LaTeXTemplate: Example Notes
My Notes for Linear Algebra Midterm

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Disclaimer

This review *probably* does not cover everything discussed in Chapters 1 through 3 in detail. This is just a quick discussion of some important details about the essentials of linear algebra.

1 Linear Equations in Linear Algebra

1.1 System of Linear Equations

A linear equation is defined as an equation with $n \geq 1$ variables in the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

A *system of linear equations* is a collection of one or more linear equations with the same variables

$$x_1, x_2, \ldots, x_n$$

involved:

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= c_1 \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= c_2 \\ &\vdots \\ m_1x_1 + m_2x_2 + \cdots + m_nx_n &= c_m \end{aligned} \tag{2}$$

Solution Set Types

A system has a *solution set* which refers to the set of points where the system of linear equations are *all consistent*. Two systems are *equivalent* if they have the *same solution set*. A system of linear equations can have three different types of solutions sets:

1. No solution: **Inconsistent**
2. Unique/one solution: **Consistent**
3. Infinitely many solutions: **Consistent**

It is a little important as this shows us how a system of linear equations *cannot have, say, 2 solutions*.

1.1.1 The Entrance of Matrices

We can use matrices to *represent* a system of linear equations. Let's take our previous system of linear equations:

$$\begin{aligned}
a_1x_1 + a_2x_2 + \cdots + a_nx_n &= c_1 \\
b_1x_1 + b_2x_2 + \cdots + b_nx_n &= c_2 \\
&\vdots \\
m_1x_1 + m_2x_2 + \cdots + m_nx_n &= c_m
\end{aligned} \tag{3}$$

We can write this, as an *augmented matrix* like this:

$$\begin{bmatrix}
a_1 & a_2 & \cdots & a_n & c_1 \\
b_1 & b_2 & \cdots & b_n & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_1 & m_2 & \cdots & m_n & c_m
\end{bmatrix} \tag{4}$$

Just like how we are solving the system of linear equations, we can solve the matrix—in an, arguably, easier manner. The matrix is equivalent to the original matrix as long as one does the following types of operations:

Elementary Row Operations

- **Replacement**: Replace a row by the sum of itself and another row.
- **Interchange**: Interchange two rows.
- **Scaling**: Multiply a row by a constant.

Now that we understand a little bit more about linear algebra, let's ask the fundamental questions of linear algebra:

Fundamental Questions of Linear Algebra

- **Existence**: Does the system of linear equations have at least one solution?
- **Uniqueness**: Does the system of linear equations have exactly one solution?

1.2 Row Reduction & Echelon Forms

Definition 1.1: Row Echelon Form

We have these following requirements for a matrix to be in (row) echelon form:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Visualization of an echelon matrix would be:

$$\begin{bmatrix} 0 & \square & \star & \star & \star & \star & \star \\ 0 & 0 & \square & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \square & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & \square \end{bmatrix} \quad \text{where} \begin{cases} \star & : \text{any number, including } 0 \\ \square & : \text{any nonzero number} \end{cases} \tag{5}$$

Definition 1.2: Reduced Row Echelon Form

We have these following requirements for a matrix to be in reduced row echelon form:

1. The matrix is in row echelon form.
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only entry in its column.

Visualization of an echelon matrix would be:

$$\begin{bmatrix} 0 & 1 & 0 & \star & 0 & \star & 0 \\ 0 & 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where} \begin{cases} \star & : \text{any number, including } 0 \end{cases} \tag{6}$$

We are also going to need to define what *pivot position* and *pivot column* is:

Definition 1.3: Pivot Position & Pivot Column

To help us understand the significance of the reduced row echelon matrix, we are also going to define the following two terms:

a) **Pivot Position**: A position of the leading 1 in the reduced echelon form of A.
b) **Pivot Column**: A column of the leading 1 in the reduced echelon form of A.

These lead us to the following theorem:

Theorem 1.1: Uniqueness of Reduced Row Echelon Form

Each matrix is equivalent to **one and only one reduced echelon matrix**.

Theorem 1.2: Existence & Uniqueness Theorem of Matrices

A linear system is **consistent** if and only if it has no columns of the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & b \end{bmatrix}$$

Since if a matrix is consistent, either it has a unique solution, or it has infinitely many solutions. How do we know which one it is? Here you go:

a) **Unique**: The matrix has a unique solution if it has no free variables.

b) **Infinite**: The matrix has infinitely many solutions if it has at least one free variable.

1.3 Vector Equations

A matrix with only one column is called a **column vector** or simply a **vector**. The set of all vectors with n entries is called $\mathbb{R}^n$.

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ in $\mathbb{R}^n$ and given the scalars $c_1, c_2, \dots, c_k$, the vector $\mathbf{y}$ defined by;

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{7}$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ with **weights** $c_1, c_2, \dots, c_k$.

Remark 1.1: Vector Equations & Augmented Matrices

A vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$$

has the same solution set as the augmented matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathbf{b} \end{bmatrix}$$

Definition 1.4: Span of Vectors

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$. Then, the **set of all linear combinations of** $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$$

and is called the **subset of** $\mathbb{R}^n$ **spanned by** $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$. In other words, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is the collection of the vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

with $c_1, c_2, \dots, c_n$ scalars.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

Definition 1.5: Matrix Multiplication with Vectors

If A is an $m \times n$ matrix, with columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, and if $\mathbf{x} \in \mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \tag{8}$$

Of course, $A\mathbf{x}$ is **defined** only and only if #columns(A) = #rows$(\mathbf{x})$.

I think you already know what this takes us to:

Theorem 1.3: Matrix Equations, Vector Equations, and Augmented Matrices

This builds up to our previous remark (Vector Equations & Augmented Matrices). A matrix equation $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix, with the columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{n}$, and if $\mathbf{b} \in \mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$$

which has the same solution set as the augmented matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathbf{b} \end{bmatrix}$$

Existence of Solutions

The theorem above leads us to a very important fact: the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ **is a linear combination of the columns of** A.

Building onto our remark (Vector Equations & Augmented Matrices) and our theorem above (Matrix Equations, Vector Equations, and Augmented Matrices), we get the following theorem:

Theorem 1.4: Relation between Linear Combination & Solution Set

Let A be an $m \times n$ matrix. Then the following statements are either all true or all false:

a) For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b) Each $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

c) The columns of A span $\mathbb{R}^m$.

d) A has a pivot position in every row.

Theorem 1.5: Properties of Matrix-Vector Product

If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and c is a scalar, then

a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

b) $A(c\mathbf{u}) = cA\mathbf{u}$

1.5 Solution Sets of Linear Systems

Definition 1.6: Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$ where A is an $m \times n$ matrix and $\mathbf{0} \in \mathbb{R}^m$. This system always has the **trivial solution** of $\mathbf{x} = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^n$. The important question, therefore, is whether the equation has a **nontrivial solution**—the solution where $\mathbf{x} \neq \mathbf{0}$. The reason being:

Nontrivial Solution
The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has *at least 1 free variable*.

We can use the **parametric vector equations** to denote the solution vectors $\mathbf{x}$:

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v}$$

Definition 1.7: Nonhomogeneous Linear Systems

Nonhomogeneous linear systems are linear systems in the form $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix and $\mathbf{b} \neq \mathbf{0} \in \mathbb{R}^m$.

Theorem 1.6: Solution Set to Nonhomogeneous Linear Systems

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b}$ given, and let $\mathbf{p}$ be a solution ($A\mathbf{p} = \mathbf{b}$). Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Quick Proof

We are given

$$A\mathbf{p} = \mathbf{b} \text{ and } A\mathbf{v}_h = \mathbf{0}$$

Then,

$$A\left(\mathbf{p} + \mathbf{v}_h\right) = \underbrace{A\mathbf{p}}_{\mathbf{b}} + \underbrace{A\mathbf{v}_h}_{\mathbf{0}} = \mathbf{b}$$

which indeed is a solution to $A\mathbf{x} = \mathbf{b}$.

1.6 Applications of Linear Algebra (*Redacted*)

1.7 Linear Independence

Definition 1.8: Linear Independence v. Dependence

An indexed set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_p\mathbf{v}_p = \mathbf{0}$$

has only **trivial solution**. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, c_2, \ldots, c_p$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$$

where there exists **at least one** $c_i \neq 0$ **for all** $i \in \{1, 2, \ldots, p\}$.

With all of our knowledge of *vectors* and *matrices*, let's observe:

Remark 1.2: Linear Independence: Vector & Matrix Equations

Let A be an $m \times n$ matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and let $\mathbf{x} \in \mathbb{R}^n$. Then,

a) RREF of $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathbf{0} \end{bmatrix}$, **has no free variables** if the vectors are **linearly independent**.

b) The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ has only the **trivial solution** $\mathbf{x} = \mathbf{0}$ if the vectors are **linearly dependent**.

c) The columns of A are **linearly independent** if only and only if $A\mathbf{x} = \mathbf{0}$ has only the **trivial solution**.

These discussions will lead to three new theorems, all of them essentially pointing to the same statement.

Theorem 1.7: Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors, some $\mathbf{v}_i \in S$ is a linear combination of the others.

Warning*: This doesn't mean that* ***all the vectors*** *in S are linear combinations of the preceding vectors.*

Theorem 1.8: Linear Dependence in $\mathbb{R}^n$

If a set contains more vectors than there are entries in each vector, for $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ where $k > n$, the vectors in the set are linearly dependent.

Theorem 1.9: Zero Vector in a Set S

If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \in \mathbb{R}^n$ contains the zero vector, then the set is *linearly dependent*.

Quick Proof

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ with $\mathbf{0} \in S$. Since

$$\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n \tag{9}$$

for $x_1 = x_2 = \cdots = x_n = 0$, the set is linearly dependent.

1.8 Introduction to Linear Transformations

Definition 1.9: Linear Transformations

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x} \in \mathbb{R}^n$ a vector $T(\mathbf{x}) \in \mathbb{R}^m$. We also call $\mathbb{R}^n$ the **domain** and $\mathbb{R}^m$ the **codomain** of T, which we use the notation $T : \mathbb{R}^n \to \mathbb{R}^m$ to denote it.

Let $T(\mathbf{x}) = A\mathbf{x}$ be a linear transformation that maps $\mathbb{R}^n$ to $\mathbb{R}^m$, which means that the matrix A is an $m \times n$ matrix. Then, we sometimes also denote this linear transformation as $\mathbf{x} \mapsto A\mathbf{x}$.

Properties of Linear Transformations

A transformation T is **linear** if

a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T

b) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and $\mathbf{u}$ in the domain of T

Remark 1.3: Outcomes of Properties of Linear Transformations

a) A linear transformation T *always* maps $\mathbf{0}$ to $\mathbf{0}$ in the domain of T:

$$T(\mathbf{0}) = \mathbf{0}$$

I wrote this fact in one of my proofs in one of the previous assignments—it is very powerful to know.

b) For all scalars c and d, and all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T, we have

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

1.9 The Matrix of a Linear Transformation

Honestly, I think this is the subsection of the chapter where you understand the power of linear algebra. Basically, let's say you have a linear transformation that is represented geometrically or described in words. As a mathematician, you want to have a formula $T(\mathbf{x})$ to describe this transformation. What linear algebra tells us to that is that if you know what a linear transformation does to the identity matrix $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$, then you know the linear transformation matrix A where $\mathbf{x} \mapsto A\mathbf{x}$ is your linear transformation T. The following theorem indicates this fact:

Theorem 1.10: The Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

We will, from now on, call the matrix A the **standard matrix for the linear transformation** $T(\mathbf{x})$.

We will define two new concepts as we are discussing linear transformations:

Definition 1.10: Onto & One-to-One

1. **Onto (*Existence*)**: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ *in* $\mathbb{R}^n$. In other words, T is onto $\mathbb{R}^m$ if the range of T is the entirety of $\mathbb{R}^m$.

   Remember!
   If $m > n$ where $T : \mathbb{R}^n \to \mathbb{R}^m$, then T is not onto $\mathbb{R}^m$—it is impossible. A linear transformation can have a dimension of at most n.

2. **One to One (*Uniqueness*)**: A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at most one* $\mathbf{x} \in \mathbb{R}^n$.

   Remember!
   T is one-to-one if and only if, for each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all.

These discussions will lead to a new theorem:

Theorem 1.11: Uniqueness of a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

These will lead us to our next theorem:

Theorem 1.12: Linear Independence on Existence & Uniqueness

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then,

a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

b) T is one-to-one if and only if the columns of A are linearly independent.

There four special transformations that can be visualized in 2D (which I won't for now):

a) **Reflections**

b) **Contractions & Expansions**

c) **Shears**

d) **Projections**

1.10 Linear Models in Business, Science, and Engineering (*Redacted*)

2 Matrix Algebra

2.1 Matrix Operations

If A is an $m \times n$ matrix, then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.

The **diagonal entries** in an $m \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ are $a_{11}, a_{22}, \ldots$ and form the **main diagonal** of A. A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are 0, for instance I_n.

Theorem 2.1: Properties of Scalar Sums and Products of Matrices

Let A, B, and C be matrices of the same size, and let r and s be scalars.

a) $A + B = B + A$
b) $(A + B) + C = A + (B + C)$
c) $A + 0 = A$
d) $r(A + B) = rA + rB$
e) $(r + s)A = rA + sA$
f) $(r + s)A = rA + sA$
g) $r(sA) = (rs)A$

Definition 2.1: Matrix Multiplication

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p$:

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots A\mathbf{b}_p \end{bmatrix}$$

Meaning Behind Multiplication
Essentially, the matrix multiplication AB corresponds to composition of linear transformations.

Theorem 2.2: Properties of Matrix Multiplication

Let A be an $n \times m$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a) **Associativity**: $A(BC) = (AB)C$
b) **Distributivity**: $A(B + C) = AB + AC$
c) **Commutativity**: $(B + C)A = BA + CA$
d) **Scalar Multiplication**: $r(AB) = (rA)B = A(rB)$
e) **Identity**: $I_m A = A = AI_n$

We also define powers of matrices and transposes of matrices:

Definition 2.2: Powers of a Matrix

Let A be an $n \times n$ matrix and k be a positive integer. Then,

$$A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

Definition 2.3: Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix denoted by A^T, whose columns are formed from the corresponding rows of A.

$$A = \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & r & q & s & t \end{bmatrix} \implies A^T = \begin{bmatrix} a & f & k & p \\ b & g & l & r \\ c & h & m & q \\ d & i & n & s \\ e & j & o & t \end{bmatrix}$$

Theorem 2.3: Properties of Transpose

Let A and B denote matrices whose sizes are appropriate for the indicated sums and products.

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(rA)^T = rA^T$

d) $(AB)^T = B^T A^T$

2.2 The Inverse of a Matrix

An $n \times n$ matrix is said to be **invertible** if there is an $n \times n$ matrix B such that $AB = I_n$.

Definition 2.4: Inverse of a Matrix

Let A be an $n \times n$ matrix. Then the inverse of the matrix, denoted by A^{-1}, is the matrix such that $A^{-1}A = I_n$ and $AA^{-1} = I_n$.

a) **Singular Matrix**: A matrix that is not invertible.

b) **Non-singular Matrix**: A matrix that is invertible.

Theorem 2.4: Inverse of a 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\det A = ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\underbrace{ad - bc}_{\det A}} \begin{bmatrix} d & -b \\ -a & c \end{bmatrix}$$

The definition of the inverse matrix will lead to very fancy stuff, one of them being finding solutions for the matrix equation $A\mathbf{x} = \mathbf{b}$.

Theorem 2.5: Solutions of Matrix Equations Using Inverse

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the *unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$.

Quick Proof

Let A be an invertible matrix and we want to solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned} \tag{10}$$

With all of our knowledge involved in transposes and inverses of matrices, we can write the useful properties of the matrices:

Theorem 2.6: Properties of Invertible Matrices

a) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b) If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is

$$(AB)^{-1} = B^{-1}A^{-1}$$

c) If A is an invertible matrix, then A^T is also an invertible matrix and

$$(A^T)^{-1} = (A^{-1})^T$$

Definition 2.5: Elementary Matrices

An **elementary matrix** is obtained by performing a single elementary row operation on an identity matrix. There are three types of elementary matrices:

1. **Row Swap**: An elementary matrix that swaps two rows. (ex: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ swaps R_1 and R_2)
2. **Row Addition**: An elementary matrix that adds a multiple of one row to another row (ex: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ adds k times R_2 to R_1)
3. **Row Scaling**: An elementary matrix that multiplies one row by a constant (ex: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ scales R_1 by k)

***Note**: An elementary matrix is always invertible.*

We will use this definition of elementary matrices to find the inverse of a matrix. In order to do so, we need the next theorem:

Theorem 2.7: Invertible Matrix A is Row Equivalent to I_n

An $n \times n$ matrix A is **invertible only and only if A is row equivalent to I_n**, and in this case, the sequence of elementary row operations that reduces A to I_n also transforms I_n to A.

Quick Proof

Let A be an invertible matrix and we know that:

$$\underbrace{E_p E_{p-1} \cdots E_1}_{\text{Elementary Matrices}} A = I_n$$

Then,

$$\underbrace{E_p E_{p-1} \cdots E_1}_{\text{Elementary Matrices}} \underbrace{AA^{-1}}_{I_n} = \underbrace{I_n A^{-1}}_{A^{-1}}$$

simplifies to

$$E_p E_{p-2} \cdots E_1 I_n = A^{-1}$$

So, now that we know the sequence to row reduce the matrix A to I_n, when applied to I_n, leads to A^{-1}, we can use this fact to find an algorithm!

Algorithm for Finding the Inverse of a Matrix

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, (say, there is a free variable which directly indicates that the matrix is not invertible), the matrix is not invertible.

2.3 Characterizations of Invertible Matrices

This subsection is the point where all of our knowledge from the first section and the second section merge into a very important theorem:

Theorem 2.8: Invertible Matrix Theorem

Let A be an invertible matrix. Then the following statements are equivalent. That is, for given A, the statements are either all true, or all false.

a) A is invertible.

b) A is row equivalent to I_n.

c) A has n pivot positions.

d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e) The columns of A form a linearly independent set.

f) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$ (*Repetition of "onto"*).

h) The columns of A span $\mathbb{R}^n$ (*Repetition of "one-to-one"*).

i) The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix C such that $CA = I$ (*Repetition of "inverse"*).

k) There is an $n \times n$ matrix D such that $AD = I$ (*Repetition of "inverse"*).

l) A^T is an invertible matrix.

We know from the previous section that matrix multiplications are essentially compositions of linear transformations. So, we define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ invertible if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, which takes us to our next theorem:

Theorem 2.9: Invertible Linear Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be a standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

a) $S(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

b) $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

The rest of the section was not part of the exam, and therefore is omitted.

3 Determinants

3.1 Introduction to Determinants

Definition 3.1: Determinant of a Matrix

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} a_{1j} \det A_{1j} \end{aligned} \tag{11}$$

where the equation is called **cofactor expansion across the first row**. We are going to talk in detail what *cofactor expansion* is in the following theorem.

Theorem 3.1: Cofactor Expansion Across a Row or Down a Column

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. We will denote the (i, j)**-th cofactor** of A by C_{ij}. That is,

$$C_{ij} = (-1)^{i+j} \det A_{ij} \tag{12}$$

The expansion across the i-th row using the cofactors is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \tag{13}$$

and the cofactor expansion down the j-th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \tag{14}$$

Theorem 3.2: Determinant of a Triangular Matrix

If A is a triangular matrix, then the determinant is the product of the entries on the main diagonal.

3.2 Properties of Determinants

Theorem 3.3: Row Operations & Their Effect on Determinants

Let A be a square matrix $(n \times n)$. Then,

a) **Row Swap**: If two rows of A are interchanged/swapped to produce a matrix B, then $\det B = -\det A$.

b) **Row Addition**: If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

c) **Row Multiplication**: If one row of A is multiplied by k to produce B, then $\det B = k \det A$.

Theorem 3.4: Determinant & Inversible Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det A \neq 0$.

Unlike the previous sections, where we already pretty much focused on the effects of row operations on matrices, now we will also take a look at the column operations:

Theorem 3.5: Column Operations & Their Effect on Determinants

If A is an $n \times n$ matrix, then $\det A^T = \det A$, and therefore, the column operations, which essentially corresponds to operations done on the transpose of A and transpose it back to A have the same effects as the row operations done on A.

Here comes a very important property of determinants:

Theorem 3.6: Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = \det A \det B$.