account for non-negligible size).

Unity®

https://unity.com/



Tools for simulation

Proximal Policy Optimization is used to train the agents (players)

- Actor-critic (online policy) Reinforcement Learning: critic estimates the value function, actor updates the policy distribution.
- Better than A2C and TRPO in most continuous control environments.

Hyperparameters

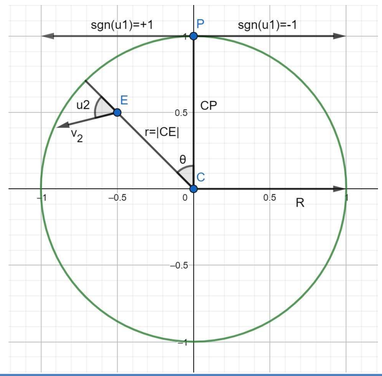
- Batch size: 512
- Buffer size: 5120
- Learning rate: 3.0×10^{-4}, decreasing linearly
- Maximum steps for each team: 10^6
- Neural network: 3-layer MLP with 512 neurons in each layer
- Discount factor: $\gamma = 0.99$

Self-play (for non-cooperative agents with inverse reward)

- Snapshots: taken every 20,000 steps
- Training team switches: every 100,000 steps
- Opponent's policy changes: every 2,000 steps, 50% chance using latest policy.

Original Problem: Setup

Let $l = \|PE\|$, $m = \frac{\overrightarrow{CE} \cdot \overrightarrow{CP}}{\|CE\|\|CP\|}$.



	Pursuer	Evader
Instantiation	Randomly on the circle	At origin
Observation	$\mathbf{s_P} = (x_\mathbf{P}, z_\mathbf{P}, x_\mathbf{E}, z_\mathbf{E}, l, m)$	$\mathbf{s_P} = (x_\mathbf{E}, z_\mathbf{E}, x_\mathbf{P}, z_\mathbf{P}, l, m)$
Action	$a_\mathbf{P} \in [-1,1]$, speed + direction	$a_\mathbf{E} \in [-1,1]$, direction
Reward	$+m$ at each step $+1000$ if successfully catches $\mathbf{E}$ -1000 if $\mathbf{E}$ escapes	$-m$ at each step $+1000$ if successfully escapes -1000 if caught

Given the Actions, position of $\mathbf{P}$ and velocity of $\mathbf{E}$:

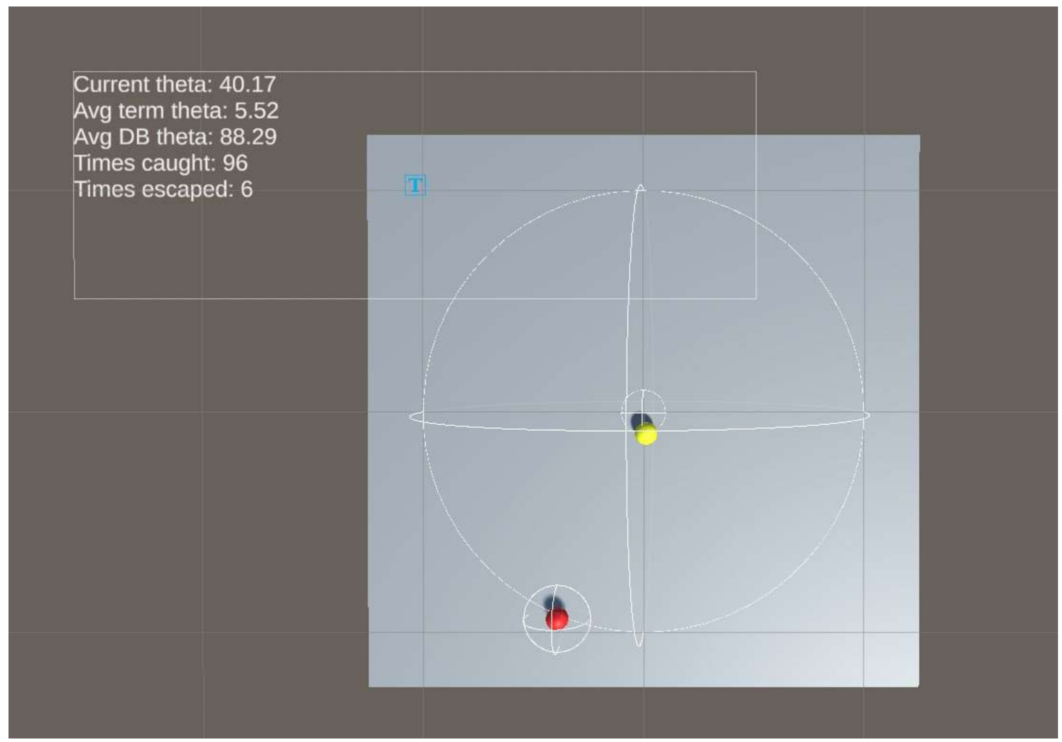
$$\mathbf{x_P} = (10\cos(\theta_\mathbf{P} + a_\mathbf{P} dt), 10\sin(\theta_\mathbf{P} + a_\mathbf{P} dt))$$
$$\mathbf{v_E} = (v_2\cos(\theta_\mathbf{E} + a_\mathbf{E}\pi), v_2\sin(\theta_\mathbf{E} + a_\mathbf{E}\pi))$$

Original Problem: Results

When $v_{\mathbf{E}} < 0.217 v_{\mathbf{P}}$:
Note that $a_{\mathbf{P}}$ is the angular velocity, the actual speed is $v_{\mathbf{P}} = a_{\mathbf{P}} R = 10 a_{\mathbf{P}}$.

So, we can take $v_{\mathbf{E}} = 1 < 0.217 \cdot 10 = 2.17$

Red is pursuer. Blue is evader.

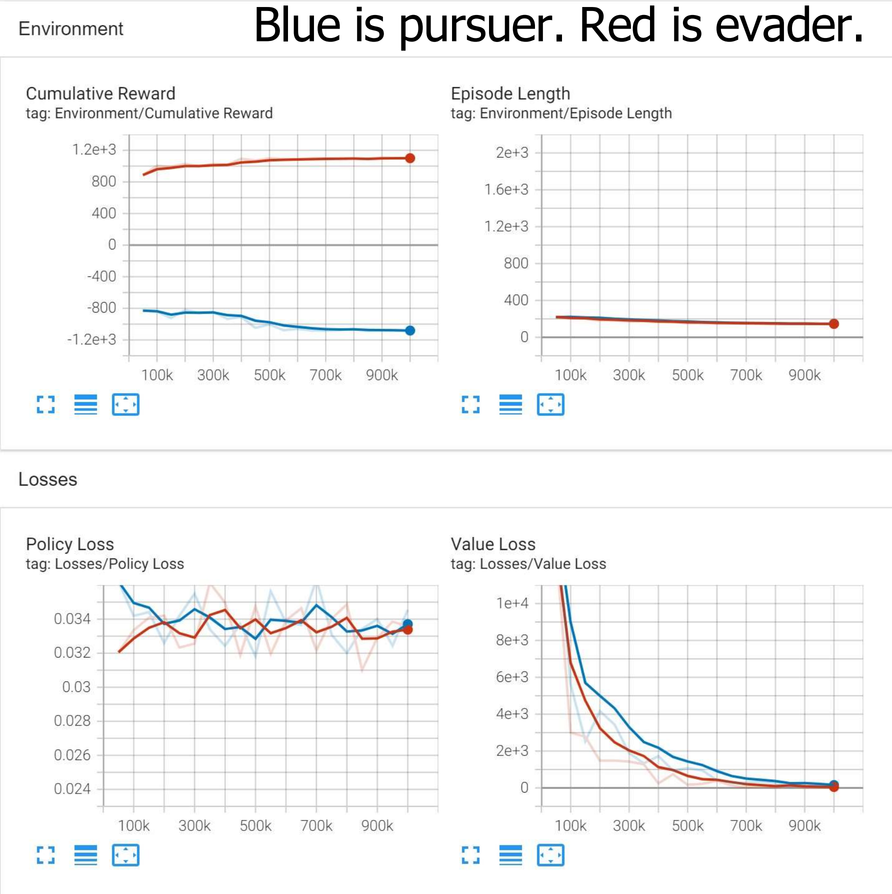


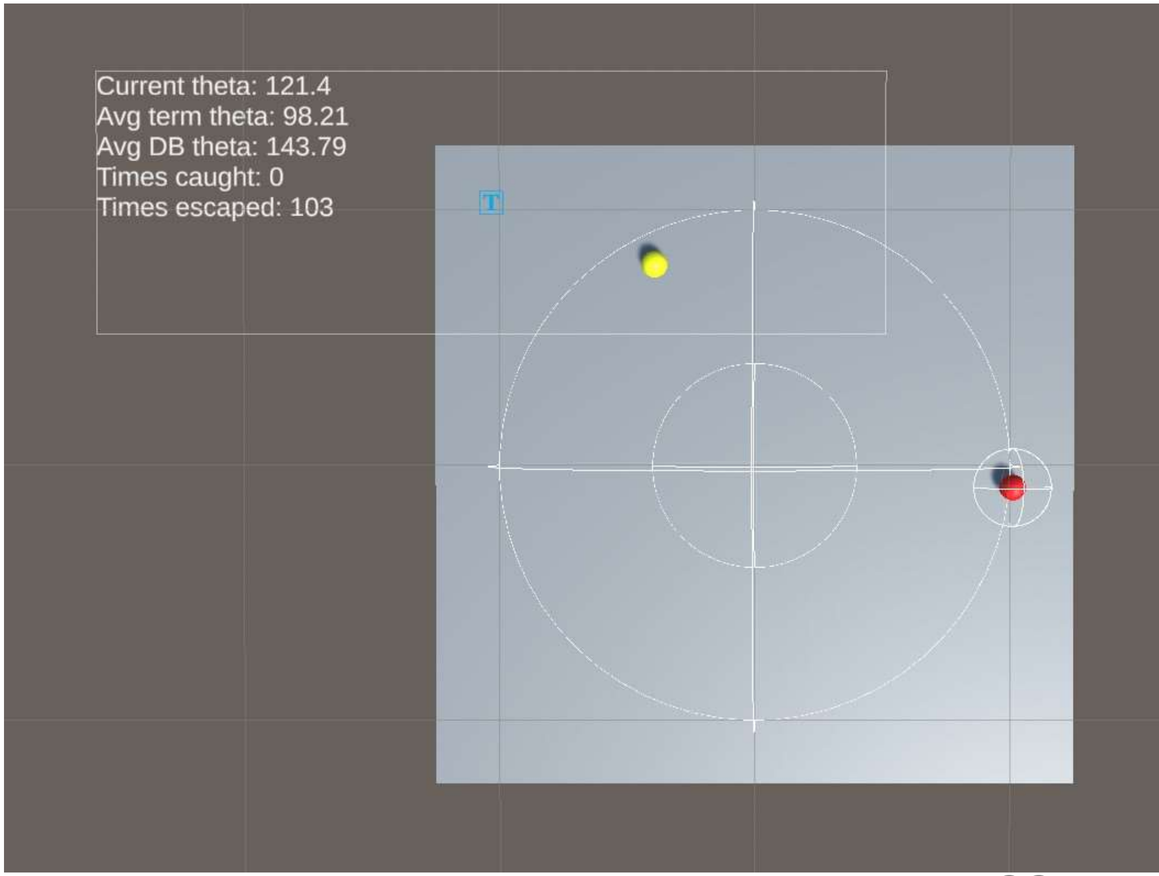
Original Problem: Results

When $v_{\mathrm{E}} > 0.217 v_{\mathrm{P}}$:

Take $v_{\mathrm{E}} = 4$

Blue is pursuer. Red is evader.

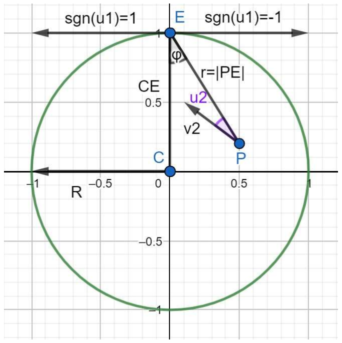




Inverse Problem: Setup

Let $r = \|PE\|$, $m = \frac{\overrightarrow{CE} \cdot \overrightarrow{CP}}{\|CE\|\|CP\|}$, $\theta = \text{atan2}\left(\overrightarrow{PE}\right)$.

$\mathbf{P}$ will be hard reset to origin if $\|\mathbf{x_P}\| > 10$.



	Pursuer	Evader
Instantiation	At origin	Randomly on the circle
Observation	$\mathbf{s_P} = (x_\mathbf{P}, z_\mathbf{P}, x_\mathbf{E}, z_\mathbf{E}, r, m)$	$\mathbf{s_P} = (x_\mathbf{E}, z_\mathbf{E}, x_\mathbf{P}, z_\mathbf{P}, r, m)$
Action	$a_\mathbf{P} \in [0.5,1.0] \times [-1,1]$, speed + direction	$a_\mathbf{E} \in [-1,1]$, speed + direction
Reward	$+0.5 - \frac{r}{20}$ at each step $+1000$ if successfully catches $\mathbf{E}$ -1000 if $\mathbf{E}$ survives for 15 sec	$+\frac{r}{20} - 0.5$ at each step $+1000$ if survives for 15 sec -1000 if caught

Given the Actions, velocity of $\mathbf{P}$ and position of $\mathbf{E}$:

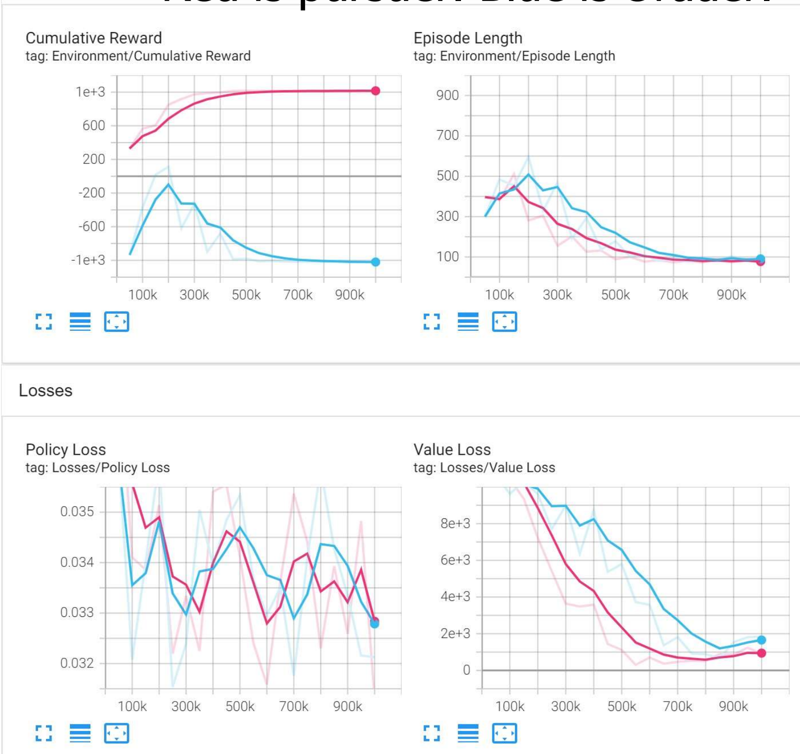
$$\mathbf{v_P} = (v_2 a_{\mathbf{P}1} \cos(\theta + a_{\mathbf{P}2}\pi), v_2 a_{\mathbf{P}1} \sin(\theta + a_{\mathbf{P}2}\pi))$$
$$\mathbf{x_E} = (10\cos(\theta_\mathbf{E} + a_\mathbf{E} dt), 10\sin(\theta_\mathbf{E} + a_\mathbf{E} dt))$$

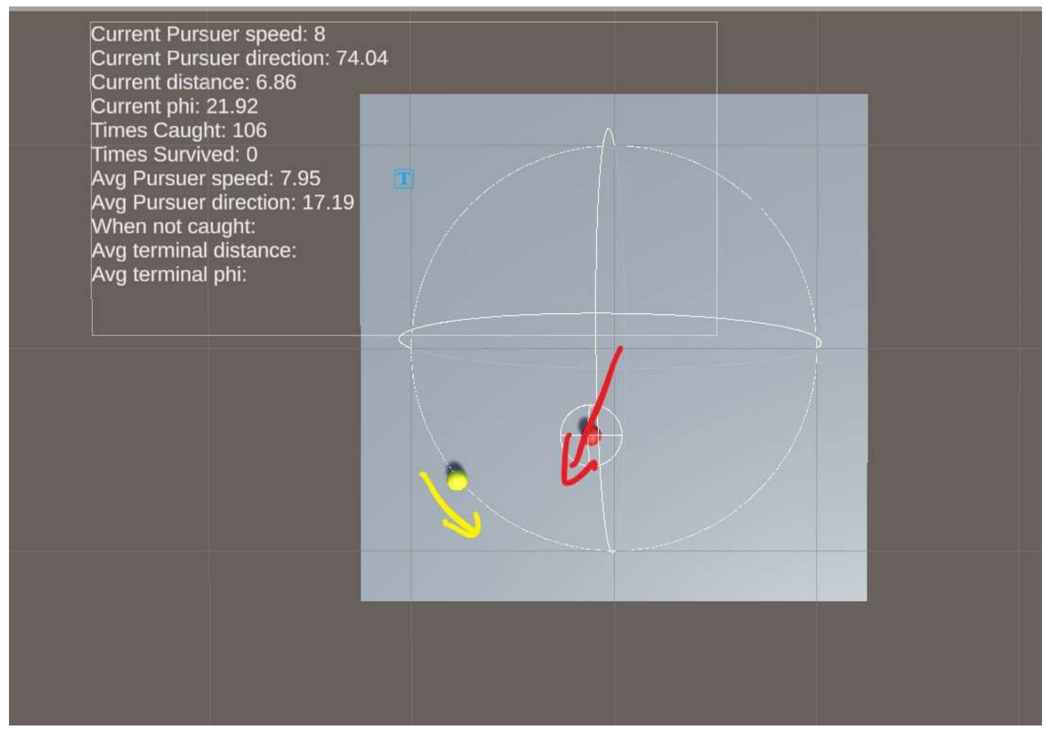
Inverse Problem: Results

Now, we don't know the optimal strategies for the agents.
So, for $v_{\mathbf{P}}$, instead of a constant speed, it is used as the maximum speed.

Take $v_{\mathbf{P}} = 8 = 0.8 v_{\mathbf{E}}$, this means that $\mathbf{P}$ can have speed $4 \sim 8$ based on my setup.

Red is pursuer. Blue is evader.

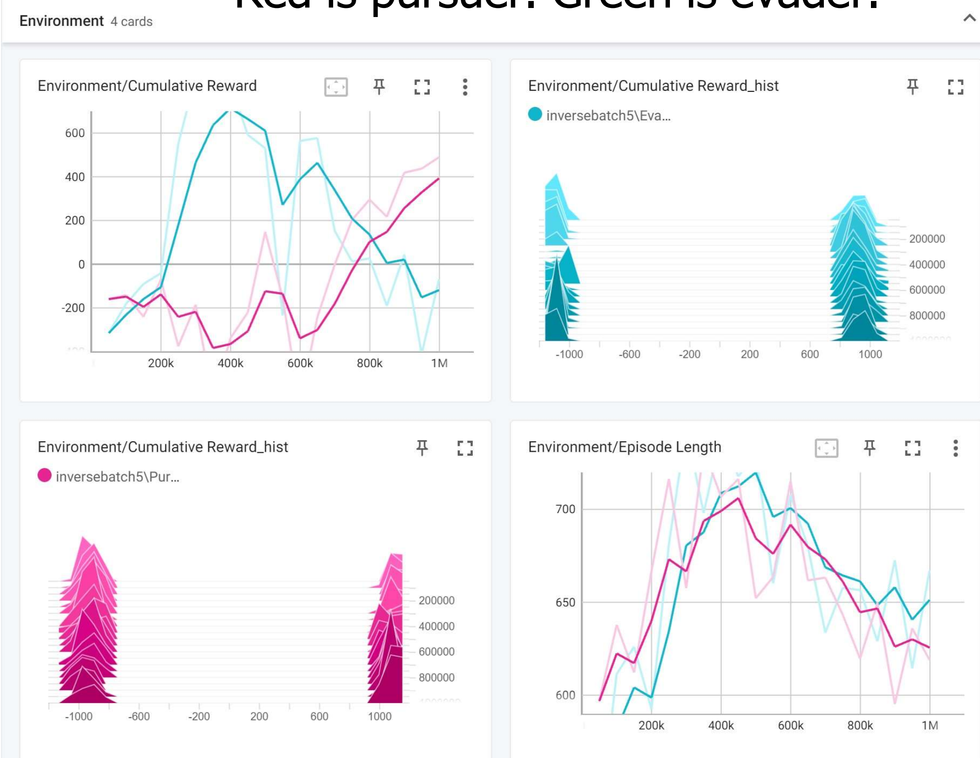


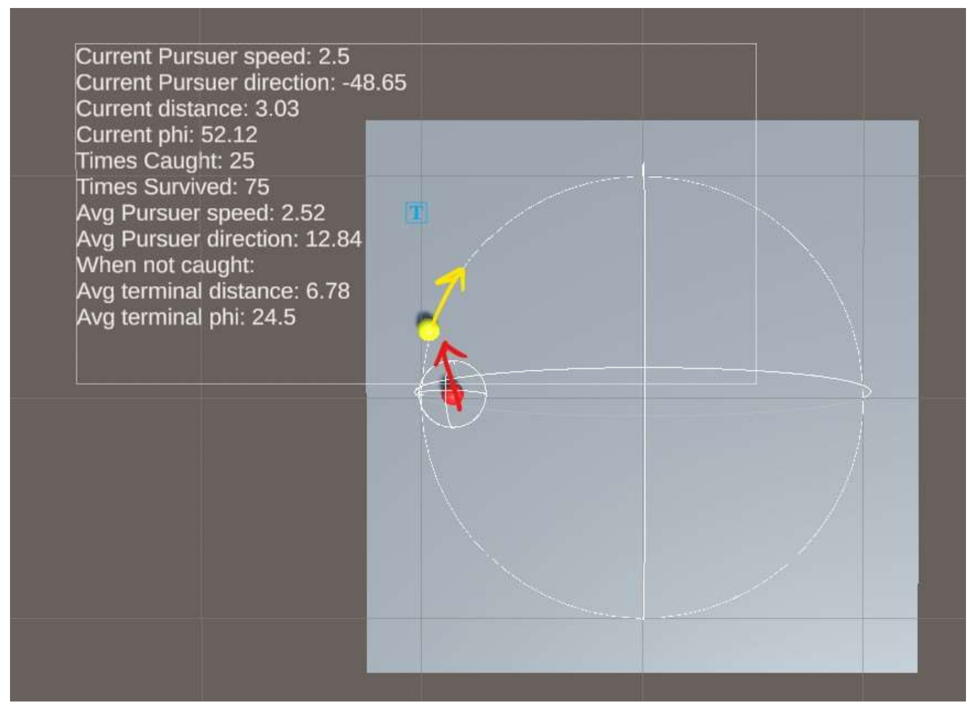


Inverse Problem: Results

Take $v_{\mathbf{P}} = 5 = 0.5v_{\mathbf{E}}$, this means that $\mathbf{P}$ can have speed $2.5{\sim}5$ based on my setup.

Red is pursuer. Green is evader.





Conclusion

What I have achieved:

- Analysis of “The Lady in the Lake” problem and its inverse problem.
- Reinforcement Learning simulation to verify the results of the original problem and analyze some behaviors of the inverse problem under different velocity conditions and different winners.

Limitations in simulation:

- Non-negligible size of the agents
- Relaxed terminal condition, can be improved by using collision detection.
- Potentially asynchronous calculation of angles, causing the game simulation to be not exactly zero-sum.
- Agents might learn better if the actions are speeds (v_x, v_z) directly instead of velocity + angle.

Limitations of the project:

- Unable to analytically solve the inverse problem. May need a different setup.
- Not tested with information advantages in either party.
- $v_{\mathbf{E}}$ can also be non-constant in the original problem for analysis.

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Q&A

Questions?