2.2 The Babuška-Brezzi theory

In this section we establish the well-posedness of variational formulations of the Babuška-Brezzi type posed in Banach spaces. We begin by collecting a couple of useful lemmas from linear functional analysis, which will be employed in what follows.

2.2.1 Preliminaries

The following result, which employs a classical result from linear operator theory, is taken from [7, Lemma A.36], but we slightly modify the second part of its proof.

Lemma 2.3. *Let X and Y be Banach spaces, and let $A \in \mathcal{L}(X,Y)$. The following assertions are equivalent*:

i) *$R(A)$ is closed,*
ii) *there exists a constant $\alpha > 0$ such that for each $y \in R(A)$ there exists $z \in X$ satisfying*

$$y = A(z) \qquad \text{and} \qquad \alpha\,\|z\| \,\leqslant\, \|y\|\,.$$

Proof. Under the assumption i), it is clear that $R(A)$ becomes a Banach space as well, and thus a direct application of the open mapping theorem to the surjective operator $A \in \mathcal{L}\big(X, R(A)\big)$ yields the existence of $r > 0$ such that $B_{R(A)}(0,r) \subseteq A\big(B_X(0,1)\big)$. Then, given $y \in R(A)$, $y \neq 0$, there holds $\frac{r}{2}\frac{y}{\|y\|} \in B_{R(A)}(0,r)$, so that employing the foregoing inclusion we deduce the existence of $x \in B_X(0,1)$ such that $A(x) = \frac{r}{2}\frac{y}{\|y\|}$. In this way, denoting $z = \frac{2\|y\|}{r}\,x$, we obtain $A(z) = y$ and $\|z\| \leqslant \frac{2}{r}\|y\|$, which proves ii) with $\alpha = \frac{r}{2}$. Conversely, assuming ii), and given $x \in X$ such that $A(x) \neq 0$, we know from the hypothesis that there exists $z \in X$ such that $A(x) = A(z)$ and $\alpha\,\|z\| \leqslant \|A(x)\|$. Hence, using that $x - z \in N(A)$, we get

$$\|A(x)\| \,\geqslant\, \alpha\,\|z\| \,=\, \alpha\,\|x - (x-z)\| \,\geqslant\, \alpha\,\mathrm{dist}\big(x, N(A)\big)\,,$$

which, according to a classical characterization result, implies that $R(A)$ is closed.

∎

The following lemma, which makes use of the foregoing one and the reflexivity concept, and which is originally proved in [1], is taken now from [7, Lemma A.42], but similarly as for the previous one, we present an alternative proof for its converse implication.

Lemma 2.4. *Let X and Y be Banach spaces and let $A \in \mathcal{L}(X,Y)$ be a surjective operator. Then, there exists $\alpha > 0$ such that*

i) *for each $y \in R(A) = Y$ there exists $z \in X$ such that $A(z) = y$, and $\alpha\,\|z\| \leqslant \|y\|$, which, in turn, implies that*
ii) $\|A'(G)\|_{X'} \,\geqslant\, \alpha\,\|G\|_{Y'} \qquad \forall\, G \in Y'$.

Conversely, if X is reflexive, i) *follows from* ii) *with the same constant α.*

Proof. Certainly, the surjectivity of A guarantees that $R(A)$ is closed, so that applying Lemma 2.3 we deduce the existence of a constant $\alpha > 0$ with which i) is satisfied. Then, for each $y \in Y$, $y \neq 0$, there exists $z \in X$ such that $A(z) = y$ and $\alpha\,\|z\| \leqslant \|y\|$, thanks to which, given $G \in Y'$, there holds

$$\frac{|G(y)|}{\|y\|} \;=\; \frac{|G\big(A(z)\big)|}{\|y\|} \;=\; \frac{|A'(G)(z)|}{\|y\|} \;\leqslant\; \frac{1}{\alpha}\,\|A'(G)\|\,,$$

and hence

$$\|G\| = \sup_{\substack{y \in Y \\ y \neq \theta}} \frac{|G(y)|}{\|y\|} \leqslant \frac{1}{\alpha} \|A'(G)\|,$$

which proves ii). Conversely, let us assume that X is reflexive and that ii) is valid with a constant $\alpha > 0$. Equivalently, this means that $A' \in \mathcal{L}(Y', X')$ is injective and of closed range, and thus $R(A') = N(A)^\circ$. It follows from this identity and ii) that $A' : Y' \longrightarrow N(A)^\circ$ is bijective and that $\|(A')^{-1}\| \leqslant \frac{1}{\alpha}$. Moreover, $A'' : \big(N(A)^\circ\big)' \longrightarrow Y''$ is also bijective and

$$\|(A'')^{-1}\| = \|\big((A')^{-1}\big)'\| = \|(A')^{-1}\| \leqslant \frac{1}{\alpha}.$$

Now, given $y \in Y$, we consider $\mathcal{J}_Y(y) \in Y''$, for which there exists $\mathcal{F}_0 \in \big(N(A)^\circ\big)'$ such that $A''(\mathcal{F}_0) = \mathcal{J}_Y(y)$. Since $N(A)^\circ$ is a subspace of X', the classical Hahn-Banach theorem guarantees the existence of $\mathcal{F} \in X''$ such that $\mathcal{F}\big|_{N(A)^\circ} = \mathcal{F}_0$ and $\|\mathcal{F}\|_{X''} = \|\mathcal{F}_0\|_{(N(A)^\circ)'}$. Thus, denoting $z = J_X^{-1}(\mathcal{F}) \in X$, we find that for each $G \in Y'$ there holds

$$\begin{aligned} G(y) &= \mathcal{J}_Y(y)(G) = A''(\mathcal{F}_0)(G) = (A')'(\mathcal{F}_0)(G) = \mathcal{F}_0(A'(G)) \\ &= \mathcal{F}(A'(G)) = \mathcal{J}_X(z)(A'(G)) = A'(G)(z) = G(A(z)), \end{aligned}$$

from which it follows, thanks to a consequence of the Hahn-Banach theorem again, that $y = A(z)$. Finally, we have

$$\begin{aligned} \|z\| &= \|\mathcal{J}_X(z)\| = \|\mathcal{F}\|_{X''} = \|\mathcal{F}_0\|_{(N(A)^\circ)'} \\ &= \|(A'')^{-1}(\mathcal{J}_Y(y))\|_{(N(A)^\circ)'} \leqslant \|(A'')^{-1}\| \, \|y\| \leqslant \frac{1}{\alpha} \|y\|, \end{aligned}$$

which completes i) and ends the proof.

■

2.2.2 Continuous analysis

In this section we study the well-posedness of the classical variational formulations of the Babuška-Brezzi type, in which two bilinear forms are involved and the unknowns and test functions spaces coincide.

2.2.2.1 The operator equation

Let $(H, \|\cdot\|_H)$ and $(Q, \|\cdot\|_Q)$ be real Banach spaces, and let $a : H \times H \to R$ and $b : H \times Q \to R$ be bounded bilinear forms. Then, given $F \in H'$ and $G \in Q'$, we consider the problem: Find $(\sigma, u) \in H \times Q$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= F(\tau) \qquad \forall\, \tau \in H \\ b(\sigma, v) \qquad\qquad &= G(v) \qquad \forall\, v \in Q. \end{aligned} \tag{2.21}$$

Now, let $\mathcal{A} \in \mathcal{L}(H, H')$ and $\mathcal{B} \in \mathcal{L}(H, Q')$ be the linear and bounded operators induced by a and b, respectively, that is