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Bayesian Estimation and Inference in R

Yun-Xiao Li

Fri 13^{th} Jun, 2025

What?

What are Bayesian estimation and Bayesian inference?

What?

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- **Bayesian estimation** is a way of using data and prior knowledge to estimate unknown values, like the mean or proportion in a population;

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- **Bayesian estimation** is a way of using data and prior knowledge to estimate unknown values, like the mean or proportion in a population;
- **Bayesian inference** is a broader term. It's the whole approach of using Bayes' Theorem to update beliefs and make decisions in the face of uncertainty.

Why?

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Why should I learn the Bayesian method?

- Mirrors real-life thinking and is easy to understand (to some extent);
- Makes use of prior knowledge;
- Gives direct answers to real questions (not like p-values);

How?

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- Level 1: get the intuitive idea of what's going on;

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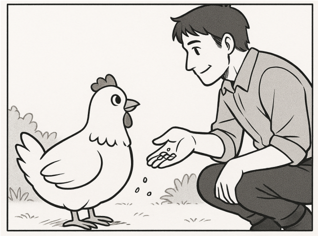
How can I learn the Bayesian method?

- Level 1: get the intuitive idea of what's going on;
- Level 2: know how to implement it (in R);
- Level 3: understand the mathematical principle;

Outline

Bayes Rule

Bayes Rule
Example



Bayes Rule
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- Two fundamental perceptions of Bayesian inference:

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  - You have a prior belief;

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- Two fundamental perceptions of Bayesian inference:
  - You have a prior belief;
  - New observations update this belief;

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Definition (Conditional probability)

Let A stands for an event or a hypothesis, and D stands for the observation, then the probability of event A occurring given that D is observed is

$$p(A|D) = \frac{p(AD)}{p(D)}.$$

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- $p(D)$: The chance of observing the man holding an axe;
- $P(AD)$: The chance of observing the man with an axe and the chicken being killed;
- What is the value of $P(AD)$ and $P(D)$?

Let's warm up the brain!

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$$p(D|A)p(A) = p(DA)$$

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Theorem (Bayes' Theorem)

$$p(A|D) = \frac{p(D|A)p(A)}{p(D)}.$$

- $P(A)$ is the **prior** belief of the chance that the man will kill the chicken;
- $P(D|A)$ is the **likelihood** of observing the man with an axe if the man decides to kill the chicken.

Let's decompose $P(D)$!

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Bayes Rule
Bayes' Theorem

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Example

Assume the incidence rate of a disease is 1/1000. A particular test for whether someone has this disease is 95% sensitive, meaning that if the patient has the disease, this test has a 95% probability of being positive. This test, however, has a 1% probability of being positive even though the patient does not suffer from this disease.

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- *Bob got a positive result for this test.*

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Hints:

- $P(A)$ is the chance of having the disease;
- $p(D|A)$ is the chance of having a positive result if the person has the disease;

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- *Bob got a positive result for this test.*

- $P(A) = \frac{1}{1000}$;
- $P(D|A) = 0.95$;
- $P(D|\neg A) = 0.01$.

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- $P(A) = \frac{1}{1000}$;
- $P(D|A) = 0.95$;
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$$P(A|D) = \frac{0.95 \times 0.001}{0.95 \times 0.001 + 0.01 \times 0.999} \approx 0.087$$

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- *Bob got a negative result in the second test.*

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- *Bob got a positive result for this test.*
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$$P(A|D) = \frac{0.05 \times 0.087}{0.05 \times 0.087 + 0.99 \times 0.913} \approx 0.0048$$

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- *Bob got a positive result for this test.*
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An alternative solution:

$$P(A|D) = \frac{0.95 \times 0.05 \times 0.001}{0.95 \times 0.05 \times 0.001 + 0.01 \times 0.99 \times 0.999} \approx 0.0048$$

What if A is not binary?

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Theorem (Bayes' Theorem)

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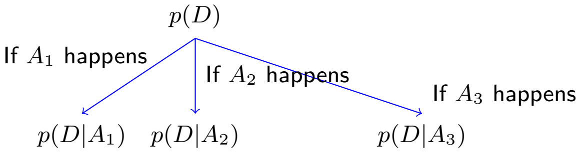

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Theorem (Bayes' Theorem)

If $A = \{A_1, A_2, \cdots, A_N\}$ *is an event set, then the probability of event* $A_i \in A$ *occurring given that* D *is observed is*

$$p(A_i|D) = \frac{p(D|A_i)p(A_i)}{\sum_{i=1}^{N} p(D|A_i)p(A_i)}.$$

Bayes Rule

Summary

- The main idea of the Bayes rule is: Prior $+$ Observation (Likelihood) $=$ Posterior

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- The mathematical form of this formula is

$$p(A_i|D) = \frac{p(D|A_i)p(A_i)}{\sum_{i=1}^{N} p(D|A_i)p(A_i)}$$

Bayesian Estimation: Examples of Coins

Example

We flipped a coin six times and got the observation $\{1, 0, 1, 1, 0, 1\}$*, where 1 represents heads, while 0 means tails. What is the probability that the coin will come up heads?*

- Find a probabilistic description of the observation. — Likelihood function

Bayesian Estimation: Examples of Coins

A Discrete Prior

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Denote The underlying probability of heads as θ. This idea can be written formally as

$$p(y = 1|\theta) = \theta.$$

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Therefore,

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The probability of the observation $Y = \{1, 0, 1, 1, 0, 1\}$ (i.e., the likelihood function) is

$$p(Y|\theta) = \theta^4(1 - \theta)^2.$$

Bayesian Estimation: Examples of Coins

A Discrete Prior

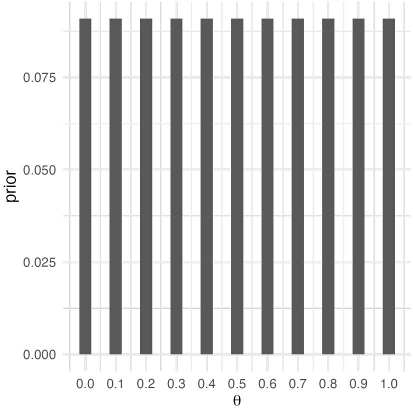
- Find a probability distribution of the parameter(s). — Prior distribution

Bayesian Estimation: Examples of Coins

A Discrete Prior

- Find a probability distribution of the parameter(s). — Prior distribution

We start with a prior that θ is equally likely to be 0, 0.1, 0.2, ..., 0.9, 1.



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A Discrete Prior

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- Following Bayes' theorem, we can reallocate credibility across the values of the parameter(s). — Posterior Distribution

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$$p(\theta_i|Y) = \frac{p(Y|\theta_i)p(\theta_i)}{\sum_i^N p(Y|\theta_i)p(\theta_i)}$$

Bayesian Estimation: Examples of Coins

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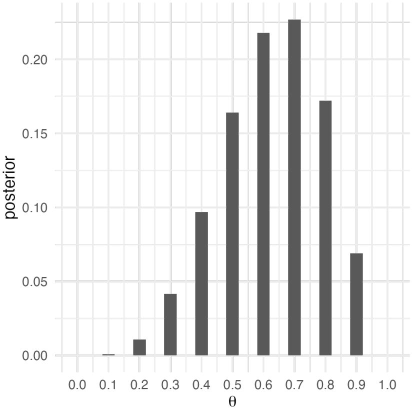
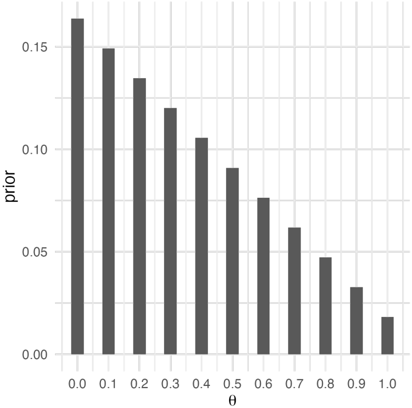


Bayesian estimation follows three steps:

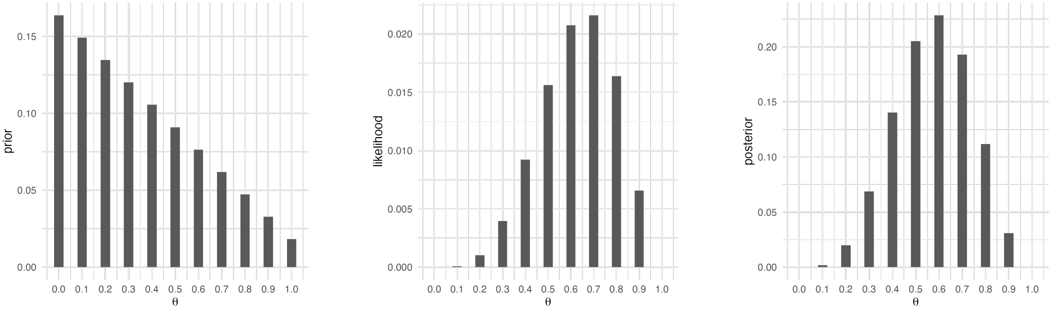
1. Find a probabilistic description of the observation. — Likelihood function;
2. Find a probability distribution of the parameter(s). — Prior distribution;
3. Following Bayes' theorem, we can reallocate credibility across the values of the parameter(s). — Posterior Distribution.

Let's try a new discrete prior!



Bayesian Estimation: Examples of Coins

A Discrete Prior



Bayesian Estimation: Examples of Coins

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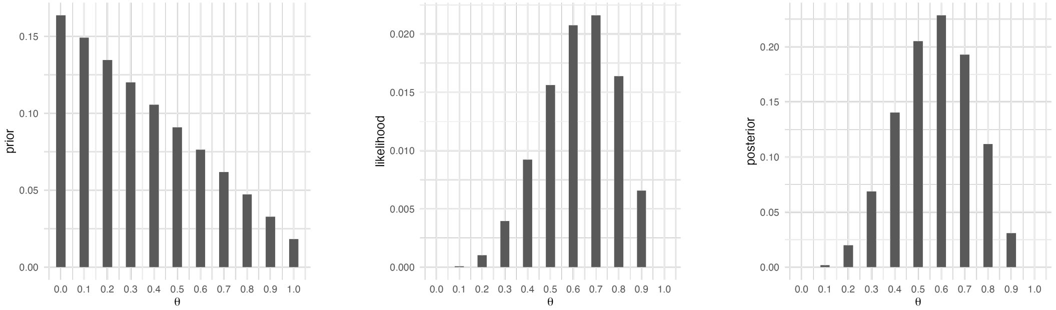

- The shape of the posterior is the combination of the shape of the prior and the likelihood.

Let's think about a more reasonable prior: a Beta distribution.

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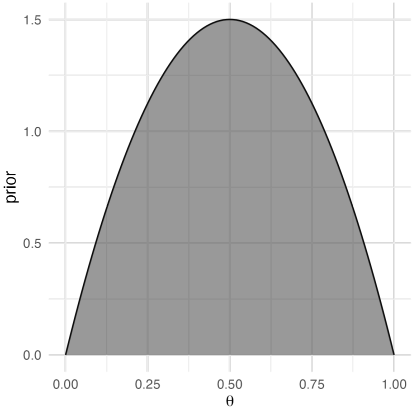

Fig. 1: The density function of $Beta(2,2)$.

Bayesian Estimation: Examples of Coins

A Continuous Prior

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- Prior: $\theta \sim Beta(2, 2)$;
- Likelihood:

Bayesian Estimation: Examples of Coins

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Bayesian Estimation: Examples of Coins

A Continuous Prior

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Bayesian Estimation: Examples of Coins

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- To "calculate" the posterior distribution, we used the Markov Chain Monte Carlo (MCMC) method.

Bayesian Estimation: Examples of Coins

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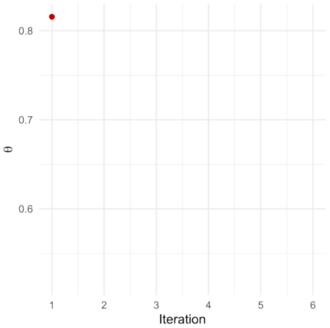

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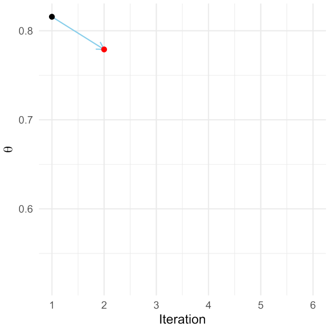

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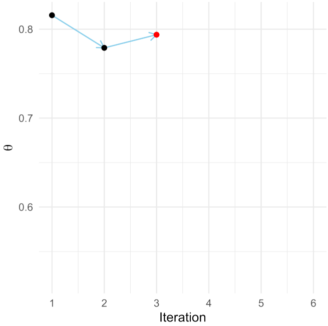

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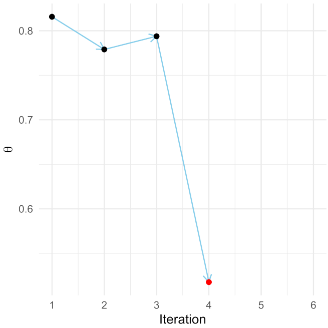

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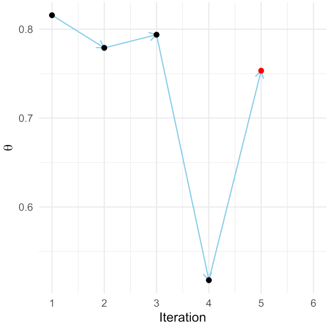

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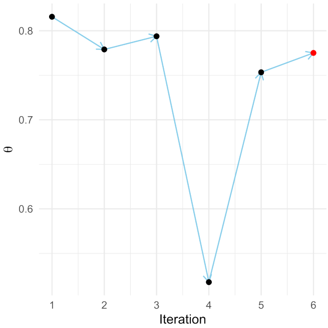

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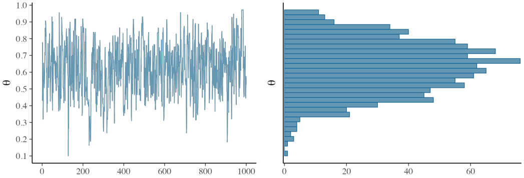

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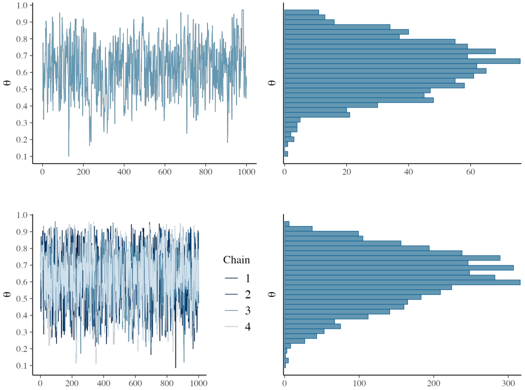
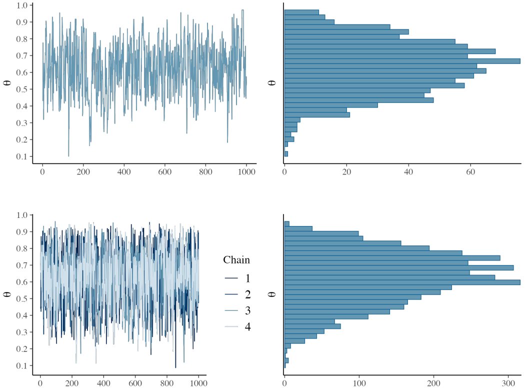


Bayesian Estimation: Examples of Coins

A Continuous Prior



- The MCMC is implemented in a `.stan` file.

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What is a `.stan` file?

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```
data {
 // The observations
}

parameters {
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}

model {
 // The prior and likelihood function
}
```

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- Comments are indicated by a double slash "//" instead of by a number sign "#."

We will write two `.stan` files:

- A “Hello world” example (See *hello_world.stan*);
- The Bayesian estimation of the coin example (See *coin.stan*).

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- The posterior distribution $p(\theta|D)$ can be approximated by MCMC, which is implemented in Stan;

Bayesian one-sample t-test

Bayesian one-sample t-test

An Energy Bar Example

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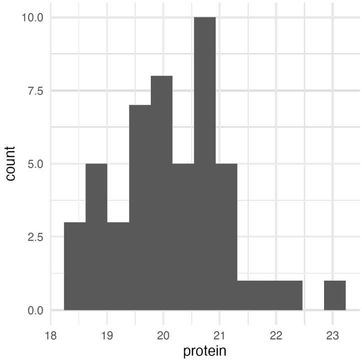
Example

There is a type of energy bar that claims each bar contains 20 grams of protein. We collected 50 samples and tested the protein content of this product. We want to know whether the label is accurate.



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Bayesian one-sample t-test

The Frequentist t-test

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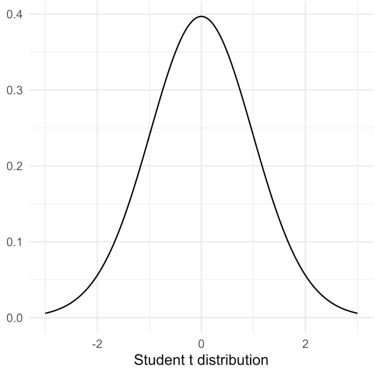

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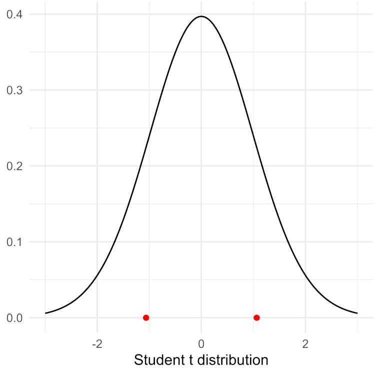

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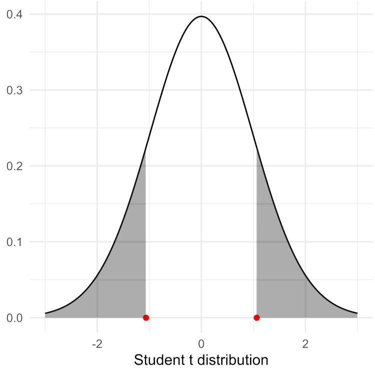

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- The t-distribution shows us what kind of T values we'd expect to see from random sampling, if the energy bar label is completely accurate.

The p-value means the probability of obtaining the results as extreme as (or more extreme than) those observed, **assuming the null hypothesis is true**.

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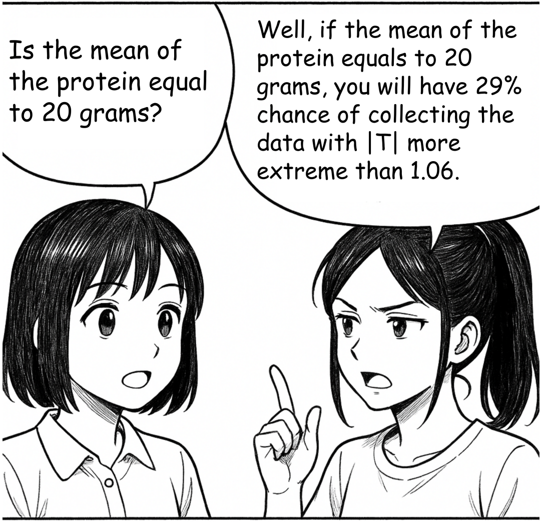
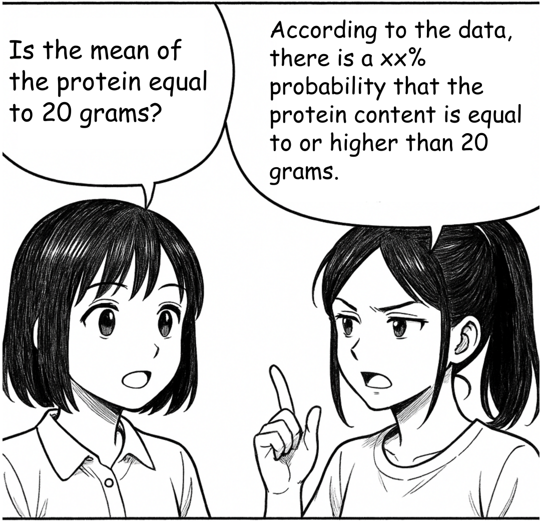
$$p = P(t < -|T| \;|\mu = 20) + P(t > |T| \;|\mu = 20)$$

Bayesian one-sample t-test

The Frequentist t-test





Is the mean of the protein equal to 20 grams?
According to the data, there is a xx% probability that the protein content is equal to or higher than 20 grams.

- Likelihood function:

- Prior:

Bayesian one-sample t-test

The Bayesian Estimation

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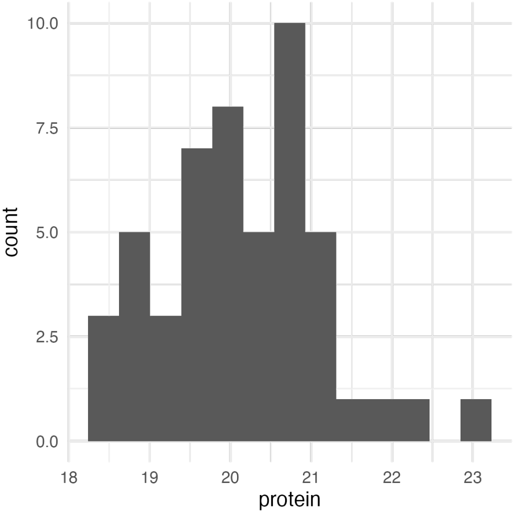

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$$L(D|\mu, \sigma) \sim N(\mu, \sigma^2)$$

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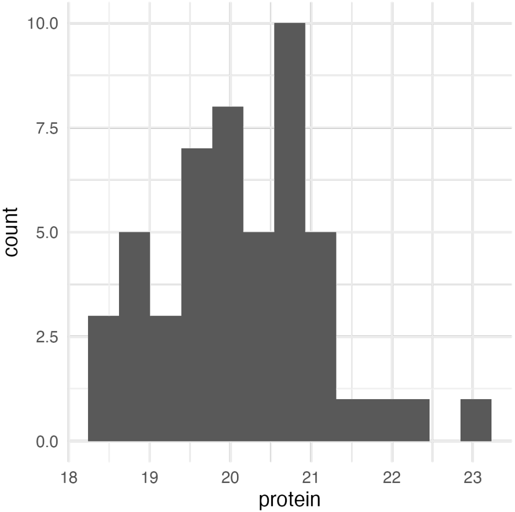

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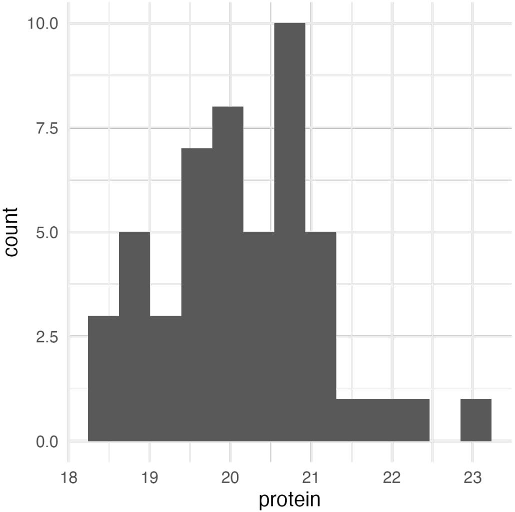
$$\mu \sim N(20, 10)$$
$$\sigma \sim N(0, 10)T[0, \infty]$$



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The Bayesian Estimation

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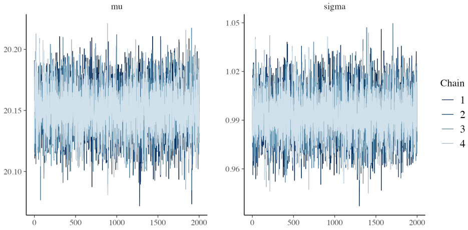
- The trace plot:



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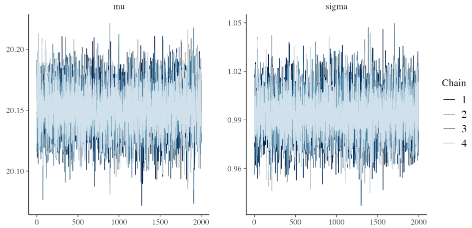
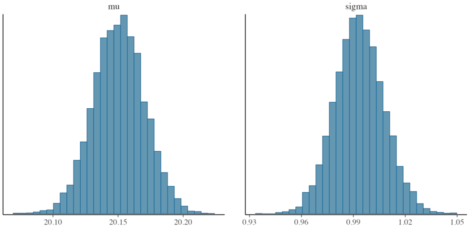


- The posterior distribution:



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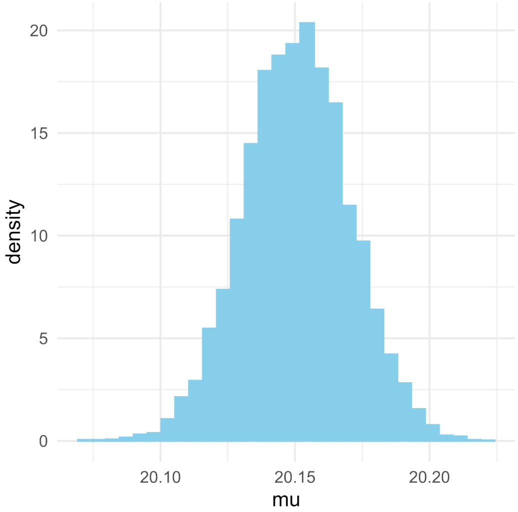
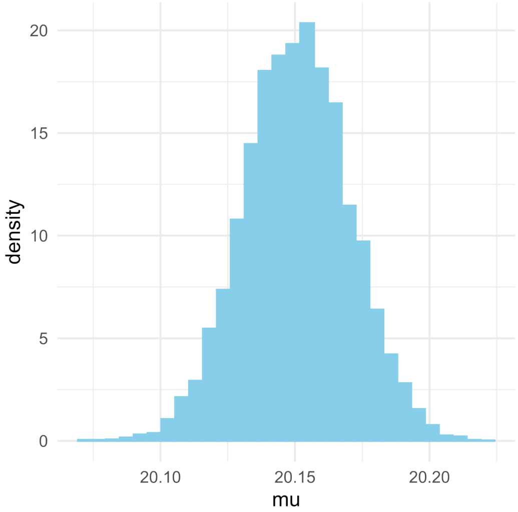
The Bayesian Estimation



How do we summarise the posterior distribution?

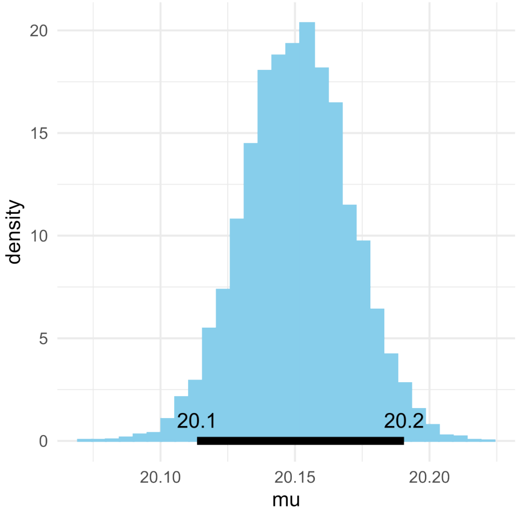
Bayesian one-sample t-test

The Bayesian Estimation



How do we summarise the posterior distribution?

- HDI (highest density interval): A 95% HDI is the shortest interval that covers the 95% mass of the distribution.



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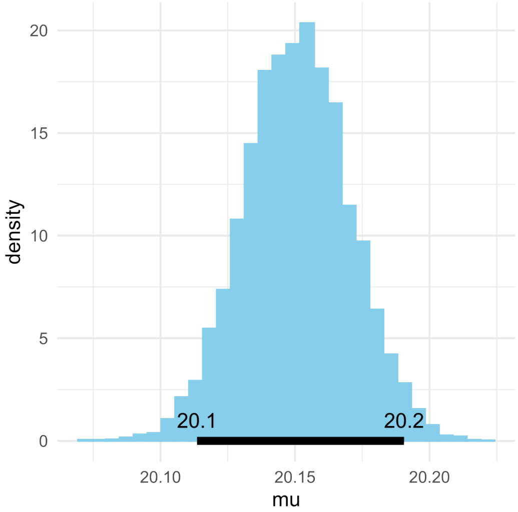
- The 95% CI means that if we did this experiment over and over, 95% of those intervals would include the true mean.
- The 95% HDI means that there is a 95% probability that the true mean lies in this interval, given our data and assumptions.



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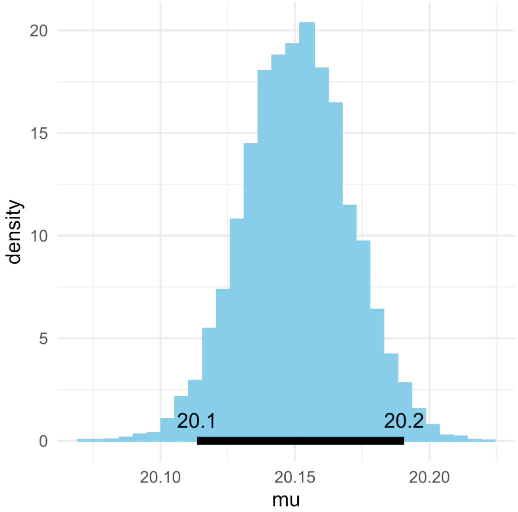

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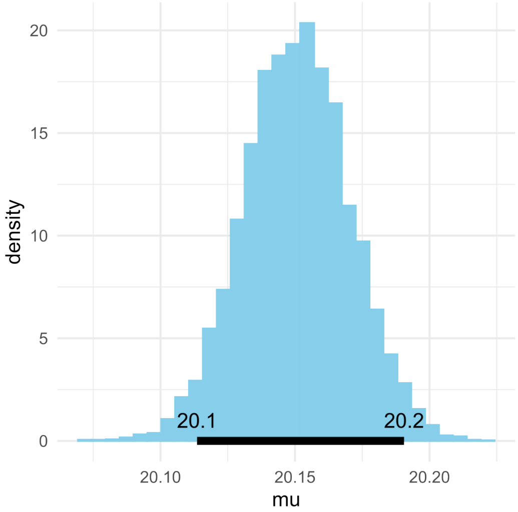
The value of 20 grams is excluded from the 95% HDI.



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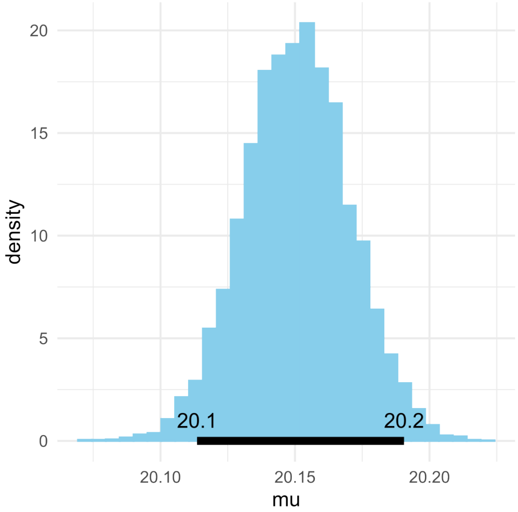

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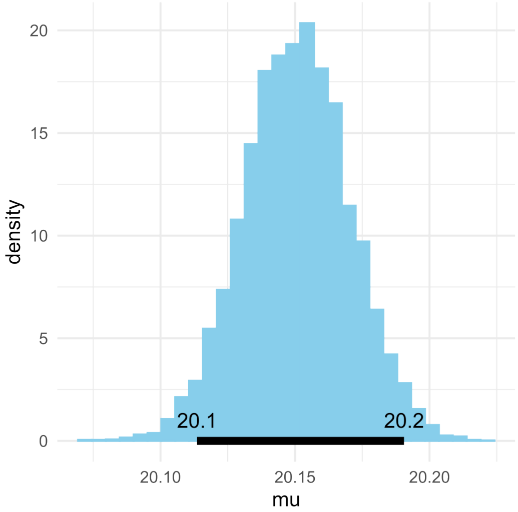
Not yet.



How do we conclude?

The value of 20 grams is excluded from the 95% HDI. Can we reject the null hypothesis now?

Not yet. We also want to know whether the difference between the posterior distribution and the null hypothesis is big *enough*.


One more thing...
Adam Tow | AllThingsD.com

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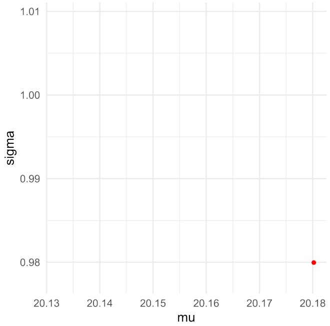
- Cohen (1988) noted that the effect is "small" when $d = 0.2$. In other words, when $d \leq 0.2$, the estimate of μ is practically equivalent to the null value μ_0.

Bayesian one-sample t-test

The Bayesian Inference



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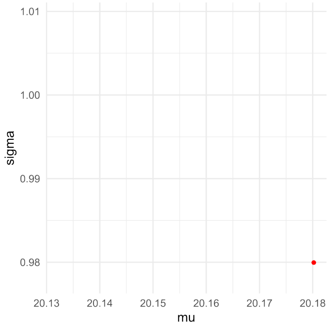

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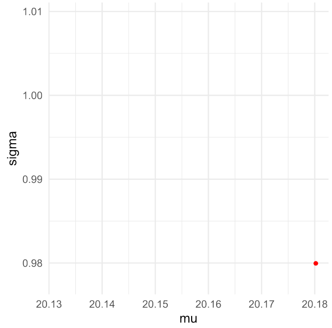

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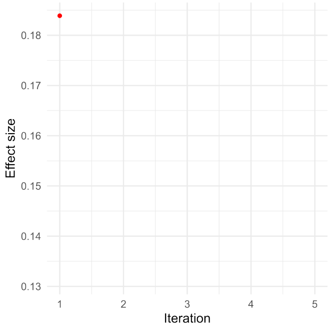

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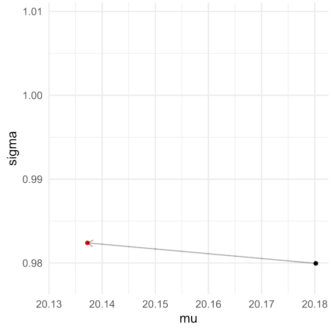

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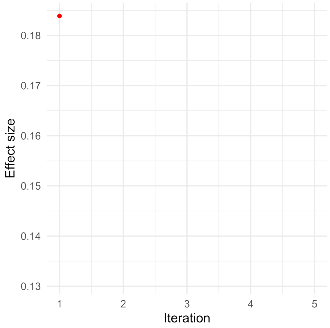

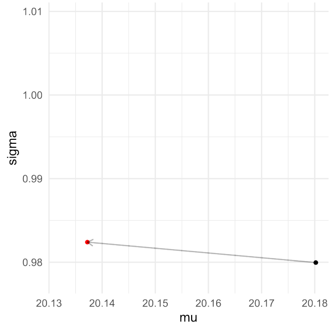

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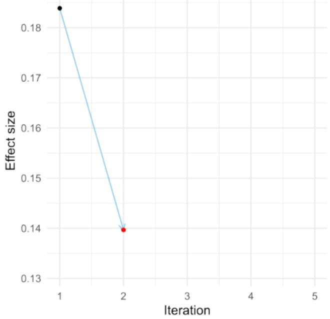

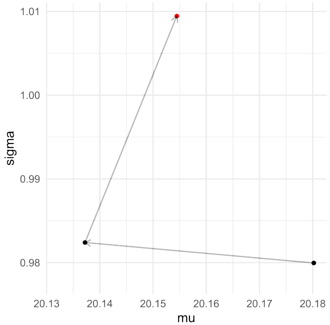

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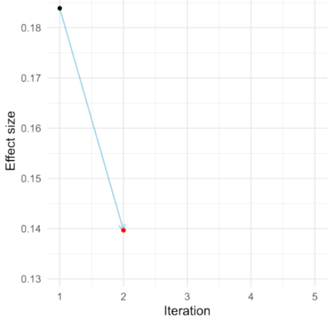

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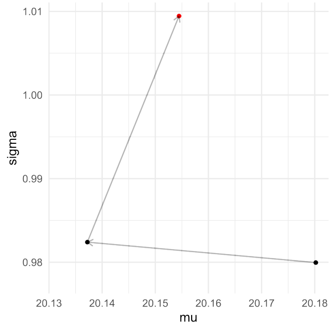

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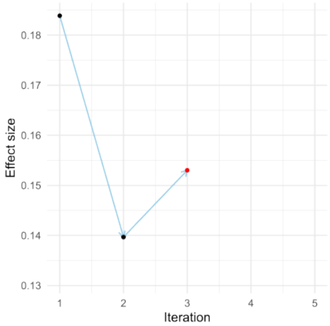

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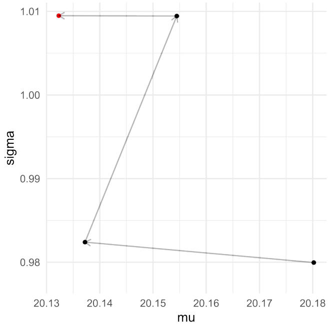

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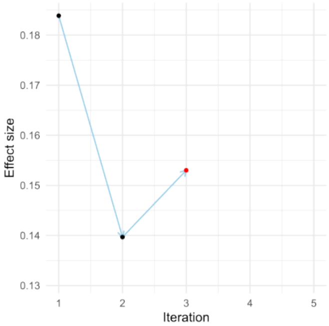

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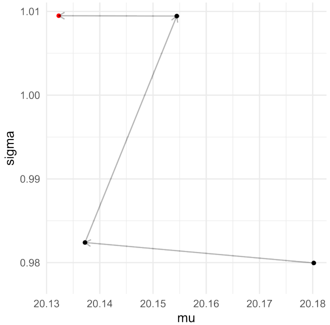

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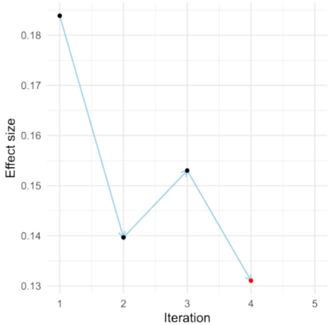

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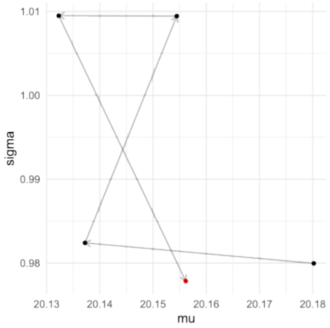

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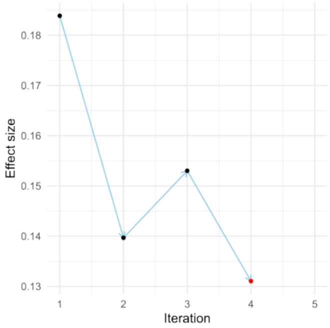

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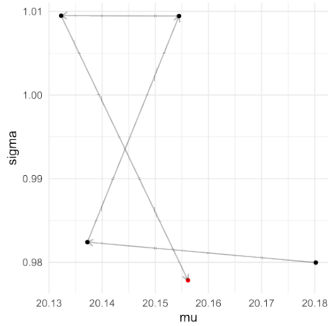

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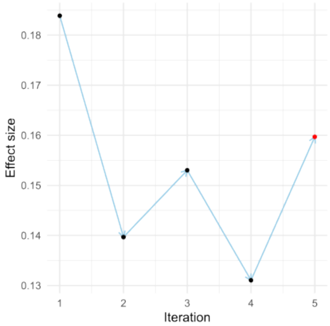

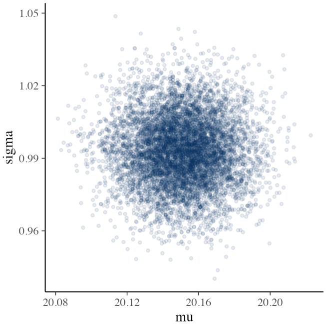

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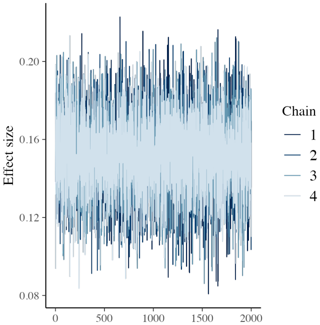

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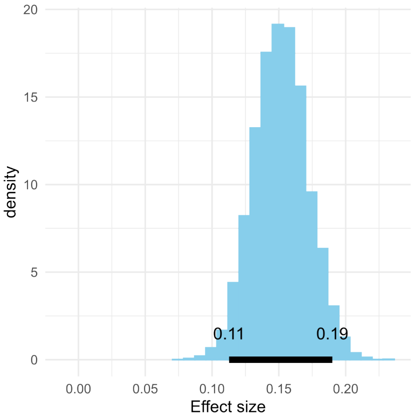

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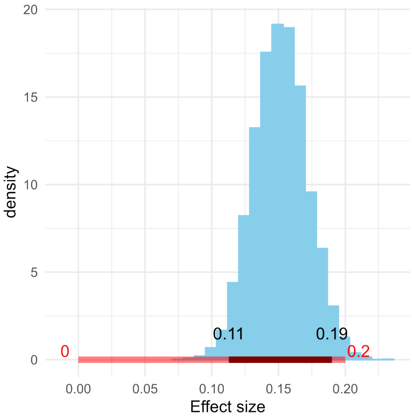

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- 95% HDI lies entirely inside the ROPE: the parameter value is declared to be accepted;
- 95% HDI lies outside the ROPE: the parameter value is declared to be not credible, or rejected.
- 95% HDI and ROPE overlap: current data are insufficient to yield a clear conclusion.

- The idea of the p-values does not answer the question we are interested in;

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- Bayesian estimation can provide a more direct answer to our question;
- (Advanced topic) Practically, we need 95% HDI and ROPE to test the hypotheses.

Thanks!

References I

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Lawrence Erlbaum Associates.

Kruschke, J. K. (2012).Bayesian estimation supersedes the t test.. *Journal of Experimental Psychology: General*, *142*(2), 573–603. https://doi.org/10.1037/a0029146