FUNDAMENTALS OF ARTIFICIAL INTELLIGENCE
Exercise 5: Logical Agents - Solutions Winter Semester 2022/23
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Problem 5.1: Syntax of first-order logic

Recall the formal syntax of first-order logic, especially the definition of terms and sentences.

Problem 5.1.1: *(Taken from [2] Exercise 2.1)*
Let $\mathcal{F} = \{d, f, g\}$ be a set of symbols, where d is a constant, f a function symbol with two arguments, and g a function symbol with three arguments. Which of the following expressions are terms over $\mathcal{F}$? (You may assume that x, y, z are symbols for variables.)

1. $g(d, d)$
2. $f(x, g(y, z), d)$
3. $g(x, f(y, z), d)$
4. $g(x, h(y, z), d)$

Solution:

1. This **is not** a term, because g needs three arguments.
2. This **is not** a term, because f needs two arguments.
3. This **is** a term, because
   - g is a function;
   - x is a variable, hence a term;
   - $f(y, z)$ is a term, because
     - f is a function;
     - y is a variable, hence a term;
     - z is a variable, hence a term;
   - d is a constant, hence a term.
4. This **is not** a term because symbol h is not in $\mathcal{F}$

Problem 5.1.2: *(Taken from [2] Exercise 2.2)*
Let m be a constant, f a function symbol with one argument, and S and B two predicate symbols, each with two arguments. Which of the following expressions are sentences in first-order logic? Specify a reason for failure for expressions which are not. (You may assume that x, y, z are symbols for variables.)

1. $S(m, x)$
2. $B(m, f(m))$
3. $B(B(m, x), y)$
4. $B(x, y) \Rightarrow [\exists z\, S(z, y)]$
5. $S(x, y) \Rightarrow S(y, f(f(x)))$

Solution:

1. This **is** a sentence, because
   - S is a predicate;
   - m is a constant, hence a term;
   - x is a variable, hence a term.
2. This **is** a sentence, because
   - B is a predicate;
   - m is a constant, hence a term;
   - $f(m)$ is a term, because
     - f is a function;
     - m is a constant, hence a term.
3. This **is not** a sentence, because the first argument of B — that is, $B(m, x)$ — is not a <u>term</u>, but <u>predicate</u>, because
   - B is a predicate;
   - m is a constant, hence a term;
   - x is a variable, hence a term.
4. This **is** a sentence, because
   - $B(x, y)$ is a sentence, because
     - B is a predicate;
     - x, y are variables, hence terms.
   - $\exists z\ S(z, y)$ is a sentence, because
     - z, y are variables, hence terms;
     - S is predicate.
5. This **is** a sentence, because
   - $S(x, y)$ is a sentence, because
     - S is a predicate;
     - x, y are variables, hence terms.
   - $S(y, f(f(x)))$ is a sentence, because
     - S is a predicate;
     - y is a variable, hence a term;
     - $f(f(x))$ is a term, because
       - f is a function;
       - $f(x)$ is a term, because
         - f is a function;
         - x is a variable, hence a term.

Problem 5.2: Universal and existential quantifiers

Let us abbreviate the predicate "x is taking the Bus" by $B(x)$, and "x has a Ticket" by $T(x)$. Suppose also that we only consider the universe of discourse where there are only three people: Alice, Bob, and Charlie.

Universal quantifiers

Problem 5.2.1: For each of our protagonists, **Table 1** lists whether they have a ticket, and whether they take the bus. Suppose that we want to formalize the sentence "All people who take the bus have a ticket". Given the characteristics in the table below, do you think the sentence evaluates to true? Or false?

Table 1

x	$B(x)$	$T(x)$
Alice	True	True
Bob	False	True
Charlie	False	False

Solution: The sentence evaluates to true according to these conditions, since this sentence applies only to people that take the bus — which, in this case, is only Alice. Since she does have a ticket, the sentence is true.

Problem 5.2.2: Now, consider the formula

$$\forall x \quad B(x) \wedge T(x).$$

By using the extended interpretation for universal quantifiers (in other words, the transformation to propositional logic), decide whether this formalization is true or false. Do you think this formula is a faithful formalization of "All people who take the bus have a ticket"? If not, what would be a better formula?

Solution: This formula evaluates to false. To see this, we apply the extended interpretation of universal quantifiers, which yields the sentence

$$B(\text{Alice}) \wedge T(\text{Alice}) \quad \wedge \quad B(\text{Bob}) \wedge T(\text{Bob}) \quad \wedge \quad B(\text{Charlie}) \wedge T(\text{Charlie}).$$

Since B(Bob) and B(Charlie) are false, the whole extended interpretation is also false. Therefore, this formalization is not a faithful representation of our sentence. The correct formalization would be

$$\forall x \quad B(x) \Rightarrow T(x).$$

Existential quantifiers

Problem 5.2.3: We now look instead at the truth assignments given in **Table 2.** Suppose that we now want to formalize the sentence "Some people who take the bus have a ticket"[1]. Given the characteristics in the table below, do you think the sentence evaluates to true? Or false?

Table 2

x	$B(x)$	$T(x)$
Alice	False	True
Bob	False	True
Charlie	False	False

[1]This has to be understood as "There is **at least** one person that takes the bus, who has a ticket".

Solution: The sentence evaluates to false if applied on the specific case of Alice, Bob and Charlie, since no one is taking the bus, no matter whether they have a ticket or not.formalization

Problem 5.2.4: Now, consider the formula

$$\exists x \quad B(x) \Rightarrow T(x).$$

By using the extended interpretation for existential quantifiers, decide whether this formalization is true or false. Do you think this formula is a faithful formalization of "Some people who take the bus have a ticket"? If not, what would be a better formula?

Solution: This formula evaluates to True. To see this, we apply the extended interpretation of existential quantifiers, which yields the sentence

$$(B(\text{Alice}) \Rightarrow T(\text{Alice})) \quad \vee \quad (B(\text{Bob}) \Rightarrow T(\text{Bob})) \quad \vee \quad (B(\text{Charlie}) \Rightarrow T(\text{Charlie})).$$

Since $B(\text{Alice})$ is False and $T(\text{Alice})$ is True, $B(\text{Alice}) \Rightarrow T(\text{Alice})$ is True, and thus so is the whole extended interpretation. Therefore, this formalization is not a faithful representation of our sentence. The correct formalization would be $\exists x \quad B(x) \wedge T(x)$.

Problem 5.3: Formalization to First Order Logic

Problem 5.3.1: *(Taken from [2] Exercise 2.1)*
Use the predicates

$$\begin{aligned} A(x, y) &: \quad x \text{ admires } y \\ P(x) &: \quad x \text{ is a professor} \end{aligned}$$

and the constant

$$d : \quad \text{Dan}$$

to translate the following sentences into first-order logic:

1. Dan admires every professor.
2. Some professor admires Dan.
3. Dan admires himself.

Solution:

1. Dan admires every professor.
$$\forall x \quad P(x) \Rightarrow A(d, x)$$
2. Some professor admires Dan.
$$\exists x \quad P(x) \ \wedge \ A(x, d)$$
3. Dan admires himself.
$$A(d, d)$$

Problem 5.3.2: *(Taken from [2] Exercise 2.1)*
Use the predicates

$$\begin{aligned} A(x, y) &: \quad x \text{ attended } y \\ S(x) &: \quad x \text{ is a student} \\ L(x) &: \quad x \text{ is a lecture} \end{aligned}$$

to translate the following sentences into first-order logic:

1. No student attended every lecture.
2. No lecture was attended by every student.
3. No lecture was attended by any student.

Solution:

1. No student attended every lecture

$$\neg\exists x \forall y \quad S(x) \ \wedge \ (L(y) \Rightarrow A(x, y))$$

   or equivalently, "Every student has a lecture he or she does not attend"

$$\forall x \exists y \quad S(x) \Rightarrow (L(y) \ \wedge \ \neg A(x, y))$$

2. No lecture was attended by every student

$$\neg\exists x \forall y \quad L(x) \ \wedge \ (S(y) \Rightarrow A(y, x))$$

   or equivalently, "Every lecture must have some student misses the lecture"

$$\forall x \exists y \quad L(x) \Rightarrow (S(y) \ \wedge \ \neg A(y, x))$$

3. No lecture was attended by any student

$$\neg\exists x \exists y \quad L(x) \ \wedge \ S(y) \ \wedge \ A(y, x)$$

   or equivalently, "Every lecture is missed by every student"

$$\forall x \forall y \quad L(x) \Rightarrow (S(y) \Rightarrow \neg A(y, x))$$

   or equivalently

$$\forall x \forall y \quad (L(x) \ \wedge \ S(y)) \Rightarrow \neg A(y, x)$$

Problem 5.3.3: *(Taken from [1])*
Let h stand for Holmes (Sherlock Holmes) and m for Moriarty (Professor Moriarty). Let us abbreviate "x can trap y" by $T(x, y)$. Give the symbolic rendition of the following:

1. Holmes can trap everyone who can trap Moriarty.
2. Holmes can trap everyone whom Moriarty can trap.
3. Holmes can trap everyone who can be trapped by Moriarty.
4. If someone can trap Moriarty, then Holmes can.
5. If everyone can trap Moriarty, then Holmes can.
6. Everyone who can trap Holmes can trap Moriarty.
7. No one can trap Holmes unless that person can trap Moriarty.
8. Everyone can trap someone who cannot trap Moriarty.
9. Everyone who can trap Holmes can trap everyone whom Holmes can trap.

Solution:

1. Holmes can trap everyone who can trap Moriarty.

$$\forall x \quad T(x,m) \Rightarrow T(h,x)$$

2. Holmes can trap everyone whom Moriarty can trap.

$$\forall x \quad T(m,x) \Rightarrow T(h,x)$$

3. Holmes can trap everyone who can be trapped by Moriarty.

$$\forall x \quad T(m,x) \Rightarrow T(h,x)$$

4. If someone can trap Moriarty, then Holmes can.

$$(\exists x \quad T(x,m)) \Rightarrow T(h,m)$$

5. If everyone can trap Moriarty, then Holmes can.

$$(\forall x \quad T(x,m)) \Rightarrow T(h,m)$$

6. Everyone who can trap Holmes can trap Moriarty.

$$\forall x \quad T(x,h) \Rightarrow T(x,m)$$

7. No one can trap Holmes unless that person can trap Moriarty.

$$\forall x \quad \neg T(x,m) \Rightarrow \neg T(x,h)$$

   or equivalently

$$\neg \exists x \quad \neg T(x,m) \wedge T(x,h)$$

8. Everyone can trap someone who cannot trap Moriarty.

$$\forall x\, \exists y \quad \neg T(y,m) \wedge T(x,y)$$

9. Everyone who can trap Holmes can trap everyone whom Holmes can trap.

$$\forall x \quad T(x,h) \Rightarrow (\forall y \quad T(h,y) \Rightarrow T(x,y))$$

   or equivalently

$$\forall x,y \quad T(x,h) \Rightarrow (T(h,y) \Rightarrow T(x,y))$$

   or equivalently

$$\forall x,y \quad T(x,h) \wedge T(h,y) \Rightarrow T(x,y)$$

References

[1] D. Gries and F. B. Schneider. *A Logical Approach to Discrete Math*. Springer-Verlag New York, Inc., New York, NY, USA, 1993.

[2] M. Huth and M. D. Ryan. *Logic in computer science - modelling and reasoning about systems (2. ed.)*. Cambridge University Press, 2004.