FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 3: CSP – Problems Winter Semester 2022/23
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# 1 Presented Problems

## Problem 3.1: Turning $n$-ary constraints into binary constraints

(from *Russell & Norvig 3ed.* q. 7.6) Suppose that we have $CSP = (X, D, E^1)$ with

$$\begin{aligned} X &= \{A, B, C\}, \\ D &= \{\mathrm{dom}(A), \mathrm{dom}(B), \mathrm{dom}(C)\}, \\ E &= \{\langle (A, B, C),\ A + B = C\rangle\}, \end{aligned}$$

where dom$(A)$, dom$(B)$, and dom$(C)$ denote the domain of variable $A$, $B$, and $C$, respectively, and each domain can be $\{0, 1, \ldots, 9\}$ for example.

**Problem 3.1.1**: Draw the constraint hypergraph for the CSP. In this case, a hypergraph is a graph with two types of nodes. The first type of node represents the *variables*, depicted by ◯, and the second type of node represents the constraint, depicted by □. Based on the number of variables involved, what is the type of the constraint?

**Problem 3.1.2**: We can eliminate the higher-order constraint in $E$ by replacing the constraint node □ with a new variable node $Z$. (We denote this new CSP as $CSP'$.) What is the domain for variable $Z$? (Hint: The domain for variable $Z$ can be ordered pairs of other values.) What is the new constraint set $E'$ after introducing the new variable $Z$?

**Problem 3.1.3**: Modify $CSP'$ such that it only contains binary constraints and formally express the new $CSP'' = (X'', D'', E'')$.

**Problem 3.1.4**: Taking inspiration from previous solutions, how can you generally turn a $n$-ary constraint into binary constraints?

## Problem 3.2: Arc consistency and backtracking search for binary constraints

Consider the constraint satisfaction problem in Fig. 1. According to the picture, we have $CSP = (X, D, E)$ with

$$\begin{aligned} X &= v_1, v_2, v_3, v_4, v_5, \\ D &= \{\mathrm{dom}(v_1), \mathrm{dom}(v_2), \mathrm{dom}(v_3), \mathrm{dom}(v_4), \mathrm{dom}(v_5)\}, \\ E &= \{\langle (v_1, v_2),\ v_2 = v_1 + 1\rangle, \\ &\quad \langle (v_1, v_3),\ v_1 \neq v_3\rangle \\ &\quad \langle (v_2, v_3),\ v_2 \neq v_3\rangle, \\ &\quad \langle (v_3, v_4),\ v_3 \neq v_4\rangle, \\ &\quad \langle (v_3, v_5),\ v_3 \neq v_5\rangle, \\ &\quad \langle (v_4, v_5),\ v_4 \neq v_5\rangle, \\ &\quad \langle (v_1, v_5),\ v_1 \neq v_5\rangle\}, \end{aligned}$$

where dom$(v_1)$, dom$(v_2)$, dom$(v_3)$, dom$(v_4)$ and dom$(v_5)$ denote the domain of variable $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$, respectively, and each domain is initially $\{2, 3, 4\}$. Note that all constraints in the graph are binary constraints.

**Problem 3.2.1**: Sort the variables once by their domain size (i.e. number of remaining values) and once by their degree (i.e. number of constraints on other unassigned variables).

[1] the symbol E is taken from German word *Einschränkung*.

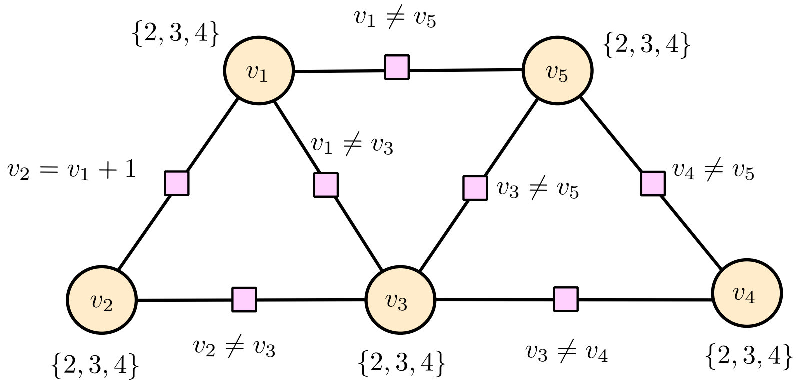



Figure 1: Constraint graph for Problem 3.2

**Problem 3.2.2**: Perform backtracking search by hand to solve the CSP problem: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose the variable with the smaller lower index. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, set the value to 3; if this is not possible choose the lowest value. After each assignment, perform forward checking as inference. Backtrack if you find an inconsistency.

**Problem 3.2.3**: Perform backtracking search again, but with a different inference: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose the variable with the smaller lower index. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, set the value to 3; if this is not possible choose the lowest value. After each assignment, perform the arc consistency algorithm. Backtrack if you find an inconsistency.

**Problem 3.2.4**: Consider the CSP in Fig. 1 at its initial state. Is the CSP arc consistent? Is this a convenient initial condition if we plan to apply backtracking search? Apply the arc consistency algorithm to the CSP as a preprocessing step: Initialize the queue with all arcs of the CSP. Is the CSP arc consistent now?

**Problem 3.2.5**: Perform backtracking search after the preprocessing step of the previous task: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, choose one randomly. After each assignment, perform arc consistency algorithm. Backtrack if you find an inconsistency. Assume the data structure of the queue is a set, i.e., if we add an element to the queue which is already in the queue, the element will not be added a second time (each element is unique).

**Problem 3.2.6**: For each of the previous performances of backtracking search with a different inference (forward checking, arc consistency, arc consistency after preprocessing), compare the number of iterations and the number of times you needed to backtrack.

# 2 Additional Problems

## Problem 3.3: Arc consistency and backtracking search for binary constraints

Consider again the constraint satisfaction problem from Problem 3.2 described in Fig. 1.

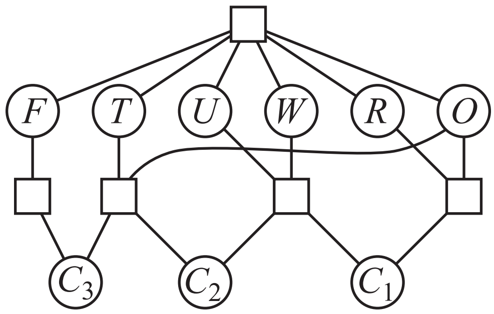
**Problem 3.3.1**: Perform backtracking search by hand with forward checking as inference after preprocessing with the arc consistency algorithm: Apply the arc consistency algorithm to the CSP initializing the queue with all arcs, then start backtracking search. Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign; if there is a tie, choose randomly. After each assignment, perform forward checking as inference. Backtrack if you find an inconsistency.

## Problem 3.4: Solving a CSP by hand performing backtracking search with minimum-remaining-values (MRV) and degree heuristics, least-constraining-value heuristics, and forward checking

(from *Russell & Norvig 3ed.* q. 7.5) Suppose that we have the cryptarithmetic problem as shown in Fig. 2.

$$\begin{array}{r} T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$

(a)




(b)

Figure 2: (a) A cryptarithmetic problem. Each letter stands for a distinct digit; the aim is to find a substitution of digits for letters such that the resulting sum is arithmetically correct. (b) The constraint hypergraph for the cryptarithmetic problem, showing the `Alldiff` constraint (square box at the top) as well as the column addition constraints (four square boxes in the middle). The variables $C1$, $C2$, and $C3$ represent the carry digits for the three columns.

We model the cryptarithmetic problem as $CSP = (X, D, C)$ with

$$\begin{aligned}
X &= \{F, T, U, W, R, O, C_1, C_2, C_3\} \\
D &= \{\text{numbers}, \ldots, \text{numbers}, \text{binary}, \text{binary}, \text{binary}\} \\
C &= \{\langle (O, R, C_1), \quad O + O = R + 10 \cdot C_1 \rangle, \\
&\quad \langle (U, W, C_1, C_2), \quad C_1 + W + W = U + 10 \cdot C_2 \rangle, \\
&\quad \langle (O, T, C_2, C_3), \quad C_2 + T + T = O + 10 \cdot C_3 \rangle, \\
&\quad \langle (C_3, F), \quad C_3 = F \rangle, \\
&\quad \langle (F, T, U, W, R, O), \quad \texttt{Alldiff}(F, T, U, W, R, O) \rangle\},
\end{aligned}$$

where numbers $= \{0, 1, 2, \ldots, 9\}$ and binary $= \{0, 1\}$.

**Problem 3.4.1**: Replace all boxes which correspond to higher-order constraints by binary constraints. Use the approach of Problem 3.1 and introduce variables such as $X_1$, $X_2$, and $X_3$, etc.

**Problem 3.4.2**: Sort the variables once by their domain size (i.e. number of remaining values) and once by their degree (i.e. number of constraints on other unassigned variables).

**Problem 3.4.3**: Perform backtracking search to solve the cryptarithmetic problem: Determine which variable to expand next by applying the minimum-remaining-values (MRV) heuristic; if there is a tie, use degree heuristics; if there is a tie again, choose one randomly. Use least-constraining-value heuristics to decide which value to assign. After each assignment, perform forward checking. Backtrack if you find an inconsistency.