TEOREMA 3.5 (**The Bramble-Hilbert Lemma**) *Let m and k be non-negative integers such that $0 \le m \le k+1$, and let $\Pi \in \mathcal{L}(H^{k+1}(S), H^m(S))$ such that $\Pi(p) = p \quad \forall\, p \in \mathbb{P}_k(S)$. Then there exists $C := C(\Pi, S) > 0$ such that*

$$\|v - \Pi(v)\|_{m,S} \le C\, |v|_{k+1,S} \qquad \forall\, v \in H^{k+1}(S).$$

DEMOSTRACIÓN. Given $v \in H^{k+1}(S)$ and $p \in \mathbb{P}_k(S)$ we have

$$v - \Pi(v) = (v+p) - \Pi(v+p) = (I - \Pi)(v+p),$$

which, using that $I \in \mathcal{L}(H^{k+1}(S), H^m(S))$ since $0 \le m \le k+1$, implies

$$\|v - \Pi(v)\|_{m,S} \le \|I - \Pi\|\, \|v + p\|_{k+1,S} \qquad \forall\, p \in \mathbb{P}_k(S),$$

and therefore

$$\|v - \Pi(v)\|_{m,S} \le \|I - \Pi\| \inf_{p \in \mathbb{P}_k(S)} \|v + p\|_{k+1,S} = \|I - \Pi\|\, \|[v]\|_{k+1,k,S}.$$

This inequality and the Deny-Lions Lemma (cf. Theorem 3.4) complete the proof.

□

On the other hand, the following two lemmas provide equivalence relationships between Sobolev spaces defined on affine-equivalent and Piola-equivalent domains.

LEMA 3.12 *Let S and $\widehat{S}$ be compact and connected sets of $\mathbb{R}^n$ with Lipschitz-continuous boundaries, and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the affine mapping given by $F(\widehat{x}) = B\,\widehat{x} + b \quad \forall\, \widehat{x} \in \mathbb{R}^n$, with $B \in \mathbb{R}^{n\times n}$ invertible and $b \in \mathbb{R}^n$, such that $S = F(\widehat{S})$. In turn, let m be a non-negative integer and let $v \in H^m(S)$. Then, $\widehat{v} := v \circ F \in H^m(\widehat{S})$ and there exists $C := C(m,n) > 0$ such that*

$$|\widehat{v}|_{m,\widehat{S}} \le \widehat{C}\, \|B\|^m\, |\det B|^{-1/2}\, |v|_{m,S}. \tag{3.14}$$

Conversely, if $\widehat{v} \in H^m(\widehat{S})$ and we let $v = \widehat{v} \circ F^{-1}$, then $v \in H^m(S)$ and there exists $\widehat{C} := \widehat{C}(m,n) > 0$ such that

$$|v|_{m,S} \le \widehat{C}\, \|B^{-1}\|^m\, |\det B|^{1/2}\, |\widehat{v}|_{m,\widehat{S}}\,. \tag{3.15}$$

DEMOSTRACIÓN. We use that $C^m(\overline{S})$ is dense in $H^m(S)$. Then, given $v \in C^m(\overline{S})$ and a multi-index α with $|\alpha| = m$, we have $\widehat{v} := v \circ F \in C^m\big(\overline{\widehat{S}}\big)$ and

$$\partial^\alpha\, \widehat{v}(\widehat{x}) = D^m\, \widehat{v}(\widehat{x})(e_{\beta_1}, e_{\beta_2}, \cdots, e_{\beta_m}) \qquad \forall\, \widehat{x} \in \widehat{S}\,,$$

where $\{e_{\beta_1}, e_{\beta_2}, \cdots, e_{\beta_m}\} \subseteq \{\vec{e_1}, \vec{e_2}, \cdots, \vec{e_n}\}$, the canonical basis of $\mathbb{R}^n$. It follows that

$$|\partial^\alpha \widehat{v}(\widehat{x})| \leq \sup_{\substack{\|\xi_i\| \leq 1 \\ i \in \{1,2,\cdots,m\}}} |D^m \widehat{v}(\widehat{x})(\xi_1, \xi_2, \cdots, \xi_m)| =: \|D^m \widehat{v}(\widehat{x})\|,$$

and hence

$$\begin{aligned} |\widehat{v}|^2_{m,\widehat{S}} &= \int_{\widehat{S}} \sum_{|\alpha|=m} |\partial^\alpha \widehat{v}(\widehat{x})|^2 \, d\widehat{x} \\ &\leq \sum_{|\alpha|=m} \int_{\widehat{S}} \|D^m \widehat{v}(\widehat{x})\|^2 \, d\widehat{x} \qquad (3.16) \\ &= C_1(m,n) \int_{\widehat{S}} \|D^m \widehat{v}(\widehat{x})\|^2 \, d\widehat{x}, \end{aligned}$$

where $C_1(m,n) := \text{card}\{\alpha : \; |\alpha| = m\}$. Now, utilizing the chain rule and the fact that $DF(\widehat{x}) \equiv B \;\; \forall \widehat{x} \in \mathbb{R}^n$, we deduce that

$$D^m \widehat{v}(\widehat{x})(\xi_1, \xi_2, \cdots, \xi_m) = D^m v(F(\widehat{x}))(B\xi_1, B\xi_2, \cdots, B\xi_m)$$

for all $(\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, from which, denoting $x = F(\widehat{x})$, we get

$$\|D^m \widehat{v}(\widehat{x})\| := \sup_{\substack{\|\xi_i\| \leq 1 \\ i \in \{1,2,\cdots,m\}}} |D^m v(x)(B\xi_1, B\xi_2, \cdots, B\xi_m)|$$

$$= \|B\|^m \sup_{\substack{\|\xi_i\| \leq 1 \\ i \in \{1,2,\cdots,m\}}} \left| D^m v(x) \left(\frac{B\xi_1}{\|B\|}, \frac{B\xi_2}{\|B\|}, \cdots, \frac{B\xi_m}{\|B\|} \right) \right|$$

$$\leq \|B\|^m \sup_{\substack{\|\lambda_i\| \leq 1 \\ i \in \{1,2,\cdots,m\}}} |D^m v(x)(\lambda_1, \lambda_2, \cdots, \lambda_m)| = \|B\|^m \|D^m v(x)\|.$$

In this way, employing also (3.11), we find from (3.16) that

$$|\widehat{v}|^2_{m,\widehat{S}} \leq C_1(m,n) \|B\|^{2m} \int_{\widehat{S}} \|D^m v(F(\widehat{x}))\|^2 \, d\widehat{x}$$

$$= C_1(m,n) \|B\|^{2m} |\det \mathrm{B}|^{-1} \int_S \|D^m v(x)\|^2 \, dx,$$

and since

$$\|D^m v(x)\| \leq C_2(n) \max_{|\alpha|=m} |\partial^\alpha v(x)| \leq C_2(n) \sum_{|\alpha|=m} |\partial^\alpha v(x)|$$

we obtain

$$|\widehat{v}|^2_{m,\widehat{S}} \leq C_3(m,n) \|B\|^{2m} |\det \mathrm{B}|^{-1} |v|^2_{m,S},$$

which proves (3.14) for $v \in C^m(\overline{S})$. Analogously, exchanging the roles of S and $\widehat{S}$, and using F^{-1} instead of F, we have (3.15) for all $\widehat{v} \in C^m\big(\overline{\widehat{S}}\big)$.

Similarly, for each $p \le m$ there hold

$$|\widehat{v}|_{p,\widehat{S}} \le C(p,n)\,\|B\|^p\,|\det B|^{-1/2}\,|v|_{p,S}$$

and

$$|v|_{p,S} \le C(p,n)\,\|B^{-1}\|^p\,|\det B|^{1/2}|\widehat{v}|_{p,\widehat{S}}$$

for all $v \in C^p(\overline{S})$ with $\widehat{v} := v \circ F \in C^p(\overline{\widehat{S}})$, which implies the existence of constants $C_i = C_i(m,n,B)$, $i \in \{1,2\}$, such that

$$C_1\,\|\widehat{v}\|_{m,\widehat{S}} \le \|v\|_{m,S} \le C_2\,\|\widehat{v}\|_{m,\widehat{S}} \qquad \forall\, v \in C^m(\overline{S}). \tag{3.17}$$

Now, given $v \in H^m(S)$, we consider a sequence $\{v_j\}_{j\in\mathbb{N}} \subseteq C^m(\overline{S})$ such that $\|v_j - v\|_{m,S} \stackrel{j\to\infty}{\longrightarrow} 0$. Thus, we obtain from (3.17) that

$$\|\widehat{v}_j - \widehat{v}_k\|_{m,\widehat{S}} \le C_1^{-1}\,\|v_j - v_k\|_{m,S},$$

from which we deduce the existence of $\widehat{v} \in H^m(\widehat{S})$ such that $\|\widehat{v}_j - \widehat{v}\|_{m,\widehat{S}} \stackrel{j\to\infty}{\longrightarrow} 0$. Moreover, it is easy to see that this limit $\widehat{v}$ is independent of the chosen sequence, and hence we can define the operator

$$\begin{array}{ccl} H^m(S) & \longrightarrow & H^m(\widehat{S}) \\ v & \longrightarrow & \widehat{v} := \text{“}v \circ F\text{”}. \end{array}$$

Finally, taking limit in the inequality (3.14) with $v = v_j$, that is

$$|\widehat{v}_j|_{m,\widehat{S}} \le \widehat{C}\,\|B\|^m\,|\det B|^{-1/2}\,|v_j|_{m,S}\,,$$

we arrive at

$$|\widehat{v}|_{m,\widehat{S}} \le \widehat{C}\,\|B\|^m\,|\det B|^{-1/2}\,|v|_{m,S}\,,$$

which shows (3.14) $\forall\, v \in H^m(S)$. Analogously we prove (3.15) $\forall\, \widehat{v} \in H^m(\widehat{S})$.

□

Lema 3.13 *Let S and $\widehat{S}$ be compact and connected sets of $\mathbb{R}^n$ with Lipschitz-continuous boundaries, and let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the affine mapping given by $F(\widehat{x}) = B\,\widehat{x} + b \quad \forall\, \widehat{x} \in \mathbb{R}^n$, with $B \in \mathbb{R}^{n\times n}$ invertible and $b \in \mathbb{R}^n$, such that $S = F(\widehat{S})$. In turn, let*