FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE

Exercise 8: Bayesian networks – Solutions Winter Semester 2022/23

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Problem 8.1:

a. ii, iii, iv and v.
   For v., consider the Markov blanket of D.
   Remember that a node in the Bayesian network is conditionally independent from all other nodes given its parents, children and children's parents (this evidence is the **Markov blanket** of the node).

b. $\mathbf{P}(D,TP,CC,UP,PP,W) = \mathbf{P}(D)\,\mathbf{P}(TP)\,\mathbf{P}(CC)\,\mathbf{P}(UP|D,TP)\,\mathbf{P}(PP|CC,UP)\,\mathbf{P}(W|CC,UP,PP)$.

c. To calculate these values we can directly use the equation derived in **b.**. In fact, we have to evaluate it using the requested values of the random variables:
   $P(\neg d,tp,cc,\neg up,pp,w) = P(\neg d)\,P(tp)\,P(cc)\,P(\neg up|\neg d,tp)\,P(pp|cc,\neg up)\,P(w|cc,\neg up,pp) =$
   $= 0.75 \cdot 0.35 \cdot 0.3 \cdot 0.9 \cdot 0.9 \cdot 0.9 = 0.05740875$.
   $P(\neg d,tp,cc,\neg up,\neg pp,w) = P(\neg d)\,P(tp)\,P(cc)\,P(\neg up|\neg d,tp)\,P(\neg pp|cc,\neg up)\,P(w|cc,\neg up,\neg pp) =$
   $= 0.75 \cdot 0.35 \cdot 0.3 \cdot 0.9 \cdot 0.1 \cdot 0.5 = 0.00354375$.

d. The required probability is $P(w|tp,\neg d,pp)$.
   To calculate this we can use enumeration:
   $\mathbf{P}(W|tp,\neg d,pp) = \alpha P(\neg d)\,P(tp)\sum_{UP}\mathbf{P}(UP|\neg d,tp)\sum_{CC}\mathbf{P}(CC)\,\mathbf{P}(pp|CC,UP)\,\mathbf{P}(W|CC,UP,pp)$.

$$\begin{aligned}
P(w|tp,\neg d,pp) = {} & \alpha P(\neg d)\,P(tp)\,\{P(up|\neg d,tp)\,[P(cc)\,P(pp|cc,up)\,P(w|cc,up,pp) \\
& + P(\neg cc)\,P(pp|\neg cc,up)\,P(w|\neg cc,up,pp)] \\
& + P(\neg up|\neg d,tp)\,[P(cc)\,P(pp|cc,\neg up)\,P(w|cc,\neg up,pp) \\
& + P(\neg cc)\,P(pp|\neg cc,\neg up)\,P(w|\neg cc,\neg up,pp)]\} \\
= {} & \alpha \cdot 0.75 \cdot 0.35\{0.1[0.3 \cdot 0.5 \cdot 0.4 + 0.7 \cdot 0.05 \cdot 0.05] + 0.9[0.3 \cdot 0.9 \cdot 0.9 + 0.7 \cdot 0.1 \cdot 0.15]\} \\
\approx {} & \alpha \cdot 0.0615,
\end{aligned}$$

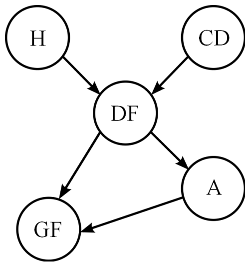
$$\begin{aligned}
P(\neg w|tp,\neg d,pp) = {} & \alpha P(\neg d)\,P(tp)\,\{P(up|\neg d,tp)\,[P(cc)\,P(pp|cc,up)\,P(\neg w|cc,up,pp) \\
& + P(\neg cc)\,P(pp|\neg cc,up)\,P(\neg w|\neg cc,up,pp)] \\
& + P(\neg up|\neg d,tp)\,[P(cc)\,P(pp|cc,\neg up)\,P(\neg w|cc,\neg up,pp) \\
& + P(\neg cc)\,P(pp|\neg cc,\neg up)\,P(\neg w|\neg cc,\neg up,pp)]\} \\
= {} & \alpha \cdot 0.75 \cdot 0.35\{0.1[0.3 \cdot 0.5 \cdot 0.6 + 0.7 \cdot 0.05 \cdot 0.95] + 0.9[0.3 \cdot 0.9 \cdot 0.1 + 0.7 \cdot 0.1 \cdot 0.85]\} \\
\approx {} & \alpha \cdot 0.0237,
\end{aligned}$$

   $\alpha = \frac{1}{0.0615+0.0237} \approx 11.7371$,
   $P(w|tp,\neg d,pp) \approx 11.7371 \cdot 0.0615 \approx 0.7218$.

e. The random variable EF has a direct influence on the race. We can reasonably assume that when there is an engine failure during the race, the pilot does not win. Another reasonable assumption is that a competitive car is also more reliable with a direct influence on the random variable EF. In conclusion, we introduce the node EF as a parent of W and child of CC.

Problem 8.2:

a. The corresponding network is the following:



b. Using enumeration:

$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{P}(CD) \sum_{DF} \mathbf{P}(gf|\neg a, DF)\mathbf{P}(\neg a|DF) \sum_H \mathbf{P}(DF|CD,H)\mathbf{P}(H)$.

$$\begin{aligned}
P(cd|\neg a, gf) &= \alpha P(cd)\{P(gf|df,\neg a)P(\neg a|df)[P(df|cd,h)P(h) + P(df|cd,\neg h)P(\neg h)] \\
&\quad + P(gf|\neg df,\neg a)P(\neg a|\neg df)[P(\neg df|cd,h)P(h) + P(\neg df|cd,\neg h)P(\neg h)]\} \\
&= \alpha \cdot 0.6\{0.4 \cdot 0.3[0.15 \cdot 0.5 + 0.01 \cdot 0.5] + 0.05 \cdot 0.75[0.85 \cdot 0.5 + 0.99 \cdot 0.5]\} \\
&\approx \alpha \cdot 0.0265,
\end{aligned}$$

$$\begin{aligned}
P(\neg cd|\neg a, gf) &= \alpha P(\neg cd)\{P(gf|df,\neg a)P(\neg a|df)[P(df|\neg cd,h)P(h) + P(df|\neg cd,\neg h)P(\neg h)] \\
&\quad + P(gf|\neg df,\neg a)P(\neg a|\neg df)[P(\neg df|\neg cd,h)P(h) + P(\neg df|\neg cd,\neg h)P(\neg h)]\} \\
&= \alpha \cdot 0.4\{0.4 \cdot 0.3[0.99 \cdot 0.5 + 0.1 \cdot 0.5] + 0.05 \cdot 0.75[0.01 \cdot 0.5 + 0.9 \cdot 0.5]\} \\
&\approx \alpha \cdot 0.0330,
\end{aligned}$$

$\alpha = \frac{1}{0.0265+0.0330} \approx 16.8067$,

$\mathbf{P}(CD|\neg a, gf) \approx 16.8067 \cdot \begin{bmatrix} 0.0265 \\ 0.0330 \end{bmatrix} \approx \begin{bmatrix} 0.4454 \\ 0.5546 \end{bmatrix}$.

c. Using variable elimination:

$$\mathbf{P}(CD|\neg a, gf) = \alpha \underbrace{P(CD)}_{f_1(CD)} \sum_{DF} \underbrace{P(gf|\neg a, DF)}_{f_2(DF)} \underbrace{P(\neg a|DF)}_{f_3(DF)} \sum_H \underbrace{P(DF|CD,H)}_{f_4(DF,CD,H)} \underbrace{P(H)}_{f_5(H)}.$$

In the following notation we represent a $2 \times 2 \times 2$ matrix as $\{[2 \times 2]\ [2 \times 2]\}$.

$$\begin{aligned}
\mathbf{f_4}(DF,CD,H) &= \left\{\ \mathbf{P}(DF|CD,h) \quad \mathbf{P}(DF|CD,\neg h)\ \right\} \\
&= \left\{ \begin{bmatrix} P(df|cd,h) & P(df|\neg cd,h) \\ P(\neg df|cd,h) & P(\neg df|\neg cd,h) \end{bmatrix} \begin{bmatrix} P(df|cd,\neg h) & P(df|\neg cd,\neg h) \\ P(\neg df|cd,\neg h) & P(\neg df|\neg cd,\neg h) \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} 0.15 & 0.99 \\ 0.85 & 0.01 \end{bmatrix} \begin{bmatrix} 0.01 & 0.1 \\ 0.99 & 0.9 \end{bmatrix} \right\},
\end{aligned}$$

$\mathbf{f_1}(CD) = \begin{bmatrix} P(cd) & P(\neg cd) \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$, $\mathbf{f_2}(DF) = \begin{bmatrix} P(gf|\neg a, df) \\ P(gf|\neg a, \neg df) \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.05 \end{bmatrix}$,

$\mathbf{f_3}(DF) = \begin{bmatrix} P(\neg a|df) \\ P(\neg a|\neg df) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.75 \end{bmatrix}$, $\mathbf{f_5}(H) = \begin{bmatrix} P(h) & P(\neg h) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$.

After defining the factors, we can write:

$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{f_1}(CD) \times \sum_{DF} \mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \sum_H \mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H)$.

In the following we use the pointwise product and we sum out variables.

$\mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H) = \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.050 \\ 0.495 & 0.450 \end{bmatrix} \right\}$,

$\mathbf{f_6}(DF,CD) = \sum_H \mathbf{f_4}(DF,CD,H) \times \mathbf{f_5}(H) = \sum_H \left\{ \begin{bmatrix} 0.075 & 0.495 \\ 0.425 & 0.005 \end{bmatrix} \begin{bmatrix} 0.005 & 0.050 \\ 0.495 & 0.450 \end{bmatrix} \right\} = \begin{bmatrix} 0.080 & 0.545 \\ 0.920 & 0.455 \end{bmatrix}$,

$\mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \mathbf{f_6}(DF,CD) = \begin{bmatrix} 0.0096 & 0.0654 \\ 0.0345 & 0.0171 \end{bmatrix}$,

$\mathbf{f_7}(CD) = \sum_{DF} \mathbf{f_2}(DF) \times \mathbf{f_3}(DF) \times \mathbf{f_6}(DF,CD) = \sum_{DF} \begin{bmatrix} 0.0096 & 0.0654 \\ 0.0345 & 0.0171 \end{bmatrix} = \begin{bmatrix} 0.0441 & 0.0825 \end{bmatrix}$.

And finally we get:

$\mathbf{P}(CD|\neg a, gf) = \alpha \mathbf{f_1}(CD) \times \mathbf{f_7}(CD) \approx \alpha \begin{bmatrix} 0.0265 & 0.0330 \end{bmatrix}$,

$\alpha = \frac{1}{0.0265+0.0330} \approx 16.8067$,

$\mathbf{P}(CD|\neg a, gf) \approx 16.8067 \cdot \begin{bmatrix} 0.0265 & 0.0330 \end{bmatrix} \approx \begin{bmatrix} 0.4454 & 0.5546 \end{bmatrix}$.

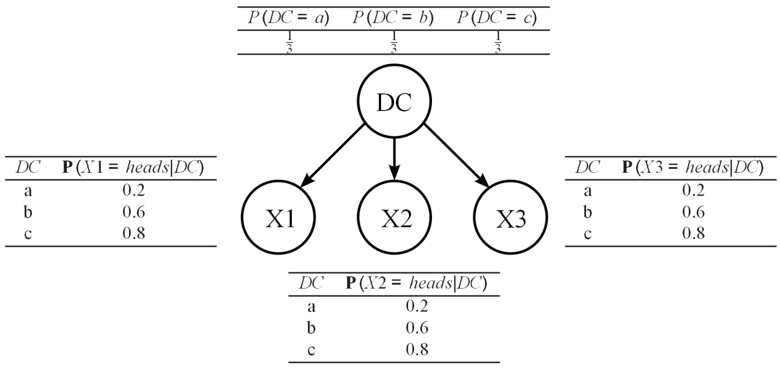
d. Besides the last calculation to complete the normalization that both methods require, we have:

	Enumeration	Variable Elimination
additions	6	6
multiplications	18	16

In this case we do not have a significant improvement using variable elimination. The improvement becomes more significant for larger Bayesian networks.

Problem 8.3:

a. The network is composed of four random variables: The coin that has been drawn DC with domain $< a, b, c >$ and the results of the three flips $X1$, $X2$ and $X3$ with domains $< heads, tails >$.



b. The required probability is $\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails)$.
We first write the full joint probability:
$\mathbf{P}(DC, X1, X2, X3) = \mathbf{P}(X1|DC)\,\mathbf{P}(X2|DC)\,\mathbf{P}(X3|DC)\,\mathbf{P}(DC)$.
The result is directly obtained introducing the evidence and using normalization:
$\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails) =$
$= \alpha\mathbf{P}(X1 = heads|DC)\,\mathbf{P}(X2 = heads|DC)\,\mathbf{P}(X3 = tails|DC)\,\mathbf{P}(DC) =$

$$= \alpha \begin{bmatrix} 0.2 \\ 0.6 \\ 0.8 \end{bmatrix} \times \begin{bmatrix} 0.2 \\ 0.6 \\ 0.8 \end{bmatrix} \times \begin{bmatrix} 0.8 \\ 0.4 \\ 0.2 \end{bmatrix} \times \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \approx \alpha \begin{bmatrix} 0.0107 \\ 0.0480 \\ 0.0427 \end{bmatrix},$$

$\alpha = \frac{1}{(0.0107+0.0480+0.0427)} \approx 9.8619329$,

$$\mathbf{P}(DC|X1 = heads, X2 = heads, X3 = tails) \approx 9.8619329 \begin{bmatrix} 0.0107 \\ 0.0480 \\ 0.0427 \end{bmatrix} = \begin{bmatrix} 0.1055 \\ 0.4734 \\ 0.4211 \end{bmatrix}.$$

The coin that is most likely to have been drawn is b.