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GENERALIZED INF-SUP CONDITIONS FOR CHEBYSHEV SPECTRAL APPROXIMATION OF THE STOKES PROBLEM*

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Abstract. A mixed problem and its approximation in an abstract framework are considered. They are proved to be well posed if and only if several inf-sup conditions are satisfied. These results are applied to the Stokes equations in a square, formulated in Chebyshev weighted Sobolev spaces and their approximations. Two kinds of spectral discretizations are analyzed: a Galerkin method and a collocation method at the Chebyshev nodes.

Résumé. On étudie un problème mixte abstrait, et son approximation, qui sont bien posés si et seulement si plusieurs conditions inf-sup sont vérifiées. Ces résultats sont appliqués aux équations de Stokes dans un carré, formulées dans les espaces de Sobolev avec poids de Tchebychev, et à leur approximation par deux types de discrétisations spectrales: méthode de Galerkin et méthode de collocation aux points de Tchebychev.

Key words. Navier-Stokes equations, spectral methods, Chebyshev collocation, inf-sup condition

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1. Introduction. In the last few years, a number of algorithms using spectral collocation methods have been successfully implemented to solve the incompressible Stokes equations in a domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$

$$\begin{aligned} -\nu\Delta\mathbf{u}+\operatorname{grad} p=\mathbf{f} \quad \text{in } \Omega,\\ \operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

provided with appropriate boundary conditions: periodic, Dirichlet, or mixed. We refer to [28] and to [10] for a detailed bibliography. In most of them the collocation points in the nonperiodic directions are the nodes of a Gauss-Lobatto quadrature formula associated with the Chebyshev polynomials. The use of the Fast Fourier Transform then allows a rapid computation of the derivatives and the nonlinear terms. However, as far as we know, the only theoretical justifications of some of these algorithms are achieved with the Chebyshev nodes replaced by the Legendre nodes (see, e.g., [6], [7]). The aim of this paper is the numerical analysis of a collocation method involving the Chebyshev nodes, for Dirichlet boundary conditions and when the domain Ω is the square $]-1,1[^2$. Corresponding numerical experiments can be found in [23], [22].

The extension from Legendre to Chebyshev methods presents essentially two difficulties.

(1) In the variational formulation of both the continuous and the discrete problems, the classical Sobolev spaces have to be replaced by weighted Sobolev spaces with the Chebyshev weight. Several trace theorems in these spaces, as well as some regularity results for the Dirichlet problem for the Laplacian in a square, are needed for our study; these are recent [5].

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(2) Due to the Chebyshev weight, the analysis of the Stokes problem, which we obtain by neglecting the nonlinear terms in (1.1), and of its approximation involves a variational formulation of the form

$$\begin{aligned} &a(\mathbf{u}, \mathbf{v}) + b_1(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ &b_2(\mathbf{u}, q) = 0. \end{aligned} \tag{1.2}$$

In this system, the bilinear forms b_1 and b_2 are distinct, in contrast to the classical saddle-point problem first studied in [8]. Nicolaides [25] has considered such a problem in an abstract framework and proved that it is well posed if several inf-sup conditions are satisfied: two for the form a and one for each of the forms b_1 and b_2. We prove here that these conditions are necessary. Next, we analyze a fairly general discretization which extends the one of [25] and is required by the collocation method; we state general error estimates that can be applied to a number of numerical methods. Our hope is that this abstract variational formulation will be used for other equations, whether or not they involve weighted spaces.

In the particular case of the discretization of the Stokes problem by spectral methods, we have to exhibit the spurious modes of the pressure in order to obtain a well-posed problem, i.e., the modes that cancel the discrete gradient. Thus, we derive an appropriate choice for the space of discrete pressures, and we can prove that the inf-sup conditions are satisfied. The results we obtain are very similar to the corresponding results for the Legendre methods [7]. The convergence estimates will be extended to the Navier–Stokes equations in a forthcoming paper (see also [4]).

An outline of the paper is as follows. Section 2 is devoted to an abstract variational problem with inf-sup conditions. In § 3, we state a variational formulation of the Stokes problem in weighted Sobolev spaces with the Chebyshev weight. In § 4, we study a Galerkin spectral method to discretize this problem. The spectral collocation method for the Stokes problem is analyzed in § 5. Two possible extensions are given in § 6.

Notation. The norm of any Banach space E is denoted by $\|\cdot\|_E$, while $\langle \cdot, \cdot \rangle$ is the duality pairing between E and its dual space E'. For any pair (E, F) of Banach spaces, $\mathscr{L}(E, F)$ represents the space of continuous linear mappings from E into F. We mean by $A \otimes B$ the tensor product of any sets A and B in a Banach space, while $A^{\otimes 2}$ is the tensor product of A with itself.

For any domain Δ in $\mathbb{R}^d$ and for any real number s, we use the classical Hilbertian–Sobolev spaces $H^s(\Delta)$, with norm $\|\cdot\|_s$.

In all that follows, $c, c' \cdots$ are generic constants, independent of the discretization.

2. An abstract variational system and its approximation.

2.1. The continuous case. Let X_i and M_i $(i=1,2)$ be real reflexive Banach spaces. We assume we are given three continuous bilinear forms, $a: X_2 \times X_1 \to \mathbb{R}$ and $b_i: X_i \times M_i \to \mathbb{R}$ $(i=1,2)$. For any given f in X_1' and g in M_2', we consider the following problem. Find (u, p) in $X_2 \times M_1$ such that

$$\begin{aligned} &\forall v \in X_1 \quad a(u, v) + b_1(v, p) = \langle f, v \rangle, \\ &\forall q \in M_2 \quad b_2(u, q) = \langle g, q \rangle. \end{aligned} \tag{2.1}$$

In order to study this problem, let us introduce the linear operators $A \in \mathscr{L}(X_2, X_1')$ and $B_i \in \mathscr{L}(X_i, M_1')$ $(i=1,2)$ associated with the forms a and b_i by the relations

$$\forall u \in X_2, \quad \forall v \in X_1, \quad \langle Au, v \rangle = a(u, v), \tag{2.2}$$

$$\forall u \in X_i, \quad \forall q \in M_i, \quad \langle B_i u, q \rangle = b_i(u, q); \tag{2.3}$$

we denote by $B_i^T \in \mathscr{L}(M_i, X_i')$ $(i=1,2)$ the adjoint operator of B_i.

For any g in M_i' $(i=1,2)$, we define the closed affine space

$$K_i(g)=\{v\in X_i;\ \forall q\in M_i,\ b_i(v,q)=\langle g,q\rangle\}, \tag{2.4}$$

and we note that the subspace $K_i=K_i(0)$ is the kernel of the operator B_i. Moreover we introduce the linear mapping $\Pi: X_1'\to K_1'$ defined for each $f\in X_1'$ by

$$\forall v\in K_1,\quad \langle \Pi f, v\rangle=\langle f, v\rangle. \tag{2.5}$$

Finally, we denote by K_1° the polar set of K_1, i.e., the kernel of the mapping Π.

Let us define the operator $\Lambda: X_2\times M_1\to X_1'\times M_2'$ by the relation

$$\Lambda(u,p)=(Au+B_1^Tp,\ B_2u); \tag{2.6}$$

then (2.1) is equivalent to

$$\Lambda(u,p)=(f,g). \tag{2.7}$$

We can prove the following theorem.

THEOREM 2.1. *The operator Λ is an isomorphism from $X_2\times M_1$ onto $X_1'\times M_2'$ if and only if the following conditions (C_0) and (C_i) $(i=1,2)$ are satisfied:*

(C_0) $\Pi A: K_2\to K_1'$ *is an isomorphism;*
(C_i) *There exists a constant $\beta_i>0$, such that for any $q\in M_i$, $\|B_i^Tq\|_{X_i'}\geqq \beta_i\|q\|_{M_i}$.*

Proof. We assume first that conditions (C_0) and (C_i) $(i=1,2)$ are satisfied, and we prove that Λ is an isomorphism. Given f in X_1' and g in M_2', let us observe that by (C_2) and Theorem 0.1 of [8] there exists $\bar u$ in X_2 such that $B_2\bar u=g$ and $\|\bar u\|_{X_2}\leqq \beta_2^{-1}\|g\|_{M_2'}$. Moreover by (C_0) there exists a unique u_0 in K_2 such that $\Pi Au_0=\Pi f-\Pi A\bar u$, and we have

$$\|u_0\|_{X_2}\leqq c(\|f\|_{X_1'}+\|g\|_{M_2'}).$$

Let us set $u=\bar u+u_0$. The element $f-Au$ belongs to K_1°; hence by (C_1) and the Closed Range Theorem, there exists a unique p in M_1 such that $B_1^Tp=f-Au$. Thus we have proved that Λ is onto. It is straightforward to check that Λ is one to one, and since it is continuous because A, B_1^T, and B_2 are, we conclude that Λ is an isomorphism by using the Open Mapping Theorem.

Conversely, let us suppose that Λ is an isomorphism. For each q in M_1, we have $\Lambda(0,q)=(B_1^Tq,0)$; hence, from the continuity of Λ^{-1}, we obtain

$$\|q\|_{M_1}\leqq \|\Lambda^{-1}\|\cdot\|B_1^Tq\|_{X_1'},$$

which is (C_1). On the other hand, for each g in M_2', let us set $(w,q)=\Lambda^{-1}(0,g)$. Then the mapping: $g\to w$ is continuous from M_2' into X_1 and is such that $B_2w=g$. Thus we have (C_2), using Theorem 0.1 of [8]. Now we prove that ΠA is an isomorphism. Given f in K_1', let us set $(w,q)=\Lambda^{-1}(f,0)$. Then w belongs to K_2 and satisfies

$$\Pi Aw=\Pi f-\Pi B_1^Tq=\Pi f=f,$$

since ΠB_1^T is equal to 0. Thus ΠA is onto. Finally, let w in K_2 be such that $\Pi Aw=0$; by (C_1) and the Closed Range Theorem, there exists q in M_1 such that $B_1^Tq=-Aw$. Thus $\Lambda(w,q)$ is equal to $(0,0)$; hence, w is equal to 0, i.e., ΠA is one to one. This concludes the proof of the theorem. $\square$

Remark 2.1. By well-known results of functional analysis, we can express conditions (C_0) and (C_i) $(i=1,2)$ in a variational form; this was the form used in [25] in order to prove the well-posedness of problem (2.1). More precisely, (C_0) is equivalent to the following condition (see, e.g., [2]): there exists a constant $\alpha_1>0$ such that

$$\forall u \in K_2 \quad \sup_{v\in K_1} \frac{a(u,v)}{\|v\|_{X_1}} \geqq \alpha_1 \|u\|_{X_2} \tag{2.8}$$

and

$$\forall v \in K_1\backslash\{0\} \quad \sup_{u\in K_2} a(u,v) > 0. \tag{2.9}$$

By the Open Mapping Theorem this is also equivalent to the existence of a constant $\alpha_2>0$ such that

$$\forall v \in K_1 \quad \sup_{u\in K_2} \frac{a(u,v)}{\|u\|_{X_2}} \geqq \alpha_2 \|v\|_{X_1} \tag{2.10}$$

and

$$\forall u \in K_2\backslash\{0\} \quad \sup_{v\in K_1} a(u,v) > 0. \tag{2.11}$$

If K_1 (or K_2) is a finite-dimensional space, then the relation (2.9) or (2.11) can be replaced by the requirement that

$$\dim K_1 = \dim K_2. \tag{2.12}$$

Similarly, we can write the condition (C_i) $(i=1,2)$ equivalently as follows: there exists a constant $\beta_i>0$ such that

$$\forall q \in M_i \quad \sup_{v\in X_i} \frac{b_i(v,q)}{\|v\|_{X_i}\|q\|_{M_i}} \geqq \beta_i > 0. \tag{2.13$_i$}$$

These are the forms under which the conditions (C_0) and (C_i) $(i=1,2)$ are usually checked in the applications.

Following the proof of Theorem 2.1, we can easily estimate the norm of the inverse isomorphism Λ^{-1} in term of the constants associated with the forms a and b_i. Denoting by γ the norm of a:

$$\gamma = \sup_{u\in X_2, v\in X_1} \frac{a(u,v)}{\|u\|_{X_2}\|v\|_{X_1}}, \tag{2.14}$$

we have the following result.

COROLLARY 2.1. *Assume that hypotheses* (2.8), (2.9), *and* (2.13)$_i$ $(i=1,2)$ *hold. Then, the solution* (u,p) *of problem* (2.1) *satisfies the following estimates*:

$$\|u\|_{X_2} \leqq \alpha_1^{-1}\|f\|_{X_1'} + \beta_2^{-1}(1+\alpha_1^{-1}\gamma)\|g\|_{M_2'}, \tag{2.15}$$

$$\|p\|_{M_1} \leqq \beta_1^{-1}(1+\alpha_1^{-1}\gamma)\|f\|_{X_1'} + \beta_1^{-1}\beta_2^{-1}\gamma(1+\alpha_1^{-1}\gamma)\|g\|_{M_2'}. \tag{2.16}$$

2.2. The finite-dimensional approximation. We want to approximate problem (2.1) by a finite-dimensional system. To this end, we introduce a discretization parameter $\delta>0$, and we assume we are given closed subspaces $X_{i\delta}$ and $M_{i\delta}$ $(i=1,2)$ contained in X_i and M_i, respectively. The continuous bilinear forms a and b_i are approximated by three continuous bilinear forms $a_\delta: X_{2\delta}\times X_{1\delta}\to\mathbb{R}$ and $b_{i\delta}: X_{i\delta}\times M_{i\delta}\to\mathbb{R}$ $(i=1,2)$.

For any f_δ in X_1' and g_δ in M_2', we consider the following approximation of problem (2.1). Find (u_δ, p_δ) in $X_{2\delta}\times M_{1\delta}$ such that

$$\begin{aligned} &\forall v_\delta \in X_{1\delta} \quad a_\delta(u_\delta, v_\delta) + b_{1\delta}(v_\delta, p_\delta) = \langle f_\delta, v_\delta\rangle, \\ &\forall q_\delta \in M_{2\delta} \quad b_{2\delta}(u_\delta, q_\delta) = \langle g_\delta, q_\delta\rangle. \end{aligned} \tag{2.17}$$

The forms a_δ and $b_{i\delta}$ must satisfy the necessary and sufficient conditions for problem (2.17) to be well posed, as stated in Theorem 2.1. For any g in M_i' $(i=1,2)$, we introduce the affine subspace

$$K_{i\delta}(g)=\{v_\delta\in X_{i\delta};\ \forall q_\delta\in M_{i\delta},\ b_{i\delta}(v_\delta,q_\delta)=\langle g,q_\delta\rangle\}, \tag{2.18}$$

and we set $K_{i\delta}=K_{i\delta}(0)$. We make the following assumptions:

(a) There exists a constant $\alpha_{1\delta}>0$ such that

$$\forall u_\delta\in K_{2\delta}\quad \sup_{v_\delta\in K_{1\delta}}\frac{a_\delta(u_\delta,v_\delta)}{\|v_\delta\|_{X_1}}\geqq \alpha_{1\delta}\|u_\delta\|_{X_2}, \tag{2.19}$$

$$\forall v_\delta\in K_{1\delta}\backslash\{0\}\quad \sup_{u_\delta\in K_{2\delta}} a_\delta(u_\delta,v_\delta)>0; \tag{2.20}$$

if $K_{1\delta}$ and $K_{2\delta}$ are finite-dimensional, this second condition can be equivalently replaced by

$$\dim K_{1\delta}=\dim K_{2\delta}. \tag{2.21}$$

(b) There exists a constant $\beta_{i\delta}>0$ $(i=1,2)$ such that

$$\forall q_\delta\in M_{i\delta}\quad \sup_{v_\delta\in X_{i\delta}}\frac{b_{i\delta}(v_\delta,q_\delta)}{\|v_\delta\|_{X_i}\|q_\delta\|_{M_i}}\geqq\beta_{i\delta}>0. \tag{2.22$_i$}$$

Moreover, we denote by

$$\gamma_\delta=\sup_{u_\delta\in X_{2\delta},v_\delta\in X_{1\delta}}\frac{a_\delta(u_\delta,v_\delta)}{\|u_\delta\|_{X_2}\|v_\delta\|_{X_1}} \tag{2.23}$$

the norm of the form a_δ.

COROLLARY 2.2. *Assume that hypotheses* (2.19), (2.20), *and* (2.22)$_i$ $(i=1,2)$ *hold. Then problem* (2.17) *has a unique solution* (u_δ,p_δ). *Moreover the solution* (u_δ,p_δ) *satisfies the following estimates*:

$$\|u_\delta\|_{X_2}\leqq\alpha_{1\delta}^{-1}\|f_\delta\|_{X_1'}+\beta_{2\delta}^{-1}(1+\alpha_{1\delta}^{-1}\gamma_\delta)\|g_\delta\|_{M_2'}, \tag{2.24}$$

$$\|p_\delta\|_{M_1}\leqq\beta_{1\delta}^{-1}(1+\alpha_{1\delta}^{-1}\gamma_\delta)\|f_\delta\|_{X_1'}+\beta_{1\delta}^{-1}\beta_{2\delta}^{-1}\gamma_\delta(1+\alpha_{1\delta}^{-1}\gamma_\delta)\|g_\delta\|_{M'_2}. \tag{2.25}$$

2.3. Error estimates. We assume now that a solution (u,p) to problem (2.1) and a solution (u_δ,p_δ) to problem (2.17) exist. We want to derive a general error estimate between them.

First, we estimate how an element of $K_i(g)$ can be approximated by an element of $K_{i\delta}(g_\delta)$, for any g and g_δ given in M_i'.

PROPOSITION 2.1. *Assume that hypothesis* (2.22)$_i$ *holds. For any element v in $K_i(g)$, the following estimate is satisfied*:

$$\begin{aligned}&\inf_{w_\delta\in K_{i\delta}(g_\delta)}\|v-w_\delta\|_{X_i}\\&\leqq c(1+\beta_{i\delta}^{-1})\left[\inf_{v_\delta\in X_{i\delta}}\left\{\|v-v_\delta\|_{X_i}+\sup_{q_\delta\in M_{i\delta}}\frac{(b_i-b_{i\delta})(v_\delta,q_\delta)}{\|q_\delta\|_{M_i}}\right\}+\sup_{q_\delta\in M_{i\delta}}\frac{\langle g-g_\delta,q_\delta\rangle}{\|q_\delta\|_{M_i}}\right].\end{aligned} \tag{2.26}$$

Proof. Let v_δ be any element in $X_{i\delta}$. By (2.22)$_i$ and Theorem 0.1 of [8], there exists z_δ in $X_{i\delta}$ such that

$$\forall q_\delta\in M_{i\delta}\quad b_{i\delta}(z_\delta,q_\delta)=b_{i\delta}(v_\delta,q_\delta)-\langle g_\delta,q_\delta\rangle, \tag{2.27}$$

$$\|z_\delta\|_{X_i}\leqq\beta_{i\delta}^{-1}\sup_{q_\delta\in M_{i\delta}}\frac{b_{i\delta}(z_\delta,q_\delta)}{\|q_\delta\|_{M_i}}. \tag{2.28}$$

Clearly, the element $w_\delta = v_\delta - z_\delta$ belongs to $K_{i\delta}(g_\delta)$ and satisfies

$$\|v - w_\delta\|_{X_i} \leqq \|v - v_\delta\|_{X_i} + \|z_\delta\|_{X_i}. \tag{2.29}$$

Thanks to (2.27), we write

$$\forall q_\delta \in M_{i\delta} \quad b_{i\delta}(z_\delta, q_\delta) = -b_i(v - v_\delta, q_\delta) - (b_i - b_{i\delta})(v_\delta, q_\delta) + \langle g - g_\delta, q_\delta\rangle;$$

hence, using (2.28) and the continuity of b_i, we obtain

$$\|z_\delta\|_{X_i} \leqq \beta_{i\delta}^{-1}\left\{c\|v - v_\delta\|_{X_i} + \sup_{q_\delta \in M_{i\delta}} \frac{(b_i - b_{i\delta})(v_\delta, q_\delta)}{\|q_\delta\|_{M_i}} + \sup_{q_\delta \in M_{i\delta}} \frac{\langle g - g_\delta, q_\delta\rangle}{\|q_\delta\|_{M_i}}\right\}.$$

The last inequality, together with (2.29), gives Proposition 2.1.

Next, we derive the main error estimate for $u - u_\delta$.

THEOREM 2.2. *Assume that hypothesis* (2.19) *holds. Then the solutions* (u, p) *of* (2.1) *and* (u_δ, p_δ) *of* (2.17), *satisfy the following estimate*:

$$\begin{aligned} \|u - u_\delta\|_{X_2} \leqq c(1 + \alpha_{1\delta}^{-1})\Bigg[&(1 + \gamma_\delta) \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u - w_\delta\|_{X_2} \\ &+ \inf_{v_\delta \in X_{2\delta}} \left\{(1 + \gamma_\delta)\|u - v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a - a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}}\right\} \\ &+ \inf_{q_\delta \in M_{1\delta}} \left\{\|p - q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1 - b_{1\delta})(z_\delta, q_\delta)}{\|z_\delta\|_{X_1}}\right\} + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f - f_\delta, z_\delta\rangle}{\|z_\delta\|_{X_1}}\Bigg]. \end{aligned} \tag{2.30}$$

Proof. Let w_δ be any element in $K_{2\delta}(g_\delta)$. By (2.19), we have

$$\|u_\delta - w_\delta\|_{X_2} \leqq \alpha_{1\delta}^{-1} \sup_{z_\delta \in K_{1\delta}} \frac{a_\delta(u_\delta - w_\delta, z_\delta)}{\|z_\delta\|_{X_1}}. \tag{2.31}$$

But, thanks to (2.1), (2.17), and (2.18), for any z_δ in $K_{1\delta}$, we can write

$$\begin{aligned} a_\delta(u_\delta - w_\delta, z_\delta) &= -a_\delta(w_\delta, z_\delta) + \langle f_\delta, z_\delta\rangle \\ &= a(u, z_\delta) - a_\delta(w_\delta, z_\delta) + b_1(z_\delta, p) - \langle f - f_\delta, z_\delta\rangle. \end{aligned}$$

Now, let v_δ be any element in $X_{2\delta}$ and q_δ be any element in $M_{1\delta}$. By (2.18), we obtain

$$a_\delta(u_\delta - w_\delta, z_\delta) = a(u, z_\delta) - a_\delta(w_\delta, z_\delta) + b_1(z_\delta, p - q_\delta) + (b_1 - b_{1\delta})(z_\delta, q_\delta) - \langle f - f_\delta, z_\delta\rangle;$$

hence,

$$\begin{aligned} a_\delta(u_\delta - w_\delta, z_\delta) = a(u - v_\delta, z_\delta) &+ a_\delta(v_\delta - w_\delta, z_\delta) + (a - a_\delta)(v_\delta, z_\delta) + b_1(z_\delta, p - q_\delta) \\ &+ (b_1 - b_{1\delta})(z_\delta, q_\delta) - \langle f - f_\delta, z_\delta\rangle. \end{aligned} \tag{2.32}$$

Using (2.23) and the continuity of a and b_1 in the last formula, we deduce

$$\begin{aligned} \|u_\delta - w_\delta\|_{X_2} \leqq \alpha_{1\delta}^{-1}\Bigg[&\gamma_\delta \|u - w_\delta\|_{X_2} + \left\{(\gamma + \gamma_\delta)\|u - v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a - a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}}\right\} \\ &+ \left\{c\|p - q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1 - b_{1\delta})(z_\delta, q_\delta)}{\|z_\delta\|_{X_1}}\right\} + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f - f_\delta, z_\delta\rangle}{\|z_\delta\|_{X_1}}\Bigg], \end{aligned}$$

which gives the theorem. □

Now, we indicate a remarkable case in which estimate (2.30) can be simplified. The result follows easily from (2.32).

COROLLARY 2.3. *Assume that hypothesis* (2.19) *holds and that*

$$K_{1\delta} \subset K_1. \tag{2.33}$$

Then the solutions (u, p) *of* (2.1) *and* (u_δ, p_δ) *of* (2.17), *satisfy the following estimate*:

$$\begin{aligned} \|u-u_\delta\|_{X_2} \leqq c(1+\alpha_{1\delta}^{-1})\Bigg[(1+\gamma_\delta) \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \\ + \inf_{v_\delta \in X_{2\delta}} \left\{ (1+\gamma_\delta)\|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}} \right\} \\ + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta, z_\delta \rangle}{\|z_\delta\|_{X_1}} \Bigg]. \end{aligned} \tag{2.34}$$

Finally, an error estimate for $p-p_\delta$ is derived in a standard way [4], [8], [16].

THEOREM 2.3. *Assume that hypotheses* (2.19) *and* $(2.22)_1$ *hold. Then the solutions* (u, p) *of* (2.1) *and* (u_δ, p_δ) *of* (2.17), *satisfy the following estimate*:

$$\begin{aligned} \|p-p_\delta\|_{M_1} \leqq c(1+\beta_{1\delta}^{-1})(1+\alpha_{1\delta}^{-1})(1+\gamma_\delta)\Bigg[(1+\gamma_\delta) \inf_{w_\delta \in K_{2\delta}(g_\delta)} \|u-w_\delta\|_{X_2} \\ + \inf_{v_\delta \in X_{2\delta}} \left\{ (1+\gamma_\delta)\|u-v_\delta\|_{X_2} + \sup_{z_\delta \in X_{1\delta}} \frac{(a-a_\delta)(v_\delta, z_\delta)}{\|z_\delta\|_{X_1}} \right\} \\ + \inf_{q_\delta \in M_{1\delta}} \left\{ \|p-q_\delta\|_{M_1} + \sup_{z_\delta \in X_{1\delta}} \frac{(b_1-b_{1\delta})(z_\delta, q_\delta)}{\|z_\delta\|_{X_1}} \right\} + \sup_{z_\delta \in X_{1\delta}} \frac{\langle f-f_\delta, z_\delta \rangle}{\|z_\delta\|_{X_1}} \Bigg]. \end{aligned} \tag{2.35}$$

3. Weighted variational formulation of the Stokes problem. We are interested in solving the Stokes problem in the domain $\Omega =]-1, 1[^2$; thus, given a force field $\mathbf{f}$ in Ω and a viscosity $\nu > 0$, we look for a velocity field $\mathbf{u}$ and a pressure p (defined up to an additive constant) such that the following equations are satisfied:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \operatorname{grad} p = \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \end{aligned} \tag{3.1}$$

together with a Dirichlet boundary condition on the boundary $\partial\Omega$ of Ω. We will first study the homogeneous case

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.2}$$

Since we want to analyze a spectral Chebyshev approximation of the Stokes problem, we first give a variational formulation of (3.1), (3.2) in terms of weighted Sobolev spaces, the weight being precisely the Chebyshev one in each variable. Then we use Theorem 2.1 to prove the well posedness of the variational problem. Finally, we extend the results to the case of nonhomogeneous boundary conditions.

3.1. The weighted spaces and the homogeneous Stokes problem. Let us briefly recall some basic material about weighted spaces of Chebyshev type (for further details, see, e.g., [10], [20]). If $\rho(\zeta) = (1-\zeta^2)^{-1/2}$ denotes the Chebyshev weight on the interval $]-1, 1[$, let

$$L_\rho^2(-1, 1) = \left\{ \varphi :]-1, 1[\to \mathbb{R}; \int_\Omega \varphi^2(\zeta)\rho(\zeta)\, d\zeta < +\infty \right\}$$

be the Lebesgue space associated with the measure $\rho(\zeta)\,d\zeta$, provided with the inner product

$$(\varphi, \psi)_\rho = \int_\Omega \varphi(\zeta)\psi(\zeta)\rho(\zeta)\,d\zeta \tag{3.3}$$

and the norm $\|\cdot\|_{0,\rho} = (\cdot,\cdot)_\rho^{1/2}$.

A scale of weighted Sobolev spaces is defined as follows: for any integer $m \geqq 0$, $H_\rho^m(-1,1)$ is the subspace of $L_\rho^2(-1,1)$ of the functions such that their distributional derivatives of order $\leqq m$ belong to $L_\rho^2(-1,1)$; it is a Hilbert space for the inner product associated with the norm

$$\|\varphi\|_{m,\rho} = \left(\sum_{k=0}^{m} |\varphi|_{k,\rho}^2\right)^{1/2}, \tag{3.4}$$

where

$$|\varphi|_{k,\rho} = \|d^k\varphi/d\zeta^k\|_{0,\rho}. \tag{3.5}$$

For a real number $s = m + \alpha$, $0 < \alpha < 1$, we define $H_\rho^s(-1,1)$ as the interpolation space between $H_\rho^m(-1,1)$ and $H_\rho^{m+1}(-1,1)$ of index α (cf. [18]); we denote its norm by $\|\cdot\|_{s,\rho}$.

Finally, we can apply a rotation and a translation to define similar Sobolev spaces on any segment of length 2 in $\mathbb{R}^2$. We use the same notation as before to indicate them, as well as their norms.

The generic point in the square Ω will be denoted by $\mathbf{x} = (x, y)$. We introduce the vertices $\mathbf{a}_J$, $J \in \mathbb{Z}/4\mathbb{Z}$, of $\bar{\Omega}$ (where $\mathbf{a}_{J+1}$ follows $\mathbf{a}_J$ counterclockwise), and call Γ_J the edge with vertices $\mathbf{a}_{J-1}$ and $\mathbf{a}_J$; for any edge Γ_J, $J \in \mathbb{Z}/4\mathbb{Z}$, $\mathbf{n}_J$ is the unit outward normal to Ω on Γ_J and $\boldsymbol{\tau}_J$ the unit vector orthogonal to $\mathbf{n}_J$, directed counterclockwise (see Fig. 3.1).

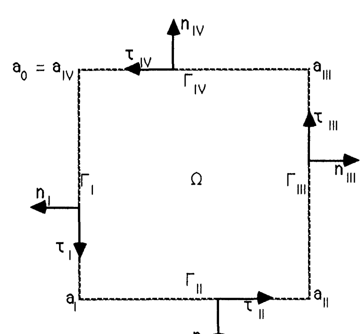

FIG. 3.1. *The square* Ω.

The Chebyshev weight on Ω is defined as $\omega(\mathbf{x}) = \rho(x)\rho(y)$. Let

$$L_\omega^2(\Omega) = \left\{ v : \Omega \to \mathbb{R};\ \int_\Omega \varphi^2(\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x} < +\infty \right\}$$

be the Lebesgue space associated with the measure $\omega(\mathbf{x})\,d\mathbf{x}$, provided with the inner product

$$(\varphi, \psi)_\omega = \int_\Omega \varphi(\mathbf{x})\psi(\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x} \tag{3.6}$$

and the norm $\|\cdot\|_{0,\omega} = (\cdot,\cdot)_\omega^{1/2}$.

Next, a scale of weighted Sobolev spaces is defined as follows: for any integer $m \geqq 0$, $H^m_\omega(\Omega)$ is the subspace of $L^2_\omega(\Omega)$ of the functions such that their distributional derivatives of order $\leqq m$ belong to $L^2_\omega(\Omega)$; it is a Hilbert space for the inner product associated with the norm

$$\|\varphi\|_{m,\omega} = \left(\sum_{k=0}^{m} |\varphi|^2_{k,\omega} \right)^{1/2}, \tag{3.7}$$

where

$$|\varphi|_{k,\omega} = \left(\sum_{j=0}^{k} \|\partial^k \varphi / \partial x^j \partial y^{k-j}\|^2_{0,\omega} \right)^{1/2}. \tag{3.8}$$

For a real number $s = m + \alpha$, $0 < \alpha < 1$, we define $H^s_\omega(\Omega)$ as the interpolation space between $H^m_\omega(\Omega)$ and $H^{m+1}_\omega(\Omega)$ of index α; we denote its norm by $\|\cdot\|_{s,\omega}$.

Being concerned with homogeneous Dirichlet boundary conditions, for any integer $m \geqq 1$, we consider the closed subspace of the functions of $H^m_\omega(\Omega)$ which vanish on the boundary $\partial\Omega$ together with all their derivatives of order up to $m-1$ (the traces being defined in the sense of [18]); this space, denoted by $H^m_{\omega,0}(\Omega)$, is the closure of $\mathcal{D}(\Omega)$ under the norm of $H^m_\omega(\Omega)$ (see [5, Prop. II.9]). Due to the Poincaré inequality, an equivalent norm on $H^m_{\omega,0}(\Omega)$ is the seminorm $|\cdot|_{m,\omega}$. The dual space of $H^m_{\omega,0}(\Omega)$ will be denoted by $H^{-m}_\omega(\Omega)$; whenever the space $L^2_\omega(\Omega)$ is identified to its dual space, we have, for instance,

$$H^{-1}_\omega(\Omega) = \{f + \partial g/\partial x + \partial h/\partial y, (f, g, h) \in [L^2_\omega(\Omega)]^3\}. \tag{3.9}$$

Now, we go back to the Stokes problem (3.1), (3.2). Assume that $\mathbf{f}$ belongs to $[H^{-1}_\omega(\Omega)]^2$. A weighted variational formulation of (3.1) is obtained by requiring that the first equation in (3.1) is satisfied in $[H^{-1}_\omega(\Omega)]^2$, and the second equation in $L^2_\omega(\Omega)$. In order to satisfy (3.2) also, we define the space

$$X = X_1 = X_2 = [H^1_{\omega,0}(\Omega)]^2, \tag{3.10}$$

in which we look for the velocity. Since the pressure p is defined up to an additive constant and $\operatorname{div} \mathbf{u}$ has zero average in Ω, we introduce the closed subspaces

$$M_1 = \left\{ q \in L^2_\omega(\Omega);\ \int_\Omega q(\mathbf{x})\omega(\mathbf{x})\, d\mathbf{x} = 0 \right\} \tag{3.11}$$

and

$$M_2 = \left\{ q \in L^2_\omega(\Omega);\ \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0 \right\}, \tag{3.12}$$

in which, respectively, we look for the pressure, and we choose the test functions to enforce the divergence-free condition; this choice will be justified later.

Thus, for any $\mathbf{f}$ in X', we consider the following variational formulation of the Stokes problem (3.1), (3.2). Find $(\mathbf{u}, p)$ in $X \times M_1$ such that

$$\begin{aligned} &\forall \mathbf{v} \in X \quad \nu \int_\Omega \operatorname{grad} \mathbf{u} \cdot \operatorname{grad}(\mathbf{v}\omega)\, d\mathbf{x} - \int_\Omega \operatorname{div}(\mathbf{v}\omega) p\, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v} \rangle, \\ &\forall q \in M_2 \quad \int_\Omega \operatorname{div} \mathbf{u}\, q\omega\, d\mathbf{x} = 0. \end{aligned} \tag{3.13}$$

This formulation will turn out to be a particular case of the abstract variational system (2.1), provided we define the bilinear forms $a: X \times X \to \mathbb{R}$ and $b_i: X \times M_i \to \mathbb{R}$ $(i=1,2)$, respectively, by

$$a(\mathbf{u}, \mathbf{v}) = -\nu\langle \Delta \mathbf{u}, \mathbf{v}\rangle = \nu \int_\Omega \operatorname{grad} \mathbf{u} \cdot \operatorname{grad}(\mathbf{v}\omega)\, d\mathbf{x}, \tag{3.14}$$

$$b_1(\mathbf{v}, q) = \langle \mathbf{v}, \operatorname{grad} q\rangle = -\int_\Omega \operatorname{div}(\mathbf{v}\omega) q\, d\mathbf{x}, \tag{3.15}$$

$$b_2(\mathbf{v}, q) = -(\operatorname{div} \mathbf{v}, q)_\omega = -\int_\Omega \operatorname{div} \mathbf{v}\, q\omega\, d\mathbf{x}. \tag{3.16}$$

Note that we have

$$K_1 = \{\mathbf{v} \in X;\ \operatorname{div}(\mathbf{v}\omega) = 0 \text{ in } \Omega\} \tag{3.17}$$

and

$$K_2 = \{\mathbf{v} \in X;\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}. \tag{3.18}$$

We have at once the following result.

PROPOSITION 3.1. *The forms $a: X \times X \to \mathbb{R}$ and $b_i: X \times M_i \to \mathbb{R}$ $(i=1,2)$ are continuous.*

Proof. It is a straightforward consequence of the continuity of the mapping: $\varphi \to \omega^{-1} \operatorname{grad}(\varphi\omega)$ from $H^1_{\omega,0}(\Omega)$ into $L^2_\omega(\Omega)$. In order to prove this result, we write

$$\omega^{-1} \operatorname{grad}(\varphi\omega) = \operatorname{grad} \varphi + \varphi \begin{bmatrix} x/(1-x^2) \\ y/(1-y^2) \end{bmatrix}$$

and, due to Hardy's inequality [24, Chap. 6, Lemma 2.1], the terms $\varphi x/(1-x^2)$ and $\varphi y/(1-y^2)$ can be bounded by the norm of φ in $H^1_{\omega,0}(\Omega)$ (see [12, Lemma 1.2] for details). $\square$

3.2. The inf-sup condition for a. Let us first deal with the form a. For some technical reasons, we introduce the weighted Sobolev spaces relative to the inverse of the Chebyshev weight: for any real number $s \geqq 0$, the spaces $H^s_{1/\rho}(-1,1)$ are defined in the same way as the spaces $H^s_\rho(-1,1)$ with ρ replaced by $1/\rho$ and provided with the norm $\|\cdot\|_{s,1/\rho}$. The spaces $H^s_{1/\omega}(\Omega)$ are defined in the same way as the $H^s_\omega(\Omega)$ with ω replaced by $1/\omega$ and provided with norm $\|\cdot\|_{s,1/\omega}$; for any integer $m \geqq 0$, we denote by $H^m_{1/\omega,0}(\Omega)$ the closure of $\mathscr{D}(\Omega)$ under the norm $\|\cdot\|_{m,1/\omega}$. We recall the following result [5].

LEMMA 3.1. *For any integer $m \geqq 0$, the mapping: $\varphi \to \omega^{1/2}\varphi$ is an isomorphism from $H^m_{\omega,0}(\Omega)$ onto $H^m_0(\Omega)$ and from $H^m_0(\Omega)$ onto $H^m_{1/\omega,0}(\Omega)$.*

Now, we are in a position to prove the following proposition.

PROPOSITION 3.2. *There exists a constant $\alpha > 0$ such that*

$$\forall \mathbf{u} \in K_2 \quad \sup_{\mathbf{v} \in K_1} \frac{a(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_{1,\omega}} \geqq \alpha \|\mathbf{u}\|_{1,\omega}, \tag{3.19}$$

$$\forall \mathbf{v} \in K_1 \quad \sup_{\mathbf{u} \in K_2} \frac{a(\mathbf{u}, \mathbf{v})}{\|\mathbf{u}\|_{1,\omega}} \geqq \alpha \|\mathbf{v}\|_{1,\omega}. \tag{3.20}$$

Proof. Let us start with (3.19). If $\mathbf{u}$ belongs to K_2, then there exists φ in $H^2_0(\Omega)$ such that $\mathbf{u} = \operatorname{curl} \varphi$ (see, e.g., [16, Chap. 1, Thm. 3.1]); since $\mathbf{u}$ is in X, φ belongs in fact to $H^2_{\omega,0}(\Omega)$. Define $\mathbf{v}=\omega^{-1}\operatorname{curl}(\varphi\omega)$. By Lemma 3.1, $\varphi\omega$ belongs to $H^2_{1/\omega,0}(\Omega)$; hence $\mathbf{v}$ belongs to $H^1_{\omega,0}(\Omega)$, with

$$\|\mathbf{v}\|_{1,\omega}\leqq c\|\mathbf{u}\|_{1,\omega}.$$

Moreover,

$$a(\mathbf{u},\mathbf{v})=\nu\int_\Omega(\operatorname{curl}\mathbf{u})(\operatorname{curl}(\mathbf{v}\omega))\,d\mathbf{x}=\nu\int_\Omega\Delta\varphi\Delta(\varphi\omega)\,d\mathbf{x}.$$

According to [21, Lemma 3.2], there exists a constant $c>0$ such that

$$\int_\Omega\Delta\varphi\Delta(\varphi\omega)\,d\mathbf{x}\geqq c\|\varphi\|^2_{2,\omega}.$$

We conclude that (3.19) holds.

In order to check (3.20), we use a similar argument. If $\mathbf{v}$ belongs to K_2, then there exists ψ in $H^2_{1/\omega,0}(\Omega)$ such that $\mathbf{v}\omega=\operatorname{curl}\psi$. We set $\mathbf{u}=\operatorname{curl}(\psi\omega^{-1})$, so that $\mathbf{u}$ belongs to K_1, and we conclude as before. □

3.3. The inf-sup condition for b_i $(i=1,2)$. Next, we check the inf-sup condition $(2.13)_i$ for the forms b_i $(i=1,2)$. To this end, we use some trace and regularity results in weighted Sobolev spaces, due to [5].

PROPOSITION 3.3. *There exist constants $\beta_i>0$ $(i=1,2)$ such that*

$$(3.21)\qquad \forall q\in M_i\quad \sup_{\mathbf{v}\in X}\frac{b_i(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}}\geqq\beta_i\|q\|_{0,\omega}.$$

Proof. We begin with the case $i=2$. Let q be any function in M_2. Due to Theorem III.1 of [5], the solution φ of

$$-\Delta\varphi=q\quad\text{in }\Omega,$$
$$\varphi=0\quad\text{on }\partial\Omega$$

satisfies

$$(3.22)\qquad \|\varphi\|_{2,\omega}\leqq c\|q\|_{0,\omega}.$$

We want to find a function ψ in $H^2_\omega(\Omega)$ such that

$$(3.23)\qquad \forall J\in\mathbb{Z}/4\mathbb{Z},\quad \frac{\partial\psi}{\partial\tau_J}=-\frac{\partial\varphi}{\partial n_J}\quad\text{and}\quad\frac{\partial\psi}{\partial n_J}=\frac{\partial\varphi}{\partial\tau_J}=0\quad\text{on }\Gamma_J.$$

To this end, we set

$$(3.24)\qquad \psi_I(\mathbf{a}_I)=0,$$

and, for $J=$ II, III, IV, I successively,

$$(3.25)\qquad \forall\mathbf{x}\in\Gamma_J\quad \psi_J(\mathbf{x})=\psi_{J-1}(\mathbf{a}_{J-1})-\int_{\Gamma_J}\left(\frac{\partial\varphi}{\partial n_J}\right)\chi([\mathbf{a}_{J-1},\mathbf{x}])\,d\sigma,$$

where $\chi([\mathbf{a}_{J-1},\mathbf{x}])$ is the characteristic function of $[\mathbf{a}_{J-1},\mathbf{x}]$ in Γ_J. By Theorem II.4 of [5], $\partial\varphi/\partial n_J$ belongs to $H^{3/4}_\rho(\Gamma_J)$ and satisfies

$$(3.26)\qquad \forall J\in\mathbb{Z}/4\mathbb{Z}\quad \begin{aligned}&(\partial\varphi/\partial n_{J+1})(\mathbf{a}_J)=(\partial\varphi/\partial\tau_J)(\mathbf{a}_J)=0,\\&(\partial\varphi/\partial n_J)(\mathbf{a}_J)=-(\partial\varphi/\partial\tau_{J+1})(\mathbf{a}_J)=0.\end{aligned}$$

Hence, ψ_J is in $L^2_\rho(\Gamma_J)$ and $\partial\psi_J/\partial\tau_J = -\partial\varphi/\partial n_J$ is in $H^{3/4}_\rho(\Gamma_J)$, so that ψ_J belongs to $H^{7/4}_\rho(\Gamma_J)$ (see [5, Lemma II.4]). Moreover, since q is in M_2, we have

$$\sum_{J\in\mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} \left(\frac{\partial\varphi}{\partial n_J}\right) d\sigma = \int_\Omega \Delta\varphi \, d\mathbf{x} = -\int_\Omega q \, d\mathbf{x} = 0;$$

hence, the functions ψ_J satisfy

$$\forall J \in \mathbb{Z}/4\mathbb{Z} \quad \psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J).$$

We also have by (3.26)

$$\forall J \in \mathbb{Z}/4\mathbb{Z} \quad (\partial\psi_J/\partial\tau_J)(\mathbf{a}_{J-1}) = (\partial\psi_J/\partial\tau_J)(\mathbf{a}_J) = 0.$$

Thanks to Theorem II.4 of [5], there exists a function ψ in $H^2_\omega(\Omega)$ such that

$$\psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = 0 \quad \text{on } \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z},$$

and which satisfies

$$\|\psi\|_{2,\omega} \leqq c \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\psi_J\|_{7/4,\rho} \leqq c\|\varphi\|_{2,\omega}. \tag{3.27}$$

Finally, we set $\mathbf{v} = \operatorname{grad}\varphi + \operatorname{curl}\psi$. Thanks to (3.23) and (3.27), the function $\mathbf{v}$ belongs to X and satisfies

$$\|\mathbf{v}\|_{1,\omega} \leqq c\|q\|_{0,\omega}. \tag{3.28}$$

Moreover, we have $\operatorname{div}\mathbf{v} = q$, so that $b_2(\mathbf{v}, q) = \int_\Omega q^2\omega(\mathbf{x})\, d\mathbf{x}$. This formula, together with (3.28), implies (3.21) in the case $i = 2$.

Next, we consider the case $i = 1$. Given any function q in M_1, denote by φ the solution of the Dirichlet problem

$$-\Delta\varphi = q\omega \quad \text{in } \Omega,$$

$$\varphi = 0 \quad \text{on } \partial\Omega.$$

Since $q\omega$ is in $L^2_{1/\omega}(\Omega)$, Theorem III.1 of [5] implies that φ belongs to $H^2_{1/\omega}(\Omega)$, with

$$\|\varphi\|_{2,1/\omega} \leqq c\|q\|_{0,\omega}. \tag{3.29}$$

Next, we build up a function ψ in $H^2_{1/\omega}(\Omega)$ such that again (3.23) holds. First, we define its trace on $\partial\Omega$ by (3.24) and (3.25). By Theorem II.2 of [5], $\partial\varphi/\partial n_J$ belongs to $H^{1/4}_{1/\rho}(\Gamma_J) \subset L^1(\Gamma_J)$; thus ψ_J is well defined and belongs to $H^{5/4}_{1/\rho}(\Gamma_J)$. Moreover, since q is in M_1, ψ_J satisfies

$$\forall J \in \mathbb{Z}/4\mathbb{Z} \quad \psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J).$$

Applying again Theorem II.2 of [5], we see that there exists a function ψ in $H^2_{1/\omega}(\Omega)$ satisfying

$$\psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = 0 \quad \text{on } \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z}$$

and such that

$$\|\psi\|_{2,1/\omega} \leqq c \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\psi_J\|_{5/4,1/\rho} \leqq c\|\varphi\|_{2,1/\omega}. \tag{3.30}$$

The function $\tilde{\mathbf{v}} = \operatorname{grad}\varphi - \operatorname{curl}\psi$ belongs to X and satisfies

$$\|\tilde{\mathbf{v}}\|_{1,1/\omega} \leqq c\|q\|_{0,\omega}. \tag{3.31}$$

Finally, we set $\mathbf{v} = \tilde{\mathbf{v}}\omega^{-1}$. By Lemma 3.1,

$$\|\mathbf{v}\|_{1,\omega} \leqq c\|\tilde{\mathbf{v}}\|_{1,1/\omega},$$

while on the other hand div $(\mathbf{v}\omega)$ is equal to div $\tilde{\mathbf{v}} = -q\omega$, so that $b_1(\mathbf{v}, q) = \int_\Omega q^2\omega(x)\,dx$. This proves the proposition. □

Remark 3.1. The proof of Proposition 3.3 uses the conditions $\int_\Omega q(\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x} = 0$ and $\int_\Omega q(\mathbf{x})\,d\mathbf{x} = 0$ in (3.11) and (3.12) the definition of M_1 and M_2, respectively.

3.4. The existence and uniqueness results. Thanks to Propositions 3.2 and 3.3, we can apply Theorem 2.1 to the variational problem (3.13), and obtain the main result of this section.

THEOREM 3.1. *For each* $\mathbf{f}$ *in* X', *there exists a unique variational solution* $(\mathbf{u}, p)$ *in* $X \times M_1$ *to the Stokes problem* (3.1), (3.2). *Moreover, the following inequality is satisfied for a constant* $c > 0$:

$$\|\mathbf{u}\|_{1,\omega} + \|p\|_{0,\omega} \leqq c\|\mathbf{f}\|_{X'}. \tag{3.32}$$

Remark 3.2. In the Stokes problem (3.1), (3.2), the pressure is defined up to an additive constant. In the variational problem (3.13), this constant is fixed by the nonstandard condition

$$\int_\Omega p(\mathbf{x})\omega(\mathbf{x})\,d\mathbf{x} = 0. \tag{3.33}$$

We conclude this section with the case of nonhomogeneous Dirichlet boundary conditions, which we write in the form

$$\mathbf{u} = \boldsymbol{\varphi}_J \quad \text{on } \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z}. \tag{3.34}$$

THEOREM 3.2. *For each* $\mathbf{f}$ *in* X' *and for each* $(\boldsymbol{\varphi}_J)_{J\in\mathbb{Z}/4\mathbb{Z}}$ *in* $\Pi_{J\in\mathbb{Z}/4\mathbb{Z}} H^{3/4}_\rho(\Gamma_J)$ *satisfying*

$$\forall J \in \mathbb{Z}/4\mathbb{Z}, \quad \boldsymbol{\varphi}_J(\mathbf{a}_J) = \boldsymbol{\varphi}_{J+1}(\mathbf{a}_J), \tag{3.35}$$

$$\sum_{J\in\mathbb{Z}/4\mathbb{Z}} \int_{\Gamma_J} \boldsymbol{\varphi}_J, \qquad \mathbf{n}_J\,d\sigma = 0, \tag{3.36}$$

there exists a unique variational solution $(\mathbf{u}, p)$ *in* $[H^1_\omega(\Omega)]^2 \times M_1$ *to the Stokes problem* (3.1), (3.34). *Moreover, the following inequality is satisfied for a constant* $c > 0$

$$\|\mathbf{u}\|_{1,\omega} + \|p\|_{0,\omega} \leqq c\left(\|\mathbf{f}\|_{X'} + \sum_{J\in\mathbb{Z}/4\mathbb{Z}} \|\boldsymbol{\varphi}_J\|_{3/4,\rho}\right). \tag{3.37}$$

Proof. In order to apply Theorem 3.1, we look for a function $\mathbf{u}_b$ in $H^1_\omega(\Omega)$ such that

$$\operatorname{div} \mathbf{u}_b = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u}_b = \boldsymbol{\varphi}_J \quad \text{on } \Gamma_J, \quad J \in \mathbb{Z}/4\mathbb{Z}, \tag{3.38}$$

or equivalently, set $\mathbf{u}_b = \operatorname{curl} \psi$, for a function ψ in $H^2_\omega(\Omega)$ such that

$$\forall J \in \mathbb{Z}/4\mathbb{Z} \quad \frac{\partial\psi}{\partial\tau_J} = \boldsymbol{\varphi}_J \cdot \mathbf{n}_J \quad \text{and} \quad \frac{\partial\psi}{\partial n_J} = -\boldsymbol{\varphi}_J \cdot \boldsymbol{\tau}_J \quad \text{on } \Gamma_J. \tag{3.39}$$

To this end, we set

$$\psi_I(\mathbf{a}_I) = 0, \tag{3.40}$$

and, for $J = $ II, III, IV, I successively,

$$\forall \mathbf{x} \in \Gamma_J, \quad \psi_J(\mathbf{x}) = \psi_{J-1}(\mathbf{a}_{J-1}) + \int_{\Gamma_J} (\boldsymbol{\varphi}_J \cdot \mathbf{n}_J)\chi([\mathbf{a}_{J-1}, \mathbf{x}])\,d\sigma, \tag{3.41}$$

where $\chi([\mathbf{a}_{J-1}, \mathbf{x}])$ is the characteristic function of $[\mathbf{a}_{J-1}, \mathbf{x}]$ in Γ_J. For $J \in \mathbb{Z}/4\mathbb{Z}$, the pair $(\psi_J, -\boldsymbol{\varphi}_J \cdot \boldsymbol{\tau}_J)$ belongs to $H^{7/4}_\rho(\Gamma_J) \times H^{3/4}_\rho(\Gamma_J)$ and, due to (3.35) and (3.36), we have

$$\begin{aligned} &\psi_J(\mathbf{a}_J) = \psi_{J+1}(\mathbf{a}_J),\\ \forall J \in \mathbb{Z}/4\mathbb{Z}, \quad &(\partial\psi_J/\partial\tau_J)(\mathbf{a}_J) = -(\boldsymbol{\varphi}_{J+1} \cdot \mathbf{n}_{J+1})(\mathbf{a}_J),\\ &(\boldsymbol{\varphi}_J \cdot \mathbf{n}_J)(\mathbf{a}_J) = (\partial\psi_{J+1}/\partial\tau_{J+1})(\mathbf{a}_J). \end{aligned}$$

It follows from Theorem II.4 of [5] that there exists ψ in $H^2_\omega(\Omega)$ satisfying

$$\forall J \in \mathbb{Z}/4\mathbb{Z} \quad \psi = \psi_J \quad \text{and} \quad \partial\psi/\partial n_J = -\boldsymbol{\varphi}_J \cdot \boldsymbol{\tau}_J \quad \text{on } \Gamma_J,$$

whence (3.39); moreover, we have

$$\|\psi\|_{2,\omega} \leqq c \sum_{J \in \mathbb{Z}/4\mathbb{Z}} \|\varphi_J\|_{3/4,\rho}.$$

Next, we define $\mathbf{u}_b = \operatorname{curl} \psi$ and, using Theorem 3.1, we consider the unique solution $(\tilde{\mathbf{u}}, p)$ of problem (3.13) with $\langle \mathbf{f}, \mathbf{v} \rangle$ replaced by $\langle \mathbf{f}, \mathbf{v} \rangle - a(\mathbf{u}_b, \mathbf{v})$. Clearly, the pair $(\mathbf{u} = \tilde{\mathbf{u}} + \mathbf{u}_b, p)$ is the solution of the Stokes problem (3.1), (3.34) and satisfies (3.37).

4. A Galerkin method for the Stokes problem. Henceforth, we fix an integer $N \geqslant 3$. In this section, we will study the convergence properties of a spectral Galerkin approximation to problem (3.13). The main difficulties of the subsequent analysis of the collocation method are already present here. Our study will be based on the abstract framework of § 2.

4.1. The discrete problem. We look for an approximate solution of (3.13), the components of which are algebraic polynomials of degree $\leqq N$ in each variable. This solution will be defined by a Galerkin method using the variational formulation of § 3. A suitable orthogonal basis in order to study the algorithm consists of the Chebyshev polynomials of the first kind.

Let us begin by recalling some basic material about Chebyshev expansions. We denote by $T_m(x)$, $m = 0, 1, \cdots$, the Chebyshev polynomials of the first kind. They are defined by $T_m(x) = \cos(m \operatorname{Arcos} x)$, and satisfy the orthogonality relation

$$\text{(4.1)} \qquad \int_{-1}^{1} T_m(\zeta) T_n(\zeta)(1-\zeta^2)^{-1/2}\, d\zeta = c_m(\pi/2)\delta_{mn}, \qquad m \geqq 0, \quad n \geqq 0,$$

where c_0 is equal to 2 and c_m is equal to 1 for $m \geqq 1$, and δ_{mn} denotes the Kronecker symbol. Moreover, for any $m \geqq 1$, the Chebyshev polynomials satisfy the differential equation

$$\text{(4.2)} \qquad \forall \zeta \in]-1, 1[\quad ((1-\zeta^2)^{1/2} T'_m(\zeta))' + m^2 T_m(\zeta)(1-\zeta^2)^{-1/2} = 0$$

and the relations

$$\text{(4.3)} \qquad \forall \zeta \in]-1, 1[\quad T_{m+1}(\zeta) = 2\zeta T_m(\zeta) - T_{m-1}(\zeta),$$

$$\text{(4.4)} \qquad \forall \zeta \in]-1, 1[\quad T_m(\zeta) = T'_{m+1}(\zeta)/2(m+1) - T'_{m-1}(\zeta)/2(m-1).$$

Let $P_N(-1, 1)$ denote the space of the algebraic polynomials of degree $\leqq N$ in one variable, restricted to the interval $]-1, 1[$. Each φ in $P_N(-1, 1)$ can be expanded as $\varphi(\zeta) = \sum_{m=0}^{N} \hat{\varphi}_m T_m(\zeta)$, with

$$\text{(4.5)} \qquad \hat{\varphi}_m = (2/\pi c_m) \int_{-1}^{1} \varphi(\zeta) T_m(\zeta)(1-\zeta^2)^{-1/2}\, d\zeta, \qquad 0 \leqq m \leqq N.$$

$P^\circ_N(-1, 1)$ will be the subspace $P_N(-1, 1) \cap H^1_0(-1, 1)$ of the polynomials vanishing at the end points $\zeta = \pm 1$.

Next, we denote by $P_N(\Omega)=[P_N(-1,1)]^{\otimes 2}$ the space of the algebraic polynomials in $\mathbb{R}^2$ which are of degree $\leqq N$ in each variable. Finally, we set $P_N^\circ(\Omega)=P_N(\Omega)\cap H_0^1(\Omega)$.

Let us now introduce the Galerkin approximation to problem (3.13). We look for the approximate velocity $\mathbf{u}^N$ in the space $X_N=[P_N^\circ(\Omega)]^2$ and for the approximate pressure p^N in $P_N(\Omega)$. We consider the following problem. Find $(\mathbf{u}^N, p^N)$ in $X_N\times P_N(\Omega)$ such that

$$\begin{aligned}&\forall \mathbf{v}\in X_N\quad \nu\int_\Omega \operatorname{grad}\mathbf{u}^N\cdot\operatorname{grad}(\mathbf{v}\omega)\,d\mathbf{x}-\int_\Omega \operatorname{div}(\mathbf{v}\omega)p^N\,d\mathbf{x}=\langle\mathbf{f},\mathbf{v}\rangle,\\ &\forall q\in P_N(\Omega)\quad \int_\Omega \operatorname{div}\mathbf{u}^N q\omega\,d\mathbf{x}=0.\end{aligned}\tag{4.6}$$

However, as it will be discussed in the next section, in order to have a well posed discrete problem, we must restrict the space of the pressures to a proper subspace M_{1N} of $P_N(\Omega)$. Similarly we restrict the space of test functions for the divergence-free condition to a proper subspace M_{2N} of $P_N(\Omega)$. Then, we obtain a particular case of the abstract approximate problem (2.17), if we set $X_{1\delta}=X_{2\delta}=X_N$ provided with the norm of X, $M_{1\delta}=M_{1N}$ and $M_{2\delta}=M_{2N}$ provided with the norm of $L^2_\omega(\Omega)$, and if the forms a_δ and $b_{i\sigma}$ $(i=1,2)$ are, respectively, the forms a and b_i $(i=1,2)$ defined in (3.14) to (3.16).

In order to apply the abstract convergence results of § 2, we have to choose the finite-dimensional spaces M_{1N} and M_{2N} for the pressure, in such a way that the inf-sup conditions for the above forms hold. In the next section, we characterize those pressures which cannot satisfy such a condition for the forms b_1 and b_2.

4.2. The spurious modes of the pressure. Spurious or "parasitic" modes of the pressure are those components of the numerical pressure that are not controlled by the discrete equations which approximate the Stokes system. Parasitic modes in finite difference or finite element methods have long been investigated (see, e.g., [16]). Y. Morchoisne [23] first brought attention to this problem in spectral methods. The characterization of spurious modes for various spectral methods of mixed Fourier-Legendre or fully Legendre type has been carried out in [6], [7].

Our aim is to characterize the polynomials q in $P_N(\Omega)$ which satisfy, respectively, for $i=1$ or 2, the following condition:

$$\forall \mathbf{v}\in X_N\quad b_i(\mathbf{v},q)=0.\tag{4.7$_i$}$$

Let Z_{iN} denote the subspace of all q in $P_N(\Omega)$ for which $(4.7)_i$ holds. It is clear that such polynomials cannot verify $(2.13)_i$. Moreover, if M_{1N} contains a nontrivial element of Z_{1N}, the pressure in the solution of the discrete problem is not unique.

Let us first deal with the form b_1 defined by (3.15). We recall that the polynomials T'_m, $m=1,\cdots,N+1$, form a basis of $P_N(-1,1)$ which, due to (4.1) and (4.2), is orthogonal with respect to the inner product

$$(u,v)_{1/\rho}=\int_{-1}^1 u(\zeta)v(\zeta)(1-\zeta^2)^{1/2}\,d\zeta.\tag{4.8}$$

Precisely, we have

$$\int_{-1}^1 T'_m(\zeta)T'_n(\zeta)(1-\zeta^2)^{1/2}\,d\zeta=\left(\frac{\pi}{2}\right)m^2\delta_{mn},\qquad m\geqq 0,\quad n\geqq 0.\tag{4.9}$$

LEMMA 4.1. *The dimension of the subspace Z_{1N} is equal to* 8.

Proof. Let us expand each q in Z_{1N} as $q(\mathbf{x})=\sum_{m,n=0}^{N}\hat{q}_{mn}T_m(x)T_n(y)$. Condition $(4.7)_1$ is equivalent to the conditions

$$\forall v\in P_N^\circ(\Omega)\quad \int_\Omega\left(\frac{\partial(v\omega)}{\partial x}\right)q\,d\mathbf{x}=0 \tag{4.10}$$

and

$$\forall v\in P_N^\circ(\Omega)\quad \int_\Omega\left(\frac{\partial(v\omega)}{\partial y}\right)q\,d\mathbf{x}=0. \tag{4.11}$$

Let us consider first (4.10). A basis in $P_N^\circ(\Omega)$ is given by the polynomials

$$v_{mn}(\mathbf{x})=(1-x^2)T'_m(x)[T_n(y)-T_{\alpha(n)}(y)],\qquad 1\leqq m\leqq N-1,\quad 2\leqq n\leqq N, \tag{4.12}$$

where $\alpha(n)$ is equal to 0 if n is even and to 1 if n is odd (we use here the fact that $T_m(\pm1)=(\pm1)^m$). From equation (4.2) and the orthogonality condition (4.1), it is clear that (4.10) is equivalent to the set of relations

$$\hat{q}_{mn}-c_{\alpha(n)}\hat{q}_{m\alpha(n)}=0,\quad 1\leqq m\leqq N-1,\quad 2\leqq n\leqq N. \tag{4.13}$$

Working out condition (4.11) in a perfectly symmetric way, we end up with another set of relations, namely

$$\hat{q}_{mn}-c_{\alpha(m)}\hat{q}_{\alpha(m)n}=0,\quad 2\leqq m\leqq N,\quad 1\leqq n\leqq N-1. \tag{4.14}$$

The relations (4.13) and (4.14) provide an orthogonal basis for Z_{1N}. This is given by the four modes $T_0(x)T_0(y)$, $T_N(x)T_0(y)$, $T_0(x)T_N(y)$ and $T_N(x)T_N(y)$ (since the corresponding coefficients do not appear in (4.13) and (4.14)), and by four other polynomials, the nonzero Chebyshev coefficients of which are, respectively, $\hat{q}_{01}$, $\hat{q}_{10}$, $\hat{q}_{11}$, and $\hat{q}_{02}$ plus the coefficients defined from these by the relations (4.13) and (4.14). This proves the lemma. □

The next lemma allows us to find another basis for Z_{1N}, which is easier to handle.

LEMMA 4.2. *The set of all elements in $P_N(-1,1)$ which are orthogonal with respect to the inner product $(\cdot,\cdot)_\rho$ to the space $P_N^\circ(-1,1)$ is the subspace of dimension* 2 *spanned by* $\{T'_N, T'_{N+1}\}$.

Proof. Each polynomial φ in $P_N(-1,1)$ satisfying $\varphi(\pm1)=0$ can be written as $\varphi(\zeta)=(1-\zeta^2)\tilde{\varphi}(\zeta)$, where $\tilde{\varphi}$ belongs to $P_{N-2}(-1,1)$. Expanding $\tilde{\varphi}$ according to the basis $\{T'_m\}_{1\leqq m\leqq N-1}$ and using (4.9), we see that $\tilde{\varphi}$ is orthogonal to T'_N and T'_{N+1}. On the other hand, the subspace $P_N^\circ(-1,1)=\{\varphi\in P_N(-1,1);\ \varphi(\pm1)=0\}$ has codimension 2; hence the lemma is proved. □

COROLLARY 4.1. *The set of all elements in $P_N(\Omega)$ which are orthogonal with respect to the inner product $(\cdot,\cdot)_\omega$ to the space $\{v\in P_N(\Omega);\ v(\pm1,\pm1)=0\}$ is the subspace of dimension* 4 *spanned by* $\{T'_N, T'_{N+1}\}^{\otimes 2}$.

Proof. Each polynomial v in $P_N(\Omega)$ satisfying $v(\pm1,\pm1)=0$ can be written as $v=w+z$, with

$$\forall(x,y)\in\bar{\Omega}\quad w(\pm1,y)=z(x,\pm1)=0$$

(take for instance $w(x,y)=((1+y)/2)v(x,1)+((1-y)/2)v(x,-1)$ and $z=v-w$). Hence, it follows from Lemma 4.2 that $\{T'_N, T'_{N+1}\}^{\otimes 2}$ is orthogonal to $\{v\in P_N(\Omega);\ v(\pm1,\pm1)=0\}$. On the other hand, since this space has codimension 4, the corollary is proved. □

Noting that, if $\mathbf{v}$ belongs to X_N, then $\omega^{-1}\operatorname{div}(\mathbf{v}\omega)$ belongs to $P_N(\Omega)$ and is equal to 0 in $(\pm1,\pm1)$, and recalling the proof of Lemma 4.1, we obtain the following characterization.

PROPOSITION 4.1. *The subset* Z_{1N} *is the vector space of dimension* 8 *spanned by* $\{T_0, T_N\}^{\otimes 2}$ *and by* $\{T'_N, T'_{N+1}\}^{\otimes 2}$.

Now we consider the form b_2 defined by (3.16). From Corollary V.I of [7], denoting by L_m, $m=0, 1, \cdots$, the Legendre polynomials, we know the following lemma.

LEMMA 4.3. *The range of* X_N *by the divergence operator is the subspace* D_N *of* $P_N(\Omega)$ *of codimension* 8 *defined by*

$$D_N=\left\{r\in P_N(\Omega);\ r(\pm 1, \pm 1)=0 \quad \text{and} \quad \forall q\in\{L_0, L_N\}^{\otimes 2},\ \int_\Omega r(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}=0\right\}. \tag{4.15}$$

We can now characterize the space Z_{2N}. Let us introduce the polynomial q_N in $P_N(-1, 1)$ defined by

$$\forall \varphi\in P_N(-1, 1)\quad \int_{-1}^{1} q_n(\zeta)\varphi(\zeta)\rho(\zeta)\,d\zeta=\int_{-1}^{1}\varphi(\zeta)\,d\zeta, \tag{4.16}$$

i.e., q_N is the orthogonal projection of $1/\rho$ onto $P_N(-1, 1)$ for the scalar product $(\cdot, \cdot)_\rho$.

PROPOSITION 4.2. *The subset* Z_{2N} *is the vector space of dimension* 8 *spanned by* $\{q_N, T_N\}^{\otimes 2}$ *and by* $\{T'_N, T'_{N+1}\}^{\otimes 2}$.

Proof. The subspace Z_{2N} is precisely the orthogonal space to D_N in $P_N(\Omega)$ with respect to the inner product $(\cdot, \cdot)_\omega$. It follows that Z_{2N} has dimension 8. Next, from (4.15) and Lemma 4.2, we deduce that Z_{2N} contains the subspace spanned by $\{T'_N, T'_{N+1}\}^{\otimes 2}$. In order to obtain the remaining components, we have to translate the orthogonality relation in (4.16) in terms of the inner product $(\cdot, \cdot)_\omega$. To this purpose, we observe that, for any φ in $P_N(-1, 1)$, if λ is the coefficient of ζ^N in φ, we have [15, § 1.13]

$$\int_{-1}^{1}\varphi(\zeta)L_N(\zeta)\,d\zeta=\lambda\int_{-1}^{1}\zeta^N L_N(\zeta)\,d\zeta=\lambda 2^N(N!)^2/(2N)!(N+\tfrac{1}{2})$$

$$(\varphi, T_N)_\omega=\lambda(\zeta^N, T_N)_\omega=\lambda\pi 2^{-N},$$

so that

$$\forall\varphi\in P_N(-1, 1)\quad \int_{-1}^{1}\varphi(\zeta)L_N(\zeta)\,d\zeta=[2^N(N!)^2/(2N)!(N+\tfrac{1}{2})](\varphi, T_N)_\omega. \tag{4.17}$$

Using this property and the definition (4.16) of q_N, we deduce from (4.15) that Z_{2N} contains the subspace spanned by $\{q_N, T_N\}^{\otimes 2}$.

Finally, let us check that the polynomials T'_N, T'_{N+1}, q_N, and T_N are linearly independent (which implies that the eight elements of $\{q_N, T_N\}^{\otimes 2}\cup\{T'_N, T'_{N+1}\}^{\otimes 2}$ are linearly independent). Assume that

$$\lambda_1 q_N+\lambda_2 T_N+\mu_1 T'_N+\mu_2 T'_{N+1}=0.$$

Using (4.4), we have

$$\lambda_1 q_N=(\lambda_2/2(N-1))T'_{N-1}-\mu_1 T'_N-(\mu_2+\lambda_2/2(N+1))T'_{N+1}.$$

Since N is $\geqq 3$, $T'_1=1$ is orthogonal to T'_{N-1}, T'_N, and T'_{N+1} with respect to $(\cdot, \cdot)_{1/\rho}$, so that

$$0=\lambda_1\int_{-1}^{1} q_N(\zeta)(1-\zeta^2)^{1/2}\,d\zeta=\lambda_1\int_{-1}^{1}(1-\zeta^2)\,d\zeta=4\lambda_1/3,$$

whence $\lambda_1=0$. Since the T'_k's are mutually orthogonal with respect to $(\cdot, \cdot)_{1/\rho}$, we obtain $\lambda_2=\mu_1=\mu_2=0$, at once. □

Remark 4.1. In this work, we are not concerned with the Stokes problem in the cube $]-1,1[^3$. However, note that, in the three-dimensional case, the dimension of the space Z_{iN} $(i=1,2)$ depends on N: it is equal to $12N+4$. Indeed, by the same techniques as in [7, § 5], it is an easy matter to prove that Z_{1N} (respectively, Z_{2N}) is spanned by $\{T_0, T_N\}^{\otimes 3}$(respectively, $\{q_N, T_N\}^{\otimes 3}$) and by the $(12N-4)$ independent modes of

$$[\{T'_N, T'_{N+1}\}\otimes\{T'_N, T'_{N+1}\}\otimes P_N(-1,1)]\cup[\{T'_N, T'_{N+1}\}\otimes P_N(-1,1)\otimes\{T'_N, T'_{N+1}\}]$$
$$\cup[P_N(-1,1)\otimes\{T'_N, T'_{N+1}\}\otimes\{T'_N, T'_{N+1}\}].$$

4.3. An inf-sup condition for the forms b_i $(i=1,2)$. The characterization carried out in the previous section suggests the most direct choice of the test and trial spaces M_{1N} and M_{2N} for the pressure. Precisely, we choose for M_{iN} the orthogonal complement to Z_{iN} in $P_N(\Omega)$ $(i=1,2)$, i.e.,

$$M_{iN}=\{q\in P_N(\Omega);\ \forall r\in Z_{iN}, (q,r)_\omega=0\}. \tag{4.18$_i$}$$

Due to $(4.7)_i$, M_{iN} is contained in M_i $(i=1,2)$ and characterized as

$$M_{1N}=\{\omega^{-1}\operatorname{div}(\mathbf{v}\omega), \mathbf{v}\in X_N\} \tag{4.19}$$

and

$$M_{2N}=\{\operatorname{div}\mathbf{v}, \mathbf{v}\in X_N\}. \tag{4.20}$$

In order to check the inf-sup conditions $(2.22)_i$ for b_i over the space $X_N\times M_{iN}$ $(i=1,2)$, we apply an abstract result of tensor algebra due to [3, Chap. V, Appendix B], which we recall here. Let H be a Hilbert space, and denote by $(\cdot,\cdot)$ its inner product and by $\|\cdot\|$ the corresponding norm. For any pair of planes A and B in H, we define the gap between A and B as the quantity

$$\delta(A,B)=\inf\{\|\varphi-\psi\|;\ \varphi\in A, \psi\in B \text{ and } \|\varphi\|=\|\psi\|=1\}. \tag{4.21}$$

Setting

$$\eta(A,B)=\sup\{(\varphi,\psi);\ \varphi\in A, \psi\in B \text{ and } \|\varphi\|=\|\psi\|=1\}, \tag{4.22}$$

we have

$$\delta^2(A,B)=2(1-\eta(A,B)). \tag{4.23}$$

LEMMA 4.4. *Let (A,B) be a pair of planes in H, the intersection of which is $\{0\}$. Any element q in $H^{\otimes 2}$ which belongs to $(A^{\otimes 2})^\perp$ and to $(B^{\otimes 2})^\perp$ can be written*

$$q=r+s, \tag{4.24}$$

where r and s belong to $A^\perp\otimes B^\perp$ and $B^\perp\otimes A^\perp$, respectively, and satisfy

$$\sup\{\|r\|,\|s\|\}\leqq(1+2/\delta(A,B))^2\|q\|. \tag{4.25}$$

PROPOSITION 4.3. *Let the space M_{1N} be defined by $(4.18)_1$. There exists a constant $\tilde\beta_1>0$ independent of N such that*

$$\forall q\in M_{1N}\quad \sup_{\mathbf{v}\in X_N}\frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}}\geqq\tilde\beta_1 N^{-2}\|q\|_{0,\omega}. \tag{4.26}$$

Proof. In the space $H=P_N(-1,1)$ provided with the inner product $(\cdot,\cdot)_\rho$, define $A=\{T_0, T_N\}$ and $B=\{T'_N, T'_{N+1}\}$, and recall that, due to Lemma 4.2, $B^\perp$ is the space $(1-\zeta^2)P_{N-2}(-1,1)$. According to Proposition 4.1, M_{1N} is equal to $(A^{\otimes 2})^\perp\cap(B^{\otimes 2})^\perp$.

Since N is $\geqq 3$, $A \cap B$ is $\{0\}$; hence, by the previous lemma, each q in M_{1N} can be written as in (4.24), with

$$r(x, y) = (1 - y^2) \sum_{m=1}^{N-1} a_m(y) T_m(x), \qquad a_m \in P_{N-2}(-1, 1), \tag{4.27}$$

and

$$s(x, y) = (1 - x^2) \sum_{n=1}^{N-1} b_n(x) T_n(y), \qquad b_n \in P_{N-2}(-1, 1). \tag{4.28}$$

Using (4.2), we write

$$\begin{aligned} r\omega &= (1 - y^2)^{1/2} \sum_{m=1}^{N-1} a_m(y) T_m(x) (1 - x^2)^{-1/2} \\ &= (\partial/\partial x) \left[(1 - x^2)^{1/2} (1 - y^2)^{1/2} \sum_{m=1}^{N-1} a_m(y) T'_m(x)/m^2 \right] = \partial(v\omega)/\partial x, \end{aligned}$$

where v, defined by

$$v(\mathbf{x}) = -(1 - x^2)(1 - y^2) \sum_{m=1}^{N-1} a_m(y) T'_m(x)/m^2,$$

belongs to $P_N^\circ(\Omega)$. Moreover, we have

$$\|\partial v/\partial x\|_{0,\omega} \leqq c \|\partial(v\omega)/\partial x\|_{0,1/\omega} = c \|r\omega\|_{0,1/\omega} \leqq c' \|r\|_{0,\omega},$$

and, by the Poincaré inequality, $\|v\|_{0,\omega} \leqq c \|r\|_{0,\omega}$. Using the following inverse inequality [11, Lemma 2.1], which is valid for any real numbers r and s, $0 \leqq r \leqq s$,

$$\forall \varphi \in P_N(\Omega) \quad \|\varphi\|_{s,\omega} \leqq c N^{2(s-r)} \|\varphi\|_{r,\omega}, \tag{4.29}$$

we can control the norm $\|\cdot\|_{0,\omega}$ of $\partial v/\partial y$ by

$$\|\partial v/\partial y\|_{0,\omega} \leqq c N^2 \|r\|_{0,\omega}.$$

Working out in a symmetric way for the component s of q, we end up with an element $\mathbf{v} = (v, w)$ of X_N satisfying

$$\omega^{-1} \operatorname{div}(\mathbf{v}\omega) = -q \tag{4.30}$$

(so that $b_1(\mathbf{v}, q) = \|q\|_{0,\omega}^2$) and

$$\|\mathbf{v}\|_{1,\omega} \leqq c N^2 (\|r\|_{0,\omega} + \|s\|_{0,\omega}). \tag{4.31}$$

It remains to estimate the gap (4.21) between A and B. To this end, set $\varphi = \lambda_1 T_0 + \lambda_2 T_N$ in A and $\psi = \mu_1 T'_N + \mu_2 T'_{N+1}$ in B. From (4.1) and (4.9), we see that

$$\|\varphi\|_{0,\rho}^2 = \pi(\lambda_1^2 + \lambda_2^2/2) \quad \text{and} \quad \|\psi\|_{0,\rho}^2 = \pi(N^3 \mu_1^2 + (N+1)^3 \mu_2^2). \tag{4.32}$$

Moreover, we have

$$(T_0, T'_m)_\rho = 0 \quad \text{if } m \text{ is even and } (T_0, T'_m)_\rho = \pi m \quad \text{if } m \text{ is odd,}$$

$$(T_N, T'_N)_\rho = 0 \quad \text{and} \quad (T_N, T'_{N+1})_\rho = \pi(N+1),$$

so that

$$|(\varphi, \psi)_\rho| \leqq \pi(|\lambda_1|\,|\mu_1| N + |\lambda_1|\,|\mu_2|(N+1) + |\lambda_2|\,|\mu_2|(N+1)). \tag{4.33}$$

From (4.32) and (4.33), we conclude that

$$\forall \varphi \in A, \quad \forall \psi \in B, \quad |(\varphi, \psi)_\rho| \leqq c N^{-1/2} \|\varphi\|_{0,\rho} \|\psi\|_{0,\rho}. \tag{4.34}$$

Since for $N \geqq 3$ the polynomials T_0, T_N, T'_N, and T'_{N+1} are linearly independent, $\delta(A, B)$ is greater than 0. Then, due to (4.22), (4.23), and (4.34), it is bounded from below independently of N. Hence, the proposition follows from (4.30), (4.31), and (4.25). □

PROPOSITION 4.4. *Let the space M_{2N} be defined by* $(4.18)_2$. *There exists a constant $\tilde{\beta}_2 > 0$ independent of N such that*

$$\forall q \in M_{2N} \quad \sup_{\mathbf{v} \in X_N} \frac{b_2(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\omega}} \geqq \tilde{\beta}_2 N^{-2} \|q\|_{0,\omega}. \tag{4.35}$$

Proof. In the space $H = P_N(-1, 1)$ provided with the inner product $(\cdot, \cdot)_\rho$, we set now $A = \{q_N, T_N\}$ and $B = \{T'_N, T'_{N+1}\}$; due to the definition (4.16) of q_N, $A^\perp$ is the subspace of $P_{N-1}(-1, 1)$ of the polynomials having zero average on $(-1, 1)$. From Proposition 4.2, we see that M_{2N} is equal to $(A^{\otimes 2})^\perp \cap (B^{\otimes 2})^\perp$. Thus, by Lemma 4.4, each q in M_{2N} can be expanded as in (4.24) with

$$r(x, y) = (1 - y^2) \sum_{m=1}^{N-1} a_m(y) L_m(x), \qquad a_m \in P_{N-2}(-1, 1), \tag{4.36}$$

and

$$s(x, y) = (1 - x^2) \sum_{n=1}^{N-1} b_n(x) L_n(y), \qquad b_n \in P_{N-2}(-1, 1) \tag{4.37}$$

(here, we find it is more appropriate to use a Legendre expansion for r and s). Following the proof of Proposition 4.3, but using the differential equation satisfied by the Legendre polynomials (see also [7, § V]), we define an element $\mathbf{v} = (v, w)$ in X_N such that

$$\operatorname{div} \mathbf{v} = -q \tag{4.38}$$

(so that $b_2(\mathbf{v}, q) = \|q\|^2_{0,\omega}$) and

$$\|\mathbf{v}\|_{1,\omega} \leqq cN^2 (\|r\|_{0,\omega} + \|s\|_{0,\omega}). \tag{4.39}$$

In order to estimate the gap between A and B, let $\varphi = \lambda_1 q_N + \lambda_2 T_N$ and $\psi = \mu_1 T'_N + \mu_2 T'_{N+1}$ be arbitrary elements in A and B, respectively. By (4.16) and (4.4) we have

$$(q_N, T_m)_\rho = \int_{-1}^{1} T_m(\zeta)\, d\zeta = [T_{m+1}/2(m+1) - T_{m-1}/2(m-1)]_{-1}^{1},$$

so that

$$(q_N, T_m)_\rho = 0 \quad \text{if } m \text{ is odd} \quad \text{and} \quad (q_N, T_m)_\rho = 2/(1 - m^2) \quad \text{if } m \text{ is even}.$$

Thus $\|\varphi\|^2_{0,\omega} \geqq c(\lambda_1^2 + \lambda_2^2)$. Moreover,

$$(q_N, T'_m)_\rho = \int_{-1}^{1} T'_m(\zeta)\, d\zeta = 0 \quad \text{if } m \text{ is even} \quad \text{and} \quad (q_N, T'_m)_\rho = 2 \quad \text{if } m \text{ is odd}.$$

As in the proof of Proposition 4.3, we have again (4.34); hence $\delta(A, B)$ is bounded from below independently of N. Thus the proposition is proved. □

Remark 4.2. Unfortunately, the constants involved in the inf-sup conditions (4.26) and (4.35) are not independent of N. However, we will see that this does not corrupt the accuracy of the approximation of the velocity in the homogeneous case.

4.4. The inf-sup condition for the form *a*. Let us denote here by K_{iN} $(i=1,2)$ the discrete kernels

$$K_{iN}=\{\mathbf{v}\in X_N;\ \forall q\in M_{iN},\ b_i(\mathbf{v},q)=0\}. \tag{4.40}$$

According to $(4.7)_i$ and $(4.18)_i$, we actually have

$$K_{1N}=\{\mathbf{v}\in X_N\,;\ \operatorname{div}(\mathbf{v}\omega)=0 \text{ in } \Omega\},$$

$$K_{2N}=\{\mathbf{v}\in X_N\,;\ \operatorname{div}\mathbf{v}=0 \text{ in } \Omega\},$$

or equivalently

$$K_{iN}=K_i\cap X_N \qquad (i=1,2), \tag{4.41}$$

where continuous kernels K_i are defined in (3.17) and (3.18). Moreover, by Propositions 4.1 and 4.2, we have

$$\dim K_{1N}=\dim K_{2N}. \tag{4.42}$$

Thus, by Remark 2.1, it is enough to check the inf-sup condition (2.19) for the form a.

PROPOSITION 4.5. *There exists a constant $\tilde\alpha>0$ independent of N such that*

$$\forall\mathbf{u}\in K_{2N}\quad \sup_{\mathbf{v}\in K_{1N}}\frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{v}\|_{1,\omega}}\geqq\tilde\alpha\|\mathbf{u}\|_{1,\omega}. \tag{4.43}$$

Proof. Let $\mathbf{u}=(u,v)$ be an element of K_{2N}. By (4.41), there exists φ in $H^2_{\omega,0}(\Omega)$ such that $\mathbf{u}=\operatorname{curl}\varphi$. Since

$$\varphi(x,y)=-\int_{-1}^{x}v(\xi,y)\,d\xi=\int_{-1}^{y}u(x,\eta)\,d\eta,$$

φ necessarily belongs to $P_N(\Omega)$ (see also [26]). Thus the element $\mathbf{v}=\omega^{-1}\operatorname{curl}(\varphi\omega)$ used in the proof of Proposition 3.2 to check the continuous inf-sup condition is indeed an element of $K_1\cap X_N=K_{1N}$, whence the result.

4.5. A convergence estimate. Above we have proved that the abstract assumptions (2.19), (2.21), and $(2.22)_i$ $(i=1,2)$ are fulfilled with the present choice of spaces for the discrete velocity and pressure. Applying Corollary 2.2, we derive the following result.

THEOREM 4.1. *For each integer $N\geqq 3$, the Galerkin approximation* (4.6) *to the Stokes problem* (3.1), (3.2) *has a unique solution $(\mathbf{u}^N,p^N)$ in $X_N\times M_{1N}$, where M_{1N} is defined by* $(4.18)_i$. *Moreover, the following inequality is satisfied*:

$$\|\mathbf{u}^N\|_{1,\omega}+N^{-2}\|p^N\|_{0,\omega}\leqq c\|\mathbf{f}\|_{X'} \tag{4.44}$$

for a constant $c>0$ independent of N.

We can obtain a convergence estimate for the velocity using the abstract error estimate (2.34). To this end, note that by (4.41), the inclusion (2.33) holds. Thus we get

$$\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\leqq c\inf_{\mathbf{v}_N\in K_{2N}}\|\mathbf{u}-\mathbf{v}_N\|_{1,\omega}. \tag{4.45}$$

We now recall a general result of polynomial approximation theory in the Chebyshev norms. This result is due to [20], except for the case $m=0$ (where $H^0_{\omega,0}(\Omega)$ is the space $L^2_\omega(\Omega)$ and Π^0_N is the truncation of the Chebyshev series) for which we refer to [11].

LEMMA 4.5. *For each integer $m\geqq 0$, there exists a projection operator Π^m_N: $H^m_{\omega,0}(\Omega)\to P_N(\Omega)\cap H^m_{\omega,0}(\Omega)$ such that, for $0\leqq r\leqq m\leqq s$,*

$$\forall\varphi\in H^s_{\omega,0}(\Omega)\quad \|\varphi-\Pi^m_N\varphi\|_{r,\omega}\leqq c(r,m,s)N^{r-s}\|\varphi\|_{s,\omega}. \tag{4.46}$$

Using the previous lemma we can show that divergence-free vector fields can be approximated by divergence-free polynomial fields with an optimal error estimate in the weighted Sobolev scale.

LEMMA 4.6. *For each* $\mathbf{v}$ *in* K_2, *there exists an element* $Q_N\mathbf{v}$ *of* X_N *satisfying* $\operatorname{div}(Q_N\mathbf{v})=0$ *in* Ω *such that, if* $\mathbf{v}$ *belongs to* $H^s_\omega(\Omega)$ *for a real number* $s\geqq 1$,

$$\|\mathbf{v}-Q_N\mathbf{v}\|_{1,\omega}\leqq cN^{1-s}\|\mathbf{v}\|_{s,\omega}. \tag{4.47}$$

Proof. A general proof, which applies to vector fields in $\mathbb{R}^3$ as well, has been given in [27]. For the convenience of the reader here we give a simpler proof, which, however, holds in the two-dimensional case only. Each $\mathbf{v}$ satisfying the assumptions of the lemma can be written $\mathbf{v}=\operatorname{curl}\varphi$, with φ in $H^2_{\omega,0}(\Omega)\cap H^{s+1}_\omega(\Omega)$. Define φ_N in $P_N(\Omega)\cap H^2_{\omega,0}(\Omega)$ as $\varphi_N=\Pi^2_N\varphi$ and set $Q_N\mathbf{v}=\operatorname{curl}\varphi_N$. Then (4.47) is a direct consequence of (4.46). □

Using (4.47), we derive from (4.45) the following convergence result.

THEOREM 4.2. *Assume that the solution* $(\mathbf{u}, p)$ *of the Stokes problem* (3.1), (3.2) *belongs to* $[H^s_\omega(\Omega)]^2\times H^{s-1}_\omega(\Omega)$ *for a real number* $s\geqq 1$. *Then the approximate velocity* $\mathbf{u}^N$, *as defined in Theorem* 4.1, *satisfies the convergence estimate*

$$\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\leqq cN^{1-s}\|\mathbf{u}\|_{s,\omega} \tag{4.48}$$

for a constant $c>0$ *independent of* N.

4.6. Computation of the pressure. Unfortunately, it is not possible to establish a similar optimal result for the pressure, if the discrete space of pressures is defined by $(4.18)_i$. As a matter of fact, it follows from Proposition 4.1 and Corollary 4.1 that the elements of M_{1N} are polynomials which vanish at the four corners of the domain, while the exact pressure does not need to satisfy this condition. Thus, spectral accuracy in the pressure cannot be achieved.

The remedy consists of choosing as discrete space of pressures another supplementary space to Z_{1N} in the space $P_N(\Omega)$, which exhibits better approximation properties to the functions of M_1 and, at the same time, which fulfills an inf-sup condition asymptotically not worse than (4.26).

From a general point of view, the new space $\tilde{M}_{1N}$ of discrete pressures can be defined by introducing a new subspace $\tilde{Z}_{1N}$ of dimension 8 (which will be a suitable perturbation of Z_{1N}) and then setting

$$\tilde{M}_{1N}=\{q\in P_N(\Omega);\ \forall r\in\tilde{Z}_{1N}, (q,r)_\omega=0\}. \tag{4.49}$$

The space $\tilde{M}_{1N}$ will allow the exact pressure to be approximated at a spectral rate if there exists a real number λ, $0<\lambda<1$, such that

$$P_{[\lambda N]}(\Omega)\cap M_1\subset\tilde{M}_{1N}. \tag{4.50}$$

($[\lambda N]$ is the integral part of λN.) This means that the elements of $\tilde{Z}_{1N}$ should be orthogonal to $P_{[\lambda N]}(\Omega)\cap M_1$.

On the other hand, as far as the inf-sup condition for b_{1N} is concerned, let $\pi_N:\tilde{M}_{1N}\to M_{1N}$ denote the orthogonal projection onto M_{1N} with respect to the inner product $(\cdot,\cdot)_\omega$. We make the assumption that there exists a constant $c>0$ independent of N such that

$$\forall q\in\tilde{M}_{1N}\quad \|q\|_{0,\omega}\leqq c\|\pi_N q\|_{0,\omega}. \tag{4.51}$$

PROPOSITION 4.6. *Let* $\tilde{M}_{1N}$ *be a subspace of* $P_N(\Omega)$ *such that hypothesis* (4.51) *holds. There exists a constant* $\tilde{\beta}_1>0$ *independent of* N *such that*

$$\forall q\in\tilde{M}_{1N}\quad \sup_{\mathbf{v}\in X_N}\frac{b_1(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}}\geqq\tilde{\beta}_1 N^{-2}\|q\|_{0,\omega}. \tag{4.52}$$

Proof. Due to the definition $(4.18)_1$ of M_{1N}, it follows that we have for all q in $\tilde{M}_{1N}$

$$\forall \mathbf{v} \in X_N \quad b_1(\mathbf{v}, q) = b_1(\mathbf{v}, \pi_N q), \tag{4.53}$$

from which we deduce

$$\sup_{\mathbf{v}\in X_N} \frac{b_1(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\omega}} \geqq \sup_{\mathbf{v}\in X_N} \frac{b_1(\mathbf{v}, \pi_N q)}{\|\mathbf{v}\|_{1,\omega}} \geqq \tilde{\beta}_1 N^{-2} \|\pi_N q\|_{0,\omega}.$$

The result follows by using (4.51).

Now let $\tilde{\mathbf{K}}_{1N}$ denote the kernel defined as in (4.40), with M_{1N} replaced by $\tilde{M}_{1N}$. If (4.51) holds, $\tilde{\mathbf{K}}_{1N}$ still satisfies (4.41), thanks to (4.53). Hence, both (4.42) and Proposition 4.5 are valid. Applying Corollary 2.2, we obtain the same result as in Theorem 4.1. □

THEOREM 4.3. *For each integer $N \geqq 3$, the Galerkin approximation* (4.6) *to the Stokes problem* (3.1), (3.2) *has a unique solution $(\mathbf{u}^N, p^N)$ in $X_N \times \tilde{M}_{1N}$, where $\tilde{M}_{1N}$ satisfies the hypothesis* (4.51). *Moreover, the following inequality is satisfied*:

$$\|\mathbf{u}^N\|_{1,\omega} + N^{-2}\|p^N\|_{0,\omega} \leqq c\|\mathbf{f}\|_{X'} \tag{4.54}$$

for a constant $c > 0$ independent of N.

Remark 4.3. It follows from (4.53) that, if $(\mathbf{u}^N, p^N)$ is the solution of (4.6) in $X_N \times \tilde{M}_{1N}$, then $(\mathbf{u}^N, \pi_N p^N)$ is the solution of (4.6) in $X_N \times M_{1N}$. In particular, the discrete velocity is independent of the choice of the space of pressures.

We can obtain a convergence estimate for both the velocity and the pressure.

THEOREM 4.4. *Assume that hypotheses* (4.50) *and* (4.51) *hold and that the solution $(\mathbf{u}, p)$ of the Stokes problem* (3.1), (3.2) *belongs to $[H^s_\omega(\Omega)]^2 \times H^{s-1}_\omega(\Omega)$ for a real number $s \geqq 1$. Then the approximate solution $(\mathbf{u}^N, p^N)$ in $X_N \times \tilde{M}_{1N}$, as defined in Theorem* 4.3, *satisfies the convergence estimates*

$$\|\mathbf{u} - \mathbf{u}^N\|_{1,\omega} \leqq cN^{1-s}\|\mathbf{u}\|_{s,\omega}, \tag{4.55}$$

$$\|p - p^N\|_{0,\omega} \leqq cN^{3-s}(\|\mathbf{u}\|_{s,\omega} + \|p\|_{s-1,\omega}) \tag{4.56}$$

for a constant $c > 0$ independent of N.

Proof. It is a straightforward consequence of (2.38) and (2.39), if we note that, due to (4.50),

$$\inf_{q_N \in \tilde{M}_{1N}} \|p - q_N\|_{0,\omega} \leqq \|p - \Pi^0_{[\lambda N]}p\|_{0,\omega} \leqq cN^{1-s}\|p\|_{s-1,\omega}.$$

Now, we give an example of choice of the space $\tilde{M}_{1N}$ (or, equivalently, of $\tilde{Z}_{1N}$) for which (4.50) and (4.51) hold. The argument is an adaptation of § V.3 of [7].

Let us recall that Z_{1N} is spanned by $\{T_0, T_N\}^{\otimes 2}$ and $\{T'_N, T'_{N+1}\}^{\otimes 2}$. By (4.4), the polynomials $s_N = T'_N$ and $t_N = T'_{N+1}$ admit the Chebyshev expansions

$$s_N = \sum_{m=0}^{N-1} \alpha_m T_m = 2N(T_{N-1} + T_{N-3} + \cdots + T_{N-(2j+1)} + \cdots), \tag{4.57}$$

$$t_N = \sum_{n=0}^{N} \beta_n T_n = 2(N+1)(T_N + T_{N-2} + \cdots + T_{N-2k} + \cdots). \tag{4.58}$$

Let us define the polynomials

$$\tilde{s}_N = \sum_{\lambda N < m < N} \alpha_m T_m = 2N(T_{N-1} + T_{N-3} + \cdots + T_{m_0}), \tag{4.59}$$

where m_0 is the smallest integer $> \lambda N$ for which α_{m_0} is $\neq 0$, and

$$\tilde{t}_N = \sum_{\lambda N < n < N} \beta_n T_n = 2(N+1)(T_{N-2} + T_{N-4} + \cdots + T_{n_0}), \tag{4.60}$$

where n_0 is the smallest integer $>\lambda N$ for which β_{n_0} is $\neq 0$. Finally, let us set

$$\tilde{Z}_{1N} = \text{Span}\,[\{T_0, T_N\}^{\otimes 2} \cup \{\tilde{s}_N, \tilde{t}_N\}^{\otimes 2}] \tag{4.61}$$

and define $\tilde{M}_{1N}$ by (4.49).

As in Proposition V.3 of [7], we can prove that this space $\tilde{M}_{1N}$ satisfies the hypotheses (4.50) and (4.51) (see also [4, Prop. IV.7]). Thus, we can apply our algorithm, and we do obtain the estimates (4.55) and (4.56).

5. A collocation method for the Stokes problem. In this section, we will study the convergence properties of a spectral collocation approximation to the Stokes problem (3.1). This method can be extended to the full Navier–Stokes equations (see § 6.2).

5.1. The discrete problem. We consider the simplest Chebyshev collocation approximation of the Stokes problem (3.1). This scheme uses a single grid for both the momentum and the continuity equation, the grid being given by the Cartesian product of the Gauss–Lobatto points in one space variable for the Chebyshev measure $\rho(\zeta)\,d\zeta$.

More precisely, for a fixed $N \geqq 3$, let us set for $0 \leqq j \leqq N$

$$\zeta_j = \cos(j\pi/N) \quad \text{and} \quad \rho_j = \pi/\bar{c}_j N, \tag{5.1}$$

(with $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_j = 1$ for $1 \leqq j \leqq N-1$). These are, respectively, the nodes and the weights of the Gauss–Lobatto integration formula for the Chebyshev weight, which is exact for polynomials of degree $\leqq 2N-1$, i.e.,

$$\forall \varphi \in P_{2N-1}(-1,1) \quad \int_{-1}^{1} \varphi(\zeta)\rho(\zeta)\,d\zeta = \sum_{j=0}^{N} \varphi(\zeta_j)\rho_j. \tag{5.2}$$

It will be useful in the sequel to recall that the internal quadrature nodes are the zeros of the polynomial T'_N, i.e.,

$$\forall j,\ 1 \leqq j \leqq N-1 \quad T'_N(\zeta_j) = 0. \tag{5.3}$$

We also recall that, as a consequence of (5.2), the bilinear form

$$\forall (\varphi, \psi) \in [\mathscr{C}^0([-1,1])]^2 \quad (\varphi, \psi)_{\rho,N} = \sum_{j=0}^{N} \varphi(\zeta_j)\psi(\zeta_j)\rho_j, \tag{5.4}$$

is a scalar product on $P_N(-1,1)$ and that the associated norm is uniformly equivalent to the norm $\|\cdot\|_{0,\rho}$ since we have [11, § 3]

$$\forall \varphi \in P_N(-1,1) \quad \|\varphi\|_{0,\rho} \leqq (\varphi, \varphi)_{\rho,N}^{1/2} \leqq \sqrt{2}\|\varphi\|_{0,\rho}. \tag{5.5}$$

Finally, for each function φ in $\mathscr{C}^0(\bar{\Omega})$, we shall denote by $\mathscr{I}_N\varphi$ the unique polynomial in $P_N(\Omega)$ which interpolates φ at the nodes defined in (5.1), i.e., such that

$$\forall j,\ 0 \leqq j \leqq N \quad \mathscr{I}_N\varphi(\zeta_j) = \varphi(\zeta_j). \tag{5.6}$$

Next we consider the Cartesian product of the points defined in (5.1) (see Fig. 5.1).

$$\Xi_N = \{\mathbf{x} = (\zeta_j, \zeta_k),\ 0 \leqq j, k \leqq N\}, \tag{5.7}$$

as well as the corresponding weights

$$\forall \mathbf{x} = (\zeta_j, \zeta_k) \in \Xi_N \quad \omega_{\mathbf{x}} = \rho_j \rho_k. \tag{5.8}$$

The bilinear form

$$\forall (\varphi, \psi) \in [\mathscr{C}^0(\bar{\Omega})]^2 \quad (\varphi, \psi)_{\omega,N} = \sum_{x \in \Xi_N} \varphi(\mathbf{x})\psi(\mathbf{x})\omega_x, \tag{5.9}$$

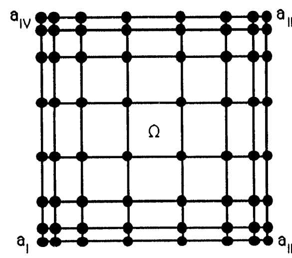

FIG. 5.1. *The set Ξ_N of collocation nodes* (*for* $N=7$).

is actually an inner product on $P_N(\Omega)$. The associated norm, which we denote by $\|\cdot\|_{\omega,N}$, is uniformly equivalent to the norm $\|\cdot\|_{0,\omega}$; more precisely we have [11, § 3]

$$\forall\varphi\in P_N(\Omega)\quad \|\varphi\|_{0,\omega}\leqq\|\varphi\|_{\omega,N}\leqq 2\|\varphi\|_{0,\omega}. \tag{5.10}$$

Finally, for each function f in $\mathscr{C}^0(\bar\Omega)$, we will denote by $\mathscr{I}_N\varphi$ the unique polynomial in $P_N(\Omega)$ which interpolates φ at the nodes defined in (5.7), i.e.,

$$\forall x\in\Xi_N\quad \mathscr{I}_N\varphi(x)=\varphi(x). \tag{5.11}$$

Note that

$$\forall\psi\in P_N(\Omega)\quad (\varphi-\mathscr{I}_N\varphi,\psi)_{\omega,N}=0. \tag{5.12}$$

Now let us introduce the collocation approximation to problem (3.1), in the homogeneous case (3.2). We look for the approximate velocity $\mathbf{u}^N$ in the space $X_N=[P_N^0(\Omega)]^2$ and for the approximate pressure p^N in $P_N(\Omega)$. We assume here that $\mathbf{f}$ belongs to $[\mathscr{C}^0(\bar\Omega)]^2$. We consider the following problem. Find $(\mathbf{u}^N, p^N)$ in $X_N\times P_N(\Omega)$ such that

$$\begin{aligned}&-\nu\Delta\mathbf{u}^N(\mathbf{x})+(\operatorname{grad}p^N)(\mathbf{x})=\mathbf{f}(\mathbf{x})\quad\text{for }\mathbf{x}\in\Xi_N\cap\Omega,\\&(\operatorname{div}\mathbf{u}^N)(\mathbf{x})=0\quad\text{for }\mathbf{x}\in\Xi_N.\end{aligned}\tag{5.13}$$

In order to discuss the well posedness of this problem, as well as its convergence properties, we now give a variational formulation of (5.13) which fits into the abstract scheme (2.17). Thus we introduce the following bilinear forms $a_N: X_N\times X_N\to\mathbb{R}$ and $b_{iN}: X_N\times P_N(\Omega)\to\mathbb{R}$ $(i=1,2)$ defined by

$$a_N(\mathbf{u},\mathbf{v})=-\nu(\Delta\mathbf{u},\mathbf{v})_{\omega,N}, \tag{5.14}$$

$$b_{1N}(\mathbf{v},q)=(\mathbf{v},\operatorname{grad}q)_{\omega,N}, \tag{5.15}$$

and

$$b_{2N}(\mathbf{v},q)=-(\operatorname{div}\mathbf{v},q)_{\omega,N}. \tag{5.16}$$

Let us note that if q belongs to $P_N(\Omega)$, grad q belongs to $[P_{N-1}(-1,1)\otimes P_N(-1,1)]\times[P_N(-1,1)\otimes P_{N-1}(-1,1)]$. Hence, from (5.2) we deduce that for any $\mathbf{v}=(v,w)$ in X_N

$$\begin{aligned}b_{1N}(\mathbf{v},q)=&\int_{-1}^{1}\rho(x)\,dx\sum_{k=0}^{N}\left(\frac{\partial q}{\partial x}\right)(x,\zeta_k)v(x,\zeta_k)\rho_k\\&+\sum_{j=0}^{N}\rho_j\int_{-1}^{1}\left(\frac{\partial q}{\partial y}\right)(\zeta_j,y)w(\zeta_j,y)\rho(y)\,dy.\end{aligned}$$

Let us integrate by parts the first term in the x-direction and the second one in the y-direction. Noting that $\omega^{-1}\operatorname{div}(\mathbf{v}\omega)$ belongs to $[P_{N-1}(-1,1)\otimes P_N(-1,1)]+[P_N(-1,1)\otimes P_{N-1}(-1,1)]$ and recalling (5.2) we obtain

$$b_{1N}(\mathbf{v}, q)=(\omega^{-1}\operatorname{div}(\mathbf{v}\omega), q)_{\omega,N}. \tag{5.17}$$

Using the same argument we also have

$$a_N(\mathbf{u},\mathbf{v})=\nu(\operatorname{grad}\mathbf{u}, \omega^{-1}\operatorname{grad}(\mathbf{v}\omega))_{\omega,N}. \tag{5.18}$$

Thus the bilinear forms a_N and b_{iN} $(i=1,2)$ are discrete approximations of the forms a and b_i $(i=1,2)$ defined in (3.14) to (3.16).

PROPOSITION 5.1. *Problem* (5.13) *is equivalent to the following variational one: Find* $(\mathbf{u}^N, p^N)$ *in* $X_N\times P_N(\Omega)$ *such that*

$$\begin{aligned}&\forall\mathbf{v}\in X_N\quad a_N(\mathbf{u}^N,\mathbf{v})+b_{1N}(\mathbf{v}, p^N)=(\mathbf{f},\mathbf{v})_{\omega,N},\\&\forall q\in P_N(\Omega)\quad b_{2N}(\mathbf{u}^N, q)=0.\end{aligned} \tag{5.19}$$

Proof. The first equation in (5.19) is obtained by taking the inner product in $\mathbb{R}^2$ of the first equation in (5.13) by $\mathbf{v}(\mathbf{x})\omega_x$, for $\mathbf{x}$ in $\Xi_N\cap\Omega$, and summing up over the points of Ξ_N (let us recall that, since $\mathbf{v}$ belongs to X_N, it is equal to 0 at any point of Ξ_N of the boundary of Ω). Similarly, the second equation in (5.19) is obtained by multiplying the second equation in (5.13) by $q(\mathbf{x})\omega_{\mathbf{x}}$ and summing up over all $\mathbf{x}$ in Ξ_N. Conversely (5.13) follows from (5.19) using as test functions the Lagrange basis in X_N and $P_N(\Omega)$ associated to the set of points Ξ_N. $\square$

In analogy to the Galerkin approximation, we expect that the spaces of trial and test pressures have to be restricted to proper subspaces M_{1N} and M_{2N} of $P_N(\Omega)$ so that the bilinear forms a_N and b_{iN} $(i=1,2)$ satisfy the inf-sup conditions (2.19), (2.20), and $(2.22)_i$. Then, we obtain another particular case of the abstract approximate problem (2.17), if we set, as before, $X_{1\delta}=X_{2\delta}=X_N$ provided with the norm of X, $M_{1\delta}=M_{1N}$ and $M_{2\delta}=M_{2N}$ provided with the norm $\|\cdot\|_{0,\omega}$, and if the forms a_δ and $b_{i\delta}$ $(i=1,2)$ are, respectively, the forms a_N and b_{iN} $(i=1,2)$ defined in (5.14), (5.15), and (5.16).

Hereafter we will characterize the spurious modes for the pressure, and we will indicate again one choice of the spaces M_{1N} and M_{2N} that leads to a "spectral" rate of convergence for both the velocity field and the pressure.

5.2. The spurious modes of the pressure. The characterization of the parasitic modes for the discrete pressure can be carried out by suitably modifying the proofs given in § 4.2 in the case of the Galerkin approximation. However, we prefer to follow a different strategy, which consists of transferring the results obtained therein into the context of a collocation approximation. To this end, let us define the operators $R_N: P_N(-1,1)\to P_N(-1,1)$ by

$$\forall\psi\in P_N(-1,1)\quad (R_N\varphi,\psi)_{\rho,N}=(\varphi,\psi)_\rho; \tag{5.20}$$

then $S_N: P_N(\Omega)\to P_N(\Omega)$ by

$$\forall\psi\in P_N(\Omega)\quad (S_N\varphi,\psi)_{\omega,N}=(\varphi,\psi)_\omega. \tag{5.21}$$

By (5.5) and (5.10), both R_N and S_N are isomorphisms, the norms of which can be bounded independently of N if we endow $P_N(-1,1)$ and $P_N(\Omega)$, respectively, with the norms $\|\cdot\|_{0,\rho}$ and $\|\cdot\|_{0,\omega}$.

Now, let Z_{iN} denote the subspace of all q in $P_N(\Omega)$ for which we have

$$\forall\mathbf{v}\in X_N\quad b_{iN}(\mathbf{v}, q)=0. \tag{5.22$_i$}$$

In order to characterize the spaces Z_{iN}, we introduce the polynomials r_0 and r_N of $P_N(-1, 1)$ satisfying

$$\forall j, 0 \leqq j \leqq N \quad r_0(\zeta_j) = \delta_{0j} \quad \text{and} \quad r_N(\zeta_j) = \delta_{Nj}, \tag{5.23}$$

and the polynomial

$$q_N^* = R_N q_N, \tag{5.24}$$

where q_N is defined in (4.16). It is an easy matter to check that q_N^* satisfies the relation

$$\forall \varphi \in P_N(-1, 1) \quad (q_N^*, \varphi)_{\rho,N} = \int_{-1}^{1} \varphi(\zeta)\, d\zeta. \tag{5.25}$$

PROPOSITION 5.2. *The subset Z_{1N} is the vector space of dimension* 8 *spanned by* $\{T_0, T_N\}^{\otimes 2}$ *and by* $\{r_0, r_N\}^{\otimes 2}$.

Proof. Let q in P_N satisfy $(5.22)_1$. Then $S_N^{-1} q$ is such that

$$\forall \mathbf{v} \in X_N \quad b_1(\mathbf{v}, S_N^{-1} q) = 0;$$

hence, by Proposition 4.1, q belongs to the space spanned by $\{R_N T_0, R_N T_N\}^{\otimes 2}$ and $\{R_N T'_N, R_N T'_{N+1}\}^{\otimes 2}$. By (5.2) we have $R_N T_0 = T_0$ and $R_N T'_N = T'_N$, while a direct computation shows that (see, e.g., [11, § 3])

$$(T_N, T_N)_{\rho,N} = 2(T_N, T_N)_\rho, \tag{5.26}$$

so that $R_N T_N = 2 T_N$. Let us check that $R_N T'_{N+1}(\zeta) = ((N+1)/N)\zeta T'_N(\zeta)$. Differentiating (4.3) and using (4.4), we obtain

$$T'_{N+1}(\zeta) = ((N+1)/N)(\zeta T'_N(\zeta) + N T_N(\zeta)). \tag{5.27}$$

Thus the relation

$$((N+1)/N)(\zeta T'_N, \psi)_{\rho,N} = (T'_{N+1}, \psi)_\rho$$

holds for all ψ in $P_{N-1}(-1, 1)$, as a consequence of (5.2) and (4.1). For $\psi = T_N$, the identity follows from (4.4), (5.2), (5.26), and (5.27).

Hence we have proven that $Z_{1N} = \text{Span}\,[\{T_0, T_N\}^{\otimes 2} \cup \{T'_N, \zeta T'_N\}^{\otimes 2}]$. Finally we easily derive from (4.2) and (5.3) that

$$r_0(\zeta) = (-1)^N((1+\zeta)/2N^2) T'_N(\zeta) \quad \text{and} \quad r_N(\zeta) = ((1-\zeta)/2N^2) T'_N(\zeta). \quad \square \tag{5.28}$$

Following the same lines we can prove the analogous result to Proposition 4.2.

PROPOSITION 5.3. *The subspace Z_{2N} is the vector space of dimension* 8 *spanned by* $\{q_N^*, T_N\}^{\otimes 2}$ *and by* $\{r_0, r_N\}^{\otimes 2}$ *(see Fig.* 5.2).

	$i = 1$	$i = 2$
Form b_i	$\{T_0, T_N\}^{\otimes 2} \cup \{T'_N, T'_{N+1}\}^{\otimes 2}$	$\{q_N, T_N\}^{\otimes 2} \cup \{T'_N, T'_{N+1}\}^{\otimes 2}$
Form b_{iN}	$\{T_0, T_N\}^{\otimes 2} \cup \{r_0, r_N\}^{\otimes 2}$	$\{q_N^*, T_N\}^{\otimes 2} \cup \{r_0, r_N\}^{\otimes 2}$

FIG. 5.2. *The parasitic modes for the forms b_i and b_{iN} $(i = 1, 2)$.*

5.3. An inf-sup condition for the forms b_{iN} ($i = 1, 2$). In order to satisfy the discrete inf-sup condition $(2.22)_i$ for the forms b_{iN} $(i = 1, 2)$, according to Propositions 5.2 and

5.3, we can choose as spaces M_{iN} $(i=1,2)$ any supplementary space to Z_{iN} in $P_N(\Omega)$, i.e., any subspace of $P_N(\Omega)$ such that

$$\text{codim } M_{iN}=8, \qquad M_{iN}\cap Z_{iN}=\{0\}. \tag{5.29}$$

If this condition is fulfilled, the existence of a constant β_{iN} for which $(2.22)_i$ holds is ensured by the finite dimension of the spaces M_{iN}. As in the previous case of approximation by Galerkin spectral method, the choice

$$M_{iN}=\{q\in P_N(\Omega);\ \forall r\in Z_{iN},\ (q,r)_{\omega,N}=0\}, \tag*{$(5.30)_i$}$$

leads to a minimal constant and is useful to prepare a better choice. Alternative characterizations for $(5.30)_i$ follow from Propositions 5.2 and 5.3: we have as in the Galerkin approximation (see (4.19) and (4.20))

$$\begin{aligned} M_{1N}&=\{\omega^{-1}\operatorname{div}(\mathbf{v}\omega),\ \mathbf{v}\in X_N\}\\ &=\{q\in P_N(\Omega);\ \forall r\in\{T_0,T_N\}^{\otimes 2},\ (q,r)_\omega=0 \quad\text{and}\quad q(\pm1,\pm1)=0\}, \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} M_{2N}&=\{\operatorname{div}\mathbf{v},\ \mathbf{v}\in X_N\}\\ &=\{q\in P_N(\Omega);\ \forall r\in\{q_N,T_N\}^{\otimes 2},\ (q,r)_\omega=0 \quad\text{and}\quad q(\pm1,\pm1)=0\}. \end{aligned} \tag{5.32}$$

We are now able to precisely state the constant β_{iN} for this choice of spaces M_{iN} $(i=1,2)$.

PROPOSITION 5.4. *Let the space M_{iN} be defined by $(5.30)_i$. There exists a constant $\tilde{\beta}_i>0$ independent of N such that*

$$\forall q\in M_{iN}\quad \sup_{\mathbf{v}\in X_N}\frac{b_{iN}(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\omega}}\geqq\tilde{\beta}_i N^{-2}\|q\|_{0,\omega}. \tag*{$(5.33)_i$}$$

Proof. Consider the case $i=1$ first. We know from the proof of Proposition 4.3 that, for each q in M_{1N}, there exists $\mathbf{v}$ in X_N such that $-\omega^{-1}\operatorname{div}(\mathbf{v}\omega)=q$ and $\|\mathbf{v}\|_{1,\omega}\leqq cN^2\|q\|_{0,\omega}$ (see (4.30) and (4.31)). Since the discrete inner product induces a norm uniformly equivalent to the norm $\|\cdot\|_{0,\omega}$ on $P_N(\Omega)$ (see (5.10)), we have

$$b_{1N}(\mathbf{v},q)=(q,q)_{\omega,N}\geqq c\|q\|_{0,\omega}^2,$$

whence $(5.33)_i$ holds. The case $i=2$ follows similarly now using the proof of Proposition 4.4. □

As established in § 4.6, the choice $(5.30)_i$ is not well suited to approximate the space of pressures. Here again an alternative choice $\tilde{M}_{1N}$ for the pressure space can be given. In order to ensure the exact pressure in M_1 to be approximated within spectral accuracy by an element in $\tilde{M}_{1N}$, we require that there exists a real number λ, $0<\lambda<1$, such that

$$P_{[\lambda N]}(\Omega)\cap M_1\subset\tilde{M}_{1N}. \tag{5.34}$$

Furthermore, in order to retain the compatibility between $\tilde{M}_{1N}$ and X_N, we also require that the orthogonal projection $\pi_N^*:\tilde{M}_{1N}\to M_{1N}$ onto M_{1N} with respect to the inner product $(\cdot,\cdot)_{\omega,N}$ must satisfy

$$\forall q\in\tilde{M}_{1N}\quad \|q\|_{0,\omega}\leqq c\|\pi_N^* q\|_{0,\omega}. \tag{5.35}$$

This yields, as in § 4, the following inf-sup condition.

PROPOSITION 5.5. *Let $\tilde{M}_{1N}$ be a subspace of $P_{[\lambda N]}(\Omega)$ such that hypothesis* (5.35) *holds. There exists a constant $\tilde{\beta}_1 > 0$ independent of N such that*

$$\forall q \in \tilde{M}_{1N} \quad \sup_{\mathbf{v} \in X_N} \frac{b_{1N}(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,\omega}} \geqq \tilde{\beta}_1 N^{-2} \|q\|_{0,\omega}. \tag{5.36}$$

We refer to § 4 to the proof of Proposition 5.5 and for the discussion of the existence of spaces $\tilde{M}_{1N}$ satisfying the hypotheses (5.34) and (5.35); indeed the space $\tilde{M}_{1N}$ defined by (4.49) and (4.61) works (see also § 5.6 for some considerations on the implementation of the method).

5.4. The inf-sup condition for the form a_N. Let us assume here that M_{iN} $(i = 1, 2)$ is any supplementary space in $P_N(\Omega)$ of the space Z_{iN} defined in $(5.22)_i$, i.e., satisfies (5.29). Setting

$$K_{iN} = \{\mathbf{v} \in X_N \,;\, \forall q \in M_{iN},\, b_{iN}(\mathbf{v}, q) = 0\}, \tag{5.37}$$

by the definition of Z_{iN}, we actually have for all $\mathbf{v}$ in K_{iN}

$$\forall q \in P_N(\Omega) \quad b_{iN}(\mathbf{v}, q) = 0;$$

thus, as in the Galerkin method,

$$K_{iN} = K_i \cap X_N \qquad (i = 1, 2). \tag{5.38}$$

It is readily seen that in this case we have also $\dim K_{1N} = \dim K_{2N}$. Thus, by Remark 2.1, it is enough to check the inf-sup condition (2.19) for the form a_N over $K_{2N} \times K_{1N}$.

PROPOSITION 5.6. *There exists a constant $\tilde{\alpha} > 0$ independent of N such that*

$$\forall \mathbf{u} \in K_{2N} \quad \sup_{\mathbf{v} \in K_{1N}} \frac{a_N(\mathbf{u}, \mathbf{v})}{\|\mathbf{v}\|_{1,\omega}} \geqq \tilde{\alpha} \|\mathbf{u}\|_{1,\omega}. \tag{5.39}$$

Proof. Let $\mathbf{u} = (u, v)$ be an element of K_{2N}. As in the proof of Proposition 4.5, we deduce from (5.38) that there exists φ in $P_N(\Omega) \cap H^2_{\omega,0}(\Omega)$ such that

$$\mathbf{u} = \operatorname{curl} \varphi. \tag{5.40}$$

Let us set again $\mathbf{v} = \omega^{-1} \operatorname{curl}(\varphi\omega)$ and recall (see the proof of Proposition 3.2) that

$$\|\mathbf{v}\|_{1,\omega} \leqq c \|\mathbf{u}\|_{1,\omega}. \tag{5.41}$$

Setting now, as in the continuous case, $\chi = \Delta\varphi$, we have by the definition (5.14) of the form a_N

$$a_N(\mathbf{u}, \mathbf{v}) = -\nu(\Delta \mathbf{u}, \mathbf{v})_{\omega,N} = -\nu(\operatorname{curl} \chi, \omega^{-1} \operatorname{curl}(\varphi\omega))_{\omega,N}$$

or more explicitly

$$a_N(\mathbf{u}, \mathbf{v}) = -\nu(\partial\chi/\partial x, \omega^{-1}\partial(\varphi\omega)/\partial x)_{\omega,N} - (\partial\chi/\partial y, \omega^{-1}\partial(\varphi\omega)/\partial y)_{\omega,N}. \tag{5.42}$$

On the other hand, recalling that the Chebyshev weight ω is the product of the Chebyshev weights in each direction, i.e., $\omega(\mathbf{x}) = \rho(x)\rho(y)$, we have

$$\begin{aligned} &-(\partial\chi/\partial x, \omega^{-1}\partial(\varphi\omega)/\partial x)_{\omega,N} \\ &\quad = -\sum_{k=0}^{N} \rho_k \left\{ \sum_{j=0}^{N} (\partial\chi/\partial x)(\zeta_j, \zeta_k)[(\partial\varphi/\partial x)(\zeta_j, \zeta_k) + \varphi(\zeta_j, \zeta_k)\rho'(\zeta_j)/\rho(\zeta_j)]\rho_j \right\}. \end{aligned}$$

By (5.2) and (5.3) it follows that

$$
\begin{aligned}
-(\partial\chi/\partial x, \omega^{-1}\partial(\varphi\omega)/\partial x)_{\omega,N} &= -\sum_{k=0}^{N} \rho_k\left[\int_{-1}^{1} (\partial\chi/\partial x)(x,\zeta_k)(\partial(\varphi\rho)/\partial x)(x,\zeta_k)\,dx\right]\\
&= \sum_{k=0}^{N} \rho_k\left[\int_{-1}^{1} (\partial^2\chi/\partial x^2)(x,\zeta_k)\varphi(x,\zeta_k)\rho(x)\,dx\right]\\
&= \sum_{j=0}^{N}\sum_{k=0}^{N} (\partial^2\chi/\partial x^2)(\zeta_j,\zeta_k)\varphi(\zeta_j,\zeta_k)\rho_j\rho_k.
\end{aligned}
$$

The term in (5.42) containing the y-derivative can be handled similarly. Thus, we have proved that

$$
a_N(\mathbf{u},\mathbf{v}) = (\Delta^2\varphi, \varphi)_{\omega,N}. \tag{5.43}
$$

By Lemma 3.2 of [21], there exists a constant $c>0$ independent of N such that

$$
\forall\varphi\in P_N(\Omega)\cap H^2_{\omega,0}(\Omega)\quad (\Delta^2\varphi,\varphi)_{\omega,N}\geqq c\|\varphi\|^2_{2,\omega}.
$$

Then the proposition follows from (5.40), (5.41), and (5.43). □

5.5. A convergence estimate. In this section we consider the collocation approximation (5.13) to the homogeneous Stokes problem (3.1), (3.2). Let us recall that $X_N=[P^0_N(\Omega)]^2$.

For an appropriate choice of the spaces of pressures and of test functions, we know from Propositions 5.6, 5.4, and 5.5 that the bilinear forms a_N, b_{1N} and b_{2N} restricted to these finite-dimensional subspaces satisfy the inf-sup conditions (2.19), (2.21), and $(2.22)_i$ $(i=1,2)$ for suitable constants. Moreover, it follows from (5.16), (5.17), (5.18), and (5.10) that a_N and b_{iN} $(i=1,2)$ are uniformly continuous over $X_N\times X_N$ and $X_N\times M_{iN}$, respectively.

THEOREM 5.1. *For each integer $N\geqq 3$, the collocation approximation* (5.13) *to the Stokes problem* (3.1), (3.2) *has a unique solution $(\mathbf{u}^N, p^N)$ in $X_N\times M_{1N}$, where M_{1N} is defined by* $(5.30)_i$ (*respectively in $X_N\times\tilde{M}_{1N}$, where $\tilde{M}_{1N}$ satisfies the hypothesis* (5.35)). *Moreover, the following inequality is satisfied*:

$$
\|\mathbf{u}^N\|_{1,\omega}+N^{-2}\|p^N\|_{0,\omega}\leqq c\|\mathbf{f}\|_{[\mathscr{C}^0(\bar\Omega)]^2}, \tag{5.44}
$$

for a constant $c>0$ independent of N.

Let us now consider the convergence of the approximation.

THEOREM 5.2. *Assume that the solution* $(\mathbf{u}, p)$ *of the Stokes problem* (3.1), (3.2) *belongs to $[H^s_\omega(\Omega)]^2\times H^{s-1}_\omega(\Omega)$ for a real number $s\geqq 1$, and the data* $\mathbf{f}$ *belong to $[H^\sigma_\omega(\Omega)]^2$ for a real number $\sigma>1$. Then the approximate velocity $\mathbf{u}^N$, as defined in Theorem* 5.1, *satisfies the convergence estimate*

$$
\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\leqq c(N^{1-s}\|\mathbf{u}\|_{s,\omega}+N^{-\sigma}\|\mathbf{f}\|_{\sigma,\omega}) \tag{5.45}
$$

for a constant c independent of N.

Proof. Let us first remark that

$$
\forall\mathbf{v}\in[P^\circ_{N-1}(\Omega)]^2\quad\forall\mathbf{z}\in X_N\quad (a-a_N)(\mathbf{v},\mathbf{z})=0; \tag{5.46}
$$

indeed, the product $\mathbf{vz}$ is an element of $P_{2N-1}(\Omega)$ and the discrete integration formula in the definition (5.14) of a_N is exact (see (5.2)). By (5.38), we can apply Corollary 2.3 with $\mathbf{v}_\delta=\mathbf{w}_\delta=Q_{N-1}\mathbf{u}$ (the divergence-free polynomial approximation to $\mathbf{u}$, the existence of which is guaranteed by Lemma 4.6) to get the following error estimate for the velocity:

$$
\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\leqq c\left(\|\mathbf{u}-Q_{N-1}\mathbf{u}\|_{1,\omega}+\sup_{\mathbf{z}\in X_N}\frac{(\mathbf{f},\mathbf{z})_\omega-(\mathbf{f},\mathbf{z})_{\omega,N}}{\|\mathbf{z}\|_{1,\omega}}\right). \tag{5.47}
$$

Due to Lemma 4.6, it is sufficient to bound the error on the data $\mathbf{f}$. We have for any $\mathbf{z}$ in X_N

$$|(\mathbf{f},\mathbf{z})_\omega-(\mathbf{f},\mathbf{z})_{\omega,N}|\leqq|(\mathbf{f},\mathbf{z})_\omega-(\Pi^0_{N-1}\mathbf{f},\mathbf{z})_\omega|+|(\Pi^0_{N-1}\mathbf{f},\mathbf{z})_{\omega,N}-(\mathcal{J}_N\mathbf{f},\mathbf{z})_{\omega,N}|,$$

where Π^0_{N-1} is the orthogonal projection operator onto $[P_{N-1}(\Omega)]^2$ with respect to $(\cdot,\cdot)_\omega$, and $\mathcal{J}_N$ is the interpolation operator at the collocation nodes defined in (5.7). Hence we get by (5.10)

$$\forall \mathbf{z}\in X_N\quad |(\mathbf{f},\mathbf{z})_\omega-(\mathbf{f},\mathbf{z})_{\omega,N}|\leqq(\|\mathbf{f}-\Pi^0_{N-1}\mathbf{f}\|_{0,\omega}+\|\mathbf{f}-\mathcal{J}_N\mathbf{f}\|_{0,\omega})\|\mathbf{z}\|_{0,\omega}. \tag{5.48}$$

The first term on the right-hand side can be estimated by (4.46), while the interpolation operator satisfies the following inequality ([11, Thm. 3.1]), valid for any real numbers r and s, $s>1$ and $0\leqq r\leqq s$,

$$\forall\varphi\in H^s_\omega(\Omega)\quad \|\varphi-\mathcal{J}_N\varphi\|_{r,\omega}\leqq cN^{2r-s}\|\varphi\|_{s,\omega}. \tag{5.49}$$

This ends the proof of the theorem. □

Estimate (5.45) is optimal with respect to the regularity of the solution and of the data. As far as the pressure is concerned, we use the same argument as for Theorem 4.4.

THEOREM 5.3. *Assume that hypotheses* (5.34) *and* (5.35) *hold and that the solution* $(\mathbf{u},p)$ *of the Stokes problem* (3.1), (3.2) *belongs to* $[H^s_\omega(\Omega)]^2\times H^{s-1}_\omega(\Omega)$ *for a real number* $s\geqq 1$, *and the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma>1$. *Then the approximate pressure* p^N *in* $\tilde{M}_{1N}$, *as defined in Theorem* 5.1, *satisfies the convergence estimate*

$$\|p-p^N\|_{0,\omega}\leqq c\{N^{3-s}(\|\mathbf{u}\|_{s,\omega}+\|p\|_{s-1,\omega})+N^{2-\sigma}\|\mathbf{f}\|_{\sigma,\omega}\} \tag{5.50}$$

for a constant c independent of N.

5.6. Concluding remarks. We complete § 5.5 with some considerations on the practical implementation of the Chebyshev collocation scheme (5.13). As a matter of fact, we have indicated a pair of spaces M_{1N} and M_{2N} for which the collocation scheme is well posed and guarantees spectral accuracy. Now, it remains to exhibit a precise set of algebraic equations, as well as of unknowns for the pressure, which corresponds to the scheme and which is efficiently implementable.

To this end, let S_c denote the set of the four corners of the square Ω, and let $\hat{S}$ be a set of four collocation points in $\Xi_N\backslash S_c$ satisfying the following property:

$$\det(q_L(\mathbf{x}_J))\neq 0,\qquad 1\leqq J,L\leqq 4, \tag{5.51}$$

where $\mathbf{x}_J$ runs through $\hat{S}$ and q_L runs through $\{q^*_N,T_N\}^{\otimes 2}$ (the polynomial q^*_N is defined in (5.25)).

PROPOSITION 5.7. *Assume that hypothesis* (5.51) *holds. To apply the collocation scheme* (5.13) *is equivalent to solving the following linear system. Find* $(\mathbf{u}^N,p^N)$ *in* $X_N\times M_{1N}$, *where* M_{1N} *is defined by* $(5.30)_i$ (*or in* $X_N\times\tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (5.35)) *such that*

$$\begin{aligned}&-\nu\Delta\mathbf{u}^N(\mathbf{x})+(\operatorname{grad}p^N)(\mathbf{x})=\mathbf{f}(\mathbf{x})\quad \textit{for } \mathbf{x}\in\Xi_N\cap\Omega,\\&(\operatorname{div}\mathbf{u}^N)(\mathbf{x})=0\quad \textit{for } \mathbf{x}\in\Xi_N\backslash\{S_c\cup\hat{S}\}.\end{aligned} \tag{5.52}$$

Proof. Of course, any solution of (5.13) is a solution of (5.52). Conversely, let $(\mathbf{u}^N,p^N)$ satisfy (5.52). Since $\mathbf{u}^N$ vanishes on $\partial\Omega$, $\operatorname{div}\mathbf{u}^N$ is equal to 0 at the four corners of Ω. Moreover, by Proposition 5.3, we know that $b_{2N}(\mathbf{u}^N,q)$ is equal to 0 for all q in $\{q^*_N,T_N\}^{\otimes 2}$. By (5.52) this relation becomes

$$\forall q\in\{q^*_N,T_N\}^{\otimes 2}\quad \sum_{\mathbf{x}\in\hat{S}}(\operatorname{div}\mathbf{u}^N)(\mathbf{x})q(\mathbf{x})\omega_{\mathbf{x}}=0.$$

Thanks to (5.51), we obtain $(\operatorname{div} \mathbf{u}^N)(\mathbf{x})=0$ for any $\mathbf{x}\in \hat{S}$. We conclude that $\mathbf{u}^N$ is divergence-free in Ω; hence $(\mathbf{u}^N, p^N)$ satisfies (5.13) (and (5.19)). □

As far as the choice of degrees of freedom for the pressure is concerned, it seems impractical to find a subset of collocation points in the domain that uniquely determines the polynomials of $\tilde{M}_{1N}$ (i.e., which forms a unisolvent set for $\tilde{M}_{1N}$). It is more convenient to determine the discrete pressure through the complete set of collocation points in the domain (or, equivalently, to retain all the modes for the pressure). This means that the algebraic system that must actually be solved is underspecified. Once a solution of this system is obtained in some way, it will yield a "good" velocity field and a "good" pressure gradient at the collocation points (and only there!). In order to get a "good" pressure, i.e., the pressure satisfying an estimate like (5.50), we must extract from the computed pressure its component along $\tilde{M}_{1N}$. This can be done by taking the orthogonal projection of the computed pressure onto $\tilde{M}_{1N}$ with respect to $(\cdot,\cdot)_\omega$. Other techniques of filtering the spurious modes have been successfully applied (see, e.g., [22]).

6. Two possible extensions.

6.1. The nonhomogeneous case. Let us now consider the approximation of the nonhomogeneous Stokes problem (3.1), (3.34) by a collocation method. We will suppose that this problem is well posed, i.e., that (3.35) and (3.36) hold.

Hereafter, we assume that the space M_{iN} is defined by $(5.30)_i$ and that the space $\tilde{M}_{1N}$ satisfies (5.35). After the analysis of the homogeneous case, we propose the following formulation of the discrete problem. Find $(\mathbf{u}^N, p^N)$ in $[P_N(\Omega)]^2\times M_{1N}$ (respectively, in $[P_N(\Omega)]^2\times \tilde{M}_{1N}$) such that

$$\begin{aligned}
&\forall \mathbf{v}\in X_N \quad a_N(\mathbf{u}^N,\mathbf{v})+b_{1N}(\mathbf{v},p^N)=(\mathbf{f},\mathbf{v})_{\omega,N},\\
&\forall q\in M_{2N} \quad b_{2N}(\mathbf{u}^N,q)=0,\\
&\mathbf{u}^N(\mathbf{x})=\varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Xi_N\cap\Gamma_J,\quad J\in\mathbb{Z}/4\mathbb{Z}.
\end{aligned}\tag{6.1}$$

Remark 6.1. This formulation is not as direct as the formulation (5.13) of the homogeneous case. Indeed, we have in mind discretizing the equations in a collocation way; hence we would like to satisfy each of the equations at a suitable set of points of Ξ_N. More precisely we would like to solve the following problem. Find $(\mathbf{u}^N, p^N)$ in $[P_N(\Omega)]^2\times M_{1N}$ (respectively, in $[P_N(\Omega)]^2\times \tilde{M}_{1N}$) such that

$$\begin{aligned}
&-\nu\Delta\mathbf{u}^N(\mathbf{x})+(\operatorname{grad} p^N)(\mathbf{x})=\mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Xi_N\cap\Omega,\\
&(\operatorname{div}\mathbf{u}^N)(\mathbf{x})=0 \quad \text{for } \mathbf{x}\in\Xi_N,\\
&\mathbf{u}^N(\mathbf{x})=\varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Xi_N\cap\Gamma_J,\quad J\in\mathbb{Z}/4\mathbb{Z}.
\end{aligned}\tag{6.2}$$

The last problem is clearly equivalent to the variational formulation. Find $(\mathbf{u}^N, p^N)$ in $[P_N(\Omega)]^2\times M_{1N}$ (respectively, in $[P_N(\Omega)]^2\times \tilde{M}_{1N}$) such that

$$\begin{aligned}
&\forall \mathbf{v}\in X_N, \quad a_N(\mathbf{u}^N,\mathbf{v})+b_{1N}(\mathbf{v},p^N)=(\mathbf{f},\mathbf{v})_{\omega,N},\\
&\forall q\in P_N(\Omega) \quad b_{2N}(\mathbf{u}^N,q)=0,\\
&\mathbf{u}^N(\mathbf{x})=\varphi_J(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Xi_N\cap\Gamma_J,\quad J\in\mathbb{Z}/4\mathbb{Z}.
\end{aligned}\tag{6.3}$$

However, it follows from § 5.3 that the relation $b_{2N}(\mathbf{u}^N, q)=0$ can only be satisfied for all q in M_{2N} and generally not for all q in $P_N(\Omega)$. Indeed, since $\mathbf{u}^N$ is a polynomial, we would derive from (6.3): div $\mathbf{u}^N=0$ exactly. In particular this imposes five conditions for $\mathbf{u}^N$ at the boundary:

$$(\operatorname{div}\mathbf{u}^N)(\mathbf{x})=0 \quad \text{for } \mathbf{x} \text{ corner of } \Omega, \tag{6.4}$$

$$\sum_{J\in\mathbb{Z}/4\mathbb{Z}}\int_{\Gamma_J}\mathbf{u}^N\cdot\mathbf{n}_J\,d\sigma=0. \tag{6.5}$$

These equations solely depend upon the values of $\mathbf{u}^N$ at the boundary, and hence upon $\mathscr{I}_N\boldsymbol{\varphi}_J$. In general these relations are not satisfied.

Example. Assume that m is even, $2\leqq m\leqq N$, and let us choose as boundary conditions

$$\forall J\in\mathbb{Z}/4\mathbb{Z}\quad \boldsymbol{\varphi}_J=(L_m(\zeta),\,L_m(\zeta)). \tag{6.6}$$

(Note that the Legendre polynomial of degree m has by definition a zero-average.) These conditions fulfill the hypotheses (3.35) and (3.36). Thus, if $\mathbf{f}$ is chosen in X', the continuous problem is well posed. But, condition (6.4) does not hold since $\mathscr{I}_N\boldsymbol{\varphi}_J$ coincide with $\boldsymbol{\varphi}_J$ for any $J\in\mathbb{Z}/4\mathbb{Z}$ and div $\mathbf{u}^N$ at $\mathbf{a}_I$, for instance, is equal to $L_m'(-1)-L_m'(1)=-m(m+1)$. In this example, the continuous problem has a solution in the weak sense only, since the divergence of the velocity is not zero at the four corners. However, even if the exact velocity field is divergence-free at the four corners, condition (6.4) need not be satisfied; in fact, the Lagrange interpolation operator will not generally preserve the boundary values of the first derivative.

Hence, problem (6.2) is generally unsolvable and the formulation (6.1) is used instead. Besides, we can check the following proposition.

PROPOSITION 6.1. *Any solution* $(\mathbf{u}^N,p^N)$ *of problem* (6.1) *in* $[P_N(\Omega)]^2\times M_{1N}$ (*respectively, in* $[P_N(\Omega)]^2\times\tilde{M}_{1N}$) *satisfies the collocation equation*

$$-\nu\Delta\mathbf{u}^N(\mathbf{x})+(\operatorname{grad}p^N)(\mathbf{x})=\mathbf{f}(\mathbf{x})\quad \textit{for } \mathbf{x}\in\Xi_N\cap\Omega. \tag{6.7}$$

Remark 6.2. Due to (5.32), solving the equation

$$\forall q\in M_{2N}\quad b_{2N}(\mathbf{u}^N,q)=0$$

in (6.1) is equivalent to the minimization of $\|\operatorname{div}\mathbf{u}^N\|_{\omega,N}$; this condition is implemented in practice. We refer to [22] for details and numerical results in the nonhomogeneous case.

The main results are stated in the following theorem and proved in [4].

THEOREM 6.1. *For each integer* $N\geqq 3$, *the collocation approximation* (6.1) *to the Stokes problem* (3.1), (3.34) *has a unique solution* $(\mathbf{u}^N,p^N)$ *in* $X_N\times M_{1N}$, *where* M_{1N} *is defined by* $(5.30)_i$ (*respectively, in* $X_N\times\tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (5.35)).

Moreover, assume that the solution $(\mathbf{u},p)$ *of the Stokes problem* (3.1), (3.34) *belongs to* $[H^s_\omega(\Omega)]^2\times H^{s-1}_\omega(\Omega)$ *for a real number* $s\geqq1$, *that the data* $\mathbf{f}$ *belong to* $[H^\sigma_\omega(\Omega)]^2$ *for a real number* $\sigma>1$, *and that the boundary data* $\boldsymbol{\varphi}_J$, $J\in\mathbb{Z}/4\mathbb{Z}$, *belong to* $H^\tau_\rho(\Gamma_J)$ *for a real number* $\tau\geqq\frac{3}{4}$. *Then the approximate velocity* $\mathbf{u}^N$ *satisfies the convergence estimate*

$$\|\mathbf{u}-\mathbf{u}^N\|_{1,\omega}\leqq c\left(N^{3-s}\|\mathbf{u}\|_{s,\omega}+N^{-\sigma}\|\mathbf{f}\|_{\sigma,\omega}+N^{7/2-\tau}\sum_{J\in\mathbb{Z}/4\mathbb{Z}}\|\boldsymbol{\varphi}_J\|_{\tau,\rho}\right) \tag{6.8}$$

for a constant c independent of N. If the assumptions (5.34) *and* (5.35) *hold, the approximate pressure* p^N *in* $\tilde{M}_{1N}$ *satisfies the convergence estimate*

$$\|p-p^N\|_{0,\omega}\leqq c\left\{N^{5-s}(\|\mathbf{u}\|_{s,\omega}+\|p\|_{s-1,\omega})+N^{2-\sigma}\|\mathbf{f}\|_{\sigma,\omega}+N^{11/2-\tau}\sum_{J\in\mathbb{Z}/4\mathbb{Z}}\|\boldsymbol{\varphi}_J\|_{\tau,\rho}\right\} \tag{6.9}$$

for a constant c independent of N.

6.2. The Navier–Stokes equations. Now we are interested in the approximation of the full Navier-Stokes equations on the domain Ω by a collocation method. From now on, we will denote by $\mathbf{x}=(x_1, x_2)$ the generic point of Ω, and by w_1 and w_2 the components of any vector $\mathbf{w}$ in $\mathbb{R}^2$. Given a force field $\mathbf{f}$ in Ω and a viscosity $\nu \geqq 0$, the problem is to find a velocity field $\mathbf{u}=(u_1, u_2)$ and a pressure p solution of

$$\begin{aligned} &-\nu\Delta\mathbf{u}+\operatorname{grad} p+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{f} \quad \text{in } \Omega, \\ &\operatorname{div}\mathbf{u}=0 \quad \text{in } \Omega, \end{aligned} \tag{6.10}$$

such that $\mathbf{u}$ satisfies the following homogeneous boundary conditions:

$$\mathbf{u}=\mathbf{0} \quad \text{on } \partial\Omega. \tag{6.11}$$

We mean by $(\mathbf{u}\cdot\nabla)\mathbf{u}$ the sum $\sum_{1\leqq i\leqq 2} u_i(\partial\mathbf{u}/\partial x_i)$.

We are going to introduce a collocation problem to approximate the Navier-Stokes equations, by using the same nodes as in the linear case. Let us recall that, for a fixed integer $N\geqq 3$, $X_N=[P_N^\circ(\Omega)]^2$. Henceforth, we still assume that M_{iN} $(i=1,2)$ is defined by $(5.30)_i$ and that $\tilde{M}_{1N}$ satisfies the hypothesis (5.35).

Before writing the problem, let us consider the nonlinear term in (6.10). We notice that, for any function $\mathbf{w}$ such that $\operatorname{div}\mathbf{w}=0$ in Ω, we have

$$\sum_{1\leqq i\leqq 2} w_i(\partial\mathbf{w}/\partial x_i)=\sum_{1\leqq i\leqq 2}\partial(w_i\mathbf{w})/\partial x_i. \tag{6.12}$$

The two forms are equivalent for the continuous problem, but generally not for the discrete problems. For reasons of numerical stability as well as to reduce the computation cost, it seems more convenient to choose the second expression. Obviously, if a function $\mathbf{w}$ of class $\mathscr{C}^0$ is known only by its values at the nodes $\mathbf{x}$ in Ξ_N (see (5.7) for the definition of Ξ_N), it is easy to derive the values of $w_i\mathbf{w}$ at the same nodes, whence $\mathscr{I}_N(w_i\mathbf{w})$. The pseudospectral approximation $\partial(w_i\mathbf{w})/\partial x_i$ consists in differentiating this interpolation function, i.e., to compute $\partial\mathscr{I}_N(w_i\mathbf{w})/\partial x_i$.

Now assume that the force $\mathbf{f}$ is given in $[\mathscr{C}^0(\Omega)]^2$. Due to the previous remark, the collocation problem is the following one. Find $(\mathbf{u}^N, p^N)$ in $X_N\times M_{1N}$ such that

$$\begin{aligned} &-\nu\Delta\mathbf{u}^N(\mathbf{x})+(\operatorname{grad} p^N)(\mathbf{x})+\sum_{1\leqq i\leqq 2}\left(\frac{\partial(\mathscr{I}_N(u_i^N\mathbf{u}^N))}{\partial x_i}\right)(\mathbf{x})=\mathbf{f}(\mathbf{x}) \quad \text{for } \mathbf{x}\in\Xi_N\cap\Omega, \\ &(\operatorname{div}\mathbf{u}^N)(\mathbf{x})=0 \quad \text{for } \mathbf{x}\in\Xi_N. \end{aligned} \tag{6.13}$$

The following results will be proved in a forthcoming paper (see also [4]).

THEOREM 6.2. *Assume that there exists a solution* $(\mathbf{u}, p)$ *of the Navier-Stokes equations* (6.10), (6.11) *such that the operator* $\mathbf{1}+\mathbf{TDG}(\mathbf{u})$ *is an isomorphism of* X; *assume, moreover, that it belongs to* $[H^s(\Omega)]^2\times H^{s-1}(\Omega)$ *for a real number* $s>1$ *and that the data* $\mathbf{f}$ *belong to* $[H_\omega^\sigma(\Omega)]^2$ *for a real number* $\sigma>1$. *For N large enough, problem* (6.13) *admits a solution* $(\mathbf{u}^N, p^N)$ *in* $X_N\times M_{1N}$, *where* M_{1N} *is defined by* $(5.30)_i$ (*respectively, in* $X_N\times\tilde{M}_{1N}$, *where* $\tilde{M}_{1N}$ *satisfies the hypothesis* (5.35)). *Moreover, the approximate velocity* $\mathbf{u}^N$ *satisfies the convergence estimate*

$$\|\mathbf{u}-\mathbf{u}^N\|_{0,\omega}\leqq c(\mathbf{u})N^{1-s}+c(\mathbf{f})N^{-\sigma} \tag{6.14}$$

for constants $c(\mathbf{u})$ *and* $c(\mathbf{f})$ *independent of N. If the assumptions* (5.34) *and* (5.35) *hold, the approximate pressure* p^N *in* $\tilde{M}_{1N}$ *satisfies the convergence estimate*

$$\|p-p^N\|_{0,\omega}\leqq c(\mathbf{u}, p)N^{3-s}+c(\mathbf{f})N^{2-\sigma} \tag{6.15}$$

for constants $c(\mathbf{u}, p)$ *and* $c(\mathbf{f})$ *independent of N.*

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