

# Chapter 3

## RAVIART-THOMAS SPACES

In this chapter we introduce the Raviart-Thomas spaces, which constitute the most classical finite element subspaces of  $H(\text{div}; \Omega)$ , and prove their main interpolation and approximation properties. Several aspects of our analysis follow the approaches from [16], [50], and [52].

### 3.1 Preliminary results

In what follows,  $\Omega$  is a bounded and connected domain of  $\mathbb{R}^n$ ,  $n \in \{2, 3\}$ , with polyhedral boundary  $\Gamma$ , and  $\mathcal{T}_h$  is a triangularization of  $\bar{\Omega}$ . More precisely,  $\mathcal{T}_h$  is a finite family of triangles (in  $\mathbb{R}^2$ ) or tetrahedra (in  $\mathbb{R}^3$ ), such that:

$$\text{i) } \bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K.$$

$$\text{ii) } \overset{\circ}{K} \neq \emptyset \quad \forall K \in \mathcal{T}_h.$$

$$\text{iii) } \overset{\circ}{K}_i \cap \overset{\circ}{K}_j = \emptyset \quad \forall K_i, K_j \in \mathcal{T}_h, K_i \neq K_j.$$

$$\text{iv) } \text{If } F = K_i \cap K_j, \quad K_i, K_j \in \mathcal{T}_h, \quad K_i \neq K_j, \text{ then } F \text{ is a common face, a common side, or a common vertex of } K_i \text{ and } K_j.$$

$$\text{v) } \text{diam}(K) =: h_K \leq h \quad \forall K \in \mathcal{T}_h.$$

In addition, to each  $\mathcal{T}_h$  we associate a fixed reference polyhedron  $\widehat{K}$ , which can or can't belong to  $\mathcal{T}_h$ , and a family of affine mappings  $\{T_K\}_{K \in \mathcal{T}_h}$  such that

- a)  $T_K : \mathbb{R}^n \rightarrow \mathbb{R}^n, T_K(\hat{x}) = B_K \hat{x} + b_K \quad \forall \hat{x} \in \mathbb{R}^n$ , with  $B_K \in \mathbb{R}^{n \times n}$  invertible, and  $b_K \in \mathbb{R}^n$ .
- b)  $K = T_K(\hat{K}) \quad \forall K \in \mathcal{T}_h$ .

One usually considers  $\hat{K}$  as the unit simplex, that is the triangle with vertices  $(1,0)$ ,  $(0,1)$ , and  $(0,0)$  in  $\mathbb{R}^2$ , or the tetrahedron with vertices  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,0,1)$  and  $(0,0,0)$  in  $\mathbb{R}^3$ .

Throughout the rest of this section we demonstrate a sequence of results characterizing the spaces  $H^1(\Omega)$  and  $H(\text{div}; \Omega)$  in terms of their local behaviours on the elements of the triangularization  $\mathcal{T}_h$ . In what follows,  $\langle \cdot, \cdot \rangle_{\partial K}$  denotes the duality between  $H^{-1/2}(\partial K)$  and  $H^{1/2}(\partial K)$  for each  $K \in \mathcal{T}_h$ . In turn, we omit the symbol  $\gamma_{\boldsymbol{\nu}}$  to denote the respective normal traces, and simply write, when no confusion arises,  $\boldsymbol{\tau} \cdot \boldsymbol{\nu} \quad \forall \boldsymbol{\tau} \in H(\text{div}; \Omega)$  and  $\boldsymbol{\tau} \cdot \boldsymbol{\nu}_K \quad \forall \boldsymbol{\tau} \in H(\text{div}; K)$ , where  $\boldsymbol{\nu}_K$  is the normal vector to  $\partial K$ . Similarly, we omit the symbol  $\gamma_0$  and just write  $v|_{\Gamma}$  (or only  $v$ ) for  $v \in H^1(\Omega)$ , and  $v|_{\partial K}$  (or only  $v$ ) for  $v \in H^1(K)$ .

LEMA 3.1 *Define the spaces  $X := \left\{ v \in L^2(\Omega) : v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h \right\}$  and  $H_0(\text{div}; \Omega) := \left\{ \boldsymbol{\tau} \in H(\text{div}; \Omega) : \boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma \right\}$ . Then*

$$H^1(\Omega) = \left\{ v \in X : \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = 0 \quad \forall \boldsymbol{\tau} \in H_0(\text{div}; \Omega) \right\}.$$

DEMOSTRACIÓN. We proceed by double inclusion. Let  $v \in X$  such that

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = 0 \quad \forall \boldsymbol{\tau} \in H_0(\text{div}; \Omega).$$

Since  $v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h$ , we have for each  $\boldsymbol{\tau} \in H_0(\text{div}; \Omega)$  that

$$\int_K \boldsymbol{\tau} \cdot \nabla v = - \int_K v \text{div } \boldsymbol{\tau} + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K},$$

which yields

$$\sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\tau} \cdot \nabla v = - \int_{\Omega} v \text{div } \boldsymbol{\tau}.$$

In particular, for  $\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n \subseteq H_0(\text{div}; \Omega)$  the above identity becomes

$$\langle \nabla v, \boldsymbol{\tau} \rangle_{[\mathcal{D}'(\Omega)]^n \times [\mathcal{D}(\Omega)]^n} = \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\tau} \cdot \nabla v = \int_{\Omega} \boldsymbol{\tau} \cdot w,$$

where  $\langle \cdot, \cdot \rangle_{[\mathcal{D}'(\Omega)]^n \times [\mathcal{D}(\Omega)]^n}$  stands for the distributional pairing of  $[\mathcal{D}'(\Omega)]^n$  and  $[\mathcal{D}(\Omega)]^n$ , and  $w \in [L^2(\Omega)]^n$  is given by  $w|_K = \nabla(v|_K) \quad \forall K \in \mathcal{T}_h$ . This proves that  $\nabla v = w$  in  $[\mathcal{D}'(\Omega)]^n$ , and hence  $v \in H^1(\Omega)$ .

Conversely, let  $v \in H^1(\Omega)$ . It is clear that  $v \in X$  since obviously  $v \in L^2(\Omega)$  and  $v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h$ . Now, given  $\boldsymbol{\tau} \in H_0(\text{div}; \Omega)$ , we utilize the Green identity (1.50) (cf. Lemma 1.4) in  $H(\text{div}; \Omega)$  and  $H(\text{div}; K) \quad \forall K \in \mathcal{T}_h$ , to deduce that

$$\begin{aligned} 0 &= \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, v \rangle_\Gamma = \int_\Omega \boldsymbol{\tau} \cdot \nabla v + \int_\Omega v \text{div } \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\tau} \cdot \nabla v + \int_\Omega v \text{div } \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \left\{ - \int_K v \text{div } \boldsymbol{\tau} + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} \right\} + \int_\Omega v \text{div } \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K}, \end{aligned}$$

which completes the proof.  $\square$

An immediate consequence of the above theorem is given by the following result.

**LEMA 3.2** *Let  $X := \{v \in L^2(\Omega) : v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ . Then:*

$$H^1(\Omega) = \left\{ v \in X : \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = 0 \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n \right\}.$$

**DEMOSTRACIÓN.** The proof follows by employing Lemma 3.1, the inclusion  $[C_0^\infty(\Omega)]^n \subseteq H_0(\text{div}; \Omega)$ , the fact that  $\boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n, \quad \forall K \in \mathcal{T}_h$ , and the identity (cf. (1.45))

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v \quad \forall v \in H^1(K), \quad \forall \boldsymbol{\tau} \in [H^1(K)]^n.$$

We omit further details.  $\square$

In order to further simplify the characterization of  $H^1(\Omega)$  given by the previous lemmas, we need the following technical result.

**LEMA 3.3** *Let  $K_i, K_j \in \mathcal{T}_h$  be adjacent polyhedra with common face/side  $F$  and let  $z \in L^2(F)$  such that  $\int_F z \rho = 0 \quad \forall \rho \in C_0^\infty(K_i \cup K_j)$ . Then  $z = 0$  on  $F$ .*

DEMOSTRACIÓN. Using that  $C_0^\infty(F)$  is dense in  $L^2(F)$ , it suffices to show that  $\int_F z\varphi = 0 \quad \forall \varphi \in C_0^\infty(F)$ . To this end, let  $G$  be a perpendicular line to  $F$ , and let  $x = (x_1, x_2, \dots, x_n)$  be the representation of a coordinate system with  $(x_1, x_2, \dots, x_{n-1}) \in F$ ,  $x_n \in G$ , and the origin given by the intersection point of  $F$  and  $G$  (which can be assumed to be the barycenter of  $F$ ). Then, given  $\varphi \in C_0^\infty(F)$ , we can construct, via regularization techniques, a function  $\psi \in C_0^\infty(G)$  such that  $\psi(0) = 1$  and so that  $\text{sop } \varphi \times \text{sop } \psi$  is contained in the interior of  $K_i \cup K_j$ . Hence, defining the function  $\rho(x) := \varphi(x_1, x_2, \dots, x_{n-1})\psi(x_n)$ , we have that  $\rho \in C_0^\infty(K_i \cup K_j)$  and  $\rho|_F = \varphi$ , which implies that  $0 = \int_F z\rho = \int_F z\varphi$ , thus finishing the proof.  $\square$

We are able now to prove the following theorem.

**TEOREMA 3.1** *Let  $X := \{v \in L^2(\Omega) : v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ . Then:*

$$H^1(\Omega) = \left\{ v \in X : v|_{K_i} - v|_{K_j} = 0 \quad \text{in } L^2(F) \right. \\ \left. \forall K_i, K_j \in \mathcal{T}_h \text{ that are adjacent with common face/side } F \right\}.$$

DEMOSTRACIÓN. Let  $v \in X$  such that  $v|_{K_i} - v|_{K_j} = 0 \quad \text{in } L^2(F) \quad \forall K_i, K_j \in \mathcal{T}_h$  that are adjacent with common face/side  $F$ . Then, given  $\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n$  we have  $\boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0$  in  $\Gamma$  and hence

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = \sum_{F \in I_h(\Omega)} \int_F (v|_{K_{i,F}} - v|_{K_{j,F}}) \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_{i,F}},$$

where  $I_h(\Omega)$  is the set of interior faces/sides of  $\mathcal{T}_h$ , and  $K_{i,F}$  and  $K_{j,F}$  are the adjacent polyhedra with common face/side  $F$ . Note here that  $\boldsymbol{\nu}_{K_{i,F}} = -\boldsymbol{\nu}_{K_{j,F}}$ . It follows that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = 0 \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n,$$

which, thanks to Lemma 3.2, implies that  $v \in H^1(\Omega)$ .

Conversely, let  $v \in H^1(\Omega)$ . It is clear from Lemma 3.2 that

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = 0 \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n.$$

In particular, given  $\boldsymbol{\tau} \in [C_0^\infty(K_i \cup K_j)]^n$ , with  $K_i, K_j \in \mathcal{T}_h$  adjacent with common face/side  $F$ , we obtain

$$\begin{aligned} 0 &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = \int_F (v|_{K_i} - v|_{K_j}) \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i} \\ &= \int_F (v|_{K_i} - v|_{K_j}) \boldsymbol{\nu}_{K_i} \cdot \boldsymbol{\tau}, \end{aligned}$$

from which, applying Lemma 3.3 to a non-null component of  $\boldsymbol{\nu}_{K_i}$ , we deduce that  $v|_{K_i} - v|_{K_j} = 0$  in  $L^2(F)$ . □

Our next goal is to characterize the space  $H(\operatorname{div}; \Omega)$  in terms of the local behaviours. We begin with the following lemma, which constitutes a kind of dual result to Lemma 3.1.

LEMA 3.4 *Let  $Y := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^n : \boldsymbol{\tau}|_K \in H(\operatorname{div}; K) \quad \forall K \in \mathcal{T}_h \right\}$ . Then*

$$H(\operatorname{div}; \Omega) = \left\{ \boldsymbol{\tau} \in Y : \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = 0 \quad \forall v \in H_0^1(\Omega) \right\}.$$

DEMOSTRACIÓN. We proceed by double inclusion. Let  $\boldsymbol{\tau} \in Y$  such that

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = 0 \quad \forall v \in H_0^1(\Omega).$$

Since  $\boldsymbol{\tau}|_K \in H(\operatorname{div}; K) \quad \forall K \in \mathcal{T}_h$ , we have for each  $v \in H_0^1(\Omega)$

$$\int_K v \operatorname{div} \boldsymbol{\tau} = - \int_K \boldsymbol{\tau} \cdot \nabla v + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K},$$

which gives

$$\sum_{K \in \mathcal{T}_h} \int_K v \operatorname{div} \boldsymbol{\tau} = - \int_\Omega \boldsymbol{\tau} \cdot \nabla v.$$

In particular, for  $v \in C_0^\infty(\Omega) \subseteq H_0^1(\Omega)$ , the above identity reduces to

$$\langle \operatorname{div} \boldsymbol{\tau}, v \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \sum_{K \in \mathcal{T}_h} \int_K v \operatorname{div} \boldsymbol{\tau} = \int_\Omega v z,$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}$  is the distributional pairing of  $\mathcal{D}'(\Omega)$  and  $\mathcal{D}(\Omega)$ , and  $z \in L^2(\Omega)$  is given by  $z|_K = \operatorname{div}(\boldsymbol{\tau}|_K) \quad \forall K \in \mathcal{T}_h$ . This shows that  $\operatorname{div} \boldsymbol{\tau} = z$  in  $\mathcal{D}'(\Omega)$ , and hence  $\boldsymbol{\tau} \in H(\operatorname{div}; \Omega)$ .

Conversely, let  $\boldsymbol{\tau} \in H(\text{div}; \Omega)$ . It is clear that  $\boldsymbol{\tau} \in Y$  since obviously  $\boldsymbol{\tau} \in [L^2(\Omega)]^n$  and  $\boldsymbol{\tau}|_K \in H(\text{div}; K) \quad \forall K \in \mathcal{T}_h$ . Thus, given  $v \in H_0^1(\Omega)$ , we first utilize the Green identity (1.50) (cf. Lemma 1.4) in  $H(\text{div}; \Omega)$  and  $H(\text{div}; K) \quad \forall K \in \mathcal{T}_h$ , and proceed next as in the second part of the proof of Lemma 3.1, to conclude that

$$0 = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, v \rangle_\Gamma = \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K},$$

which completes the proof. □

The following theorem is consequence of the above lemma and the technical result given by Lemma 3.3.

**TEOREMA 3.2** *Let  $Z := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^n : \boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall K \in \mathcal{T}_h \right\}$ . Then*

$$H(\text{div}; \Omega) \cap Z = \left\{ \boldsymbol{\tau} \in Z : \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i} + \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_j} = 0 \quad \text{in } L^2(F) \right. \\ \left. \forall K_i, K_j \in \mathcal{T}_h \text{ that are adjacent with common face/side } F \right\}.$$

**DEMOSTRACIÓN.** Let  $\boldsymbol{\tau} \in Z$  such that  $\boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i} + \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_j} = 0$  in  $L^2(F) \quad \forall K_i, K_j \in \mathcal{T}_h$  that are adjacent with common face/side  $F$ . Then, given  $v \in H_0^1(\Omega)$ , we use that  $\boldsymbol{\tau} \cdot \boldsymbol{\nu}_K \in L^2(\partial K)$  since  $\boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall K \in \mathcal{T}_h$ , and employ the same notation of Theorem 3.1, to deduce that

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = \sum_{F \in \mathcal{I}_h(\Omega)} \int_F \left( \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i, F} + \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_j, F} \right) v = 0,$$

which, thanks to Lemma 3.4, yields  $\boldsymbol{\tau} \in H(\text{div}; \Omega)$ .

Conversely, let  $\boldsymbol{\tau} \in H(\text{div}; \Omega) \cap Z$ . It follows again from Lemma 3.4, that

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = 0 \quad \forall v \in H_0^1(\Omega).$$

In particular, for  $v \in C_0^\infty(K_i \cup K_j)$ , where  $K_i, K_j \in \mathcal{T}_h$  are adjacent polyhedra with common face/side  $F$ , we find that

$$0 = \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K v = \int_F \left( \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i} + \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_j} \right) v,$$

and hence, in virtue of Lemma 3.3, we conclude that  $\boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_i} + \boldsymbol{\tau} \cdot \boldsymbol{\nu}_{K_j} = 0$  in  $L^2(F)$ . □