MAT1000/1001 Real Analysis

1 Measure Theory

1.1 Motivation

Consider the Riemann integral. Given $f : [a, b] \to \mathbb{R}$, $\eta : (x_0, ..., x_n)$, $a = x_0 < \cdots < x_n = b$, a partition, $|\eta| = \sup |x_{k+1} - x_k|$, then $\int_a^b f dx = \lim_{|\eta|\to 0} \overline{S}_\eta(f) = \lim_{|\eta|\to 0} \underline{S}_\eta(f)$ if limits exist and equal.

For $g = \chi_{\mathbb{Q}\cap[0,1]} = \begin{cases} 1, x \in \mathbb{Q} \cap [0, 1] \\ 0, \text{ otherwise} \end{cases}$, since $\overline{S}_\eta(g) = 1$ and $\underline{S}_\eta(g) = 0$, g is not Riemann integrable.

Consider the Fourier series: $f(x) = \frac{1}{2}a_0 + \sum_{k=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ makes sense if $\int \sum_n f_n = \sum_n \int f_n$ (limits are interchangeable.)

Lebesgue's contribution:

1. Define measure before integral
2. works well with limits to an extent

1.2 Measure

Let X be a set, $\mathcal{P}(X) = 2^X$ be the power set of X, $\mathcal{E} \subset \mathcal{P}(X)$ is a family/collection/class.

***Definition:* 1.1: Ring**

R is a ring of sets if $A, B \in R \Rightarrow A \cup B, A \setminus B \in R$.

It follows that R is a ring, then $A, B \in R \Rightarrow A \cap B \in R$, $A \triangle B = (A \setminus B) \cup (B \setminus A) \in R$ (symmetric difference).
If we take $\triangle$ as addition, $\cap$ as multiplication, then $(R, \triangle, \cap)$ is a ring, with $(R, \triangle)$ being a group.

***Definition:* 1.2: Algebra**

A non-empty $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra if $A, B \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}, A \cap B \in \mathcal{A}$.

Lemma 1. *A ring R is an algebra if $X \in R$.*

***Definition:* 1.3: Jordan Measurable Set**

The Jordan Measurable Set is $J_0 = \{E \subset \mathbb{R} : \chi_E \text{ is Riemann integrable}\}$.

Lemma 2. *J_0 is a ring.*

Proof. Following linearity and properties of Riemann integrals, $\chi_{E\cap F} = \chi_E \chi_F$, $\chi_{E\setminus F} = \chi_E - \chi_{E\cap F}$ $\square$

Note: $J = \left\{E : E \in J_0 \text{ or } E^C \in J_0\right\}$ is an algebra.

***Definition:* 1.4: Jordan Measure**

The Jordan measure is defined as $m(E) = \begin{cases} \int \chi_E, & \text{if } E \in J_0 \\ \infty, & \text{if } E^C \in J_0 \end{cases}$

***Definition:* 1.5: Finitely Additive Measure**

Let R be a ring, $\mu : R \to [0, \infty]$ is a finite additive measure if

1. $\mu(\emptyset) = 0$
2. $A_1, ..., A_n$ pairwise disjoint, then $\mu\left(()\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$

Remark 1. We consider infinity as a number with the following properties

1. $a \pm \infty = \pm\infty$
2. $a \cdot \infty = \begin{cases} \infty, a > 0 \\ 0, a = 0 \\ -\infty, a < 0 \end{cases}$

Lemma 3. *Let R be a ring, μ be a finite additive measure, $A \in R$, then $R|_A = \{A \cap E, E \subset R\}$, $\mu|_A(E) = \mu(A \cap E)$, $R|_A$ is an algebra on A ($R|_A$ is R restricted on A)*

***Theorem:* 1.1: Inclusion-Exclusion Principle**

Let $\mu : \mathcal{A} \to [0, \infty]$ be a finite additive measure. $\mathcal{A}$ is an algebra, $E_1, .., E_n \in \mathcal{A}$, then

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) - \sum_{1\le i<j\le n} \mu(E_i \cap E_j) + \cdots + (-1)^{n+1}\mu\left(\bigcap_{i=1}^{n} E_i\right)$$

Proof. The equation is equivalent to

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) + \sum_{I\subset\{1,...,n\}, I\neq\emptyset} (-1)^{|I|}\mu\left(\bigcap_{i\in I} E_i\right) = 0.$$

Given $J \subset \{1, ..., n\}$, define $E_J = \bigcap_{j\in J} E_j \cap \left(\bigcap_{j\in J^C} E_j^C\right)$, it generates all possible differences. All E_js are different.

Claim: $\mu(E_J) + \sum_{I\subset\{1,...,n\}, I\neq\emptyset} (-1)^{|I|}\mu\left(\bigcap_{i\in I} E_i \cap E_j\right) = 0 \forall J \neq \emptyset$.

Note if $I \cap J^C \neq \emptyset$, then $\bigcap_{i\in I} E_i \cap E_j = \emptyset$, and if $I \subset J$, then $E_J \subset \bigcap_{i\in I} E_i$.

$$
\begin{aligned}
\text{LHS} &= \mu(E_J) + \sum_{I\subset J, I\neq\emptyset} \mu(E_J)(-1)^{|I|} \\
&= \mu(E(J))\left[1 + \sum_{k=1}^{|J|}\binom{|J|}{k}(-1)^k\right] \\
&= \mu(E(J))(1+(-1)) = 0 \text{ (By Binomial Theorem)}
\end{aligned}
$$

□

Lemma: 1.1: Properties of Finitely Additive Measure

Let R be a ring, and μ a finitely additive measure, then

1. Monotonicity: $A, B \in R$, $A \subset B \Rightarrow \mu(A) \leq \mu(B)$
2. Subadditivity: $A_i \in R$, $i = 1, ..., n \Rightarrow \mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$

Proof. (Subadditivity) Let $B_1 = A_1$, $B_k = \bigcup_{i=1}^{k} A_i \setminus \bigcup_{i=1}^{k-1} A_i$. Then $\cup B_i = \cup A_i$, but $B_i \subset A_i$.

$$
\begin{aligned}
\mu(\cup A_i) &= \mu(\cup B_i) \\
&= \sum \mu(B_i) \ B_i \text{ are disjoint} \\
&\leq \sum \mu(A_i) \text{ subset}
\end{aligned}
$$

□

1.3 Sigma algebra and Measures

Definition: 1.6: σ-algebra

$\mathcal{M}$ is a σ-algebra if $\mathcal{M}$ is an algebra and $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$. *i.e.* Countable union of sets in $\mathcal{M}$ is still in $\mathcal{M}$.
$\mathcal{M}$ is a measurable set.

Definition: 1.7: σ-additive Measure

$\mu : \mathcal{M} \to [0, \infty]$ is a (σ-additive) measure if

1. $\mu(\emptyset) = 0$
2. $E_i \in M$, $i \geq 1$ pairwise disjoint, then $\mu(\cup E_i) = \sum \mu(E_i)$.

Remark 2. Jordan measurable sets J is not a σ-algebra. The counter example is the characteristic funciton.

Lemma: 1.2:

An algebra $\mathcal{A}$ is a σ-algebra if it is closed under countable disjoint union. ($E_i \in A, i \geq 1$ pairwise disjoint $\Rightarrow \cup E_i \in \mathcal{A}$)

Proof. Let $F_1 = E_1$, $F_k = \bigcup_{i=1}^{k} E_i \setminus \bigcup_{i=1}^{k-1} E_i, k \geq 2$. Then $\cup F_i = \cup E_i \in \mathcal{A}$. F_i is disjoint, satisfying condition, $\cup E_i \in \mathcal{A}$ by definition of σ-algebra. $\square$

Examples: $\{\emptyset, X\}, \mathcal{P}(X), \{E : E \text{ or } E^C \text{ is at most countable}\}$ are σ-algebra.

Proposition: 1.1: σ-algebra Generator

$\mathcal{E} \subset \mathcal{P}(X)$, then $\mathcal{M}(\mathcal{E}) = \bigcap \{\mathcal{S} \text{ is a } \sigma - \text{algebra} : \mathcal{E} \subset \mathcal{S}\}$ is a σ-algebra generated by $\mathcal{E}$. Note that $\{\mathcal{S}\} \in \mathcal{P}(\mathcal{P}(X))$.

Proof. If $E_i \in \mathcal{M}(\mathcal{E})$, $i \geq 1$, then $\forall \mathcal{S}$ a σ-algebra, $\mathcal{E} \subset \mathcal{S}$, we have $E_i \in S, i \geq 1$. Then $\cup E_i \in \mathcal{S}$. Take intersection over all $\mathcal{S}$ to get $\cup E_i \in \mathcal{M}(\mathcal{E})$. Similarly for E^C. $\square$

Remark 3. Same proof works for other classes such as rings and algebras.

Remark 4. Given $\mathcal{E}$, $\mathcal{E}_1 = \left\{ \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} E_{i,j} : E_{i,j} \text{ or } E_{i,j}^C \in \mathcal{E} \right\} \subset \mathcal{M}(\mathcal{E})$.

$\mathcal{E}_2 = \left\{ \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} E_{i,j} : E_{i,j} \text{ or } E_{i,j}^C \in \mathcal{E}_1 \right\} \subset \mathcal{M}(\mathcal{E})$. This continuous to infinity
But in general $\cup \mathcal{E}_i \neq \mathcal{M}(\mathcal{E})$.

Remark 5. $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2)$ if and only if $\mathcal{E}_1 \subset \mathcal{M}(\mathcal{E}_2)$ and $\mathcal{E}_2 \subset \mathcal{M}(\mathcal{E}_1)$ by previous remark.

Example: $(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$

Definition: 1.8: Semi-ring

A non-empty class $\mathcal{E}$ is a semi-ring if $E, F \subset \mathcal{E} \Rightarrow E \setminus F$ is a finite union of elements in $\mathcal{E}$.
$\mathcal{E}$ is a semi-ring, then $\left\{ \bigcup_{i=1}^{n} E_i : E_i \in \mathcal{E} \right\}$ is a ring.

Examples (measures):

1. Trivial: $\mu(A) = 0, \forall A$
2. $0 - \infty$: $\mu(A) = \infty$ if $A \neq \emptyset$, $\mu(\emptyset) = 0$
3. Dirac: at $x_0 \in X$, $\delta_{x_0}(A) = \begin{cases} 1, \text{ if } x_0 \in A \\ 0, \text{ if } x_0 \notin A \end{cases}$
4. Counting: $\mu(A) = |A|$ (cardinality of A)

Lemma: 1.3: Continuity of Measures

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. If $A_1 \subset A_2 \subset \cdots, A_i \in \mathcal{M}$, $(A_i \nearrow \cup A_i)$, then $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$
2. If $A_1 \supset A_2 \supset \cdots, A_i \in \mathcal{M}$, $(A_i \searrow \cap A_i)$, $\mu(A_1) < \infty$, then $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$

Proof. (2) Note that $\infty > \mu(A_1) = \mu(\bigcup_{i=1}^{\infty}(A_i \backslash A_{i+1}) \cup \bigcap_{i=1}^{\infty} A_i) \geq \sum_{i=1}^{\infty}(A_i \backslash A_{i+1})$ by subadditivity in Lemma 1.1, and since $A_i \setminus A_{i+1}$ are all disjoint.
Therefore, $\lim_{n\to\infty} \sum_{i=n}^{\infty} \mu(A_i \setminus A_{i+1}) = 0$ (convergent series).
$\mu(A_n) = \mu(\bigcup_{i=n}^{\infty}(A_i \setminus A_{i+1}) \cup \cap A_i) = \mu(\bigcup_{i=n}^{\infty}(A_i \setminus A_{i+1}) + \mu(\cap A_i)$. Then, $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_{i=1}^{\infty} A_i)$, since the first term is 0. □

Remark 6. $A_n \nearrow A$, then $\chi_{A_n} \nearrow \chi_A$, and $A_n \searrow A \Rightarrow \chi_{A_n} \searrow \chi_A$.

Definition: 1.9: Limsup and Liminf of sets

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i = \{x : \exists n_k \to \infty \text{ s.t. } x \in A_{n_k}\}$$
$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i = \{x : \exists N \text{ s.t. } x \in A_n \forall n \geq N\}.$$

Definition: 1.10: Measurable and Measure Space

$(X, \mathcal{M})$ is a measurable space
$(X, \mathcal{M}, \mu)$ is a measure space

Lemma: 1.4: Properties of Measures

1. Finite if $\mu(X) < \infty$
2. σ-finite if $\exists A_i \in \mathcal{M}$, $i > 1$ s.t. $X = \cup A_i$ and $\mu(A_i) < \infty$
3. Semi-finite: $\forall A \in \mathcal{M}$, $\exists B \neq \emptyset \subset A$, $B \in \mathcal{M}$ s.t. $\mu(B) < \infty$ (for any measurable set, there is a finite-measurable subset)

Example: Counting measure on $\mathcal{P}$ is semi-finite, but not σ-finite.

Lemma: 1.5: Pseudo-distance

Let $(X, \mathcal{M}, \mu)$ be a measurable space, $d : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ s.t. $d(A, B) = \mu(A \triangle B)$ is a pseudo-distance s.t.

1. $\mu(A \triangle A) = 0$
2. $\mu(A \triangle B) = \mu(B \triangle A)$
3. $\mu(A \triangle B) \leq \mu(A \triangle C) + \mu(C \triangle B)$

Definition: 1.11: μ-null Set

Let $(X, \mathcal{M}, \mu)$ be a measurable space, N is a μ-null set if $\exists A \in \mathcal{M}$, $\mu(A) = 0$ and $N \subset A$.

Definition: 1.12: Completion of Measurable Set

$\overline{\mathcal{M}_\mu} = \sigma-$algebra generated by $\mathcal{M} \cup \{\mu - \text{null sets}\}$ is the completion of $\mathcal{M}$ over μ.

$$\overline{\mathcal{M}_\mu} = \{E \cup N : E \subset \mathcal{M} \text{ and } N \text{ is } \mu - \text{null}\}$$

Proof. Countable union of $E \cup N$ will be countable union of E and countable union of N which has measure 0. $\bar{\mu}(E \cup N) = \mu(E)$ for completion of μ. □

1.4 Construction of Measures

Consider the Ring $R_0 = \left\{ \bigcup_{i=1}^{n} (a_i, b_i], a_i \leq b_i \in \mathbb{R} \right\}$.

Lemma: 1.6: Standard Representation

For any $A \in R_0$, $\exists a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$, s.t. $A = \bigcup_{i=1}^{n} (a_i, b_i]$

Proof. (Sketch) Existence: merge intervals whose closure intersect.
Uniqueness: By induction on intervals □

Definition: 1.13: Lebesgue Measure

The Lebesgue measure is $m : R_0 \to [0, \infty]$, $m(A) = \sum_{i=1}^{n} (b_i - a_i)$, where A is given by standard representation.

Proof. We focus on Additivity. There are several cases

1. Let $(a, b] = \bigcup_{i=1}^{n} (a_i, b_i]$ be disjoint union. Order the intervals s.t. $a_1 < b_1 = a_2 < b_2 \cdots = a_n < b_n$.

   Then $m((a, b]) = \sum_{i=1}^{n} m((a_i, b_i])$

2. Let $A \in R_0$, $A = \bigcup_{j=1}^{m} J_j$, $J_j = (c_j, d_j]$ disjoint union. Let $A = \bigcup_{i=1}^{n} I_i$ in standard representation. $I_i = \bigcup_{j \text{ s.t. } J_j \subset I_i} J_j$. Then $m(A) = \sum_i m(I_i) = \sum_i \sum_{j, J_j \subset I_i} m(J_j) = \sum m(J_j)$.
3. $A = \bigcup_{i=1}^{n} A_i$ be disjoint union, $A_i \in R_0$. $\bigcup_j I_{i,j}$ be standard representation. Then $m(A) = \sum_{i,j} m(I_{i,j}) = \sum_i m(A_i)$.

□

Definition: 1.14: Premeasure

Let $\mathcal{E} \subset \mathcal{P}(X)$, $\emptyset \in \mathcal{E}$ be an arbitrary class. $\mu_0 : \mathcal{E} \to [0, \infty]$ is a premeasure if
1. $\mu_0(\emptyset) = 0$
2. if $E_i \in \mathcal{E}$, $i \geq 1$ disjoint and $\cup E_i \in \mathcal{E}$, then $\sum \mu_0(E_i) = \mu_0(\sum E_i)$.

Proposition: 1.2:

The Lebesgue measure m is a premeasure on R_0

Proof. It suffices to consider $I = (a, b] = \bigcup_{i=1}^{\infty} I_i = \cup(a_i, b_i]$ disjoint union.

By Subadditivity from Lemma 1.1, $m(I) \geq m(\bigcup_{i=1}^{n} I_i) = \sum_{i=1}^{n} m(I_i)$. Take $n \to \infty$, we get $m(I) \geq \sum_{i=1}^{\infty} m(I_i)$

For $\epsilon > 0$, we have

$$\bigcup_{i=1}^{\infty}(a - 2^{-i}\epsilon, b + 2^{-i}\epsilon] \supset \bigcup_{i=1}^{\infty}(a - 2^{-i}\epsilon, b + 2^{-i}\epsilon) \supset \bigcup_{i=1}^{\infty}(a, b] \supset [a + \epsilon, b]$$

By Heine-Borel Theorem, $\exists n$ s.t. $\bigcup_{i=1}^{n}(a - 2^{-i}\epsilon, b + 2^{-i}\epsilon] \supset (a, b]$

$$\begin{aligned}
b - (a + \epsilon) &= m((a + \epsilon, b]) \\
&\leq \sum_{i=1}^{n} m((a_i - 2^{-i}\epsilon, b_i + 2^{-i}\epsilon]) \\
&\leq \sum_{i=1}^{\infty}(b_i - a_i + 2 \cdot 2^{-i}\epsilon) \\
&= \sum_{i=1}^{\infty} m(I_i) + 2\epsilon
\end{aligned}$$

Take $\epsilon \to 0$, $m(I) \leq \sum_{i=1}^{\infty} m(I_i)$. □

Definition: 1.15: Outer Measure

$mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure if

1. $\mu^*(\emptyset) = 0$
2. $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum \mu^*(A_i)$
3. $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$.

Theorem: 1.2: Construction of Outer Measure

Let $\mathcal{E} \subset \mathcal{P}(X)$ be a class with $\emptyset \in \mathcal{E}$, and $\exists A_i \in \mathcal{E}$, $i \geq 1$ s.t. $X \subset \cup A_i$. $\mu_0 : \mathcal{E} \to [0, \infty]$ with $\mu_0(\emptyset) = 0$. Then $\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : E_i \in \mathcal{E} \text{ and } A \subset \cup E_i \right\}$ is an outer measure.
Furthermore, if $\mathcal{E}$ is a ring R and μ_0 is a premeasure, then $\mu^*(E) = \mu_0(E)$ for all $E \in \mathcal{E}$.

Proof. Monotonocity is clear from inf.

Subadditivity: Let $E_i \subset X$, $i \geq 1$.. For $\epsilon > 0$, $\mu^*(E_i) > \sum_{j=1}^{\infty} \mu_0(A_{i,j}) - 2^{-i}\epsilon$, where $A_{i,j} \in \mathcal{E}$, $\bigcup_{j=1}^{\infty} A_{i,j} \supset E_i$.
Since $\cup_j A_{i,j}$ covers E_i, the measure should be larger.

$$\mu^*(\bigcup_i E_i) \leq \sum_{i,j} \mu_0(A_{i,j}) = \sum_i \sum_j \mu_0(A_{i,j}) < \sum_i \mu^*(E_i) + \sum_i 2^{-i}\epsilon = \sum_i \mu^*(E_i) + \epsilon$$

Restriction: If $E \in \mathcal{E} = R$, $\mu^*(E) \leq \mu_0(E)$ is trivial.

Suppose $A_i \in R$, $E \subset \cup A_i$. Define $B_1 = A_1$, $B_n = \bigcup_{i=1}^{n} A_i \setminus \bigcup_{i=1}^{n-1} A_i$.

$\mu_0(E) = \mu_0(\bigcup_{i=1}^{\infty}(E \cap B_i)) = \sum_{i=1}^{\infty} \mu_0(E \cap B_i)$, since B_i are disjoint.

Then $\mu_0(E) \leq \sum_{i=1}^{\infty} \mu_0(A_i)$, because $E \cap B_i \subset A_i$.

Taking inf over all possible coverings, we get $\mu_0(E) \leq \mu^*(E)$. $\square$

Theorem: 1.3: Caratheodory Criterion

Let μ^* be an outer measure, A is μ^*-measurable if $\forall E \subset X$, $\mu^*(E \cap A) + \mu^*(E \cap A^C) = \mu^*(E)$.
Let $\mathcal{M}_{\mu^*}$ be μ^*-measurable sets, $\mathcal{M}_{\mu^*}$ is a σ-algebra, and $\mu^*|_{\mathcal{M}_{\mu^*}}$ is closed under countable union.

Proof. Clearly, $A \in \mathcal{M}_{\mu^*} \Leftrightarrow A^C \in \mathcal{M}_{\mu^*}$. We only need to show that $\mathcal{M}_{\mu^*}$ is closed under countable union.

(1) Closed under finite union:
Let $A, B \in \mathcal{M}_{\mu^*}$, we want to show $A \cup B \in \mathcal{M}_{\mu^*}$.
Let $E \subset X$.

$$\begin{aligned}
\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^C) &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^C) + \mu^*(E \cap B^C \cap A^C) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^C) \\
&= \mu^*(E)
\end{aligned}$$

Therefore, $A \cup B \in \mathcal{M}_{\mu^*}$.

(2) μ^* is additive:
Take $A, B \in \mathcal{M}_{\mu^*}$, $A \cap B = \emptyset$.

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B \cap A) + \mu^*(A \cup B \cap A^C) = \mu^*(A \cup B)$$

Also, $\mu^*(E \cap (A \cup B)) = \mu^*(E \cap A) + \mu^*(E \cap B)$, $\mu^*|E$ is additive on $\mathcal{M}_{\mu^*}|_E$.

(3) $\mathcal{M}_{\mu^*}$ is a σ-algebra:

Take $A_i \in \mathcal{M}_{\mu^*}$, $i \geq 1$, pairwise disjoint. Denote $A_\infty = \bigcup_{i=1}^{\infty} A_i$.

We want to show that $\mu^*(E \cap A_\infty) + \mu^*(E \cap A_\infty^C) = \mu^*(E), \forall E \subset X$.

If $\mu^*(E \cap A_\infty) = \infty$, then it is trivial.

Assume $\mu^*(E \cap A_\infty) < \infty$.

$$\infty > \mu^*(E \cap A_\infty) \geq \mu^*(E \cap \bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu^*(E \cap A_i)$$

Therefore, $\sum_{i=1}^{\infty} \mu^*(E \cap A_i)$ converges. Hence $\lim_{n\to\infty} \sum_{i=n+1}^{\infty} \mu^*(E \cap A_i) = 0$

$$\begin{aligned}\mu^*(E \cap A_\infty) + \mu^*(E \cap A_\infty^C) &\leq \mu^*(E \cap \bigcup_{i=1}^{n} A_i) + \mu^*(E \cap \bigcup_{i=n+1}^{\infty} A_i) + \mu^*(E \cap (\bigcup_{i=1}^{n} A_i)^C) \\ &= \mu^*(E) + \epsilon\end{aligned}$$

Also, $\mu^*(E \cap A_\infty) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i)$ by taking limit in the equation above.

$\mu^*(N) = 0 \Rightarrow N \in \mathcal{M}_{\mu^*}$ ☐

***Proposition:* 1.3:**

Assume μ^* is induced by μ_0 on R. $R \subset \mathcal{M}_{\mu^*}$, then $\mathcal{M}(R) \subset \mathcal{M}_{\mu^*}$. ($\mathcal{M}(R)$ is the σ-algebra generated by R)

Proof. Let $A \in R, E \subset X$. Let $B_i \in R$, $i \geq 1$, $E \subset \cup_i B_i$

Because $\cup B_i$ is a covering of E, $\mu^*(A \cap E) \leq \sum_{i=1}^{\infty} \mu_0(A \cap B_i)$.

Similarly, $\mu^*(A^C \cap E) \leq \sum_{i=1}^{\infty} \mu_0(A \cap B_i)$. Therefore, by additivity:

$$\mu^*(A \cap E) + \mu^*(A^C \cap E) \leq \sum_{i=1}^{\infty} [\mu_0(A \cap B_i) + \mu_0(A^C \cap B_i)] \leq \sum_{i=1}^{\infty} \mu_0(B_i)$$

Take infimum over covering B_i, $\mu^*(A \cap E) + \mu^*(A^C \cap E) \leq \mu^*(E)$. ☐

Proposition: 1.4: Construction of Measures

1. Define a premeasure μ_0 on a ring R
2. Extend to outer measure $\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : \cup A_i \supset E, A_i \in R\right\}$
3. Define $\mathcal{M}_{\mu^*}$, $\mu^*|_{\mathcal{M}_{\mu^*}}$ is a measure.
4. $\mathcal{M}(R) \subset \mathcal{M}_{\mu^*}$

Proposition: 1.5: Property of μ Constructed from μ_0

$A \in \mathcal{M}_{\mu^*}$, $\forall \epsilon > 0$, $\exists B_i \in R$, $i \geq 1$ s.t. $A \subset \cup_i B_i$, $\mu(A) \leq \mu(\cup_i B_i) \leq \mu(A) + \epsilon$. If μ_0 is σ-finite, then $\mu(\cup_i B_i \setminus A) < \epsilon$.

Proof. By definition, $\exists B_i$ s.t. $A \subset \cup_i B_i$. Since A is measurable, and by definition of infimum, $\mu(A) = \mu^*(A) \geq \sum_i \mu_0(A_i) - \epsilon$.
If $\mu(A) < \infty$, then $\mu(\cup B_i \setminus A) < \epsilon$ holds.
If μ_0 is σ-finite, then $\exists E_k \in R$, $X \subset \cup_k E_k$, $\mu(E_k) < \infty$.
Let $B_{k,j} \in R$ s.t. $A_k = A \cap E_k \subset \cup_j B_{k,j}$. Then $\mu(\cup_j B_{k,j} \setminus A_k) < 2^{-k}\epsilon$.
Then $\mu(\bigcup_{k,j} B_{k,j} \setminus A) \leq \sum_k \mu(\bigcup_j B_{k,j} \setminus A) \leq \sum_k \mu(\bigcup_j B_{k,j} \setminus A_k) < \sum 2^{-k}\epsilon = \epsilon$. ∎

Corollary 1. *Let $A \in \mathcal{M}_{\mu^*}$, $\mu(A) < \infty$. $\forall \epsilon > 0$, $\exists B_1, ..., B_n \in R$. Then $\mu(A \triangle \bigcup_{i=1}^{n} B_i) < \epsilon$*

Proof. $\exists B_i \in R$ disjoint s.t. $A \subset \cup_i B_i$. $\mu(\bigcup_{i=1}^{\infty} B_i \setminus A) < \frac{\epsilon}{2}$.
Since $\bigcup_{i=1}^{n} B_i \setminus A \nearrow \bigcup_{i=1}^{\infty} B_i \setminus a$ and $\bigcup_{i=1}^{\infty} B_i \setminus \bigcup_{i=1}^{n} B_i \searrow \emptyset$ by continuity,
$\exists n$ s.t. $\mu(\bigcup_{i=1}^{n} B_i \setminus A) < \frac{\epsilon}{2}$, $\mu(\bigcup_{i=1}^{\infty} B_i \setminus \bigcup_{i=1}^{n} B_i) < \frac{\epsilon}{2}$. Then

$$\mu(A \triangle \bigcup_{i=1}^{n} B_i) = \mu(A \setminus \bigcup_{i=1}^{n} B_i) + \mu(\bigcup_{i=1}^{n} B_i \setminus A) \leq \mu(\bigcup_{i=1}^{\infty} B_i \setminus \bigcup_{i=1}^{n} B_i) + \mu(\bigcup_{i=1}^{n} B_i \setminus A) < \epsilon$$

∎

Corollary 2. *$\forall A \in \mathcal{M}_{\mu^*}$, μ_0 σ-finite, $\exists B \in \mathcal{M}(R)$ s.t. $A \subset B$ and $\mu(B \setminus A) = 0$.*

Theorem: 1.4: Uniqueness of Extension

Let μ_0 be a σ-finite premeasure on R, ν a measure on $\mathcal{M}(R)$ s.t. $\nu|_R = \mu_0|_R$. Then $\nu|_{\mathcal{M}(R)} = \mu^*|_{\mathcal{M}(R)} = \mu|_{\mathcal{M}(R)}$.

Proof. Show $\nu(A) \leq \mu(A)$:
Take covering B_i of $A \in \mathcal{M}(R) \subset \mathcal{M}_{\mu^*}$. $\nu(A) \leq \sum_i \nu(B_i) = \sum_i \mu_0(B_i)$.
Take infimum over covering, we get $\nu(A) \leq \mu^*(A) = \mu(A)$.

Show $\mu(A) \le \nu(A)$:
Assume $\mu(A) < \infty$. $\forall \epsilon > 0$, $\exists E \in R$ s.t. $\mu(A \triangle E) < \epsilon$.
Note that $\mu(E) \ge \mu(E \cap A) = \mu(A) - \mu(A \setminus E)$.

$$\begin{aligned}\nu(A) &= \nu(A \cap E) = \nu(E) - \nu(E \setminus A) \\ &\ge \mu(E) - \mu(E \setminus A) \\ &\ge \mu(A) - \mu(A \setminus E) - \mu(E \setminus A) > \mu(A) - \epsilon.\end{aligned}$$

When μ_0 is σ-finite, use $2^{-k}\epsilon$ to prove. □

***Proposition:* 1.6: Completion of Measure Space**

Suppose μ_0 is σ-finite, $(\mathcal{M}_{\mu^*}, \mu)$ is the completion of $(\mathcal{M}(R), \mu)$.

Proof. Let $\bar{\mathcal{M}}(R)$ be the completion of $\mathcal{M}(R)$ over μ. We want to show that $\bar{\mathcal{M}}(R) = \mathcal{M}_{\mu^*}$. We prove this by showing that they share the same μ-null set.

Let $N \in \bar{\mathcal{M}}(R)$, $\exists E \in \mathcal{M}(R)$ s.t. $\mu(E) = 0$, $N \subset E$. So $\mu^*(N) = 0$. $N \in \mathcal{M}_{\mu^*}$ by Theorem 1.3.

Let $N \in \mathcal{M}_{\mu^*}$, $\mu^*(N) = 0$, $\exists E \in \mathcal{M}(R)$ s.t. $N \subset E$, $\mu(E \setminus N) = 0$, $N \in \bar{\mathcal{M}}(R)$.

Let $A \in \mathcal{M}_{\mu^*}$. $\exists B \in \mathcal{M}(R)$, $A^C \subset B$, $\mu(B \setminus A^C) = \mu(B \cap A) = 0$
$A = (A \setminus B) \cup (B \cap A) \in \bar{\mathcal{M}}(R)$. (union of measurable core and a null set.) □

1.5 Lebesgue and Lebesgue-Stieltjes Measures

Let $R_0 = \bigcup_i (a_i, b_i]$, $m((a,b]) = b - a$.

***Theorem:* 1.5: Borel Measure**

Let $\Sigma = \{A : A \text{ is a sigma algebra cotaining all subsets of } \mathbb{R}\}$. (*e.g.* $\mathcal{P}(\mathbb{R}) \in \Sigma$) Define $\mathcal{B} = \bigcap_{A \in \Sigma} A \subset \mathcal{P}(\mathbb{R})$. Then $\mathcal{B}$ is the smallest σ-algebra containing all subsets of $\mathbb{R}$. This is the Borel Measure.

***Definition:* 1.16: Lebesgue Measure**

Letbesgue measure m is the unique extension of m on R_0 to $\mathcal{B}_{\mathbb{R}}$ (the Borel σ-algebra). $\mathcal{L}_{\mathbb{R}}$ is the completion of $\mathcal{B}_{\mathbb{R}}$ over m.

***Theorem:* 1.6: Properties of Lebesgue Measure**

Let m be Lebesgue measure on $\mathbb{R}$. Then

1. Finite approximation: If $A \in \mathcal{L}_{\mathbb{R}}$ and $m(A) < \infty$, $\forall \epsilon > 0$, $\exists I_1, ..., I_n$ intervals, $m(A \triangle \bigcup_{i=1}^{n} I_i) < \epsilon$.
2. $A \in \mathcal{L}_{\mathbb{R}}$, $\forall \epsilon > 0$, $\exists U$ an open set s.t. $A \subset U$, $m(U \setminus A) < \epsilon$.
3. $\exists G_\delta$-set (Countable intersection of open sets) B s.t. $A \subset B$ and $m(B \setminus A) = 0$.

Definition: 1.17: Lebesgue-Stieltjes (L-S) Measure

Let $F : \mathbb{R} \to \mathbb{R}$ be right continuous and non-decreasing. Define $\mu_F((a, b]) = F(b) - F(a)$. If $b_n \searrow b$, then $\lim\limits_{n\to\infty} \mu_F((a, b_n]) = \mu_F((a, b])$. If $a_n \nearrow a$, then $\lim\limits_{n\to\infty} \mu_F((a_n, b]) = \mu_F((a, b])$. μ_F is a pre-measure.
Lebesgue-Stieltjes measure F is the unique extention of μ_F to $\mathcal{B}_{\mathbb{R}}$.

Example:

1. $H_{x_0}(x) = \begin{cases} 1, x \geq x_0 \\ 0, x < x_0 \end{cases}$, $\mu_{H_{x_0}} = \delta_{x_0}$ (the Dirac measure). This represents point mass.
2. f Riemann integrable, $f \geq 0, \displaystyle\int_{-\infty}^{\infty} f < \infty$. $F(x) = \displaystyle\int_{-\infty}^{x} f(t)dt$, $\mu_F([a, b]) = \displaystyle\int_a^b f(t)dt$. This represents smooth density.
3. Singular non-atomic ($\mu(\{x\}) = 0$ and not integrable):
Consider the Cantor set $C = \displaystyle\bigcap_k \bigcup_{i_0,\ldots,i_{k-1}} I_{i_0,\ldots,i_{k-1}}$, $\mu^*(C) \leq 2^k 3^{-k} \to 0$.
This creates the Devil Stair case function. F_C inductively defined in $[0, 1] \setminus C$. F_C is uniformly continuous on $[0, 1] \setminus C$.
Extend to $[0, 1]$ by continuity. Let μ_C be LS-measurable of F_C, then $\mu_C(\{x\}) = 0$, $\mu_C(C) = 1$, μ_C is nowhere differentiable on C.

Remark 7. Let μ be a Borel-measure s.t. $\mu([a, b]) < \infty$ for $-\infty < a < b < \infty$. $F(x) = \begin{cases} \mu((0, x]), x > 0 \\ -\mu((x, 0]), x < 0 \end{cases}$

Theorem: 1.7: Translation Invariance of Lebesgue Measure

$\forall E \in \mathcal{L}_{\mathbb{R}}$, $\forall x \in \mathbb{R}$, $x + E \in \mathcal{L}_{\mathbb{R}}$, and $m(x + E) = m(E)$.

Proof. Let $E \in \mathcal{L}_{\mathbb{R}}$, $B_i \in R_0$ s.t. $E_i \subset \displaystyle\bigcup_{i=1}^{\infty} B_i$.
$m^*(E + x) \leq \sum_i m(B_i + x) = \sum_i m(B_i)$. Taking infimum over coverings, we get $m^*(x + E) \leq m(E)$.
Get reverse by $E = (E + x) + (-x)$. □

1.6 Vitali's Example

Axiom of Choice: Let $F : A \to \mathcal{P}(X)$ s.t. $F(a) \neq \emptyset$. Then $\exists f : A \to X$ s.t. $f(a) \in F(a)$.

Corollary 3. *If $\mathcal{E} \subset \mathcal{P}(X)$ is a mutually disjoint non-empty sets, then $\exists A \subset X$ s.t. $A \cap E$ is a singleton for $E \in \mathcal{E}$.*

Lemma: 1.7: Rotation Map

Let $R_\alpha : [0, 1) \to [0, 1)$ be $R_\alpha(x) = (x + \alpha) \mod 1$. $\forall E \in \mathcal{L}_{\mathbb{R}}|_{[0,1)}$, $m(R_\alpha(E)) = m(E)$.

Proof. The interval $[0, 1 - \alpha)$ is mapped to $[\alpha, 1)$ and $[1 - \alpha, 1)$ is mapped to $[0, \alpha)$. Each of them preseres measure by Theorem 1.7 then apply Theorem 1.3. □

Definition: 1.18:

$$R_\alpha^j(x) = \begin{cases} \underbrace{R_\alpha \circ \cdots \circ R_\alpha}_{j}(x), j \geq 1 \\ x, j = 0 \\ R_\alpha^{-1} \circ \cdots R_\alpha^{-1}(x), j < 0 \end{cases}. \text{ The orbit } \mathcal{O}(x) = \left\{ R_\alpha^j(x) : j \in \mathbb{Z} \right\}.$$

Lemma 4. *$x \sim y \Leftrightarrow x \in \mathcal{O}(y)$ is an equivalent relation*

Proof. Identity: $x \in \mathcal{O}(x)$
Symmetry: $x \in \mathcal{O}(y)$, then $\exists j$ s.t. $x = R_\alpha^j(y)$, $\Rightarrow y = R_\alpha^{-1}(x)$, $y \in \mathcal{O}(x)$.
Transitivity is similar. □

$\mathcal{E} = \{\mathcal{O}(x) : x \in [0, 1)\}$ is a partition (mutually disjoint and union is the whole space).

Lemma 5. *$R_\alpha^j(x) \neq R_\alpha^k(x)$, $j \neq k \Leftrightarrow \alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

Proof. If $R_\alpha^j(x) = R_\alpha^k(x)$, then $R_\alpha^{j-k}(x) = x$, $x + (j - k)\alpha = x \mod 1$, $\alpha \in \mathbb{Q}$. □

Theorem: 1.8:

Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, by axiom of choice, $\exists E_v$ s.t. $E_v \cap \mathcal{O}(x)$ is a singleton. $E_v \notin \mathcal{L}_\mathbb{R}$.

Proof. Suppose $E_v \in \mathcal{L}_\mathbb{R}$, $[0, 1) = \bigcup_{j \in \mathbb{Z}} R_\alpha^j(E_v)$ and $R_\alpha^j(E_v) \cap R_\alpha^k(E_v) = \emptyset$, $\forall j \neq k$. $\forall x \in [0, 1)$, $\exists j \in \mathbb{Z}$ s.t. $R_\alpha^j(x) \in E_v$, so $x \in R_\alpha^{-j}(x)(E_v)$.
If $z \in R_\alpha^j(E_v) \cap R_\alpha^k(E_v)$, then $R_\alpha^{-j}(z), R_\alpha^{-k}(z) \in E_v$, $j = k$
$m([0, 1)) = \sum_{j \in \mathbb{Z}} m(R_\alpha^j(E_v)) = \sum_{j \in \mathbb{Z}} m(E_v)$.
If $m(E_v) = 0$, then $\sum m(E_v) = 0$. If $m(E_v) = \alpha$, then $\sum m(E_v) = \infty$. Contradition. □

1.7 Measurable Mappings

Proposition: 1.7:

Let $T : X \to Y$ and $\mathcal{N}$ is a σ-algebra on Y. Then $T^{-1}(\mathcal{N}) = \left\{ T^{-1}(E) : E \in \mathcal{N} \right\}$ is a σ-algebra.

Proof. Using the properties of preimages. $T^{-1}(E^C) = (T^{-1}(E))^C$ and $T^{-1}(\cup E_i) = \cup T^{-1}(E_i)$. □

Definition: 1.19: Measurable Mappings

Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces. $T : X \to Y$ is measurable if $T^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{N}$. Equivalently, $T^{-1}(\mathcal{N}) \subset \mathcal{M}$.

Proposition: 1.8:

If $\mathcal{E}$ generates $\mathcal{N}$, then $T : X \to Y$ is measurable $\Leftrightarrow T^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{E}$.

Proof. Construct $\mathcal{F} = \{E \in N : T^{-1}(E) \in \mathcal{M}\}$. $\mathcal{F}$ is a σ-algebra by Proposition 1.7.
By assumption, $\mathcal{E} \subset \mathcal{F}$, hence $\mathcal{N} = \mathcal{M}(\mathcal{E}) \subset \mathcal{F}$. □

Definition: 1.20: Measurable Functions

Let $(X, \mathcal{M})$ be measurable space, $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable if f is measurable as mapping of $(X, \mathcal{M})$ to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Definition: 1.21: Borel and Lebesgue Measurable

$f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if it is $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$-measurable. f is Lebesgue measurable if it is $(\mathbb{R}, \mathcal{L}_{\mathbb{R}})$-measurable.

Proposition: 1.9:

$f : X \to \mathbb{R}$ is $(X, \mathcal{M})$ measurable $\Leftrightarrow f^{-1}((-\infty, a]) \in \mathcal{M}$, $\forall a \in \mathbb{R}$ (or any generating sets for Borel σ-algebra, *i.e.* any intervals or singleton sets)

Proposition: 1.10:

Any continous function $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable.

Proposition: 1.11:

Let $E \in \mathcal{B}_{\mathbb{R}}$, $f : E \to \mathbb{R}$ is increasing, then f is $\mathcal{B}_{\mathbb{R}}|_E$-measurable

Proof. For $a \in \mathbb{R}$, let $t = \sup\{f^{-1}(x) : x \leq a\} = \sup f^{-1}((-\infty, a])$, then $f^{-1}((-\infty, a]) = E \cap (-\infty, t] \in \mathcal{B}_{\mathbb{R}}|_E$. □

Consider the increasing function $F_C(C) = [0, 1]$. The Cantor set C is Borel, so F_C is Borel. F_C is not 1-1, but $\exists A$ countable s.t. $F|_{C \setminus A}$ is 1-1, $A =$ all base-3 finite decimals. $C \setminus A$ makes all intervals in C open. Also $f = (F_C|_{C \setminus A})^{-1}$ is Borel.

Theorem: 1.9:

$\mathcal{L}_{\mathbb{R}} \setminus \mathcal{B}_{\mathbb{R}} \neq \emptyset$

Proof. Let E be the Vitali set, $f = (F_C|_{C \setminus A})^{-1}$, $F = f(E)$, $m^*(F) = 0$, hence $F \in \mathcal{L}_{\mathbb{R}}$, $E = f^{-1}(F)$ is not Borel, hence F is not Borel. □

2 Integration

In the following discussions, fix $(X, \mathcal{M})$ a measurable space, $\overline{\mathbb{R}} = [-\infty, \infty]$, $\mathcal{B}_{\overline{\mathbb{R}}} = \{E : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$.

2.1 Measurable Functions

***Definition:* 2.1: Borel Functions**

$f : X \to \overline{\mathbb{R}}$ is Borel if $f^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{B}_{\overline{\mathbb{R}}}$ and if and only if $f^{-1}([-\infty, a)) = \{x : f(x) < a\} \in \mathcal{M}, \forall a \in \overline{\mathbb{R}}$.

***Proposition:* 2.1: Properties of Measurable Functions**

Let $f, g : X \to \mathbb{R}$ be $\mathcal{M}$-measurable, then the following functions are $\mathcal{M}$-measurable.

1. $f + g$
2. $af, a \in \mathbb{R}$
3. f^2
4. fg

Proof. In general, we consider $f^{-1}([-\infty, a)) = \{x : f(x) < a\}$.

1. $\{x : f(x) + g(x) < a\} = \bigcup_{t \in \mathbb{Q}} \{x : f(x) < t\} \cap \{x : g(z) < a - t\} \in \mathcal{M}$ since f, g are measurable.
2. $\{x : af(x) < b, a \in \mathbb{R}\} = \bigcup_{a \in \mathbb{R}} \left\{x : f(x) < \frac{b}{a}, a \neq 0\right\} \in \mathcal{M}$
3. $\left\{x : f^2(x) < a\right\} = \begin{cases} \emptyset, a \leq 0 \\ \{-\sqrt{a} < f < \sqrt{a}\}, f > 0 \end{cases} \in \mathcal{M}$
4. Use the identity: $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ with 1,2,3.

□

***Proposition:* 2.2: Measurable Functions with inf and sup**

Let $f_j : X \to \overline{\mathbb{R}}$ be measurable, $j \in \mathbb{N}$. Then so are

1. $\sup\limits_j f_j$
2. $\inf\limits_j f_j$
3. $\limsup\limits_j f_j$
4. $\liminf\limits_j f_j$

Proof. We only need to show 1, and 2,3,4 directly follows

$$\left\{x : \sup_j f_j(x) < a\right\} = \bigcap_j \{x : f_j(x) < a\} \in \mathcal{M}$$

□

Corollary 4. *If f_j is measurable, then $\left\{x : \lim\limits_{j \to \infty} f_j(x) \text{ is finite}\right\}$ is measurable.*

Proof. By definition, $\lim = \limsup = \liminf$ when $\limsup = \liminf$. □

Corollary 5. *If f is measurable, then $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$ are measurable.*

2.2 Simple Functions

Definition: 2.2: Simple Functions

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}, a_i \in \mathbb{R}, E_i \in \mathcal{M}$$

Simple functions are measurable and $\phi(x)$ is finite. Suppose $\phi(X) = \{b_1, ..., b_k\}$, then $\phi(x) = \sum_{j=1}^{k} b_j \chi_{\phi^{-1}(\{b_j\})}$ is the *standard representation*. Same as a_i distinct, E_i disjoint, and $a_i = 0$ is included.

Definition: 2.3: Integration of Simple Functions

Let μ be a measure. Let $\phi \geq 0$ be a simple function. Define $\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$ if $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ in standard representation. uppose $E \in \mathcal{M}$, $\int_E \phi d\mu = \int \chi_E \phi d\mu$.

Note: $\phi \geq 0$ requirement screens out the possibility of $\infty - \infty$.

Proposition: 2.3: Properties of Integration of Simple Functions

Let $\phi, \psi \geq 0$ be simple functions. Then

1. $\int \phi + \psi = \int \phi + \int \psi$
2. $\int a\phi = a \int \phi$ for $a \neq 0$
3. $\phi \leq \psi \Leftrightarrow \int \phi \leq \int \psi$.

Proof. Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $\psi = \sum_{j=1}^{k} b_j \chi_{F_j}$ be standard representation. Then $\phi + \psi = \sum_{i,j} (a_i + b_j) \chi_{E_i \cap F_j}$

$$\begin{aligned}
\int \phi + \psi &= \sum_{c \in (\phi+\psi)(X)} c\mu\left(\{x : \phi + \psi = c\}\right) \\
&= \sum_{c} c \sum_{i,j, a_i + b_j = c} \mu(E_i \cap F_j) \\
&= \sum_{i,j} (a_i + b_j) \mu(E_i \cap F_j) \\
&= \sum_{i,j} a_i \mu(E_i \cap F_j) + \sum_{i,j} b_j \mu(E_i \cap F_j) = \int \phi + \int \psi
\end{aligned}$$

□

Proposition: 2.4:

Let $\phi \geq 0$ be simple, then $E \mapsto \int_E \phi d\mu$ is a measure.

Proof. $\int_E \chi_F d\mu = \mu(E \cap F) = \mu|_F(E)$ is a measure.

Since ϕ is a finite linear combination of different χ_F, $E \mapsto \int_E \phi d\mu$ is a measure. $\square$

Proposition: 2.5:

Let $f : X \to \overline{\mathbb{R}}$ be measurable, $f \geq 0$. Then $\exists \phi_n : \phi_n \geq 0$ simple s.t. $\phi_n \nearrow f$ pointwise. Moreover, convergence is uniform over all sets on which f is bounded.

Proof. Let $E_{n,j} = \left\{x : f(x) \in (j2^{-n}, (j+1)2^{-n}]\right\}, j = 0, ..., 4^n - 1, F_n = \{x : f(x) > 2^n\}$.
$\phi_n = \sum_j (j2^{-n})\chi_{E_{n,j}} + 2^n \chi_{F_n}$ is a finite sum, thus simple.
If $n < m$, and $E_{n,j} \cap E_{m,k} \neq \emptyset$, then $(k2^{-m}, (k+1)2^{-m}] \subset (j2^{-n}, (j+1)2^{-n}]$, *i.e.* $E_{m,k} \subset E_{n,j}$.
If $x \in E_{n,j} \cap E_{m,k}$, then $\phi_n(x) \leq \phi_m(x)$.
On $F_N^C = \left\{x : f(x) \leq 2^N\right\}$, $\phi_n \to f$ uniformly, $\bigcup_n F_n^C = \{x : f(x) < \infty\}$. $\square$

Definition: 2.4: Integrals

Let $f : X \to [0, \infty]$ be measurable, denote $f \in \mathcal{L}^+(X, \mathcal{M})$. Define $\int f d\mu = \sup_{0 \leq \phi \leq f, \phi \text{ simple}} \int \phi d\mu$.
$\int_E f d\mu = \int_X \chi_E f d\mu$.

Theorem: 2.1: Monotone Convergence

Let $f_n \in \mathcal{L}^+$, $f_n \nearrow f$, then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$.

Proof. To show equality, we prove two inequalities

Since $f_n \leq f$, $\int f_n \leq \int f$, by definition of sup. By Proposition 2.2, $\int f \geq \lim_{n\to\infty} \int f d\mu$.

Let ϕ be a simple function s.t. $0 \leq \phi \leq f$. The idea is to make $\phi < f$, then $\phi \leq f_n < f$.
Fix $a \in (0, 1)$. Let $E_n = \{x : f_n(x) \geq \alpha\phi(x)\}$, then $E_n \nearrow X$.
Since $E \mapsto \int_E \alpha\phi$ is a measure,

$$\int_X \alpha\phi d\mu = \lim_{n\to\infty} \int_{E_n} \alpha\phi d\mu = \lim_{n\to\infty} \int \alpha\chi_{E_n}\phi d\mu \leq \lim_{n\to\infty} \int f_n d\mu$$

Therefore $\alpha \int \phi \leq \lim_{n\to\infty} \int f_n$. Take sup over $0 \leq \phi \leq f$ and $\alpha \to 1$, we get $\int f \leq \lim_{n\to\infty} \int f_n$. $\square$

***Proposition:* 2.6:**

Let $f, g \in \mathcal{L}^+(X, \mathcal{M})$, then

1. $\int f + g = \int f + \int g$
2. $E \mapsto \displaystyle\int_E f d\mu$ is a measure
3. $\int af = a \int f$
4. $f \leq g \Rightarrow \int f \leq \int g$
5. If $f_n \in \mathcal{L}^+$, then $\displaystyle\int \sum_n f_n = \sum_n \int f_n$

Proof. For 1, take ϕ_n, ψ_n simple s.t. $\phi_n \nearrow f$, and $\psi_n \nearrow g$.
By Theorem 2.1, $\displaystyle\int f + g = \lim_{n\to\infty} \int \phi_n + \psi_n \overset{\text{By Prop 2.3}}{=} \lim_{n\to\infty} \int \phi_n + \int \psi_n = \int \phi + \int \psi$.

For 2, $E_n \nearrow E$, $\chi_{E_n} f \nearrow \chi_E f$. By Theorem 2.1, $\int \chi_{E_n} f \to \int \chi_E f$. Therefore $E \mapsto \displaystyle\int_E f d\mu$ is a measure. □

***Lemma:* 2.1: Fatou Lemma**

Let $f_n \in \mathcal{L}^+$, then

$$\int \liminf_n f_n \leq \liminf_n \int f_n$$

Proof. Since $\inf_{i\geq n} f_i \nearrow \liminf_n f_n$ by definition, apply Theorem 2.1, $\displaystyle\lim_{n\to\infty} \int \inf_{i\geq n} f_i = \int \liminf_n f_n$.
But $\displaystyle\liminf_n \int f_i = \lim_{n\to\infty} \inf_{i\geq n} \int f_i \geq \lim_{n\to\infty} \int \inf_{i\geq n} f_i$. □

Example:

1. $f_n = \frac{1}{n}\chi_{[0,n)}$, $\liminf f_n = 0$, but $\int f_n = 1$ (Escape through width)
2. $f_n = n\chi_{(0,\frac{1}{n}]}$, $f_n \to 0$ pointwise, but $\int f_n = 1$ (Escape through height)
3. $f_n = \chi_{[n,n+1)}$, $f_n \to 0$ pointwise, but $\int f_n = 1$ (Escape through non-compactness)

***Lemma:* 2.2: Markov Inequality**

Let $f \in \mathcal{L}^+$, $t > 0$, then $\mu(\{x : f(x) > t\}) \leq \frac{1}{t} \int f d\mu$

Proof. $t\chi_{\{x:f(x)>t\}} \leq f$. Integrate both sides. □

Corollary 6. *If $f \in \mathcal{L}^+$, $\int f d\mu < \infty$, then $\mu((x : f(x) = \infty)) = 0$.*

***Definition:* 2.5: Equality Almost Everwhere**

$f = g$ μ a.e. (almost everywhere) if $\exists E \in \mathcal{M}$, $\mu(E) = 0$ s.t. $f(x) = g(x)$, $\forall x \in E^C$.

Lemma: 2.3:

Let $f, g \in \mathcal{L}^+$.

1. If $f = g$ μ a.e., then $\int f = \int g$
2. $\int f d\mu = 0 \Leftrightarrow f = 0$ μ a.e.

Proof. (1) Let ϕ be simple s.t. $0 \leq \phi \leq f$, E be s.t. $\mu(E) = 0$ and $f|_{E^C} = g|_{E^C}$.

$$\int \phi = \int_{E^C} \phi \leq \int_{E^C} f = \int_{E^C} g \leq \int g$$

Take sup in ϕ, we get $\int f \leq \int g$
Similarly, we have $\int g \leq \int f$, then $\int f = \int g$

(2) $\mu\left(\left\{x : f(x) > \frac{1}{n}\right\}\right) \leq n \int f d\mu = 0$
$\mu(\{x : f(x) > 0\}) = \sup_n \mu\left(\left\{x : f(x) > \frac{1}{n}\right\}\right) = 0$ ☐

2.3 Integrals of General Functions

Let $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$, $f = f^+ - f^-$

Definition: 2.6: Integration of General Function

$f : X \to \mathbb{R}$ be measurable and integrable or $f \in L^1$ if $\int f^+ < \infty$ and $\int f^- < \infty$.

$$\int f = \int f^+ - \int f^-$$

If $f \in L^1$, then f is finite a.e.

Definition: 2.7: Integration of Complex Function

$f : X \to \mathbb{C}$ is measurable or integrable if $\mathrm{Re} f$ and $\mathrm{Im} f$ are both measurable/integrable.

Proposition: 2.7: Properties of General Integration

Let $f, g : X \to \mathbb{R}$ be L^1. Then so are $f + g$, $af, a \in \mathbb{R}$

1. $\int f + g = \int f + \int g$
2. $\int af = a \int f$

Proof.

$$\int |f + g| \leq \int (f + g)^+ + (f + g)^- \leq 2 \int |f| + 2 \int |g|$$

Also, $\int |f| \leq \int f^+ + \int f^-$, so $f + g \in L^1$.
Since $f + g = (f + g)^+ - (f + g)^- = f^+ - f^- + g^+ - g^-$, we get $(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+$.
Integrate both sides and apply linearity of positive functions, we can prove the equality. ☐

Corollary 7. *The same holds for complex valued functions*

Proposition: 2.8:

If $f : X \to \mathbb{C}$ is integrable, then $\left|\int f d\mu\right| \leq \int |f| \, d\mu$

Proof. Since $\int f d\mu \in \mathbb{C}$, $\exists \theta \in \mathbb{R}$ s.t. $e^{-i\theta} \int f d\mu = \left|\int f d\mu\right|$. Then

$$\begin{aligned}\left|\int f d\mu\right| &= e^{-i\theta} \int f d\mu = \int e^{-i\theta} f d\mu \text{ (By Linearity)} \\ &= \int \operatorname{Re}(e^{-i\theta} f) d\mu \\ &\leq \int \left|e^{-i\theta} f\right| d\mu \\ &\leq \int |f| d\mu\end{aligned}$$

□

Remark 8. We can rewrite some definitions/lemmas with complex-valued functions

1. $f \in L^1 \Leftrightarrow \int |f| d\mu < \infty$
2. $\mu(\{x : |f(x)| > t\}) \leq \frac{1}{t} \int |f| d\mu$

Proposition: 2.9:

1. If $f \in L^1$, then $\{x : |f(x)| > 0\}$ is σ-finite.
2. $f \in L^1$, if $\displaystyle\int_E f d\mu = 0 \ \forall E \in \mathcal{M}$, then $f = 0$ a.e.

Proof. 1) take countable union with Markov inequality

2) for $f : X \to \mathbb{R}$, $\displaystyle\int f^+ = \int_{\{x: f(x) \geq 0\}} f = 0$, then $f^+ = 0$ a.e. Same for f^-. □

Definition: 2.8: L^1-Functions

$$L^1(\mu) = \left\{ f \text{ measurable}, \ \int |f| d\mu < \infty \right\} / \sim,$$

where $f \sim g \Leftrightarrow f = g \ \mu$ a.e.
$\|f\|_1 = \int |f| d\mu$ is a norm.

Definition: 2.9: a.e. Convergence

$f_n \to f \ \mu$ a.e. if $\exists E \in \mathcal{M}$, $\mu(E) = 0$ s.t. $f_n(x) \to f(x), \forall x \in E^C$.

Theorem: 2.2: Dominated Convergence Theorem

Let $f_n \to f \ \mu$ a.e. and $\exists g \in L^+ \cap L^1$ s.t. $|f_n| \leq g$, then $f \in L^1$ and $\displaystyle\lim_{n \to \infty} \int f_n = \int f$

Proof. It sufficies to consider only positive functions $f_n \in L^+$.
Assume convergence is pointwise. Then

$$\int f = \int \lim_{n\to\infty} f_n \underset{\text{Convergence}}{=} \int \liminf_{n\to\infty} f_n \underset{\text{Fatou}}{\leq} \liminf_{n\to\infty} f_n,$$

and

$$\int g - f = \int \liminf_{n\to\infty}(g - f_n) \leq \liminf_{n\to\infty} \int (g - f_n) = \int g - \limsup_{n\to\infty} \int f_n$$

Therfore $\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n$. We get $\lim_{n\to\infty} \int f_n = \int f$. □

Corollary 8. *With the same assumptions in Theorem 2.2, we have* $\|f_n - f\|_1 \to 0$

Proof. $|f| \leq g$ a.e. $|f_n - f| \leq 2g$ and $|f_n - f| \to 0$ a.e. Apply Theorem 2.2. □

Remark 9. g must be fixed for Theorem 2.2 to work.

***Proposition:* 2.10:**

Suppose $f \in L^1$, then there exist simple functions $\phi_n \in L^1$ s.t. $|\phi_n| \leq |f|$ and $\phi_n \to f$ μ a.e. Convergence is uniform on the set for which f is bounded.

Remark 10. $\phi_n = \sum_{i=1}^{k} a_i \chi_{E_i}$, where $\mu(E_i) < \infty$ for $a_i \neq 0$. Then by Theorem 2.2, there exists simple $\phi_i \in L^1$ s.t. $\|\phi_n - f\|_1 \to 0$

***Proposition:* 2.11: Absolute Continuity**

Let $f \in L^1$, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\int_E |f| d\mu < \epsilon$ for all E s.t. $\mu(E) < \delta$.

Proof. Gien $\epsilon > 0$, there exists simple ϕ s.t. $\int |\phi - f| d\mu < \frac{\epsilon}{2}$.
All simple functions are bounded. Then $\int_E |\phi| d\mu \leq \mu(E) \sup |\phi|$.
Choose $\delta < \frac{\epsilon}{2\sup|\phi|}$, we get $\int_E |f| d\mu \leq \int_E |\phi - f| d\mu + \int_E |\phi| d\mu < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

2.4 Modes of Convergence

***Theorem:* 2.3: Egoroff**

Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $f_n \to f$ μ.a.e., f_n measurable. Then $\forall \epsilon > 0$, $\exists E$, $\mu(E) < \epsilon$ s.t. $f_n|_{E^C} \to f|_{E^C}$ uniformly.

Proof. Let $E_{n,m} = \left\{ x : \sup_{k\geq n} |f_k(x) - f(x)| > \frac{1}{m} \right\}$, $E_{n,m} \searrow \cap_n E_{n,m}$.
$f_n \to f$ μ.a.e. $\Leftrightarrow \mu(\cap_n E_{n,m}) = 0$, $\forall m$ by Definition 2.9.
Given $\mu(X) < \infty$, $\mu(\cap_n E_{n,m}) = \lim_{n\to\infty} \mu(E_{n,m})$.
Given $\epsilon > 0$, choose n_m s.t. $\mu(E_{n_m,m}) < \epsilon 2^{-m}$

Let $F = \cap_m (E_{n_m,m})^C$.
For $x \in F$, $\forall m$, $k \geq n_m$, $|f_k(x) - f(x)| \leq \frac{1}{m}$, *i.e.* $f_n \to f$ uniformly on F.
$\mu(F^C) = \mu(\cup_m E_{n_m,m}) < \sum 2^{-m}\epsilon = \epsilon$. □

Two modes of convergence:

1. $f_n \to f$ μ a.e.
2. $f_n \to f$ in L^1

Note that these two do not imply each other.

Example: Consider the sequence constructed by $\chi_{[0,\frac{1}{n}]}, \chi_{[\frac{1}{n},\frac{2}{n}]}, \ldots, \chi_{[\frac{n-1}{n},1]}$. It converges in L^1, but piecewise diverges.

***Definition:* 2.10: Cauchy Sequence**

If f_n is Cauchy in measure, then there exists a subsequence n_k s.t. $f_{n_k} \to f$ μ a.e. and $f_n \to f$ in measure.

Lemma 6. *$f_n \to f$ in $L^1 \Rightarrow f_n \to f$ in measure.*

Proof. By Markov's inequality. □

***Theorem:* 2.4:**

If f_n is Cauchy in measure, then there exists a subsequence n_k s.t. $f_{n_k} \to f$ μ a.e. and $f_n \to f$ in measure.

Proof. Choose n_k s.t. $\mu\left(\left\{x : |f_j(x) - f_l(x)| > 2^{-k}\right\}\right) < 2^{-k}$ for all $j, l \geq n_k$. (Take a subsequence to accelerate convergence)
Define $E_k = \left\{x : \left|f_{n_k}(x) - f_{n_{k+1}}(x)\right| > 2^{-k}\right\}$ and $F_m = \cap_{k\geq m} E_k^C$. Assume $m \leq j < k$.
If $x \in F_m$, then $\left|f_{n_k}(x) - f_{n_j}(x)\right| \leq \sum_{l=j}^{k-1} \left|f_{n_{l+1}}(x) - f_{n_l}(x)\right| \leq \sum_{l=j}^{k-1} 2^{-l} = 2^{-j+1}$.
f_{n_k} is Cauchy, hence $f_{n_k}(x) \to f(x)$, $\forall x \in F_m$. Same holds for $F = \cup_m F_m$.
Now we show that $\mu(F^C) = 0$, this can be seen from $\mu(F_m^C) = \mu(\cup_{k\geq m} E_k) \leq \sum_{k\geq m} 2^{-k} = 2^{-m+1}$.

$$\left|f(x) - f_{n_j}(x)\right| = \lim_{k\to\infty} \left|f_{n_k}(x) - f_{n_j}(x)\right| \leq 2^{-j+1}$$

Fpr $l \geq n_k$, note $|f - f_l| \leq |f - f_{n_k}| + |f_{n_k} - f_l|$.

$$\begin{aligned}
\mu\left(\left\{x : |f(x) - f_l(x)| > 2 \cdot 2^{-k}\right\}\right) &\leq \mu\left(\left\{x : |f(x) - f_{n_k}(x)| > 2^{-k}\right\}\right) \\
&\quad + \mu\left(\left\{x : |f_{n_k}(x) - f_l(x)| > 2^{-k}\right\}\right) \\
&\leq 2^{-k+1} + 2^{-k} \to 0
\end{aligned}$$

as $k \to \infty$. □

***Proposition:* 2.12:**

If $f_n \to f$ μ a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

Proof. By Theorem 2.3, $\forall\delta > 0$, there exists E s.t. $\mu(E) < \delta$ and $f_n \to f$ uniformly on E^C. *i.e.* $\forall\epsilon > 0$, $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N, x \in E^C$.

$$\mu\left(\{x : |f_n(x) - f(x)\} > \epsilon\right) \leq \mu(E) < \delta \Rightarrow \limsup_n \mu\left(\{x : |f_n(x) - f(x)| > \epsilon\}\right) < \delta$$

When the measure is infinite, $f_n(x) = \frac{|x|}{n}$ cannot converge in measure. □

Theorem: 2.5: Completeness of L^1

If f_n is Cauchy in L^1, then $\exists f \in L^1$ s.t. $f_n \to f$ in L^1.

Proof. Choose n_k s.t. $\left\|f_{n_k} - f_{n_{k+1}}\right\|_1 < 2^{-k}$. By restricting to a subsequence s.t. $f_{n_k} \to f$ μ a.e.
Take $g = |f_{n_1}| + \sum_{k=1}^{\infty} |f_{n_{k+1}} - f_{n_k}|$. By Theorem 2.1, $\displaystyle\int g = \int |f_{n_1}| + \sum_{k=1}^{\infty} \int |f_{n_{k+1}} - f_{n_k}| < \infty$.
Also, $|f_{n_j}| \leq |f_{n_1}| + \sum_{k=1}^{j-1} |f_{n_{k+1}} - f_{n_k}| \leq g$.
By Theorem 2.2, $\|f_{n_k} - f\|_1 \to 0$ as $k \to \infty$.

$$\lim_{k\to\infty} \sup_{m \geq n_k} \|f_m - f\|_1 \leq \; lim_{k\to\infty} \left(\sup_{m\geq n_k} \|f_m - f_{n_k}\|_1 + \|f_{n_k} - f\|_1\right) = 0$$

□

2.5 Lebesgue Integral

Theorem: 2.6: Properties of Lebesgue Integral

Let $f \in L^1(\mathbb{R}, m)$

1. $\forall\epsilon > 0$, there exists ϕ simple s.t. $\int |\phi - f| dm < \epsilon$
2. $\forall\epsilon > 0$, there exists a step function $h = \sum_{i=1}^{n} a_i \chi_{I_i}$ where I_i are intervals s.t. $\int |h - f| < \epsilon$.
3. $\forall\epsilon > 0$, $\exists g \in C_C(\mathbb{R})$ (continuous compact support) s.t. $\int |f - g| < \epsilon$.

Proof. 1 is proved for abstract measure.

2.Let ϕ be simple function s.t. $\int |\phi - f| < \frac{\epsilon}{2}$. $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ satisfies $m(E_i) < \infty$ for $a_i \neq 0$.

Let $h = \sum_{i,j} a_i \chi_{I_{i,j}}$. χ_{E_i} can be approximated by $\bigcup_{j=1}^{k} I_{i,j}$ disjoint.

$$\int |\chi_{E_i} - \sum \chi_{I_{i,j}}| dm = m\left(E_i \triangle \bigcup_{j=1}^{k} I_{i,j}\right) < 2^{-i-1}\epsilon$$

□

Remark 11. 2 links Lebesgue integrals with Riemann integrals. Lebesgue is completion of Riemann. 3 is true for $g \in C_C^\infty(\mathbb{R})$.

2.6 Product Measure

Definition: 2.11: Monotone Class

$\mathcal{E} \subset \mathcal{P}(X)$ is a monotone class if $E_n \in \mathcal{E}$, $E_n \nearrow E \Rightarrow E \in \mathcal{E}$, and $E_n \in \mathcal{E}, E_n \searrow E \Rightarrow E \in \mathcal{E}$.

Lemma 7. *An algebra that is also a monotone class is a σ-algebra.*

Theorem: 2.7: Monotone Class

Let $\mathcal{A}$ be an algebra. Then the monotone class generated by $\mathcal{A}$ is equal to $\mathcal{M}(\mathcal{A})$.

Proof. Let $\mathcal{E}$ denote the generated monotone class by $\mathcal{A}$.
Given $E \subset X$, define $\mathcal{F}(E) = \left\{A \in \mathcal{E} : A^C \in \mathcal{E} \text{ and } A \cap E \in \mathcal{E}\right\}$.
We want to show that $\mathcal{F}(E) = \mathcal{E}$ for all $E \in \mathcal{E}$.

Claim: $\mathcal{F}(E)$ is a monotone class.
Let $A_n \in \mathcal{F}(E)$, $A_n \nearrow A$. Then $A_n \cap E \in \mathcal{E}$, adn $A_n \cap E \nearrow A \cap E$, then $A \cap E \in \mathcal{E}$ as monotone class.
For decreasing sequence, take A_n^C.

Suppose $E \in \mathcal{A}$, $\mathcal{A} \subset \mathcal{F}(E)$, then $\mathcal{F}(E)$ is monotone $\Rightarrow \mathcal{E} \subset \mathcal{F}(E)$, and $\mathcal{F}(E) = \mathcal{E}$.

Suppose $E \in \mathcal{E}$, by the previous argument, $E \subset \mathcal{F}(A)$ for all $A \in \mathcal{A}$, then $A \in \mathcal{F}(E)$. Hence $\mathcal{A} \subset \mathcal{F}(E)$, $\mathcal{E} \subset \mathcal{F}(E)$. □

Definition: 2.12: Product σ-algebra

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. $M \times N = \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}$, $M \otimes N = \ll M \times N \gg$ is σ-algebra generated by $M \times N$.

Lemma: 2.4:

Let $\mathcal{M}, \mathcal{N}, \mathcal{Q}$ be σ-algebras, then $(\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{Q} = \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{Q})$.

Proof. Let $C \in \mathcal{Q}$, $\mathcal{E}(C) = \{E \in \mathcal{M} \otimes \mathcal{N} : E \times C \in \ll \mathcal{M} \times \mathcal{N} \times \mathcal{Q} \gg\}$.
$\mathcal{E}(C)$ is a σ-algebra and $\mathcal{M} \times \mathcal{N} \subset \mathcal{E}(C)$. Hence $\mathcal{E}(C) = \mathcal{M} \times \mathcal{N}$.
Then $(\mathcal{M} \otimes \mathcal{N}) \times \mathcal{Q} \subset \ll \mathcal{M} \times \mathcal{N} \times \mathcal{Q} \gg$, same for $(\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{Q}$.
Therefore, $(\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{Q} = \ll \mathcal{M} \otimes \mathcal{N} \otimes \mathcal{Q} \gg$.
Similarly, $\mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{Q}) = \ll \mathcal{M} \otimes \mathcal{N} \otimes \mathcal{Q} \gg$. □

Remark 12. $\otimes_{i=1}^n \mathcal{M}_i$ is well-defined.
$(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$.

Lemma: 2.5: Measure of Product Space

Let $(A_i \times B_i)$ be pairwise disjoint and $A \times B = \bigcup_{i=1}^{\infty} A_i \times B_i$. Then $(\mu \times \nu)(A \times B) \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i)$.

Proof. Let $x \in X$, $E \subset X \times Y$, $E_x = \{y \in Y : (x, y) \in E\}$ projection onto X.
Disjoint implies that $(A \times B)_x = \cup_i (A_i \times B_i)_x \subset Y$. If $x \in A$, then $\nu(B) = \nu((A \times B)_x) = \sum_{i=1}^{\infty} \nu((A_i \times B_i)_x) = \sum_{i=1}^{\infty} \nu(B_i)\chi_{A_i}(x)$.

If $x \notin A$, $\nu((A \times B)_x) = 0$. Therefore, $\chi_A(x)\nu(B) = \sum_{i=1}^{\infty} \nu(B_i)\chi_{A_i}(x)$.

Integrate w.r.t. μ and by Theorem 2.1, $(\mu \times \nu)(A \times B) \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i)$. ☐

Corollary 9. *Let $A_i \times B_i$ be pairwise disjoint, then $(\mu \times \nu)\left(\bigcup_{i=1}^{n} A_i \times B_i\right) = \sum_{i=1}^{n} \mu(A_i)\nu(B_i)$ is an additive measure on $\mathcal{A}(\mathcal{M} \times \mathcal{N})$, algebra generated by $\mathcal{M} \times \mathcal{N}$.*

Definition: 2.13: Product Measure

The product measure $\mu \times \nu$ is the restriction of $(\mu \times \nu)^*$ on $\mathcal{M} \otimes \mathcal{N}$.

Remark 13. This extension is unique if μ, ν are σ-finite.

Definition: 2.14: Projection

Let $E \subset X \times Y$, $x \in X$, $y \in Y$. Define $E_x = \{y : (x, y) \in E\}$ projection of E onto X, $E^y = \{x : (x, y) \in E\}$ projection of E onto Y. Let $f : X \times Y \to \mathbb{R}$, $f_x : Y \to \mathbb{R}$, $f^y : X \to \mathbb{R}$, then $f_x(y) = f(x, y) = f^y(x)$.

Lemma: 2.6:

1. If $E \in \mathcal{M} \otimes \mathcal{N}$, then $\forall x, y$, $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$
2. If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable

Proof. (1) Fix $x \in X$. Let $\mathcal{E} = \{E \in \mathcal{M} \otimes \mathcal{N} : E_x \in \mathcal{N}\}$, $\mathcal{M} \times \mathcal{N} \in \mathcal{E}$ and $\mathcal{E}$ is a σ-algebra. Hence $\mathcal{E} = \mathcal{M} \otimes \mathcal{N}$.

(2) Note that $f_x^{-1}((-\infty, a)) = (f^{-1}((-\infty, a)))_x$ and $f^{-1}((-\infty, a))$ is measurable. ☐

Theorem: 2.8: Tonelli Theorem for Sets

Let $E \in \mathcal{M} \otimes \mathcal{N}$, μ, ν are σ-finite. Then

1. $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable
2. $(\mu \times \nu)(E) = \displaystyle\int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y)$

Proof. Let $\mathcal{E} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{all statements hold}\}$, $M \times N \subset \mathcal{E}$.
We want to show that $\mathcal{E}$ is a σ-algebra.

Claim: $\mathcal{A}(\mathcal{M} \times \mathcal{N}) \subset \mathcal{E}$ and $\mathcal{E}$ is a monotone class.
Assume μ, ν are finite, $E_n \in \mathcal{E}$ s.t. $E_n \nearrow E$.
Let $f_n(y) = \nu((E_n)^y)$, then $f_n \nearrow f$ s.t. $f(y) = \nu(E^y)$.

$$\int \mu((E_n)^y) d\nu(y) = \int \lim_{n\to\infty} f_n(y) d\nu = \lim_{n\to\infty} \int \mu((E_n)^y) d\nu = \lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E)$$

For $E_n \searrow E$, consider E_n^C. Therefore $\mathcal{E} = \mathcal{M} \otimes \mathcal{N}$.

For σ-finite case, $X = \cup X_n, Y = \cup Y_n$, disjoint unions s.t. $\mu(X_n), \nu(Y_n) < \infty$.

$$\begin{aligned}
(\mu \times \nu)(E) &= \sum_{m,n} (\mu \times \nu)(E \cap (X_n \times Y_m)) \\
&= \sum_{m,n} \int \nu(E \cap (X_n \times Y_m)_x) d\mu(x) \\
&= \sum_{m,n} \int \chi_{X_n}(x) \nu(E_x \cap Y_m) d\mu(x) \\
&= \int \sum_{m,n} \chi_{X_n}(x) \nu(E_x \cap Y_m) d\mu(x) \text{ (By Theorem 2.1)} \\
&= \int \sum_{n} \chi_{X_n}(x) \nu(E_x) d\mu \\
&= \int \nu(E_x) d\mu
\end{aligned}$$

□

***Theorem:* 2.9: Tonelli-Fubini**

Let μ, ν be σ-finite measures

1. (Tonelli) For $f \in L^+(\mathcal{M} \otimes \mathcal{N})$, $g(x) = \int f(x,y) d\nu(y)$ and $h(y) = \int f(x,y) d\mu(x)$ are measurable, then $\int f d\mu \times \nu = \int g(x) d\mu = \int h(y) d\nu$
2. (Fubini) The same holds for $f \in L^1(\mu \times \nu)$

Proof. Take ϕ_n simple s.t. $\phi_n \nearrow f$. Apply Theorem 2.8. □

***Proposition:* 2.13: Layered Cake**

Let $f \in L^+$, then

$$\int f d\mu = \int_{[0,\infty)} \mu\left(\{x : f(x) > t\}\right) dm(t)$$

Proof.

$$\begin{aligned}
\int f(x) d\mu(x) &= \int \int_{[0,f(x))} (t) dm(t) d\mu(x) \\
&= \int \chi_{\{(x,t): 0 \le t < f(x)\}} d\mu \times m \\
&= \int \int \chi_{\{x: f(x) > t\}} d\mu dm(t) \\
&= \int \mu\left(\{x : f(x) > t\}\right) dm(t)
\end{aligned}$$

□

2.7 Infinite Product Measures

Let $(X_\alpha, \mathcal{M}_\alpha, \mu_\alpha)$, $\alpha \in A$ be a class of possibly infinity many measure spaces. Define cylinders $C^{E_1,\dots,E_n}_{\alpha_1,\dots,\alpha_n} = \left\{X \in \prod_{\alpha \in A} X_\alpha : X_{\alpha_i} \in E_i, E_i \in \mathcal{M}_{\alpha_i}\right\}$. Define the projection $\pi_\alpha = \prod \pi^{-1}_{\alpha_i}(E_i)$.

Definition: 2.15: Infinite Product Measures

The tensor product of σ-algebra on possibly infinity many measure spaces is $\otimes_{\alpha \in A}\mathcal{M}_\alpha = \mathcal{M}(C^{E_1,\dots,E_n}_{\alpha_1,\dots,\alpha_n})$.

The measure on the σ-algebra is $\left(\prod\limits_{\alpha \in A} \mu_\alpha\right)(C^{E_1,\dots,E_n}_{\alpha_1,\dots,\alpha_n}) = \prod\limits_{i=1}^{n} \mu_{\alpha_i}(E_i) \prod\limits_{\alpha \notin \{\alpha_1,\dots,\alpha_n\}} \mu_\alpha(X_\alpha)$. However, $\prod\limits_{\alpha \notin \{\alpha_1,\dots,\alpha_n\}} \mu_\alpha(X_\alpha)$ is not always defined unless $\mu_\alpha(X_\alpha) = 1, \forall \alpha$ (probability measure).

Proposition: 2.14:

$\prod\limits_{\alpha} \mu_\alpha$ is a premeasure on $\mathcal{A}(C^{E_1,\dots,E_n}_{\alpha_1,\dots,\alpha_n})$.

Proof. It suffices to show if $E_n \in \mathcal{A}(C^{E_1,\dots,E_n}_{\alpha_1,\dots,\alpha_n})$, $E_n \searrow \emptyset$, then $\prod\limits_{\alpha} \mu_\alpha(E_n) \to 0$ as $n \to \infty$.

WLOG, assume A is countable, rename s.t. $A = \mathbb{N}$, the product measure is $\prod\limits_{i=1}^{\infty} \mu_i$.

Assume towards a contradiction, $\exists \epsilon > 0$ s.t. $\prod_{i=1}^{\infty} \mu_i > \infty$.
Define the slicing $E^{y_1,\dots,y_n} = \left\{(x_i)_{i=n+1}^{\infty} : (y_1,\dots,y_n,x_{n+1},\dots) \in E\right\}$, $\mu^{(n)} = \prod_{i=n+1}^{\infty} \mu_i$. Apply Theorem 2.8 to $\prod_{i=1}^{n} \mu_i \times \mu^{(n)}$.
Define $D^1_n = \left\{x_1 \in X_1 : \mu^{(1)}(E_n^{x_1}) > \frac{\epsilon}{2}\right\}$ the slice on the first dimension at x_1 with measure at least $\frac{\epsilon}{2}$. D^1_n is non-empty because E_n is bounded below.

Apply Theorem 2.9,

$$
\begin{aligned}
\epsilon &< \int_{X_1} \mu^{(1)}(E_n^x)d\mu_1(x) = \int_{D^1_n} + \int_{X_1 \setminus D^1_n} \\
&\leq \mu(D^1_n) + \frac{\epsilon}{2}, \text{ since } \mu^{(1)}(E_n^x) \leq 1
\end{aligned}
$$

This gives $\mu(D^1_n) > \frac{\epsilon}{2}$.

Let $D_1 = \cap_n D^1_n$, then $\mu_1(D_1) > \frac{\epsilon}{2}$. Pick $y_1 \in D_1$. Then inductively, suppose $y_1,\dots,y_k$ are defined. Let $D^{k+1}_n = \left\{x_k \in X_k : \mu^{(k+1)}(E_n^{y_1,\dots,y_k,x_{j+1}}) > 2^{-k-1}\epsilon\right\}$. Then $D_{k+1} = \cap_n D^{k+1}_n$ has $\mu_{k+1}(D_{k+1}) > 2^{-k-1}\epsilon$. Pick $y_{k+1} \in D_{k+1}$.

Claim: $\forall n, (y_k) \in E_n$. $\exists K$ s.t. $E_n = E_{n,K} \times \prod_{i=K+1}^{\infty} X_i$. If $\mu^{(k+1)}(E_n^{y_1,\dots,y_k}) > 0$, then it needs to be 1. Therefore, $y_1,\dots,y_K$ + arbitrary sequence is in E_n ☐

2.8 Digression

Definition: 2.16: Push Forward

Let $(X, \mathcal{M}), (Y, \mathcal{N})$ be measurable space, μ a measure on X, $T : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable. The push forward of μ by T is $T_*\mu = \mu \circ T^{-1}$ where $\mu \circ T^{-1}$ where $\mu \circ T^{-1}(E) = \mu(T^{-1}(E))$ for $E \in \mathcal{N}$. $T_*\mu$ is a measure on $(Y, \mathcal{N})$.

Proposition: 2.15: Integration of Push Forward Measure

Let $f \in L^+(\mathcal{N})$, then $f \circ T \in L^+(\mu)$,

$$\int f d(\mu \circ T^{-1}) = \int f \circ T d\mu$$

$$\begin{tikzcd}
(X, \mathcal{M}, \mu) \arrow[r, "T"] \arrow[dr, "f \circ T"'] & (Y, \mathcal{N}, \mu \circ T^{-1}) \arrow[d, "f"] \\
 & \mathbb{R}
\end{tikzcd}$$

Proof.

$$(X, \mathcal{M}) \xrightarrow{T} (Y, \mathcal{N}) \xrightarrow{f} (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$$

$\int \chi_E \circ T d\mu = \int \chi_{T^{-1}(E)} d\mu$. If T is invertible and T^{-1} is measurable, then $\mu \circ T$ is a measure. $\square$

Definition: 2.17: Lebesgue Measure on $\mathbb{R}^n$

The Lebesgue measure on $\mathbb{R}^n$ is defined as $m_{\mathbb{R}^n} = m \times m \times \cdots m$ where m is the Lebesgue measure on $\mathbb{R}$. The Borel σ-algebra is $\mathcal{B}_{\mathbb{R}^n} = \mathcal{M}(\{\text{ open sets}\}) = \mathcal{M}(\{\text{open balls}\}) = \mathcal{M}(\{\text{rectangles}\}) = \mathcal{B}_{\mathbb{R}} \otimes \cdots \mathcal{B}_{\mathbb{R}}$. $m_{\mathbb{R}^n}$ is the unique extension of Jordan measure on rectangles.

Proposition: 2.16: Properties of Lebesgue Measure ($\mathbb{R}^n$)

Let m be n-dim Lebesgue measure.

1. Scaling and translation invariance: Let $\lambda \neq 0, a \in \mathbb{R}^n$, $T_\lambda(x) = \lambda x$, $\tau_a(x) = x + a$, then $m \circ T_\lambda = |\lambda| m$, $m \circ \tau_a = m$.
2. Inner and outer regularity (same as 1D Lebesgue measure)
3. For $m(E) < \infty$, $\exists$ rectangles $R_i, i = 1, ..., k$ s.t. $m\left(E \triangle \bigcup_{i=1}^k R_i\right) < \epsilon$
4. Let $f \in L^1$, $\forall \epsilon > 0$, there exists simple, step (sum of change of rectangles), C_c (compactly supported continuous functions) functions approximating f in L^1 up to error ϵ.

Proposition: 2.17: Change of Variable (Linear maps)

Let $A \in \mathrm{GL}(n, \mathbb{R})$ be $n \times n$ non-singular matrix, $T_A(x) = Ax$. Then $m \circ T_A = |\det A| m$. If $f \in L^+$ or L^1, then

$$\int f dm = |\det A| \int f \circ T_A dm$$

Proof. $m \circ T_A = |\det A| m$ because it holds on rectangles (by Riemann integration)

$$\int f \circ T_A dm = \int f d(m \circ T_A^{-1}) = \frac{1}{|\det A|} \int f dm$$

$\square$

Proposition: 2.18: Change of Variable

Let $\Omega \subset \mathbb{R}^n$ be open, $G : \Omega \to \mathbb{R}^n$ is a C^1-diffeomorphism (homeomorphism, C^1 and $\det DG \neq 0$, where DG is the Jacobian of G). If $f \in L^+(G(\Omega))$ or $L^1(G(\Omega))$, then

$$\int_{G(\Omega)} f dm = \int_\Omega f \circ G |\det DG| dm$$

Proof. It suffices to show that for $E \in \mathcal{B}_{\mathbb{R}^n}$, $m(G(E)) = \int_E |\det DG| dm$.

Define $Q_r(x) = \{y \in \mathbb{R}^n : \|y - x\|_\infty \leq r\}$ a rectangle.

Lemma 8. *Let Ω_1 be compactly contained in Ω, then $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $0 < r < \delta$, $x \in \Omega_1$, $G(Q_r(x)) \subset DG(x)(Q_{(1+\epsilon)r}(x)) + G(x)$. Then $m(G(Q_r(x))) \leq (1+\epsilon)^n |\det DG(x)| m(Q_r(x))$.*

Proof. $\left\|(DG(x))^{-1}(G(x,z) - G(x) - DG(x)z)\right\| = o(\|z\|)$ as $\|z\| \to 0$ uniformly with $x \in \Omega_1$. Also note that $\|z\|_\infty \leq \|z\| \leq \sqrt{n}\, \|z\|_\infty$. $\square$

Lemma 9. *If Q is a cube, then $m(G(Q)) \leq \int_Q |\det DG| dm$*

Proof. Slice Q into small pieces, $Q = \bigcup_{i=1}^K Q_i$ disjoint union,

$$\begin{aligned}
m(G(Q)) &= \sum_{i=1}^K m(G(Q_i)) \\
&\leq \sum_{i=1}^K (1+\epsilon)^n |\det DG(x_i)| m(Q_i) \\
&= \sum_{i=1}^K (1+\epsilon)^n \int |\det DG(x_i)| \chi_Q \\
&\overset{\text{uniformly}}{\to} (1+\epsilon)^n \int |\det DG(x)| \chi_Q(x).
\end{aligned}$$

$\square$

Lemma 10. *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is L-Lipschitz, i.e. $\|F(x) - F(y)\| \leq L\|x - y\|$, $\forall x, y$, then $\forall E \subset \mathbb{R}^n$, $m^*(F(E)) \leq L^n m^*(E)$.*

On any Ω_1 compact subset of Ω. G is L-Lipschitz, where $L = \sup\limits_{x \in \Omega_1} \|DG(x)\|$, then $m(G(E)) \leq L^n m(E)$, $E \in \mathcal{L}_{\mathbb{R}^n}$.

For the theorem, we only need to prove for $E \in \mathcal{L}_{\mathbb{R}^n}$, $E \subset \Omega_1$ compact subset of Ω, $m(G(E)) \leq \displaystyle\int_E |\det DG| dm$. $\forall \epsilon > 0$, there exists cube $Q_1, ..., Q_k$ disjoint s.t. $m\left(E \triangle \displaystyle\bigcup_{i=1}^k Q_i\right) < \epsilon$.

$$\begin{aligned}
m(G(E)) &\leq m(G(\cup_{i=1}^k Q_i)) + m(G(E \setminus \cup_{i=1}^k Q_i)) \\
&\leq \int_{\cup_{i=1}^k Q_i} |\det DG| dm + L^n \epsilon \\
&\leq \int_E |\det DG| dm + \sup |\det DG| m(\cup_i Q_i \setminus E) + L^n \epsilon \\
&\leq \int_E |\det DG| dm + \epsilon \cdot \text{const.}
\end{aligned}$$

$\square$

3 Signed Measure and Differentiation

3.1 Signed Measure

We consider the measurable space $(X, \mathcal{M})$.

Example: If $f \in L^+$, then $E \mapsto \int_E f d\mu$ is a measure. If $f \in L^1$, then $E \mapsto \int_E f d\mu$ is a countably-additive set function.

***Definition:* 3.1: Signed Measure**

A signed measure is a set function $\nu : \mathcal{M} \to \bar{\mathbb{R}}$ s.t.

1. ν only takes one of $\pm\infty$ as values
2. $\nu(\emptyset) = 0$
3. ν is countably additive, $\nu(\cup_i E_i) = \sum_i \nu(E_i)$ for E_i disjoint and limit exists.

***Definition:* 3.2: Positive/Negative Set**

E is a positive set for ν if $\nu(F) \geq 0$ for all $F \subset E$, $F \in \mathcal{M}$. Similarly, we define a negative set. E is a null set if it is both a positive set and a negative set.

Remark 14. If E is a positive set, then $\nu|_E$ is a measure.

***Lemma:* 3.1: Properties of Signed Measure**

1. If $E \subset F$, $|\nu(F)| < \infty$, then $|\nu(E)| < \infty$.
2. If $A_n \nearrow A$, then $\lim\limits_{n\to\infty} \nu(A_n) = \nu(A)$
3. If $A_n \searrow A$, and $|\nu(A_i)| < \infty$, then $\lim\limits_{n\to\infty} \nu(A_n) = \nu(A)$

Proof. 1. Suppose $|\nu(F)| < \infty$, but $\nu(E) = \infty$, then $\nu(F \setminus E) = -\infty$, violates the definition.

2,3 are the same as positive measure. □

***Lemma:* 3.2:**

Suppose ν does not take $-\infty$ as a value. Then if $F_0 \in \mathcal{M}$, $\nu(F_0) \leq 0$, $\exists F \subset F_0$ s.t. F is negative and $\nu(F) \leq \nu(F_0)$.

Proof. Assume that $\nu(F) \leq \nu(F_0)$, F is not negative. Since $F_0 \subset F_0$, F_0 is not negative.

Let $k_0 = \inf\left\{k \in \mathbb{N} : \exists G_0 \subset F_0 \text{s.t. } \nu(G_0) \geq \dfrac{1}{k}\right\} < \infty$. $\exists G_0$ s.t. $G_0 \subset F_0$ and $\nu(G_0) \geq \frac{1}{k_0}$. $F_1 = F_0 \setminus G_0$, then $\nu(F_1) = \nu(F_0) - \nu(G_0) \leq \nu(F_0)$.

Inductively, we construct F_n, k_{n-1}, F_n is not negative. Define $k_n = \inf\left\{k \in \mathbb{N} : \exists G_n \subset F_n \text{s.t. } \nu(G_n) \geq \dfrac{1}{k}\right\}$. Then $F_{n+1} = F_n \setminus G_n$.

Let $F = \cap_n F_n$,

$$\nu(F) = \lim_{n\to\infty} \nu(F_n) = \nu(F_0) - \sum_{n=0}^{\infty} \nu(G_n) \leq \nu(F_0) - \sum_{n=0}^{\infty} \frac{1}{k_n}$$

Then $\sum_{n=0}^{\infty} \frac{1}{k_n} \leq \nu(F_0) - \nu(F) < \infty$, $k_n \to \infty$, then F is negative. Contradiction. $\square$

Theorem: 3.1: Hahn Decomposition

Let ν be a signed measure. Then $\exists E^+ \in \mathcal{M}, E^- = (E^+)^C$. $\forall F \in \mathcal{M}$, $\nu(F \cap E^+) \geq 0$, $\nu(F \cap E^-) \leq 0$. Call Hatin decomposition. If $F^\pm$ is another decomposition, then $F^+ \triangle E^+$ and $F^- \triangle E^-$ are null.

Proof. WLOG, $\nu(E) > -\infty, \forall E \in \mathcal{M}$.
Let $\beta = \inf\{\nu(E) : E \text{ is a negative set}\}$, $\beta \leq 0$ always exist. Then there exists a negative sequence s.t. $\nu(E_n) \to \beta$.
Since union of negative sets are negative, let $F_n = \cup_i E_i$, then $E^- = \cup_n F_n$, $\nu(E^- \cap E) = \lim_{n\to\infty} \nu(F_n \cap E) \leq 0$.
$\beta = \nu(E^-) > -\infty$, E^- is negative.

Let $E^+ = (E^-)^C$. Assume E^+ is not positive. Then $\exists F_0 \subset E^+$ s.t. $\nu(F_0) < 0$. By Lemma 3.2, $\exists F \subset F_0$ s.t. $\nu|_F \leq 0$, $\nu(F) \leq \nu(F_0) < 0$.
Now $E^- \cup F$ is negative and $\nu(E^- \cup F) < \beta$ Contradiction.
Uniqueness: $E^+ \setminus F^+$ and $F^+ \setminus E^+$ are both positive and negative, therefore null sets. $\square$

Theorem: 3.2: Jordan Decomposition

If ν is a signed measure, then $\exists \nu^+, \nu^-$ both positive measures s.t. $\nu = \nu^+ - \nu^-$. The decomposition is unique *i.e.* if $\nu = \mu^+ - \mu^-$, then $\mu^\pm = \nu^\pm$ as set functions.

Definition: 3.3: Total Variation

Let ν be a signed measure. Then $|\nu| = \nu^+ + \nu^-$ is a positive measure, called total variation. ν is finite (σ-finite) if $|\nu|$ is.

Remark 15. $M(X) = \{\nu \text{ signed measures} : |\nu|(X) < \infty\}$ is a normed vector space with $\|\nu\| = |\nu|(X)$.

Definition: 3.4: Singular Measures

Let ν_1, ν_2 be signed measures, $\nu_1 \perp \nu_2$ (ν_1 is singular of ν_2) if there exists $E \in \mathcal{M}$ s.t. $\nu_1|_{E^C} = 0$ and $\nu_2|_E = 0$.

Definition: 3.5: Absolute Continuous in Measures

Let ν be a signed measure, and μ be a measure. $\nu \ll \mu$ (absolute continuous) if $\mu(E) = 0 \Rightarrow \nu(E) = 0$.
If ν_1, ν_2 are signed measures, $\nu_1 \ll \nu_2$ if $\nu_1 \ll |\nu_2|$.

Let $f \in L^1$, $\nu_f(E) = \int_E f d\mu$, then $\nu_f \ll \mu$, because integral on measure zero sets are zero.

Suppose $f = f^+ - f^-$, and if at least one of $\int f^\pm$ is finite, then we can define $\int f = \int f^+ - \int f^-$.

Lemma: 3.3:

Let ν, μ be finite measures, $\nu \ll \mu$. Suppose $\nu \neq 0$, then $\exists \epsilon > 0$ and $A \in \mathcal{M}$ s.t. $\nu(A) > 0$ and $(\nu - \epsilon\mu)|_A \geq 0$.

Proof. For each n, let $\nu_n = \nu - \frac{1}{n}\mu$. Let E_n^+ be the positive part of Hahn decomposition of ν_n. We want to show that $\nu(\cup_n E_n^+) > 0$. We consider $\nu((\cup_n E_n^+)^C) = \nu(\cap_n E_n^-) = 0$ Let $E \subset\subset \cap_n E_n^-$, then $\nu(E) - \frac{1}{n}\mu(E) \le 0$.
This gives $\nu(E) \le 0$, $\nu(E) = 0$. Then $\nu(\cup_n E_n^+) > 0$, $\nu(E_n) > 0$ for some n, *i.e.* $\nu - \frac{1}{n}\mu|_{E_n} \ge 0$. □

Theorem: 3.3: Radon-Nikodym

Let ν be a signed measure, μ a measure, both σ-finite, $\nu \ll \mu$. Then $\exists f$ measurable s.t. $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$. (The integral has to be defined.)

Proof. Assume ν is positive (apply decompositions) and both ν, μ are finite (for σ-finite, take countable unions).
Let $K = \left\{g \in L^+ : \int_E g d\mu < \nu(E), \forall E \in \mathcal{M}\right\}$. We want to find a maximal element of K. $K \neq \emptyset$ because $0 \in K$, and $\sup_{g \in K} \int g d\mu \le \nu(X) < \infty$.

Let $g_1, g_2 \in K$, then $g = \max(g_1, g_2) \in K$. Let $E = \{x : g_1(x) > g_2(x)\}$,

$$\int_A g d\mu = \int_{A \setminus E} g d\mu + \int_{A \cap E} g d\mu \le \nu(A \setminus E) + \nu(A \cap E) = \nu(A)$$

Let $g_n \in K, g_n \nearrow g$, then $g \in K$ by Theorem 2.1.

Take $g_n \in K$ s.t. $\displaystyle\int g_n \to \sup_{g \in K} \int g$ and $g_n \nearrow f$. We want to show that $\nu_f(E) = \int_E f d\mu = \nu(E)$.
Since $\nu_f \le \nu$, $\nu - \nu_f \ge 0$. Suppose by contradiction, $\nu - \nu_f \neq 0$.
Apply Lemma 3.3, $\exists \epsilon > 0, A \in \mathcal{M}$ s.t. $(\nu - \nu_f)(A) > 0$ and $(\nu - \nu_f - \epsilon\mu)|_A \ge 0$. Therefore, $(\nu - \nu_f)|_A \ge \epsilon\mu|_A > 0$, because $\nu - \nu_f \ll \mu$ and $(\nu - \nu_f)(A) > 0$.

Then $\nu|_A \ge (\nu_f + \epsilon\mu)|_A$. Let $g = f + \epsilon\chi_A$, $g \in K$.

$$\begin{aligned}\int_E g d\mu &= \int_E (f + \epsilon\chi_A) d\mu \\ &= \nu(E \setminus A) + (\nu_f + \epsilon\mu)(E \cap A) \\ &\le \nu(E \setminus A) + \nu(E \cap A) = \nu(E).\end{aligned}$$

But $\displaystyle\int_E f d\mu = \int f + \epsilon\mu(A) > \sup_{h \in K} \int h$. Contradiction. □

Notation: We denote f satisfying Theorem 3.3 as $f = \frac{d\nu}{d\mu}$ or $f d\mu = d\nu$.

Corollary 10. *Let $\nu \ll \mu$ be both finite. Then $\forall\epsilon > 0, \exists\delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu|(E) = \int_E \left|\frac{d\nu}{d\mu}\right| d\mu < \epsilon$.*

Proof. Assume by contradiction that $\exists E_n$ s.t. $\mu(E_n) \to 0$, but $\nu(E_n) \ge \delta > 0$.
$\exists n_k \to \infty$ s.t. $\sum \mu(E_{n_k}) < \infty$. The set $E = \limsup_k E_{n_k}$ gives a contradiction, because $\mu(E) = 0$ and $\nu(E) > 0$. □

Remark 16. Recall that if $f \in L^1$, then $\forall\epsilon > 0$, $\exists\delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E f d\mu < \epsilon$. The absolute continuous definitions are equivalent.

Theorem: 3.4: Lebesgue Decomposition Theorem

Let ν be a signed measure, μ a measure. Both σ-finite. Then we can decompose $\nu = \nu_1 + \nu_2$ s.t. $\nu)1 \perp \mu, \nu_2 \ll \mu$ and by Theorem 3.3, $\nu = \nu_1 + f d\mu$.

Proof. Assume μ, ν finite and positive. Let $\rho = \mu + \nu$, then $\mu \ll \rho, \nu \ll \rho$.
Let $f_1 = \frac{d\mu}{d\rho}, f_2 = \frac{d\nu}{d\rho}$, then

$$\int_E (f_1 + f_2) d\rho = \int_E d\mu + \int_E d\nu = \mu(E) + \nu(E) = \rho(E) = \int_E d\rho$$

Then $f_1 + f_2 = 1$ ρ-a.e.
Let $F = \{x : f_2(x) = 1\}$, $\nu_1 = \nu|_F$, $\nu_2 = \nu|_{F^C}$.

$$\mu(F) = \int_F f_1 d\rho = \int_F (1 - f_2) d\rho = 0 \Rightarrow \nu_1 \perp \mu$$

Suppose $\mu(A \cap F^C) = 0$,

$$\begin{aligned} \int_{A\cap F^C} 1 d\nu = \nu_2(A) = \nu(A \cap F^C) &= \int_{A\cap F^C} f_2 d\rho \\ &= \int_{A\cap F^C} f_2 d\mu + \int_{A\cap F^C} f_2 d\nu \\ &= \int_{A\cap F^C} f_2 d\nu. \end{aligned}$$

So $0 = \int_{A\cap F^C} (1 - f_2) d\nu$, but $f_2 = 0$ a.e. on F^C. This gives $\nu(A \cap F^C) = 0$, $\nu \ll \mu$. ☐

Theorem: 3.5: Measurable Change of Variable

Let ν, μ be positive σ-finite $\nu \ll \mu$, $f \in L^1(\nu)$, then $\displaystyle\int f d\nu = \int f \frac{d\nu}{d\mu} d\mu$.

Proof. Start with simple function, then L^+ function, and finally L^1 functions. ☐

3.2 Complex Measures

Definition: 3.6: Complex Measures

$\nu : \mathcal{M} \to \mathbb{C}$ is a complex measure if σ-additive and finite. $\nu = \nu_R + i\nu_I$, where ν_R, ν_I are signed measures. $\frac{d\nu}{d\mu} = \frac{d\nu_R}{d\mu} + i\frac{d\nu_I}{d\mu}$. $\nu \ll \mu$ if and only if $\nu_R \ll \mu$ and $\nu_I \ll \mu$.

Proposition: 3.1:

Let ν be a complex measure and μ_1, μ_2 be measures s.t. $\nu \ll \mu_1, \mu_2$. Then

$$\int_E \frac{d\nu}{d\mu_1} d\mu_1 = \int_E \frac{d\nu}{d\mu_2} d\mu_2$$
$$\int_E \left|\frac{d\nu}{d\mu_1}\right| d\mu_1 = \int_E \left|\frac{d\nu}{d\mu_2}\right| d\mu_2.$$

Proof. Let $\rho = \mu_1 + \mu_2$, $f_1 = \frac{d\nu}{d\mu_1}$, $f_2 = \frac{d\nu}{d\mu_2}$.

$$\int f_1 \frac{d\mu_1}{d\rho} d\rho = \int_E f_1 d\mu_1 = \nu(E) = \int_E f_2 d\mu_2 = \int_E f_2 \frac{d\mu_2}{d\rho} d\rho$$

Hence $f_1 \frac{d\mu_1}{d\rho} = f_2 \frac{d\mu_2}{d\rho}$, ρ-a.e, and $|f_1| \frac{d\mu_1}{d\rho} = |f_2| \frac{d\mu_2}{d\rho}$, ρ-a.e.
Integrate on E again, and we get the results. □

Definition: 3.7: Complex Measure and Norm

We can then define $|\nu|(E) = \int_E \left|\frac{d\nu}{d\mu}\right| d\mu$ for any $\nu \ll \mu$. A canonical choice is $|\nu| = |\nu_R| + |\nu_I|$.
Define $\|\nu\| = |\nu|(X)$. This is a norm on $M(X, \mathcal{M}) = \{\text{all complex/signed measures on } (X, \mathcal{M})\}$.

Proposition: 3.2: Properties of Complex Measures

Let ν be a complex measure.

1. $|\nu(E)| \le |\nu|(E)$
2. $\nu \ll |\nu|$, $\left|\frac{d\nu}{d|\nu|}\right| = 1$, $|\nu|$-a.e.
3. If $L^1(\nu) = L^1(|\nu_R|) \cap L^1(|\nu_I|)$, then $L^1(\nu) = L^1(|\nu|)$, and $\left|\int f d\nu\right| = \left|\int f d\nu_R + i \int f d\nu_I\right| \le \int |f| \, d|\nu|$.

3.3 Lebesgue Differentiation

Question: Let $E \in \mathcal{B}_{\mathbb{R}^n}$. How dense is E?

- There exists positive measures s.t. we have no where dense sets.
- $\forall \epsilon > 0$, $\exists I$ s.t. $\frac{m(E \cap I)}{m(I)} > 1 - \epsilon$.

Theorem: 3.6: Lebesgue Density

Let $E \in \mathcal{B}_{\mathbb{R}^n}$. Then $\exists F \in \mathcal{B}_{\mathbb{R}^n}$ s.t. $m(F) = 0$ and for all $x \in E \setminus F$ (*i.e.* for a.e. $x \in E$), $\lim_{r \to 0^+} \frac{m(E \cap B_r(x))}{m(B_r(x))} = 1$

Proof. By applying to □

Definition: 3.8: Locally Integrable Functions

$f \in L^1_{loc}(\mathbb{R}^n, m)$ if $f\chi_{Br(0)} \in L^1(B_r(0), m)$, *i.e.* f restricted to a bounded region is integrable.

Definition: 3.9: Average Function

Let $f \in L^1_{loc}$, for $r > 0$, the average function is

$$A_r f(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y) dm(y)$$

Lemma: 3.4:

Let $f \in L^1(\mathbb{R}^n)$ $A_r f(x)$ is jointly continuous in (r, x) on the set $r \neq 0$. If $g \in C(\mathbb{R}^n)$, then $\lim_{r \to 0^+} A_r g(x) = g(x)$.

Proof. Let $(x_n, r_n) \to (x, r)$, $r \to 0$.

$$|A_{r_n} f(x_n) - A_r f(x)| \leq \frac{1}{m(B_r(x))} \left| \frac{m(B_r(x))}{m(B_{r_n}(x_n)) \int_{B_{r_n}(x_n)} f - \int_{B_r(x)} f} \right|$$

$$\leq \frac{1}{m(B_r(x))} \left(\mathcal{O}\left(\frac{|m(B_r(x)) - m(B_{r_n}(x_n))|}{m(B_{r_n}(x_n))} \right) \|f\|_1 + \int_{B_{r_n}(x_n) \triangle B_r(x)} |f| \right) \to 0.$$

The second integral is by absolute continuity of $B_{r_n}(x_n) \triangle B_r(x)$. $\square$

Lemma: 3.5: Vitalli's 3r

Let C be a finite collection of balls in $\mathbb{R}^n$. Then there exists a subcollection C_1 pairwise-disjoint, and

$$\bigcup_{B_r(x) \in C} B_r(x) \subset \bigcup_{B_r(x) \in C_1} B_{3r}(x)$$

Proof. Pick $B_{r_1}(x_1)$ s.t. r_1 is the largest. Inductively, pick $B_{r_k}(x_k)$ to be the largest ball disjoint from $B_{r_1}(X_1), ..., B_{r_{k-1}}(x_{k-1})$.
Let $B_r(x) \in C$, $k = \max\{j : r_j \geq r\}$. Then $B_r(x)$ intersects some $B_{r_j}(x_j), 1 \leq j \leq k$. Otherwise:

1. If $B_{r_k}(x_k)$ is the last item, then we should add another ball to C_1.
2. Or $r_{k+1} < r$, we should set r_{k+1} to r

Both are contradiction. Therefore, $B_r(x) \subset B_{3r_k}(x_k)$ $\square$

Definition: 3.10: Hardy-Littlewood (H-L) Maximum Function

Let $f \in L^1_{loc}$, the Hardy-Littlewood (H-L) maximum function is

$$Mf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f|$$

$Mf(x)$ is Borel-measurable.

Theorem: 3.7: Hardy-Littlewood Maximum Inequality

$\exists c > 0$ s.t. $\forall \lambda > 0$, $f \in L^1$, $m(\{x : |Mf(x)| > \lambda\}) < c\lambda^{-1} \|f\|_1$.

Remark 17. Markov inequality is $m(\{x : |f(x)| > \lambda\}) < \lambda^{-1} \|f\|_1$. H-L is saying that Mf has the same decay upto a constant. Define $[f]_1 = \sup_{\lambda > 0} \lambda m(\{x : |f(x)| > \lambda\})$, then $[f]_1 \leq \|f\|_1$. Also $[Mf]_1 \leq c\|f\|_1$.

Proof. $\forall x \in K$, $\exists r(x) > 0$ s.t. $\displaystyle\int_{B_{r(x)}(x)} |f| \geq \lambda m(B_{r(x)}(x))$.
Let $\mathcal{C}$ be a finite subcover of $\left\{B_{r(x)}(x)\right\}_{x \in K}$. Choose it to be the 3r cover. Then

$$\begin{aligned} m(K) &\leq m\left(\bigcup_{B_r(x)\in\mathcal{C}} B_{3r}(x)\right) \\ &\leq 3^n m\left(\bigcup_{B_r(x)\in\mathcal{C}} B_r(x)\right) = \sum_{B_r(x)\in\mathcal{C}} 3^n m(B_r(x)) \\ &\leq \sum_{\mathcal{C}} 3^n \lambda^{-1} \int_{B_r(x)} |f| = 3^n\lambda^{-1}\int_{\cup B_r(x)} |f| \\ &\leq 3^n \lambda^{-1} \|f\|_1 \end{aligned}$$

□

Theorem: 3.8: Lebesgue Differentiation I

Let $f \in L^1_{loc}$, then for m-a.e. x,

$$\lim_{r\to 0^+} A_r f(x) = f(x)$$

Proof. It suffices to show the theorem for $f \in L^1$.
Let $E_\lambda = \left\{x : \left|\lim_{r\to 0^+} A_r f(x) - f(x)\right| > \lambda\right\}$. $\forall \epsilon > 0$, $\exists g \in C(\mathbb{R}^n)$ and $\|f-g\|_1 < \epsilon$.

$$E_\lambda \subset \left\{x : \limsup_{r\to 0^+} |A_r f - A_r g| > \frac{\lambda}{3}\right\} \cup \left\{x : \limsup_{r\to 0^+} |A_r g - g| > \frac{\lambda}{3}\right\} \cup \left\{x : |f-g| > \frac{\lambda}{3}\right\}$$

Call these three sets E_1, E_2, E_3.
$E_1 \subset \left\{x : \sup_{r>0} A_r|f-g| > \frac{\lambda}{3}\right\}$, so $m(E_1) \leq 3\lambda^{-1} C \|f-g\|_1 \leq 3C\lambda^{-1}\epsilon$ by Theorem 3.7. $E_2 = \emptyset$ by Lemma 3.4.
$m(E_3) \leq 3\lambda^{-1}\|f-g\|^{-1} \leq 3\lambda^{-1}\epsilon$ by Lemma 2.2. Since ϵ is arbitrary, $m(E_\lambda) = 0$. □

Theorem: 3.9: Lebesgue Differentiation II

$f \in L^1_{loc}$, then for m-a.e. x,

$$\lim_{r\to 0^+} \frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y) - f(x)|\, dm(y) = 0$$

Proof. Apply Theorem 3.8 to $|f(x) - a|$, where $a \in \mathbb{R}$. Then $\exists F_a \in \mathcal{B}_{\mathbb{R}^n}$ s.t. if $x \notin F_a$, then $\lim_{r\to 0^+} A_r|f - a|(x) = |f(x) - a|$.
Let $E = \cup_a F_a$, $\forall x \in E^C$, $\epsilon > 0$, choose $a \in \mathbb{Q}$, s.t. $|f(x) - a| < \epsilon$.

$$\limsup_{r\to 0^+} \frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y) - f(x)| \leq \lim_{r\to 0^+}\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(x) - a| + |f(x) - a| \leq 2|f(x) - a| \leq 2\epsilon$$

□

Corollary 11. *Let $f \in L^1_{loc}(\mathbb{R})$, then $\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_x^{x+\epsilon} f(y) dm(y) = f(x)$ for m-a.e. x (Newton-Leibniz formula)*

Proof. For $\epsilon > 0$, we have

$$\frac{1}{\epsilon} \int_x^{x+\epsilon} |f(y) - f(x)| dm(y) \leq 2 \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} |f(y) - f(x)| dm(y) \to 0$$

By Theorem 3.9 □

Corollary 12. *Let $\nu \ll m$, ν finite on bounded setm then $\lim_{r \to 0^+} \frac{\nu(B_r(x))}{m(B_r(x))} = \frac{d\nu}{dm}$, m-a.e. x.*

Recall the change of variable formula

$$\begin{aligned}
\int_{G(\Omega)} f dm &= \int_{G(\Omega)} f \circ G \circ G^{-1} dm = \int_\Omega f \circ G dm \circ G \\
&= \int_\Omega f \circ G \frac{dm \circ G}{dm} dm = \int_\Omega f \circ G |\det DG| dm,
\end{aligned}$$

where $m \circ G \ll m$.

Lemma 11. $\lim_{r \to 0^+} \frac{m \circ G(B_r(x))}{m(B_r(x))} = \frac{dm \circ G}{dm} = |\det DG|$ *a.e. x.*

***Definition:* 3.11: Borel Regular**

A Borel measure μ on $\mathbb{R}^n$ is Borel regular if
1. $\mu(K) < \infty$ for all K compact sets.
2. Outer regular: $\mu(E) = \inf \{\mu(G) : E \subset G, G \text{ open}\}$

***Lemma:* 3.6: Inner Regularity**

A Borel regular measure is always inner regular, *i.e.*

$$\mu(E) = \sup \{\mu(K) : K \subset E, K \text{ compact}\}$$

***Theorem:* 3.10:**

Suppose μ is Borel regular and $\mu \perp m$. Then $\lim_{r \to 0^+} \frac{\mu(B_r(x))}{m(B_r(x))} = 0$ Lebesgue m-a.e. x.

Proof. Since $\mu \perp m$, $\exists E \in \mathcal{B}_{\mathbb{R}^n}$ s.t. $m(E^C) = 0$, $\mu(E) = 0$.
Let $E_\alpha = \left\{ x \in E : \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{m(B_r(x))} > \alpha \right\}$. Consider $K \subset E_\alpha$ compact.
$\forall \epsilon > 0, \exists U \supset E_\alpha$ s.t. $\mu(U) < \epsilon$.
$\forall x \in K, \exists r(x) > 0$ s.t. $\mu(B_{r(x)}(x)) > \alpha m(B_{r(x)}(x))$ and $B_{r(x)}(x) \subset U$.

Let $\mathcal{C}$ be a subcover and $\mathcal{C}_1$ be a $3r$ subcover of $\left\{B_{r(x)}(x)\right\}_{x\in K}$.

$$\begin{aligned}
m(K) &\leq m\left(\bigcup_{B_r(x)\in\mathcal{C}_1} B_{3r}(x)\right)\\
&\leq \sum_{\mathcal{C}_1} 3^n m(B_r(x)) \leq \alpha^{-1}3^n\sum_{\mathcal{C}_1}\mu(B_r(x))\\
&\leq \alpha^{-1}3^n\mu(\cup_{\mathcal{C}_1}B_r(x)) \leq \alpha^{-1}3^n\mu(U)\\
&< \alpha^{-1}3^n\epsilon.
\end{aligned}$$

Therefore, $m(K)=0$ □

***Definition:* 3.12:**

Say $E_r(x)$ shrinks nicely to x if $\exists\alpha>0$, $E_r(x)\subset B_r(x)$, $m(E_r(x))\geq \alpha m(B_r(x))$.

All Theorem of ratio of limits we stated hold with $B_r(x)$ replaced by $E_r(x)$.

Now, we restrict to $\mathbb{R}$.
If $F:\mathbb{R}\to\mathbb{R}$ is increasing and right continuous, we can define μ_F a Lebesgue-Stieltjes (L-S) measure.
If μ is Borel finite on bounded set, then $F_\mu = \begin{cases}\mu((0,x]), x>0\\ 0, x=0\\ -\mu((x,0]), x<0\end{cases}$ is right continuous and increasing.

Corollary 13. *All Borel measures finite on bounded sets are regular*

***Theorem:* 3.11:**

Let F be right continuous and increasing, then $F'(x)$ exists Lebesgue-a.e. x

Proof. Let μ_F be the L-S measure.

$$\lim_{h\to 0^+}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0^+}\frac{\mu_F((x,x+h])}{m((x,x+h])}$$

Let $\mu_F = fdm+\lambda$, where fdm is absolute continuous, and $\lambda\perp m$. Then $\displaystyle\lim_{h\to 0^+}\frac{F(x+h)-F(x)}{h} = f$ m-a.e. x.
Same proof applies to $h\to 0^-$ □

***Theorem:* 3.12: Rademacher Theorem in $\mathbb{R}$**

If F is Lipschitz, then F' exists Lebesgue-a.e.

Proof. If F is L-Lipschitz, then $F(x)+Lx$ is increasing. Then apply Theorem 3.11. □

***Theorem:* 3.13:**

If F is increasing, then
1. $F(x+)=F(x-)$ for at most countably many x
2. If $G(x)=F(x+)$, $G'(x)=F'(x)$ Lebesgue-a.e. *i.e.* increasing functions are a.e. continuous

Proof. (1) Fix $N > 0$.

$$\sum_{x\in(-N,N)} F(x+) - F(x-) = \sup_{E\subset(-N,N),\text{finite}} \sum_E F(x+) - F(x-) \le F(N) - F(-N) < \infty$$

This implies $\{x : x \in (-N,N), F(x+) - F(x-) > 0\}$ is at most countable. Then take countable unions.

(2) Let $D = \{x : F(x+) > F(x-)\}$, $H(x) = F(x+) - F(x)$. We want to show $H'(x) = 0$ Lebesgue-a.e. x. $\exists a(x), x \in D$ s.t. $H(t) = \sum_{x\in D} a(x)\chi_{\{x\}}(t)$.

Let $\lambda = \sum_{x\in D} a(x)\delta_x$, $\lambda \perp m$.

$$\lim_{h\to 0^+} \frac{H(x+h) - H(x)}{h} = \lim_{h\to 0^+} \frac{\lambda((x,x+h])}{m((x,x+h])} = 0$$

Lebesgue-a.e. x. □

Let μ be Borel regular, then $\mu = \mu_d + \mu_{sc} + \mu_{ac}$.

1. $\mu_d = \sum_i \chi_i\delta_{xi}$ is signular point mass. F_d is increasing step function right continous.
2. μ_{sc} is singular continous part. F_{sc} is continous like devil's staircase.
3. $\mu_{ac} \ll m$. $F_{ac} = c + \int_0^x f dm$.

3.4 Bounded Variation and Absolute Continuous

Definition: 3.13: Bounded Variation

For $F : \mathbb{R} \to \mathbb{R}$ or $\mathbb{C}$, define the variation:

$$V_{[a,b]}F = \sup_{a<x_0<\cdots<x_n=b} \sum_{k=0}^{n-1} |F(x_{k+1}) - F(x_k)|$$

Total variation:

$$T_F(x) = \sup_{a=x_0<\cdots<x_n=x} \sum_{k=0}^{n-1} |F(x_{k+1}) - F(x_k)| = \sup_{a<x} V_{[a,x]}F$$

$F \in BV$ (bounded variation) if $\sup_x T_F(x) = \sup_{a<b} V_{[a,b]}F < \infty$. We also define $F \in BV([a,b])$ if it is bounded variation on $[a,b]$.
$[a,b] \subset [c,d] \Rightarrow V_{[a,b]}F \le V_{[c,d]}F$, $V_{[a,b]}F + V_{[b,c]}F = V_{[a,c]}F$.

Example:

1. If F is increasing, $V_{[a,b]}F = F(b) - F(a)$.
2. If F is L-Lipschitz, $|F(x) - F(y)| < L|x-y|$, then $V_{[a,b]}F \le L|b-a|$
3. If $F(x) = \int_{-\infty}^x f(t)dt$, $f \in L^1(\mathbb{R})$, then $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|\, dt, V_{[a,b]}F \le \int |f|$.

Proposition: 3.3:

Let ν be a complex measure on $\mathbb{R}$, $F(x) = \nu((-\infty, x])$, then $F \in BV$, and $T_F(x) = |\nu|(-\infty, x]$.

Remark 18. This characterize all right-continuous functions. They are all bounded variation.

Proof.

$$\sum_{k=0}^{n-1} |F(x_{k+1}) - F(x_k)| = \sum_{k=0}^{n-1} |\nu(x_k, x_{k+1}]| \leq \sum_{k=0}^{n-1} |\nu|(x_k, x_{k+1}] \leq |\nu|(-\infty, x_n]$$

Note that $|\nu|(E) = \sup\left\{\sum_{k=1}^{n} |\nu(E_i)| : E \text{ disjoint union of } E_i\right\}$. Therefore,

$$\begin{aligned}
|\nu|(a, b] &= \sup\left\{\sum |\nu(E_i)| : (a, b] = \cup E_i \text{ disjoint}\right\} \\
&= \sup\left\{\sum |\nu(I_i)|, (a, b] = \cup U_i \text{ disjoint half open intervals}\right\} \text{ (Approximate by Intervals)} \\
&= V_{[a,b]}F
\end{aligned}$$

□

Lemma: 3.7:

If $F \in BV$ and right continuous (RC), then $T_F(x)$ is also RC.

Proof. Since F is RC, for $x \in \mathbb{R}$, $\epsilon > 0$, $\exists \delta > 0$ s.t. $|F(x + y) - F(x)| < \frac{\epsilon}{2}$ for $y \in (0, \delta)$.

If $F \in BV$, $\exists x = x_0 < \cdots < x_n = x + \delta$,

$$\begin{aligned}
V_{[x,x+\delta]}F &< \sum_{k=0}^{n-1} |F(x_{k+1}) - F(x_k)| + \frac{\epsilon}{2} \\
&= |F(x_1) - F(x_0)| + \sum_{k=1}^{n-1} |F(x_{k+1}) - F(x_k)| + \frac{\epsilon}{2} \\
&\leq |F(x_1) - F(x_0)| + V_{[x_1,x+\delta]}F + \frac{\epsilon}{2}
\end{aligned}$$

Hence, $V_{[x,x_1]}F \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$. If $y \in (0, \delta = x_1 - x_0)$, then $T_F(x + y) - T_F(x) \leq V_{[x,x_1]}F < \epsilon$. □

Proposition: 3.4:

1. $F : \mathbb{R} \to \mathbb{R}, F \in BV \Leftrightarrow$ there exists increasing BV functions F_1, F_2 s.t. $F = F_1 - F_2$, Canonial: $F = \frac{1}{2}(F + T_F) - \frac{1}{2}(T_F - F)$
2. F has one-sided limits and if $G(x) = F(x+)$, $G' = F'$ a.e., F is continuous at at most countably many points. This also works for complex functions $F : \mathbb{R} \to \mathbb{C}$.

Definition: 3.14: Normalized Bounded Variation

$F \in NBV$ (normalized bounded variation) if $F \in BV$, F is RC and $F(-\infty) = 0$

Theorem: 3.14:

Let $F : \mathbb{R} \to \mathbb{C}$ be NBV. Then there exists a complex measure ν s.t. $\nu((-\infty, x]) = F(x)$

Definition: 3.15: Absolute Continuous

$F : \mathbb{R} \to \mathbb{C}$ is absolutely continuous on $[a, b]$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $I_k = (a_k, b_k)$ are disjoint intervals $k = 1, ..., n$, $I_k \subset [a, b]$, $\displaystyle\sum_{k=1}^{n} |b_k - a_k| < \delta \Rightarrow \sum_{k=1}^{n} |F(b_k) - F(a_k)| < \epsilon$. We say that $F \in AC[a, b]$.

Proposition: 3.5:

1. $AC[a, b] \subset BV[a, b]$
2. $AC[a, b] \Rightarrow$ Uniform continuous. The converse is false $e.g. x^2 \sin \frac{1}{x}$ is uniformly continuous but not absolutely continuous.
3. Lipschitz continuous $\Rightarrow$ absolute continuous, but not conversely.

Proposition: 3.6:

$F \in BV[a, b]$, extend F to $\mathbb{R}$ by $F(x) = F(a), x < a$, $F(x) = F(b), x > b$. Then $F \in AC[a, b] \Leftrightarrow \nu_F \ll m$

Proof. ($\Rightarrow$) Let ϵ, δ be as in Definition 3.15. Take E s.t. $m(E) = 0$, $K \subset E$ compact.
Then $\exists I_i = (a_i, b_i]$ s.t. $K \subset \cup I_i$ dis joint union and $\sum |b_i - a_i| < \delta$.
Then $\nu_F(K) \leq \sum \nu_F(I_i) = \displaystyle\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon$, $\nu_F(E) = 0$ by regularity.

($\Leftarrow$) Integral is bounded by L^1 norm. □

Theorem: 3.15: Fundamental Theorem of Calculus

The following are equivalent (TFAE):

1. $F \in AC[a, b]$
2. $\exists f \in L^1([a, b])$ s.t. $F(x) - F(a) = \displaystyle\int_a^x f(t)dt$
3. F' exists a.e. and $F' \in L^1$, $F(x) - F(a) = \displaystyle\int_a^x F'(t)dt$.

Proof. $3 \Rightarrow 2$ is clear, let $f = F'$.
$2 \Rightarrow 1$, Let $\nu(E) = \int_E f dm$, $\nu \ll m$, use Proposition 3.6.
$1 \Rightarrow 3$, $F' = \frac{d\nu_F}{dm}$. □

Remark 19. F' exists a.e. does not imply $F' \in L^1$. *e.g.* $x^2 \sin \frac{1}{x}$

Proposition: 3.7: Standard Results from Calculus

1. F is L-Lipschitz $\Leftrightarrow$ $F \in AC$ and $|F'| \leq L$ a.e. x
2. $f, g \in AC[a,b]$, then $\int_a^b fg' = fg|_a^b - \int_a^b f'g$
3. Let $\phi \in AC[a,b]$ increasing. If $f \in L^1([\phi(a), \phi(b)])$, then $\int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f \circ \phi(y)\phi'(y)dy$.

Proof. $\phi^{-1}(c,d] = (\sup \phi^{-1}(c), \sup \phi^{-1}(d)]$. Let $\mu_\phi(a,b] = \phi(b) - \phi(a)$.
Then $\mu_\phi \circ \phi^{-1}(c,d] = \mu_\phi(\sup \phi^{-1}(c), \sup \phi^{-1}(d)] = d - c$ by continuity.
So $\mu_\phi \circ \phi^{-1} = m$ by uniqueness of extension.

$$\begin{aligned}
\int_{[\phi(a),\phi(b)]} f dm &= \int_{[\phi(a),\phi(b)]} f d\mu_\phi \circ \phi^{-1} \\
&= \int_{[a,b]} f \circ \phi d\mu_\phi \\
&= \int_{[a,b]} f \circ \phi \frac{d\mu_\phi}{dm} dm = \int_{[a,b]} f \circ \phi \phi' dm
\end{aligned}$$

□

Theorem: 3.16: IBP on BV Functions

Let $F \in NBV$, G is continuous and BV, then

$$\int_{(a,b]} F d\mu_G + \int_{(a,b]} G d\mu_F = FG|_a^b$$

Proof.

$$\begin{aligned}
\int_{(a,b]} F(x) - F(a) d\mu_G(x) &= \int_{(a,b]} \int_{(a,x]} d\mu_F(t) d\mu_G(x) \\
&= \int_{(a,b]} \int_{[t,b]} d\mu_G(x) d\mu_F(t) \\
&= \int_{(a,b]} G(b) - G(t) d\mu_F(t) \text{ by continuity}
\end{aligned}$$

Rearrange to get the desired result. □

4 Basic Functional Analysis

4.1 Topology

Definition: 4.1: Topology

A Topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X such that:
1. $\emptyset, X \in \mathcal{T}$
2. $U_\alpha \in \mathcal{T} \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$
3. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

$\mathcal{E}$ is a base of $\mathcal{T}$ if $\mathcal{T} = \{\text{arbitrary union of sets in } \mathcal{E}\}$; $U \in \mathcal{T} \Leftrightarrow U$ is arbitrary union of elements in base $\mathcal{E}$.

Example: In a metric space, $\{B_r(x)\}_{x \in X}$ is a base for its topology.

Proposition: 4.1:

$\mathcal{E}$ is a base of some topology if and only if
1. $\forall x \in X$, $\exists U \in \mathcal{E}$ s.t. $x \in U$
2. $U, V \in \mathcal{E}$, $\forall x \in U \cap V$, $\exists W$ s.t. $x \in W \subset U \cap V$.

$\mathcal{T}(\mathcal{E})$ has $\{\text{finite intersection of elements in } \mathcal{E}\}$ as base.

Definition: 4.2: Neighborhood

U is a neighborhood of $x \in X$ if $\exists O \in \mathcal{T}$ s.t. $x \in O \subset U$.

Definition: 4.3: Neighborhood Base

The class $\mathcal{N}_x \subset \mathcal{P}(X)$ is a neighborhood base at x if
1. $\forall U \in \mathcal{N}_x$, $x \in U$
2. $\forall U, V \in \mathcal{N}_x$, $\exists W \subset \mathcal{N}_x$ s.t. $W \subset U \cap V$
3. $\forall x \in U$, $\exists W \in \mathcal{N}_x$ neighborhood base s.t. $W \subset U$.

Definition: 4.4: Countable Space

1. 1st countable space $\Leftrightarrow$ countable neighborhood base at every x
2. 2nd countable (separable) $\Leftrightarrow$ Countable dense sets

Definition: 4.5: Hausdorff Space

In a Hausdorff space X, $\forall x, y \in X$, $\exists U, V$ open, $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition: 4.6: Locally Compact

Locally compact $\Rightarrow$ there exists a precompact neighborhood base.

4.2 Banach Space

Definition: 4.7: Normed Vector Space

Given vector space X on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A seminorm is $p : X \to [0, \infty)$ s.t.

1. $p(x+y) \leq p(x) + p(y)$
2. $p(\alpha x) = |\alpha| p(x)$

A norm is a seminorm with $p(x) = 0 \Rightarrow x = 0$.
A normed vector space is $(X, \|\cdot\|)$, the metric induced is $d(x,y) = \|x-y\|$. $x \mapsto \|x\|$ is continous, $x \mapsto x+y, x \mapsto \alpha x$, $\alpha \neq 0$ are homeomorphisms.

Definition: 4.8: Banach Space

A Banach space $(\mathcal{X}, \|\cdot\|)$ is a complete normed vector space. *i.e.* All Cauchy sequences converge.

Proposition: 4.2:

$(X, \|\cdot\|)$ is complete if and only if absolute convergence of series implies convergence, *i.e.* $\sum_{n=1}^{\infty} \|x_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} x_n$ converges.

Proof. $(\Leftarrow)$ Suppose absolute convergence of series implies convergence. Let x_n be Cauchy, n_k be a subsequence s.t. $\left\|x_{n_{k+1}} - x_{n_k}\right\| < 2^{-k}$. Then

$$\sum_{k=0}^{\infty} x_{n_{k+1}} - x_{n_k} < \infty \Rightarrow y_m = \sum_{k=0}^{m} (x_{n_{k+1}} - x_{n_k}) = x_{n_m} - x_{n_0} \to x - x_{n_0}$$

for some x. Then

$$0 = \lim_{n\to\infty} \lim_{m\to\infty} \|x_n - x_m\| = \lim_{n\to\infty} \|x_n - x\|$$

Therefore, all Cauchy sequence converges. □

Corollary 14. *$L^1(\mu)$ is complete*

Proof. Suppose $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$. Then by Theorem 2.1, $\sum_{n=1}^{\infty} |f_n| \in L^1$.
Then $\sum_{n=1}^{\infty} |f_n| < \infty$ a.e. x. $\sum_{n=1}^{\infty} f_n = f$.
$\left\|\sum_{n=1}^{m} f_n - \sum_{n=1}^{\infty} f_n\right\| \to 0$ as $m \to \infty$. Therefore, all Cauchy sequence converges. □

Definition: 4.9: Basic Functional Spaces

$C(X)$: the set of continuous functions $f : \mathbb{R} \to \mathbb{F}$.
$C_b(X)$: set of bounded functions in $C(X)$, $\left\{ \|f\| = \sup_x |f(x)| \right\}$.
$C_0(X) = \{f \in C(X) : \{x : |f(x)| > \epsilon\}$ is compact$\}$, *i.e.* $f \in C_0(X)$ if f goes to 0 at ∞.
$C_b(X)$ is a Banach space, with the norm defined by $\|f\| = \sup_x |f(x)|$.

Proposition: 4.3:

$C_0(X)$ is a closed subspace of $C_b(X)$, hence is a Banach space with $\|\cdot\|$ of $C_b(X)$.

Proof. Suppose $f_n \in C_0(X), f_n \to f \in C_b$.
$\forall \epsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \sup_x |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Then

$$\{x : |f(x)| > \epsilon\} \subset \left\{ x : |f_N(x) - f(x)| \geq \frac{\epsilon}{2} \right\} \cup \left\{ x : |f_N(x)| \geq \frac{\epsilon}{2} \right\}$$

is compact. Therefore $f \in C_0(X)$. ☐

Definition: 4.10: Compactly Supported Function

The support of the function is $\mathrm{supp} f = \overline{\{x : f(x) \neq 0\}}$. The compactly supported function if $C_C(X) = \{f \in C(X) : \mathrm{supp} f$ is compact$\}$

Lemma: 4.1: Urysohn's Lemma

Let X be a locally compact Hausdorff (LCH) space. $K \subset U$, K compact, U open. Then $\exists f \in C_C(X)$ $0 \leq f \leq 1$ s.t. $f|_K = 1$, $\mathrm{supp} f \subset U$.

Proposition: 4.4:

If X is LCH, then $\overline{C_C(X)} = C_0(X)$

Proof. Let $f \in C_0(X)$, $U_n = \left\{ x : |f(x)| > \frac{1}{n} \right\}$, $K_n = \overline{U_n} = \left\{ x : |f(x)| \geq \frac{1}{n} \right\}$.
By Lemma 4.1, $\exists g_n$ s.t. $g_n|_{K_n} = 1$, $\mathrm{supp} g_n \subset U_{n-1}$, $f_n = g_n f$. Then

$$\|f_n - f\| = \|(g_n - 1)f\| < \frac{1}{n} \to 0$$

$g_n - 1 = 0$ on K_n^C and $|f(x)| < \frac{1}{n}$ on K_n^C. ☐

Examples of Banach spaces:

1. C^k with $\|f\|_{C^k} = \sum_{i=0}^{k} \left\| f^{(i)} \right\|_{C_b(X)}$, where f is s.t. ith derivative $f^{(i)} \in C_b(X)$ for $i = 0, ..., k$.
2. $AC[a, b]$ with $\|f\|_{AC} = |f(a)| + \left\| f' \right\|_1$

4.3 Basic Properties of Banach Spaces

Proposition: 4.5: Bounded Linear Operators

Let $(X, \|\cdot\|_1), (Y, \|\cdot\|_2)$ be normed vector spaces. $T : X \to Y$ be a linear mapping. TFAE.

1. T is continous.
2. T is continous at 0
3. T is bounded *i.e.* $\exists C > 0$ s.t. $\|Tx\|_2 \leq C \|x\|_1$.

Proof. 1⇒2 is clear

2⇒3: $\exists r > 0$ s.t. $T(B_r(0)) \subset B_1(0)$ *i.e.* $\|x\|_1 < r \Rightarrow \|Tx\| < 1$.
$\forall x \in X$, let $\alpha = \frac{1}{\|x\|}\frac{r}{2}$. Then $\|\alpha x\| = \frac{r}{2} < r$. Hence $\|T(\alpha x)\| < 1$ or $\|Tx\| < \frac{1}{\alpha}\frac{2}{r}\|x\|$.

3⇒1: $\|Tx - Ty\| \leq C \|x - y\|$ □

Definition: 4.11: Operator Norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|.$$

Corollary 15. *Given two norms $\|\cdot\|_1, \|\cdot\|_2$ on X, they define the same topology if $\exists c_1, c_2 > 0$ s.t. $c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$.*

Proof. Let $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ s.t. $I(x) = x$, I, I^{-1} both bounded. □

Proposition: 4.6: Bounded Linear Operator Space

Let $L(X, Y)$ be the space of all bounded linear maps with norm $\|T\|$. Suppose Y is complete, then $L(X, Y)$ is complete.

Proof. Let T_n be Cauchy, then $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \|x\| \to 0$ as $n, m \to \infty$.
Hence $\{T_n(x)\}$ is Cauchy, therefore converges to limit denoted $T(x)$. □

Corollary 16. $L(X, \mathbb{R})$ *is complete*

4.4 Finite Dimensional Spaces

Proposition: 4.7:

All norms on a finite dimensional space $\mathcal{X}$ are equivalent.

Proposition: 4.8:

Pick $e_1, ..., e_n$ a basis, $\|x\|_\infty = \sup |a_i|$. If $x = \sum a_i e_i$, then any $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.

Proof. Let $I : (\mathcal{X}, \|\cdot\|_\infty) \to (\mathcal{X}, \|\cdot\|)$.

$$\left\|\sum_{i=1}^n a_i e_i\right\| \leq \sum_{i=1}^n |a_i| \|e_i\| \leq \|x\|_\infty \sum_{i=1}^n \|e_i\|$$

Therefore, I is bounded.

$\partial B_\infty = \{x : \|x\|_\infty = 1\}$ is compact in $\|\cdot\|_\infty$. Also, it is compact in $\|\cdot\|$.
Since $0 \notin \partial B_\infty$, $\exists r > 0$ s.t. $B_r(0) \cap \partial B_\infty = \emptyset$.
Given $x \in B_r(0) \setminus \{0\}$, $x / \|x\|_\infty \in \partial B_\infty$.

$$\|x / \|x\|_\infty\| \geq r$$
$$r > \|x\| \geq r \|x\|_\infty$$
$$\Rightarrow \|x\|_\infty < 1 \Rightarrow x \in B_\infty.$$

This implies I^{-1} is bounded. ☐

Corollary 17. *Any finite dimensional subspace of normed vector space X is closed.*

Proof. If $\mathcal{M}$ is finite dimensional. $(\mathcal{M}, \|\cdot\|) \cong (\mathbb{R}^n, \|\cdot\|)$.
In particular $(\mathcal{M}, \|\cdot\|)$ is complete. Hence $\mathcal{M}$ is closed in $\mathcal{X}$. ☐

Theorem: 4.1: Riesz Lemma

Let $\mathcal{X}$ be a normed vector space, $\mathcal{M}$ be a closed subspace. Then $\forall \alpha \in (0,1)$, $\exists \|x\| = 1$ s.t. $d(x, \mathcal{M}) = \sup_{y \in \mathcal{M}} \|x - y\| > \alpha$ or $\sup_{\|x\|=1} d(x, \mathcal{M}) = 1$.

Proof. We argue by contradiction. Suppose $\exists \alpha \in (0,1)$ s.t. $\sup_{\|x\|=1} d(x, \mathcal{M}) \leq \alpha$ or $\|x\| = 1 \Rightarrow x \in \bigcup_{y \in \mathcal{M}} \overline{B}_\alpha(y)$.

Consider λx with $\lambda \in (0,1)$.

$$d(\lambda x, \mathcal{M}) = \lambda d(x, \lambda^{-1}\mathcal{M}) \leq d(x, \mathcal{M})$$

Hence $\overline{B}_1(0) \subset \bigcup_{y \in \mathcal{M}} y + \overline{B}_\alpha(0) = M + \alpha \overline{B}_1(0)$.

$\alpha \overline{B}_1(0) \subset \alpha \mathcal{M} + \alpha^{\in} \overline{\mathcal{B}}_\infty(\prime) = \mathcal{M} + \alpha^{\in} \overline{\mathcal{B}}_\infty(\prime)$. Then

$$\overline{B}_1 \subset \mathcal{M} + \alpha \overline{B}_1 = \mathcal{M} + \mathcal{M} + \alpha^2 \overline{B}_1(0) = \mathcal{M} + \alpha^2 \overline{B}_1(0) \subset \cdots$$

Let $\overline{B}_1 \subset \bigcap_{k \geq 1} \mathcal{M} + \alpha^k \overline{B}_1(0) = \mathcal{M}$, because $\mathcal{M}$ is closed.

Contradiction, since $y \in \mathcal{X} \setminus \mathcal{M}$, $y / \|y\| \in \overline{B}_1 \subset \mathcal{M}$. ☐

Proposition: 4.9:

Any locally compact normed vector space is finite dimensional.

Proof. Let $\mathcal{X}$ be locally compact. Then $\overline{B}$ is compact. There exists a finite subcover of $\{x + B_\alpha(0), x \in \overline{B}\}$, $x_1 + B_\alpha, \ldots, x_N + B_\alpha$.
Let $M = \operatorname{span}\{x_1, \ldots, x_N\}$, M is closed.

If $M \neq \mathcal{X}$, then $M + \overline{B}_\alpha \supset \overline{B}_1$. Contradicts Theorem 4.1. ☐

4.5 Linear Functionals

Definition: 4.12: Linear Functionals

A linear functional is a mapping $\mathcal{L}(\mathcal{X}, \mathbb{F}) = \mathcal{X}^*$, the dual space of $\mathcal{X}$. $\mathbb{F}$ can be $\mathbb{R}$ or $\mathbb{C}$. If $\mathcal{X}$ is a Banach space, $\|f\| = \sup_{x \neq 0} \dfrac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|$.

Question: Does $\mathcal{X}^*$ contain any nontrivial elements?

Definition: 4.13: Sublinear Functionals

$q : \mathcal{X} \to \mathbb{R}$ is sublinear if

1. $q \geq 0$
2. $q(x+y) \leq q(x) + q(y)$
3. Positive homogenous: $\forall \lambda > 0, q(\lambda x) = \lambda q(x)$

Remark 20. Semi-norms p are sublinear.

Definition: 4.14: Partial order

An order $<$ on $\mathcal{X}$ is s.t.

1. $x < x$
2. $x < y$ and $y < z \Rightarrow x < z$
3. $x < y$ and $y < x \Rightarrow x = y$.

Definition: 4.15: Linearly Ordered Set

A set A is linearly ordered if $\forall x \in A$, either $x < y$ or $y < x$.

Definition: 4.16: Upperbound and Max Element

y is an upperbound of A if and only if $\forall x \in A, x < y$.
y is a max element of A if $\forall x \in A, y < x \Rightarrow y = x$.

Lemma: 4.2: Zorn's Lemma

Let X be partially ordered. Suppose any linearly ordered set has an upperbound, then X has a max element.

Theorem: 4.2: Hahn-Banach over $\mathbb{R}$

Let $\mathcal{X}$ be a vector space over $\mathbb{R}$, q sublinear on $\mathcal{X}$, and $\mathcal{M}$ a subspace. If $f_0 : \mathcal{M} \to \mathbb{R}$ linear s.t. $f_0(x) \leq q(x), \forall x \in \mathcal{M}$, then $\exists f : \mathcal{X} \to \mathbb{R}$ a linear extension, $f|_{\mathcal{M}} = f_0|_{\mathcal{M}}$, $f(x) \leq q(x), \forall x \in \mathcal{X}$.

Proof. Let $\mathcal{N}$ be a subspace of $\mathcal{X}$, g be linear on $\mathcal{N}$. Define the partial order on the pair $(g, \mathcal{N})$ s.t. $g|_{\mathcal{N}} \leq q|_{\mathcal{N}}$, $g|_{\mathcal{M}} = f_0$.
$(g_1, \mathcal{N}_1) < (g_2, \mathcal{N}_2) \Leftrightarrow \mathcal{N}_1 \subset \mathcal{N}_2$ and $g_2|_{\mathcal{N}_1} = g_1|_{\mathcal{N}_1}$.
By Lemma 4.2, $\exists (g, \mathcal{N})$ s.t. $g|_{\mathcal{N}} \leq q|_{\mathcal{N}}$ and $(g, \mathcal{N})$ is maximal.

We want to show $\mathcal{N} = \mathcal{X}$.

Suppose $\mathcal{N} \subsetneq \mathcal{X}$. Define

$$\overline{q}(t) = \inf_{y \in \mathcal{N}} (q(y + tx_0) - q(y))$$

Let $\lambda > 0$,

$$\begin{aligned}\overline{q}(\lambda t) &= \inf_{y \in \mathcal{N}} (q(y + t\lambda x_0) - g(y)) \\ &= \lambda \inf_{y \in \mathcal{N}} \left(q\left(\frac{y}{\lambda} + tx_0\right) - g\left(\frac{y}{\lambda}\right)\right) \\ &= \lambda \overline{q}\end{aligned}$$

$\forall y_1, y_2 \in \mathcal{N}, t_1, t_2 \in \mathbb{R}$:

$$\begin{aligned}&q(y_1 + t_1x_0) - g(y_1) + q(y_2 + t_2x_0) - g(y_2) \\ &\geq q(y_1 + y_2 + (t_1 + t_2)x_0) - g(y_1 + y_2) \\ &\geq \overline{q}(t_1 + t_2).\end{aligned}$$

Then $\overline{q}(t) = \begin{cases} at, t < 0 \\ bt, t \geq 0, \end{cases}$, where $a \leq b$. Define

$$\begin{aligned}g * (y + tx_0) &= g(y) + at \\ &\leq g(y) + \overline{q}(t) \leq q(y + tx_0)\end{aligned}$$

$(g^*, \mathcal{N} + tx_0)$ is an extension of $(g, \mathcal{N})$, contradicting maximality of $(g, \mathcal{N})$. $\square$

Geometric Interpretation of Theorem 4.2. $f(x) \leq q(x), \forall x \Leftrightarrow \{x : q(x) \leq 1\} \subset \{x : f(x) \leq 1\}$. LHS is a convex set and RHS is a half space.

Corollary 18. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{R}$. $x_0 \neq 0$. $\exists f \in \mathcal{X}^*$ s.t. $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

Proof. Define $f_0 : \operatorname{span}\{x_0\} \to \mathbb{R}$. $f_0(tx_0) = t\|x_0\| \leq \|tx_0\|$.
By Theorem 4.2, $\exists f : \mathcal{X} \to \mathbb{R}$ s.t. $f(x) \leq \|x\|$, and $f(x_0) = \|x_0\|$. This is because $f(-x) \leq \|x\|$ and $f(x) \geq -\|x\|$. $\square$

Remark 21. $\mathcal{X}^*$ separates points. $\forall x \neq y \in \mathcal{X}$, $\exists f \in \mathcal{X}^*$ s.t. $f(x) \neq f(y)$.

Complex Vector Spaces If $\mathcal{X}$ is a vector space over $\mathbb{C}$, then $\mathcal{X}$ is also a vector space over $\mathbb{R}$.

***Proposition:* 4.10: Properties of Complex Vector Spaces**

Let $\mathcal{X}$ be a vector space over $\mathbb{C}$.

1. If $f : \mathcal{X} \to \mathbb{R}$ is linear, then $\tilde{f}(x) = f(x) - if(ix)$ is linear $\mathcal{X} \to \mathbb{C}$.
2. If $g : \mathcal{X} \to \mathbb{C}$ is linear, then $g(x) = \mathrm{Re}g(x) - i\mathrm{Re}g(ix)$.
3. If p is a semi-norm, $f : \mathcal{X} \to \mathbb{R}$, then $f(x) \leq p(x), \forall x \Leftrightarrow |f(x)| \leq p(x), \forall x \Leftrightarrow |\tilde{f}(x)| \leq p(x), \forall x$.

Proof. (3) Let $z \in \mathbb{C}$. Define $\operatorname{sgn}(z) = \begin{cases} 0, z = 0 \\ \frac{z}{|z|}, z \neq 0 \end{cases}$. Then $\overline{\operatorname{sgn}(z)}z = |z|$.
Suppose $f(x) \leq p(x)$. Let $\alpha = \operatorname{sgn}\tilde{f}(x)$. Then

$$\begin{aligned}|\tilde{f}(x)| &= \overline{\alpha}\tilde{f}(x) \\ &= \tilde{f}(\overline{\alpha}x) = \mathrm{Re}\tilde{f}(\overline{\alpha}x) \\ &= f(\overline{\alpha}x) \leq p(\overline{\alpha}x) = p(x)\end{aligned}$$

□

Theorem: 4.3: Hahn-Banach over $\mathbb{C}$

$\mathcal{X}$ is a complex vector space. p is a semi-norm. $\mathcal{M}$ is a subspace. $f_0 : \mathcal{M} \to \mathbb{C}$ is linear s.t. $|f_0(x)| \leq p(x)$. Then there exists a linear extension $f : \mathcal{X} \to \mathbb{C}$ s.t. $|f(x)| \leq p(x)$.

Proof. Extend $\mathrm{Re} f_0$ to $\mathcal{X}$ as a $\mathbb{R}$-functional. Reconstruct the $\mathbb{C}$-functional. □

4.6 Dual Space

Definition: 4.17: Dual Space

$\mathcal{X}^*$ is a dual of a normed vector space $\mathcal{X}$ if
1. $\forall x_0$, $\exists \|f\| = 1$ s.t. $|f(x_0)| = \|x_0\|$
2. $\mathcal{X}^*$ separate points.

Example:

1. $l^1 = \left\{(x_n)_{n\in\mathbb{N}} : \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty\right\}$. $(l^1)^* = \left\{f(x) = \sum a_i x_i : \sup |a_n| < \infty\right\}$. Notice:
$$(l^1)^* = l^\infty = \{(x_n) : \sup |x_n| = \|x\|_\infty < \infty\}$$
2. If ϕ is measurable and $\sup |\phi| < \infty$, then $f \mapsto \int f\phi d\mu$, $L^1(\mu) \to \mathbb{F}$ is in $(L^1)^*$. $\left|\int f\phi d\mu\right| \leq \int |f\phi| d\mu \leq \sup |\phi| \int |f| d\mu$
3. Let μ be a finite signed/complex Borel measure, then $f \mapsto \int f d\mu$, $C_b(X) \to \mathbb{F}$ is bounded.

Definition: 4.18: Double Dual

$\mathcal{X}^{**} = (\mathcal{X}^*)^*$. Given $x \in \mathcal{X}$, define $\hat{x}(f) = f(x)$.
$$\|\hat{x}\| = \sup_{\|f\|=1} |\hat{x}(f)| = \sup_{\|f\|=1} |f(x)| \leq \|x\|$$
$x \mapsto \hat{x}$ is bounded. Called the canonical embedding $X \to X^{**}$.

Theorem: 4.4:

Let X be a normed vector space. $x \mapsto \hat{x}$ ($\hat{x}(f) = f(x)$) satisfies $\hat{x} \in \mathcal{X}^{**}$ and $\|\hat{x}\|_{\mathcal{X}^{**}} = \|x\|_{\mathcal{X}}$.

Proof. By Definition 4.18, $\|\hat{x}\|_{\mathcal{X}^{**}} = \sup_{\|f\|_{\mathcal{X}^*}} |f(x)| \leq \sup_{\|f\|=1} \|f\| \, \|x\| = \|x\|$.

By Theorem 4.2, $\exists f_0$ s.t. $\|f_0\| = 1$, $|f_0(x)| = \|x\| \geq \|x\|$. □

Remark 22. $x \mapsto \hat{x}$ is an isometry.

Definition: 4.19: Reflexive Vector Space

$\mathcal{X}$ is called reflexive if the cannonical embedding is onto, $\mathcal{X} \cong \mathcal{X}^{**}$.

1. Finite dimensional spaces are reflexive.

2. If $\mathcal{X}$ is a Hilbert space, *i.e.* $\|x\| = \langle \cdot, \cdot \rangle$ is an inner product, then $\mathcal{X}$ is reflexive.

Definition: 4.20: Weak Topology

Weak topology on $\mathcal{X}$ has the subbase $\{y : |f(y) - f(x)| < r\}_{x \in \mathcal{X}, f \in \mathcal{X}^*, r>0}$. It is a Hausdorff topology, because $x, y \in \mathcal{X}$ can be separated.
Weak-$*$ topology on $\mathcal{X}^*$ is $\{g : |g(x) - f(x)| < r\}_{x \in \mathcal{X}, f \in \mathcal{X}^*, r>0}$.

1. $x_n \to x$ in weak topology if and only if $f(x_n) \to f(x)$ for all $f \in \mathcal{X}^*$ (projection by functionals converge)

2. $f_n \to f$ in weak-$*$ topology if and only if $f_n(x) \to f(x), \forall x \in \mathcal{X}$ (pointwise convergence)

Theorem: 4.5: Alaoglu

The closed unit ball $\overline{B}$ in $\mathcal{X}^*$ w.r.t. $\|\cdot\|$ is compact in weak-$*$ topology.

Corollary 19. *If $\mathcal{X}$ is reflexive, then $\overline{B}$ is weakly compact.*

5 L^p Spaces

Definition: 5.1: Convex Function

Let I be an open interval, $\phi : I \to \mathbb{R}$ is conveex if

$$\phi((1-\lambda)x+\lambda y) \leq (1-\lambda)\phi(x)+\lambda\phi(y), \lambda \in (0,1), x, y \in I$$

ϕ is strictly convex if equality $\Rightarrow x = y$.

Proposition: 5.1: Properties of Convex Functions

Let ϕ be convex.

1. $y \mapsto \frac{\phi(y)-\phi(x)}{y-x}$ is increasing (convex functions are absolutely continuous)
2. $\forall x$, $\exists \beta \in \mathbb{R}$ s.t. $\phi(y)-\phi(x) \geq \beta(y-x), \forall y$, β is called *subderivative* or *subdifferential.* If ϕ is strictly convex, then equality $\Rightarrow y = x$. (If a convex function coincides with linear function, then it must be linear.)

Proof. Let $l_y = \frac{\phi(y)-\phi(x)}{y-x}$. Then $\left[\sup\limits_{y<x} l_y, \inf\limits_{y>x} l_y\right]$ is non-trivial. Any β in this interval is a subdifferential.

If $\phi(y) = \phi(x) + \beta(y-x)$, then

$$(1-\lambda)\phi(x)+\lambda\phi(y) = \phi(x)+\beta\lambda(y-x) \leq \phi((1-\lambda)x+\lambda y)$$

Together with the definition of convexity, we get a strict equality. □

Theorem: 5.1: Jensen's Inequality

Let $\phi : I \to \mathbb{R}$be convex, $\mu(X) = 1$ is a probability measure, $f : X \to I$ is integrable, then $\phi\left(\int f d\mu\right) \leq \int \phi \circ f d\mu$. If ϕ is strictly convex, then equality $\Rightarrow f =$const μ-a.e.

Intuition: if $\lambda_i \in (0,1), \sum \lambda_i = 1$, then $\phi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \phi(x_i)$.

Let $X = \{1, ..., n\}, f(i) = x_i, \mu(\{i\}) = \lambda_i$, then $\phi\left(\int f d\mu\right) \leq \int \phi \circ f d\mu$ and pointmass approximates any probability measure.

Proof. Let $t = \int f d\mu \in I$, $\exists \beta > 0$ s.t. $\phi(f(x)) - \phi(t) \geq \beta(f(x)-t)$. Integrate both sides w.r.t. μ, $\int f d\mu = t \int d\mu = t$. Then $\int \phi \circ f d\mu - \phi\left(\int f d\mu\right) \geq 0$.

If ϕ is strictly convex, $g = \phi \circ f - \phi(t) - \beta(f(x)-t) \geq 0$. Equality $\Rightarrow \int g = 0$, $\Rightarrow g = 0$ a.e. $\Rightarrow f = t$ a.e. □

Definition: 5.2: L^p Spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space, $p > 0$.

$$L^p(\mu) = \left\{ f \text{ measurable} : \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} < \infty \right\} / \sim \text{a.e. equality}$$

Typically, $p \geq 1$.
L^p is a vector space.

Proof.

$$\begin{aligned}
\|f+g\|_p^p = \int |f+g|^p d\mu &\leq \int (2 \max\{|f|, |g|\})^p \, d\mu \\
&\leq 2^p \int \max\{|f|^p, |g|^p\} \\
&\leq 2^p \int |f|^p + |g|^p = 2^p \left(\|f\|_p^p + \|g\|_p^p \right)
\end{aligned}$$

Therefore, L^p is a vector space. □

Remark 23. If $p \in (0,1)$, $\|\cdot\|_p$ is not a semi-norm, because triangle inequality fails.

Proof. Let $a, b \in [0, \infty)$, $p \in (0,1)$, then

$$\left(\frac{a}{a+b} \right)^{1/p} + \left(\frac{b}{a+b} \right)^{1/p} < \frac{a}{a+b} + \frac{b}{a+b} = 1 \Rightarrow a^{1/p} + b^{1/p} < (a+b)^{1/p}$$

Let $E, F \in \mathcal{M}$, $\mu(E), \mu(F) \in (0, \infty)$, $\mu(E \cap F) = 0$, then

$$\|\chi_E + \chi_F\|_p = \left(\int \chi_{E \cup F} \right)^{1/p} = (\mu(E) + \mu(F))^{1/p} > \mu(E)^{1/p} + \mu(F)^{1/p} = \|\chi_E\|_p + \|\chi_F\|_p$$

□

Theorem: 5.2: Convergence Theorem in L^p

If $|f_n| \nearrow |f|$, then by Theorem 2.1, $\|f_n\|_p \nearrow \|f\|_p$. If $f_n \to f$ a.e. and $|f_n| \geq g \in L^p$, then $f_n \to f$ in L^p by Theorem 2.2.

Definition: 5.3: Duality of L^p

$p, q \in [1, \infty]$ are conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$ or $q = \frac{p}{p-1}$.

Lemma: 5.1: Generalized Geometric Inequality

If $a, b > 0$, $\lambda \in (0,1)$, then $a^{1-\lambda} b^{\lambda} \leq (1-\lambda)a + \lambda b$ with equality if and only if $a = b$.

Proof. $\log x$ is concave.

$$\log((1-\lambda)a + \lambda b) \geq (1-\lambda) \log a + \lambda \log b$$

Take exponentials on both sides, get equality if and only if $a = b$. □

Theorem: 5.3: Holder's Inequality

Let $p, q \in (1, \infty)$ be conjugate, $\frac{1}{p} + \frac{1}{q} = 1$, f, g be measurable, then

$$\int |fg| \leq \|f\|_p \|g\|_q$$

If $\|f\|_p \|g\|_q < \infty$, then equality holds if and only if $|f|^p$ and $|g|^q$ are related by a scalar multiple.

Proof. Conclusion is trivial if either $\|f\|_p, \|g\|_q$ are in $\{0, \infty\}$.
Assume $\|f\|_p, \|g\|_q \in (0, \infty)$. Let $f_1 = \frac{f}{\|f\|_p}$, $g_1 = \frac{g}{\|g\|_q}$.

$$\begin{aligned}
\int |f_1 g_1| &= \int (|f_1|^p)^{1/p} (|g_1|^q)^{1/q} \\
&\leq \int \frac{1}{p}|f_1|^p + \frac{1}{q}|g_1|^q \text{ By Lemma 5.1} \\
&= \frac{1}{p}\int |f_1|^p + \frac{1}{q}\int |g_1|^q = \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

Since $\frac{1}{p}|f_1|^p + \frac{1}{q}|g_1|^q - |f_1 g_1| \geq 0$. If integral is zero, then $\frac{1}{p}|f_1|^p + \frac{1}{q}|g_1|^q - |f_1 g_1| = 0$ a.e. and $|f_1|^p = |g_1|^q$ a.e. $\square$

Theorem: 5.4: Cauchy-Schwarz Inequality

Let $p, q = 2$, by Theorem 5.3

$$\int |fg| \leq \left(\int |f|^2\right)^{1/2} \left(\int |g|^2\right)^{1/2}$$

Theorem: 5.5: Minkowski's Inequality

Let $p \in [1, \infty)$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof.

$$\begin{aligned}
\|f + g\|_p^p &= \int |f + g|^p = \int |f + g||f + g|^{p-1} \\
&\leq \int |f||f + g|^{p-1} + \int |g||f + g|^{p-1} \\
&\text{Apply Theorem 5.3 with } q = \frac{p}{p-1} \\
&\leq \|f\|_p \int \left(|f + g|^{(p-1)q}\right)^{1/q} + \|g\|_p \int \left(|f + g|^{(p-1)q}\right)^{1/q} \\
&= \|f\|_p \|f + g\|_p^{p/q} + \|g\|_p \|f + g\|_p^{p/q} \\
&= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1}
\end{aligned}$$

Cancel $\|f + g\|_p^{p-1}$ on both sides to conclude. $\square$

Corollary 20. $\|\cdot\|_p$ *is a norm.*

***Theorem:* 5.6:**

$(L^p, \|\cdot\|_p)$ is complete for $p \in [1, \infty)$.

Proof. It suffices to show $M = \sum_{n=1}^{\infty} \|f_n\|_p < \infty \Rightarrow \sum_{n=1}^{m} f_n \to \sum_{n=1}^{\infty} f_n$.

By Theorem 5.5, $\left\| \sum_{n=1}^{\infty} |f_n| \right\|_{L^p} \leq M$.

By Theorem 2.1, $\left\| \sum_{n=1}^{\infty} |f_n| \right\|_p \leq M, g = \sum_{n=1}^{\infty} |f_n| \in L^p$ and $\sum_{n=1}^{m} f_n \to \sum_{n=1}^{\infty} f_n$ a.e.

By Theorem 2.2, $\sum_{n=1}^{m} f_n \to \sum_{n=1}^{\infty} f_n$ in L^p. □

***Proposition:* 5.2: Simple Functions in L^p**

For $p \in [1, \infty)$, define L^p simple functions by

$$\Sigma = \left\{ \sum_{i=1}^{n} a_i \chi_{E_i} : a_i \neq 0 \Rightarrow \mu(E_i) < \infty \right\}$$

Σ is dense in L^p.

Proof. If $f \in L^p$, then $\exists \phi_n$ simple s.t. $|\phi_n| \leq |f|$, $\phi_n \to f$ a.e. By Theorem 2.2, $\phi_n \to f$ in L^p. □

Remark 24. If $(X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{B}, m)$, we can approximate Σ by $C_C(\mathbb{R}^n)$ or $C_C^\infty(\mathbb{R}^n)$ etc.

***Definition:* 5.4: L^∞ Norm**

Let f be measurable, define the infinity norm as

$$\|f\|_\infty = \inf \{a \in \mathbb{R} : \mu(\{|f(x)| > a\}) = 0\}$$

with the convention $\inf \emptyset = \infty$.

$$\text{esssup}\,|f| = \inf \left\{ \sup_{x \in E^C} |f(x)| : \mu(E) = 0 \right\}$$

***Proposition:* 5.3:**

1. $\mu(\{|f(x)| > \|f\|_\infty\}) = 0$, and infimum exists
2. $\|f\|_\infty = \text{esssup}|f|$

Proof. 1. $a_n \searrow \|f\|_\infty$, use continuity of measure.

2. If $a > \|f\|_\infty$, then $E = \{x : \{f(x) > a\}$ is null$\}$.
Hence $\text{esssup}|f| \leq a$, $\text{esssup}|f| \leq \|f\|_\infty$.

If $a < \|f\|_\infty$, then $\mu(\{x : |f(x)| > a\}) > 0$.
$\forall \mu(E) = 0,\ \sup_{x \in E^C} |f(x)| \geq a$. Hence $\operatorname{esssup}|f| \geq a$. $\square$

Proposition: 5.4: Properties of L^∞ Space

Define the L^∞ space with measure μ as

$$L^\infty(\mu) = \{f : \|f\|_\infty < \infty\} / \sim$$

1. Theorem 5.3 with $p = 1, q = \infty$: $\displaystyle\int |fg| \leq \|f\|_1 \|g\|_\infty$
2. $\|\cdot\|_\infty$ is a norm
3. $\|\cdot\|_\infty$ is complete
4. Simple functions (not necessarily L^1) are dense in L^∞
5. Let $p \in [1, \infty)$, $L^p \cap L^\infty$ is dense in L^p, but not dense in L^∞ in general.

Proposition: 5.5: Relations between L^p Functions

Let $t > 0$, define $f_t = \chi_{[0,1]} x^{-t}$, $g_t = \chi_{(1,\infty)} x^{-t}$. For $p > 0$,

1. $f_t \in L^p \Leftrightarrow \displaystyle\int_0^1 x^{-tp} dx < \infty \Leftrightarrow tp \in (0, 1)$. Blowups precludes larger L^p spaces, $p < q$, $L^q \not\subset L^p$
2. $g_t \in L^p \Leftrightarrow \displaystyle\int_1^\infty x^{-tp} dx < \infty \Leftrightarrow tp > 1$. Longtail precludes small L^p spaces, $p < q$, $L^p \not\subset L^q$.

Lemma: 5.2: Monotonicity in Sepecial Cases

If $\mu(X) < \infty$, then if $0 < p < q \leq \infty$, $\|f\|_p \leq \mu(X)^{\frac{1}{p} - \frac{1}{q}} \|f\|_q$, hence $L^q \subset L^p$.

Proof. 1. By Theorem 5.1. If $q = \infty$,

$$\|f\|_p^p = \int |f|^p d\mu \leq \int \|f\|_\infty^p d\mu = \mu(X) \|f\|_\infty^p$$

If $q < \infty$, assume WLOG $\|f\|_q < \infty$. Then

$$\begin{aligned}
\|f\|_p^p = \int |f|^p d\mu = \mu(X) \int |f|^p \frac{d\mu}{\mu(X)} &= \mu(X) \int (|f|^q)^{p/q} \frac{d\mu}{\mu(X)} \\
&x^{p/q} \text{ is concave, by Theorem 5.1} \\
&\leq \mu(X) \left(\int |f|^q \frac{d\mu}{\mu(X)} \right)^{p/q} \\
&= \mu(X) \left(\|f\|_q^q \mu^{-1}(X) \right)^{p/q} \\
&= \mu(X)^{1-p/q} \|f\|_q^q
\end{aligned}$$

Similarly by Theorem 5.3,

$$\|f\|_p^p = \int |f|^p d\mu = \int (|f|^q)^{p/q} d\mu$$
$$\text{set } \frac{1}{s} = \frac{p}{q}, \frac{1}{t} = 1 - \frac{p}{q}$$
$$= \int (|f|^q)^{p/q} 1 d\mu \leq \left| \int |f|^q \right|^{1/s} \mu(X)^{1/t}$$

□

Definition: 5.5: l^p Space

For $p \in (0, \infty)$

$$l^p = L^P(\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{counting measure}) = \left\{ (x_n) : \|x\| = \left(\sum |x_n|^p \right)^{1/p} < \infty \right\}$$

Lemma: 5.3: Monotonicty of l^p

If $0 < p < q \leq \infty$, then $\|x\|_q \leq \|x\|_p$, $L_p \subset L_q$.

Proof. If $q = \infty$,

$$\left(\sum |x_n|^p \right)^{1/p} \geq \sup_n (|x_n|^p)^{1/p} = \|x\|_\infty$$

If $0 < p < q < \infty$,

$$\|x\|_q = \left(\sum |x_n|^q \right)^{1/q} \leq \left(\sum |x_n|^p \|x\|_\infty^{q-p} \right)^{1/q} = \|x\|_\infty^{1-\frac{p}{q}} \|x\|_p^{\frac{p}{q}} \leq \|x\|_p^{1-\frac{p}{q}} \|x\|_p^{\frac{p}{q}} = \|x\|_p$$

□

Theorem: 5.7: Interpolation

Let $0 < p < q < r \leq \infty$. Let $\lambda \in (0,1)$ s.t. $\frac{1}{q} = (1-\lambda)\frac{1}{r} + \lambda\frac{1}{p}$, *i.e.* $\lambda = \frac{\frac{1}{q} - \frac{1}{r}}{\frac{1}{p} - \frac{1}{r}}$. Then $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$. Therefore, $L^p \cap L^r \subset L^q$.

Proof.

$$\|f\|_p^p = \int |f|^q = \int |f|^{\lambda q} |f|^{(1-\lambda)q}$$

Set $\lambda q s = p, (1-\lambda)qt = r, \frac{1}{s} + \frac{1}{t} = \frac{\lambda q}{p} + \frac{(1-\lambda)q}{r} = 1$. Then Apply Theorem 5.3, we get

$$\|f\|_p^p \leq \left(\int |f|^p \right)^{1/s} \left(\int |f|^r \right)^{1/t}$$
$$= \|f\|_p^{p/s} \|f\|_r^{r/t} = \|f\|_p^{\lambda q} \|f\|_r^{(1-\lambda)q}$$

Then take q-th root on both sides. □

Lemma: 5.4:

Let $0 < p < q < r \le \infty$. Then $L^q \subset L^p + L^r$.

Proof. Let $f = \chi_{\{|f(x)| \le 1\}} f + \chi_{\{|f(x)|>1\}} f := f_1 + f_2$. $f_1 \in L^r$, $f_2 \in L^p$. $\square$

5.1 Dual L^p-Spaces

dual-lp Let $p \in [1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $g \in L^q$. Define $\phi_g : L^p \to \mathbb{F}$, $f \mapsto \int fg d\mu$. Then

$$\begin{aligned} \|\phi_g\| &= \sup_{\|f\|_p = 1} |\phi_g| = \sup_{\|f\|_p=1} \left| \int fg \right| \\ &\le \sup_{\|f\|_p=1} \|f\|_p \|g\|_q = \|g\|_q \end{aligned}$$

Theorem: 5.8:

Let $p \in [1,\infty)$, then for $g \in L^q$, $\|\phi_g\| = \|g\|_q$, same holds if $p = 1$ and μ is semi-finite.

Proof. Idea: we need equality in Theorem 5.3 or $|f|^q = \lambda |g|^q$

Suppose $p \in (1,\infty)$, $\|g\|_q = 1$ after normalization. Set $f = \overline{\operatorname{sgn}(g)} |g|^{q/p}$ where $\frac{q}{p} = p - 1$.
Then $\left| \int fg \right| = \int |g|^q = \|g\|_q^q = 1$. This means that $\|\phi_g\| \ge 1$.

If $p = 1$, $\forall \epsilon > 0$, define $E = \{x : |g(x)| > \|g\|_\infty - \epsilon\}$. E has positive measure.
Then $\exists F \subset E$ s.t. $\mu(F) \in (0,\infty)$. Let $f = \frac{1}{\mu(F)} \chi_F \overline{\operatorname{sgn}(g)}$.
$\|f\|_1 = 1$ and $\displaystyle\int fg = \frac{1}{\mu(F)} \int_F |g| \ge \|g\|_\infty - \epsilon$. Hence $\|\phi_g\| \ge \|g\|_\infty$. $\square$

Remark 25. $g \mapsto \phi_g$, $L^q \to (L^q)^*$ is an isometry (preserves norms).

Theorem: 5.9:

Let g be measurable, $p \in [1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, and
1. $S_g = \{g(x) \neq 0\}$ is σ-finite
2. $fg \in L^1$ for all $f \in \Sigma$ (L^p simple functions)
3. $M_q(g) = \sup \left\{ \left| \int fg \right| : f \in \Sigma, \|f\|_p = 1 \right\} < \infty$

i.e. bounded linear operators on a dense subset. Then $g \in L^q$ and $\|g\|_q = M_q(g)$.

Proof. Claim: Suppose $\exists E$ s.t. $\mu(E) < \infty$, $f|_{E^C} = 0$, $f \in L^\infty \cap L^p$ and $\|f\|_p = 1$. Then $|\int fg| \le M_q(g)$.

Take $f_n \in \Sigma$ s.t. $f_n \to f$ a.e. and $|f_n| \nearrow |f|$. Then $\left| \int fg \right| \le M_q(g)$ by Theorem 2.1 and 2.2.

Supose $q < \infty$, since S_g is σ-finite, $\exists E_n \nearrow S_g$, $\mu(E_n) < \infty$.
Let $\phi_n \in \Sigma$, $\phi_n \to g$ a.e. $|\phi_n| \nearrow |g|$. Let $g_n = \chi_{E_n} \phi_n$, $f_n = \frac{|g_n|^{q-1} \overline{\operatorname{sgn}(g)}}{\|g_n\|_q^{q/p}}$.
Then $\|f_n\|_p = 1$ and $f_n \in L^\infty$ supported on E_n.

$$M_q(g) \ge \left| \int f_n g \right| = \frac{\int |g_n|^{q-1} |g|}{\|g_n\|_q^{q/p}} \ge \frac{\int |g_n|^{q-1}}{\|g_n\|_q^{q/p}} = \|g_n\|_q$$

Take $n \to \infty$, by Theorem 2.1, $M_q(g) = \|g\|_q$.

Let $q = \infty$. For $\epsilon > 0$, we want to show $\mu(\{x : |g(x)| > M_\infty(g) + \epsilon\}) = 0$.
Suppose not, $\exists E \subset \{x : |g(x)| > M_\infty(g) + \epsilon\}$ s.t. $\mu(E) \in (0, \infty)$. Set $f = \frac{1}{\mu(E)} \chi_E \overline{\text{sgn}(g)}$.
$M_\infty(g) \geq |\int fg| \geq M_\infty + \epsilon$ Contradiction. $\square$

Corollary 21. $\|g\|_q = \sup\left\{\left|\int fg\right| : \|f\|_p = 1\right\}$

Theorem: 5.10:

Suppose $p \in (1, \infty)$ or $p = 1$ and μ is σ-finite. Then if $\phi \in (L^p)^*$, $\exists g \in L^q$ s.t. $\phi = \phi_g$.

Proof. Suppose $\mu(E) < \infty$. Given A_n pairwise disjoint. $\sum_{n=1}^{\infty} \chi_{A_n} \in L^p$, by Theorem 2.2, $\sum_{n=1}^{m} \chi_{A_n} \xrightarrow{L^p} \sum_{n=1}^{\infty} \chi_{A_n}$.
Then $\phi(\sum \chi_{A_n}) = \sum \phi(\chi_{A_n})$. Hence the additivity is satisfied and $\nu(E) = \phi(\chi_E)$ is a complex meausre.
$\nu \ll \mu$, since $\mu(E) = 0 \Leftrightarrow \chi_E = 0$ a.e.
Let $g = \frac{d\nu}{d\mu}$, $\int_E g = \phi(\chi_E)$.
Hence, $\forall f \in \Sigma$, $\int fg = \phi(f)$, $\|g\|_q = M_q(g) \leq \|\phi\|$.

Suppose μ is σ-finite. *i.e.* $\exists X_n \nearrow X$ s.t. $\mu(X_n) < \infty$. $\phi_n = \phi(\chi_{X_n} f)$ is bounded on $L^p(X_n, \mu|_{X_n})$.
$\exists g_n \in L^p(X_n, \mu)$ s.t. $\int_{X_n} g_m f = \phi(\chi_{X_n} f) = \int_{X_n} g_n f$, so $X_n g_m = g_n$.
Let $g_n \to g$, $\|\phi\| \geq \|g_n\|_q \to \|g\|_q$. If $f \in L^p$ and $\|f\|_p = 1$, then

$$\int fg = \lim \int \chi_{X_n} fg = \lim \int fg_n \leq \lim \|g_n\|_q = \|g\|_q\,,$$

and $\|\phi\| = \sup_{\|f\|=1} \int fg \leq \|g\|_q$.

Now consider $p > 1$, μ arbitrary.
For every σ-finite E, $\exists g_E$ s.t. $\phi(\chi_E f) = \int fg$ for $f \in L^p$.
Let $M = \sup\left\{\|g_E\|_q : E \text{ is } \sigma\text{-finite}\right\}$. Take E_n s.t. $\|g_{E_m}\| \nearrow M$, $F = \cup E_n$, $g_{\cup_{n=1}^m E_n} \to g_F$.
Let A be any σ-finite set, then $g_{A\cup F} = g_F + g_{A\setminus F}$.

$$\phi(\chi_{A\cup F} f) = \phi(\chi_F f) + \phi(\chi_{A\setminus F} f) = \int g_F f + \int g_{A\setminus F} f$$
$$\|g_{A\cup F}\|_q^q = \|g_F\|_q^q + \|g_{A\setminus F}\|_q^q$$

Then $\|g_{A\setminus F}\|_q^q = 0$, otherwise contradicts maximality of F.

Finally, if $f \in L^p$, then $S_f = \{x : f(x) \neq 0\}$ is σ-finite, $\int fg_F = \int_{S_f} fg_F = \int_F fg_F$. $\square$

Definition: 5.6: Weak Convergence

Let $p \in [1, \infty)$, $f_n \to f$ weakly in L^p if and only if $\int f_n g \to \int fg$ for all $g \in L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 22. *If $p \in (1, \infty)$, L^p is reflexive.*

Lemma: 5.5: Riemann-Lebesgue

$\forall E$ s.t. $m(E) < \infty$, $\int_E \sin(nx) \to 0$ as $n \to \infty$. Hence $\forall g \in \Sigma$, $\int_E g \sin(nx) \to 0$ as $n \to \infty$. Same applies for all $g \in L^q$, $q \in [1, \infty)$.

Remark 26. $\sin(nx) \to 0$ weakly in $L^p(\mathbb{R})$ for $p \in (1, \infty)$ and $\sin(nx) \to 0$ weakly in $L^1([0, 2\pi)$. However, $\sin(nx)$ diverges a.e.

The Spaces L^1 and L^∞. The dual of L^1 is L^∞, but the dual of L^∞ has elements that are not L^1. We use two examples from l^1 and l^∞ to illustrate.

Example: Consider $\phi_N : l^\infty \to \mathbb{R}$, $\phi_N((x_n)) = \dfrac{1}{N}\displaystyle\sum_{n=1}^{N} x_n$, $\|\phi_N\| \leq 1$, $\phi_N \in (l^\infty)^*$.

By Theorem 4.5, $\exists n_k \to \infty$ s.t. $\phi_{n_k} \to \phi$ in weak-$*$, *i.e.* $\phi_N(x) \to \phi(x)$ for all $x \in l^\infty$.

For every coordinate vector e_n, $\phi(e_n) = \displaystyle\lim_{k\to\infty} \frac{1}{n_k} 1 = 0$, $\phi(\mathbf{1}) = 1$.

Consider $E \mapsto \phi(\chi_E)$, well-defined for all finite sets, but cannot be extended to a σ-additive set function. Therefore, $\phi \notin l^1$.

Example: Let $c_0 \left\{x \in l^\infty : \displaystyle\lim_{n\to\infty} x_n = 0\right\}$, $\phi \in (c_0)^*$, $\|\phi\| = 1$ and $a_n = \phi(e_n)$. Then

$$\phi\left((x_k)_{k=1}^N\right) = \left|\sum_{k=1}^{N} a_k x_k\right| \leq \sum_{k=1}^{N} |a_k| \, \|x_k\|_\infty$$

By choosing x_k, equality holds. Then $\sum |a_k| \leq 1$. This means that $\phi(x) = \int_{\mathbb{N}} ax$, for $a \in L^1(\mathbb{N}) = l^1$. Therefore, $(c_0)^* \cong l^1$.

***Definition:* 5.7:**

$C_0(X) \subset \underline{C_b(X)}$ s.t. $\forall \epsilon, \{x : f(x) \geq \epsilon\}$ is compact. If X is locally compact Hausdorff (LCH), then $C_0(X) = \overline{C_C(X)}$, $\mathcal{M}(X, \mathcal{B}) = \{$all finite signed /complex measures on $(X, \mathcal{B})\}$. $l^1 = \mathcal{M}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

6 Radon Measure

Assume X is LCH. If μ is a positive measure, finite on compact sets, then $I_\mu(f) = \int f d\mu$ for any $f \in C_C(X)$.

Definition: 6.1: Positive Integration Functional

The linear functional $I : C_C(X) \to \mathbb{R}$ is positive if $I(f) \geq 0$ for $f \geq 0$.

Remark 27. $f \geq g \Rightarrow I(f) \geq I(g)$.

Definition: 6.2: Radon Measure

A Borel measure μ is Radon if

1. $\mu(K) < \infty$, $\forall K$ compact.
2. Outer regular: $\mu(E) = \inf\{\mu(U) : E \subset U\}$, U open.
3. Inner regular for open sets: $\mu(U) = \sup\{\mu(K) : K \subset U\}$ for U open, K compact.

Definition: 6.3: Subordinate

Say $f \prec U$ for U open if $0 \leq f \leq 1$, $f \in C_C$ and $\text{supp} f \subset U$.

Theorem: 6.1: Riesz Representation Theorem for Positive Functionals

Let X be LCH, $I : C_C(X) \to \mathbb{R}$ be positive, then there exists a unique Radon measure μ s.t. $I = I_\mu$.

Proof. The proof consists of 4 steps:

1. Uniqueness
2. Construct μ
3. μ is Radon
4. $I = I_\mu$

□

Lemma: 6.1:

Let μ, ν be Radon, then $I_\mu = I_\nu \Rightarrow \mu = \nu$.

Proof. Let U be open. μ is Radon, then $\exists K_n \subset U$ s.t. $\mu(K_n) \to \mu(U)$.
Since X is LCH, by Lemma 4.1, $\exists f_n \prec U$ s.t. $f_n|_{K_n} = 1$.

$$\mu(K_n) = \int \chi_{K_n} d\mu \leq \int f_n d\mu = I_\mu(f_n) = I_\nu(f_n) = \int f_n d\nu \leq \nu(U)$$

As $n \to \infty$, $\mu(U) \leq \nu(U)$. By symmetry, we get $\mu(U) = \nu(U)$. Then $\mu = \nu$ by outer regularity. □

Theorem: 6.2: Partition of Unity

Let X be LCH, K compact, $\{U_\alpha\}$ covers K. Then there exists $U_1, ..., U_n$ and $f_1, ..., f_n$ with $f_i \prec U_i$ s.t. $\sum f_i|_K = 1$.

Construct μ . Let $\mu(U) = \sup\{I(f) : f \prec U\}$.
For any $E \subset X$, U open, let $\mu^*(E) = \inf\{\mu(U) : E \subset U\}$. We want to show that μ^* is an outer measure.
Let $U = \bigcup_{n=1}^{\infty} U_n$. Take $f \prec U$.

Since $\{U_n\}$ convers $\text{supp}(f)$, by Theorem 6.2, $\exists V_1, ..., V_k \in \{U_n\}$ and $f_i \prec V_i$ s.t. $\sum_{i=1}^{k} f_i|_{\text{supp} f} = 1$.

$$I(f) \le I\left(\sum_{i=1}^{k} f_i\right) = \sum_{i=1}^{k} I(f_i) \le \sum_{i=1}^{k} \mu(V_i) \le \sum_{n=1}^{\infty} \mu(U_n)$$

Take supremum in f, then $\mu(U) \le \sum \mu(U_n)$. Therefore, μ^* is subadditive from $2^{-k}\epsilon$ proof.

Let $\mu = \mu^*|_{\mathcal{M}_{\mu^*}}$, we want to show that $\mathcal{B}_X \subset \mathcal{M}_{\mu^*}$.
Let U be open, $E \subset X$.

1. $\forall \epsilon > 0$, $\exists V$ open s.t. $\mu(V) < \mu^*(E) + \epsilon$, $E \subset V$.
2. Let $g \prec U \cap V$ s.t. $I(g) > \mu(U \cap V) - \epsilon$. Let $K = \text{supp}(g)$
3. $\exists f \prec V \setminus K$ s.t. $I(f) > \mu(V \setminus K) - \epsilon$, $f + g \prec V$.

Then

$$\begin{aligned}
\mu^*(U \cap E) + \mu^*(E \setminus U) &\le \mu(U \cap V) + \mu(V \setminus U) \\
&\mu(U \cap V) + \mu(V \setminus K) \\
&\le I(f) + I(g) + 2\epsilon = I(f + g) + 2\epsilon \\
&\le \mu(U) + 2\epsilon \le \mu * (E) + 3\epsilon.
\end{aligned}$$

***Lemma:* 6.2:**

1. $f \in C_C$, $f|_K \ge 1$, K compact, then $I(f) \ge \mu(K)$
2. $f \in C_C$, $0 \le f \le 1$, then $I(f) \le \mu(\text{supp}(f))$
3. μ is Radon.

Proof. 1. Let $U_\epsilon = \{x : f(x) > 1 - \epsilon\}$. U_ϵ is open and $K \subset U_\epsilon$. Let $g \prec U_\epsilon$.
$f \ge fg > (1 - \epsilon)g$, then $I(f) \ge (1 - \epsilon)I(g)$. Take supremum over g. $I(f) \ge (1 - \epsilon)\mu(U_\epsilon) \ge (1 - \epsilon)\mu(K)$.

2. Let U be open s.t. $\text{supp}(f) \subset U$, $f \prec U$, hence $I(f) \le \mu(U)$.
By construction, for K compact, $\mu(K) = \mu^*(K) = \inf\{\mu(U) : K \subset U\}$, so $I(f) \le \mu(\text{supp}(f))$.

3. μ is outer regular by definition. Let U be open. $\mu(U) = \sup\{I(f) : f \prec U\}$. $\forall \epsilon$, $\exists f \prec u$ s.t.

$$\mu(\text{supp}(f)) \ge I(f) \ge \mu(U) - \epsilon$$

Then $\sup\{\mu(K) : K \subset U\} \ge \mu(U)$. $\square$

Finally, we show $I = I_\mu$.
It suffices to consider $\|f\| = 1$, $f \ge 0$. Let $E_i = \left\{x : |f(x)| \ge \frac{i}{n}\right\}$ be Lebesgue slices.

Let $f_i = \begin{cases} 0, x \notin E_i \\ \frac{1}{n}, x \in E_{i+1} \\ f(x) - \frac{i}{n}, x \in E_i \setminus E_{i+1} \end{cases}$. Then $f = \sum_{i=0}^{n-1} f_i$.

$$\begin{aligned}
&\frac{1}{n}\mu(E_{i+1}) \leq I(f_i) \leq \frac{1}{n}\mu(E_i) \\
&\sum_{i=0}^{n-1} \mu(E_{i+1}) \leq I(f) = \sum_{i=0}^{n-1} \frac{1}{n} I(f_i) \leq \sum_{i=0}^{n-1} \mu(E_i) = \sum_{j=0}^{n-1} \mu(E_j \setminus E_{j+1}) \leq \int \left(f(x) - \frac{1}{n}\right) d\mu
\end{aligned}$$

Remark 28. Take Riemann integral I, then I is a positive functional and the Lebesgue measure m is the corresponding Radon measure.

Remark 29. Baire σ-algebra $\mathcal{B}_0$ is the smallest σ-algebra s.t. all $C_C(X)$ functions are measurable. The preimage of a compact set under $C_C(X)$ functions are G_δ compact sets, so $\mathcal{B}_0$ is generated by all G_δ compact sets. If X is separable, then all compact sets are G_δ. We don't require Borel σ-algebra for Theorem 6.1, $\mathcal{B}_0$ is enough.

***Proposition:* 6.1:**

Let μ be Radon, then μ is inner regular at all σ-finite sets.

Proof. Consider finite sets $\mu(E) < \infty$.
Let U be open s.t. $E \subset U$, $\mu(U \setminus E) < \epsilon$.
There exists F compact, $F \subset U$ s.t. $\mu(U \setminus F) < \epsilon$.
$\exists V$ open, $U \setminus E \subset V$ s.t. $\mu(V) - \mu(U \setminus E) < \epsilon$.
Let $K = F \setminus U \subset E$, K is compact.

$$\begin{aligned}
\mu(E \setminus K) &= \mu(E) - \mu(F \setminus V) \\
&= \mu(E) - \mu(F) + \mu(F \cap V) \\
&\leq \mu(E) - \mu(U) + \mu(V) + \epsilon \\
&\leq 2\epsilon.
\end{aligned}$$

□

***Proposition:* 6.2:**

Suppose X is σ-compact (countable union of compact sets), then any Borel measures finite on compact sets is Radon and Borel regular.

Proof. Let μ be Borel, $\mu(K) < \infty$ for all K compact.
Let $I_\mu(f) = \int f d\mu$. By Theorem 6.1, $\exists \nu$ s.t. $\int f d\mu = \int f d\nu$ for all $f \in C_C(X)$.
Both μ, ν are inner regular at open sets.
$\forall U$ open, $\exists K_n$ compact s.t. $K_n \nearrow U$.
Then $\exists f_n \prec U$ s.t. $f_n|_{K_n} = 1$,

$$\mu(U) = \lim_{n\to\infty} \mu(K_n) \leq \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int f_n d\nu \leq \nu(U)$$

Similarly, $\nu(U) \leq \mu(U)$. Hence $\mu(U) = \nu(U)$ for any U open.
Let $E \in \mathcal{B}_X$, $\nu(E) < \infty$. By Prop. 6.1, $\exists K_n \subset E \subset U_n$, K_n compact, U_n open s.t. $\nu(U_n \setminus K_n) \to 0$.
Therefore, $\mu(U_n \setminus K_n) \to 0$. μ is inner regular as well. Hence $\mu = \nu$ by Theorem 6.1. □

Definition: 6.4: Singed/Complex Radon Measure

$M(X,\mathbb{R}), M(X,\mathbb{C})$ are the finite signed/complex Radon measures on X. $C_C(X)$ is incomplete, $C_0(X)$ is complete, but $(C_C(X))^* \cong (C_0(X))^*$ isometrically isomorphic.

Proposition: 6.3:

If $I \in (C_C(X,\mathbb{R}))^*$, then $\exists I_\pm \in (C_C(X,\mathbb{R}))^*$, $I_\pm$ is positive and $I = I_+ - I_-$.

Proof. If $f \geq 0$. Let $I^+ = \sup\{I(g) : 0 \leq g \leq f\}$. I^+ is positive. $I^- = I^+ - I$ is positive by definition. Need to show that I^+ is linear on $f \geq 0$ and extends linearly to all functions by Theorem 6.1.

If $I \in (C_C(X,\mathbb{C}))^*$, write $I = (\mathrm{Re}I)^+ - (\mathrm{Re}I)^- + i(\mathrm{Im}I)^+ - i(\mathrm{Im}I)^-$, then $I = I_\nu$ for $\nu \in M(X,\mathbb{C})$. □

Theorem: 6.3: Riesz Representation Theorem 2

The mapping $\mu \mapsto I_\mu$ is an isometric isomorphism $M(X,\mathbb{C}) \to (C_X(X,\mathbb{C}))^*$.

Proof.

$$\left|\int f d\mu\right| \leq \int |f| d|\mu| \leq \|f\| \, |\mu|(X) = \|f\| \, \|\mu\|$$

To see equality can be reached, write $d\mu = hd|\mu|$. Approximate h by simple functions over open sets, then by C_C functions. □

Theorem: 6.4: Krylov-Bogolyubov

Let X be a compact metric space (guaranteed to be LCH), $T : X \to X$ be continuous. Then there exists an invariant probability measure $\mu(E) = \mu(T^{-1}E) = T_X\mu(E)$ for all E Borel.

Proof. Let ν be any probability measure,

$$\mu_n = \frac{1}{n}\left(\nu + \nu \circ T^{-1} + \nu \circ T^{-2} + \cdots + \nu \circ T^{-(n-1)}\right) = \frac{1}{n}\sum_{k=0}^{n-1} T_X^{k-1}\nu$$

Unit ball of $M(X)$ is weak-$*$ compact. There exists $n_k \to \infty$ s.t. $\mu_{n_k} \to \mu$ weak-$*$. Let $f \in C(X)$,

$$\begin{aligned}\int f \circ T d\mu - \int f d\mu &= \lim_{k\to\infty} \frac{1}{n_k} \int \sum_{j=0}^{n_k-1} f \circ T^{j+1} d\nu - \frac{1}{n_k} \int \sum_{j=0}^{n_k-1} f \circ T^j d\nu \\ &= \lim_{k\to\infty} \frac{1}{n_k} \int \left(f \circ T^{n_k} - f\right) d\nu = 0\end{aligned}$$

□

7 Functional Analysis (MAT1001)

Theorem: 7.1: Inverse Function Theorem

Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^n$ a C^1 map, $x_0 \in U$. If $Df(x_0)$ is invertible, then f is locally bijective. *i.e.* $\exists \epsilon, \delta > 0$ s.t. $\forall y \in B_\delta(f(x_0))$, $\exists! x \in B_\epsilon(x_0)$ s.t. $f(x) = y$.

Recall some examples of Banach spaces:

- Finite dimensional normed vector spaces
- L^p space
- $BC^k(\mathbb{R}^n) = \left\{ f \in C^k(\mathbb{R}^n) : \|f\|_{C^k} = \sum_{i=0}^{k} \sup_x |f^{(i)}(x)| < \infty \right\}$.

Given Banach spaces X, Y, a linear map $L : X \to Y$ is bounded if $\exists C > 0$ s.t. $\|Lx\|_Y \leq C \|x\|_X$. $\|L\|_{op}$ =inf of all such C. $\mathcal{L}(X, Y) = \{\text{bounded linear maps } L : X \to Y\}$. L is invertible if L^{-1} exists and is bounded.

Definition: 7.1: Frechet Derivative

Let X be a Banach space, $U \subset X$ be open, $F : U \to Y$ be a continuous map. $x_0 \in U$. We say F is Frechet differentiable at x_0 if $\exists L \in \mathcal{L}(X, Y)$ s.t. $\|F(x_0 + h) - F(x_0) - Lh\| = o(\|h\|)$ as $\|h\| \to 0$. In this case, $L = DF(x_0)$ is the Frechet derivative of F at x_0. F is C^1 if the map $x \mapsto DF(x)$ is continuous.

Theorem: 7.2: Inverse Function Theorem (Banach Space)

Let X be a Banach space, $U \subset \mathbb{R}^n$, $F : U \to Y$ be a C^1 map, $x_0 \in U$, then $DF(x_0)$ is invertible.

Proof. WLOG, let $x_0 = F(x_0) = 0$, $U = B_\epsilon(x_0)$. Let $y \in B_\delta(F(x_0))$. We want to solve for $F(x) = y$ for $x \in U$.
Idea: use Newton's iteration. Let $x_0 = 0$, $L = DF(x_0)$.

$$\begin{aligned} x_{n+1} &= x_n + L^{-1}(y - F(x_n)) \\ x_{n+1} - x_n &= x_n - x_{n-1} - L^{-1}(F(x_n) - F(x_{n-1})) \\ &= L^{-1}\left[L(x_n - x_{n-1}) - (F(x_n) - F(x_{n-1}))\right] \end{aligned}$$

Apply MVT, we get $F(x_n) - F(x_{n-1}) = DF(\hat{x}_n)(x_n - x_{n-1})$ for some $\hat{x}_n$.
Then $x_{n+1} - x_n = L^{-1}(L - DF(\hat{x}_n))(x_n - x_{n-1})$.
Let $A = \left\|L^{-1}\right\|$, then $\|x_{n+1} - x_n\| \leq A \|L - DF(\hat{x}_n)\| \|x_n - x_{n-1}\|$.
Since $x_0 = 0$ and $\|y\| \leq \delta$, $\|x_1\| = \left\|L^{-1}y\right\| \leq A\delta$.
Pick ϵ s.t. $\sup_{x \in U} \|L - DF(x)\| \leq \dfrac{1}{2A}$, then pick $\delta < \frac{\epsilon}{2A} \Rightarrow \|x_1\| < \frac{\epsilon}{2}$.
$\Rightarrow \|x_2 - x_1\| < \frac{\epsilon}{4} \Rightarrow \|x_3 - x_2\| < \frac{\epsilon}{8} \Rightarrow \cdots$
x_n converges as a Cauchy sequence in Banach space. $\square$

Theorem: 7.3: Baire Category Theorem

Let X be a complete metric space, then

1. $U_i \subset X$ are open dense subsets for $i = 1, 2, ...$, then $\bigcap_{i=1}^{\infty} U_i$ is dense
2. If $X = \bigcup_{i=1}^{\infty} C_i$, $C_i \subset X$ subsets, then at least one of $\overline{C_i}$ has non-empty interior

Proof. 1. Let $U_i \subset X$ be open and dense. Let $W \subset X$ be open and non-empty, $\exists x_1 \in W \cap U_1$, since U_1 is dense in X. $\exists r_1 > 0$, $\overline{B(x_1, r_1)} \subset W \cap U_1$. Let $W_2 = B(x_1, r_1)$. Iteratively, we can find $x_k \in W \cap U_1 \cap U_2 \cap \cdots \cap U_k$, and $W_{k+1} = B(x_k, r_k) \subset W \cap U_1 \cap \cdots \cap U_k$.
Make sure we pick $r_k \to 0$, so we get a Cauchy sequence x_k. Since X is complete, $x_k \to x_\infty$ exists. Since $x_n \in \overline{B(x_k, r_k)}$ for $n > k$, $x_\infty \in \overline{B(x_k, r_k)} \subset B(x_{k-1}, r_{k-1})$, $x_\infty \in W \cap \bigcup_{i=1}^{\infty} U_i$.

2. Assume $X = \bigcup_{i=1}^{\infty} C_i$ s.t. C_is are nowhere dense, *i.e.* $\overline{C_i}$ has empty interior, then $\overline{C_i}^C$ are open and dense. $\bigcap_{i=1}^{\infty} \overline{C_i}^C$ is dense, but it also needs to be empty. Contradiction. □

Definition: 7.2: Open Map

Let X, Y be topological spaces, $f : X \to Y$ is open if $f(U)$ is open for all open $U \subset X$.

Examples

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not open, because $f((-1, 1)) = [0, 1)$ is not open.
2. Any smooth curve $f : [0, 1] \to \mathbb{R}^2$ is not open, because the image does not contain open sets.
3. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x$ is open. Check a set of basis, $f(B_r(x, y)) = B_r(x)$.

Theorem: 7.4: Open Mapping Theorem

Let $T : X \to Y$ be a bounded linear operator between Banach spaces. If T is surjective, then T is an open map.

Proof. Assume $T \in \mathcal{L}(X, Y)$ is surjective. It suffices to check that $T(B_r(x))$ contains an open ball $B_r(T(x))$.

$$U = \bigcup_{x \in U} B_r(x) \Rightarrow T(U) = \bigcup_{x \in U} T(B_r(x)) \supset \bigcup_{x \in U} B_r(T(x)) \supset T(U)$$

By linearity, $T(B_r(x)) = T(x + B_r(0)) = Tx + rT(B_1(0))$.
We now show that $\exists B(0, c) \subset T(B_1(0))$.

1. $B(0, c) \subset \overline{T(B_1(0))}$.

T is surjective, so $\bigcup_{n=1}^{\infty} T(B_n(0)) = Y$.
By Theorem 7.3, one of $\overline{T(B_n(0))} \supset B(y, r)$, so $\overline{T(B_1(0))} \supset B(y, r)$ by linearity.
Notice $\overline{T(B_1(0))}$ is symmetric ($x \in \overline{T(B_1(0))} \Leftrightarrow -x \in \overline{T(B_1(0))}$) and convex ($x, y \in \overline{T(B_1(0))} \Rightarrow tx + (1 -$

$t)y \in \overline{T(B_1(0))}$).
Symmetry $\Rightarrow B(-y,r) \in \overline{T(B_1(0))}$, convexity $\Rightarrow B(0,\frac{1}{2}r) = \frac{1}{2}(-y + B(y,r)) \subset \overline{T(B_1(0))}$.

2. $B(0,c) \subset T(B_1(0))$.
By 1., $B(0,c) \subset \overline{T(B_1(0))}$, by scaling, $B(0,cr) \subset \overline{T(B_r(0))}$.
Take $y \in B(0,c)$, $\forall \epsilon > 0$, $\exists x \in B_1(0)$ s.t. $\|y - Tx\| < \epsilon$.
Pick $\epsilon = \frac{c}{2}$, $\exists x_1 \in B_1(0)$ s.t. $\|y - Tx_1\| < \frac{c}{2}$, $y_1 = y - Tx_1 \in B(0,\frac{c}{2})$.
$\Rightarrow \forall \epsilon > 0$, $\exists B_{\frac{1}{2}}(0)$ s.t. $\|y_1 - Tx\| < \epsilon$. Pick $\epsilon = \frac{c}{4}$, $\exists x_2 \in B_{\frac{1}{2}}(0)$ s.t. $\|y_1 - Tx_2\| < \frac{c}{4}$.
Repeat, $x_k \in B_{\frac{1}{2^{k-1}}}(0)$ s.t. $\|y_{k-1} - Tx_k\| < \frac{c}{2^k}$, $y_k = y_{k-1} - Tx_k \to 0$.
$\Rightarrow y_k = y - T(x_1 + x_2 + \cdots + x_k) \Rightarrow \|x_k\| \leq \frac{1}{2^{k-1}}$, $\|\sum x_k\| \leq 2$, $Tx = y$ for $x \in B_2(0)$.
By rescaling, $B(0,c) \subset T(B_1(0))$. ☐

Definition: 7.3: Closed Map

Let X, Y be Banach spaces, $T : X \to Y$ is closed if $\Gamma(T) = \text{graph}(T) = \{(x, Tx) : x \in X\} \subset X \times Y$ is a closed set. *i.e.* if $\exists (x_k, Tx_k)$ s.t. $x_k \to x$ and $Tx_k \to y$, then $(x,y) \in \Gamma(T)$, $y = Tx$.

Continuous maps are always closed, but the converse is not necessarily true.

Theorem: 7.5: Closed Graph Theorem

Let X, Y be Banach spaces. $T : X \to Y$ is linear, then T is bounded if and only if T is closed.

Proof. Assume T is closed. Consider $(X, \|\cdot\|_T)$ with $\|x\|_T = \|x\| + \|Tx\|$ graph norm.
T is closed $\Rightarrow (X, \|\cdot\|_T)$ is complete.
Because x_k is Cauchy in $\|\cdot\|_T \Leftrightarrow x_k$ is Cauchy and Tx_k is Cauchy $\Rightarrow x_k \to x$ and $Tx_k \to y$. Closedness $\Rightarrow$ $y = Tx$.
Since $\|x\| \leq \|x\|_T$, $Id : (X, \|\cdot\|_T) \to (X, \|\cdot\|)$ is bounded. Id is bijective, by Theorem 7.4, $\exists C > 0$, $\|x\|_T \leq C\|x\|$ and $\|Tx\| \leq C\|x\|$, the map is invertible and the inverse is bounded. ☐

Theorem: 7.6: Uniform Boundedness Theorem

Let X, Y be Banach spaces, $\mathcal{A} \subset \mathcal{L}(X,Y)$ collection of bounded operators, then the following are equivalent:

1. $\forall x \in X$, $\sup\limits_{T \in \mathcal{A}} \|Tx\| < \infty$
2. $\sup\limits_{T \in \mathcal{A}} \|T\| < \infty$

Proof. $2 \Rightarrow 1$ is direct from definition.

$1 \Rightarrow 2$. Let $E_n \subset X$, $E_n = \bigcap\limits_{T \in \mathcal{A}} T^{-1}(\overline{B(0,n)}) = \{x \in X : \|Tx\| \leq n, \forall T \in \mathcal{A}\}$.

Then E_n is closed and $\bigcup\limits_{n=1}^{\infty} E_n = X$ by 1.
By Theorem 7.3, one of E_n contains $\overline{B(x,r)}$.
Notice E_n is symmetric and convex, $E_n \supset B(0,r)$.
$\Rightarrow \forall x \in B(0,r')$, $\|Tx\| \leq n$, $\forall T \in \mathcal{A}$, $\|T\| \leq \frac{n}{r'}$. ☐

Corollary 23. *Pointwise limit of continous linear maps are continuous.*

Proof. $T_i \in \mathcal{L}(X,Y)$, $\forall x \in X$, $T_i x \to Tx$ as $i \to \infty$.
By Theorem 7.6, $\|T_i\| \le B \Rightarrow \forall x \in X$, $\|T_i x\| \le B\|x\| \Rightarrow \forall x \in X$, $\|Tx\| \le B\|x\|$. □

Recall the definition of linear functionals in Definition 4.12. A linear functional is a linear bounded map $f : X \to \mathbb{R}$ or $\mathbb{C}$, with $(X, \|\cdot\|)$ a normed vector space. $f \in \mathcal{L}(X, \mathbb{R})$.
Example:

1. $X = L^1(\mathbb{R})$, $I_{a,b} : X \to \mathbb{R}$ by $I_{a,b}(f) = \int_a^b f(x)dx$
2. $X = C^0([-1,1])$, $ev_0(f) = f(0)$, $|f(0)| \le \|f\|_{C^0} = \sup\limits_{x\in[-1,1]} |f(x)|$
3. $X = C^1([0,1])$, $\|f\|_{C^1} = \|f\|_{C^0} + \sup\limits_{x\in[-1,1]} |f'(x)|$, $Lf = f'(0)$.

Hahn-Banach Theorem (Theorem 4.2) guarantees existence of many linear functionals.

Corollary 24. *If X is a normed vector space and $f : M \to \mathbb{R}$ is a bounded linear functional on $M \subset X$, then f can be extended to $F \in \mathcal{L}(X, \mathbb{R})$ s.t. $\|f\| = \|F\|$.*

The dual of X is $X^* = \{\text{bounded linear functionals}\} = \mathcal{L}(X, \mathbb{R})$. $\|f\|_{X^*} = \sup\limits_{x\neq 0\in X} \dfrac{|f(x)|}{\|x\|}$.

Fact: X^* is always complete, hence Banach.

Let X^{**} denote the double dual, $X \subset X^{**}$ is an isometric embedding. (See Prop 4.4)

Recall Definition 4.19 (Reflexive Vector Spaces)
Example:

1. Finite dimensional spaces are always reflexive.
2. $(L^p(\mathbb{R}))^* = L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p < \infty$, L^p is reflexive.
3. $(L^1(\mathbb{R}))^* = L^\infty(\mathbb{R})$, but $L^1 \subsetneqq (L^\infty(\mathbb{R}))^*$

***Definition:* 7.4: Completion of Vector Space**

$\overline{X} \subset X^{**}$ is the completion of X.

Note: the completion of L^1 is L^1 itself.

Recall Heine-Borel: $K \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. But if X is infinite dimensional Banach space, $\overline{B_1}$ (closed unit ball) is non-compact.
Example: $X = C^0$ or L^p. $f_i \in X$ s.t. $\|f_i\| \le 1$, $0 < f_i < 1$ for $x \in (i-1, i+1)$, and 0 everywhere else. There is no converging subsequence. The space is bounded but not compact.
Solution is to change the topology

7.1 Weak Topology

Point set topology is $\sigma \subset 2^X$ closed under union and intersection operations.
Let $f_\alpha : X \to Y_\alpha$, $\alpha \in A$, then the topology generated by f_α is the smallest topology s.t. f_α is continuous. σ is generated by $f_\alpha^{-1}(U)$ for U open. *i.e.* we take all these sets, close it under finite intersections and then arbitrary unions.

In this topology, $x_k \to x_\infty$ if and only if $f_\alpha(x_k) \to f_\alpha(x_\infty)$ for all α.

Definition: 7.5: Weak Topology

Weak topology is the weakest topology s.t. $f \in X^* \Rightarrow f$ is continous (all bounded linear functionals are continuous)

$x_k \to x$ in weak topology, $f(x_k) \to f(x), \forall f \in X^*$.

f in the previous example converges in the weak topology of L^p.

Proof. Assume $p > 1$, then $(L^p)^* = L^q$.
Let $f \in L^p$, $g \in L^q$, $\langle f, g\rangle = \int fg$

$$|\langle f_i, g\rangle| = \left|\int gf_i\right| \leq \int_{[i,i+1]} |g| \to 0$$

□

In general, $x_k \to x$ strongly $\Rightarrow x_k \to x$ weakly, but not the other way around.

Notation: $x_k \rightharpoonup x$ for weak convergence.

Definition: 7.6: Weak* Topology

Weak* topology is the topology generated by $x \in X \subset X^{**}$. $f_i \in X^*$ converge in the weak* topology if and only if $f_i(x) \to f(x), \forall x \in X$ (pointwise convergence of functions)

Remark 30. In general, weak*<weak<strong on X^*, but if X is reflexive, then weak*=weak.

Theorem: 7.7: Tychonoff's Theorem

$X = \prod_{\alpha \in A} X_i$. If X_i is compact, then X is compact w.r.t. the product topology, where product topology is the weakest topology s.t. all projection maps $p : X \to X_i$ are continuous. *i.e.* elements in X are $\{x_\alpha\}_{\alpha \in A}$, $x_\alpha \in X_\alpha$.

Theorem: 7.8: Banach-Alaoglu

$\overline{B_1} \subset X^*$ is always compact in weak* topology.

Proof. Consider $\mathbb{R}^X = \{f : X \to \mathbb{R}\}$, $X^* \subset \mathbb{R}^X$.
Topology inherited from product topology on $\mathbb{R}^X$ is just weak*topology.

$$\begin{aligned}
\overline{B_1} &= \{f : X \to \mathbb{R}, f \text{ linear}, |f(x)| \leq \|x\| \text{ for all } x \in X\} \\
&= \bigcap_{x \in X} \{f : X \to \mathbb{R}, f \text{ linear}, |f(x)| \leq \|x\|\} \\
&\subset \bigcap_{x \in X} \hat{x}^{-1}([-\|x\|, \|x\|]) \\
&\subset \prod_{x \in X} [-\|x\|, \|x\|]
\end{aligned}$$

□

In practice, this version of Banach-Alaoglu is not easy to use.
Recall: If X, a topological space, is metrizable, then compact = sequentially compact.

Definition: 7.7: Separable Space

X is separable if there exists countable dense subsets.

Example: finite dimensional spaces and $L^p(\mathbb{R})$ $(p < \infty)$ are separable.

Theorem: 7.9: Improved Banach-Alaoglu

If X is separable, then $\overline{B_1} \subset X^*$ is sequentially compact.

Proof. $\{x_k\}_{k=1}^{\infty} \subset X$ is countable and dense, $f_i \in X^*$ s.t. $|f_i(x)| \leq \|x\|, \forall x \in X$.
For any $x \in X$, there exists subsequence $f_i \in X^*$ s.t. $f_i(x) \to a_i$.
By diagonalization argument, assume this happens at any x_k, $k = 1, 2, ...$
In fact, we can assume f_i converges to f_∞ on $V = \text{span}\{x_1, ..., x_k, ...\} \subset X$ and f_∞ extends to $f_\infty : X \to \mathbb{R}$ with $|f_\infty(x)| \leq \|x\|$.
$f_i \to f_\infty$ converges on a dense subset $V \Rightarrow f_i \to f_\infty, \forall x \in X$ by equicontinuity. □

8 Hilbert Space

Definition: 8.1: Pre-Hilbert Space

Let V be a vector space ($\mathbb{R}$ or $\mathbb{C}$). A pre-Hilbert space is $(V, \langle\cdot,\cdot\rangle : V \times V \to \mathbb{C})$ s.t.

1. $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$, $\forall a, b \in \mathbb{C}, x, y, z \in \mathbb{C}$
2. $\overline{\langle x, y\rangle} = \langle y, x\rangle$
3. $\langle x, x\rangle \geq 0$, and 0 is attained if and only $x = 0$.

Example:

1. $\mathbb{C}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}$
2. $L^2(\mathbb{R} \to \mathbb{C})$, $\langle f, g\rangle = \int f\overline{g}$ (L^2 inner product)
3. $C_0^\infty(\mathbb{R})$, $\langle f, g\rangle = \int f\overline{g} + \int f'\overline{g}'$ (H^1 inner product)

Theorem: 8.1: Cauchy-Schwartz Inequality

Let V be a pre-Hilbert space, $x, y \in V$,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$

Proof. Assume WLOG, $\langle x, x\rangle = \langle y, y\rangle = 1$ and $|\langle x, y\rangle| = \langle x, y\rangle$

$$0 \leq \langle x - ty, x - ty\rangle = \langle x, x\rangle - 2t\langle x, y\rangle + t^2\langle y, y\rangle = 1 - 2t|\langle x, y\rangle| + t^2$$

Minimum attained at $t = |\langle x, y\rangle|$, so

$$0 \leq 1 - |\langle a, b\rangle|^2 \Rightarrow |\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$

□

Corollary 25. *$\|x\| = \langle x, x\rangle^{\frac{1}{2}}$ is a norm*

Proof.

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x\rangle + \langle y, y\rangle + \langle x, y\rangle + \langle y, x\rangle \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

□

Corollary 26. *For a fixed $y \in V$, $\langle\cdot, y\rangle : x \to \langle x, y\rangle$ is a bounded linear functional and $\|\langle\cdot, y\rangle\|_{V^*} = \|y\|_V$*

Definition: 8.2: Hilbert Space

$(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space if it is complete.

Example: $C_0^\infty(\mathbb{R})$ is incomplete, but we can take its completion $H^1 = \overline{(C_0^\infty(\mathbb{R}), \langle\cdot,\cdot\rangle_{H^1})} \subset L^2$.

8.1 Orthogonality and Orthonormality

Definition: 8.3: Orthogonality

Let $x, y \in H$,

1. $x \perp y$ if $\langle x, y\rangle = 0$
2. If $E \subset H$, $E^\perp = \{f \in H : f \perp g, \forall g \in E\}$ the orthogonal complement of E is a closed subspace.

Theorem: 8.2: Parallelogram Law

$$\|x-y\|^2 + \|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proposition: 8.1:

If $E \subset H$ is a closed subspace, then $H = E \oplus E^\perp$ *i.e.* $\forall f \in H$, $f = g + h$, $\forall g \in E$, $h \in E^\perp$ uniquely.

Proof. Given $f \in H$, $D = \inf_{g\in E} \|f-g\|$.
Find g_n s.t. $D_n = \|f-g_n\| \searrow D$ as $n \to \infty$.
Let $x = f - g_n, y = f - g_m$.

$$\begin{aligned}
\|g_n - g_m\|^2 &= 2\left(\|f-g_n\|^2 + \|f-g_m\|^2\right) - \|2f - g_n - g_m\|^2 \\
&= 2(D_n^2 + D_m^2) - 4\left\|f - \frac{1}{2}(g_n+g_m)\right\|^2 \\
&\le 4D_n^2 - 4D_m^2 \to 0
\end{aligned}$$

The final line assumes $D_n > D_m$, $\frac{1}{2}(g_n + g_m) \to g_n, g_m$ as $n, m \to \infty$.
Therefore, g_n is Cauchy. It has a convergent subsequence, so the infimum is attained.

For any $g' \in E$, $D = \|f-g\| \le \|f-g'\|$
$f = g + (f-g)$, let $h = f - g$, we want to show that $h \in E^\perp$.
Take $u \in E$, consider $\langle h, u\rangle$.
Define $f(t) = \|h+tu\|^2 = \|h\|^2 + 2t\langle h, u\rangle + t^2\|u\|^2$.
$\|h+tu\| = \|f-g+tu\| = \|f-(g-tu)\|$, so we require $f'(0) = 0$ to achieve infimum at 0.
$f'(t) = 2\langle h,u\rangle + 2t\|u\|^2$, $f'(0) = 2\langle h,u\rangle = 0$, so $\langle h,u\rangle = 0$, $h \in E^\perp$. □

Corollary 27. *If E is closed, then $(E^\perp)^\perp = E$*

Theorem: 8.3: Riesz Representation

Given $l \in \mathcal{H}^*$, there exists a unique $g \in \mathcal{H}$ s.t. $l = l_g$ where $l_g(f) = \langle f, g\rangle$ and $\|l_g\|_{\mathcal{H}^*} = \|g\|_H$

Proof. Uniqueness is from linearity of inner product.

For existence: Let $E = \ker(l) \subset H$, E is a closed subspace, then by Proposition 8.1, $\mathcal{H} = E \oplus E^\perp$.
If $l \neq 0$, then $\exists z \in E^\perp$ s.t. $\|z\| = 1$ and $\langle l, z\rangle = 0$.
Claim: $E^\perp = \mathbb{R}z$, *i.e.* $E^\perp$ is spanned by a single vector, $E^\perp$ is 1DD.
If $x \in H$, then $x - \frac{l(x)}{l(z)}z \in E$, because $l\left(x - \frac{l(x)}{l(z)}z\right) = l(x) - \frac{l(x)}{l(z)}l(z) = 0$. Hence $x = \left(x - \frac{l(x)}{l(z)}z\right) + \frac{l(x)}{l(z)}z$.
Take $g \in \mathbb{R}z$ s.t. $\langle z, g\rangle = l(z)$. □

Corollary 28. *$\mathcal{H} \cong \mathcal{H}^*$, conjugate linear identification. All Hilbert spaces are reflexive, i.e. $\mathcal{H} \cong \mathcal{H}^{**}$.*

***Definition:* 8.4: Orthonormal Sets**

$\{u_\alpha\}_{\alpha\in\mathcal{A}} \subset H$ is orthonormal if $\langle u_\alpha, u_\beta\rangle = \delta_{\alpha\beta}$.

***Theorem:* 8.4: Gram-Schmidt**

Given $\{x_i\}_{i=1}^\infty$ linearly independent vectors in $\mathcal{H}$, we can construct $\{u_i\}_{i=1}^\infty$ an orthonormal set s.t. span $\{x_1, ..., x_k\}$ = span $\{u_i, ..., u_k\}$ by the following procedure:

1. $u_1 = \frac{x_1}{\|x_1\|}$
2. $v_2 = x_2 - \langle x_2, u_1\rangle u_1$, $u_2 = \frac{v_2}{\|v_2\|}$
3. $v_{i+1} = x_{i+1} - \sum_{j=1}^{i} \langle x_{i+1}, u_j\rangle u_j$, $u_{i+1} = \frac{v_{i+1}}{\|v_{i+1}\|}$

***Definition:* 8.5: Orthonormal Basis**

If $\mathcal{H}$ is a Hilbert space, an orthonormal set $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ is an orthonormal basis if $\langle x, u_\alpha\rangle = 0, \forall \alpha \in A$ $\Rightarrow x = 0$.

***Theorem:* 8.5: Bessel's Inequality**

Let $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal set. Then given $x \in \mathcal{H}$, $\{\alpha \in \mathcal{A} : \langle x, u_\alpha\rangle \neq 0\}$ is countable and $\sum_{\alpha\in\mathcal{A}} |\langle x, u_\alpha\rangle|^2 \leq \|x\|^2$.

Proof. Suffice to prove for A finite. For infinite case, take supremum. □

***Theorem:* 8.6:**

Let $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ be an orthonormal set in $\mathcal{H}$. Then the following are equivalent:

1. If $x \in \mathcal{H}$ s.t. $\langle x, u_\alpha\rangle = 0, \forall\alpha$, then $x = 0$.
2. $\|x\|^2 = \sum_{\alpha\in\mathcal{A}} |\langle x, u_\alpha\rangle|^2, \forall x \in \mathcal{H}$
3. $x = \sum_{\alpha\in\mathcal{A}} \langle x, u_\alpha\rangle u_\alpha, \forall x \in \mathcal{H}$.

Proof. 2)⇒ 1) because if 2 holds and every $\langle x, u_\alpha\rangle = 0$, we must have each component of x.

3)⇒ 2) by definition and $\langle u_\alpha, u_\alpha\rangle = 1$.

1)⇒ 3) Start with $x \in \mathcal{H}$.
By Theorem 8.5, $\langle x, u_\alpha\rangle \neq 0$ for countably many α and $\sum_{\alpha\in A} \langle x, u_\alpha\rangle u_\alpha = \hat{x}$
$\Rightarrow \langle \hat{x}, u_\alpha\rangle = \langle x, u_\alpha\rangle, \forall\alpha \in A$.
So $x = \hat{x}$ by 1). □

***Theorem:* 8.7:**

Every Hilbert space $\mathcal{H}$ has an orthonormal basis

Proof. By Zorn's Lemma on all orthonormal sets. □

Example: Orthonormal basis for Hilbert space:

1. $l^2(\mathbb{R})$: pathological (functions supported on discrete set)
2. $l^2(\mathbb{N})$: countable orthonormal basis.

Theorem: 8.8: Separable Hilbert Space

A Hilbert space $\mathcal{H}$ has a countable orthonormal basis $\Leftrightarrow$ $\mathcal{H}$ is separable ($\exists \{x_i\}_{i=1}^{\infty} \subset \mathcal{H}$ dense)

Proof. ($\Rightarrow$) If $\mathcal{H}$ admits countable orthonormal basis, then there exists a unitary map $\mathcal{H} \to l^2(\mathbb{N})$, $\sum_{i=1}^{\infty} a_i u_i \mapsto f(i) = a$. *i.e.* it is invertible, bounded and preserves inner product (isomorphism)
Since $l^2(\mathbb{N})$ is separable, then $\mathcal{H}$ is separable.

($\Leftarrow$) Let $\{x_i\}_{i=1}^{\infty} \subset \mathcal{H}$ be dense.
We can construct a linearly independent subset $\{x_{i_k}\}_{k=1}^{\infty} \subset \{x_i\}_{i=1}^{\infty}$ with the same span (finite linear combinations)
Apply Theorem 8.4 to get $\{u_i\}_{i=1}^{\infty}$ an orthonormal set with $\operatorname{span}\{u_i\}_{i=1}^{\infty} = \operatorname{span}\{x_i\}_{i=1}^{\infty}$. Therefore, $\overline{\operatorname{span}\{u_i\}_{i=1}^{\infty}} = H$.
Therefore, if $\langle x, u_i \rangle = 0, \forall i = 1, 2, ...$, then $\langle x, y \rangle = 0, \forall y \in H \Rightarrow x = 0$, so $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis by Theorem 8.6. □

Corollary 29. *Every separable Hilbert space is unitarily equivalent to* $l^2(\mathbb{N})$.

Example: $L^2(\mathbb{R}) \cong l^2(\mathbb{N})$, $L^2(S^1) \cong l^2(\mathbb{N})$.
$L^2(S^1)$ has an orthonormal basis given by $\theta \in [0, 2\pi]/\sim$, $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(k\theta), \frac{1}{\sqrt{\pi}} \cos(k\theta)\right\}_{k=1}^{\infty}$

8.2 Operators

Hilbert spaces are Banach spaces, so operators carry forward.

Definition: 8.6: Adjoint Operator

Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the adjoint of A if $\langle Ax, y \rangle_{\mathcal{H}_2} = \langle x, A^* y \rangle_{\mathcal{H}_1}$.

Theorem: 8.9:

A^* always exist and $\|A^*\| = \|A\|$.

Proof. Given $y \in \mathcal{H}_2$, consider $l_y : \mathcal{H}_1 \to \mathbb{R}, x \mapsto \langle Ax, y \rangle_{\mathcal{H}_2}$ which is a bounded linear functional.
By Theorem 8.3, $l_y(x) = \langle x, z \rangle$ for some $z \in \mathcal{H}_1$, define $A^* y = z$.
We can check that $y \mapsto z = A^* y$ is linear.

$$\|A^* y\| = \|z\| = \|l_y\|_{\mathcal{H}_1^*} \leq \|A\| \, \|y\| \Rightarrow \|A^*\| \leq \|A\|$$

But $A^{**} = A$, so by the same argument $\|A\| = \|A^{**}\| \leq \|A^*\|$. □

Example:

1. $A : \mathbb{R}^n \to \mathbb{R}^m$, $A^* = A^T$

2. $A : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $Ae_i = \displaystyle\sum_{j=1}^{\infty} a_i^j e_j$, A^* is represented by $(a^*)^i_j = a^j_i$

3. $A : L^2([0,1]) \to L^2([0,1])$, sometimes A is defined by a kernel $K_A(x,y)$.

$$Af(y) = \int_0^1 K_A(x,y) f(x) dx \Rightarrow K_{A^*}(x,y) = K_A(y,x)$$

Adjoint is like conjugate transpose in infinite dimensions.

***Definition:* 8.7: Compact Operator**

$K : X \to Y$ a bounded linear operator between Banach spaces is compact if $K(B_1)$ is compact in Y *i.e.* if $x_i \in X, \|x_i\| = 1$, then $K(x_i)$ has a convergent subsequent.

Example:

1. If Y is finite dimensional, then any bounded operator is compact
2. $Id : X \to X$ is not compact if $\dim(X) = \infty$
3. $T : C^0([0,1]) \to C^0([0,1])$, $Tf(x) = \int_0^x f(s)ds$, $(Tf)' = f$
4. The same operator $T : C^0([0,1]) \to C^1([0,1])$ is bounded but not compact
5. Inclusion map: $C^1([0,1]) \to C^0([0,1])$ is compact

Proof. 3) $\sup |f_i| \leq 1$. Let $g_i(x) = Tf_i(x)$, $g_i(0) = 0$, $g_i'(x) = f_i(x)$.
g_i is bounded and equicontinuous. By Arzela-Ascoli, there exists a convergent subsequence, so the operator is compact.

4) we cannot extract the convergent subsequence in C^1, because the norm in C^1 is more restrictive on derivatives. $\square$

Fact: If K is compact and T is bounded, then $K \circ T$ and $T \circ K$ are compact.

Remark 31. Compact operators form a closed subspace $K(X \times Y) \subset \mathcal{L}(X \times Y)$.

***Lemma:* 8.1:**

If A is self-adjoint $(A = A^*)$, then $\|A\| = \sup\limits_{\|x\|=1} |\langle Ax, x\rangle|$.

Proof.

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle| \geq \sup_{\|x\|=1} |\langle Ax, x\rangle| = K$$

$$\begin{aligned}
\langle Ax, y\rangle &= \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle\right) \\
&\leq \frac{K}{4}\left(\|x+y\|^2 + \|x-y\|^2\right) = \frac{K}{2}\left(\|x\|^2 + \|y\|^2\right) \\
&\leq K
\end{aligned}$$

$\square$

Theorem: 8.10: Spectral Theorem for Compact Self-Adjoint Operators

If $A : \mathcal{H} \to \mathcal{H}$ is a compact, self-adjoint operator ($A = A^*$), then there exists $\lambda_1, \lambda_2, ...,$ $\lim_{k\to\infty} \lambda_k = 0$ and eigenvectors e_i s.t. $Ae_i = \lambda_i e_i$ which form an orthonormal basis for $(\ker A)^\perp$. $\mathcal{H}$ admits orthogonal decomposition $\mathcal{H} = \ker A \oplus \bigoplus_{i=1}^{\infty} N_{\lambda_i}$ where N_{λ_i} are finite dimensional eigenspaces and $\ker(A)$ is possibly infinity-dimensional.

Proof. Step 1. $\lambda_1 = \pm \|A\|$ is an eigenvalue.

By Lemma 8.1. Now $\|A\| = \sup_{\|x\|=1} \|Ax\|$. Take $\|x_n\| = 1$ and $\lim_{n\to\infty} \langle Ax_n, x_n\rangle \to \|A\| = \lambda_1$. Xince $\|Ax_n\|^2 \le \|A\|^2 \|x_n\|^2 = \lambda^2 \|x_n\|^2$.

$$0 \le \|Ax_n - \lambda x_n\|^2 = \|Ax_n\|^2 + \lambda^2 \|x_n\|^2 - 2\lambda \langle Ax_n, x_n\rangle \le \epsilon \to 0$$

Then $Ax_n - \lambda x_n \to 0$, and $Ax_n \to x$, $\lambda x_n \to x$, $Ax = \lambda x$.

Step 2. A compact $\Rightarrow N_{\lambda_1}$ is finite dimensional. $\mathcal{H} = N_{\lambda_1} \oplus N_{\lambda_1}^\perp$

N_{λ_1} is closed by definition and $A|_{N_{\lambda_1}} = \lambda_1 Id$. By compactness of A, $A|_{N_{\lambda_1}}$ is compact, but Id is compact if and only if it is finite dimensional. Therefore N_{λ_1} must be finite dimensional and $H = N_{\lambda_1} \oplus N_{\lambda_1}^\perp$.

Step 3. A is self-adjoint $\Rightarrow A : N_{\lambda_1}^\perp \to N_{\lambda_1}^\perp$.

Repeat with $\mathcal{H}_2 = N_{\lambda_1}^\perp$.
Step 4: show $\lambda_i \to 0$ and what's leftover is $\ker A$.

Let $\mathcal{H}_2 = N_{\lambda_1}^\perp \cap N_{-\lambda_1}^\perp$, $\|A|_{\mathcal{H}_2}\| = |\lambda_2|$. Repeating the process, we get $\mathcal{H} \supset \mathcal{H}_2 \supset \mathcal{H}_3 \supset \cdots$.
If $x \in \mathcal{H}_k$, then $\frac{\|Ax\|}{\|x\|} \le |\lambda_k| \to 0$. Thus, $\bigcap_{k=1}^{\infty} \mathcal{H}_k = \mathrm{Ker}(A)$ $\square$

Remark 32. Compactness is important. Typically, a bounded self-adjoint operator may not be diagonalizable.

9 L^p-space on Operators/Functionals

Let $(X, \mathcal{M}, \mu)$ be a measure space, $f : X \to \mathbb{R}$ or $\mathbb{C}$ be a measurable function.

The L^p-norm is $\left(\int_X |f|^p d\mu\right)^{1/p} = \|f\|_{L^p}$ for $1 < p < \infty$. $\|f\|_{L^\infty} = \text{esssup}_{x\in X}|f(x)|$ (Definition 5.4)

Facts:

1. $(L^p, \|\cdot\|_p)$ is a Banach space
2. $(\mathbb{R}^n, \|p\|) \cong \{n\text{-points with discrete measure}\}$

Balls in R^2:

1. L^1: $B = \{|x| + |y| \le 1\}$
2. L^2: $B = \{x^2 + y^2 \le 1\}$
3. L^∞: $B = \{|x|, |y| \le 1\}$

L^p restricts the order of growth:

1. $L^p([0, 1])$, $f(x) = \frac{1}{x^\alpha} \in L^p \Leftrightarrow \alpha < \frac{1}{p}$
2. $L^p([1, \infty))$, $f(x) = \frac{1}{x^\alpha} \in L^p \Leftrightarrow \alpha > \frac{1}{p}$

Theorem 5.7 gives some convexity results. Taking logs:

$$\frac{1}{q} = \frac{1-\lambda}{r} + \frac{\lambda}{p} \Rightarrow \log \|f\|_q \le (1-\lambda)\|f\|_r + \lambda \|f\|_p$$

i.e. $\frac{1}{q} \mapsto \log \|f\|_q$ is convex.

Often, one needs to not only interpolate between functions, but also between operators. Consider

$$Tf(x) = \int K(x, y) f(y) d\nu(y)$$

Is this operator $T : L^p \to L^q$ bounded?

e.g. Let $A : \mathbb{R}^n \to \mathbb{R}^m$. Suppose $\|A\|_{p_0\to q_0} \le K_0$, $\|A\|_{p_1\to q_1} \le K_1$. Can we get bounds for $\|A\|_{p\to q}$ for other (p, q)?

***Lemma:* 9.1: Hadamard 3-line**

Let $\phi : \{0 < \text{Re}(z) < 1\} \to \mathbb{C}$ be holomorphic and bounded. If $|\phi(z)| \le M_0$ for $\text{Re}(z) = 0$ and $|\phi(z)| \le M_1$ for $\text{Re}(z) = 1$, then $|\phi(z)| \le M_0^{1-\theta} M_1^\theta$ on $\text{Re}(z) = \theta \in [0, 1]$

Proof. Let $\phi_z(z) = \phi(z) M_0^{z-1} M_1^z e^{\epsilon z(z-1)}$. $\phi_z(z)$ is holomorphic.

$$|\phi_z(z)| = |\phi(z)| M_0^{\text{Re}(z)-1} M_1^{\text{Re}(z)} e^{\epsilon \text{Re}(z(z-1))}$$

Let $z = \theta + it$. Then $\text{Re}(z) = \theta$, $\text{Re}(z(z-1)) = \theta(\theta-1) - t^2$. As $t \to \infty$, $\phi_z(z) \to 0$. Therefore, maximum is attained in the interior. By maximum modulus principle, $|\phi_z(z)| \le 1$ either on $\text{Re}(z) = 0$ or $\text{Re}(z) = 1$. Thus, $|\phi(z)| \le M_0^{1-\theta} M_1^\theta$. □

Lemma: 9.2:

Let g be measurable function. If $M = \sup_{\|f\|_p = 1, f \text{ simple}} \left\{ \left| \int fg d\mu \right| \right\}$ is finite, then $f \in L^q$, $\|g\|_q = M$.

Theorem: 9.1: Riesz-Thorin

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. If $A : L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(Y) + L^{q_1}(Y)$ s.t. $A : L^{p_0}(X) \to L^{q_1}(Y)$ and $A : L^{p_1}(X) \to L^{q_1}(Y)$ are bounded, then for $\theta \in (0,1)$, $A : L^{p_\theta}(X) \to L^{q_\theta}(Y)$ is bounded for $\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$ and $\frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_0}$. Moreover, if $\|A\|_{p_0 \to q_0} \leq K_0$ and $\|A\|_{p_1 \to q_1} \leq K_1$, then $\|A\|_{p_\theta \to q_\theta} \leq K_0^{1-\theta} K_1^\theta$.
$\left(\frac{1}{p}, \frac{1}{q}\right) \mapsto \log \|A\|_{p \to q}$ is convex.

Proof. 1) $p_0 = p_1$. Follows Theorem 5.7.

$$\|Af\|_{q_\theta} \leq \|Af\|_{q_1}^\theta \|Af\|_{q_0}^{1-\theta} \leq K_1^\theta K_2^{1-\theta} \|f\|_{p_1}^\theta \|f\|_{p_0}^{1-\theta} \leq K_1^\theta K_2^{1-\theta} \|f\|_{p_\theta}$$

2) $p_0 \neq p_1$. It sufficies to prove for simple functions.
Let $f : X \to \mathbb{C}$ be simple, $f = \sum_{i=1}^n a_i \chi_{E_i}$, where $\mu(E_i) < \infty$. Assume $\|f\|_{p_\theta} = 1$ by rescaling.
We want to bound $\|Af\|_{q_\theta}$. It sufficies by Lemma 9.2 to bound $\int_Y Afg d\nu$ for all simple function g with $\|g\|_{q'_\theta} = 1$ where $\frac{1}{q_\theta} + \frac{1}{q'_\theta} = 1$.

Extend f, g to depend on z, $f_z = \sum_{k=1}^n |a_k|^{\frac{u(z)}{u(\theta)}} e^{i\alpha_k} \chi_{E_k}$, where $u(z) = \frac{1-z}{p_0} + \frac{z}{p_1}$, $\frac{u(z)}{u(\theta)}$ is holomorphic.
If $g = \sum_{k=1}^m b_k \chi_{F_k}$, then $g_z = \sum_{k=1}^n |b_k|^{\frac{1-v(z)}{1-v(\theta)}} e^{i\beta_k} \chi_{F_k}$, where $v(z) = \frac{1-z}{q_0} + \frac{z}{q_1}$.

Let $\phi(z) = \displaystyle\int_Y Af_z g_z d\nu$. Then

$$\phi(z) = \sum_{k=1}^n \sum_{j=1}^m |a_k|^{\frac{u(z)}{u(\theta)}} |b_j|^{\frac{1-v(z)}{1-v(\theta)}} (e^{i\alpha_k} e^{i\beta_j} A_{kj})$$

On $\mathrm{Re}(z) = 0$, $|a_k|^{\frac{u(z)}{u(\theta)}} = |a_k|^{p_\theta\left(\frac{1-iy}{p_0} + \frac{iy}{p_1}\right)} = |a_k|^{\frac{p_\theta}{p_0}} |a_k|^{ip_\theta y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}$,
so $|f_{iy}| \leq |f|^{\frac{p_\theta}{p_0}}$, $\|f_{iy}\|_{p_\theta} \leq \|f\|_{p_\theta} = 1$
Similarly, $|g_{iy}| \leq |g|^{\frac{q'_\theta}{q_0}}$, $\|g_{iy}\|_{q'_\theta} \leq \|g\|_{q_\theta} = 1$.
Boundedness $\Rightarrow |\phi(z)| \leq K_0$ on $\{\mathrm{Re}(z) = 0\}$
Similarly, $|\phi(z)| \leq K_1$ on $\{\mathrm{Re}(z) = 1\}$.
By Lemma 9.1, $|\phi(\theta)| \leq K_0^{1-\theta} K_1^\theta$, $\left| \displaystyle\int_Y Afg d\nu \right| \leq K_0^{1-\theta} K_1^\theta$.

To extend to all functions, note that the set of simple functions is dense. ☐

Theorem: 9.2: Young's Inequality

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, $K : X \times Y \to \mathbb{R}$ be measurable function, $Tf(x) = \displaystyle\int_Y K(x,y) f(y) d\nu(y)$. Assume $\displaystyle\int_Y |K(x,y)| d\nu(y) \leq M$ for all x and $\displaystyle\int_X |K(x,y)| d\mu(x) \leq M$ for all y. Then if $f \in L^p(Y)$, then $Tf(x)$ is defined a.e. and $\|Tf\|_{L^p(X)} \leq M \|f\|_{L^p(Y)}$ for $p \in [1, \infty]$.

Proof. When $p = \infty$,

$$\|Tf(x)\|_\infty \le \left(\int |K(x,y)|d\nu\right) \|f\|_\infty ,$$

so $\|Tf\|_\infty \le M \|f\|_\infty$.

When $p = 1$, by Theorem 2.9,

$$\int |Tf(x)| = \int_X \left| \int_Y K(x,y)f(y)d\nu \right| d\mu \le M \int |f(y)|d\nu$$

By Theorem 9.1, $\|T\|_{p\to p} \le M$. □

Application: $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin(kx), \frac{1}{\sqrt{\pi}}\cos(kx)\right\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2([0,\infty))$. Equivalently, $\left\{e^{i2\pi kx}\right\}_{k=-\infty}^{\infty}$ is an orthonormal bais for $L^2_{\mathbb{C}}([0,1])$. Every $f \in L^2([0,1])$ can be written as

$$f \overset{L^2}{=} \sum_{k=-\infty}^{\infty} a_k e^{i2\pi kx}$$

Consider the Fourier operator $\mathcal{F} : L^2([0,1]) \to l^2(\mathbb{Z})$, $\mathcal{F}(f) = (a_k)$, $a_k = \int_0^1 f(x)e^{-i2\pi kx}dx$. $\mathcal{F}$ is a unitary map and preserves inner product (isomorphism).

When is $\mathcal{F}$ bounded from $L^p([0,1])$ to $L^q(\mathbb{Z})$?

***Theorem:* 9.3: Hausdorff-Young Inequality**

$$\|\mathcal{F}f\|_{L^q(\mathbb{Z})} \le \|f\|_{L^p([0,1])}$$

for $q \ge 2$ and $\frac{1}{p} + \frac{1}{q} = 1$

Proof. To show this with Theorem 9.1, we just need to show for the endpoints $\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(1, 0)$. $\left(\frac{1}{2}, \frac{1}{2}\right)$ is known. For $(1, 0)$, consider $\mathcal{F} : L^1([0,1]) \to L^\infty(\mathbb{Z})$:

$$|a_k| \le \int_0^1 |f(x)||e^{-i2\pi kx}|dx \le \|f\|_{L^1}$$

□

Remark 33. Sometimes the endpoint estimates fail, so Riesz-Thorin interpolation does not work. We need a new interpolation theorem with weaker control at endpoints.

9.1 Weak L^p

***Definition:* 9.1: Distribution Function**

Let $f : X \to [-\infty, \infty]$, the distribution function of f is

$$\lambda_f(t) = \mu\left(\{x \in X : |f(x)| > t\}\right),$$

which is the volume of super level sets

Definition: 9.2: Weak-L^p

The weak-L^p if f is $[f]_p = \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}}$. and the vector space weak-$L^p = \{f : [f]_p < \infty\}$ (functions that decay faster than $\frac{1}{t^p}$)

Note: $[f]_p$ is not a norm, because the triangle inequality is not satisfied.
weak-L^p contains slightly more functions than L^p

Theorem: 9.4: Chebyshev Inequality

$[f]_p \leq \|f\|_p$, *i.e.* $\lambda_f(t) \leq \frac{\|f\|_p^p}{t}$

Proof.

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_{\{|f|>t\}} t^p d\mu = t^p \lambda_f(t)$$

□

Example:

1. $f(t) = \frac{1}{t^\alpha}$ on $[0,1]$, $\lambda_f(t) = t^{-\frac{1}{\alpha}}$, $f \in$weak-L^p if and only if $\sup_t t\left(t^{-\frac{1}{\alpha}}\right)^{\frac{1}{p}} < \infty$ or $p \leq \frac{1}{\alpha}$
2. $f(t) = \frac{1}{t^\alpha}$ on $[1,\infty)$, $f \in$weak-L^p if and only if $p \geq \frac{1}{\alpha}$

$f(x) = \frac{1}{x}$ is a weak-L^1 function.

Proposition: 9.1:

$$\int_X |f|^p d\mu = p\int_0^\infty t^{p-1}\lambda_f(t)dt$$
$$\|f\|_{L^\infty} = \inf\{t : \lambda_f(t) = 0\}$$

The vector space weak-$L^\infty = L^\infty$

Proof. By FTC, $|f|^p = p\int_0^{|f(x)|} t^{p-1}dt$.

$$\begin{aligned}
\int_X |f|^p d\mu &= p\int_X \int_0^{|f(x)|} t^{p-1} dt d\mu(x) \\
&= p\int_X \int_0^\infty \chi_{\{t<|f|\}} t^{p-1} dt d\mu(x) \\
&= p\int_0^\infty \int_X \chi_{t<|f|} t^{p-1} d\mu(x) dt \\
&= p\int_0^\infty t^{p-1}\lambda_f(t)dt
\end{aligned}$$

□

Theorem: 9.5: Properties of $\lambda_f(t)$

1. $\lambda_f(t)$ is decreasing and right continuous
2. $|f| \leq |g| \Rightarrow \lambda_f(t) \leq \lambda_g(t)$
3. $|f_n| \nearrow |f| \Rightarrow \lambda_{f_n}(t) \nearrow \lambda_f(t)$
4. $\lambda_{f+g}(t) \leq \lambda_f\left(\frac{t}{2}\right) + \lambda_g\left(\frac{t}{2}\right)$, because $\{|f+g| \geq t\} \subset \left\{|f| \geq \frac{t}{2}\right\} \cup \left\{|g| \geq \frac{t}{2}\right\}$

Theorem: 9.6: Interpolation of Weak-L^p Functions

Let $p_0 < p_1 \in [1, \infty]$. If $[f]_{p_0} \leq K_0$, $[f]_{p_1} \leq K_1$, then $f \in L^p$ for $p \in (p_0, p_1)$ and $\|f\|_p \leq CK_0^{1-\theta}K_1^{\theta}$, where $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_0}$. $C = C(p, p_0, p_1)$ and $C \to \infty$ as $\theta \to 0$ or 1.

Proof. By Proposition 9.1

$$\|f\|_p^p = \int_X |f|^p d\mu = p\int_0^\infty t^{p-1}\lambda_f(t)dt = p\int_0^A t^{p-1}\lambda_f(t)dt + p\int_A^\infty t^{p-1}\lambda_f(t)dt$$

By Definition 9.2, $[f]_{p_0} \leq K_0 \Rightarrow \lambda_f(t) \leq \frac{K_0^{p_0}}{t^{p_0}}$, similarly, $\lambda_f(t) \leq \frac{K_1^{p_1}}{t^{p_1}}$

$$\begin{aligned}\|f\|_p^p &\leq p\int_0^A t^{p-1-p_0}K_0^{p_0}dt + p\int_A^\infty t^{p-1-p_1}K_1^{p_1}dt \\ &= \frac{p}{p-p_0}K_0^pA^{p-p_0} + \frac{p}{p_1-p}K_1^pA^{p-p_1}\end{aligned}$$

Taking derivative, best bound is achieved when $A = K_1^{\frac{p_1}{p_1-p_0}}K_0^{-\frac{p_0}{p_1-p_0}}$. Then

$$\|f\|_p \leq \left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{\frac{1}{p}} K_0^{1-\theta}K_1^{\theta}$$

So, $C = \left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{\frac{1}{p}}$. $\square$

Definition: 9.3: Weakly-bounded Operators

Let $T : \Sigma_X(X) \to L(Y)$ ($\Sigma_X(X)$ is the set of simple functions, $L(Y)$ is the set of measurable functions).

1. T is of strong type (p, q) if $\|Tf\|_{L^q(Y)} \leq C\|f\|_{L^p(X)}$
2. T is of weak type (p, q) if $[Tf]_q \leq C\|f\|_{L^p(X)}$, *i.e.* $\lambda_{Tf}(t) \leq \frac{C^q\|f\|_p^p}{t^q}$

Definition: 9.4: Sublinear Operators

$T : X \to Y$ is sublinear if $|T(x+y)| \leq |Tx| + |Ty|$ and $|T(cx)| \leq c|Tx|$.

Lemma: 9.3: Minkowski Inequality for Integrals

Let $f_x(y) = f(x,y)$, the following two inequalities are equivalent

$$\left\| \int_X f(x,y) d\mu \right\|_{L^p(Y)} \leq \int_X \|f(x,y)\|_{L^p(Y)} d\mu(x)$$

$$\left(\int_Y \left| \int_X f(x,y) d\mu(x) \right|^p d\nu(y) \right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p d\nu(y) \right)^{\frac{1}{p}} d\mu(x)$$

Lemma: 9.4:

If f is measurable and $f = g_A + h_A$, where $h_A = f\chi_{\{|f| \leq A\}} + A\mathrm{sgn}(f)\chi_{\{|f|>A\}}$, $h_A \in [-A, A]$, g_A is the rest, then

$$\int_X |g_A|^p \leq p \int_A^\infty t^{p-1} \lambda_f(t) dt$$

$$\int_X |h_A|^p = p \int_0^A t^{p-1} \lambda_f(t) dt$$

Proof. By definition, $\lambda_{h_A}(t) = \begin{cases} \lambda_f(t), t \leq A, 0, t > A \end{cases}$, $\lambda_{g_A}(t) = \lambda_f(t + A)$. Then

$$\int |h_A|^p = p \int_0^\infty t^{p-1} \lambda_{h_A}(t) dt = p \int_0^A t^{p-1} \lambda_f(t) dt$$

$$\int |g_A|^p = p \int_0^\infty t^{p-1} \lambda_{g_A}(t) dt = p \int_A^\infty (t - A)^{p-1} \lambda_f(t) dt \leq p \int_A^\infty t^{p-1} \lambda_f(t) dt$$

□

Theorem: 9.7: Marcinkiewicz Interpolation

Let $p_0, p_1, q_0, q_1 \in [1, \infty]$, $p_0 \leq q_0$, $p_1 \leq q_1$ and $q_0 \neq q_1$. Let $T : L^{p_0}(X) + L^{p_1}(X) \to L(Y)$ be a sub-linear operator. If T is of weak type (p_0, q_0) and (p_1, q_1), then it is of strong type (p_θ, q_θ). Moreover, if $\lambda_{Tf}(t) \leq \frac{K_0^{q_0} \|f\|_{p_0}^{q_0}}{t^{q_0}}$ and $\lambda_{Tf}(t) \leq \frac{K_1^{q_1} \|f\|_{p_1}^{q_1}}{t^{q_1}}$, then $\|Tf\|_{q_\theta} \leq C \|f\|_{p_\theta}$. $C = C(\theta, p_0, p_1, q_0, q_1, K_0, K_1)$ and $C \to \infty$ as $\theta \to 0$ or 1

Remark 34. Theorem 9.7 does not imply Theorem 9.1, due to restrictions on p_0, q_0, p_1, q_1 and bound may fail (infinity) on boundary.

10 Fourier Analysis

Goal: Decompose f into its frequencies

Consider the heat equation $\begin{cases} \partial_t u = \partial_x^2 u \\ u(0,x) = u_0(x) \end{cases}$. Assume $u_0(x) = u_0(x + 2\pi)$ *i.e.* a cicular rod.

If $u_0(x) = \sin(kx)$, then $u(t,x) = e^{-k^2 t}\cos(kx)$. Similarly, if $u_0(x) = \cos(kx)$, then $u(t,x) = e^{-k^2 t}\cos(kx)$. Linearity implies:

$$\begin{aligned} u_0 &= \sum_{n=1}^{N} a_{k_n}\sin(k_n x) + b_{k_n}\cos(k_n x) \\ \Rightarrow u(t,x) &= \sum_{n=1}^{N} e^{-k_n^2 t}\left(a_{k_n}\sin(k_n x) + b_{k_n}\cos(k_n x)\right) \end{aligned}$$

Fourier claimed that any u_0 can be decomposed into infinite sum of sine and cosine with

$$\begin{aligned} a_k &= \frac{1}{\sqrt{\pi}}\int_0^{2\pi} u_0(x)\sin(2\pi kx)dx \\ b_k &\frac{1}{\sqrt{\pi}}\int_0^{2\pi} u_0(x)\cos(2\pi kx)dx \end{aligned}$$

10.1 Fourier Series

***Definition:* 10.1: Fourier Transform**

Let $f : S^1 \to \mathbb{C} \in L^1(S^1)$, where S^1 is unit circle parametrized from $[0,1]$. The Fourier transform of f is $\hat{f} : \mathbb{Z} \to \mathbb{C}$ s.t.

$$\hat{f}(k) = \int_0^1 f(x)e^{-i2\pi kx}dx$$

The transform map $\Lambda : L^1(S^1) \to L^\infty(\mathbb{Z})$ is bounded, and $\Lambda : L^2(S^1) \to L^2(\mathbb{Z})$ is an isomorphism, so inverse exists. By Theorem 9.1, $\Lambda : L^p(S^1) \to L^q(\mathbb{Z})$ is bounded if $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

***Definition:* 10.2: Inverse Fourier Transform**

$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{i2\pi kx}$$

This is the Fourier Series representation of f.

By Theorem 9.3, $\left\|\hat{f}\right\|_{L^q(\mathbb{Z})} \leq \|f\|_{L^p(S^1)}$.

Remark 35. S^1 is compact, $\mu(S^1) < \infty$, $L^1(S^1) \supset L^2(S^1) \supset L^\infty(S^1)$. $\mathbb{Z}$ is discrete, $L^1(\mathbb{Z}) \subset L^2(\mathbb{Z}) \subset L^\infty(\mathbb{Z})$.

Definition: 10.3: Convolution

$$f * g(x) = \int_{S^1} f(x-y)g(y)dy$$

Theorem: 10.1: Properties of Fourier Transform

1. Translation: $\tau_t f(x) = f(x-t)$, then $\hat{\tau_t f}(k) = \hat{f}(k)e^{-i2\pi kt}$
2. Differentiation: $f \in C^1(S^1) \Rightarrow f' : S^1 \to \mathbb{C} \in L^1(S^1)$, $\hat{f}'(k) = (i2\pi k)\hat{f}(k) \in L^2(\mathbb{Z})$
3. Riemann-Lebesgue: if $f \in L^1(S^1)$, then $\lim_{|k|\to\infty} \hat{f}(k) = 0$. The closed set $C_0(\mathbb{Z}) = \left\{f : \mathbb{Z} \to \mathbb{C} : f \in L^1(S^1) \Rightarrow \lim_{|k|\to\infty} \hat{f}(k) = 0\right\} \subsetneq L^\infty$. $\Lambda(L^1) \subset C_0 \subsetneq L^\infty$.
4. $\hat{f * g}(k) = \hat{f}(k)\hat{g}(k)$
5. $\hat{fg}(k) = \displaystyle\sum_{k=-\infty}^{\infty} \hat{f}(k-n)\hat{g}(n)$

Proof. 1)

$$\begin{aligned}
\hat{\tau_t f}(k) &= \int_0^1 f(x-t)e^{-i2\pi kt}dx \\
&= \int_{-t}^{1-t} f(x')e^{-i2\pi k(x'+t)}dx' = \int_0^1 f(x')e^{-i2\pi k(x'+t)}dx' \text{ Change of Variable} \\
&= \int_0^1 f(x')e^{-i2\pi kx'}e^{-i2\pi kt}dx' \\
&= \hat{f}(k)e^{-i2\pi kt}
\end{aligned}$$

2)

$$\begin{aligned}
\hat{f}'(k) &= \int_0^1 f'(x)e^{-i2\pi kx}dx \\
&= -\int_0^1 f(x)(e^{-i2\pi kx})'dx \text{ IBP} \\
&= i2\pi k\hat{f}(k)
\end{aligned}$$

3) If $f \in L^2(S^1)$, then $\hat{f} \in L^2(\mathbb{Z}) \subset C_0(\mathbb{Z})$. If $f \in L^1(S^1)$, then $\exists f_i \in L^2$ s.t. $f_i \to f$ in L^1. Therefore, $\left\|\hat{f}_i - \hat{f}\right\|_\infty \le \|f_i - f\|_1 \to 0$.

4)

$$\begin{aligned}
\hat{f * g}(k) &= \int_{S^1}\int_{S^1} f(x-y)g(y)dy e^{-i2\pi kx}dx \\
&= \int_{S^1} g(y)\int_{S^1} f(x-y)e^{-i2\pi kx}dxdy \\
&= \int_{S^1} g(y)\hat{f}(k)e^{-i2\pi ky}dy \text{ by 1)} \\
&= \hat{f}(k)\hat{g}(k)
\end{aligned}$$

□

Corollary 30. *If $f \in C^l$, then $|k|^l \hat{f}(k) \in L^2(\mathbb{Z})$. i.e. if f is l times differentiable, then $\hat{f}(k)$ should decay as $\mathcal{O}\left(\frac{1}{|k|^l}\right)$*

Remark 36. From 4) and 5), Fourier transform exchanges multiplication with convolution.

10.1.1 Fourier Series on Torus

Let $T^n = \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$ be torus.

$$L^1(T^n) = \left\{ f : T^n \to \mathbb{C} : \int_{T^n} |f| < \infty \right\}$$

Notation: If $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, $x = (x_1, ..., x_n) \in T^n$, then $k \cdot x = k_1x_1 + \cdots + k_nx_n$.
Given $f \in L^1(T^n)$, $\hat{f} : \mathbb{Z}^n \to \mathbb{C}, \hat{f}(k) = \displaystyle\int_{T^n} f(x)e^{-i2\pi k\cdot x}dx$.
All properties are carried forward.

***Proposition:* 10.1:**

If (X, μ), (Y, ν) are σ-finite, $\{f_n\}$ is an orthonormal basis for $L^2(X)$ and $\{g_m\}$ is an orthonormal basis for $L^2(Y)$, then $\{f_n g_m\}$ is an orthonormal basis for $L^2(X \times Y)$.

Since $T^n = S^1 \times \cdots \times S^1$, then $\left\{e^{i2\pi k\cdot x}\right\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(T^n)$.

***Theorem:* 10.2: Parseval's Identity**

$\Lambda : L^2(T^n) \to L^2(\mathbb{Z}^n)$ is an isomorphism with inverse

$$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{i2\pi k\cdot x}$$

10.2 Funtions on R^n

$C_C^\infty(\mathbb{R}^n)$ (compactly supported smooth functions) is dense in L^p for $p \in [1, \infty]$, but not true for $p = \infty$, L^p requires decay, $f(x) = 1 \in L^\infty$, but cannot be compactly supported.

Translation: $\tau_y f(x) = f(x - y)$, composition: $\tau_y\tau_z f = \tau_{y+z} f$

Convolution: $f * g(x) = \displaystyle\int_{\mathbb{R}^n} f(x - y)g(y)dy$ is the moving average of g w.r.t. f.
Example: $f = \frac{1}{|B_r|}\chi_{B_r}$, for any $g(x)$, $f * g(x) = \int_{B_r(x)} g(y)dy$.

***Proposition:* 10.2: Properties of Convolution**

Assuming all integrals exist

1. Commutative: $f * g = g * f$
2. Associative: $(f * g) * h = f * (g * h)$
3. $(f + g) * h = f * h + g * h$
4. $\tau_z(f * g) = \tau_z f * g + f * \tau_z g$
5. $\text{supp}(f * g) \subset A = \{z + y : z \in \text{supp}(f), y \in \text{supp}(g)\}$. If f, g have compact support, then so does $f * g$.

Theorem: 10.3: Young's Convolution Inequality

If $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, then

$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

Remark 37. Equality typically does not hold, we can have a factor < 1 depending on p.

Proof. Fix $f \in L^1(\mathbb{R}^n)$, then $g \mapsto f * g$ is a linear operator.

When $p = \infty$,

$$\|f * g\|_\infty \leq \int_{\mathbb{R}^n} |f(x-y)||g(y)|dy \leq \|f\|_{L^1} \|g\|_{L^\infty}$$

When $p = 1$,

$$\begin{aligned}
\|f * g\|_{L^1} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)dy \right| dx \\
&\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)|dxdy \\
&= \|f\|_{L^1} \int_{\mathbb{R}^n} |g(y)|dy = \|f\|_{L^1} \|g\|_{L^1}
\end{aligned}$$

By Theorem 9.1, we get inequality for $1 < p < \infty$. □

Proposition: 10.3:

If $f \in L^p$, $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \in [1, \infty]$, then $f * g$ is bounded and uniformly continuous. If $p \in (1, \infty)$, then $f * g \in C_0 = \left\{ f \in C(\mathbb{R}^n) : \lim_{|x| \to \infty} |f(x)| = 0 \right\}$

Proof. By Theorem 5.3, $\|f * g\|_{L^\infty} \leq \|f\|_{L^p} \|g\|_{L^q}$.

$$\|\tau_y(f * g) - f * g\|_{L^\infty} = \|\tau_y f * g - f * g\|_{L^\infty} = \|(\tau_y f - f) * g\|_{L^\infty} \leq \|\tau_y f - f\|_{L^p} \|g\|_{L^q} \to 0$$

When $p \in (1, \infty)$, pick $f_i, g_j \in C_C$ s.t. $f_i \to f$ in L^p, $g_j \to g$ in L^q. Then $f_i * g_i \in C_C(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$.

Now we show that $f_i * g_i \to f * g$ uniformly $\Rightarrow$ $f * g \in C_0$.

$$\begin{aligned}
\|f_i * g_i - f * g\|_\infty &= \|f_i * g_i - f_i * g + f_i * g - f * g\|_\infty \\
&\leq \|f_i * (g_i - g)\|_\infty + \|(f_i - f) * g\|_\infty \\
&\leq \|f_i\|_p \|g_i - g\|_q + \|f_i - f\|_p \|g\|_q \text{ By Theorem 5.3} \\
&\to 0
\end{aligned}$$

as $i \to \infty$. □

Summarize:

1. $C_C * C_C = C_C$
2. $L^p * L^1 \subset L^p$ (Young's inequality)
3. $L^\infty * L^1 \subset C_{\text{uniformly continuous, bounded}} \subset L^\infty$

4. $L^p * L^q \subset C_0 \subset L^\infty$ if $p \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$
5. $L^p * L^q \subset L^r$ if $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$ (Generalized Young's inequality)

How does convolution behave w.r.t. differentiation?

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{\tau_{-he_i} f - f}{h}$$

If everything exists $f, g \in C_C^\infty$,

$$\frac{\partial}{\partial x_i}(f*g)(x) = \lim_{h\to 0} \frac{\tau_{-he_i}(f*g) - f*g}{h} = \frac{\partial f}{\partial x_i} * g = f * \frac{\partial g}{\partial x_i}$$

Theorem 10.3 $\Rightarrow \left\| \frac{\partial f}{\partial x_i}(f*g)\right\|_\infty \leq \left\|\frac{\partial f}{\partial x_i}\right\|_\infty \|g\|_1$.

Approximation gives $f \in C^1, g \in L^1 \Rightarrow f*g \in C^1$. Pick $f_n \to f$ in C^1, $g_n \to g$ in L^1:

$$\left\| \frac{\partial}{\partial x_i}(f_k * g_k) - \frac{\partial f}{\partial x_i} * g \right\|_\infty \to 0$$

Therefore, $\frac{\partial}{\partial x_i}(f_k * g_k) \to \frac{\partial f}{\partial x_i} * g$ in C^0, $f_k * g_k \to f*g$ in C^0.

Notation: $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

If $x = (x_1, ..., x_n)$, then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ monomial of degree α, $\alpha! = \prod_{i=1}^n \alpha_i!$.

Product rule: $\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \partial^\beta f \partial^\gamma g$

Taylor's formula: $f(x) = \sum_{|\alpha|\leq k} (\partial^\alpha f)(x_0) \frac{(x-x_0)^\alpha}{\alpha!} + R_k$

***Theorem:* 10.4:**

$C^k * L^1 \subset C^k$. If $f \in C^k$, $g \in L^1$, then $f*g \in C^k$, $\partial^\alpha(f*g) = \partial^\alpha f * g$ for $|\alpha| \leq k$.

***Definition:* 10.4: Approximate Identity**

Let $\phi \in L^1(\mathbb{R}^n)$, rescale by t, $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$.

$$\int_{\mathbb{R}^n} \phi_t = \int_{\mathbb{R}^n} \phi = a$$

independent of t.

***Theorem:* 10.5: Properties of Approximate Identity**

1. If $f \in L^p$, $p \in [1,\infty)$, then $f * \phi_t \to af$ in L^p as $t \to 0$
2. If f is bounded and uniformly continuous, then $f*\phi_t \to af$ uniformly as $t \to 0$
3. If $f \in L^\infty(\mathbb{R}^n) \cap C^0(U)$ where $U \subset \mathbb{R}^n$ is open, then $f * \phi_t \to af$ uniformly in any K compact subset of U

Proof.

$$\begin{aligned}
f * \phi_t - af &= \int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_t(y)dy \\
&= \int_{\mathbb{R}^n} (f(x-tz) - f(x))\phi(z)dz \ (y = tz) \\
&\leq \int_{\mathbb{R}^n} |\tau_{tz} f - f(x)||\phi(z)|dz
\end{aligned}$$

For 1, by Theorem 5.5, we get

$$\|f * \phi_t - at\|_{L^p} \leq \int_{\mathbb{R}^n} \|\tau_{tz} f - f\|_{L^p} |\phi(z)|dz \to 0$$

For 2, 3, use L^∞. □

Corollary 31. *C_C^∞ is dense in L^p for $p \in [1, \infty)$*

***Theorem:* 10.6:**

If $|\phi(x)| \leq C(1 + |x|)^{-n-\epsilon}$ for $\epsilon > 0$,*i.e.* $\phi \in L^1$, and $f \in L^p$, $p \in [1, \infty]$, then $f * \phi_t(x) \to af(x)$ pointwise a.e. in Lebesgue set.

Proof. Lebesgue set $= \left\{ x : \lim_{r \to 0} \frac{1}{r^n} \int_{B_r} |f(x-y) - f(x)|dy = 0 \right\}$. It has full measure.

$$\begin{aligned}
|f * \phi_t(x) - af(x)| &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)||\phi_t(y)|dy \\
&= \int_{B_r(0)} |f(x-y) - f(x)||\phi_t(y)|dy + \int_{\mathbb{R}^n \setminus B_r(0)} |f(x-y) - f(x)||\phi_t(y)|dy
\end{aligned}$$

$|f(x-y) - f(x)|$ is small near zero in integral sense, $\phi_t(y)$ concentrates around 0.

On $\mathbb{R}^n \setminus B_r(0)$, $|\phi_t(x)| \leq \frac{C}{t^n} \left(1 + \left|\frac{x}{t}\right|\right)^{-n-\epsilon} \leq Ct^\epsilon (t + |x|)^{-n-\epsilon}$.
By Theorem 5.3,

$$\int_{\mathbb{R}^n \setminus B_r(0)} |f(x-y) - f(x)||\phi_t(y)|dy \leq \|f(x-y) - f(x)\|_{L^p} \|\phi_t\|_{L^q(\mathbb{R}^n \setminus B_r(0))} \to 0$$

On $B_r(0)$,

$$\begin{aligned}
\int_{B_r(0)} |f(x-y) - f(x)||\phi_t(y)|dy &\leq \sum_{k=1}^{N} \int_{\{2^{-k}r \leq |y| \leq 2^{-k+1}r\}} |f(x-y) - f(x)| \underbrace{|\phi_t(y)|}_{\leq t^\epsilon (2^k r)^{-n-\epsilon}} dy \\
&\quad + \int_{B_t} |f(x-y) - f(x)| \underbrace{|\phi_t(y)|}_{\leq t^{-n}} dy
\end{aligned}$$

Here $N \sim \log \frac{r}{t}$.
Choose r small s.t. $\frac{1}{t^n} \int_{B_t} |f(x-y) - f(x)|dy < \delta$ for $t < r$.

$$\leq C\delta + \sum_{k=1}^{N} t^\epsilon (2^{-k} r)^{-n-\epsilon} r^n 2^{(-k+1)n} \delta \leq C_1 \delta$$

□

10.3 Fourier Series of Real Valued Functions

Let $\mathcal{F}f(\xi) = \hat{f}(\xi) = \displaystyle\int_{\mathbb{R}^n} f(x)e^{-i2\pi\xi\cdot x}dx$, $\mathcal{F}: L^1 \to L^\infty$ is bounded.

***Proposition:* 10.4:**

1. $f \in L^1 \Rightarrow \hat{\tau_y f}(\xi) = e^{-i2\pi\xi\cdot y}\hat{f}(\xi)$
2. If $A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and linear, $f \in L^1$, $f_A(x) = f(Ax)$, then $\hat{f}_A(\xi) = |\det A^{-1}|\hat{f}\left(A^{-T}\xi\right)$
3. $f, g \in L^1 \Rightarrow \hat{f * g} = \hat{f}\hat{g}$
4. $f, g \in L^1 \Rightarrow \displaystyle\int_{\mathbb{R}^n} f\hat{g} = \int_{\mathbb{R}^n} g\hat{f}$
5. $x^\alpha f = x_1^{\alpha_1}\cdots x_n^{\alpha_n} f \in L^1$ for all $|\alpha| \le j$, then $\hat{f} \in C^k$, $\partial^\alpha \hat{f} = [(-2\pi i x)^\alpha f]\,\hat{}$.
6. If $f \in C^k$ and $\partial^\alpha f \in L^1$ for $|\alpha| < k$ and $\partial^\alpha f \in C_0$ for $|\alpha| \le k-1$, then $\hat{\partial^\alpha f}(\xi) = (i2\pi\xi)^\alpha \hat{f}(\xi)$.
7. Riemann Lebesgue: $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$, where $C_0(\mathbb{R}^n) = \left\{ f: \text{ unifromly continuous with } \lim_{|x|\to\infty} f(x) = 0 \right\}$

Proof. 2) Let $y = Ax$ for change of variable:

$$\hat{f}_A(\xi) = \int_{\mathbb{R}^n} f(Ax)e^{-i2\pi\xi\cdot x}dx = \int_{\mathbb{R}^n} f(y)e^{-i2\pi\xi A^{-1}y}\frac{dy}{|\det A|} = |\det A^{-1}|\hat{f}(A^{-T}\xi)$$

5) By Induction. If $f \in L^1$, then $x_1 f \in L^1$, and

$$\frac{\partial \hat{f}}{\partial \xi_1}(\xi) = \frac{\partial}{\partial \xi_1}\int_{\mathbb{R}^n} f(x)e^{-i2\pi\xi\cdot x}dx = \int_{\mathbb{R}^n} f(x)e^{-i2\pi\xi\cdot x}(-i2\pi x_1)dx = [(-2\pi i x)^\alpha f]\,\hat{}$$

6) Assume $k = 1, n = 1$,

$$\int_{-\infty}^{\infty} f'(x)e^{-i2\pi\xi x}dx = -\int_{-\infty}^{\infty} f(x)(e^{-i2\pi\xi x})'dx + [f(x)e^{-i2\pi\xi x}]_{-\infty}^{\infty}$$

Since $f(x) \in C_0$, the second term is 0, so we get

$$-\int_{-\infty}^{\infty} f(x)(e^{-i2\pi\xi x})'dx = \int_{-\infty}^{\infty} (i2\pi\xi)f(x)e^{-i2\pi\xi x}dx = (i2\pi\xi)\hat{f}(\xi)$$

For $n > 1$, same calculation on each 1D slice, for $k > 1$, apply induction.

7) C_0 is a closed strict subset of L^∞.
Since $C_C^\infty(\mathbb{R}^n) \subset L^1$ is dense, and $\mathcal{F}: L^1 \to L^\infty$ is a continuous linear map, it suffices to show that $\mathcal{F}(C_C^\infty) \subset C_0 \subset L^\infty$.
If $f \in C_C^\infty(\mathbb{R}^n)$, then $\forall\alpha$, $x^\alpha f \in C_C^\infty$, by 3), $\hat{f} \in C^\infty(\mathbb{R}^n)$.
and $\forall\alpha$, $\partial^\alpha f \in C_C^\infty$, by 6, $\hat{f}(\xi) \le \frac{C_k}{(1+|\xi|)^k}$, so $f \in C_0$. □

Corollary 32. *1.* $f_t(x) = f(tx)$, $\hat{f}_t(\xi) = t^{-n}\hat{f}(t^{-1}\xi)$

2. If A is orthogonal, $A^{-1} = A^T$, $\hat{f}_T(\xi) = \hat{f}(A\xi)$

3. If $f \in C^k$, $\partial^\alpha f \in L^1$ *and* $\partial^\alpha f \in C_0$, *then* $|\hat{f}(\xi)| \le \frac{C}{(1+|\xi|)^k}$

Definition: 10.5: Schwartz Space

Given $f \in C_0^\infty$,

$$\hat{f} \in S := \left\{ g \in C^\infty : |x^\beta \partial^\alpha g| \leq C, \forall \alpha, \beta \right\}$$

S is called Schwartz space. It is the space of functions (with their derivatives) that have super polynomial decay.

Example: $C_C^\infty \subset S$, $e^{-|x|^2}, p(x)e^{-|x|^2} \in S$.

Proposition: 10.5: Properties of Schwartz Space

1. $C_C^\infty \subset S \subset L^p$, $\forall p$
2. $f \in S \Rightarrow \partial^\alpha f \in S$
3. $f \in S \Rightarrow x^\alpha f \in S$ (can be multiplied by any polynomials)
4. $f, g \in S \Rightarrow fg \in S$
5. $f \in S \Rightarrow \mathcal{F}(S) \subset S$ (Schwartz space is preserved under Fourier transform)
6. $f, g \in S \Rightarrow f * g \in S$

Examples:
1) $f = \chi_{[0,1]}$ (not smooth, but compactly supported).

$$\hat{f}(\xi) = \int_0^1 e^{-i2\pi\xi x} dx = \frac{1}{-i2\pi\xi} e^{-i2\pi x}|_0^1 = \frac{e^{-i2\pi\xi} - 1}{-i2\pi\xi}$$

This is a smooth function and decay $\sim \frac{1}{|\xi|}$.

2) $f(x) = e^{-\pi|x|^2}$, $f \in S$

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} e^{-\pi|x|^2 - i2\pi\xi\cdot x} dx \\
&= \int_{\mathbb{R}^n} e^{-\pi\left(\sum_{j=1}^n x_j^2 + i2\xi_j x_j\right)} dx \\
&= \prod_{j=1}^n \int_{\mathbb{R}} e^{-\pi(x_j^2 + i2\xi_j x_j)} dx_j \\
&= \prod_{j=1}^n e^{-\pi\xi_j^2} = e^{-\pi|\xi|^2}
\end{aligned}$$

It is a fixed point under Fourier transform. $\mathcal{F}(e^{-\pi|x|^2}) = e^{-\pi|\xi|^2}$, $\mathcal{F}(e^{-\pi t^2|x|^2}) = t^{-n} e^{-\frac{\pi|\xi|^2}{t^2}}$.

3) $f(x) = \frac{1}{\pi(1+x^2)}$ (Possion kernel), analytic and decays $\sim \frac{1}{|x|^2}$

$$\hat{f}(\xi) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{e^{-i2\pi\xi x}}{1+x^2} dx = \frac{1}{\pi} 2\pi i \mathrm{Res}(\pm i) = e^{-2\pi|\xi|}$$

(If $\xi > 0$, we take the UHP contour, otherwise, take LHP)
Note that $\hat{f}(\xi)$ is non-smooth at $\xi = 0$

$$\mathcal{F}\hat{f}(x) = \int_{-\infty}^\infty \hat{f}(\xi) e^{-i2\pi\xi x} dx = \int_0^\infty e^{-2\pi(1+ix)\xi} d\xi + \int_{-\infty}^0 e^{-2\pi(-1+ix)\xi} d\xi = \frac{1}{\pi(1+x^2)}$$

4) $f(x) = xe^{-\pi x^2} = \left[-\frac{1}{2\pi}e^{-\pi x^2}\right]'$

$$\hat{f} = (i2\pi\xi)\left[\widehat{-\frac{1}{2\pi}e^{-\pi x^2}}\right](\xi) = -i\xi e^{-\pi\xi^2}$$

$$\mathcal{F}(\mathcal{F}(f)) = (-i)^2 f = -f$$

5) $f(x) = x^2 e^{-\pi x^2}$, $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$

Recall for $f : T^n \to \mathbb{R} \in L^2$, we have the Fourier inversion formula $f = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{i2\pi kx}$, $\hat{\hat{f}}(x) = f(-x)$.

For $f(\xi) \in L^1(\mathbb{R})$, we define

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)e^{i2\pi\xi\cdot x}d\xi = \hat{f}(-x)$$

Theorem: 10.7: Fourier Inversion

If $f, \hat{f} \in L^1$, then $\check{\hat{f}} = f$ a.e.

Proof. Direct integration does not work, because $f(y)e^{i2\pi\xi x}e^{-i2\pi\xi y} \notin L^1(\mathbb{R}^n_y \times \mathbb{R}^n_\xi)$

Consider the approximate identity $\phi_t(x) = t^{-n}e^{-\frac{\pi|x|^2}{t^2}} = \widehat{e^{-\pi t^2|\xi|^2}}(x)$

$$\begin{aligned}
f(x) &\overset{\text{a.e.}}{=} \lim_{t\to 0} \phi_t * f(x) \\
&= \lim_{t\to 0} \int \phi_t(x-y)f(y)dy \\
&= \lim_{t\to 0} \int \tau_x\phi_t(y)f(y)dy \\
&= \lim_{t\to 0} \int \widehat{e^{-\pi t|\xi|^2}e^{2\pi i\xi x}}(y)f(y)dy \\
&= \lim_{t\to 0} \int e^{-\pi t|\xi|^2}e^{i2\pi\xi x}\hat{f}(\xi)d\xi \text{ since } \int \hat{f}g = \int f\hat{g} \\
&= \int e^{i2\pi\xi x}\hat{f}(\xi)d\xi \text{ by Theorem 2.2} \\
&= \check{\hat{f}}(x)
\end{aligned}$$

□

Corollary 33. *1. If $f, \hat{f} \in L^1$, then $\mathcal{F}(\mathcal{F}(f))(x) = f(-x)$*

2. If $f, \hat{f} \in L^1$, then $f \in C_0$

3. If $f \in L^1$ s.t. $\hat{f} = 0$, then $f = 0$

4. $\mathcal{F}(S) = S$

Theorem: 10.8: Plancherel's Theorem

If $f, g \in S$, then $\langle f, g\rangle = \left\langle \hat{f}, \hat{g}\right\rangle$, *i.e.* $\int f\overline{g} = \int \hat{f}\overline{\hat{g}}$.

Proof. If $g \in S$, then $\check{g} \in S$ and $\overline{\hat{g}} = \check{\bar{g}}$. Therefore, $\langle f, g\rangle = \langle f, \check{\hat{g}}\rangle = \left\langle \hat{f}, \hat{g}\right\rangle$. ∎

Corollary 34. *Since S is dense in L^2, there exists extention $\tilde{\mathcal{F}} : L^2 \to L^2$ which agrees with $\mathcal{F}$ on $L^1 \cap L^2$.*

Corollary 35. *If $f, g \in L^2$, $(\hat{f} \cdot \hat{g})\check{} = f * g \in C_0$.*

Proof.

$$\begin{aligned}
g * f &= \int g(x-y)f(y)dy \\
&= \int f \cdot \overline{h}\ h(y) = \overline{g(x-y)} \\
&= \int \hat{f}\overline{\hat{g}} \text{ By Theorem 10.8} \\
&= \int \hat{f}\hat{g}e^{2\pi i\xi\cdot x}d\xi \\
&= (\hat{f}\hat{g})\check{}
\end{aligned}$$

∎

Summary: $\mathcal{F} : L^1 + L^2 \to C_0 + L^2 \subset L^\infty + L^2$

1. $\|\mathcal{F}(f)\|_{L^\infty} \le \|f\|_{L^1}$
2. $\|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}$
3. $\|\mathcal{F}(f)\|_{L^p} \le \|f\|_{L^q}$ for $q \in (1, 2)$, $\frac{1}{p} + \frac{1}{q} = 1$.

10.4 Fourier Transform on Real and Torus

We now know that we can perorm Fourier transforms $\mathbb{R}^n \leftrightarrow \mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n = T^n \leftrightarrow \mathbb{Z}^n$. Can we connect these spaces?

Given $f \in L^1(\mathbb{R}^n)$, consider the projection

$$Pf = \sum_{y\in\mathbb{Z}^n} \tau_y f \in L^1([0,1]^n) = L^1(T^n)$$

Let Q_z denote the integer lattices (squares)

$$\|Pf\|_{L^1(T^n)} \le \sum_{z\in Z^n} \|f\|_{L^1(Q_z)} = \|f\|_{L^1(\mathbb{R}^n)}, \hat{Pf} \in L^\infty(\mathbb{Z}^n)$$

***Theorem:* 10.9:**

$$\hat{Pf}(k) = \hat{f}(k)$$

Proof.

$$\begin{aligned}
\hat{Pf}(k) &= \int_{T^n} Pf(x)e^{-i2\pi kx}dx \\
&= \sum_{z\in\mathbb{Z}^n} \int_{Q_z} f(x)e^{-i2\pi kx}dx = \int_{\mathbb{R}^n} f(x)e^{-i2\pi kx}dx
\end{aligned}$$

∎

Theorem: 10.10: Poisson Summation Formula

If $f \in C(\mathbb{R}^n)$ and $|f(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}}$, $|\hat{f}(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}}$, then $Pf(x) = \sum_{k\in\mathbb{Z}^n} f(x+k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{i2\pi k\cdot x}$ uniformly in T^n.

Corollary 36. *Under the same assumption, with $x = 0$, $\sum_{k\in\mathbb{Z}^n} f(k) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)$.*

10.5 Fourier Inversion on General Functions

For $f \in L^1$, we can compute the Fourier transform directly. For $f \in L^2$, we can compute the Fourier transform by L^2-approximation. How should we invert $\mathcal{F}$ for $\hat{f} \notin L^1$? The idea is to introduce a multiplier Φ to make the integral converge and take it to 1.

$$\Phi(\xi) \in L^1 \cap C_0 \Rightarrow \Phi(\xi) \in L^\infty$$
$$\Phi(0) = 1$$
$$\phi(x) = \check{\Phi}(x) \in L^1 \Rightarrow \hat{\phi} = \Phi, \int \phi = \Phi(0) = 1$$
$$\Phi(t\xi) \to 1 \text{ as } t \to 0 \text{ pointwise}$$

$\check{\Phi}(t) = t^{-n}\phi\left(\frac{x}{t}\right)$ is the approximate identity.

Theorem: 10.11: Approximate Fourier Inversion

If $f \in L^1 + L^2$, then $\hat{f} \in L^\infty + L^2$.

$$f^t(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\Phi(t\xi)e^{i2\pi\xi\cdot x}dx$$

Then $f^t = f * \phi_t$, with the following properties:

1. If $f \in L^p$, then $f^t \in L^p$, and $\left\|f^t - f\right\|_{L^p} \to 0$
2. If f is bounded and uniformly continuous, then f^t is bounded and uniformly continuous, $\sup|f^t - f| \to 0$
3. If $|\phi(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}}$, then $f^t(x) \to f(x)$ Lebesgue-a.e.

Proof. If $f = f_1 + f_2 \in L^1 + L^2$, then

$$f_2^t(x) = \int \hat{f}_2(\xi)\Phi(t\xi)e^{i2\pi\xi\cdot x}d\xi = (\hat{f}_2\hat{\phi}_t)\check{}(x) = f_2 * \phi_t(x)$$
$$\begin{aligned} f_1^t(x) &= \int \hat{f}_1(\xi)\Phi(t\xi)e^{i2\pi\xi\cdot x}d\xi \\ &= \int \hat{f}_1(\xi)\hat{\phi}_t(\xi)e^{i2\pi\xi\cdot x}dx \\ &= \int \widehat{f_1 * \phi_t}(\xi)e^{i2\pi\xi\cdot x}d\xi = f_1 * \phi_t \end{aligned}$$

□

Theorem: 10.12: Approximate Fourier Inversion for Torus

If $|\Phi(\xi)| \leq \frac{C}{(1+|\xi|)^{n+\epsilon}}$ and $|\phi(x)| \leq \frac{C}{(1+|x|)^{n+\epsilon}}$, $f \in L^1(T^n)$, then define

$$f^t(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\Phi(tk)e^{i2\pi k \cdot x}$$

with the following properties

1. If $f \in L^p(T^n)$, then $\left\| f^t - f \right\|_{L^p} \to 0$
2. If $f \in C(T^n)$, then $\sup |f^t - f| \to 0$
3. $f^t \to f$ a.e. in Lebesgue set of f (set where Theorem 3.9 holds)

Proof. Define $\psi_t(x) = \sum_{k \in \mathbb{Z}^n} \phi_t(x-k) = \sum_{k \in \mathbb{Z}^n} \Phi(tk)e^{i2\pi k \cdot x}$, the approximate identity on T^n, $\hat{\psi}_t(k) = \Phi(tk)$.

We show that $f^t = f * \psi_t$:

$$\widehat{f * \psi_t}(k) = \hat{f}(k)\hat{\psi}_t(k) = \hat{f}(t)\Phi(tk) = \hat{f}^t(k),$$

so $f^t = f * \psi_t$ $\square$

Note: $f(x) = \lim_{t \to 0} \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\Phi(tk)e^{i2\pi k \cdot x} = f * \psi_t$

Examples:
1) $\Phi(\xi) = e^{-2\pi|\xi|}$, $\phi(x) = \frac{1}{\pi(1+x^2)}$

$$\begin{aligned} f^t(x) &= \sum_{k=-\infty}^{\infty} e^{-2\pi t|k|}\hat{f}(k)e^{i2\pi kx} \\ &= \sum_{k=-\infty}^{\infty} r^{|k|}\hat{f}(k)e^{i2\pi kx} \text{ Let } r = e^{-2\pi t} \\ &= \hat{f}(0) + \sum_{k=1}^{\infty} r^k(\hat{f}(k)e^{i2\pi kx} + \hat{f}(-k)e^{-i2\pi kx}) \end{aligned}$$

It follows the form of Abel sum

Definition: 10.6: Abel Sum

If $\sum_{k=0}^{\infty} a_k$ diverges, consider $\sum_{k=0}^{\infty} a_k r^k$ for $r < 1$ and take $r \to 1$.

2) If $\Phi(\xi) = \max(1 - |\xi|, 0)$, $\phi(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2$

$$\begin{aligned}
f^t(x) &= \sum_{k=-\infty}^{\infty} \max(1 - t|k|, 0)\hat{f}(k)e^{i2\pi k\cdot x} \\
&= \sum_{|k|<\frac{1}{t}} (1 - t|k|)\hat{f}(k)e^{i2\pi k\cdot x} \\
&= \sum_{k=-\infty}^{\infty} \left(1 - \frac{|k|}{m+1}\right)\hat{f}(k)e^{i2\pi k\cdot x} \text{ Let } \frac{1}{t} = m+1 \\
&= f(0) + \frac{1}{m+1}\sum_{k=1}^{m}(m+1-k)(\hat{f}(k)e^{i2\pi kx} + \hat{f}(-k)e^{-i2\pi kx})
\end{aligned}$$

***Definition:* 10.7: Cesaro Sum**

If $\sum_{k=0}^{\infty} a_k$ diverges, let $S_n = \sum_{k=0}^{n} a_k$ be the partial sum. Consider $\frac{1}{m+1}\sum_{n=0}^{m} S_n$ and take $m \to \infty$.

Summary: If we pick $\phi = e^{-2\pi|\xi|}$, then Fourier inversion holds for $f \in L^1 + L^2$ if $\sum$ is Abel sum. If we pick $\phi = \max(1 - |\xi|, 0)$, then Fourier inversion holds for $f \in L^1 + L^2$ if $\sum$ is Cesaro sum.

10.6 Pointwise Convergence

Let $f \in C(S^1)$. Fourier inversion gives that in L^2, $f = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{i2\pi kx}$.

For a fixed $x \in S^1$, does $S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k)e^{i2\pi kx} \to f(x)$?

Idea 1: Take $\Phi(\xi) = \chi_{[-1,1]}$, scaling gives $[-m, m]$, then $f^m(x) = f * \phi = S_m f(x)$.
Here $\phi(x) = \int_{-\infty}^{\infty} \Phi(\xi)e^{i2\pi\xi x}d\xi = \frac{\sin(2\pi x)}{\pi x} = \mathcal{O}\left(\frac{1}{|x|}\right)$ which does not decay fast enough.

Idea 2: Consider $S^1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

$$\begin{aligned}
S_m f(x) &= \sum_{k=-m}^{m} \hat{f}(k)e^{i2\pi kx} = \sum_{k=-m}^{m} \int_{-1/2}^{1/2} f(y)e^{-i2\pi ky}e^{i2\pi kx}dy \\
&= \int_{-1/2}^{1/2} f(y) \sum_{k=-m}^{m} e^{i2\pi k(x-y)}dy \\
&= f * D_m(x - y)
\end{aligned}$$

Here $D_m(x) = \sum_{k=-m}^{m} e^{i2\pi kx}$ is Dirichlet kernel.

We also have a closed formula for D_m:

$$D_m(x) = \sum_{k=-m}^{m} e^{i2\pi kx} = \frac{e^{i(2m+1)\pi x} - e^{-i(2m+1)\pi x}}{e^{\pi x}e^{-i\pi x}} = \frac{\sin((2m+1)\pi x)}{\sin(\pi x)} \in C^{\infty}(S^1)$$

However, D_m does not behave like an approximate identity, $f * D_m \not\to f$ as $m \to \infty$. D_m has too many oscillations away from 0 as $m \to \infty$.
In limit: $D_m^2 \to \frac{1}{|\sin(\pi x)|^2}$, $\int_0^x D_m(t)dt \to$step function. $\|D_m\|_{L^1} \to \infty$ as $m \to \infty$.

Pointwise convergence fails, typically when function oscillates fast.

***Lemma:* 10.1:**

Let $\phi, \psi : [a,b] \to \mathbb{R}$, ϕ is monotone and right continuous, and ψ is continuous, then for $\eta \in [a,b]$

$$\int_a^b \phi(x)\psi(x)dx = \phi(a)\int_a^\eta \psi(x)dx + \phi(b)\int_\eta^b \psi(x)dx$$

***Lemma:* 10.2:**

$\exists c > 0$ s.t. for any $[a,b] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ and any $m \geq 0$, $\left|\int_a^b D_m(x)dx\right| \leq c$. Moreover, $\int_{-1/2}^0 D_m(x)dx = \int_0^{1/2} D_m(x)dx = \frac{1}{2}$

Proof. $D_m = \frac{\sin((2m+1)\pi x)}{\sin(\pi x)}$
If $0 \notin [a,b]$, then it is trivial because D_m is bounded. Otherwise, by approximation:

$$\begin{aligned}\int_a^b D_m(x)dx &= \int_a^b \frac{\sin((2m+1)\pi x)}{\pi x}dx + \int_a^b \sin((2m+1)\pi x)\left(\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right)dx \\ &= \operatorname{sinc}((2m+1)\pi b) - \operatorname{sinc}((2m+1)\pi a) + \text{const} \leq C\end{aligned}$$

□

***Theorem:* 10.13: Dirichlet**

If $f \in BV(S^1)$, *i.e.* for $x_1 \leq x_2 \leq \cdots \leq x_n$, $\sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})| < C$, then for any $x \in S^1$,

$$\lim_{m\to\infty} S_m f(x) = \frac{1}{2}(f(x+) + f(x-))$$

Proof. WLOG. Assume f is monotone non-decreasing and right continuous. Also assume $x = 0$.

$$\begin{aligned}S_m f(0) - \frac{1}{2}(f(0-) + f(0+)) &= \int_{-1/2}^{1/2} f(x)D_m(x)dx - \frac{1}{2}(f(0-) + f(0+)) \\ &= \int_{-1/2}^0 (f(x) - f(0-))D_m(x)dx + \int_0^{1/2} (f(x) - f(0+))D_m(x)dx,\end{aligned}$$

since $\int_{-1/2}^0 D_m(x)dx = \int_0^{1/2} D_m(x)dx = \frac{1}{2}$. Split the integral:

$$I = \int_{-1/2}^0 (f(x) - f(0-))D_m(x)dx = \int_{-\delta}^0 (f(x) - f(0-))D_m(x)dx + \int_{-1/2}^{-\delta} (f(x) - f(0-))D_m(x)dx$$

Consider the first part:

$$\begin{aligned}\left|\int_{-\delta}^{0}(f(x)-f(0-))D_m(x)dx\right| &\leq |f(-\delta)-f(0-)|\left|\int_{-\delta}^{0}D_m(x)dx\right| \text{By Lemma 10.1}\\ &\leq C\,|f(-\delta)-f(0-)| \text{ By Lemma 10.2}\end{aligned}$$

For the second part, because $\sin(\pi x)\neq 0$ on $[-1/2,-\delta]$,

$$\begin{aligned}\int_{-1/2}^{-\delta}(f(x)-f(0-))D_m(x)dx &= \int_{-1/2}^{-\delta}(f(x)-f(0-))\frac{\sin(2m+1)\pi x}{\sin\pi x}dx\\ &\leq C_\delta\int_{-1/2}^{-\delta}(f(x)-f(0-))\frac{e^{i(2m+1)\pi x}-e^{-i(2m+1)\pi x}}{2i}dx\\ &= \hat{g_-}(-m)+\hat{g_+}(m),\ \text{ where } g_\pm(x)=\chi_{[-1/2,-\delta]}\frac{f(x)-f(0-)}{2i}e^{\mp i\pi x}\end{aligned}$$

By Riemann-Lebesgue (Proposition 10.4), it converges to 0 as $m\to 0$. □

Corollary 37. *If f is absolutely continuous, then $S_mf(x)\to f(x)$.*

***Theorem:* 10.14: Local Convergence**

Let $(a,b)\subset S^1$ and $f\in L^1(S^1)$. If $f=0$ on (a,b), then $S_mf\to f$ in $C^0(K)$ where K is a compact subset of (a,b). It also works for smooth functions.

***Theorem:* 10.15: Gibb's Phenomenon**

Let $f:\mathbb{R}\to\mathbb{R}$ be piecewise continuously differentiable function. Suppose at some point there is a jump $f(x_0^-)-f(x_0^+)=c\neq 0$. Then for $x_m\to x_0$

$$\begin{aligned}\limsup_{m\to\infty}S_mf(x_m)&\leq f(x_0^+)+c\delta,\\ \liminf_{m\to\infty}S_mf(x_m)&\geq f(x_0^-)+c\delta,\end{aligned}$$

where $\delta\approx 0.089$.

Consider the Cesaro sum,

$$C_mf(x)=\frac{1}{m+1}\sum_{k=0}^{m}S_mf(x)=f*\left(\frac{1}{m+1}\sum_{k=0}^{m}D_k\right)(x)$$

$F_m=\dfrac{1}{m+1}\displaystyle\sum_{k=0}^{m}D_k=\left(\dfrac{\sin(2m+1)\pi x}{\sin\pi x}\right)^2\geq 0$ is the Fejer kernel. It always have pointwise convergence.

10.7 Application to PDEs

Heat Equation: Find $u(x,t)$ s.t. $\begin{cases}\partial_t u=\Delta u\\ u(0,x)=u_0(x)\end{cases}$.

Apply Fourier transform on x only,

$$\hat{u}(t,\xi)=\int_{\mathbb{R}^n}u(t,x)e^{-i2\pi\xi x}dx$$

The system then becomes:

$$\begin{cases} \partial_t \hat{u} = \sum_{k=1}^n (i2\pi\xi)^2 \hat{u}(t,\xi) = -4\pi^2|\xi|^2\hat{u}(t,\xi) \\ \hat{u}(0,\xi) = \hat{u_0}(\xi) \end{cases}$$

We convert a PDE into ODE, because derivatives of Fourier transform is converted to multiplications. Then

$$\hat{u}(t,\xi) = \exp(-4\pi^2|\xi|^2 t)\hat{u_0}(\xi)$$

$\exp(-4\pi^2|\xi|^2 t)$ is the Fourier transform of the heat kernel $G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{t}}$. Therefore,

$$u(t,x) = u_0(x) * G_t(x)$$

Theorem: 10.16:

If $u_0 \in L^p$, then $u(t,x) = u_0(x) * G_t(x)$ gives a solution to $\begin{cases} \partial_t u = \Delta u, \text{ on } \{t>0\} \times \mathbb{R}^n \\ \lim_{t\to 0} u(t,x) = u_0(x) \text{ a.e.} \\ \|u(t,\cdot) - u_0\|_{L^p} \to 0 \text{ as } t \to 0 \end{cases}$

Proof. $(\partial_t - \Delta)u = u_0 * (\partial_t - \Delta)G_t(x) = u_0 * 0 = 0$ ☐

Harmonic Function: $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\begin{cases} \Delta u + \partial_t^2 u = 0, \text{ on } \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = u_0(x) \end{cases}$.

Apply Fourier transform: $\begin{cases} \partial_t^2 \hat{u} - 4\pi^2|\xi|^2\hat{u} = 0 \\ \hat{u}(0,\xi) = \hat{u_0}(\xi) \end{cases}$. This gives the transformed solution:

$$\hat{u}(t,\xi) = C_1(\xi)\exp(2\pi|\xi|t) + C_2(\xi)\exp(-2\pi|\xi|t), \hat{u_0}(\xi) = C_1(\xi) + C_2(\xi)$$

Note that $\exp(2\pi|\xi|t)$ grows exponentially fast, so we set $C_1(\xi) = 0$.

$$\hat{u}(t,\xi) = \hat{u_0}(\xi)\exp(-2\pi|\xi|t)$$

$\exp(-2\pi|\xi|t)$ is the Fourier transform of Poisson kernel $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$, $P_t(x) = t^{-n}P_1\left(\frac{x}{t}\right)$.

$$u(t,x) = u_0(x) * P_t(x)$$

Theorem: 10.17:

If $u_0 \in L^p$, then $u(t,x) = u_0(x) * P_t(x)$ gives a solution to $\begin{cases} \partial_t^2 \hat{u} - 4\pi^2|\xi|^2\hat{u} = 0 \text{ on } \mathbb{R}_+ \times \mathbb{R}^n \\ \lim_{t\to 0} u(t,x) = u_0(x) \text{ a.e.} \\ \|u(t,\cdot) - u_0\|_{L^p} \to 0 \text{ as } t \to 0 \end{cases}$

Possion Equation: $f \in C_C^\infty(\mathbb{R}^n)$, $\Delta u = f$.

Apply Fourier transform: $-4\pi^2|\xi|^2\hat{u}(\xi) = \hat{f}(\xi)$. This gives:

$$\hat{u}(\xi) = \hat{f}(\xi)\left(-\frac{1}{4\pi^2|\xi|^2}\right)$$

We want to find a function G s.t. $\hat{G} = -\frac{1}{4\pi^2|\xi|^2}$, $G(x) = c_n \frac{1}{|x|^{n-2}}$, but $G(x) \notin L^1$ or L^2. However, we can still show, using IBP, that

$$u(x) = f * G(x) = c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

Wave Equation: $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_0'(x) \end{cases}$

Apply Fourier transform: $\begin{cases} \partial_t^2 \hat{u} + 4\pi^2|\xi|^2 \hat{u} = 0 \\ \hat{u}(0,\xi) = \hat{u_0}(\xi) \\ \partial_t \hat{u}(0,\xi) = \hat{u_0'}(\xi) \end{cases}$

$$\Rightarrow \hat{u}(t,\xi) = \cos(2\pi|\xi|t)\hat{u_0}(\xi) + \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\hat{u_0'}(\xi)$$

But there is no function whose Fourier transform is $\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}$

Remark 38. Fourier transform typically works for linear, constant coefficient PDEs, but not for nonlinear PDEs. In the case where we cannot find a Fourier inverse, we should consider a broader sense of functions.

10.8 Frechet Space and Distribution

Consider $\delta : C_C^\infty(\mathbb{R}^n) \to \mathbb{R}$ s.t. $\delta(f) = f(0)$, $\langle \delta, f \rangle = \int \delta f dx = f(0)$. We can think of it as a measure supported at $\{0\}$.

Let K be a compact set. We want to consider the space $C^\infty(K) = \cap_{k\geq 0} C^k(K)$ with $\|\cdot\|_{C^k}$ a sequence of norms $\phi_i \overset{C^\infty}{\to} \phi$ if $\|\phi_i - \phi\|_{C^k} \to 0$, $\forall k \geq 0$.

Let X be a vector space, a sequence of countably many semi-norms $\|\cdot\|_{\alpha\in\mathbb{Z}}$ defines a topology on X.

***Definition:* 10.8: Frechet Space**

$(X, \|\cdot\|_\alpha)$ is Frechet space if it is complete and non-degenerate (*i.e.* $\|x\|_\alpha = 0, \forall \alpha \in \mathbb{Z} \Rightarrow x = 0$). $\{\phi_i\}_{i=1}^\infty \subset X$ is Cauchy if and only if it is Cauchy w.r.t. $\|\cdot\|_\alpha, \forall \alpha \in \mathbb{Z}$.

Remark 39. All Banach spaces are Frechet spaces. Frechet spaces with finitely many norms can be converted to a Banach space.

Examples:

1. Define $C^\infty(\mathbb{R}^n) = \cap_{k\geq 0} C^k(\mathbb{R}^n)$ where $C^k(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \|f\|_{C^k} < \infty\}$, $C^\infty(\mathbb{R}^n)$ is a Frechet space.
2. $C_{loc}^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : f \text{ is smooth}\}$ is Frechet space with smooth convergence on compact set w.r.t. $\|\cdot\|_{C^k(\overline{B_R})}$ for $k = 0,1,2,...$ and $R = 1,2,....$ This is a semi-norm, because $f|_{\overline{B_R}} = 0$ for many functions.
3. $L_{loc}^p(\mathbb{R}^n) = \{f : f \text{ is } L^p \text{ on compact sets}\}$ w.r.t. $\|\cdot\|_{L^p(\overline{B_R})}$ for $R = 1,2,3,...$ is a Frechet space with topology of L^p convergence on compact sets.
   Similarly, $L_{loc}^p((0,1)) = \{f : (0,1) \to \mathbb{R} : f \in L^p[a,b], [a,b] \subset (0,1)\} \neq L^p(0,1)$ is a Frechet space
4. S =Schwarz space with $\|\phi\|_{\alpha,\beta} = \sup \left| x^\alpha \partial^\beta \phi \right|$ is a Frechet space.

Definition: 10.9:

If $(X, \|\cdot\|_{\alpha_i})$ and $(Y, \|\cdot\|_{\beta_i})$ are Frechet spaces, then $T : X \to Y$ is continuous if and only if $\forall \beta_i$, $\exists \alpha_{i_1}, ..., \alpha_{i_N}$ s.t. $\|Tf\|_{\beta_i} \leq C \sum_{k=1}^{N} \|f\|_{\alpha_{i_k}}$.

Examples:

1. $\partial^\alpha : C^\infty(K) \to C^\infty(K)$ and $\partial^\alpha : C^\infty_{loc}(K) \to C^\infty_{loc}(K)$ are continuous
2. The Fourier transform $\mathcal{F} : S \to S$ is a continuous isomorphism

Proof. For any α, β

$$\begin{aligned}\sup \left|\xi^\alpha \partial^\beta \hat{f}\right| &\leq \sup \left|\mathcal{F}(\partial^\alpha (x^\beta f))\right| \\ &\leq C \left\|\partial^\alpha (x^\beta f)\right\|_{L^1} \leq C \sum_{\alpha_i, \beta_i} \left\|x^{\beta_i} \partial^{\alpha_i} f\right\|_{L^1} \\ &\leq C \sum \left\|(1+|x|^2)^{\frac{n}{2}+1} x^{\beta_i} \partial^{\alpha_i} f\right\|_{L^\infty}\end{aligned}$$

The last inequality comes from $\left\|f \frac{(1+|x|^2)^{\frac{n}{2}+1}}{(1+|x|^2)^{\frac{n}{2}+1}}\right\|_{L^1} \leq \left\|f(1+|x|^2)^{\frac{n}{2}+1}\right\|_{L^\infty} \left\|\frac{1}{(1+|x|^2)^{\frac{n}{2}+1}}\right\|_{L^1}$. □

If $K \subset \mathbb{R}^n$ is compact, then $C^\infty_C(K) \subset C^\infty(\mathbb{R}^n)$ is a closed subspace and is a Frechet space.

If $U \subset \mathbb{R}^n$ is open, denote $D(U) = C^\infty_C(U) = \bigcup_{K \subset U} C^\infty_C(K)$, K is compact. We say $\phi_i \to \phi$ in $C^\infty_C(U)$ if $\phi_i, \phi \in C^\infty_C(K)$ for some $K \subset U$ compact and $\phi_i \to \phi$ in $C^\infty_C(K)$. $D(U)$ is not Frechet, but union of Frechet spaces.

A linear functional $F : D(U) \to \mathbb{R}$ is continuous if and only if $F|_{C^\infty_C(K)}$ is continous for all $K \subset U$ compact.

Definition: 10.10: Distribution

Given $U \subset \mathbb{R}^n$ open, a distribution is a continouous linear map $F : D(U) \to \mathbb{R}$. $F_i \to F$ if and only if $\forall \phi \in C^\infty_C(U), \langle F_i, \phi\rangle \to \langle F, \phi\rangle$

Examples:

1. If $F \in L^1_{loc}$,then F defines a distribution by $\langle F, \phi\rangle = \int_U \phi(x)F(x)dx$, where ϕ is a compactly supported function.

   F defines the zero distribution if and only if $F(x) = 0$ for a.e. x.

   $L^1_{loc} \subset D'(U)$ =dual space of $D(U)$=space of distributions=continuous linear functionals on $D(U)$.
2. $\langle \delta_{x_0}, \phi\rangle = \phi(x_0)$ is a distribution.
3. Any Radon measure $d\mu$ is a distribution (Radon measures are Borel measures which are finite on compact sets), δ_{x_0} is a Radon measure. $\langle d\mu, \phi\rangle = \int_U \phi d\mu$, $\left|\int_K \phi d\mu\right| \leq C_C \|\phi\|_{C^0(K)}$
4. $\langle \partial\delta_0, \phi\rangle = -\phi'(0)$. If $\phi \in C^\infty_C(K)$, $|\langle \partial\delta_0, \phi\rangle| \leq \|\phi\|_{C^1(K)}$
5. $\langle F, \phi\rangle = \displaystyle\int_U \partial^\alpha \phi(x) d\mu$ is a distribution.

Non-examples:

1. $\frac{1}{|x|^n}$ on $\mathbb{R}^n$ is not a distribution, because it is not $L^1_{loc}(\mathbb{R}^n)$
2. Non-Radon measures are typically not distributions, because they may not be finite on compact sets.

Let $f \in L^1(\mathbb{R}^n)$ s.t. $\displaystyle\int_{\mathbb{R}^n} f = 1$, $f_t(x) = t^{-n} f\left(\frac{x}{t}\right)$, then $f_t \to \delta_0$ as a distribution. *i.e.* for $\phi \in C^\infty_C(\mathbb{R}^n)$, $\displaystyle\int_{\mathbb{R}^n} f_t\phi \to \phi(0)$ as $t \to \infty$.

***Theorem:* 10.18: Properties of Distributions**

1. If $U' \subset U$, then $F \in D'(U)$ restricted to $F|_{D'(U')}$ is a distribution on U'
2. If $U_1, U_2 \subset \mathbb{R}^n$, $F_1 \in D'(U_1), F_2 \in D'(U_2)$, and $F_1|_{D'(U_1 \cap U_2)} = F_2|_{D'(U_1 \cap U_2)}$, then $F_i = F|_{D'(U_i)}$ for $F \in D'(U_1 \cup U_2)$
3. Let $F \in D'(U)$, $\text{supp}(F) = U \setminus \displaystyle\bigcup_{U' \subset U, F|_{D'(U')}=0} U'$ (subset of U where F is non-zero)

Example:

1. $\text{supp}(\delta) = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus \{0\}) = \{0\}$
2. If $F \in L^1_{loc}$, then $\text{supp}(F)$ is the usual notion of support
3. $\text{supp}(\partial\delta_0) = \{0\}$.

***Theorem:* 10.19: Operations on Distributions**

1. Addition and scalar multiplication follows vector space
2. Multiplication by $C^\infty_{loc}(U)$: Let $\eta \in C^\infty_{loc}(U)$, $F \in D'(U)$, $\langle \eta F, \phi\rangle = \langle F, \eta\phi\rangle$
3. Differetiation: Given $F \in D'(U)$, define $\partial_i F \in D'(U)$ by $\langle \partial_i F, \phi\rangle = -\langle F, \partial_i\phi\rangle$, $\langle \partial^\alpha F, \phi\rangle = (-1)^{|\alpha|}\langle F, \partial^\alpha\phi\rangle$, $\partial_i F$ is a distribution
4. $\partial_i(\eta F) = (\partial_i\eta)F + \eta(\partial_i F)$
5. $\partial_i\partial_j F = \partial_j\partial_i F$
6. $\text{supp}(\partial^\alpha F) \subset \text{supp}(F)$
7. Translation: If $F \in D'(U)$, $\tau_y F \in D'(U+y)$, $\langle \tau_y F, \phi\rangle = \langle F, \tau_y\phi\rangle$
8. Compose with linear map: If $S : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, $F \in D'(U)$, then $F \circ S \in D'(S^{-1}(U))$, $\langle F \circ S, \phi\rangle = |\det S^{-1}| \left\langle F, \phi \circ S^{-1}\right\rangle$.

For differentiation, given $f \in C^\infty_{loc}(U)$, $\partial_i f \in C^\infty_{loc}(U)$. By IBP, $\displaystyle\int_U (\partial_i f)\phi = -\int_U f(\partial_i\phi)$. Boundary term vanishes, because ϕ is compactly supported.

Examples:

1) $\langle \partial_i\delta_{x_0}, \phi\rangle = -\langle \delta_{x_0}, \delta_i\phi\rangle = -\partial_i\phi(x_0)$

2) $f(x) = \chi_{(0,\infty)} \in L^1_{loc}(\mathbb{R})$. Classically, $f'(x) = 0$ for $x \neq 0$ and does not exist at 0.

$$\left\langle f', \phi\right\rangle = -\int_{\mathbb{R}} f\phi' = -\int_0^\infty \phi' dx = -(\phi(\infty) - \phi(0)) = \phi(0) \Rightarrow f' = \delta_0$$

3) On $\mathbb{R}$, take $f(x) = \chi_{(0,\infty)} \log x \in L^1_{loc}(\mathbb{R})$. Classically, $f'(x) = \chi_{(0,\infty)}\frac{1}{x} \notin L^1_{loc}(\mathbb{R})$

$$\left\langle f', \phi\right\rangle = -\int_0^\infty \log x\phi'(x)dx = -\int_\epsilon^\infty \log x\phi'(x)dx - \int_0^\epsilon \log x\phi'(x)dx$$

$\int_0^\epsilon \log x\phi'(x)dx \to 0$ as $\epsilon \to 0$, because $\log x \in L^1_{loc}(\mathbb{R})$.
$\int_\epsilon^\infty \log x\phi'(x)dx = \int_\epsilon^\infty \frac{\phi(x)}{x}dx + \phi(\epsilon)\log\epsilon$.

4) $f(x) = \log|x|$, $f'(x) = \frac{1}{x}$, but not defined at $x = 0$

$$
\begin{aligned}
\langle f, \phi\rangle &= -\int_{-\infty}^{\infty} \log|x|\phi'(x)dx = \lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\phi(x)}{x} + \int_{-\infty}^{-\epsilon}\frac{\phi(x)}{x} + (\phi(\epsilon)-\phi(-\epsilon))\log\epsilon\right] \\
&= \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\phi(x)}{x}dx \\
&= \text{p.v.}\int_{\mathbb{R}}\frac{\phi(x)}{x}dx
\end{aligned}
$$

Definition: 10.11: Convolution of Distribution

Let $F \in D'(U)$, $\psi \in C_C^\infty(\mathbb{R}^n)$. Define $\tilde{\psi}(x) = \psi(-x)$, $\tilde{F} = F \circ (-Id)$. Then $F * \psi(x) = \left\langle F, \tau_x\tilde{\psi}\right\rangle = \langle F, \psi(x - \cdot)\rangle$ is well-defined if $\text{supp}(\psi) \subset x - U \Leftrightarrow x - \text{supp}(\psi) \subset U$. $F * \psi$ is a function on $V = \{x : x - \text{supp}(\psi) \subset U\}$.

Example: $F = \delta_0$, $F * \psi(x) = \langle\delta_0, \psi(x-0)\rangle = \psi(x)$, $\delta_0 *$ gives the identity.

Theorem: 10.20: Properties of Convolution of Distribution

1. $F * \psi \in C_{loc}^\infty(V)$
2. $\partial_i(F * \psi) = (\partial_i F) * \psi = F * (\partial_i\psi)$
3. If $\phi \in C_C^\infty(V)$, then $\langle F * \psi, \phi\rangle = \int_V F * \psi(x)\phi(x)dx = \left\langle F, \phi * \tilde{\psi}\right\rangle$

Proof. 1)

$$
\begin{aligned}
\lim_{h\to 0}\frac{\tau_{he_i}(F*\psi) - F*\psi}{h} &= F * \lim_{h\to 0}\frac{\tau_{he_i}\psi - \psi}{h} \\
&= \lim_{h\to 0}\left\langle F, \tau_x\tilde{\partial_i^h\psi}\right\rangle = \left\langle F, \tau_x\tilde{\partial_i\psi}\right\rangle \\
&= F * \partial_i\psi(x)
\end{aligned}
$$

2)

$$
\partial_i F * \psi(x) = \left\langle \partial_i F, \tau_x\tilde{\psi}\right\rangle = -\left\langle F, \tau_x\partial_i\tilde{\psi}\right\rangle = \left\langle F, \tau_x\tilde{\partial_i\psi}\right\rangle = F * \partial_i\psi
$$

3) If F is a function, then it is equivalent to Theorem 2.9:

$$
\iint F(y)\psi(x-y)\phi(y)dydx = \iint F(y)\psi(x-y)\phi(y)dxdy
$$

For F a linear functional, consider the Riemann sum

$$\phi * \tilde{\psi} = \int_V \phi(y)\psi(y-x)dy = \lim_{m\to\infty} 2^{-nm} \sum_{j=1}^{2^{nm}} \phi(y_j)\psi(y_j - x)$$

$$\begin{aligned}\left\langle F, \phi * \tilde{\psi} \right\rangle &= \lim_{m\to\infty} 2^{-nm} \sum_{j=1}^{2^{nm}} \phi(y_j) \left\langle F, \psi(y_j - \cdot) \right\rangle \\ &= \int_V \phi(y) \left\langle F, \psi(y - \cdot) \right\rangle dy \\ &= \int_V F * \psi(y)\phi(y)dy\end{aligned}$$

□

Theorem 1. *density-of-compactly-supported-smooth-func $C_C^\infty \subset D'(U)$ is dense in the weak$*$-topology.*

Proof. Let $F \subset D'(U)$

Step 1: Approximate F by compactly supported functions.
Let $K_1 \subset K_2 \subset \cdots \subset U$ be compact exhaustion of U. $\eta_i = 1$ on K_i, $\eta_i \in C_C^\infty(U)$.
$\eta_i F \to F$ in $D'(U)$ because $\langle \eta_i F, \phi \rangle = \langle F, \eta_i \phi \rangle \to \langle F, \phi \rangle$.

Step 2: Approximate $\eta_i F$ by $C_C^\infty(U)$.
Let $\psi \in C_C^\infty(B_1)$ with $\int \psi = 1$. $\psi_t = t^{-n}\psi\left(\frac{x}{t}\right)$.
Claim: $(\eta_i F) * \psi_t \in C_C^\infty(U)$ if $t << 1$, $(\eta_i F) * \psi_t \to \eta_i F$ in $D'(U)$.
$\langle \eta_i F * \psi_t, \phi \rangle = \left\langle \eta_i F, \phi * \tilde{\psi}_t \right\rangle = \langle \eta_i F, \phi \rangle$. □

Remark 40. Multiplication of distributions is ill-defined in general.

10.8.1 Subspace of Distributions

Distributions with Compact Support

$$\mathcal{E}'(U) = \left\{ F \in D'(U) : \operatorname{supp}(F) \subset K \text{ for some compact } K \subset U \right\}$$

Recall that $D'(U)$ is dual to $D(U) \subset C_C^\infty(U)$.

If $F \in \mathcal{E}'(U)$, then $\operatorname{supp}(F)$ is compact subset of U. Let $\eta = 1$ on $\operatorname{supp}(F)$ and $\eta \in C_C^\infty(U)$. Let $\phi \in C_{loc}^\infty(U)$, then $\langle F, \phi \rangle = \langle F, \eta\phi \rangle$.

Since $C_{loc}^\infty(U) \subset D(U)$, $\mathcal{E}'(U) \subset (C_{loc}^\infty(U))'$.

Lemma 12. *$C_C^\infty(U)$ is dense in $C_{loc}^\infty(U)$.*

Claim: $\mathcal{E}'(U)$ is the dual space of $C_{loc}^\infty(U)$.

Proof. Consider the two sets

$$\begin{aligned}(C_{loc}^\infty(U))' &= \left\{ F : C_{loc}^\infty \to \mathbb{R} : |\langle F, \phi \rangle| \le C \|\phi\|_{C^k(K)} \text{ for some } k > 0, K \text{ compact} \right\} \\ \mathcal{E}'(U) &= \left\{ F \in D'(U) : \operatorname{supp}(F) \subset K \text{ for some compact } K \subset U \right\}\end{aligned}$$

Firstly, $\mathcal{E}'(U) \subset (C_{loc}^\infty(U))'$ as shown above.

Now let $F \in (C^\infty_{loc}(U))'$, by definition, $|\langle F, \phi \rangle| \leq C \|\phi\|_{C^k(K)}$. Pick $\eta_K = 1$ on K, $\eta_K \in C^\infty_C(U)$, $\phi - \eta_K \phi = 0$ on K. Therefore,

$$\langle F, \phi - \eta_K \phi \rangle = 0 \Rightarrow \langle F, \phi \rangle = 0,$$

if $\text{supp}(\phi) \cap K = \emptyset$, $\phi = 0$ on K.

Therefore, $\text{supp}(F) \subset K$ is a compact subset of U, $F \in \mathcal{E}'(U)$. $\square$

Theorem: 10.21: Properties of $\mathcal{E}'(U)$

1. $C^\infty_{loc} \cdot \mathcal{E}' \subset \mathcal{E}'$
2. $\partial^\alpha \mathcal{E}' \subset \mathcal{E}'$

Recall $F * \phi(x) = \langle F, \phi(x - \cdot) \rangle$. If $F \in \mathcal{E}'(\mathbb{R}^n)$, then RHS is well-defined for $\phi \in C^\infty_{loc}(\mathbb{R}^n)$. Therefore, $F * \phi(x)$ makes sense if $\phi \in C^\infty_{loc}(\mathbb{R}^n)$.

Claim: $F * \phi(x) \in C^\infty_{loc}$ and

$$\int_{\mathbb{R}^n} F * \phi(x) \psi(x) = \left\langle F, \phi * \tilde{\psi} \right\rangle, \forall \psi \in C^\infty_C(\mathbb{R}^n)$$

Another point of view:

$$\langle F * \phi, \psi \rangle = \left\langle F, \phi * \tilde{\psi} \right\rangle,$$

where $F * \phi$ is a distribution.

When $F \in \mathcal{E}'$, then RHS makes sense even if ϕ is a distribution, $\phi * \tilde{\psi}$ is a smooth function, $F * \tilde{\psi}$ is compactly supported.
i.e. If $F \in \mathcal{E}'(\mathbb{R}^n)$, $G \in D'(\mathbb{R}^n)$, $F * G \in D'(\mathbb{R}^n)$, then

$$\langle F * G, \psi \rangle = \left\langle F, G * \tilde{\psi} \right\rangle = \left\langle G, F * \tilde{\psi} \right\rangle$$

Example: $\delta_0 \in \mathcal{E}'$,

$$\langle \delta_0 * F, \phi \rangle = \left\langle F, \delta_0 * \tilde{\phi} \right\rangle = \langle F, \phi \rangle,$$

so $\delta_0 * F = F$

Tempered Distributions

Notice $C^\infty_C(\mathbb{R}^n) \subset S(\mathbb{R}^n)$, then $S' \subset (C^\infty_C)'$.

$$\begin{aligned} S'(\mathbb{R}) &= \left\{ F : S \to \mathbb{R} : |\langle F, \phi \rangle| \leq C \sum_{i=1}^N \sup_x \left| x^{\alpha_i} \partial^{\beta_i} \phi(x) \right| \right\} \\ &= \left\{ F \in D'(\mathbb{R}^n) : \exists (\alpha_i, \beta_i), i = 1, ..., N, C \text{ s.t. } |\langle F, \phi \rangle| \leq C \sum_{i=1}^N \|\phi\|_{\alpha_i, \beta_i} \right\} \end{aligned}$$

is the space of tempered distributions.

Recall $\int \hat{f} \phi = \int f \hat{\phi}$ for $f, \phi \in S$. We can replace f with a tempered distribution. We want to use this to define the Fourier transform for distributions.

If F is tempered, then we can test against any $\phi \in S$.

Examples:
1) $\mathcal{E}'(\mathbb{R}^n) \subset S'$, so δ_0 is tempered

2) $f(x) = |x|^k$,

$$\begin{aligned}
|\langle f, \phi\rangle| &\leq \int_{\mathbb{R}^n} |x|^k |\phi(x)| dx \\
&= \left\| |x|^k \phi(x) \right\|_{L^1} \\
&\leq C \left\| (1+|x|)^{k+n+1} \phi(x) \right\|_{L^\infty} \\
&\leq C \sum_{|\alpha|<k+n+1} \|\phi\|_{\alpha,0}
\end{aligned}$$

The second inequality is because $\|f\|_{L^1} = \left\| \frac{(1+|x|^{n+1})}{(1+|x|)^{n+1}} f \right\|_{L^1} \leq \left\| \frac{1}{(1+|x|)^{n+1}} \right\|_{L^1} \left\| (1+|x|)^{n+1} f \right\|_{L^\infty}$.

3) $f(x) = e^x$ is not tempered
Let $\eta = \chi_{[0,\infty)}$, then $\eta e^{-\frac{1}{2}x}$ is a tempered distribution, but $\left\langle f, \eta e^{-\frac{1}{2}x} \right\rangle$ is unbounded.
Let ϕ_k be a Schwartz function on $[k, k+1]$, $\langle f, \phi_k \rangle \sim e^k$, but $C \|\phi_k\|_{\alpha,\beta} \sim |k|^\alpha$.

4) $f \in L^1_{loc}$, $\int_{\mathbb{R}^n} (1+|x|)^{-N} f(x) < \infty$ for some N, then f is tempered even if f is exponential pointwise

5) $F(x) = \sin(e^x) \in S'$, because sin is bounded. Its derivative $F'(x) = e^x \cos(e^x) \in S'$. Although F' has an exponential component, $\cos(e^x)$ gives cancellation. It is still tempered.

***Theorem:* 10.22: Properties of S'**

1. S' is closed under differentiation ∂^α, since S is closed under differentiation
2. Let $\eta \in C^\infty_{loc}$, η is slowly (polynomially) growing if $\forall \alpha$, $\exists n$ s.t. $|\partial^\alpha \eta|(x) \leq C(1+|x|)^N$. If $F \in S'$ and η is slowly growing, then $\eta F \in S'$, $\eta S \subset S$, $\eta S' \subset S'$

Examples: $\eta(x) = p(x)$, $\eta(x) = \sin x p(x)$, $\eta(x) = \sin(p(x))$, $\eta(x) = (1+|x|^2)^k$ are slowly growing.

Let $F \in S'$, $\psi \in S$, then the convolution $F * \psi(x) = \langle F, \psi(x - \cdot)\rangle$ is well-defined.

Claim: $F * \psi$ is smooth, slowly growing and $\langle F * \psi, \phi \rangle = \left\langle F, \phi * \tilde{\psi} \right\rangle$

Proof. We prove the slowly growing here,

$$|F * \psi(x)| = |\langle F, \psi(x - \cdot)\rangle| \leq C \sum_{i=1}^N \sup_y \left| y^{\alpha_i} \partial^{\beta_i} \psi(x-y) \right|$$

Note $|y|^\alpha \leq C(|x-y|^\alpha + |x|^\alpha)$.

$$|\langle F, \psi(x - \cdot)\rangle| \leq C \sum_i \sup_y \left| |x-y|^{\alpha_i} \partial^{\beta_i} \psi(x-y) \right| + \sup_y \left| |x|^{\alpha_i} \partial^{\beta_i} \psi(x-y) \right| \leq C(1+|x|^\alpha)$$

The same applies to derivatives, so F is slowly growing $\square$

Then S'(slowly growing) $\subset S'$, $S' * S \subset$ slowly growing $\subset S'$.

10.8.2 Fourier Transform with Tempered Distributions

If $F \in S'$, then $\left\langle \hat{F}, \phi \right\rangle = \left\langle F, \hat{\phi} \right\rangle$, $\langle \check{F}, \phi \rangle = \langle F, \check{\phi} \rangle$.

Example:

$$\left\langle \hat{\delta_\xi}, \phi \right\rangle = \left\langle \delta_\xi, \hat{\phi} \right\rangle = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i2\pi\xi x} dx = \left\langle e^{-i2\pi\xi x}, \phi(x) \right\rangle$$

Therefore, $\hat{\delta_\xi} = e^{-i2\pi\xi x} = E_\xi(x)$.

Fourier inversion holds: $\mathcal{F}^{-1}\mathcal{F}(F) = F \Rightarrow \mathcal{F}(S') = S'$

Basic properties of Fourier transforms all holds

1. $\mathcal{F}(\tau_y F) = e^{-i2\pi\xi y}\hat{F}$
2. $\mathcal{F}(\partial^\alpha F) = (i2\pi\xi)^\alpha \hat{F}$
3. $\mathcal{F}(F \circ T) = |\det T|^{-1}\hat{F} \circ (T^*)^{-1}$
4. $\mathcal{F}(F * \psi) = \hat{F}\hat{\psi}$

Note $\hat{\delta_0} = 1$, $\mathcal{F}(\partial^\alpha \delta_0) = (i2\pi\xi)^\alpha$, then $\mathcal{F}(\delta_0 + \text{derivatives})$ =polynomials, and $\mathcal{F}(\sum \delta_i + \text{derivatives}) = \sum(\text{polynomials})e^{-i2\pi\xi x}$

Poisson's equation: $\delta u = f$, $f \in S$.
If we solve $\Delta K = \delta_0$, then $u(x) = K * f(x)$, $K \in S'$ is called the fundamental solution.

$$\mathcal{F}(\Delta K) = -4\pi^2|\xi|^2\hat{K}(\xi), \mathcal{F}(\delta_0) = 1 \Rightarrow \hat{K}(\xi) = -\frac{1}{4\pi^2|\xi|^2}$$
$$K(x) = -\frac{1}{4\pi^2}\mathcal{F}(|\xi|^{-2})$$

since $|\xi|^{-2}$ is radially symmetric.

Claim: $\mathcal{F}(|\xi|^{-2}) = c_n \frac{1}{|x|^{n-2}}$

Proof. We want to show that

$$\int \frac{\hat{\phi}(x)}{|\xi|^2} = \langle |\xi|^{-2}, \mathcal{F}\phi \rangle = \langle \mathcal{F}|\xi|^{-2}, \phi \rangle = c_n \int \frac{\phi(x)}{|x|^{n-2}}$$

Trick: $|\xi|^{-2} = c\int_0^\infty te^{-\pi t^2|\xi|^2}dt$

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\hat{\phi}(\xi)}{|\xi|^2} &= c\int_{\mathbb{R}^n}\int_0^\infty te^{-\pi t^2|\xi|^2}\hat{\phi}(\xi)dtd\xi \\
&= c\int_0^\infty \int_{\mathbb{R}^n} \mathcal{F}(e^{-\pi t^2|\xi|^2})\phi(x)dxdt \\
&= c\int_0^\infty t^{1-n}\int_{\mathbb{R}^n} e^{-\pi\frac{|x|^2}{t^2}}\phi(x)dxdt \\
&= \int_{\mathbb{R}^n}\int_0^\infty t^{1-n}e^{-\pi\frac{|x|^2}{t^2}}dt\phi(x)dx \\
&= c\int_{\mathbb{R}^n} \frac{1}{|x|^{n-2}}dx
\end{aligned}$$

$\square$

This means that $\Delta\left(-\frac{c_n}{|x|^{n-2}}\right) = \delta_0$ is the fundamental solution.

To solve for $\Delta u = f, f \in S$, we have

$$u(x) = \left(-\frac{c_n}{|x|^{n-2}}\right) * f = -c_n \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

For wave equation: $\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0,x) = u_0 \\ \partial_t u(0,x) = u_0' \end{cases}$

Take Fourier transform and solve:

$$\hat{u}(t,\xi) = \cos(2\pi t|\xi|)\hat{u_0}(\xi) + \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\hat{u_0'}(\xi)$$

Let $G_t = \cos(2\pi t|\xi|)$, $P_t = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}$, then

$$u(t,x) = \mathcal{F}^{-1}G_t * u_0 + \mathcal{F}^{-1}P_t * u_0'$$

Note that G_t and P_t are tempered distributions so $u(t,x)$ is well-defined

***Proposition:* 10.6: Fourier Transform of Compactly Supported Distributions**

If $F \in \mathcal{E}'$, then $\hat{F} \in C^\infty_{loc}(\mathbb{R}^n)$ is slowly growing and $\hat{F}(\xi) = \left\langle F, e^{-i2\pi\xi x}\right\rangle$

Proof. Let $g(\xi) = \left\langle F, e^{-i2\pi\xi x}\right\rangle$, $g \in C^\infty_{loc}$ is smooth.
$\partial^\alpha g(\xi) = \left\langle F, (-i2\pi\xi)^\alpha e^{-i2\pi\xi x}\right\rangle$, g is slowly growing

To show that $g = \hat{F}$, we need to check $\langle g, \phi\rangle = \left\langle F, \hat{\phi}\right\rangle$. Note that $g(\xi)\phi(\xi) \in S$.

$$\langle g, \phi\rangle = \int_{\mathbb{R}^N} g(\xi)\phi(\xi) = \lim_{m\to\infty}\sum_{j=1}^{2^{mn}} g(\xi_j)\phi(\xi_j)\Delta\xi_j$$

$$\hat{\phi} = \int \phi(\xi)e^{-i2\pi\xi x}d\xi = \lim_{m\to\infty}\sum_{j=1}^{2^{mn}} \phi(\xi_j)e^{-i2\pi\xi_j x}\Delta\xi_j$$

$$\left\langle F, \hat{\phi}\right\rangle = \lim_{m\to\infty}\sum_{j=1}^{2^{mn}} \phi(\xi_j)\left\langle F, e^{-i2\pi\xi_j x}\right\rangle\Delta\xi_j$$

Matching terms, $g(\xi_j) = \left\langle F, e^{-i2\pi\xi_j x}\right\rangle$ □

***Theorem:* 10.23:**

If $F \in \mathcal{E}'$, then $\exists N, C_\alpha, f \in C_0$ s.t. $F = \sum_{|\alpha|\le N} C_\alpha\partial^\alpha f$. Every compactly supported distribution can be written as a sum of derivatives of continuous smooth functions

Proof. If $F \in \mathcal{E}'$, then $\hat{F} \in C^\infty_{loc}(\mathbb{R}^n)$ and grows slowly.

Then $\hat{F}(\xi) = (1+|\xi|^2)^N\hat{f}(\xi)$, where $\hat{f} \in L^1$, then $F = (1-\Delta)^N f(x)$ for $f \in C_0$. □

10.9 Hilbert Transform

Consider $K(x) = \frac{1}{x}$, $K \notin L^1_{loc}$, but can be extended to S' (tempered distribution)

$$\langle \text{p.v.}K, \phi \rangle = \text{p.v.} \int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\phi(x)}{x} dx$$

Observations:

1. $K \notin L^1_{loc}$, so the Lebesgue integral $\int \frac{\phi(x)}{x} dx$ DNE
2. K is odd, so there exists cancellation that makes p.v.K well-defined
3. p.v.K is not a Radon measure.

If K is Radon, then $|\langle \text{p.v.}K, \phi \rangle| \leq C \|\phi\|_{C^0}$. We can take ϕ_k to be a smooth approximation of a step function $f(x) = \text{sgn}(x)$, we can check $\|\phi_k\|_{C^0} \leq 1$, but $\left\langle \text{p.v.}\frac{1}{x}, \phi_k \right\rangle \to \infty$.

Definition: 10.12: Hilbert Tranform

Let $f : \mathbb{R} \to \mathbb{R}$, then the Hilbert transform of f is

$$Hf(x) = \text{p.v.}K * f(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

Question: Is H bounded $L^p \to L^p$? It is tricky because $K \notin L^1_{loc}$ and Theorem 10.3 does not apply.

Special cases:
1) $p = \infty$. It is not bounded, because $|H\phi_k|(0) \to \infty$, but $\|H\phi_k\|_{L^\infty} \leq 1$.

2) $p = 1$. It is not bounded, because p.v.$K * \delta \sim \frac{1}{x}$. Let ϕ_t be the approximate identity, $\|(\text{p.v.}K) * \phi_t\|_{L^1} \to \infty$.

3) $p = 2$. Apply Fourier transform to $Hf(x) = \text{p.v.}K * f(x)$, we get $\widehat{Hf} = \widehat{\text{p.v.}K}\hat{f}$.

Claim: $(2\pi i x)(\text{p.v.}K) = 2\pi i$ as a tempered distribution.

Take the Fourier transform:

$$\partial_\xi \left(\widehat{\text{p.v.}K} \right) = (-2\pi i)\delta_0$$

$$\Rightarrow \widehat{\text{p.v.}K} = -\pi i \text{sgn}(\xi) \in L^\infty$$

$$\Rightarrow \|Hf\|_{L^2} = \left\| \widehat{Hf} \right\|_{L^2} = \left\| \left(\widehat{\text{p.v.}K} \right) \hat{f} \right\|_{L^2} \leq \left\| \left(\widehat{\text{p.v.}K} \right) \right\|_{L^\infty} \left\| \hat{f} \right\|_{L^2} = \pi \|f\|_{L^2}$$

So H is bounded $L^2 \to L^2$.

Let $f \in S$, consider the Poisson equation $\Delta u = f$.
The solution is $u(x) = N * f(x)$, where $N(x) = \frac{c_n}{|x|^{n-2}}$ is the Newtonian potential. In Fourier space:

$$\hat{u}(\xi) = -\frac{1}{4\pi^2 |\xi|^2} \hat{f}(\xi)$$

Is there an estimate of the bound on Laplacian: $\|D_{ij}u\|_{L^p} \leq C \|f\|_{L^p}$?
$D_{ij}u(x) = (\partial_i \partial_j N) * f(x)$ where $\partial_i \partial_j N$ is the distributional derivative.

Claim: If $K(x) = \frac{\partial^2}{\partial x_i \partial x_j} \frac{c_n}{|x|^{n-2}}$ on $\mathbb{R}^n \setminus \{0\}$, then

$$\partial_i \partial_j N = \text{p.v.}K(x) + \frac{1}{n}(\delta_{ij})\delta_0(x)$$

Remark 41. p.v.K is a distribution because

1. $K(x)$ is homogenous of degree $-n$
2. $\int_{S^{n-1}} K(x) d\sigma(x) = 0$ with cancellation

Let $Tf = (\text{p.v.}K) * f$. Is T bounded $L^p \to L^p$?

1) $p = \infty$, p.v.K is not a Radon measure, so T is not bounded.

2) $p = 1$, $(\text{p.v.}K) * \delta$ is homogenous of degree $-n$, which is not L^1_{loc}. T is not bounded

3) $p = 2$: equivalent to asking whether $\widehat{\text{p.v.}K} \in L^\infty$.
Consider $\hat{u}(\xi) = -\frac{1}{4\pi^2|\xi|^2}\hat{f}(\xi)$. Take 2 derivaties:

$$\partial_i \partial_j \hat{u}(\xi) = (2\pi i \xi_i)(2\pi i \xi_j)\hat{u}(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}\hat{f}(\xi),$$

so $\widehat{\text{p.v.}K} = \frac{\xi_i \xi_j}{|\xi|^2}$, and $\|D_{ij}u\|_{L^2} \leq \|f\|_{L^2}$

***Definition:* 10.13: Singular Integral Operator**

Let $K(x) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be an operator such that
1. $K(\lambda x) = \lambda^{-n} K(x)$
2. $\int_{S^{n-1}} K(x) d\sigma(x) = 0$

This means that p.v.$K \in S'$. Then $Tf = (\text{p.v.}K) * f$ is a singular integral operator.

***Theorem:* 10.24: Calderon-Zygmund**

Let $Tf = (\text{p.v.}K) * f$. Then for $p \in (1, \infty)$,

$$\|Tf\|_{L^p} \leq C_p \|f\|_{L^p},$$

where C_p is a constant depending on p.

Proof. Strategy:

1. Prove strong (L^2, L^2) bound using Fourier
2. Prove weak $(1, 1)$ bound
3. By Theorem 9.7, we get strong (p, p) for $p \in (1, 2]$
4. By duality on adjoint operators, we get strong (p, p) for $p \in (1, \infty)$

1) Apply Fourier transform to $Tf = (\text{p.v.}K) * f$. $\widehat{Tf} = \widehat{\text{p.v.}K}\hat{f}$

Let $K_\epsilon(x) = K(x)\chi_{B_{\epsilon^{-1}} \setminus B_\epsilon}$ cutoff in an annulus $\epsilon < r < \epsilon^{-1}$. K and K_ϵ are radially symmetric. $K_\epsilon \to$ p.v.K in S' as $\epsilon \to 0$. It sufficies to show $\left|\hat{K}_\epsilon(\xi)\right| \leq C$.

Let $\xi = se_1$ coordinate direction.

$$\begin{aligned}
\hat{K}_\epsilon(\xi) &= \int_{B_{\epsilon^{-1}}\setminus B_\epsilon} K(x)e^{-i2\pi s x_1}dx \\
\text{Let } y = sx \quad &= \int_{B_{s\epsilon^{-1}}\setminus B_{s\epsilon}} K(y)e^{-i2\pi y_1}dy \\
&= \int_{B_{s\epsilon^{-1}}\setminus B_1} K(y)e^{-i2\pi y_1}dy - \int_{B_1\setminus B_{s\epsilon}} K(y)e^{-i2\pi y_1}dy
\end{aligned}$$

As $\epsilon \to 0$,

$$\left|\int_{B_1\setminus B_{s\epsilon}} K(y)e^{-i2\pi y_1}dy\right| = \left|\text{p.v.}\int_{B_1} K(y)e^{-i2\pi y_1}dy\right| \le C$$

Let $R = s\epsilon^{-1}$, then

$$\begin{aligned}
\left|\int_{B_R\setminus B_1} K(y)e^{-i2\pi y_1}dy\right| &\le C\left|\int_{B_R\setminus B_1} \frac{\partial K}{\partial y_1}(y)e^{-i2\pi y_1}dy\right| + C\left|\int_{\partial(B_R\setminus B_1)} K(y)e^{-i2\pi y_1}dy\right| \\
&\le C\int_{B_R\setminus B_1}\frac{1}{|x|^{n+1}} + C(1+R^{-1}) \le C
\end{aligned}$$

2) We want to prove the weak L^1 bound: $|\{|Tf| > \lambda\}| \le C\frac{\|f\|_{L^1}}{\lambda}$.
By replacing f with $\frac{f}{\lambda}$, we can assume $\lambda = 1$. Then we replace f by $f_t(x) = f(tx)$, $\|f_t\| = t^{-n}\|f\|_{L^1}$, $|\{|Tf_t| > 1\}| = t^{-n}|\{|Tf| > 1\}|$. So we can assume $\lambda = \|f\|_{L^1} = 1$ and show that $|\{|Tf| > 1\}| \le C$.

If f is smooth, then $|\{|Tf| > 1\}| \le C\|f\|_{L^2} \le C\|f\|_{L^1}^{1/2} \le C$.

In another extreme, $f \sim \delta$, $Tf \sim K$, $|\{|Tf| > 1\}| \le C$.

Idea: Decompose $f = g + b$, g is bounded, b contains all the spikes. $b = \sum b_i$, where b_i are localized parts with integral 0.

Calderon-Zygmund Decomposition:
Cut $\mathbb{R}^n$ into unit cubes Q. On each Q, we know $\int_Q |f| \le 1$. Cut each Q into $Q = \sum_{i=1}^{2^n} Q_i$. On Q_i, if $1 \le \int_{Q_i}|f| \le 2^n$, put Q_i in bad bin, repeat this on the leftover cubes. After infinite iterations, we get $Q_1, Q_2, \ldots$ bad cubes.
On Q_i, $1 \le \int_{Q_i}|f| \le 2^n$; On $\mathbb{R}^n \setminus \cup Q_i$, $|f| < 1$.

Let $g = \begin{cases} f, \text{ on } \mathbb{R}^n\setminus\cup Q_i \\ \int_{Q_i}, f \text{ on } Q_i \end{cases}$, $b = \begin{cases} 0, \text{ on } \mathbb{R}^n\setminus\cup Q_i \\ f - \int_{Q_i} f, \text{ on } Q_i \end{cases}$. $\|g\|_{L^1} \le \|f\|_{L^1}$, $\|b\|_{L^1} \le \|f\|_{L^1}$.

Consider the split $|\{|Tf| > 1\}| = \left|\{|Tg| > \frac{1}{2}\}\right| + \left|\{|Tb| > \frac{1}{2}\}\right|$

For the good part, $\left|\{|Tg| > \frac{1}{2}\}\right| \le c\|g\|_{L^2} \le C\|g\|_{L^1}^{1/2} \le C$.

For the bad part, $b = \sum b_i$, $b_i = b\chi_{Q_i}$, $\int_{Q_i}|b_i| \le 2^n$, $\int_{Q_i} b_i = 0$.

Claim: $\int_{\mathbb{R}^n\setminus 2Q_i}|Tb_i| \le C|Q_i|$.

By dilation, we can assume Q has size 1. Let $x \in \mathbb{R}^n \setminus 2Q$

$$
\begin{aligned}
|Tb_i(x)| &= \left| \int_Q b_i(y) K(x-y) dy \right| \\
&\leq \left| \int_Q b_i(y)(K(x-y) - K(x)) dy \right| \\
&\leq C\,|x|^{-n-1}
\end{aligned}
$$

The last inequality comes from:

$$
|K(x-y) - K(x)| \leq \int_{l: x \to y} |\nabla K| \leq C\,|x|^{-n-1}
$$

Then,

$$
\begin{aligned}
&\int_{\mathbb{R}^n \setminus \cup 2Q_i} |Tb| \leq \sum_i \int_{\mathbb{R}^n \setminus 2Q_i} |Tb_i| \leq C \sum |Q_i| \\
\Rightarrow &\left| \left\{ |Tb| > \frac{1}{2} \right\} \setminus \cup_i 2Q_i \right| \leq C \sum |Q_i| \leq C \sum_i \int_{Q_i} |f| \leq C
\end{aligned}
$$

Therefore, $\left|\left\{|Tb| > \frac{1}{2}\right\}\right| \leq C + 2\sum |Q_i|$ is bounded. $\square$

Examples:
1) $K(x) = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$ gives Hilbert transform, $\widehat{Hf}(\xi) = -i\pi \text{sgn} \hat{f}(\xi)$.

2) $K(x) = \partial_i \partial_j \frac{1}{|x|^{n-2}}$ on $\mathbb{R}^n \setminus \{0\}$. This gives the Laplace operator $\Delta u = f$, $\left\|D^2 u\right\|_{L^p} \leq C \left\|f\right\|_{L^p}$.

3) Riesz transform: $K(x) = \frac{x_i}{|x|^{n+1}}$ on $\mathbb{R}^n \setminus \{0\}$ related to Half-Laplacian.