Example Document

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1 Let's have some fun!

Theorem 1.1: Stokes' Theorem

For a surface $\mathcal{S}$ defined on a region $\mathcal{D}$ with a boundary $\mathcal{C}$, in a vector field $\vec{\mathbf{F}}$, Stokes' theorem states that:

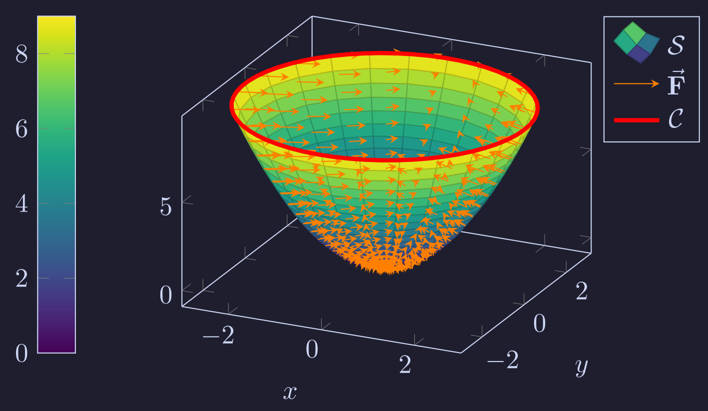
$$\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} (\nabla \times \vec{\mathbf{F}}) \cdot \mathbf{n} \, \mathrm{d}\sigma \tag{1}$$

Remark 1.1: Green's Theorem & Stokes' Theorem

Essentially, Green's theorem is a special case of Stokes' theorem. If $\mathcal{C}$ is a curve in the xy-plane, oriented clockwise, and $\mathcal{D}$ is the region in the xy-plane bounded by $\mathcal{C}$, then $\mathrm{d}\sigma = \mathrm{d}x \, \mathrm{d}y$ and $\mathbf{n} = \mathbf{k}$ and therefore,

$$\begin{aligned} (\nabla \times \vec{\mathbf{F}}) \cdot \mathbf{n} &= (\nabla \times \vec{\mathbf{F}}) \cdot (\mathbf{k}) \\ &= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \end{aligned} \tag{2}$$

Surface $\mathcal{S}$, its Boundary Curve $\mathcal{C}$, and Vector Field $\vec{\mathbf{F}}$



Interesting Fact

Stokes' theorem is named after Sir George Gabriel Stokes, who formulated it in the 19th century. It has applications in various fields, including fluid dynamics, electromagnetism, and differential geometry.

Caution

When applying Stokes' theorem, ensure that the surface $\mathcal{S}$ is smooth and oriented correctly with respect to its boundary curve $\mathcal{C}$. The orientation of $\mathcal{C}$ must be consistent with the right-hand rule applied to the normal vector $\mathbf{n}$ of the surface.

We might as well do some linear algebra while we're at it!

Theorem 1.2

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues. Assume

$$\begin{cases} S_1 &= \{v_1^{(1)}, v_2^{(1)}, \ldots, v_{d_1}^{(1)}\} \\ S_2 &= \{v_1^{(2)}, v_2^{(2)}, \ldots, v_{d_2}^{(2)}\} \\ \vdots \\ S_k &= \{v_1^{(k)}, v_2^{(k)}, \ldots, v_{d_k}^{(k)}\} \end{cases}$$

where S_i is a linearly independent set of eigenvectors corresponding to λ. Then,

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

is linearly independent.

Proof Since the linear combinations of the eigenvectors of an eigenspace are still in the eigenspace, we have that

$$\begin{cases} v_1 = a_1^{(1)} v_1^{(1)} + a_2^{(1)} v_2^{(1)} + \cdots + a_{d_1}^{(1)} v_{d_1}^{(1)} & \in \mathrm{Nul}(A - \lambda_1 I) \\ v_2 = a_1^{(2)} v_1^{(2)} + a_2^{(2)} v_2^{(2)} + \cdots + a_{d_2}^{(2)} v_{d_2}^{(2)} & \in \mathrm{Nul}(A - \lambda_2 I) \\ \vdots & \vdots \\ v_k = a_1^{(k)} v_1^{(k)} + a_2^{(k)} v_2^{(k)} + \cdots + a_{d_k}^{(k)} v_{d_k}^{(k)} & \in \mathrm{Nul}(A - \lambda_k I) \end{cases}$$

Then, we want to show that the only solution to

$$v_1 + v_2 + \cdots + v_k = 0$$

is the trivial solution $v_1 = v_2 = \cdots = v_k = 0$. Applying A to both sides, we have that

$$A(v_1 + v_2 + \cdots + v_k) = \lambda_1 v_1 + \lambda_2 v_2 + \cdots \lambda_k v_k = A0 = 0.$$

So, we want to solve the system (with $\lambda_i \neq \lambda_j$),

$$\begin{cases} v_1 + v_2 + \cdots + v_k &= 0 \\ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k &= 0 \end{cases}$$

Inductively, we can show that this implies $v_1 = v_2 = \cdots = v_k = 0$. Thus, the set S is linearly independent.

Now, let's do some probability!

Proposition 1.1: Properties of Independent Random Variables

(a) Let X_1, X_2 be independent random variables. Let $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ be functions. Then,

$$E[u_1(X_1) \cdot u_2(X_2)] = E[u_1(X_1)] \cdot E[u_2(X_2)].$$

(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables. Then, the linear combination of these random variables,

$$Y = \sum_{i=1}^{n} a_i X_i,$$

satisfies

$$E[Y] = \sum_{i=1}^{n} a_i E[X_i], \quad \operatorname{Var}(Y) = \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i).$$

Proposition 1.2

If $X \sim \Gamma(s, \lambda)$ and $Y \sim \Gamma(t, \lambda)$ are independent, then $X + Y \sim \Gamma(s + t, \lambda)$.

Proof We know that

$$f_X(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)}, \quad 0 < x < \infty,$$

and

$$f_Y(y) = \frac{\lambda e^{-\lambda y}(\lambda y)^{t-1}}{\Gamma(t)}, \quad 0 < y < \infty.$$

Now, we will find the PDF of $X + Y$ for $0 < x < \infty$ and $0 < y < \infty$:

$$\begin{aligned} f_{X+Y}(a) &= \int_0^a f_X(x) f_Y(a-x)\,\mathrm{d}x \\ &= \int_0^a \frac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} \cdot \frac{\lambda e^{-\lambda(a-x)}(\lambda(a-x))^{t-1}}{\Gamma(t)}\,\mathrm{d}x \\ &= \frac{\lambda^{s+t} e^{-\lambda a}}{\Gamma(s)\Gamma(t)} \int_0^a x^{s-a}(a-x)^{t-1}\,\mathrm{d}x \end{aligned} \tag{3}$$

Now, if we do a u-substitution with $u = \frac{x}{a}$, then $\mathrm{d}x = a\,\mathrm{d}u$. Thus,

$$\begin{aligned} f_{X+Y}(a) &= \frac{\lambda^{s+t} e^{-\lambda a}}{\Gamma(s)\Gamma(t)} \int_0^1 (au)^{s-1}(a-au)^{t-1} a\,\mathrm{d}u \\ &= \frac{\lambda^{s+t} e^{-\lambda a}}{\Gamma(s)\Gamma(t)} a^{s+t-1} \int_0^1 u^{s-1}(1-u)^{t-1}\,\mathrm{d}u \\ &= Ce^{-\lambda a} a^{s+t-1} \end{aligned} \tag{4}$$

Now, using the fact that

$$\int_{-\infty}^{\infty} f_{X+Y}(a)\,\mathrm{d}a = 1,$$

we can solve for C:

$$1 = C \int_0^{\infty} e^{-\lambda a} a^{s+t-1}\,\mathrm{d}a \tag{5}$$

Let $u = \lambda a$. Then, $\mathrm{d}u = \frac{1}{\lambda}\,\mathrm{d}a$. Thus,

$$\begin{aligned} 1 &= C \int_0^\infty e^{-u} \left(\frac{u}{\lambda}\right)^{s+t-1} \frac{1}{\lambda}\,\mathrm{d}u \\ &= C \frac{1}{\lambda^{s+t}} \int_0^\infty e^{-u} u^{s+t-1}\,\mathrm{d}u \\ &= C \frac{\Gamma(s+t)}{\lambda^{s+t}} \end{aligned} \tag{6}$$

Then,

$$C = \frac{\lambda^{s+t}}{\Gamma(s+t)}.$$

So, the PDF of $X+Y$ is given by

$$f_{X+Y}(a) = \frac{\lambda^{s+t} e^{-\lambda a} a^{s+t-1}}{\Gamma(s+t)}, \quad 0 < a < \infty.$$

This is exactly the PDF of $\Gamma(s+t, \lambda)$.

Corollary 1.1

If $X_1, X_2, \ldots, X_n \sim \operatorname{Exp}(\lambda)$ are independent, then $X_1 + X_2 + \cdots + X_n \sim \Gamma(n, \lambda)$.

Let's finish off with a Numerical Analysis problem.

Problem 1.1

Suppose we are given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and we want to fit them with the model

$$f(x) = a + be^{-x}.$$

The goal is to determine a and b by minimizing the squared error

$$E = \sum_{i=1}^n (a + be^{-x_i} - y_i)^2.$$

Rewrite this least squares fitting problem as a system of equations in the unknowns a and b.

First of all, we need to write the normal equations by setting the gradients to zero:

$$\frac{\partial E}{\partial a} = \frac{\partial}{\partial a}\left(\sum_{i=1}^n (a + be^{-x_i} - y_i)^2\right) = 2\sum_{i=1}^n (a + be^{-x_i} - y_i) = 0,$$

and likewise for b:

$$\frac{\partial E}{\partial b} = \frac{\partial}{\partial b}\left(\sum_{i=1}^n (a + be^{-x_i} - y_i)^2\right) = 2\sum_{i=1}^n (a + be^{-x_i} - y_i)e^{-x_i} = 0.$$

From the first equation, we have

$$\sum_{i=1}^n a + be^{-x_i} = \sum_{i=1}^n y_i \implies na + b\sum_{i=1}^n e^{-x_i} = \sum_{i=1}^n y_i.$$

And similarly, from the second equation, we have

$$\sum_{i=1}^n ae^{-x_i} + b\sum_{i=1}^n e^{-2x_i} = \sum_{i=1}^n y_i e^{-x_i}.$$

Then, in the matrix form, we want to find a and b such that

$$\begin{pmatrix} n & \sum_{i=1}^n e^{-x_i} \\ \sum_{i=1}^n e^{-x_i} & \sum_{i=1}^n e^{-2x_i} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n y_i e^{-x_i} \end{pmatrix}.$$

The equivalent system of equations is therefore

$$\begin{cases} na + b\sum_{i=1}^n e^{-x_i} = \sum_{i=1}^n y_i, \\ a\sum_{i=1}^n e^{-x_i} + b\sum_{i=1}^n e^{-2x_i} = \sum_{i=1}^n y_i e^{-x_i}. \end{cases}$$