The Kalman Filter

ELG 5218 - Uncertainty Evaluation in Engineering Measurements and Machine Learning

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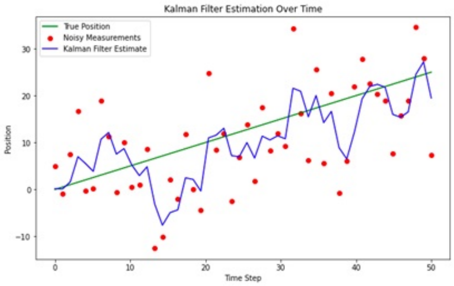
Outline

Notebook: ssm_examples.ipynb

The Filtering Problem

Scenario: A robot moving in 1D space

- **State:** position z_t (hidden, uncertain)
- **Observations:** noisy GPS measurements y_t
- **Goal:** Estimate current position from all measurements so far

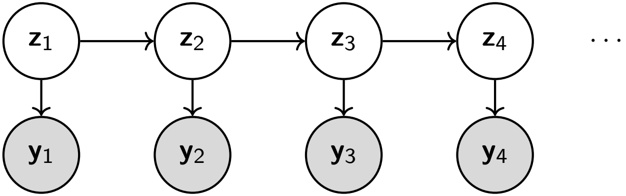


Challenge: How to optimally fuse noisy measurements with predictions from a motion model?

What is Filtering?

General Problem:

- Hidden state sequence: $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$
- Observations: $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t$
- Compute posterior: $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$



$$\begin{tikzcd}
\mathbf{z}_1 \arrow[r] \arrow[d] & \mathbf{z}_2 \arrow[r] \arrow[d] & \mathbf{z}_3 \arrow[r] \arrow[d] & \mathbf{z}_4 \arrow[d] & \cdots \\
\mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 &
\end{tikzcd}$$

Types of Inference:

- **Filtering:** $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$ — estimate current state
- **Smoothing:** $p(\mathbf{z}_t \mid \mathbf{y}_{1:T})$ — estimate past state (more data)
- **Prediction:** $p(\mathbf{z}_{t+k} \mid \mathbf{y}_{1:t})$ — forecast future

Why Kalman Filter?

The Special Case

Linear dynamics + Gaussian noise $\Rightarrow$ Exact, closed-form solution!

Advantages:

1. **Optimal:** Minimum mean squared error (MMSE) estimator
2. **Recursive:** Online processing, constant memory $O(n_z^2)$
3. **Efficient:** No integrals, just matrix operations
4. **Interpretable:** Clear predict-update structure

Historical Impact:

- Apollo lunar missions (1960s)
- GPS navigation
- Autonomous vehicles
- Economics, signal processing, control systems

Probabilistic State Space Model

Transition Model (Dynamics)

$$\mathbf{z}_t = \mathbf{F}_t \mathbf{z}_{t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t + \mathbf{q}_t, \quad \mathbf{q}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t) \tag{1}$$

Observation Model (Measurement)

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{z}_t + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t + \mathbf{r}_t, \quad \mathbf{r}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t) \tag{2}$$

Components:

- $\mathbf{z}_t \in \mathbb{R}^{n_z}$: hidden state (position, velocity, ...)
- $\mathbf{y}_t \in \mathbb{R}^{n_y}$: observation (sensor measurement)
- $\mathbf{u}_t \in \mathbb{R}^{n_u}$: control input (known)
- $\mathbf{F}_t$: state transition matrix, $\mathbf{H}_t$: observation matrix
- $\mathbf{Q}_t$: process noise covariance, $\mathbf{R}_t$: measurement noise covariance

Bayesian Filter: Recursive Structure

Goal: Compute $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$ recursively

Initialization

Start with prior: $p(\mathbf{z}_0) = \mathcal{N}(\mathbf{z}_0 \mid \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$

Prediction Step (Chapman-Kolmogorov)

Marginalize over previous state:

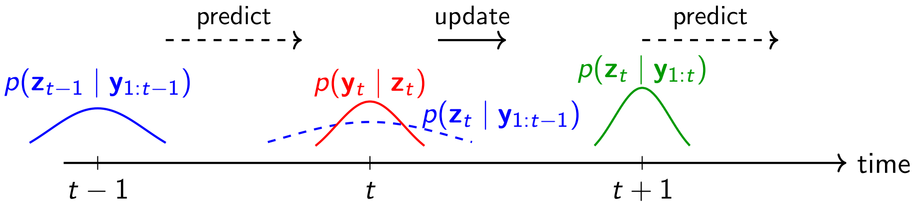
$$p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1}) = \int p(\mathbf{z}_t \mid \mathbf{z}_{t-1}) p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1}) d\mathbf{z}_{t-1} \tag{3}$$

Update Step (Bayes' Rule)

Incorporate new measurement:

$$p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = \frac{p(\mathbf{y}_t \mid \mathbf{z}_t) p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})}{p(\mathbf{y}_t \mid \mathbf{y}_{1:t-1})} \tag{4}$$

Bayesian Filter: Graphical View


predict
update
predict
$p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})$
$p(\mathbf{y}_t \mid \mathbf{z}_t)$
$p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$
$p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$
time
$t-1$
t
$t+1$

Key Insight:

- **Prediction** propagates uncertainty (wider distribution)
- **Update** incorporates evidence (narrower distribution)

Bayesian Filter Derivation (1/3): Model Assumptions

Goal: Compute $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$ recursively.

Starting point: joint probability factorization

For a state-space model (SSM), the joint distribution of states and observations up to time t factorizes as:

$$p(\mathbf{z}_{0:t}, \mathbf{y}_{1:t}) = p(\mathbf{z}_0) \prod_{k=1}^{t} p(\mathbf{z}_k \mid \mathbf{z}_{k-1})\, p(\mathbf{y}_k \mid \mathbf{z}_k).$$

Key assumptions

- **Markov property (state dynamics):**

$$p(\mathbf{z}_k \mid \mathbf{z}_{0:k-1}) = p(\mathbf{z}_k \mid \mathbf{z}_{k-1}).$$

- **Conditional independence (observations):**

$$p(\mathbf{y}_k \mid \mathbf{z}_{0:k}, \mathbf{y}_{1:k-1}) = p(\mathbf{y}_k \mid \mathbf{z}_k).$$

Bayesian Filter Derivation (2/3): Prediction

Prediction step (Chapman–Kolmogorov)

Start from the posterior at time $t-1$: $p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})$. To obtain the *prior* (predictive) at time t:

$$\begin{aligned} p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1}) &= \int p(\mathbf{z}_t, \mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})\, d\mathbf{z}_{t-1} \\ &= \int p(\mathbf{z}_t \mid \mathbf{z}_{t-1}, \mathbf{y}_{1:t-1})\, p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})\, d\mathbf{z}_{t-1} \\ &\overset{\text{Markov}}{=} \int p(\mathbf{z}_t \mid \mathbf{z}_{t-1})\, p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})\, d\mathbf{z}_{t-1}. \end{aligned}$$

Bayesian Filter Derivation (3/3): Update

Update step (Bayes' rule)

Start with Bayes' theorem (posterior at time t): $p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = \frac{p(\mathbf{y}_{1:t}|\mathbf{z}_t)\, p(\mathbf{z}_t)}{p(\mathbf{y}_{1:t})}$.

Decompose $\mathbf{y}_{1:t} = \{\mathbf{y}_{1:t-1}, \mathbf{y}_t\}$: $p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = \frac{p(\mathbf{y}_t, \mathbf{y}_{1:t-1}|\mathbf{z}_t)\, p(\mathbf{z}_t)}{p(\mathbf{y}_t, \mathbf{y}_{1:t-1})}$.

Use conditional probability: $p(\mathbf{y}_t, \mathbf{y}_{1:t-1} \mid \mathbf{z}_t) = p(\mathbf{y}_t \mid \mathbf{y}_{1:t-1}, \mathbf{z}_t)\, p(\mathbf{y}_{1:t-1} \mid \mathbf{z}_t)$.

Conditional independence assumption: $p(\mathbf{y}_t \mid \mathbf{y}_{1:t-1}, \mathbf{z}_t) = p(\mathbf{y}_t \mid \mathbf{z}_t)$.

Substitute into Bayes: $p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = \frac{p(\mathbf{y}_t|\mathbf{z}_t)\; p(\mathbf{y}_{1:t-1}|\mathbf{z}_t)\; p(\mathbf{z}_t)}{p(\mathbf{y}_t, \mathbf{y}_{1:t-1})}$.

Recognize two identities: $\frac{p(\mathbf{y}_{1:t-1}|\mathbf{z}_t)\, p(\mathbf{z}_t)}{p(\mathbf{y}_{1:t-1})} = p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$, $\qquad \frac{p(\mathbf{y}_{1:t-1})}{p(\mathbf{y}_t, \mathbf{y}_{1:t-1})} = \frac{1}{p(\mathbf{y}_t|\mathbf{y}_{1:t-1})}$.

Therefore (Bayes update): $p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = \frac{p(\mathbf{y}_t|\mathbf{z}_t)\; p(\mathbf{z}_t|\mathbf{y}_{1:t-1})}{p(\mathbf{y}_t|\mathbf{y}_{1:t-1})}$.

- **Summary:** Predict (propagate uncertainty) $\rightarrow$ Update (incorporate measurement).
- **Kalman filter case:** for linear-Gaussian models, both integrals are closed form.

Kalman Filter: The Gaussian Case

Key Property: Linear transformations of Gaussians are Gaussian!

If $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} + \mathbf{w}$ where $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{W})$, then:

$$\mathbf{y} \sim \mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T + \mathbf{W}) \tag{5}$$

Consequence for filtering:

- If $p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1}) = \mathcal{N}(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Sigma}_{t-1|t-1})$
- Then $p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$ is also Gaussian!
- And $p(\mathbf{z}_t \mid \mathbf{y}_{1:t})$ is also Gaussian!

Kalman Filter = Recursive Gaussian Propagation

We only need to track means $\boldsymbol{\mu}_{t|t}$ and covariances $\boldsymbol{\Sigma}_{t|t}$ at each step!

Notation: $t \mid t$ vs. $t \mid t-1$ (Prediction & Update)

How to read the notation

"$t \mid s$" $\implies$ time t given data up to time s.

State estimates (mean & covariance)

$$\boldsymbol{\mu}_{t|t-1} = \mathbb{E}[\mathbf{z}_t \mid \mathbf{y}_{1:t-1}], \qquad \boldsymbol{\Sigma}_{t|t-1} = \mathrm{Cov}(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$$

$$\boldsymbol{\mu}_{t|t} = \mathbb{E}[\mathbf{z}_t \mid \mathbf{y}_{1:t}], \qquad \boldsymbol{\Sigma}_{t|t} = \mathrm{Cov}(\mathbf{z}_t \mid \mathbf{y}_{1:t})$$

Predicted measurement

For linear observations $\mathbf{y}_t = \mathbf{H}_t \mathbf{z}_t + \mathbf{v}_t$:

$$\hat{\mathbf{y}}_t = \mathbb{E}[\mathbf{y}_t \mid \mathbf{y}_{1:t-1}] = \mathbf{H}_t\, \boldsymbol{\mu}_{t|t-1}$$

(Nonlinear: $\hat{\mathbf{y}}_t = \mathbb{E}[h(\mathbf{z}_t) \mid \mathbf{y}_{1:t-1}] \approx h(\boldsymbol{\mu}_{t|t-1})$.)

Mini example (1D)

Random walk + noisy measurement:
$z_t = z_{t-1} + q_t, \quad q_t \sim \mathcal{N}(0, 0.2^2),$
$y_t = z_t + v_t, \quad v_t \sim \mathcal{N}(0, 0.5^2).$
Assume at $t-1$: $\mu_{t-1|t-1} = 2.0, \qquad \Sigma_{t-1|t-1} = 0.3^2 = 0.09$

Predict: $\mu_{t|t-1} = 2.0, \qquad \Sigma_{t|t-1} = 0.09 + 0.04 = 0.13,\ \hat{y}_t = \mu_{t|t-1} = 2.0$

If measurement $y_t = 2.6$ arrives, innovation $= y_t - \hat{y}_t = 0.6$.

Update (Kalman-style):
$S = \Sigma_{t|t-1} + 0.5^2 = 0.13 + 0.25 = 0.38$
$K = \frac{\Sigma_{t|t-1}}{S} = \frac{0.13}{0.38} = 0.342$
$\mu_{t|t} = 2.0 + 0.342(0.6) = 2.205,$
$\Sigma_{t|t} = (1 - K)\, 0.13 = 0.0856$

Prediction Step

Goal: derive $p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$.

Identify Theorem 3 (Affine-Gaussian forward model):
Let $\mathbf{x} = \mathbf{z}_{t-1}$ and $\mathbf{y} = \mathbf{z}_t$. Assume

$$p(\mathbf{x}) = \mathcal{N}(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Sigma}_{t-1|t-1}), \qquad p(\mathbf{y} \mid \mathbf{x}) = \mathcal{N}(\mathbf{F}_t\mathbf{x} + \mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t, \mathbf{Q}_t).$$

This matches the dynamics $\mathbf{z}_t = \mathbf{F}_t\mathbf{z}_{t-1} + \mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t + \mathbf{q}_t$.

Apply Corollary 2 (marginal / predictive distribution):

$$p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1}) = \int p(\mathbf{z}_t \mid \mathbf{z}_{t-1})\, p(\mathbf{z}_{t-1} \mid \mathbf{y}_{1:t-1})\, d\mathbf{z}_{t-1} = \mathcal{N}(\boldsymbol{\mu}_{t|t-1}, \boldsymbol{\Sigma}_{t|t-1})$$

with

$$\boldsymbol{\mu}_{t|t-1} = \mathbf{F}_t\boldsymbol{\mu}_{t-1|t-1} + \mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t, \qquad \boldsymbol{\Sigma}_{t|t-1} = \mathbf{Q}_t + \mathbf{F}_t\boldsymbol{\Sigma}_{t-1|t-1}\mathbf{F}_t^\top.$$

Kalman Filter: Prediction (Time Update)

Goal: derive $p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1})$.

Mapping: Gaussian formulas

Use Corollary 2: $p(\mathbf{y}) = \int p(\mathbf{y} \mid \mathbf{x}) p(\mathbf{x})\, d\mathbf{x}$.

Theorem symbol	Kalman symbol
$\mathbf{x}$	$\mathbf{z}_{t-1}$
$\mathbf{y}$	$\mathbf{z}_t$
A	$\mathbf{F}_t$
b	$\mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t$
$\Sigma_{\mathbf{y}\mid\mathbf{x}}$	$\mathbf{Q}_t$
$\mu_{\mathbf{x}}$	$\boldsymbol{\mu}_{t-1\mid t-1}$
$\Sigma_{\mathbf{x}}$	$\boldsymbol{\Sigma}_{t-1\mid t-1}$

Intuition: Affine transformation shifts the mean by $\mathbf{F}_t(\cdot) + (\mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t)$ and adds uncertainty: *propagated covariance* $+\ \mathbf{Q}_t$.

Theorem 3 + Corollary 2

If $p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \mu_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ and

$p(\mathbf{y} \mid \mathbf{x}) = \mathcal{N}(\mathbf{y}; A\mathbf{x} + b, \Sigma_{\mathbf{y}\mid\mathbf{x}})$,

then the marginal (predictive) distribution is

$$p(\mathbf{y}) = \mathcal{N}\big(\mathbf{y};\ A\mu_{\mathbf{x}} + b,\ \Sigma_{\mathbf{y}\mid\mathbf{x}} + A\Sigma_{\mathbf{x}}A^\top\big).$$

Apply mapping $\Rightarrow$ Kalman prediction

$$p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1}) = \mathcal{N}\big(\mathbf{z}_t; \boldsymbol{\mu}_{t\mid t-1}, \boldsymbol{\Sigma}_{t\mid t-1}\big)$$

$$\boldsymbol{\mu}_{t\mid t-1} = \mathbf{F}_t \boldsymbol{\mu}_{t-1\mid t-1} + \mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t$$

$$\boldsymbol{\Sigma}_{t\mid t-1} = \mathbf{Q}_t + \mathbf{F}_t \boldsymbol{\Sigma}_{t-1\mid t-1} \mathbf{F}_t^\top$$

Kalman Filter: Update (1/2) – Build a Joint Gaussian

Given (from prediction): $p(\mathbf{z}_t \mid \mathbf{y}_{1:t-1}) = \mathcal{N}\big(\mathbf{z}_t; \boldsymbol{\mu}_{t|t-1}, \boldsymbol{\Sigma}_{t|t-1}\big)$.

Mapping to Theorem 3

Theorem symbol	Kalman symbol
$\mathbf{x}$	$\mathbf{z}_t$
$\mathbf{y}$	$\mathbf{y}_t$
A	$\mathbf{H}_t$
b	$\mathbf{D}_t\mathbf{u}_t + \mathbf{d}_t$
$\Sigma_{\mathbf{y}\mid\mathbf{x}}$	$\mathbf{R}_t$
$\mu_{\mathbf{x}}$	$\boldsymbol{\mu}_{t|t-1}$
$\Sigma_{\mathbf{x}}$	$\boldsymbol{\Sigma}_{t|t-1}$

Observation model (Affine-Gaussian)

$$p(\mathbf{y}_t \mid \mathbf{z}_t) = \mathcal{N}\big(\mathbf{y}_t; \mathbf{H}_t\mathbf{z}_t + \mathbf{D}_t\mathbf{u}_t + \mathbf{d}_t, \mathbf{R}_t\big).$$

Why we do this Once we have the **joint Gaussian** of $(\mathbf{z}_t, \mathbf{y}_t)$, Theorem 2 gives the **conditional** $p(\mathbf{z}_t \mid \mathbf{y}_t, \mathbf{y}_{1:t-1})$.

Theorem 3 (Affine-Gaussian

If $p(\mathbf{x}) = \mathcal{N}(\mu_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ and $p(\mathbf{y} \mid \mathbf{x}) = \mathcal{N}(A\mathbf{x} + b, \Sigma_{\mathbf{y}|\mathbf{x}})$, then $\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}$ is jointly Gaussian with block covariance structure.

Apply mapping $\Rightarrow$ joint of $(\mathbf{z}_t, \mathbf{y}_t) \mid \mathbf{y}_{1:t-1}$

$$\begin{bmatrix}\mathbf{z}_t\\ \mathbf{y}_t\end{bmatrix}\Big|\mathbf{y}_{1:t-1} \sim \mathcal{N}\left(\begin{bmatrix}\boldsymbol{\mu}_{t|t-1}\\ \hat{\mathbf{y}}_t\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{t|t-1} & \boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top\\ \mathbf{H}_t\boldsymbol{\Sigma}_{t|t-1} & \mathbf{S}_t\end{bmatrix}\right)$$

$$\hat{\mathbf{y}}_t = [\mathbf{y}_t \mid \mathbf{y}_{1:t-1}] = \mathbf{H}_t\boldsymbol{\mu}_{t|t-1} + \mathbf{D}_t\mathbf{u}_t + \mathbf{d}_t$$

$$\mathbf{S}_t = (\mathbf{y}_t \mid \mathbf{y}_{1:t-1}) = \mathbf{H}_t\boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top + \mathbf{R}_t$$

Kalman Filter: Update (2/2) – Condition the Joint

Goal: Compute the filtering posterior: $p(\mathbf{z}_t \mid \mathbf{y}_{1:t}) = p(\mathbf{z}_t \mid \mathbf{y}_t, \mathbf{y}_{1:t-1})$.

Theorem 2

For a joint Gaussian $\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}$,

$$\boldsymbol{\mu}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{x}} + \boldsymbol{\Sigma}_{\mathbf{xy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}),$$

$$\boldsymbol{\Sigma}_{\mathbf{x}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{xx}} - \boldsymbol{\Sigma}_{\mathbf{xy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yx}}.$$

Mapping to joint from Update

Theorem 2 symbol	Kalman symbol
$\mathbf{x}$	$\mathbf{z}_t$
$\mathbf{y}$	$\mathbf{y}_t$
$\boldsymbol{\mu}_{\mathbf{x}}$	$\boldsymbol{\mu}_{t\|t-1}$
$\boldsymbol{\mu}_{\mathbf{y}}$	$\hat{\mathbf{y}}_t$
$\boldsymbol{\Sigma}_{\mathbf{xy}}$	$\boldsymbol{\Sigma}_{t\|t-1}\mathbf{H}_t^\top$
$\boldsymbol{\Sigma}_{\mathbf{yy}}$	$\mathbf{S}_t$

Kalman update = Theorem 2

$$\mathbf{K}_t = \boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top \mathbf{S}_t^{-1}$$

$$\mathbf{S}_t = \mathbf{H}_t\boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top + \mathbf{R}_t$$

Define the **innovation** (prediction error):

$$\tilde{\mathbf{y}}_t = \mathbf{y}_t - \hat{\mathbf{y}}_t$$

$$(\hat{\mathbf{y}}_t = \mathbf{H}_t\boldsymbol{\mu}_{t|t-1} + \mathbf{D}_t\mathbf{u}_t + \mathbf{d}_t)$$

$$\boldsymbol{\mu}_{t|t} = \boldsymbol{\mu}_{t|t-1} + \mathbf{K}_t\,\tilde{\mathbf{y}}_t$$

$$\boldsymbol{\Sigma}_{t|t} = \boldsymbol{\Sigma}_{t|t-1} - \mathbf{K}_t\,\mathbf{S}_t\,\mathbf{K}_t^\top$$

Kalman Filter: Complete Algorithm

Initialization

$\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0$ (prior mean and covariance)

For $t = 1, 2, \ldots, T$:

Prediction Step

$$\boldsymbol{\mu}_{t|t-1} = \mathbf{F}_t \boldsymbol{\mu}_{t-1|t-1} + \mathbf{B}_t \mathbf{u}_t + \mathbf{b}_t \tag{6}$$

$$\boldsymbol{\Sigma}_{t|t-1} = \mathbf{F}_t \boldsymbol{\Sigma}_{t-1|t-1} \mathbf{F}_t^T + \mathbf{Q}_t \tag{7}$$

Update Step

$$\hat{\mathbf{y}}_t = \mathbf{H}_t \boldsymbol{\mu}_{t|t-1} + \mathbf{D}_t \mathbf{u}_t + \mathbf{d}_t \tag{8}$$

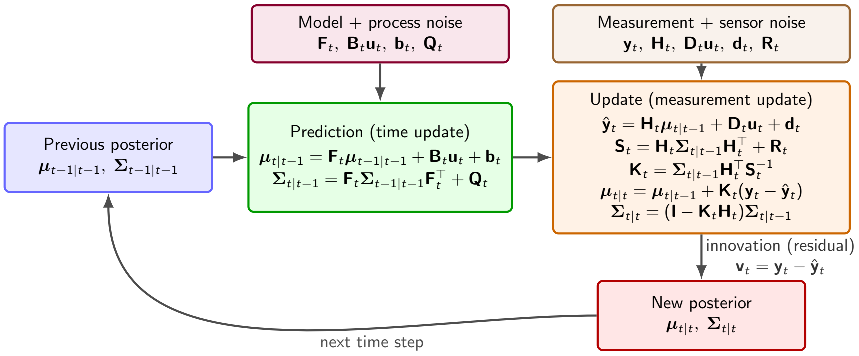
$$\mathbf{S}_t = \mathbf{H}_t \boldsymbol{\Sigma}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t \tag{9}$$

$$\mathbf{K}_t = \boldsymbol{\Sigma}_{t|t-1} \mathbf{H}_t^T \mathbf{S}_t^{-1} \tag{10}$$

$$\boldsymbol{\mu}_{t|t} = \boldsymbol{\mu}_{t|t-1} + \mathbf{K}_t (\mathbf{y}_t - \hat{\mathbf{y}}_t) \tag{11}$$

$$\boldsymbol{\Sigma}_{t|t} = (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \boldsymbol{\Sigma}_{t|t-1} \tag{12}$$

Kalman Filter Operations (One Time Step)



Model + process noise
$\mathbf{F}_t,\ \mathbf{B}_t\mathbf{u}_t,\ \mathbf{b}_t,\ \mathbf{Q}_t$
Measurement + sensor noise
$\mathbf{y}_t,\ \mathbf{H}_t,\ \mathbf{D}_t\mathbf{u}_t,\ \mathbf{d}_t,\ \mathbf{R}_t$
Previous posterior
$\boldsymbol{\mu}_{t-1|t-1},\ \boldsymbol{\Sigma}_{t-1|t-1}$
Prediction (time update)
$\boldsymbol{\mu}_{t|t-1} = \mathbf{F}_t\boldsymbol{\mu}_{t-1|t-1} + \mathbf{B}_t\mathbf{u}_t + \mathbf{b}_t$
$\boldsymbol{\Sigma}_{t|t-1} = \mathbf{F}_t\boldsymbol{\Sigma}_{t-1|t-1}\mathbf{F}_t^\top + \mathbf{Q}_t$
Update (measurement update)
$\hat{\mathbf{y}}_t = \mathbf{H}_t\boldsymbol{\mu}_{t|t-1} + \mathbf{D}_t\mathbf{u}_t + \mathbf{d}_t$
$\mathbf{S}_t = \mathbf{H}_t\boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top + \mathbf{R}_t$
$\mathbf{K}_t = \boldsymbol{\Sigma}_{t|t-1}\mathbf{H}_t^\top\mathbf{S}_t^{-1}$
$\boldsymbol{\mu}_{t|t} = \boldsymbol{\mu}_{t|t-1} + \mathbf{K}_t(\mathbf{y}_t - \hat{\mathbf{y}}_t)$
$\boldsymbol{\Sigma}_{t|t} = (\mathbf{I} - \mathbf{K}_t\mathbf{H}_t)\boldsymbol{\Sigma}_{t|t-1}$
innovation (residual)
$\mathbf{v}_t = \mathbf{y}_t - \hat{\mathbf{y}}_t$
New posterior
$\boldsymbol{\mu}_{t|t},\ \boldsymbol{\Sigma}_{t|t}$
next time step

Computational Complexity

Per time step:

- **Prediction:**
  - Mean: $O(n_z^2)$ (matrix-vector multiply)
  - Covariance: $O(n_z^3)$ (matrix-matrix multiply)
- **Update:**
  - Innovation covariance $\mathbf{S}_t$: $O(n_z^2 n_y)$
  - Matrix inverse: $O(n_y^3)$ or $O(n_z^3)$ depending on form
  - Kalman gain: $O(n_z^2 n_y)$
  - Mean and covariance update: $O(n_z^2 n_y)$

Total: $O(n_z^3)$ per time step

Memory

Constant: $O(n_z^2)$ — only need to store current $\boldsymbol{\mu}_{t|t}$ and $\boldsymbol{\Sigma}_{t|t}$

Compare to batch inference: $O(T^3 n_z^3)$ *complexity!*

The Kalman Gain: Optimal Weighting

$$\mathbf{K}_t = \mathbf{\Sigma}_{t|t-1}\mathbf{H}_t^T(\mathbf{H}_t\mathbf{\Sigma}_{t|t-1}\mathbf{H}_t^T + \mathbf{R}_t)^{-1}$$

Interpretation:

$$\boldsymbol{\mu}_{t|t} = \boldsymbol{\mu}_{t|t-1} + \mathbf{K}_t \underbrace{(\mathbf{y}_t - \hat{\mathbf{y}}_t)}_{\text{innovation}}$$

- $\mathbf{K}_t$ determines how much we trust the new measurement vs. prediction
- Innovation $\mathbf{v}_t = \mathbf{y}_t - \hat{\mathbf{y}}_t$: "surprise" in the measurement
- Large $\mathbf{K}_t \to$ trust measurement more
- Small $\mathbf{K}_t \to$ trust prediction more

Extreme Cases

- If $\mathbf{R}_t \to 0$ (perfect measurements): $\mathbf{K}_t \to \mathbf{H}_t^{-1}$, trust measurement fully
- If $\mathbf{Q}_t \to 0$ (perfect model): $\mathbf{K}_t \to \mathbf{0}$, ignore measurements

Kalman Filter: 1D Scalar Case

Simplified state-space model

$$z_t = z_{t-1} + q_t, \qquad q_t \sim \mathcal{N}(0,\ q_t)$$

$$y_t = z_t + r_t, \qquad r_t \sim \mathcal{N}(0,\ r_t)$$

What was replaced (from the general matrix KF)

- $\boldsymbol{\mu}_{t|t} \to \mu_{t|t}, \quad \boldsymbol{\Sigma}_{t|t} \to \sigma^2_{t|t}$
- $F_t = 1$ so $F_t^2\sigma^2 \to \sigma^2$
- $H_t = 1$ so $S_t = H_t^2\sigma^2 + r_t \to \sigma^2 + r_t$
- $u_t = b_t = d_t = 0$ so $\hat{y}_t = H_t\mu_{t|t-1} \to \mu_{t|t-1}$
- $\mathbf{K}_t = \Sigma H^\top S^{-1} \to K_t = \sigma^2_{t|t-1}/(\sigma^2_{t|t-1} + r_t)$

Intuition: q_t makes uncertainty grow over time (prediction), r_t controls how much we trust the measurement (update).

Complete algorithm

Initialization: $z_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$

For $t = 1, \ldots, T$**:**

Prediction (time update):

$$\mu_{t|t-1} = \mu_{t-1|t-1} \qquad \sigma^2_{t|t-1} = \sigma^2_{t-1|t-1} + q_t$$

Update (measurement update):

$$\hat{y}_t = \mu_{t|t-1} \qquad S_t = \sigma^2_{t|t-1} + r_t$$

$$K_t = \frac{\sigma^2_{t|t-1}}{\sigma^2_{t|t-1} + r_t}$$

$$\mu_{t|t} = \mu_{t|t-1} + K_t\,(y_t - \hat{y}_t)$$

$$\sigma^2_{t|t} = (1 - K_t)\,\sigma^2_{t|t-1}$$

Kalman Gain: 1D Scalar Case

For scalar z_t, y_t, the Kalman gain simplifies to:

$$K_t = \frac{\sigma^2_{t|t-1}}{\sigma^2_{t|t-1} + r_t}$$

Special cases:

- If $\sigma^2_{t|t-1} \gg r_t$ (uncertain prediction, accurate measurement):

$$K_t \approx 1 \quad \Rightarrow \quad \mu_{t|t} \approx y_t$$

- If $\sigma^2_{t|t-1} \ll r_t$ (confident prediction, noisy measurement):

$$K_t \approx 0 \quad \Rightarrow \quad \mu_{t|t} \approx \mu_{t|t-1}$$

- If $\sigma^2_{t|t-1} = r_t$ (equal uncertainty):

$$K_t = 0.5 \quad \Rightarrow \quad \mu_{t|t} = 0.5\mu_{t|t-1} + 0.5y_t$$

The Kalman gain automatically balances prediction and measurement!

Example 1: 1D Position Tracking

Setup:

- State: z_t = position (scalar)
- Dynamics: random walk with drift
- Observation: noisy position measurement

Model:

$$z_t = z_{t-1} + v\Delta t + q_t, \quad q_t \sim \mathcal{N}(0, q) \tag{13}$$

$$y_t = z_t + r_t, \quad r_t \sim \mathcal{N}(0, r) \tag{14}$$

State space form:

- $F_t = 1,\ b_t = v\Delta t,\ Q_t = q$
- $H_t = 1,\ R_t = r$

See Python notebook for implementation and visualization

Example 2: Constant Velocity Model (2D Tracking)

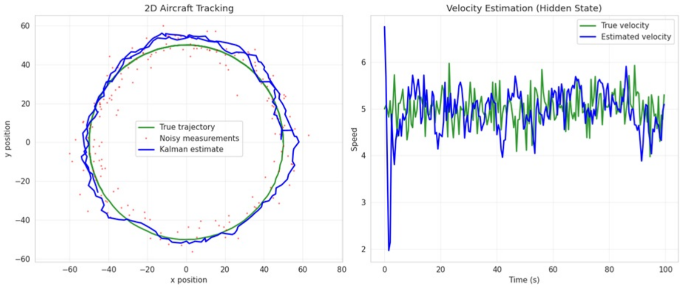
State: $\mathbf{z}_t = [x_t, \dot{x}_t, y_t, \dot{y}_t]^T$ (position + velocity in 2D)
Dynamics: Newton's laws with random acceleration

$$\mathbf{F}_t = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q}_t = q \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} & 0 & 0 \\ \frac{\Delta t^3}{2} & \Delta t^2 & 0 & 0 \\ 0 & 0 & \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} \\ 0 & 0 & \frac{\Delta t^3}{2} & \Delta t^2 \end{bmatrix} \tag{15}$$

Observation: Only position is measured

$$\mathbf{H}_t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \tag{16}$$

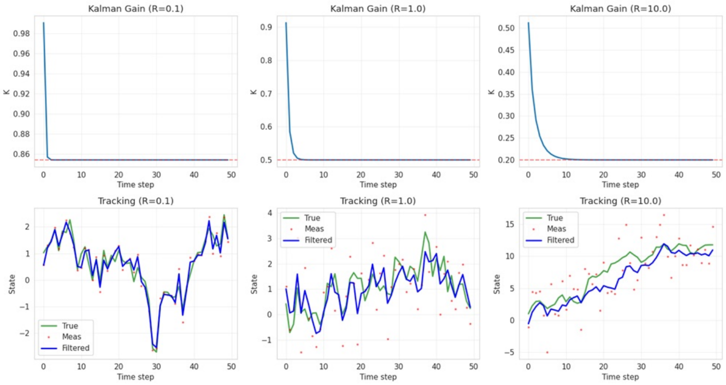
Example 2: Results



Key observations:

- Position estimate tracks ground truth well despite noisy measurements
- **Velocity is estimated despite not being measured!**
- Uncertainty (covariance) decreases as measurements accumulate
- Filter automatically adapts gain based on noise levels

Example 2: Kalman Gain Evolution



Key observations:

- Kalman gain K_t converges to steady-state value
- Initially: large gain (uncertain prior)
- After convergence: constant gain (balance prediction vs measurement)
- Variance decreases and stabilizes

Steady-State Kalman Filter: For time-invariant systems, we can pre-compute K_∞ by solving the **Discrete Algebraic Riccati Equation (DARE)**

Key Properties of Kalman Filter

1. **Optimality:**
   - Minimum Mean Squared Error (MMSE) estimator
   - Best Linear Unbiased Estimator (BLUE)
   - Maximum A Posteriori (MAP) estimator for Gaussians
2. **Consistency:**
   - Innovation $\mathbf{v}_t$ should be white noise: $\mathbf{v}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{S}_t)$
   - Covariance $\boldsymbol{\Sigma}_{t|t}$ represents true uncertainty (if model correct)
3. **Separability:**
   - Estimation and control can be designed independently
   - Basis for Linear Quadratic Gaussian (LQG) control

Limitations and Extensions

Kalman Filter Assumptions:

- Linear dynamics and observations
- Gaussian noise
- Known model parameters $(\mathbf{F}_t, \mathbf{H}_t, \mathbf{Q}_t, \mathbf{R}_t)$

What if assumptions are violated?

1. **Nonlinear systems:**
   - Extended Kalman Filter (EKF): linearize via Jacobians
   - Unscented Kalman Filter (UKF): sigma points
   - Particle Filter: Monte Carlo sampling
2. **Non-Gaussian noise:**
   - Mixture Kalman filters
   - Particle filters
3. **Unknown parameters:**
   - Expectation-Maximization (EM) algorithm
   - Adaptive Kalman filtering

Extended Kalman Filter (EKF)

Nonlinear system:

$$\mathbf{z}_t = f(\mathbf{z}_{t-1}, \mathbf{u}_t) + \mathbf{q}_t \tag{17}$$

$$\mathbf{y}_t = h(\mathbf{z}_t) + \mathbf{r}_t \tag{18}$$

Idea: Linearize around current estimate
Jacobians:

$$\mathbf{F}_t = \left.\frac{\partial f}{\partial \mathbf{z}}\right|_{\mathbf{z}=\boldsymbol{\mu}_{t-1|t-1}} \tag{19}$$

$$\mathbf{H}_t = \left.\frac{\partial h}{\partial \mathbf{z}}\right|_{\mathbf{z}=\boldsymbol{\mu}_{t|t-1}} \tag{20}$$

EKF equations: Same as Kalman filter, but use:

- $\boldsymbol{\mu}_{t|t-1} = f(\boldsymbol{\mu}_{t-1|t-1}, \mathbf{u}_t)$ instead of $\mathbf{F}_t\boldsymbol{\mu}_{t-1|t-1}$
- $\hat{\mathbf{y}}_t = h(\boldsymbol{\mu}_{t|t-1})$ instead of $\mathbf{H}_t\boldsymbol{\mu}_{t|t-1}$
- $\mathbf{F}_t, \mathbf{H}_t$ computed via Jacobians

Works well for mildly nonlinear systems; can diverge if highly nonlinear.

Deep Kalman Filters

Idea: Combine Kalman filtering with deep learning

Approach:

- Parameterize $f(\cdot), h(\cdot)$ with neural networks
- Learn from data using gradient descent
- Use Kalman filter for inference given learned model

Architectures:

1. **Variational RNN (VRNN):**
    - $p(\mathbf{z}_t \mid \mathbf{z}_{t-1})$ and $p(\mathbf{y}_t \mid \mathbf{z}_t)$ are neural networks
    - Approximate inference with amortized variational inference
2. **Neural ODE + Kalman:**
    - Continuous-time dynamics: $\frac{d\mathbf{z}}{dt} = f_\theta(\mathbf{z}, t)$
    - Use Kalman filter for discrete observations

Active research area: combining model-based and learning-based approaches

Modern multi-object visual trackers

Tracker	**Year**	**State estimator / motion model**	**Tracking highlights**
SORT	2016	Linear KF (CV)	Fast baseline; short-term linear motion
DeepSORT	2017	Linear KF (CV)	Adds deep ReID; longer occlusion tolerance
Tracktor++	2019	Detector bbox regression	Uses detector regression as motion model; +CMC/ReID
CenterTrack	2020	Learned offsets (no KF)	Tracks centers via regressed inter-frame offsets
ByteTrack	2021	Linear KF (CV)	Improves continuity by using low-score detections
BoT-SORT	2022	KF + camera motion comp.	Improved box state + CMC for stable motion
StrongSORT	2022	KF + motion/camera corr.	Upgraded DeepSORT baseline; optional smoothing/link
OC-SORT	2023	KF + ORU re-update	Observation-centric re-update reduces KF drift in occlusion
TrackFormer	2022	Track queries (Transformer)	End-to-end tracking-by-attention; no explicit KF

Summary: Why Kalman Filter Matters

Theoretical Elegance

- Optimal Bayesian solution for linear-Gaussian systems
- Closed-form recursive equations
- Beautiful mathematical structure

Practical Impact

- Real-time state estimation with constant memory
- Navigation: GPS, inertial guidance, autonomous vehicles
- Control: LQG controllers, Model Predictive Control
- Signal processing: noise reduction, prediction

Conceptual Foundation

- Template for all filtering algorithms (EKF, UKF, particle filters)
- Connects to modern ML: RNNs, VAEs, S4/Mamba models
- Embodiment of Bayesian recursive inference

Key Takeaways

1. **Predict-Update Cycle:**
    - Prediction: propagate state forward using dynamics
    - Update: incorporate new measurement via Bayes' rule
2. **Kalman Gain:**
    - Automatically balances model vs. measurement
    - Proportional to confidence in each source
3. **Optimality:**
    - MMSE estimator for linear-Gaussian systems
    - Recursive with $O(n_z^3)$ per step, $O(n_z^2)$ memory
4. **Extensions:**
    - EKF/UKF for nonlinear systems
    - RTS smoother for offline problems
    - Deep learning connections

Further Reading

Classic References:

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- S. Särkkä & L. Svensson, *Bayesian Filtering and Smoothing*, 2nd ed. (Cambridge Univ. Press, 2023).
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- Y. Pei, S. Biswas, D. S. Fussell, K. Pingali, “An Elementary Introduction to Kalman Filtering,” arXiv:1710.04055 (2017). (Also appears as an ACM tutorial article, 2019.)

Modern Extensions (SSM-based sequence models):

- A. Gu, K. Goel, C. Ré, “Efficiently Modeling Long Sequences with Structured State Spaces (S4),” ICLR 2022 (arXiv:2111.00396).
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