

Problem 6.1: The man in the painting

(The following puzzle appears in [1] question 9.10.) A man stands in front of a painting and says the following:

Brothers and sisters have I none, but that man's father is my father's son.

What is the relationship between the man in the painting and the speaker? Use the predicates

$Male(x)$: x is male.

$Father(x, y)$: x is the father of y .

$Son(x, y)$: x is a son of y .

$Parent(x, y)$: x is a parent of y .

$Child(x, y)$: x is a child of y .

$Sibling(x, y)$: x is a sibling of y

and the knowledge

- A sibling is another child of one's parents.

$$\forall x, y \quad Sibling(x, y) \Leftrightarrow x \neq y \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)$$

- Parent and child are inverse relations.

$$\forall p, c \quad Parent(p, c) \Leftrightarrow Child(c, p)$$

to solve the riddle with first-order logic.

Problem 6.1.1: Define the rule which relates *son*, *child*, and *male*. Define also the rule which relates *father* and *parent*.

Solution:

- Every son is a male child, and every male child is a son.

$$\forall s, p \quad Son(s, p) \Leftrightarrow Child(s, p) \wedge Male(s)$$

- Every father is a male parent, and every male parent is a father.

$$\forall p, c \quad Father(p, c) \Leftrightarrow Parent(p, c) \wedge Male(p)$$

Problem 6.1.2: Using the constants *Me* for the speaker and *That* for the person depicted in the painting, formalise the sentences regarding the sexes of the people in the puzzle.

Solution: From the problem definition, we know that the constant *Me* is male. Therefore, we add the following sentence to our knowledge base.

$$Male(Me)$$

From the problem definition, we also know that the person in the painting is male.

$$Male(That)$$

Problem 6.1.3: Formalise the sentences “Brothers and sisters have I none” and “That man’s father is my father’s son” in first-order logic.

Solution:

- Brothers and sisters have I none.

$$\forall x \quad \neg Sibling(x, Me) \wedge \neg Sibling(Me, x)$$

- That man’s father is my father’s son.

$$\exists f_1, f_2 \quad Father(f_1, That) \wedge Father(f_2, Me) \wedge Son(f_1, f_2)$$

Problem 6.1.4: Solve this puzzle informally and decide what is the relation between the man in the painting and the speaker.

Solution: We analyse the sentence

... but that man’s father is my father’s son.

The phrase “my father’s son” could be either “Me” or “My Sibling”. However, from previous constraint

Brothers and sisters have I none, ...

we know that it could not be “My Sibling” because it does not exist. Therefore, the phrase “my father’s son” is “Me”. Using this equality, we know from the first quote above that “That man’s father” is “Me”. Therefore, “That man” is the son of “Me”.

Problem 6.1.5: Using the resolution technique for first-order logic, prove your answer.

Solution:

We want to prove $\alpha := Son(That, Me)$. To do so, we first need to transform the different rules we are given into Conjunctive Normal Forms (CNF); this will be done in the following, and the different clauses will then be written explicitly at the end of each transformation:

1. A sibling is another child of one’s parents (\Leftarrow).

$$\forall x, y \quad Sibling(x, y) \Leftarrow (x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y)$$

(Removing implication)

$$\equiv \forall x, y \quad Sibling(x, y) \vee \neg((x \neq y) \wedge \exists p \quad Parent(p, x) \wedge Parent(p, y))$$

(Pushing \neg inwards)

$$\equiv \forall x, y \quad Sibling(x, y) \vee (x = y) \vee \forall p \quad \neg Parent(p, x) \vee \neg Parent(p, y)$$

(Dropping universal quantifier)

$$\equiv Sibling(x, y) \vee (x = y) \vee \neg Parent(p, x) \vee \neg Parent(p, y)$$

$$\boxed{Sibling(x, y) \vee (x = y) \vee \neg Parent(p, x) \vee \neg Parent(p, y)} \quad (1)$$

2. A sibling is another child of one's parents (\Rightarrow).

$$\begin{aligned}
& \forall x, y \quad \neg \text{Sibling}(x, y) \vee ((x \neq y) \wedge \exists p \quad \text{Parent}(p, x) \wedge \text{Parent}(p, y)) \\
& \quad \text{(Skolemisation)} \\
& \equiv \forall x, y \quad \neg \text{Sibling}(x, y) \vee ((x \neq y) \wedge \text{Parent}(F(x, y), x) \wedge \text{Parent}(F(x, y), y)) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv \neg \text{Sibling}(x, y) \vee ((x \neq y) \wedge \text{Parent}(F(x, y), x) \wedge \text{Parent}(F(x, y), y)) \\
& \quad \text{(Distributivity of } \vee \text{ over } \wedge) \\
& \equiv (\neg \text{Sibling}(x, y) \vee (x \neq y)) \wedge (\neg \text{Sibling}(x, y) \vee \text{Parent}(F(x, y), x)) \wedge \dots \\
& \quad \dots \wedge (\neg \text{Sibling}(x, y) \vee \text{Parent}(F(x, y), y))
\end{aligned}$$

$$\boxed{\neg \text{Sibling}(x, y) \vee (x \neq y)} \quad (2)$$

$$\boxed{\neg \text{Sibling}(x, y) \vee \text{Parent}(F(x, y), x)} \quad (3)$$

$$\boxed{\neg \text{Sibling}(x, y) \vee \text{Parent}(F(x, y), y)} \quad (4)$$

3. Parent and child are inverse relations (\Rightarrow).

$$\begin{aligned}
& \forall p, c \quad \text{Parent}(p, c) \Rightarrow \text{Child}(c, p) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall p, c \quad \neg \text{Parent}(p, c) \vee \text{Child}(c, p) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv \neg \text{Parent}(p, c) \vee \text{Child}(c, p)
\end{aligned}$$

$$\boxed{\neg \text{Parent}(p, c) \vee \text{Child}(c, p)} \quad (5)$$

4. Parent and child are inverse relations (\Leftarrow).

$$\begin{aligned}
& \forall p, c \quad \text{Child}(c, p) \Rightarrow \text{Parent}(p, c) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall p, c \quad \neg \text{Child}(c, p) \vee \text{Parent}(p, c) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv \neg \text{Child}(c, p) \vee \text{Parent}(p, c)
\end{aligned}$$

$$\boxed{\neg \text{Child}(c, p) \vee \text{Parent}(p, c)} \quad (6)$$

5. Every son is a male child (\Rightarrow).

$$\begin{aligned}
& \forall s, p \quad \text{Son}(s, p) \Rightarrow \text{Child}(s, p) \wedge \text{Male}(s) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall s, p \quad \neg \text{Son}(s, p) \vee (\text{Child}(s, p) \wedge \text{Male}(s)) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv \neg \text{Son}(s, p) \vee (\text{Child}(s, p) \wedge \text{Male}(s)) \\
& \quad \text{(Distributivity of } \vee \text{ over } \wedge) \\
& \equiv (\neg \text{Son}(s, p) \vee \text{Child}(s, p)) \wedge (\neg \text{Son}(s, p) \vee \text{Male}(s))
\end{aligned}$$

$$\boxed{\neg \text{Son}(s, p) \vee \text{Child}(s, p)} \quad (7)$$

$$\boxed{\neg \text{Son}(s, p) \vee \text{Male}(s)} \quad (8)$$

6. Every son is a male child (\Leftarrow).

$$\begin{aligned}
& \forall s, p \quad Son(s, p) \Leftarrow Child(s, p) \wedge Male(s) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall s, p \quad Son(s, p) \vee \neg(Child(s, p) \wedge Male(s)) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv Son(s, p) \vee \neg(Child(s, p) \wedge Male(s)) \\
& \quad \text{(de Morgan's rule)} \\
& \equiv Son(s, p) \vee \neg Child(s, p) \vee \neg Male(s) \\
& \quad \boxed{Son(s, p) \vee \neg Child(s, p) \vee \neg Male(s)} \tag{9}
\end{aligned}$$

7. Every father is a male parent (\Rightarrow).

$$\begin{aligned}
& \forall p, c \quad Father(p, c) \Rightarrow Parent(p, c) \wedge Male(p) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall p, c \quad \neg Father(p, c) \vee (Parent(p, c) \wedge Male(p)) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv \neg Father(p, c) \vee (Parent(p, c) \wedge Male(p)) \\
& \quad \text{(Distributivity of } \vee \text{ over } \wedge) \\
& \equiv (\neg Father(p, c) \vee Parent(p, c)) \wedge (\neg Father(p, c) \vee Male(p)) \\
& \quad \boxed{\neg Father(p, c) \vee Parent(p, c)} \tag{10}
\end{aligned}$$

$$\boxed{\neg Father(p, c) \vee Male(p)} \tag{11}$$

8. Every father is a male parent (\Leftarrow).

$$\begin{aligned}
& \forall p, c \quad Father(p, c) \Leftarrow Parent(p, c) \wedge Male(p) \\
& \quad \text{(Removing implication)} \\
& \equiv \forall p, c \quad Father(p, c) \vee \neg(Parent(p, c) \wedge Male(p)) \\
& \quad \text{(de Morgan's rule)} \\
& \equiv \forall p, c \quad Father(p, c) \vee \neg Parent(p, c) \vee \neg Male(p) \\
& \quad \text{(Dropping universal quantifier)} \\
& \equiv Father(p, c) \vee \neg Parent(p, c) \vee \neg Male(p) \\
& \quad \boxed{Father(p, c) \vee \neg Parent(p, c) \vee \neg Male(p)} \tag{12}
\end{aligned}$$

9. Brothers and sisters have I none.

$$\begin{aligned}
& \forall x \quad \neg Sibling(x, Me) \wedge \neg Sibling(Me, x) \\
& \quad \text{(Dropping quantifier)} \\
& \equiv \neg Sibling(x, Me) \wedge \neg Sibling(Me, x) \\
& \quad \boxed{\neg Sibling(x, Me)} \tag{13}
\end{aligned}$$

$$\boxed{\neg Sibling(Me, x)} \tag{14}$$

10. That man's father is my father's son.

$$\begin{aligned} & \exists f_1, f_2 \quad \text{Father}(f_1, \text{That}) \wedge \text{Father}(f_2, \text{Me}) \wedge \text{Son}(f_1, f_2) \\ & \quad \text{(Skolemisation)} \\ \equiv & \quad \text{Father}(F_1, \text{That}) \wedge \text{Father}(F_2, \text{Me}) \wedge \text{Son}(F_1, F_2) \end{aligned}$$

$$\boxed{\text{Father}(F_1, \text{That})} \quad (15)$$

$$\boxed{\text{Father}(F_2, \text{Me})} \quad (16)$$

$$\boxed{\text{Son}(F_1, F_2)} \quad (17)$$

11. Sex of the person in the painting.

$$\boxed{\text{Male}(\text{That})} \quad (18)$$

12. Sex of the person standing in front of the painting.

$$\boxed{\text{Male}(\text{Me})} \quad (19)$$

13. negation of the goal

$$\boxed{\neg \text{Son}(\text{That}, \text{Me})} \quad (20)$$

Formal proof using the Resolution Algorithm: We start with the negation of the goal:

$$\begin{aligned} & \neg \text{Son}(\text{That}, \text{Me}) \\ & \quad \text{(Rule 9 with } \{s/\text{That}, p/\text{Me}\}) \\ \rightarrow & \quad \neg \text{Child}(\text{That}, \text{Me}) \vee \neg \text{Male}(\text{That}) \\ & \quad \text{(Rule 18)} \\ \rightarrow & \quad \neg \text{Child}(\text{That}, \text{Me}) \\ & \quad \text{(Rule 5, with } \{c/\text{That}, p/\text{Me}\}) \\ \rightarrow & \quad \neg \text{Parent}(\text{Me}, \text{That}) \end{aligned}$$

Graphically, this can be represented as follows:

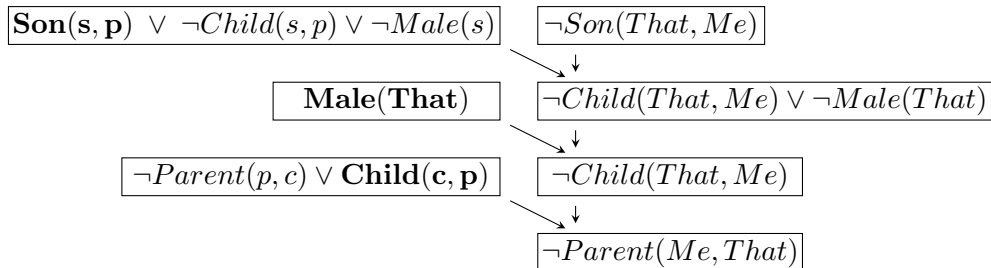


Figure 1: Resolution proof for $\neg \text{Parent}(\text{Me}, \text{That})$

We add the conclusion to our resolution clauses:

$$\boxed{\neg \text{Parent}(\text{Me}, \text{That})} \quad (21)$$

We, then start again with the clause 17, which we shall combine in the very end with 21:

$$\begin{aligned}
& \text{Son}(F_1, F_2) \\
& \quad \text{(Rule 7 with } \{s/F_1, p/F_2\}) \\
\rightarrow & \text{Child}(F_1, F_2) \\
& \quad \text{(Rule 6 with } \{c/F_1, p/F_2\}) \\
\rightarrow & \text{Parent}(F_2, F_1) \\
& \quad \text{(Rule 1 with } \{x/F_1, p/F_2\}) \\
\rightarrow & \text{Sibling}(F_1, y) \vee (F_1 = y) \vee \neg \text{Parent}(F_2, y) \\
& \quad \text{(Rule 10 with } \{p/F_2, c/y\}) \\
\rightarrow & \text{Sibling}(F_1, y) \vee (F_1 = y) \vee \neg \text{Father}(F_2, y) \\
& \quad \text{(Rule 16 with } \{y/Me\}) \\
\rightarrow & \text{Sibling}(F_1, Me) \vee (F_1 = Me) \\
& \quad \text{(Rule 13 with } \{x/F_1\}) \\
\rightarrow & (F_1 = Me) \\
& \quad \text{(Demodulation rule with rule 15)} \\
\rightarrow & \text{Father}(Me, That) \\
& \quad \text{(Rule 10 with } \{p/Me, c/That\}) \\
\rightarrow & \text{Parent}(Me, That) \\
& \quad \text{(The last clause added 21)} \\
\rightarrow & \{\}
\end{aligned}$$

Graphically, this can be represented as follows:

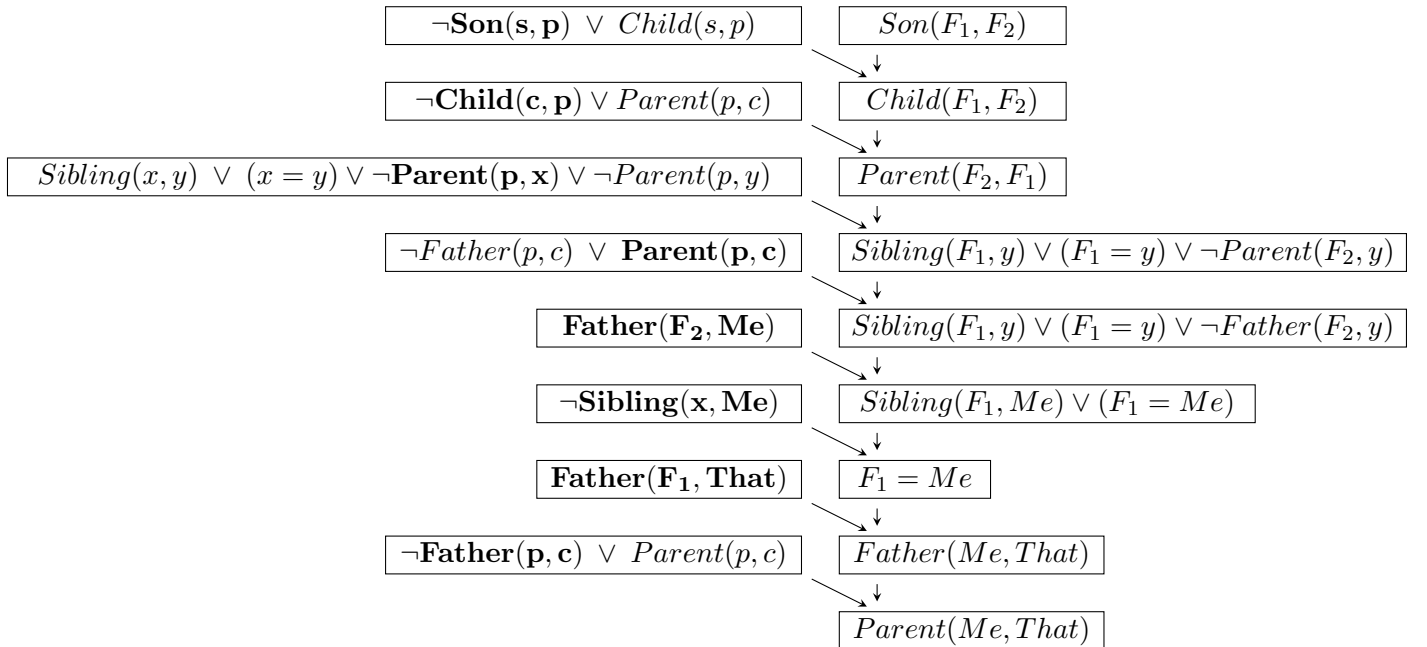


Figure 2: Resolution proof for $\text{Parent}(Me, That)$

At the end of the day, we end up with the empty clause, proving that our knowledge base does indeed entail the original claim $\text{Son}(That, Me)$.

Problem 6.2: Backward chaining

(The following exercise is taken from [1] Question 9.9.) Suppose you are given the following axioms:

1. $0 \leq 3$
2. $7 \leq 9$
3. $\forall x \quad x \leq x$
4. $\forall x \quad x \leq x + 0$
5. $\forall x \quad x + 0 \leq x$
6. $\forall x, y \quad x + y \leq y + x$
7. $\forall w, x, y, z \quad w \leq y \wedge x \leq z \Rightarrow w + x \leq y + z$
8. $\forall x, y, z \quad x \leq y \wedge y \leq z \Rightarrow x \leq z.$

Give a backward-chaining proof of the sentence $7 \leq 3 + 9$. (Be sure, of course, to use only the axioms given here, not anything else you may know about arithmetic.) Show only the steps that lead to success, not the irrelevant steps.

Solution: For the sake of simplicity, we shall not go through a “complete” backwards chaining algorithm, in the sense that we will only apply one rule at a time to go from assumption to assumption; in contrast, if we were to use the actual algorithm we would need to apply every available rule at every step.

Before we actually apply the algorithm, let us first consider a summary of the proof; in the following, the step numbers on the side indicate the steps in the algorithm:

$$\begin{array}{lcl}
 7 \leq 3 + 9 & & \\
 \Leftarrow \begin{array}{l} \text{(Rule 8 with } \{x/7, z/3 + 9\}) \\ 7 \leq y \quad \wedge \quad y \leq 3 + 9 \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 8 with } \{x/7, z/3 + 9\}) \\ 7 \leq y \quad \wedge \quad y \leq 3 + 9 \end{array}} \right\} \text{Step 1} \\
 \Leftarrow \begin{array}{l} \text{(Rule 4 with } \{x/7, y/7 + 0\}) \\ \text{True} \quad \wedge \quad 7 + 0 \leq 3 + 9 \\ \text{(True rule)} \\ 7 + 0 \leq 3 + 9 \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 4 with } \{x/7, y/7 + 0\}) \\ \text{True} \quad \wedge \quad 7 + 0 \leq 3 + 9 \\ \text{(True rule)} \\ 7 + 0 \leq 3 + 9 \end{array}} \right\} \text{Step 2} \\
 \Leftarrow \begin{array}{l} \text{(Rule 8 with } \{x/7 + 0, z/3 + 9\}) \\ 7 + 0 \leq y \quad \wedge \quad y \leq 3 + 9 \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 8 with } \{x/7 + 0, z/3 + 9\}) \\ 7 + 0 \leq y \quad \wedge \quad y \leq 3 + 9 \end{array}} \right\} \text{Step 3} \\
 \Leftarrow \begin{array}{l} \text{(Rule 6 with } \{y/0 + 7, x_6/7, y_6/0\}) \\ \text{True} \quad \wedge \quad 0 + 7 \leq 3 + 9 \\ \text{(True rule)} \\ 0 + 7 \leq 3 + 9 \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 6 with } \{y/0 + 7, x_6/7, y_6/0\}) \\ \text{True} \quad \wedge \quad 0 + 7 \leq 3 + 9 \\ \text{(True rule)} \\ 0 + 7 \leq 3 + 9 \end{array}} \right\} \text{Step 4} \\
 \Leftarrow \begin{array}{l} \text{(Rule 8 with } \{w/0, y/3, x/7, z/9\}) \\ 0 \leq 3 \quad \wedge \quad \leq 7 \leq 9 \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 8 with } \{w/0, y/3, x/7, z/9\}) \\ 0 \leq 3 \quad \wedge \quad \leq 7 \leq 9 \end{array}} \right\} \text{Step 5} \\
 \Leftarrow \begin{array}{l} \text{(Rule 1 and Rule 2)} \\ \text{True} \quad \wedge \quad \text{True} \\ \text{(True rule)} \\ \text{True} \end{array} & \left. \vphantom{\begin{array}{l} \text{(Rule 1 and Rule 2)} \\ \text{True} \quad \wedge \quad \text{True} \\ \text{(True rule)} \\ \text{True} \end{array}} \right\} \text{Steps 6 \& 7}
 \end{array}$$

We now consider how the algorithm shown in the lecture is applied in each step. For the first 3 steps, we describe in detail which operations one needs to make. We start with the assumption we would like to prove, that is to say $7 \leq 3 + 9$:

Step 1: Since our goal is to prove $7 \leq 3 + 9$, we set the variable `goals` as follows:

$$\text{goals} := \{7 \leq 3 + 9\}$$

Since this is the start of the algorithm, there are no additional substitutions to be made here:

$$q' \leftarrow \text{SUBST}(\emptyset, 7 \leq 3 + 9)$$

We now decide to use Rule 8, that is to say

$$\boxed{\forall x_8, y_8, z_8 \quad x_8 \leq y_8 \wedge y_8 \leq z_8 \Rightarrow x_8 \leq z_8}$$

To apply this on our `goals`, we need to unify the variables from $7 \leq 3 + 9$ and $x_8 \leq z_8$, which can be done by substituting $x_8/7$ and $z_8/(3 + 9)$. We store the substitutions we have to make:

$$\theta' \leftarrow \{x_8/7, z_8/3 + 9\}$$

Now, to use Rule 8, we have to assume that $x_8 \leq y_8$ and $y_8 \leq z_8$ hold, therefore we store them as our new goals for the next step:

$$\text{new_goals} \leftarrow \{x_8 \leq y_8, y_8 \leq z_8\}$$

Step 2: From the previous step, we get the following goals:

$$\text{goals} := \{x_8 \leq y_8, y_8 \leq z_8\}$$

as well as the substitutions given by θ' from the previous step; evaluating these substitutions on the first goal $x_8 \leq y_8$ is done by

$$q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9\}, x_8 \leq y_8)$$

and yields the formula $7 \leq y_8$. To continue, we wish to apply Rule 4

$$\boxed{\forall x_4, \quad x_4 \leq x_4 + 0}$$

by identifying $x_4 \leq x_4 + 0$ with our $q' = 7 \leq y_8$. This can be done by choosing $x_4/7$ and $y_8/(x_4 + 0)$:

$$\theta' \leftarrow \{x_4/7, y_8/(x_4 + 0)\}$$

Since Rule 4 does not have any conditions to be true, it does not add new goals to be proven. Therefore, our new goals consist just of the initial goals we have not treated in this step:

$$\text{new_goals} \leftarrow \{y_8 \leq z_8\}$$

Step 3: From the previous step, we get the following goals:

$$\text{goals} := \{y_8 \leq z_8\}$$

As well as the following substitutions, applied on the first (and only) goal $y_8 \leq z_8$:

$$q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9, x_4/7, y_8/x_4 + 0\}, y_8 \leq z_8)$$

This yields $q' = 7 + 0 \leq 3 + 9$. We now wish to apply Rule 8 again, i.e.

$$\boxed{\forall x'_8, y'_8, z'_8 \quad x'_8 \leq y'_8 \wedge y'_8 \leq z'_8 \Rightarrow x'_8 \leq z'_8}$$

for which we need to identify $x'_8 \leq z'_8$ with $7 + 0 \leq 3 + 9$; this can be done by using the substitutions

$$\theta' \leftarrow \{x'_8/7 + 0, z'_8/3 + 9\}$$

To use Rule 8, we need to prove that $x'_8 \leq y'_8$ and $y'_8 \leq z'_8$ holds with our substitutions, therefore we add those two sentences to our goals:

$$\text{new_goals} \leftarrow \{x'_8 \leq y'_8, y'_8 \leq z'_8\}$$

Step 4: $\text{goals} := \{x'_8 \leq y'_8, y'_8 \leq z'_8\}$
 $q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9, x_4/7, y_8/x_4 + 0, x'_8/7 + 0, z'_8/3 + 9\}, x'_8 \leq y'_8) = [7 + 0 \leq y'_8]$
 Using Rule 6: $\boxed{\forall x_6, y_6 \quad x_6 + y_6 \leq y_6 + x_6}$
 $\theta' \leftarrow \{y'_8/y_6 + x_6, x_6/7, y_6/0\}$
 $\text{new_goals} \leftarrow \{y'_8 \leq z'_8\}$

Step 5: $\text{goals} := \{y'_8 \leq z'_8\}$
 $q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9, x_4/7, y_8/x_4 + 0, x'_8/7 + 0, z'_8/3 + 9, y'_8/y_6 + x_6, \dots, x_6/7, y_6/0\}, y'_8 \leq z'_8) = [0 + 7 \leq 3 + 9]$
 Using Rule 7: $\boxed{\forall w_7, x_7, y_7, z_7 \quad w_7 \leq y_7 \wedge x_7 \leq z_7 \Rightarrow w_7 + x_7 \leq y_7 + z_7}$
 $\theta' \leftarrow \{w_7/0, y_7/3, x_7/7, z_7/9\}$
 $\text{new_goals} \leftarrow \{w_7 \leq y_7, x_7 \leq z_7\}$

Step 6: $\text{goals} := \{w_7 \leq y_7, x_7 \leq z_7\}$
 $q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9, x_4/7, y_8/x_4 + 0, x'_8/7 + 0, z'_8/3 + 9, y'_8/y_6 + x_6, x_6/7, y_6/0, w_7/0, y_7/3, \dots, x_7/7, z_7/9\}, w_7 \leq y_7) = [0 \leq 3]$
 Using Rule 1: $\boxed{0 \leq 3}$
 $\theta' \leftarrow \emptyset$
 $\text{new_goals} \leftarrow \{x_7 \leq z_7\}$

Step 7: $\text{goals} := \{x_7 \leq z_7\}$
 $q' \leftarrow \text{SUBST}(\{x_8/7, z_8/3 + 9, x_4/7, y_8/x_4 + 0, x'_8/7 + 0, z'_8/3 + 9, y'_8/y_6 + x_6, x_6/7, y_6/0, w_7/0, y_7/3, \dots, x_7/7, z_7/9\}, x_7 \leq z_7) = [7 \leq 9]$
 Using Rule 2: $\boxed{7 \leq 9}$
 $\theta' \leftarrow \emptyset$
 $\text{new_goals} \leftarrow \emptyset$

Since at the end of the algorithm we end up with the empty clause, we have proven that the knowledge base does indeed entail our original assumption $7 \leq 3 + 9$.

References

- [1] S. Russell and P. Norvig, *Artificial Intelligence: A Modern Approach*. Prentice Hall, 2010.