State Space Models

ELG 5218 - Uncertainty Evaluation in Engineering Measurements and Machine Learning

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Outline

Notebook: ssm_examples.ipynb

Motivation: The Hidden Structure Problem

Key Questions

- How do we model systems where we only observe *part* of the state?
- How do we handle noisy observations?
- How do we incorporate domain knowledge about dynamics?

Answer (SSM idea): introduce a latent state $\mathbf{z}_t$ and specify a transition model $p(\mathbf{z}_t \mid \mathbf{z}_{t-1})$ and measurement model $p(\mathbf{y}_t \mid \mathbf{z}_t)$.

Examples:

- Tracking an aircraft: observe radar position, but velocity is hidden
- Economic modeling: observe GDP, but structural factors are latent
- Robotics: observe sensor readings, but true robot state is uncertain

What Makes SSMs Powerful?

1. **Separation of concerns:** dynamics vs. observations
2. **Probabilistic framework:** principled uncertainty quantification
3. **Recursive inference:** efficient online algorithms (Kalman filter, particle filter)
4. **Generality:** unifies many models (ARIMA, HMMs, Dynamic Linear Models)

Core Insight

SSMs let us reason about *what we don't see* from *what we do see*

Classical Time Series: AR(p) Model

Autoregressive model of order p:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t \tag{1}$$

where $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$

Limitations:

- Everything is directly observed
- No natural way to incorporate external inputs
- Difficult to handle missing data
- No distinction between "true state" and "noisy measurement"

This is a state space model!

AR(2) model:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t, \qquad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2).$$

Step 1: Define the hidden state:

$$\mathbf{z}_t = \begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix}.$$

Step 2: Transition (state dynamics):

$$\mathbf{z}_t = \underbrace{\begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}}_{F} \mathbf{z}_{t-1} + \mathbf{q}_t, \qquad \mathbf{q}_t = \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}, \qquad \mathbf{q}_t \sim \mathcal{N}(\mathbf{0}, Q),$$

$$Q = \begin{bmatrix} \sigma_\epsilon^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 3: Observation equation:

$$y_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \mathbf{z}_t + r_t, \qquad r_t \sim \mathcal{N}(0, R), \qquad R = 0$$

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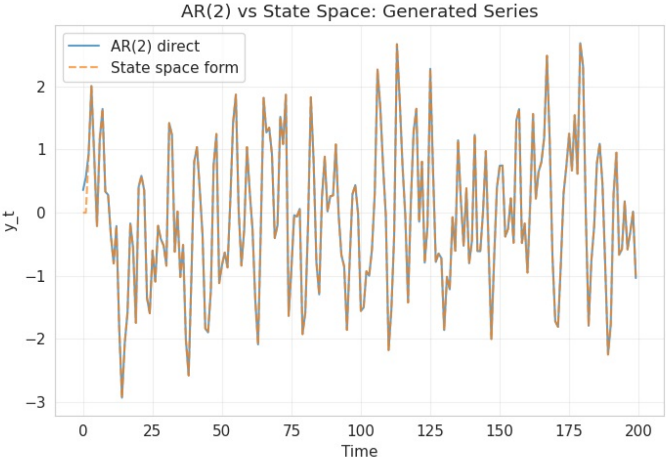
Converting AR(2) to State Space Form



General Linear State Space Model (SSM)

Transition Model (Dynamics)

$$\mathbf{z}_t = \mathbf{F}_t \mathbf{z}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t + \mathbf{q}_t, \quad \mathbf{q}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t) \tag{2}$$

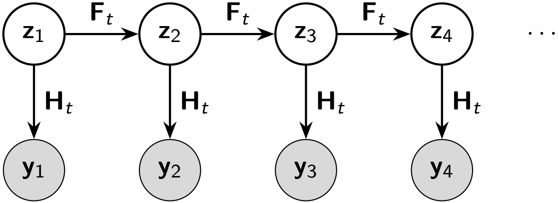
Observation Model (Measurement)

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{z}_t + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t + \mathbf{r}_t, \quad \mathbf{r}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t) \tag{3}$$

Notation (Murphy-style):

- $\mathbf{z}_t \in \mathbb{R}^{n_z}$: hidden state or latent variable
- $\mathbf{y}_t \in \mathbb{R}^{n_y}$: observation
- $\mathbf{u}_t \in \mathbb{R}^{n_u}$: control input
- $\mathbf{F}_t$: state transition matrix, $\mathbf{H}_t$: observation matrix
- $\mathbf{Q}_t$: process noise covariance, $\mathbf{R}_t$: measurement noise covariance

Graphical model



Latent states (white) evolve according to dynamics
Observations (gray) are noisy measurements of states

Continuous-Time State Space Model

Many physical systems are naturally described in continuous time:

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t) + \mathbf{w}(t) \tag{4}$$

$$\mathbf{y}(t) = \mathbf{C}_c\mathbf{z}(t) + \mathbf{v}(t) \tag{5}$$

Examples:

- Newton's laws: $\frac{d^2x}{dt^2} = F/m$
- RC circuits: $\frac{dV}{dt} = -\frac{1}{RC}V + \frac{1}{RC}V_{in}$
- Chemical reactions: $\frac{d[A]}{dt} = -k[A]$

Why Discretize?

Practical Reality:

- Computers operate in discrete time
- Sensors sample at fixed intervals (e.g., 100 Hz)
- Numerical algorithms need discrete representations

Our Goal: Transform continuous dynamics into discrete-time form:

$$\mathbf{z}_{t+1} = \mathbf{F}\mathbf{z}_t + \mathbf{B}\mathbf{u}_t + \mathbf{q}_t \tag{6}$$

Key Question

How do we choose **F** and **B** to best approximate continuous dynamics?

Method 1: Euler Discretization

Continuous: $\dot{\mathbf{z}}(t) = \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c\mathbf{u}(t)$

Approximate derivative:

$$\frac{\mathbf{z}_{t+1} - \mathbf{z}_t}{\Delta t} \approx \mathbf{A}_c\mathbf{z}_t + \mathbf{B}_c\mathbf{u}_t \tag{7}$$

Rearrange:

$$\mathbf{z}_{t+1} = (\mathbf{I} + \Delta t \cdot \mathbf{A}_c)\mathbf{z}_t + \Delta t \cdot \mathbf{B}_c\mathbf{u}_t \tag{8}$$

Result: $\mathbf{F} = \mathbf{I} + \Delta t \cdot \mathbf{A}_c, \quad \mathbf{B} = \Delta t \cdot \mathbf{B}_c$

Pros: Simple, intuitive
Cons: Can be unstable for large Δt

Method 2: Exact Discretization (Matrix Exponential)

Solve the ODE exactly:

$$\mathbf{z}(t + \Delta t) = e^{\mathbf{A}_c \Delta t}\mathbf{z}(t) + \int_0^{\Delta t} e^{\mathbf{A}_c(\Delta t - \tau)}\mathbf{B}_c\mathbf{u}(t+\tau)d\tau \tag{9}$$

For constant input $\mathbf{u}(t) = \mathbf{u}_t$:

$$\mathbf{F} = e^{\mathbf{A}_c \Delta t} \tag{10}$$

$$\mathbf{B} = \mathbf{A}_c^{-1}(e^{\mathbf{A}_c \Delta t} - \mathbf{I})\mathbf{B}_c \tag{11}$$

Pros: Numerically stable, exact for LTI systems
Cons: Requires matrix exponential computation

In Python: use `scipy.signal.cont2discrete`

Example: Spring-Mass-Damper System

Continuous dynamics:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{12}$$

State-space form: Let $\mathbf{z} = [x, \dot{x}]^T$

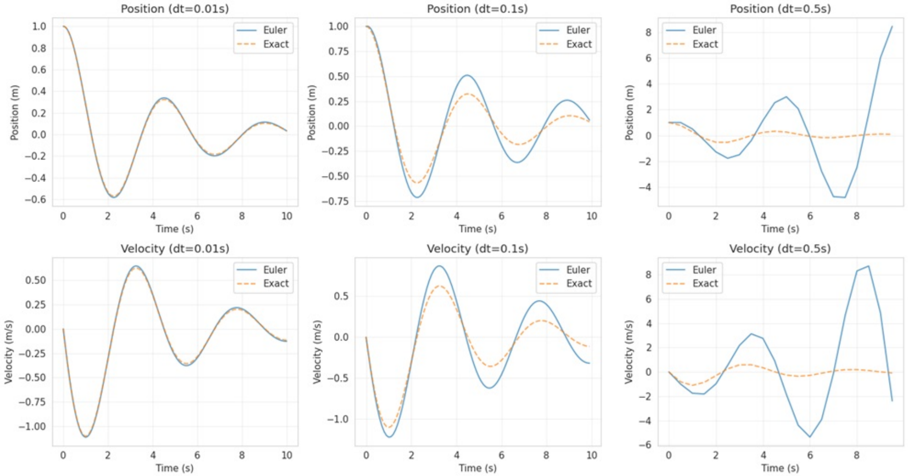
$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F(t) \tag{13}$$

After discretization (see Python notebook for numerical values):

$$\mathbf{z}_{t+1} = \mathbf{F}\mathbf{z}_t + \mathbf{B}F_t \tag{14}$$

We'll implement this in the notebook!

Example: Euler Discretzation of Spring-Mass-Damper System



Latent Variables: What are they?

Definition (informal)

A **latent variable** is a variable in the model that is **not directly observed** but helps explain the observed data.

Model: observed $\mathbf{y}$ is generated from latent $\mathbf{z}$.

Why introduce them?

Latent variables let us represent **hidden structure**:

- **Clustering/segments**: which group generated the point?
- **Dynamics**: hidden state that evolves over time.
- **Missing/uncertain quantities**: treat unknowns as random variables.

Key idea

Latent variables are **random variables**. We infer them from data:

$$p(\mathbf{z} \mid \mathbf{y}) \quad \text{(posterior over latent variables).}$$

Latent Variables vs Parameters vs Hyperparameters

Three kinds of unknowns in ML models

- **Latent variables $\mathbf{z}$**: hidden *random* quantities that can vary across data points or time.
- **Parameters $\boldsymbol{\theta}$**: fixed (but unknown) model quantities (weights, matrices) that define distributions.
- **Hyperparameters $\boldsymbol{\lambda}$**: settings that control priors/regularization/model capacity (often chosen by CV or set a priori).

How they differ (quick checklist)

Question	Latent variables	Parameters	Hyperparameters
Vary per datapoint / time?	Yes	No	No
Random variables?	Yes	Bayesian: yes / Frequentist: fixed	Fixed
Typical objective / target	Infer $p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})$	Learn $\hat{\boldsymbol{\theta}}$	Tune / set $\boldsymbol{\lambda}$

Typical inference targets (summary):

$$\underbrace{p(\mathbf{z} \mid \mathbf{y}, \boldsymbol{\theta})}_{\text{infer latent states}} \qquad \underbrace{\hat{\boldsymbol{\theta}}}_{\text{learn parameters}} \qquad \underbrace{\boldsymbol{\lambda}}_{\text{tune/set}}$$

Latent Variables in State Space Models (SSMs)

SSM viewpoint

In an SSM, the latent variable is the **hidden state sequence** $\mathbf{z}_{1:T} = (\mathbf{z}_1, \ldots, \mathbf{z}_T)$, and we observe measurements $\mathbf{y}_{1:T} = (\mathbf{y}_1, \ldots, \mathbf{y}_T)$.

What makes SSMs special? (time structure)

SSMs assume:

- **Markov dynamics**: $\mathbf{z}_t$ depends mainly on $\mathbf{z}_{t-1}$ (and input $\mathbf{u}_t$).
- **Conditional independence of observations**: $\mathbf{y}_t$ depends mainly on $\mathbf{z}_t$ (and $\mathbf{u}_t$).

These correspond to the standard transition density $p(\mathbf{z}_t \mid \mathbf{z}_{t-1})$ and observation likelihood $p(\mathbf{y}_t \mid \mathbf{z}_t)$.

Typical inference tasks in SSMs: **Filtering**: $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$ (real-time state estimate)

SSMs in Probabilistic Notation

Dynamics as a conditional distribution

Your linear state update

$$\mathbf{z}_t = \mathbf{F}_t \mathbf{z}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t + \mathbf{q}_t, \qquad \mathbf{q}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t)$$

is equivalent to:

$$\boxed{p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{u}_t) = \mathcal{N}\big(\mathbf{z}_t;\ \mathbf{F}_t \mathbf{z}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t,\ \mathbf{Q}_t\big)}$$

Measurement as a likelihood

Your observation model

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{z}_t + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t + \mathbf{r}_t, \qquad \mathbf{r}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$$

is equivalent to:

$$\boxed{p(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{u}_t) = \mathcal{N}\big(\mathbf{y}_t;\ \mathbf{H}_t \mathbf{z}_t + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t,\ \mathbf{R}_t\big)}$$

SSM as a Full Probabilistic Model (Joint Factorization)

Generative story (with inputs)

Given inputs $\mathbf{u}_{1:T}$:

1. Sample initial state: $\mathbf{z}_0 \sim p(\mathbf{z}_0)$
2. For $t = 1, \ldots, T$:
   - Sample state: $\mathbf{z}_t \sim p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{u}_t)$
   - Sample observation: $\mathbf{y}_t \sim p(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{u}_t)$

Joint distribution (key SSM factorization)

$$\boxed{p(\mathbf{z}_{0:T}, \mathbf{y}_{1:T} \mid \mathbf{u}_{1:T}) = p(\mathbf{z}_0) \prod_{t=1}^{T} p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{u}_t) \prod_{t=1}^{T} p(\mathbf{y}_t \mid \mathbf{z}_t, \mathbf{u}_t)}$$

Latent Variables Beyond State Space Models

Latent-variable models appear everywhere

- **Mixture models**: latent cluster z_n explains multimodal data.
- **Topic models**: latent topic proportions explain documents.
- **Factor analysis / PCA**: latent factors explain correlations in features.
- **VAEs**: latent code explains data via a neural decoder.
- **Missing data**: treat missing entries as latent variables and infer them.

What we do with latent variables (common inference patterns)

- **Posterior inference**: compute/approximate $p(\mathbf{z} \mid \mathbf{y})$
- **Learning parameters**: maximize $\log p(\mathbf{y} \mid \boldsymbol{\theta})$ or do Bayesian learning
- **Approximate inference**: variational methods, MCMC, particle methods (when exact is hard)

Takeaway

Classic Applications

1. **Tracking and Navigation**
   - GPS receivers: fuse satellite measurements with motion model
   - Aircraft/ship tracking: noisy radar + physics-based dynamics
2. **Signal Processing**
   - Noise reduction in sensor data
   - Smoothing and interpolation
3. **Control Systems**
   - State estimation for feedback control
   - Model Predictive Control (MPC)
4. **Econometrics**
   - Dynamic factor models
   - Estimating unobserved economic indicators

Key Takeaways

1. **SSMs separate dynamics from observations**—models what we believe happens vs. what we see
2. **Time series → SSM:** Add latent states to capture hidden structure
3. **Continuous → Discrete:** Use matrix exponential or Euler for principled discretization
4. **Kalman filter:** Optimal recursive Bayesian inference for linear-Gaussian case
5. **Extensions:** EKF, UKF, particle filters, deep SSMs bridge classical and modern ML

Philosophy

SSMs are about building models with *inductive biases*—using domain knowledge about dynamics and measurement to constrain learning

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