A Banach spaces-based mixed-primal finite element method for the coupled Brinkman-Forchheimer and nonlinear transport equations*

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Abstract

We propose and analyze a Banach spaces-based mixed-primal finite element method for the coupling of the Brinkman–Forchheimer equations with a nonlinear transport phenomenon
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Key words: Brinkman–Forchheimer equations, nonlinear transport, pseudostress-velocity formulation, fixed point theory, perturbed saddle-point, mixed finite elements

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1 Introduction

Preliminary notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector on Γ. In what follows, standard notation is adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both

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R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}\,,$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}\,.$$

Furthermore, for each $t \in [1, +\infty)$ we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,,$$

equipped with the natural norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,.$$

Additionally, we recall that, proceeding as in [3, eq. (1.43), Section 1.3.4] (see also [1, Section 4.1] and [2, Section 3.1]), one can prove that for $t \in \begin{cases} (1,+\infty] \text{ in } \mathrm{R}^2\,, \\ [\frac{6}{5},+\infty] \text{ in } \mathrm{R}^3\,, \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{v}\rangle = \int_\Omega \Big\{\boldsymbol{\tau} : \nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \quad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega) \times \mathbf{H}^1(\Omega)\,, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.1) the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$.

2 The model problem

We consider a porous medium occupying the region Ω, and assume that a viscous fluid governed by the Brinkman-Forchheimer equations flows through it, so that the sought variables are its pressure p and velocity $\mathbf{u}$. In addition, we let ϕ be the concentration of a chemical component transported by the fluid, which is advected and diffused in Ω according to the corresponding physical principle. Alternatively, ϕ could represent the temperature of the fluid, among several other possibilities. In this way, the coupled model of interest is given by the following system of partial differential equations:

$$\begin{aligned} -\,\mu\,\Delta\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} + \nabla p &= \phi\,\mathbf{f} && \text{in} \quad \Omega\,, \\ \mathrm{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,, \\ \mathrm{div}\big(\vartheta(|\nabla\phi|)\,\nabla\phi - \phi\,\mathbf{u} - f(\phi)\,\mathbf{g}\big) &= g && \text{in} \quad \Omega\,, \\ \mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi &= \phi_{\mathrm{D}} && \text{on} \quad \Gamma\,, \end{aligned} \tag{2.1}$$

where μ is the constant viscosity of the fluid, $\mathtt{D}, \mathtt{F} > 0$ are the Darcy and Forchheimer coefficients, respectively, ρ is a given number in $[3,4]$, $\vartheta : \mathrm{R}^+ \to \mathrm{R}^+$ is a nonlinear diffusivity function, f is a nonlinear flux acting in the direction of $\mathbf{g}$, which, in turn, is a constant vector pointing in the direction of gravity, $\mathbf{f}$ and g are given source functions, and $\mathbf{u}_{\mathrm{D}}$ and ϕ_{D} are Dirichlet data for $\mathbf{u}$ and ϕ, respectively. Regarding ϑ, we assume that there exist constants ϑ_1, $\vartheta_2 > 0$ such that

$$\vartheta_1 \le \vartheta(t) \le \vartheta_2 \quad \text{and} \quad \vartheta_1 \le \vartheta(t) + t\,\vartheta'(t) \le \vartheta_2 \qquad \forall\, t \in \mathrm{R}^+ . \tag{2.2}$$

In addition, f is required to be bounded and Lipschitz-continuous, which means that there exist constants f_1, f_2, $\mathrm{L}_f > 0$ such that

$$f_1 \le f(t) \le f_2 \quad \text{and} \quad |f(t) - f(s)| \,\le\, \mathrm{L}_f\,|s - t| \qquad \forall\, s,\, t \in \mathrm{R}^+ . \tag{2.3}$$

Now, due to the incompressibility of the fluid (cf. second row of (2.1)), $\mathbf{u}_{\mathrm{D}}$ must formally satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_{\mathrm{D}} \cdot \mathbf{n} \,=\, 0 . \tag{2.4}$$

On the other hand, for the uniqueness of p we look for this unknown in the space

$$\mathrm{L}^2_0(\Omega) \,:=\, \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\} .$$

Next, in order to derive a mixed-primal formulation for (2.1), we first define as an auxiliary unknown the fluid pseudostress

$$\boldsymbol{\sigma} \,:=\, \mu\,\nabla\mathbf{u} \,-\, p\,\mathbb{I} , \tag{2.5}$$

so that the first row of (2.1) becomes

$$-\,\mathbf{div}(\boldsymbol{\sigma}) \,+\, \mathtt{D}\,\mathbf{u} \,+\, \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \,=\, \phi\,\mathbf{f} . \tag{2.6}$$

Thus, taking matrix trace along with the fact that $\mathrm{tr}(\nabla\mathbf{u}) = \mathrm{div}(\mathbf{u}) = 0$, and then applying the deviatoric operator, we deduce from (2.5) that

$$p \,=\, -\frac{1}{n}\,\mathrm{tr}(\boldsymbol{\sigma}) \quad \text{and} \quad \frac{1}{\mu}\,\boldsymbol{\sigma}^{\mathrm{d}} \,=\, \nabla\mathbf{u} , \tag{2.7}$$

which are equivalent to the pair of equations formed by the incompressibility condition and (2.5). Hence, eliminating the unknown p, and computing it afterwards according to the identity provided in (2.7), the original system (2.1) can be stated, equivalently, as: Find $\boldsymbol{\sigma}$, $\mathbf{u}$, and ϕ in suitable spaces to be indicated below, such that

$$\begin{aligned}
\frac{1}{\mu}\,\boldsymbol{\sigma}^{\mathrm{d}} \;&=\; \nabla\mathbf{u} && \text{in} \quad \Omega , \\
-\,\mathbf{div}(\boldsymbol{\sigma}) \,+\, \mathtt{D}\,\mathbf{u} \,+\, \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \;&=\; \phi\,\mathbf{f} && \text{in} \quad \Omega , \\
\mathrm{div}\big(\vartheta(|\nabla\phi|)\,\nabla\phi - \phi\,\mathbf{u} \,-\, f(\phi)\,\mathbf{g}\big) \;&=\; g && \text{in} \quad \Omega , \\
\mathbf{u} = \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi \;&=\; \phi_{\mathrm{D}} && \text{on} \quad \Gamma , \\
\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;&=\; 0 .
\end{aligned} \tag{2.8}$$

3 The variational formulation

Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathrm{H} \times \mathrm{Q}$ and $(\phi, \xi) \in \mathrm{X} \times \mathrm{Y}$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}\,, \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) \,-\, \mathbf{c}_{\mathbf{u}}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_{\phi}(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathrm{Q}\,, \\ [\mathrm{A}_{\mathbf{u}}(\phi), \varphi] + \mathrm{B}(\varphi, \xi) &= \mathrm{F}_{\phi}(\varphi) \qquad \forall\, \varphi \in \mathrm{X}\,, \\ \mathrm{B}(\phi, \eta) &= \mathrm{G}(\eta) \qquad \forall\, \eta \in \mathrm{Y}\,, \end{aligned} \tag{3.1}$$

where the bilinear forms $\mathbf{a} : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$, $\mathbf{b} : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, and $\mathbf{c}_{\mathbf{z}} : \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}$, and the nonlinear operator $\mathrm{A}_{\mathbf{z}} : \mathrm{X} \to \mathrm{X}'$, for each $\mathbf{z} \in \mathrm{Q}$, and the bilinear form $\mathrm{B} : \mathrm{X} \times \mathrm{Y} \to \mathrm{R}$, and the linear functionals $\mathbf{F} : \mathrm{H} \to \mathrm{R}$, $\mathbf{G}_{\psi} : \mathrm{Q} \to \mathrm{R}$ and $\mathrm{F}_{\psi} : \mathrm{X} \to \mathrm{R}$, for each $\psi \in \mathrm{X}$, and $\mathrm{G} : \mathrm{Y} \to \mathrm{R}$, are defined as

$$\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{\mu} \int_{\Omega} \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathrm{H}\,, \tag{3.2}$$

$$\mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) := \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathrm{H} \times \mathrm{Q}\,, \tag{3.3}$$

$$\mathbf{c}_{\mathbf{z}}(\mathbf{w}, \mathbf{v}) := \mathtt{D} \int_{\Omega} \mathbf{w} \cdot \mathbf{v} \,+\, \mathtt{F} \int_{\Omega} |\mathbf{z}|^{\rho - 2} \mathbf{w} \cdot \mathbf{v} \qquad \forall\, \mathbf{w},\, \mathbf{v} \in \mathrm{Q}\,, \tag{3.4}$$

$$[\mathrm{A}_{\mathbf{z}}(\psi), \varphi] := \int_{\Omega} \vartheta(|\nabla \psi|)\, \nabla \psi \cdot \nabla \varphi - \int_{\Omega} \psi\, \mathbf{z} \cdot \nabla \varphi \qquad \forall\, \psi,\, \varphi \in \mathrm{X}\,, \tag{3.5}$$

$$\mathrm{B}(\varphi, \eta) := \langle \eta, \varphi \rangle \qquad \forall\, (\varphi, \eta) \in \mathrm{X} \times \mathrm{Y}\,, \tag{3.6}$$

$$\mathbf{F}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_{\mathrm{D}} \rangle \qquad \forall\, \boldsymbol{\tau} \in \mathrm{H}\,, \tag{3.7}$$

$$\mathbf{G}_{\psi}(\mathbf{v}) := -\int_{\Omega} \psi\, \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathrm{Q}\,, \tag{3.8}$$

$$\mathrm{F}_{\psi}(\varphi) := \int_{\Omega} f(\psi)\, \mathbf{g} \cdot \nabla \varphi \,-\, \int_{\Omega} g\, \varphi \qquad \forall\, \varphi \in \mathrm{X}\,, \tag{3.9}$$

and

$$\mathrm{G}(\eta) := \langle \eta, \phi_{\mathrm{D}} \rangle \qquad \forall\, \eta \in \mathrm{Y}\,. \tag{3.10}$$

Note here that $[\cdot, \cdot]$ represents the duality pairing between X' and X. In turn, in the definitions of B (cf. (3.6)) and G (cf. (3.10)), $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, whereas in the definition of $\mathbf{F}$ (cf. (3.7)) $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$.

4 The Galerkin scheme

5 A priori error analysis

6 Numerical results

References

[1] J. Camaño, C. Muñoz and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces*. Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[2] E. Colmenares, G.N. Gatica and S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem.* ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.

[3] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications*. SpringerBriefs in Mathematics. Springer, Cham, 2014.