量子计算作业

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1. How to unambiguously distinguish $\{|\psi_1\rangle = |0\rangle, |\psi_2\rangle = |+\rangle\}$ using POVM measurements?

解：设测量算子为 $\{E_0, E_+\}$，需要满足半正定性和完备性，即

$|E_0| \geqslant 0$，$|E_+| \geqslant 0$，$E_0 + E_+ = I$.

经过计算可得到一组测量算子 $\{E_0, E_+\} = \left\{\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\right\}$ 满足条件，且测量得到的概率结果为：

$$\langle 0|E_0|0\rangle = [1\ 0]\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = 1$$

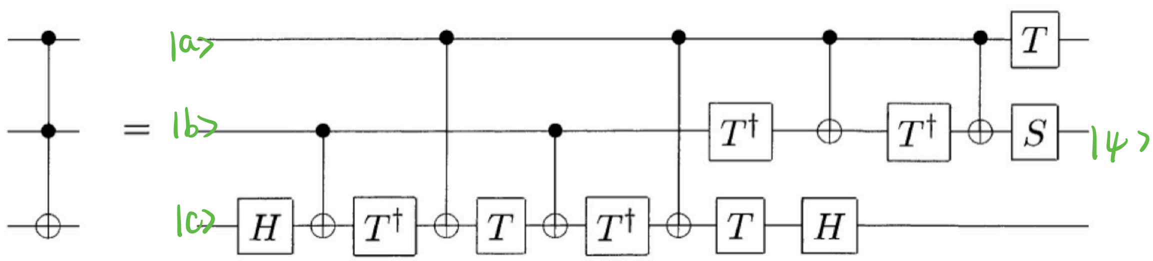
$$\langle 0|E_+|0\rangle = [1\ 0]\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = 0$$

$$\langle +|E_0|+\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = 0$$

$$\langle +|E_+|+\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = 1$$

$\therefore$ $\{E_0, E_+\}$ 可以无错区分 $|0\rangle$ 和 $|+\rangle$.

2. Verify the following circuit implements the Toffoli gate.



解：设三条线路的输入状态分别为$|a\rangle, |b\rangle, |c\rangle$，电路输出状态为$|\psi\rangle$.

$|\psi\rangle = (T|a\rangle) \otimes (SX^aT^\dagger X^aT^\dagger|b\rangle) \otimes (HTX^aT^\dagger X^bTX^aT^\dagger X^bH|c\rangle)$

已知 $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$ $T^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{i\pi}{4}} \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{2}} \end{bmatrix}$ $HH = I$

① 当$|ab\rangle = |00\rangle$时：

$|\psi\rangle = (T|0\rangle) \otimes (ST^\dagger T^\dagger|0\rangle) \otimes (HTT^\dagger TT^\dagger H|c\rangle)$

$= |0\rangle \otimes |0\rangle \otimes |c\rangle$

$= |00c\rangle$

② 当$|ab\rangle = |01\rangle$时：

$|\psi\rangle = (T|0\rangle) \otimes (ST^\dagger T^\dagger|1\rangle) \otimes (HTT^\dagger XTT^\dagger XH|c\rangle)$

$= |0\rangle \otimes (e^{\frac{i\pi}{2}} \cdot e^{-\frac{i\pi}{4}} \cdot e^{-\frac{i\pi}{4}}|1\rangle) \otimes |c\rangle$

$= |01c\rangle$

③ 当$|ab\rangle = |10\rangle$时：

$|\psi\rangle = (T|1\rangle) \otimes (SXT^\dagger XT^\dagger|0\rangle) \otimes (HTXT^\dagger TXT^\dagger H|c\rangle)$

$= (e^{\frac{i\pi}{4}} \cdot |1\rangle) \otimes (e^{-\frac{i\pi}{4}} \cdot |0\rangle) \otimes |c\rangle$

$= |10c\rangle$

④ 当$|ab\rangle = |11\rangle$时：

$|\psi\rangle = (T|1\rangle) \otimes (SXT^\dagger XT^\dagger|1\rangle) \otimes (H\underline{TXT^\dagger X}\,\underline{TXT^\dagger X}H|c\rangle)$

$= (e^{\frac{i\pi}{4}} \cdot |1\rangle) \otimes (e^{\frac{i\pi}{4}} \cdot |1\rangle) \otimes (-iHZH|c\rangle)$

$= i \cdot |11\rangle \otimes (-iX|c\rangle)$

$= |11\rangle \otimes X|c\rangle$

根据计算结果知，当且仅当$|ab\rangle = |11\rangle$时，$|c\rangle$才会翻转，故实现了Toffoli门。

3. If the state of the second register is $|\psi\rangle = \sum_u c_u|u\rangle$ not just $|u\rangle$ in the phase estimation, how to implement the 3 stages?

解：如果第2个寄存器状态变成 $\sum_u c_u|u\rangle$，那么输入变成 $|0\rangle\sum_u c_u|u\rangle$，输出变成 $\sum_u c_u|\tilde{\varphi}_u\rangle|u\rangle$。已知变换前精确估计出 φ 的 n 位的概率至少为 $1-\varepsilon$，其中 $t \geq n+\ln(2+\frac{1}{2\varepsilon})$，当输入变成 $|0\rangle\sum_u c_u|u\rangle$ 后，输入是 $|0\rangle|u\rangle$ 的概率正好是 $|c_u|^2$。后面两步的过程与前面相互独立，所以精确估计出 n 位 φ_u 的概率变成 $|c_u|^2(1-\varepsilon)$.