A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem*

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Abstract

In this paper we propose and analyze, utilizing mainly tools and abstract results from Banach spaces rather than from Hilbert ones, a new fully-mixed finite element method for the stationary Boussinesq problem with temperature-dependent viscosity. More precisely, following an idea that has already been applied to the Navier-Stokes equations and to the fluid part only of our model of interest, we first incorporate the velocity gradient and the associated Bernoulli stress tensor as auxiliary unknowns. Additionally, and differently from earlier works in which either the primal or the classical dual-mixed method is employed for the heat equation, we consider here an analogue of the approach for the fluid, which consists of introducing as further variables the gradient of temperature and a vector version of the Bernoulli tensor. The resulting mixed variational formulation, which involves the aforementioned four unknowns together with the original variables given by the velocity and temperature of the fluid, is then reformulated as a fixed point equation. Next, we utilize the well-known Banach and Brouwer theorems, combined with the application of the Babuška-Brezzi theory to each independent equation, to analyze the solvability of the continuous and discrete schemes. In particular, Raviart-Thomas spaces of order $k \geq n-1$ for the Bernoulli tensor and its vector version for the heat equation, and piecewise polynomials of degree $\leq k$ for the velocity, the temperature, and both gradients, become a feasible choice. Finally, we derive optimal a priori error estimates and provide several numerical results illustrating the performance of the fully-mixed scheme and confirming the theoretical rates of convergence.

Key words: Boussinesq equations, fully–mixed formulation, fixed point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76D05, 76R10

1 Introduction

The development of accurate and efficient new finite element methods for the Boussinesq problem, based on primal, dual-mixed, and augmented variational formulations, has been profusely addressed by the community of numerical analysts of partial differential equations in the last few decades. As it is

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well-known, this model arises from diverse phenomena in engineering sciences, and it mainly deals with the fluid motion generated by density differences due to temperature gradients. Mathematically, it consists of the Navier–Stokes equations with a buoyancy term depending on the temperature, coupled to the heat equation with a convective term depending on the velocity of the fluid, and assuming suitable boundary conditions. In addition, the corresponding viscosity of the fluid might eventually depend on the temperature as well. A subset of the most representative contributions in the above described direction, which consider either constant or variable viscosity, and even time-dependent models, can be found in [2], [3], [4], [5], [8], [11], [17], [18], [19], [25], [26], [33], [35], [36], [40], and the references therein, some of which are described in the following paragraphs.

In particular, [8] constitutes one of the first works employing the primal method in both Navier-Stokes and heat equations, thus yielding a conforming finite element method for the Boussinesq equations with the velocity, the pressure, and the temperature of the fluid as the main unknowns of the system. The topological degree theory is applied there to establish existence of solutions, and finite element spaces with the same order for the velocity and the temperature are shown to lead optimal rates of convergence. Other finite element methods based on primal formulations of the Boussinesq system, using the primitive variables and incorporating the normal heat flux through the boundary as an additional unknown, respectively, are also proposed in [35] and [36] for the case of viscosity and thermal conductivity depending on the temperature. Both works provide existence of solutions under small data assumptions, uniqueness of continuous solutions under an additional regularity hypothesis, and optimal rates of convergence of the discrete solutions. In turn, a dual-mixed approach for the respective two-dimensional model, in which the gradients of both the velocity and the temperature are also introduced as further unknowns, has been proposed in [25]. More recently, the approach from [15], which introduces a modified nonlinear pseudostress tensor involving the gradient of the velocity, the convective term and the pressure, for defining a dual-mixed formulation of the Navier-Stokes equations, is extended in [17] to derive an augmented mixed-primal variational formulation for the stationary Boussinesq model. The augmentation there, being motivated by the fact that the velocity lives in a smaller space than usual, reduces to the incorporation of suitable Galerkin type expressions arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, and aims to still obtain a strongly monotone operator for representing the fluid equations. The resulting augmented scheme for the fluid flow is coupled with a primal scheme for the convection-diffusion equation, thus yielding the aforementioned nonlinear pseudostress, the velocity, the temperature and the normal derivative of the latter on the boundary, as the main unknowns. A fixed-point setting resembling the approach first applied in [6] is then utilized to study the well-posedness of the continuous and discrete schemes in [17]. Later on, the tools from [17] are extended in [18] to propose and analyze a new augmented fully-mixed finite element method for the stationary Boussinesq problem. Additionally to what was done for the fluid equations in [17], a new vector unknown involving the temperature, its gradient and the velocity, is introduced in [18] to derive now a mixed formulation for the convection-diffusion equation, which is then suitably augmented as well.

Furthermore, and concerning other methods for models with variable viscosity, we begin by referring to [4], where a mixed-primal formulation as in [17] was considered for the case of a temperature-dependent viscosity in a pseudostress-velocity-vorticity formulation of the Boussinesq model. In this way, the same fixed-point strategies from [17] and [18] allow to derive an optimally-convergent method whenever the exact solution is smooth enough, and the data are sufficiently small, by using Raviart-Thomas and piecewise polynomials to approximate the unknowns involved. Nevertheless, the results in [4] are restricted to the 2D case only since the use of Sobolev embeddings into smaller L^{p} spaces becomes crucial for the corresponding analysis. This drawback has been recently overcomed in [5] by defining the rate of strain tensor as a new variable, thanks to which more flexibility in the reasoning is achieved, and thus a mixed-primal formulation for the n-dimensional case can be considered. The rest of the analysis is again based, among other facts, on the introduction of the pseudostress and vorticity tensors, and the incorporation of augmented Galerkin-type terms in the mixed formulation for the momentum equations. The analysis and results from [5], but considering now both the viscosity and the thermal conductivity of the fluid as temperature-dependent functions, were extended in [3] to the case of an augmented fully-mixed formulation of the n-dimensional model. This means that, in addition to the same approach from [5] for the Navier-Stokes equations, a mixed formulation for the energy model is also employed. For this purpose, the temperature gradient and a pseudoheat vector are introduced as additional variables, which together with the temperature, rate of strain, pseudostress, velocity and vorticity comprise all the unknowns of the problem.

On the other hand, and going back to dual-mixed formulations for the stationary Boussinesq model with constant viscosity, we now refer to [19], where two mixed approaches, based on a dual-mixed method developed in [31] and [32] for the Navier-Stokes equations, are proposed and analyzed. Thus, the main novelty here is in the fluid part, where, besides the velocity gradient, the authors introduce the Bernoulli stress tensor as a primary variable, which can be seen as an incomplete version of the usual stress tensor whose divergence yields the full equilibrium equation. The methods in [19] are completed with both the primal and mixed-primal approaches for the heat equation. In particular, the latter incorporates the normal component of the temperature gradient on the Dirichlet boundary as a suitable Lagrange multiplier. Both formulations mix the unknowns coming from each equation, that is they are not decoupled into fluid and heat parts, and they exhibit the same classical structure of the Navier-Stokes equations. In addition, the aforementioned detail on the Bernoulli tensor yields the necessity of a weak continuity property for some terms forming part of the main bilinear form involved. Existence of continuous and discrete solutions are derived in [19], and uniqueness as well as optimal error estimates are obtained under the assumption of sufficiently small data.

According to the above discussion, the objective of the present paper is to complement the theory developed so far and to keep contributing to the design of new finite element methods to solve the stationary n-dimensional Boussinesq equations. More precisely, we are particularly interested in the development of fully-mixed formulations not involving any augmentation procedure (as done, e.g. in [18] and [3]). To this end, we now extend the applicability of the approach employed in [19] for the fluid part of our model, to the energy equation of it. In other words, and instead of using the primal or the dual-mixed method, we now employ a modified mixed formulation in the heat equation, which consists of introducing the gradient of temperature and a vector version of the Bernouilli tensor as further unknowns. In this way, and besides eliminating the pressure, which can be approximated later on via postprocessing, the resulting mixed variational formulation does not need to incorporate any augmented term, and it yields basically the same Banach saddle-point structure for both equations. This feature constitutes a clear advantage of the method proposed here, from both the theoretical and computational point of view, since the corresponding continuous and discrete analyses for the fluid and heat models can be carried out separately and very similary. Moreover, this might very well imply the use of the same kind of finite element subspaces to approximate the unknowns from the fluid and energy equations. In particular, we are able to show that Raviart-Thomas spaces of order $k \geq n-1$ for the Bernoulli tensor and its vector version, and piecewise polynomials of degree $\leq k$ for the velocity, the temperature, and both gradients, constitute a feasible choice.

Outline

We have organized the contents of the paper as follows. The remainder of this section describes some standard notations and functional spaces. In Section 2 we introduce the model problem, define all the auxiliary variables to be employed in the setting of the fully-mixed formulation, and eliminate the pressure unknown, which, however, can be recovered later on via a postprocessing formula. The continuous formulation is derived first in Section 3, and then, by decoupling the fluid and heat equations, it is rewritten as a fixed-point operator equation. The corresponding solvability analysis is finally performed by employing some tools from linear and nonlinear functional analysis, such as the Banach version of the classical Babuška-Brezzi theory, and the Banach fixed-point theorem. Next, in Section 4 we define the Galerkin scheme with arbitrary finite element subspaces of the continuous spaces, and analyze its solvability under suitable assumptions on these discrete spaces, and following basically the same techniques employed in Section 3. In Section 5 we employ diverse tools from functional analysis to derive specific finite element subspaces satisfying the assumptions stipulated in Section 4. Indeed, our analysis makes use of equivalence and sufficiency results for inf-sup conditions holding on products of reflexive Banach spaces. In addition, the derivation is based on the availability of suitable pairs of finite element subspaces yielding stable Galerkin schemes for the usual primal formulation of the Stokes problem. As a particular example we define the explicit subspaces arising from the Scott-Vogelius pair. Some results on the Raviart-Thomas elements in Banach spaces are also recalled here since they are needed to complete the discrete analysis. This section ends with the corresponding approximation properties for the aforementioned example. In Section 6 we assume sufficiently small data to derive an a priori error estimate for our Galerkin scheme with arbitrary finite element subspaces verifying the hypotheses from Section 4. Finally, some numerical examples illustrating the performance of our fully-mixed formulation with the specific finite elements subspaces derived in Section 5, are reported in Section 7.

Preliminary notations

Let $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, be a given bounded domain with polyhedral boundary Γ, and let $\boldsymbol{\nu}$ be the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ is its dual. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Furthermore, as usual $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. Also, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$ we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i\, w_j)_{i,j=1,n}.$$

In turn, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}.$$

Next, given $p > 1$, we introduce the Banach spaces

$$\begin{aligned}\mathbf{H}(\mathrm{div}_p;\Omega) &:= \Big\{ \boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^p(\Omega) \Big\}, \\ \mathbb{H}(\mathbf{div}_p;\Omega) &:= \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^p(\Omega) \Big\},\end{aligned} \tag{1.1}$$

provided with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_p;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,p;\Omega} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,p;\Omega}\,.$$

Throughout the rest of the paper we will consider the above definitions for $p = 4/3$.

2 The model problem

The stationary Boussinesq problem consists of a system of equations where the incompressible Navier-Stokes equation is coupled with the heat equation through a convective term and a buoyancy term typically acting in opposite direction to gravity. More precisely, given a fluid occupying the region Ω, an external force per unit mass $\boldsymbol{g} \in \mathbf{L}^\infty(\Omega)$, and data $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ and $\varphi_D \in \mathrm{H}^{1/2}(\Gamma)$, the model of interest (without dimensionless numbers for readability purposes) reads: Find a velocity field $\mathbf{u}$, a pressure field p and a temperature field φ such that

$$\begin{aligned} -\mathbf{div}(2\mu(\varphi)\boldsymbol{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p &= \varphi\boldsymbol{g} && \text{in} && \Omega\,, \\ \mathrm{div}\,\mathbf{u} &= 0 && \text{in} && \Omega\,, \\ -\mathrm{div}(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi &= 0 && \text{in} && \Omega\,, \\ \mathbf{u} &= \mathbf{u}_D && \text{in} && \Gamma\,, \\ \varphi &= \varphi_D && \text{in} && \Gamma\,, \end{aligned} \tag{2.1}$$

where $\boldsymbol{e}(\mathbf{u}) := \dfrac{1}{2}\Big\{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{t}}\Big\}$ is the symmetric part of the velocity gradient $\nabla\mathbf{u}$, also known as the strain rate tensor, and $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ is a uniformly positive tensor describing the thermal conductivity of the fluid, thus allowing the possibility of anisotropy (cf. [34]). In turn, $\mu : \mathrm{R} \longrightarrow \mathrm{R}^+$ is the temperature dependent viscosity, which is assumed to be a Lipschitz-continuous and bounded from above and below function, which means that there exist constants $L_\mu > 0$ and $\mu_1, \mu_2 > 0$, such that

$$|\mu(s) - \mu(t)| \,\le\, L_\mu\,|s - t|\,, \qquad \forall\, s, t \ge 0, \tag{2.2}$$

and

$$\mu_1 \,\le\, \mu(s) \,\le\, \mu_2, \qquad \forall s \,\ge 0. \tag{2.3}$$

We observe here that, because of the incompressibility of the fluid (cf. second eq. of (2.1)) and the Dirichlet boundary condition (cf. fourth eq. of (2.1)), $\mathbf{u}_D$ must satisfy the compatibility condition $\int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\nu} = 0$. In addition, due to the first equation of (2.1), and in order to guarantee uniqueness of the pressure, this unknown will be sought in the space

$$\mathrm{L}^2_0(\Omega) \,:=\, \Big\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q \,=\, 0 \Big\}\,.$$

Next, in order to derive a fully-mixed formulation for (2.1), in which the Dirichlet boundary conditions will become natural ones, and as suggested by similar approaches in several previous papers (see, e.g.

[3], [5], [18], [19]), we now introduce the velocity gradient and the Bernoulli stress tensor as further unknowns, that is

$$\mathbf{t} := \nabla \mathbf{u} \quad \text{and} \quad \boldsymbol{\sigma} := 2\mu(\varphi)\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u}) - p\mathbf{I}\,, \tag{2.4}$$

where $\mathbf{t}_{sym} := \dfrac{1}{2}\{\mathbf{t} + \mathbf{t}^{\mathrm{t}}\}$ is the symmetric part of $\mathbf{t}$, so that the second equation of (2.4) is considered from now on as the constitutive law of the fluid. Then, noting thanks to the incompressibility condition that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u} = \mathbf{t}\mathbf{u}$, we find that the first equation of (2.1) is rewritten as

$$-\mathbf{div}\,\boldsymbol{\sigma} + \frac{1}{2}\,\mathbf{t}\mathbf{u} - \varphi\boldsymbol{g} = 0\,.$$

In addition, applying the matrix trace to the aforementioned constitutive equation and using that $\mathrm{tr}(\mathbf{t}_{sym}) = \mathrm{div}\,\mathbf{u} = 0$, we deduce that

$$p = -\frac{1}{2n}\mathrm{tr}\big(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}\big)\,, \tag{2.5}$$

which yields

$$\boldsymbol{\sigma}^{\mathrm{d}} = 2\,\mu(\varphi)\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}\,. \tag{2.6}$$

Conversely, starting from (2.5) and (2.6) we readily recover the incompressibility condition and the second equation of (2.4), whence these pair of equations are actually equivalent. Furthermore, for the heat equation we define the temperature gradient and a vector version of $\boldsymbol{\sigma}$ as auxiliary unknowns, that is

$$\widetilde{\mathbf{t}} := \nabla\varphi \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}} := \mathbb{K}\widetilde{\mathbf{t}} - \frac{1}{2}\varphi\mathbf{u}\,, \tag{2.7}$$

thanks to which the third equation of (2.1) becomes

$$-\,\mathrm{div}\,\widetilde{\boldsymbol{\sigma}} + \frac{1}{2}\mathbf{u}\cdot\widetilde{\mathbf{t}} = 0\,.$$

According to the above discussion, our model problem (2.1) is re-stated as follows: Find $(\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, \varphi, \widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\sigma}})$ in suitable spaces to be defined below such that

$$\begin{aligned}
\nabla\mathbf{u} &= \mathbf{t} && \text{in} \quad \Omega\,,\\
-\mathbf{div}\,\boldsymbol{\sigma} + \frac{1}{2}\mathbf{t}\mathbf{u} - \varphi\boldsymbol{g} &= 0 && \text{in} \quad \Omega\,,\\
2\mu(\varphi)\mathbf{t}_{sym} - \frac{1}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} &= \boldsymbol{\sigma}^{\mathrm{d}} && \text{in} \quad \Omega\,,\\
\nabla\varphi &= \widetilde{\mathbf{t}} && \text{in} \quad \Omega\,,\\
\mathbb{K}\widetilde{\mathbf{t}} - \frac{1}{2}\varphi\mathbf{u} &= \widetilde{\boldsymbol{\sigma}} && \text{in} \quad \Omega\,,\\
-\mathrm{div}\,\widetilde{\boldsymbol{\sigma}} + \frac{1}{2}\mathbf{u}\cdot\widetilde{\mathbf{t}} &= 0 && \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_D \quad \text{and} \quad \varphi &= \varphi_D && \text{on} \quad \Gamma\,,\\
\int_\Omega \mathrm{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) &= 0\,.
\end{aligned} \tag{2.8}$$

At this point we remark that, as suggested by (2.5), p is eliminated from the present formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ by using that identity. This fact justifies the introduction of the last equation in (2.8), which aims to ensure that the resulting p does belong to $\mathrm{L}^2_0(\Omega)$.

3 The continuous problem

In this section we introduce and analyze the continuous formulation of (2.8). More precisely, we first derive the associated fully-mixed scheme, and then, by decoupling the fluid and the heat equations, we rewrite it as a fixed-point operator equation. Finally, the corresponding solvability analysis is performed by employing several tools from linear and nonlinear functional analysis.

3.1 The fully-mixed formulation

We begin with the first equation of (2.8). Indeed, performing a tensor inner product with $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, integrating by parts, and using the Dirichlet condition for $\mathbf{u}$, we find that

$$\int_\Omega \boldsymbol{\tau} : \mathbf{t} \,+\, \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau\nu}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)\,, \tag{3.1}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands from now on for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Note here that the continuous injection of $\mathbf{H}^1(\Omega)$ in $\mathbf{L}^4(\Omega)$ guarantees that $\boldsymbol{\tau\nu}$ is well defined and belongs to $\mathbf{H}^{-1/2}(\Gamma)$ when $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$. In addition, we also remark that (3.1) makes sense for $\mathbf{t} \in \mathbb{L}^2(\Omega)$ and $\mathbf{u} \in \mathbf{L}^4(\Omega)$, but due to the incompressibility condition we plan to look for $\mathbf{t}$ in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \,:=\, \Big\{ \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \mathrm{tr}(\mathbf{s}) \,=\, 0 \Big\}\,.$$

In turn, the second equation of (2.8) can be rewritten as

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{1}{2}\int_\Omega \mathbf{t}\mathbf{u}\cdot\mathbf{v} \,-\, \int_\Omega \varphi\boldsymbol{g}\cdot\mathbf{v} \,=\, 0 \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{3.2}$$

whereas the properties of the deviatoric tensors allow to test the third equation of (2.8) as follows

$$\int_\Omega 2\mu(\varphi)\mathbf{t}_{sym} : \mathbf{s} \,-\, \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} \,=\, \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.3}$$

On the other hand, concerning the heat equation, we easily realize that, proceeding similarly to (3.1), (3.2), and (3.3), the corresponding testing of the fourth, fifth, and sixth equation of (2.8) is given by

$$\int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\widetilde{\mathbf{t}} \,+\, \int_\Omega \varphi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \,=\, \langle \widetilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, \varphi_D\rangle_\Gamma \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,, \tag{3.4}$$

$$-\int_\Omega \psi\,\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,+\, \frac{1}{2}\int_\Omega \psi\mathbf{u}\cdot\widetilde{\mathbf{t}} \,=\, 0 \qquad \forall\, \psi \in \mathrm{L}^4(\Omega)\,, \tag{3.5}$$

and

$$\int_\Omega \mathbb{K}\widetilde{\mathbf{t}}\cdot\widetilde{\mathbf{s}} - \frac{1}{2}\int_\Omega \varphi\mathbf{u}\cdot\widetilde{\mathbf{s}} \,=\, \int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\mathbf{s}} \qquad \forall \widetilde{\mathbf{s}} \in \mathbf{L}^2(\Omega)\,, \tag{3.6}$$

where, certainly, the Dirichlet boundary condition for φ has been employed in the derivation of (3.4). In this way, conveniently gathering all the equations (3.1) up to (3.6) we arrive at first glance to the following weak variational formulation of (2.8): Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma},\varphi,\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\sigma}}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}(\mathbf{div}_{4/3};\Omega)\times$

$\mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ such that $\displaystyle\int_\Omega \mathrm{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$ and

$$
\begin{array}{rcll}
\displaystyle -\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \frac{1}{2}\int_\Omega \mathbf{t}\mathbf{u}\cdot\mathbf{v} - \int_\Omega \varphi\boldsymbol{g}\cdot\mathbf{v} & = & 0 & \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,, \\
\displaystyle \int_\Omega 2\mu(\varphi)\mathbf{t}_{sym} : \mathbf{s} - \frac{1}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} & = & \displaystyle\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s}^{\mathrm{d}} & \forall\,\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \\
\displaystyle \int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) & = & \langle\boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma & \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,, \\
\displaystyle -\int_\Omega \psi\,\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) + \frac{1}{2}\int_\Omega \psi\mathbf{u}\cdot\widetilde{\mathbf{t}} & = & 0 & \forall\,\psi\in\mathrm{L}^4(\Omega)\,, \\
\displaystyle \int_\Omega \mathbb{K}\widetilde{\mathbf{t}}\cdot\widetilde{\mathbf{s}} - \frac{1}{2}\int_\Omega \varphi\mathbf{u}\cdot\widetilde{\mathbf{s}} & = & \displaystyle\int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\mathbf{s}} & \forall\,\widetilde{\mathbf{s}}\in\mathbf{L}^2(\Omega)\,, \\
\displaystyle \int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\widetilde{\mathbf{t}} + \int_\Omega \varphi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) & = & \langle\widetilde{\boldsymbol{\tau}}\cdot\boldsymbol{\nu}, \varphi_D\rangle_\Gamma & \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,.
\end{array}
\tag{3.7}
$$

We now consider the orthogonal descomposition (cf., e.g. [28], [37])

$$
\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}\,, \tag{3.8}
$$

where

$$
\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \left\{ \boldsymbol{\zeta}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0 \right\}, \tag{3.9}
$$

and observe, in particular, that the unknown $\boldsymbol{\sigma}$ can be uniquely decomposed, according to (3.8) and the mean value condition $\displaystyle\int_\Omega \mathrm{tr}(2\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0$, as

$$
\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 \,+\, c_0\,\mathbb{I}\,, \qquad \text{with} \quad \boldsymbol{\sigma}_0\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 \,:=\, -\frac{1}{2n|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,. \tag{3.10}
$$

In this way, and similarly as for the pressure, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that the first two equations of (3.7), that is those involving $\boldsymbol{\sigma}$, remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. In addition, thanks to the compatibility condition satisfied by the datum $\mathbf{u}_D$ and the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, we realize that testing the third equation of (3.7) against $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Consequently, from now we denote $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and instead of (3.7) consider the modified, though still equivalent formulation, given by: Find $(\mathbf{u},\mathbf{t},\boldsymbol{\sigma},\varphi,\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\sigma}})\in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ such that the six equations of (3.7) hold for all $(\mathbf{v},\mathbf{s},\boldsymbol{\tau},\psi,\widetilde{\mathbf{s}},\widetilde{\boldsymbol{\tau}})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$.

Next, in order to write the above formulation in a more suitable way for the analysis to be developed below, we now set the notations

$$
\vec{\mathbf{u}} \,:=\, (\mathbf{u},\mathbf{t})\,, \quad \vec{\mathbf{v}} \,:=\, (\mathbf{v},\mathbf{s})\,, \quad \vec{\mathbf{u}}_0 \,:=\, (\mathbf{u}_0,\mathbf{t}_0) \,\in\, \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,
$$

and

$$
\vec{\varphi} \,:=\, (\varphi,\widetilde{\mathbf{t}})\,, \quad \vec{\psi} \,:=\, (\psi,\widetilde{\mathbf{s}}) \,\in\, \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\,,
$$

with corresponding norms given by

$$
\|\vec{\mathbf{u}}\| \,=\, \|(\mathbf{u},\mathbf{t})\| \,:=\, \|\mathbf{u}\|_{0,4;\Omega} \,+\, ||\mathbf{t}||_{0,\Omega} \qquad \forall\,\vec{\mathbf{u}}\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.11}
$$

and

$$\|\vec{\varphi}\| \,=\, \|(\varphi,\widetilde{\mathbf{t}})\| \,:=\, \|\varphi\|_{0,4;\Omega} \,+\, ||\widetilde{\mathbf{t}}||_{0,\Omega} \qquad \forall\,\vec{\varphi}\in \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\,. \tag{3.12}$$

Then, the fully-mixed formulation for our stationary Boussinesq problem can be stated as: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) \in \big(\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\big)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ such that

$$\begin{aligned}
a_\varphi(\vec{\mathbf{u}},\vec{\mathbf{v}}) + c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) + b(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; F_\varphi(\vec{\mathbf{v}}) && \forall\,\vec{\mathbf{v}}\in\big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\,,\\
b(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; G(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\widetilde{a}(\vec{\varphi},\vec{\psi}) + \widetilde{c}_{\mathbf{u}}(\vec{\varphi},\vec{\psi}) + \widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\sigma}}) \;&=\; 0 && \forall\,\vec{\psi}\in\big(\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\big)\,,\\
\widetilde{b}(\vec{\varphi},\widetilde{\boldsymbol{\tau}}) \;&=\; \widetilde{G}(\widetilde{\boldsymbol{\tau}}) && \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,
\end{aligned} \tag{3.13}$$

where, given arbitrary $(\mathbf{w},\phi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, the forms a_ϕ, b, $c(\mathbf{w};\cdot,\cdot)$, $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}_{\mathbf{w}}$, and the functionals F_ϕ, G, and $\widetilde{G}$, are defined by

$$a_\phi(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,:=\, \int_\Omega 2\mu(\phi)\mathbf{t}_{sym}:\mathbf{s}\,,\quad b(\vec{\mathbf{v}},\boldsymbol{\tau}) \,:=\, -\int_\Omega\boldsymbol{\tau}:\mathbf{s} \,-\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{3.14}$$

$$c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,:=\, \frac{1}{2}\left\{\int_\Omega \mathbf{t}\mathbf{w}\cdot\mathbf{v} - \int_\Omega(\mathbf{u}\otimes\mathbf{w})^{\mathrm{d}}:\mathbf{s}^{\mathrm{d}}\right\}\,, \tag{3.15}$$

for all $\vec{\mathbf{u}} \,:=\, (\mathbf{u},\mathbf{t})$, $\vec{\mathbf{v}} \,:=\, (\mathbf{v},\mathbf{s})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, for all $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$,

$$\widetilde{a}(\vec{\varphi},\vec{\psi}) \,:=\, \int_\Omega \mathbb{K}\widetilde{\mathbf{t}}\cdot\widetilde{\mathbf{s}}\,,\quad \widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\tau}}) \,:=\, -\int_\Omega\widetilde{\boldsymbol{\tau}}\cdot\widetilde{\mathbf{s}} \,-\, \int_\Omega\psi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}})\,, \tag{3.16}$$

$$\widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) \,:=\, \frac{1}{2}\left\{\int_\Omega\psi\mathbf{w}\cdot\widetilde{\mathbf{t}} - \int_\Omega\varphi\mathbf{w}\cdot\widetilde{\mathbf{s}}\right\}\,, \tag{3.17}$$

for all $\vec{\varphi} \,:=\, (\varphi,\widetilde{\mathbf{t}})$, $\vec{\psi} \,:=\, (\psi,\widetilde{\mathbf{s}})\in\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)$, for all $\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and

$$F_\phi(\vec{\mathbf{v}}) \,:=\, \int_\Omega\phi\boldsymbol{g}\cdot\mathbf{v}\,,\quad G(\boldsymbol{\tau}) \,:=\, -\langle\boldsymbol{\tau}\,\nu,\mathbf{u}_D\rangle_\Gamma\,,\quad \widetilde{G}(\widetilde{\boldsymbol{\tau}}) \,:=\, -\langle\widetilde{\boldsymbol{\tau}}\cdot\nu,\varphi_D\rangle_\Gamma\,, \tag{3.18}$$

for all $\vec{\mathbf{v}} \,:=\, (\mathbf{v},\mathbf{s})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, for all $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, for all $\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)$.

In what follows we proceed similarly as in [17] (see also [6], [18]) and utilize a fixed point strategy to prove that problem (3.13) is well posed. More precisely, we first rewrite (3.13) in Section 3.2 as an equivalent fixed point equation in terms of an operator T. Then, in Section 3.3 we show that T is well defined, and finally in Section 3.4 we apply the classical Banach theorem to conclude that T has a unique fixed point.

3.2 The fixed point approach

We first let $S:\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\longrightarrow\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ be the operator defined by

$$S(\mathbf{w},\phi) \,:=\, (S_1(\mathbf{w},\phi),S_2(\mathbf{w},\phi)) = \vec{\mathbf{u}} \qquad \forall\,(\mathbf{w},\phi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\,,$$

where $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \big(\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\big)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} a_\phi(\vec{\mathbf{u}},\vec{\mathbf{v}}) + c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) + b(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= F_\phi(\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,,\\ b(\vec{\mathbf{u}},\boldsymbol{\tau}) &= G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \end{aligned} \tag{3.19}$$

In turn, we let $\widetilde{S}:\mathbf{L}^4(\Omega)\longrightarrow \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)$ be the operator given by

$$\widetilde{S}(\mathbf{w}) := (\widetilde{S}_1(\mathbf{w}),\widetilde{S}_2(\mathbf{w})) = \vec{\varphi} \qquad \forall\,\mathbf{w}\in\mathbf{L}^4(\Omega)\,,$$

where $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) \in \big(\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\big)\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ is the unique solution (to be confirmed below) of the problem:

$$\begin{aligned} \widetilde{a}(\vec{\varphi},\vec{\psi}) + \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) + \widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\sigma}}) &= 0 \qquad \forall\,\vec{\psi}\in\mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega)\,,\\ \widetilde{b}(\vec{\varphi},\widetilde{\boldsymbol{\tau}}) &= \widetilde{G}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \end{aligned} \tag{3.20}$$

Having introduced the mappings S and $\widetilde{S}$, we now set $T:\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\longrightarrow\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$ as

$$T(\mathbf{w},\phi) := \Big(S_1(\mathbf{w},\phi),\widetilde{S}_1(S_1(\mathbf{w},\phi))\Big) \qquad \forall\,(\mathbf{w},\phi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\,, \tag{3.21}$$

and realize that solving (3.13) is equivalent to seeking a fixed point of T, that is: Find $(\mathbf{u},\varphi)\in\mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$ such that

$$T(\mathbf{u},\varphi) = (\mathbf{u},\varphi)\,. \tag{3.22}$$

3.3 Well-definedness of the fixed point operator

In what follows we show that T is well defined, which reduces to prove that the uncoupled problems (3.19) and (3.20) defining S and $\widetilde{S}$, respectively, are well posed. To this end, we now recall the Banach version of the Babuška-Brezzi theorem in Hilbert spaces. More precisely, we have the following result (cf. [23, Theorem 2.34]).

Theorem 3.1 *Let* H *and* Q *be reflexive Banach spaces, and let* $a:\mathrm{H}\times\mathrm{H}\longrightarrow\mathrm{R}$ *and* $b:\mathrm{H}\times\mathrm{Q}\longrightarrow\mathrm{R}$ *be bounded bilinear forms with induced operators* $A\in\mathcal{L}(\mathrm{H},\mathrm{H}')$ *and* $B\in\mathcal{L}(\mathrm{H},\mathrm{Q}')$*, respectively. In addition, let* V *be the null space of* B*, and assume that*

i) *there exists* $\alpha>0$ *such that*

$$\sup_{v\in\mathrm{V}}\frac{a(u,v)}{\|v\|_{\mathrm{H}}} \ge \alpha\,\|u\|_{\mathrm{H}} \qquad \forall\,u\in\mathrm{V}\,, \tag{3.23}$$

ii) *there holds*

$$\sup_{u\in\mathrm{V}} a(u,v) > 0 \qquad \forall\,v\in\mathrm{V}, \quad v\neq\mathbf{0}\,, \tag{3.24}$$

iii) *there exists* β *such that*

$$\sup_{v\in H}\frac{b(v,\tau)}{\|v\|_{\mathrm{H}}} \ge \beta\,\|\tau\|_Q \qquad \forall\,\tau\in Q\,. \tag{3.25}$$

Then, there exits a unique $(u,\sigma) \in \mathrm{H} \times \mathrm{Q}$ *such that*

$$\begin{aligned} a(u,v) \,+\, b(v,\sigma) &= F(v) \qquad \forall\, v \in \mathrm{H}\,, \\ b(v,\tau) &= G(\tau) \qquad \forall\, \tau \in \mathrm{Q}\,, \end{aligned} \tag{3.26}$$

and the following a priori estimates hold:

$$\begin{aligned} \|u\| &\le \frac{1}{\alpha}\,\|F\| + \frac{1}{\beta}\Big(1 + \frac{\|A\|}{\alpha}\Big)\|G\|\,, \\ \|\sigma\| &\le \frac{1}{\beta}\,\Big(1 + \frac{\|A\|}{\alpha}\Big)\|F\| \,+\, \frac{\|A\|}{\beta^2}\Big(1 + \frac{\|A\|}{\alpha}\Big)\|G\|\,. \end{aligned} \tag{3.27}$$

We remark here that if the bilinear form a is elliptic on V, that is if there exists $\alpha > 0$ such that

$$a(v,v) \;\ge\; \alpha\,\|v\|^2 \qquad \forall\, v \in \mathrm{V}\,,$$

then the inequalities (3.23)-(3.24) are clearly fulfilled. Obviously, the above remains true if the ellipticity of a holds on the whole space H.

Next, in order to apply Theorem 3.1 to problems (3.19) and (3.20), we let $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ be the kernels of the operators induced by the bilinear forms b and $\widetilde{b}$, that is

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \int_\Omega \boldsymbol{\tau} : \mathbf{s} + \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \Big\}\,, \tag{3.28}$$

and

$$\widetilde{\mathbf{V}} := \Big\{ \vec{\psi} = (\psi,\widetilde{\mathbf{s}}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : \quad \int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\widetilde{\mathbf{s}} + \int_\Omega \psi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \quad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) \Big\}\,, \tag{3.29}$$

which easily yields

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \quad \nabla\mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \Big\}\,, \tag{3.30}$$

and

$$\widetilde{\mathbf{V}} := \Big\{ \vec{\psi} = (\psi,\widetilde{\mathbf{s}}) \in \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : \quad \nabla\psi = \widetilde{\mathbf{s}} \quad \text{and} \quad \psi \in \mathrm{H}^1_0(\Omega) \Big\}\,. \tag{3.31}$$

In particular, we stress that for the derivation of (3.30) we make use of the fact that the identity defining $\mathbf{V}$ is equivalent to testing it against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$.

Then, we introduce the spaces $\mathbf{H} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ and $\widetilde{\mathbf{H}} := \mathrm{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$, with norms given by (3.11) and (3.12), respectively, and readily establish the boundedness of a_ϕ, b, $\widetilde{a}$, and $\widetilde{b}$, by using the Cauchy-Schwarz inequality, and the bounds for μ (cf. (2.3)) and $\mathbb{K}$. More precisely, there hold

$$a_\phi(\vec{\mathbf{u}},\vec{\mathbf{v}}) \;\le\; 2\mu_2\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\,\phi \in \mathrm{L}^4(\Omega)\,, \quad \forall\,\vec{\mathbf{u}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{3.32}$$

$$b(\vec{\mathbf{v}},\boldsymbol{\tau}) \;\le\; \|\vec{\mathbf{v}}\|\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\vec{\mathbf{v}} \in \mathbf{H}\,, \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{3.33}$$

$$\widetilde{a}(\vec{\varphi},\vec{\psi}) \;\le\; ||\mathbb{K}||_{\infty,\Omega}\,\|\vec{\varphi}\|\,\|\vec{\psi}\| \qquad \forall\,\vec{\varphi},\,\vec{\psi} \in \widetilde{\mathbf{H}}\,, \tag{3.34}$$

and

$$\widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\tau}}) \;\le\; \|\vec{\psi}\|\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\,\vec{\psi} \in \widetilde{\mathbf{H}}\,, \quad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \tag{3.35}$$

The following lemma establishes the ellipticity of the bilinear forms a_ϕ and $\widetilde{a}$.

Lemma 3.2 *There exist positive constants α and $\widetilde{\alpha}$ such that*

$$a_{\phi}(\vec{\mathbf{v}},\vec{\mathbf{v}}) \,\geq\, \alpha\,\|\vec{\mathbf{v}}\|^2 \qquad \forall\,\phi \in \mathrm{L}^4(\Omega)\,, \quad \forall\vec{\mathbf{v}} \in \mathbf{V}\,, \tag{3.36}$$

and

$$\widetilde{a}(\vec{\psi},\vec{\psi}) \,\geq\, \widetilde{\alpha}\,\|\vec{\psi}\|^2 \qquad \forall\,\vec{\psi} \in \widetilde{\mathbf{V}}\,. \tag{3.37}$$

Proof. Given $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}) \in \mathbf{V}$ and $\phi \in \mathrm{L}^4(\Omega)$, we know from (3.30) that $\nabla\mathbf{v} = \mathbf{s}$ and $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$. Hence, applying the lower bound of μ (cf. (2.3)), the Korn inequality in $\mathbf{H}^1_0(\Omega)$, the continuous injection $\mathbf{i} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{L}^4(\Omega)$, and the Friedrichs-Poincaré inequality with constant c_p, we obtain

$$\begin{aligned} a_{\varphi}(\vec{\mathbf{v}},\vec{\mathbf{v}}) \,&=\, \int_{\Omega} 2\mu(\varphi)\mathbf{s}_{sym} : \mathbf{s}_{sym} \,\geq\, 2\mu_1\,\|\mathbf{s}_{sym}\|^2_{0,\Omega} \,=\, 2\mu_1\,\|\mathbf{e}(\mathbf{v})\|^2_{0,\Omega} \\ &\geq\, \mu_1\,|\mathbf{v}|^2_{1,\Omega} \,=\, \frac{\mu_1}{2}\,|\mathbf{v}|^2_{1,\Omega} \,+\, \frac{\mu_1}{2}\,\|\mathbf{s}\|^2_{0,\Omega} \,\geq\, \frac{\mu_1 c_p}{2\|\mathbf{i}\|^2}\,\|\mathbf{v}\|^2_{0,4;\Omega} \,+\, \frac{\mu_1}{2}\,\|\mathbf{s}\|^2_{0,\Omega}\,, \end{aligned}$$

which gives (3.36) with α depending on μ_1, c_p, and $\|\mathbf{i}\|$. The proof of (3.37), being very similar to the one of (3.36) and using that $\mathbb{K}$ is a uniformly positive definite tensor, is omitted. $\square$

We now prove that b and $\widetilde{b}$ (cf. (3.14) and (3.16)) verify the inf-sup condition (3.25) from Theorem 3.1. To this end, we first notice that a well known estimate (see, e.g. [28, Lemma 2.3]) that is valid for tensors in the space $\mathbb{H}_0(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div}_2;\Omega)$ (cf. (1.1)), can be easily extended to $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. More precisely, a slight modification of the proof of [28, Lemma 2.3] allows to show the existence of a positive constant c_1, depending only on Ω, such that

$$c_1\,\|\boldsymbol{\tau}\|^2_{0,\Omega} \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,4/3;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{3.38}$$

Then, we have the following lemma establishing the aforementioned inf-sup conditions.

Lemma 3.3 *There exist positive constants β and $\widetilde{\beta}$ such that*

$$\sup_{\substack{\vec{\mathbf{v}}\in\mathbf{H}\\ \vec{\mathbf{v}}\neq 0}} \frac{b(\vec{\mathbf{v}},\boldsymbol{\tau})}{\|\vec{\mathbf{v}}\|} \,\geq\, \beta\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{3.39}$$

and

$$\sup_{\substack{\vec{\psi}\in\widetilde{\mathbf{H}}\\ \vec{\psi}\neq 0}} \frac{\widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\tau}})}{\|\vec{\psi}\|} \,\geq\, \widetilde{\beta}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \tag{3.40}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, we denote by $\mathcal{S}(\boldsymbol{\tau})$ the supremum on the left hand side of (3.39). Then, taking in particular $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}) = (\mathbf{0},\boldsymbol{\tau}^{\mathrm{d}}) \in \mathbf{H}$, we find that

$$\mathcal{S}(\boldsymbol{\tau}) \,\geq\, \frac{b((\mathbf{0},\boldsymbol{\tau}^{\mathrm{d}}),\boldsymbol{\tau})}{\|(\mathbf{0},\boldsymbol{\tau}^{\mathrm{d}})\|} \,=\, \frac{\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega}}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}} \,=\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\,. \tag{3.41}$$

In turn, denoting by $\boldsymbol{\tau}_j$ the j-th row of $\boldsymbol{\tau}$ $\forall\,j = \overline{1,n}$, we now set $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{0}) \in \mathbf{H}$, with $\mathbf{v} := (v_j)_{j=\overline{1,n}}$ and $v_j := \mathrm{div}(\boldsymbol{\tau}_j)^{1/3} \in \mathrm{L}^4(\Omega) \quad \forall\,j = \overline{1,n}$. Then, it follows that

$$\mathcal{S}(\boldsymbol{\tau}) \,\geq\, \frac{b((\mathbf{v},\mathbf{0}),\boldsymbol{\tau})}{\|(\mathbf{v},\mathbf{0})\|} \,=\, \frac{\|\mathbf{div}(\boldsymbol{\tau})\|^{4/3}_{0,4/3;\Omega}}{\|\mathbf{div}(\boldsymbol{\tau})\|^{1/3}_{0,4/3;\Omega}} \,=\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}\,, \tag{3.42}$$

which, together with (3.41) and (3.38) imply (3.39) and complete the proof. In turn, given $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, the proof of (3.40) follows analogously by simply taking now $\vec{\psi} = (\psi, \widetilde{\mathbf{s}}) = (\mathbf{0}, \widetilde{\boldsymbol{\tau}}) \in \widetilde{\mathbf{H}}$ and $\vec{\psi} = (\psi, \widetilde{\mathbf{s}}) = (\mathrm{div}(\widetilde{\boldsymbol{\tau}})^{1/3}, \mathbf{0}) \in \widetilde{\mathbf{H}}$. Further details are not described. $\square$

Some boundedness properties of the forms $c(\mathbf{w};\cdot,\cdot)$ and $\widetilde{c}_{\mathbf{w}}$ are established next.

Lemma 3.4 *The bilinear forms $c(\mathbf{w};\cdot,\cdot) : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ and $\widetilde{c}_{\mathbf{w}} : \widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}} \to \mathrm{R}$ are bounded for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$ with boundedness constants given in both cases by $\|\mathbf{w}\|_{0,4;\Omega}$, and there hold the following additional properties:*

$$c(\mathbf{w};\vec{\mathbf{v}},\vec{\mathbf{v}}) = 0 \quad \text{and} \quad \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\varphi}) = 0 \qquad \forall\, \mathbf{w} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \quad \forall\, \vec{\varphi} \in \widetilde{\mathbf{H}}\,, \tag{3.43}$$

$$\big|c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,-\, c(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}})\big| \,\le\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\, \mathbf{w},\,\mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\mathbf{u}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{3.44}$$

$$\big|\widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\psi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\psi})\big| \,\le\, \|\mathbf{w}\|_{0,4;\Omega}\,\|\vec{\phi}-\vec{\varphi}\|\,\|\vec{\psi}\| \qquad \forall\, \mathbf{w} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\phi},\,\vec{\varphi},\,\vec{\psi} \in \widetilde{\mathbf{H}}\,, \tag{3.45}$$

$$\big|\widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) \,-\, \widetilde{c}_{\mathbf{z}}(\vec{\varphi},\vec{\psi})\big| \,\le\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\varphi}\|\,\|\vec{\psi}\| \qquad \forall\, \mathbf{w},\,\mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\varphi},\,\vec{\psi} \in \widetilde{\mathbf{H}}\,, \tag{3.46}$$

Proof. The boundedness of the forms $c(\mathbf{w};\cdot,\cdot)$ and $\widetilde{c}_{\mathbf{w}}$ follows directly from their definitions (cf. (3.15) and (3.17)) by applying Cauchy-Schwarz inequality. Similarly, the null properties from (3.43) are consequence of (3.15), (3.17), and simple algebraic computations. In particular, the one for $c(\mathbf{w};\cdot,\cdot)$ uses the identity $(\mathbf{v}\otimes\mathbf{w})^{\mathbf{d}} : \mathbf{s}^{\mathbf{d}} = (\mathbf{v}\otimes\mathbf{w}) : \mathbf{s} = \mathbf{s}\mathbf{w}\cdot\mathbf{v}$, which is valid for all $\mathbf{v}$, $\mathbf{w} \in \mathbf{L}^4(\Omega)$, and for all $\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$. Next, given $\mathbf{w}$, $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t})$, $\vec{\mathbf{v}} = (\mathbf{v},\mathbf{s}) \in \mathbf{H}$, we obtain

$$\begin{aligned}
&\big|c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,-\, c(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}})\big| \\
&\quad = \left|\frac{1}{2}\Big\{\int_\Omega \mathbf{t}\mathbf{w}\cdot\mathbf{v} - \int_\Omega (\mathbf{u}\otimes\mathbf{w})^d : \mathbf{s}^d\Big\} \,-\, \frac{1}{2}\Big\{\int_\Omega \mathbf{t}\mathbf{z}\cdot\mathbf{v} - \int_\Omega (\mathbf{u}\otimes\mathbf{z})^d : \mathbf{s}^d\Big\}\right| \\
&\quad \le\, \frac{1}{2}\Big\{\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{t}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,4;\Omega} \,+\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\Big\} \\
&\quad \le\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\|\,,
\end{aligned}$$

which proves (3.44). The inequalities (3.45) and (3.46) are derived similarly, and hence we omit the corresponding details. $\square$

We are now in position to confirm that the operator S is well-defined.

Lemma 3.5 *For each $(\mathbf{w},\phi) \in \mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, problem (3.19) has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Moreover, there exists a positive constant C_S, independent of $(\mathbf{w},\phi)$, such that*

$$\|S(\mathbf{w},\phi)\| \,:=\, \|\vec{\mathbf{u}}\| \,\le\, C_S\,\Big\{\|\phi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 \,+\, \|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{3.47}$$

Proof. Given $(\mathbf{w},\phi) \in \mathbf{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, we introduce the bilinear form $\mathcal{A}_{\mathbf{w},\phi} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ defined by

$$\mathcal{A}_{\mathbf{w},\phi}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,:=\, a_\phi(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{u}},\,\vec{\mathbf{v}} \in \mathbf{H}\,, \tag{3.48}$$

whence problem (3.19) can be reformulated as: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ such that

$$\begin{aligned}
\mathcal{A}_{\mathbf{w},\phi}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, b(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; F_\phi(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\
b(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,.
\end{aligned} \tag{3.49}$$

It follows from (3.32) and Lemma 3.4 that there holds

$$\big|\mathcal{A}_{\mathbf{w},\phi}(\vec{\mathbf{u}},\vec{\mathbf{v}})\big| \,\le\, \big(2\mu_2+\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}\| \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}}\in\mathbf{H}\,. \tag{3.50}$$

In addition, it is clear from (3.36) (cf. Lemma 3.2) and (3.43) (cf. Lemma 3.4) that $\mathcal{A}_{\mathbf{w},\phi}$ is $\mathbf{V}$-elliptic with the same constant α from (3.36). In turn, we know from (3.39) (cf. Lemma 3.3) that our bilinear form b satisfies the inf-sup condition required by Theorem 3.1. On the other hand, simple computations show (cf. (3.18)) that

$$\|F_\phi\| \,\le\, |\Omega|^{1/2}\,\|\phi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \quad\text{and}\quad \|G\| \,\le\, \|\mathbf{u}_D\|_{1/2,\Gamma}\,. \tag{3.51}$$

Hence, a straightforward application of Theorem 3.1 implies the unique solvability of (3.49) and the a priori estimate (cf. first inequality in (3.27))

$$\|S(\mathbf{w},\phi)\| \,:=\, \|\vec{\mathbf{u}}\| \,\le\, \frac{1}{\alpha}\,\|F_\phi\| + \frac{1}{\beta}\Big(1+\frac{\|\mathcal{A}_{\mathbf{w},\phi}\|}{\alpha}\Big)\|G\|\,,$$

which, together with (3.50) and (3.51), yield (3.47) with C_S depending on Ω, μ_2, α and β. $\square$

For later use in the paper we note here that, applying the second inequality from (3.27), and employing the bounds given by (3.50) and (3.51) for $\|\mathcal{A}_{\mathbf{w},\phi}\|$, and for F_ϕ and G, respectively, the a priori estimate for the second component of the solution to the problem defining S (cf. (3.19) or (3.49)), reduces to

$$\|\boldsymbol{\sigma}\| \,\le\, \left(1+\frac{2\mu_2+\|\mathbf{w}\|_{0,4;\Omega}}{\alpha}\right)\left\{\frac{|\Omega|^{1/2}}{\beta}\|\phi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \frac{2\mu_2+\|\mathbf{w}\|_{0,4;\Omega}}{\beta^2}\,\|\mathbf{u}_D\|_{1/2,\Gamma}\right\}. \tag{3.52}$$

The following lemma proves the well-posedness of (3.20), or equivalently, that $\widetilde{S}$ is well-defined.

Lemma 3.6 *For each $\mathbf{w}\in\mathbf{L}^4(\Omega)$, problem (3.20) has a unique solution $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})\in\widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$. Moreover, there exists a positive constant $C_{\widetilde{S}}$, independent of w, such that*

$$||\widetilde{S}(\mathbf{w})|| := ||\vec{\varphi}|| \,\le\, C_{\widetilde{S}}\Big\{\big(1\,+\,\|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma} \,+\, \|\mathbf{w}\|_{0,4;\Omega}\,\|\varphi_D\|_{1/2,\Gamma}\Big\}\,. \tag{3.53}$$

Proof. We proceed similarly as in the proof of Lemma 3.5. In fact, given $\mathbf{w}\in\mathbf{L}^4(\Omega)$, we let $\widetilde{\mathcal{A}}_{\mathbf{w}}:\widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}\to\mathrm{R}$ be the bilinear form defined as

$$\widetilde{\mathcal{A}}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) \,:=\, \widetilde{a}(\vec{\varphi},\vec{\psi}) \,+\, \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) \qquad \forall\,\vec{\varphi},\,\vec{\psi}\in\widetilde{\mathbf{H}}\,,$$

whence problem (3.20) can be reformulated as: Find $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})\in\widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ such that

$$\begin{aligned} \widetilde{\mathcal{A}}_{\mathbf{w}}(\vec{\varphi},\vec{\psi}) \,+\, \widetilde{b}(\vec{\psi},\widetilde{\boldsymbol{\sigma}}) \;&=\; 0 &&\forall\,\vec{\psi}\in\widetilde{\mathbf{H}}\,,\\ \widetilde{b}(\vec{\varphi},\widetilde{\boldsymbol{\tau}}) \;&=\; \widetilde{G}(\widetilde{\boldsymbol{\tau}}) &&\forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{H}(\mathrm{div}_{4/3};\Omega)\,. \end{aligned} \tag{3.54}$$

It is easy to see from (3.34) and Lemma 3.4 that $\widetilde{\mathcal{A}}_{\mathbf{w}}$ is bounded with boundedness constant given by $\|\mathbb{K}\|_{\infty,\Omega}+\|\mathbf{w}\|_{0,4;\Omega}$. In addition, (3.37) (cf. Lemma 3.2) and (3.43) (cf. Lemma 3.4) guarantee that $\widetilde{\mathcal{A}}_{\mathbf{w}}$ is $\widetilde{\mathbf{V}}$-elliptic with the same constant $\widetilde{\alpha}$ from (3.37). In turn, it is clear from (3.40) (cf. Lemma 3.3) that $\widetilde{b}$ also satisfies the inf-sup condition required by Theorem 3.1. In this way, an application again of Theorem 3.1 confirms the unique solvability of (3.54) and the a priori estimate

$$||\widetilde{S}(\mathbf{w})|| \,:=\, ||\vec{\varphi}|| \,\le\, \frac{1}{\widetilde{\beta}}\Big(1+\frac{\|\widetilde{\mathcal{A}}_{\mathbf{w}}\|}{\widetilde{\alpha}}\Big)\|\widetilde{G}\|\,,$$

from which, observing from (3.18) that $\|\widetilde{G}\| \le \|\varphi_D\|_{1/2,\Gamma}$, we conclude (3.53) with $C_{\widetilde{S}}$ depending on $\widetilde{\alpha}$ and $\widetilde{\beta}$. $\square$

Similarly as for the derivation of (3.52), we now notice that, applying again the second inequality from (3.27), and employing the aforementioned bounds for $\|\widetilde{\mathcal{A}}_{\mathbf{w}}\|$ and $\|\widetilde{G}\|$, the a priori estimate for the second component of the solution to the problem defining $\widetilde{S}$ (cf. (3.20) or (3.54)), reduces to

$$\|\widetilde{\boldsymbol{\sigma}}\| \le \left(\frac{\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}\|_{0,4;\Omega}}{\widetilde{\beta}^2}\right)\left\{1 + \frac{\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}\|_{0,4;\Omega}}{\widetilde{\alpha}}\right\}\|\varphi_D\|_{1/2,\Gamma}\,. \tag{3.55}$$

3.4 Solvability analysis of the fixed-point equation

Having proved the well-posedness of (3.19) and (3.20), thus ensuring that operators S, $\widetilde{S}$, and hence T, are well-defined, we now aim to establish the existence of a unique fixed-point of the operator T. We begin by providing suitable conditions under which T maps a ball into itself.

Lemma 3.7 *Given $r > 0$, let W be the closed ball in $\mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ with center at the origin and radius r, and assume that the data satisfy*

$$\Big\{\big(1 + \|\varphi_D\|_{1/2,\Gamma}\big)\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) + \big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\|\varphi_D\|_{1/2,\Gamma}\Big\} \le \frac{r}{C(r)}\,, \tag{3.56}$$

where $C(r) := C_S \max\big\{1, C_{\widetilde{S}}\big\}(r+1) + C_{\widetilde{S}}$, and C_S and $C_{\widetilde{S}}$ are the constants specified in Lemmas 3.5 *and* 3.6*, respectively. Then, there holds $T(W) \subseteq W$.*

Proof. Given $(\mathbf{w}, \phi) \in W$, from the definition of T (cf. (3.21)) and the a priori estimate for $\widetilde{S}$ (cf. (3.53)), we first obtain

$$\begin{aligned}\|T(\mathbf{w},\phi)\| &= \|(S_1(\mathbf{w},\phi), \widetilde{S}_1(S_1(\mathbf{w},\phi)))\| = \|S_1(\mathbf{w},\phi)\| + \|\widetilde{S}_1(S_1(\mathbf{w},\phi))\| \\ &\le \big(1 + C_{\widetilde{S}}\|\varphi_D\|_{1/2,\Gamma}\big)\|S_1(\mathbf{w},\phi)\|_{0,4;\Omega} + C_{\widetilde{S}}\big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\|\varphi_D\|_{1/2,\Gamma}\,.\end{aligned}$$

Then, bounding $\|S_1(\mathbf{w},\phi)\|_{0,4;\Omega}$ in the foregoing inequality according to the estimate (3.47), noting that both $\|\mathbf{w}\|_{0,4;\Omega}$ and $\|\phi\|_{0,4;\Omega}$ are bounded by r, and performing some minor algebraic manipulations, we arrive at

$$\|T(\mathbf{w},\phi)\| \le C(r)\Big\{\big(1 + \|\varphi_D\|_{1/2,\Gamma}\big)\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) + \big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\|\varphi_D\|_{1/2,\Gamma}\Big\},$$

which, thanks to the assumption (3.56), yields $\|T(\mathbf{w},\phi)\| \le r$ and ends the proof. $\square$

We now aim to prove that the operator T is Lipschitz continuous, for which, according to (3.21), it suffices to show that both S and $\widetilde{S}$ satisfy this property. We begin next with the corresponding result for S, for which we need to assume further regularity on the solution of the problem defining this operator. More precisely, we suppose that $\mathbf{u}_D \in \mathbf{H}^{1/2+\epsilon}(\Gamma)$ for some $\epsilon \in [1/2, 1)$ (when $n = 2$) or $\epsilon \in [3/4, 1)$ (when $n = 3$), and that for each $(\mathbf{w}, \phi) \in \mathbf{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ there holds $S(\mathbf{w},\phi) := \vec{\mathbf{u}} = (\mathbf{u}, \mathbf{t}) \in \mathbf{W}^{\epsilon,4}(\Omega) \times \big(\mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{H}^{\epsilon}(\Omega)\big)$ and

$$\|\mathbf{u}\|_{\epsilon,4:\Omega} + \|\mathbf{t}\|_{\epsilon,\Omega} \le c_S\Big\{\|\phi\|_{0,4;\Omega}\|\boldsymbol{g}\|_{\infty,\Omega} + \big(1 + \|\mathbf{w}\|_{0,4;\Omega}\big)\|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}\Big\}, \tag{3.57}$$

with a positive constant c_S independent of the given $(\mathbf{w}, \phi)$. We notice that the reason of the indicated range for ϵ will be clarified in the proof of the following lemma.

Lemma 3.8 *There exists a positive constant L_S, depending on L_μ, α, ϵ, n, and $|\Omega|$, such that*

$$\begin{aligned}&\|S(\mathbf{w},\phi) - S(\mathbf{z},\psi)\| \\ &\quad \le\; L_S\left\{\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|S(\mathbf{z},\psi)\| \;+\; \|\phi-\psi\|_{0,4;\Omega}\left(\|\boldsymbol{g}\|_{\infty,\Omega} \;+\; \|S_2(\mathbf{z},\psi)\|_{\epsilon,\Omega}\right)\right\}\end{aligned} \tag{3.58}$$

for all $(\mathbf{w},\phi)$, $(\mathbf{z},\psi) \in \mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)$.

Proof. Given $(\mathbf{w},\phi)$, $(\mathbf{z},\psi) \in \mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)$, we let $\vec{\mathbf{u}} = (\mathbf{u},\mathbf{t}) := S(\mathbf{w},\phi)$ and $\vec{\mathbf{u}}_0 = (\mathbf{u}_0,\mathbf{t}_0) := S(\mathbf{z},\psi)$ be the respective solutions of (3.19). It is clear from the corresponding second equations of (3.19) that $\vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ (cf. (3.30)), and then the $\mathbf{V}$-ellipticity of a_ϕ (cf. (3.36)) and the first equation of (3.19) applied to both $S(\mathbf{w},\phi)$ and $S(\mathbf{z},\psi)$, yield

$$\begin{aligned}\alpha\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|^2 \;&\le\; a_\phi(\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; a_\phi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\\ &=\; F_\phi(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; a_\phi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\\ &=\; F_\phi(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; F_\psi(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\\ &\quad+\; c(\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;+\; a_\psi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; a_\phi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\,.\end{aligned} \tag{3.59}$$

We now estimate the right hand side of (3.59) by separating it into three suitable terms. Inded, we first observe that

$$\big|F_\phi(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) - F_\psi(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\big| \;=\; |F_{\phi-\psi}(\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)| \;\le\; |\Omega|^{1/2}\,\|\phi-\psi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega}\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\,. \tag{3.60}$$

Then, using from (3.43) that $c(\mathbf{w};\vec{\mathbf{u}}-\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) = 0$, and applying (3.44), we find that

$$\begin{aligned}&\big|c(\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) - c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\big|\\ &\quad=\; \big|c(\mathbf{z};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) - c(\mathbf{w};\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\big|\\ &\quad\le\; \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}_0\|\,\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\,.\end{aligned} \tag{3.61}$$

Next, employing the Lipschitz continuity of μ (cf. (2.2)), and the Cauchy-Schwarz and Hölder inequalities, we deduce that

$$\begin{aligned}&\big|a_\psi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; a_\phi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\big| = \Big|2\int_\Omega \big(\mu(\psi)-\mu(\phi)\big)\mathbf{t}_{0,sym} : (\mathbf{t}-\mathbf{t}_0)\Big|\\ &\quad\le\; 2L_\mu\,\|(\psi-\phi)\mathbf{t}_{0,sym}\|_{0,\Omega}\,\|\mathbf{t}-\mathbf{t}_0\|_{0,\Omega} \;\le\; 2L_\mu\,\|\psi-\phi\|_{0,2q;\Omega}\;\|\mathbf{t}_0\|_{0,2p;\Omega}\;\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\,,\end{aligned} \tag{3.62}$$

where $p,q \in [1,\infty)$ are such that $\dfrac{1}{p}+\dfrac{1}{q} = 1$. In this way, bearing in mind the further regularity (3.57), we recall that the Sobolev embedding Theorem (cf. [1, Theorem 4.12], [23, Corollary B.43], [37, Theorem 1.3.4]) establishes the continuous injection $i_\epsilon : \mathbb{H}^\epsilon(\Omega) \to \mathbb{L}^{\epsilon^*}(\Omega)$, where $\epsilon^* = \begin{cases} \frac{2}{1-\epsilon} & \text{if } n = 2\,,\\ \frac{6}{3-2\epsilon} & \text{if } n = 3\end{cases}$. Thus, choosing p such that $2p = \epsilon^*$, there holds $\mathbf{t}_0 \in \mathbb{L}^{2p}(\Omega)$ and $\|\mathbf{t}_0\|_{0,2p;\Omega} \le \|i_\epsilon\|\,\|\mathbf{t}_0\|_{\epsilon,\Omega}$. Moreover, with this choice of $2p$, we obtain that $2q = n/\epsilon$, and hence, using that for the specified ranges of ϵ there holds $\|\psi-\phi\|_{0,n/\epsilon;\Omega} \le c(\epsilon,n,|\Omega|)\,\|\psi-\phi\|_{0,4;\Omega}$, with a positive constant $c(\epsilon,n,|\Omega|)$ depending on ϵ, n, and $|\Omega|$, (3.62) becomes

$$\big|a_\psi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0) \;-\; a_\phi(\vec{\mathbf{u}}_0,\vec{\mathbf{u}}-\vec{\mathbf{u}}_0)\big| \;\le\; 2L_\mu\,\|i_\epsilon\|\,c(\epsilon,n,|\Omega|)\,\|\psi-\phi\|_{0,4;\Omega}\;\|\mathbf{t}_0\|_{\epsilon,\Omega}\;\|\vec{\mathbf{u}}-\vec{\mathbf{u}}_0\|\,. \tag{3.63}$$

Finally, replacing (3.60), (3.61), and (3.63) back into (3.59), and then simplifying by $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|$, we get (3.58) with $L_S := \alpha^{-1} \max\big\{1\,,\, |\Omega|^{1/2}\,,\, 2L_\mu\,\|i_\epsilon\|\,c(\epsilon,n,|\Omega|)\big\}$. □

We find it important to stress at this point that in the particular, though very frequent situation in applications, in which the viscosity μ is constant, the regularity assumption (3.57) is not needed anymore. In this case, the Lipschitz-continuity estimate (3.58) reduces to

$$\|S(\mathbf{w},\phi) - S(\mathbf{z},\psi)\| \,\le\, L_S \Big\{ \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|S(\mathbf{z},\psi)\| \,+\, \|\phi-\psi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \Big\}\,, \tag{3.64}$$

for all $(\mathbf{w},\phi)$, $(\mathbf{z},\psi) \in \mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)$, with $L_S = \alpha^{-1}$.

We now focus on proving the Lipschitz-continuity of $\widetilde{S}$.

Lemma 3.9 *There exists a positive constant $L_{\widetilde{S}}$, depending on $\widetilde{\alpha}$ and $C_{\widetilde{S}}$ (cf. Lemma 3.6), such that*

$$\begin{aligned} &\|\widetilde{S}(\mathbf{w}) - \widetilde{S}(\mathbf{z})\| \\ &\qquad \le\, L_{\widetilde{S}}\,\|\mathbf{z}-\mathbf{w}\|_{0,4;\Omega} \Big\{ \big(1+\|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma} \,+\, \|\mathbf{z}\|_{0,4;\Omega}\,\|\varphi_D\|_{1/2,\Gamma} \Big\} \end{aligned} \tag{3.65}$$

for all $\mathbf{w},\ \mathbf{z} \in \mathbf{L}^4(\Omega)$.

Proof. We proceed analogously to the proof of Lemma 3.8. Indeed, given $\mathbf{w}$, $\mathbf{z} \in \mathbf{L}^4(\Omega)$, we first let $\vec{\varphi} := (\varphi, \widetilde{\mathbf{t}}) = \widetilde{S}(\mathbf{w})$ and $\vec{\phi} := (\phi,\widetilde{\mathbf{r}}) = \widetilde{S}(\mathbf{z})$ be the respective solutions of (3.20). It is clear from the corresponding second equations of (3.20) that $\vec{\varphi} - \vec{\phi} \in \widetilde{\mathbf{V}}$, and hence, employing the $\widetilde{\mathbf{V}}$-ellipticity of $\widetilde{a}$ (cf. (3.37)) and the first equation of (3.20) applied to both $\widetilde{S}(\mathbf{w})$ and $\widetilde{S}(\mathbf{z})$, we find that

$$\begin{aligned} \widetilde{\alpha}\,\|\widetilde{S}(\mathbf{w}) - \widetilde{S}(\mathbf{z})\|^2 \,&=\, \widetilde{\alpha}\,\|\vec{\varphi}-\vec{\phi}\|^2 \,\le\, \widetilde{a}(\vec{\varphi},\vec{\varphi}-\vec{\phi}) - \widetilde{a}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \\ &=\, \widetilde{c}_{\mathbf{z}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\varphi}-\vec{\phi})\,. \end{aligned}$$

Then, adding and subtracting $\widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\varphi}-\vec{\phi})$, and employing (3.43) and (3.46), we deduce that

$$\begin{aligned} \widetilde{\alpha}\,\|\widetilde{S}(\mathbf{w}) - \widetilde{S}(\mathbf{z})\|^2 \,&\le\, \widetilde{c}_{\mathbf{z}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,+\, \widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\varphi},\vec{\varphi}-\vec{\phi}) \\ &=\, \widetilde{c}_{\mathbf{z}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\varphi}-\vec{\phi},\vec{\varphi}-\vec{\phi}) \\ &=\, \widetilde{c}_{\mathbf{z}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,-\, \widetilde{c}_{\mathbf{w}}(\vec{\phi},\vec{\varphi}-\vec{\phi}) \,\le\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\phi}\|\,\|\vec{\varphi}-\vec{\phi}\| \\ &=\, \|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\widetilde{S}(\mathbf{z})\|\,\|\vec{\varphi}-\vec{\phi}\|\,. \end{aligned}$$

Finally, simplifying by $\|\vec{\varphi}-\vec{\phi}\|$ and using the estimate for $\|\widetilde{S}(\mathbf{z})\|$ provided by (3.53) (cf. Lemma 3.6), we arrive at (3.65) with $L_{\widetilde{S}} = \widetilde{\alpha}^{-1}\,C_{\widetilde{S}}$. □

As a consequence of the previous lemmas, we establish now the Lipschitz-continuity of T.

Lemma 3.10 *There exists a positive constant L_T, depending on L_S, $L_{\widetilde{S}}$, C_S, and c_S, such that*

$$\begin{aligned} &\|T(\mathbf{w},\phi) - T(\mathbf{z},\psi)\| \\ &\le\, L_T \Big\{ 1 + \Big(1 + \|\mathbb{K}\|_{\infty,\Omega} + \|\psi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} + (1+\|\mathbf{z}\|_{0,4;\Omega})\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\,\|\varphi_D\|_{1/2,\Gamma} \Big\} \\ &\times\Big(1+\|(\mathbf{z},\psi)\|\Big)\,\Big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}\Big)\,\|(\mathbf{w},\phi)-(\mathbf{z},\psi)\| \end{aligned} \tag{3.66}$$

for all $(\mathbf{w},\phi),(\mathbf{z},\psi) \in \mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)$.

Proof. According to the definition of T (cf. (3.21)) and the Lipschitz-continuity of $\widetilde{S}$ (cf. (3.65)), we first obtain that

$$\begin{aligned} \|T(\mathbf{w},\phi) - T(\mathbf{z},\psi)\| \,&=\, \|S_1(\mathbf{w},\phi) - S_1(\mathbf{z},\psi)\| \,+\, \|\widetilde{S}_1\big(S_1(\mathbf{w},\phi)\big) - \widetilde{S}_1\big(S_1(\mathbf{z},\psi)\big)\| \\ &\leq \Big\{1 + L_{\widetilde{S}}\big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma} + L_{\widetilde{S}}\|S_1(\mathbf{z},\psi)\|\,\|\varphi_D\|_{1/2,\Gamma}\Big\}\,\|S_1(\mathbf{w},\phi) - S_1(\mathbf{z},\psi)\|\,. \end{aligned} \tag{3.67}$$

In turn, the Lipschitz-continuity of S (cf. (3.58)) gives

$$\begin{aligned} &\|S_1(\mathbf{w},\phi) - S_1(\mathbf{z},\psi)\| \\ &\quad\leq\, L_S\Big\{\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|S(\mathbf{z},\psi)\| \,+\, \|\phi-\psi\|_{0,4;\Omega}\Big(\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|S_2(\mathbf{z},\psi)\|_{\epsilon,\Omega}\Big)\Big\}\,, \end{aligned} \tag{3.68}$$

whereas the a priori estimate of S (cf. (3.47)) establishes

$$\|S(\mathbf{z},\psi)\| \,\leq\, C_S\Big\{\|\psi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 \,+\, \|\mathbf{z}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,, \tag{3.69}$$

and the regularity assumption (3.57) yields

$$\|S_2(\mathbf{z},\psi)\|_{\epsilon,\Omega} \,\leq\, c_S\Big\{\|\psi\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 \,+\, \|\mathbf{z}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}\Big\}\,. \tag{3.70}$$

In this way, employing (3.69) and (3.70) in (3.68), replacing the resulting estimate in (3.67), bounding $\|\mathbf{u}_D\|_{1/2,\Gamma}$ by $\|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}$, and performing several algebraic manipulations aiming to simplify the whole writting, we are lead to (3.66) with $L_T \,:=\, L_S\,\max\big\{1, L_{\widetilde{S}}, C_S L_{\widetilde{S}}\big\}\,\max\big\{2C_S, 2c_S, 1\big\}$. $\square$

We are now in a position to establish sufficient conditions for the existence and uniqueness of a fixed-point of T (equivalently, the well posedness of the coupled problem (3.13)). More precisely, we have the following result.

Theorem 3.11 *Given $r>0$, let W be the closed ball in $\mathbf{L}^4(\Omega)\times \mathrm{L}^4(\Omega)$ with center at the origin and radius r, and assume that the data satisfy* (3.56)*, that is*

$$\Big\{\big(1 + \|\varphi_D\|_{1/2,\Gamma}\big)\,\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma}\Big\} \,\leq\, \frac{r}{C(r)}\,, \tag{3.71}$$

where the constant $C(r)$ is specified in Lemma 3.7*. In addition, define*

$$C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\varphi_D) := \Big\{1 + \Big(1 + \|\mathbb{K}\|_{\infty,\Omega} + \|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\,\|\varphi_D\|_{1/2,\Gamma}\Big\}\,, \tag{3.72}$$

and suppose that

$$L_T\,(1+r)^2\,C(\mathbb{K},\boldsymbol{g},\mathbf{u}_D,\varphi_D)\,\Big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}\Big) \,<\, 1\,. \tag{3.73}$$

Then, the operator T has a unique fixed point $(\mathbf{u},\varphi)\in W$. Equivalently, the coupled problem (3.13) *has a unique solution $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in \mathbf{H}\times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})\in \widetilde{\mathbf{H}}\times \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, with $(\mathbf{u},\varphi)\in W$. Moreover, there hold the following a priori estimates*

$$\|\vec{\mathbf{u}}\| \,\leq\, C_S\Big\{r\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1+r\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,, \tag{3.74}$$

$$\|\vec{\varphi}\| \,\leq\, C_{\widetilde{S}}\Big\{1 \,+\, \|\mathbb{K}\|_{\infty,\Omega} \,+\, r\Big\}\,\|\varphi_D\|_{1/2,\Gamma}\,, \tag{3.75}$$

$$\|\boldsymbol{\sigma}\| \,\leq\, \left(1 + \frac{2\mu_2 + r}{\alpha}\right)\,\left\{\frac{|\Omega|^{1/2}}{\beta} r\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \frac{2\mu_2 + r}{\beta^2}\,\|\mathbf{u}_D\|_{1/2,\Gamma}\right\}\,, \tag{3.76}$$

and

$$\|\widetilde{\boldsymbol{\sigma}}\| \,\leq\, \left(\frac{\|\mathbb{K}\|_{\infty,\Omega} + r}{\widetilde{\beta}^2}\right)\,\left\{1 + \frac{\|\mathbb{K}\|_{\infty,\Omega} + r}{\widetilde{\alpha}}\right\}\,\|\varphi_D\|_{1/2,\Gamma}\,. \tag{3.77}$$

Proof. We first recall from Lemma 3.7 that, under the assumption (3.71), T maps the ball W into itself. In addition, given $(\mathbf{w}, \phi)$, $(\mathbf{z}, \psi) \in W$, $\|(\mathbf{z}, \psi)\|$, $\|\mathbf{z}\|$, and $\|\psi\|$ are certainly bounded by r, and hence the estimate (3.66) yields

$$\begin{aligned} &\|T(\mathbf{w}, \phi) - T(\mathbf{z}, \psi)\| \\ &\quad \leq \; L_T\,(1+r)^2\, C(\mathbb{K}, \boldsymbol{g}, \mathbf{u}_D, \varphi_D) \Big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma} \Big) \, \|(\mathbf{w}, \phi) - (\mathbf{z}, \psi)\| \end{aligned}$$

for all $(\mathbf{w}, \phi), (\mathbf{z}, \psi) \in W$. In this way, (3.73), the foregoing inequality, and the classical Banach theorem imply the existence of a unique fixed point $(\mathbf{u}, \varphi) \in W$ of T. Thus, defining $\vec{\mathbf{u}} := S(\mathbf{u}, \varphi)$ and $\vec{\varphi} := \widetilde{S}(\mathbf{u})$, and letting $\boldsymbol{\sigma}$ and $\widetilde{\boldsymbol{\sigma}}$ be the second components of the solutions to (3.19) and (3.20) (or (3.49) and (3.54)), respectively, with $(\mathbf{w}, \phi) = (\mathbf{u}, \varphi)$, we conclude that $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) \in \mathbf{H} \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ and $(\vec{\varphi}, \widetilde{\boldsymbol{\sigma}}) \in \widetilde{\mathbf{H}} \times \mathbf{H}(\mathrm{div}_{4/3}; \Omega)$ constitute a unique solution of (3.13) with $(\mathbf{u}, \varphi) \in W$. Consequently, the estimates (3.74), (3.75), (3.76), and (3.77) follow straightforwardly from (3.47), (3.53), (3.52), and (3.55), respectively, by bounding $\|\mathbf{w}\| = \|\mathbf{u}\|$ and $\|\phi\| = \|\varphi\|$ by r. $\square$

4 The Galerkin scheme

In this section we introduce and analyze the corresponding Galerking scheme for the fully-mixed formulation (3.13). The solvability of this scheme is addressed following basically the same techniques employed throughout Section 3.

4.1 Preliminaries

Consider arbitrary finite dimensional subspaces $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{L}^4(\Omega)$, $\mathbb{H}_h^{\mathbf{t}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$, $\mathrm{H}_h^{\varphi} \subseteq \mathrm{L}^4(\Omega)$, $\mathbf{H}_h^{\widetilde{\mathbf{t}}} \subseteq \mathbf{L}^2(\Omega)$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \subseteq \mathbf{H}(\mathrm{div}_{4/3}; \Omega)$, whose specific choices will be described later on Section 5. Hereafter, h stands for the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, that is $h := \max\big\{h_K : K \in \mathcal{T}_h\big\}$, and denote

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h)\,, \quad \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h)\,, \quad \vec{\mathbf{u}}_{0,h} := (\mathbf{u}_{0,h}, \mathbf{t}_{0,h})\,,$$

as elements of $\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}$, and

$$\vec{\varphi}_h := (\varphi_h, \widetilde{\mathbf{t}}_h)\,, \quad \vec{\psi}_h := (\psi_h, \widetilde{\mathbf{s}}_h)\,,$$

as elements of $\widetilde{\mathbf{H}}_h := \mathrm{H}_h^{\varphi} \times \mathbf{H}_h^{\widetilde{\mathbf{t}}}$. In addition, from now on we denote the symmetric and skew-symmetric part of each $\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}$ by $\mathbf{s}_{h,sym}$ and $\mathbf{s}_{h,skw}$, respectively. Then, the Galerkin scheme associated with (3.13) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ such that

$$\begin{aligned} a_{\varphi_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, c(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) \;&=\; F_{\varphi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,, \\ b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) \;&=\; G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \\ \widetilde{a}(\vec{\varphi}_h, \vec{\psi}_h) \,+\, \widetilde{c}_{\mathbf{u}_h}(\vec{\varphi}_h, \vec{\psi}_h) \,+\, \widetilde{b}(\vec{\psi}_h, \widetilde{\boldsymbol{\sigma}}_h) \;&=\; 0 \qquad \forall\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,, \\ \widetilde{b}(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h) \;&=\; \widetilde{G}(\widetilde{\boldsymbol{\tau}}_h) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,. \end{aligned} \tag{4.1}$$

In order to analyze (4.1), we now follow a discrete analogue of the fixed point approach developed in Section 3.2. To this end, we first introduce the operator $S_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi} \to \mathbf{H}_h$ defined by

$$S_h(\mathbf{w}_h, \phi_h) \;:=\; (S_{1,h}(\mathbf{w}_h, \phi_h), S_{2,h}(\mathbf{w}_h, \phi_h)) \;=\; \vec{\mathbf{u}}_h \quad \forall\, (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}\,,$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ is the unique solution (to be confirmed below) of the problem

$$
\begin{aligned}
a_{\phi_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + c(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= F_{\phi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,, \\
b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,.
\end{aligned}
\tag{4.2}
$$

In turn, we also let $\widetilde{S}_h : \mathbf{H}_h^{\mathbf{u}} \to \widetilde{\mathbf{H}}_h$ be the operator given by

$$
\widetilde{S}_h(\mathbf{w}_h) := (\widetilde{S}_{1,h}(\mathbf{w}_h), \widetilde{S}_{2,h}(\mathbf{w}_h)) = \vec{\varphi}_h \qquad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}\,,
$$

where $(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ is the unique solution (to be confirmed below) of the problem

$$
\begin{aligned}
\widetilde{a}(\vec{\varphi}_h, \vec{\psi}_h) + \widetilde{c}_{\mathbf{w}_h}(\vec{\varphi}_h, \vec{\psi}_h) + \widetilde{b}(\vec{\psi}_h, \widetilde{\boldsymbol{\sigma}}_h) &= 0 \qquad \forall\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,, \\
\widetilde{b}(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h) &= \widetilde{G}(\widetilde{\boldsymbol{\tau}}_h) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,.
\end{aligned}
\tag{4.3}
$$

Hence, by introducing the operator $T_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi} \to \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ as

$$
T_h(\mathbf{w}_h, \phi_h) := \Big(S_{1,h}(\mathbf{w}_h, \phi_h), \widetilde{S}_{1,h}(S_{1,h}(\mathbf{w}_h, \phi_h)) \Big) \qquad \forall\, (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}\,, \tag{4.4}
$$

we realize that solving (4.1) is equivalent to seeking a fixed point of T_h, that is: Find $(\mathbf{u}_h, \varphi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ such that

$$
T_h(\mathbf{u}_h, \varphi_h) = (\mathbf{u}_h, \varphi_h)\,. \tag{4.5}
$$

4.2 Solvability analysis

We now aim to establish the well-posedness of problem (4.1) by analyzing the equivalent fixed-point equation (4.5). More precisely, we will apply the well-known Brouwer fixed-point theorem, which, for sake of completeness, is recalled next (cf. [16, Theorem 9.9-2]).

Theorem 4.1 *Let W be a compact and convex subset of a finite dimensional Banach space X and let $T : W \to W$ be a continuous mapping. Then T has at least one fixed-point.*

According to the above, and exactly as we did for the continuous case in Section 3.4, we begin by showing that the operators S_h and $\widetilde{S}_h$ (and hence T_h) are well defined. For this purpose, we need to introduce general hypotheses on the discrete spaces employed in (4.1). In this regard, we stress that later on we will provide specific examples satisfying these conditions. We begin with the following assumptions:

Assumption **4.1** *There exists a positive constant $\beta_{\mathtt{d}} > 0$, independent of h, such that*

$$
\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{b(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|} \geq \beta_{\mathtt{d}} \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,. \tag{4.6}
$$

Assumption **4.2** *Let $\mathbf{V}_h$ be the discrete kernel of b, that is*

$$
\mathbf{V}_h := \Big\{ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h : \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_h + \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \Big\}\,. \tag{4.7}
$$

Then, there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$
\|\mathbf{s}_{h,sym}\|_{0,\Omega} \geq C_{\mathtt{d}} \|(\mathbf{v}_h, \mathbf{s}_{h,skw})\| \qquad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h\,. \tag{4.8}
$$

Then, the discrete analogue of Lemma 3.5 is as follows.

Lemma 4.2 *For each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$, problem (4.2) has a unique solution $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$. Moreover there exists a positive constant $C_{S,\mathtt{d}}$, independent of h and $(\mathbf{w}_h, \phi_h)$, such that*

$$\|S_h(\mathbf{w}_h, \phi_h)\| \,:=\, \|\vec{\mathbf{u}}_h\| \,\leq\, C_{S,\mathtt{d}} \Big\{ \|\phi_h\|_{0,4;\Omega} \, \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} . \tag{4.9}$$

Proof. Given $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$, we let $\mathcal{A}_{\mathbf{w}_h,\phi_h} : \mathbf{H}_h \times \mathbf{H}_h \to \mathrm{R}$ be the bilinear form defined by

$$\mathcal{A}_{\mathbf{w}_h,\phi_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,:=\, a_{\phi_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, c(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{u}}_h,\ \vec{\mathbf{v}}_h \in \mathbf{H}_h \times \mathbf{H}_h \,,$$

and observe that problem (4.2) can be reformulated as: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ such that

$$\begin{aligned} \mathcal{A}_{\mathbf{w}_h,\phi_h}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) \,+\, b(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= F_{\phi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h \,, \\ b(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \,. \end{aligned} \tag{4.10}$$

We already know from (3.48) and (3.50) that $\mathcal{A}_{\mathbf{w}_h,\phi_h}$ is bounded with $||\mathcal{A}_{\mathbf{w}_h,\phi_h}|| \,\leq\, (2\mu_2 + \|\mathbf{w}_h\|_{0,4;\Omega})$. Then, given $\vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h$, we employ (4.8) (cf. Assumption 4.2) and find that

$$\begin{aligned} a_{\phi_h}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,&=\, \int_\Omega 2\mu(\phi_h)\, \mathbf{s}_{h,sym} : \mathbf{s}_{h,sym} \,\geq\, 2\mu_1 \, \|\mathbf{s}_{h,sym}\|_{0,\Omega}^2 \\ &\geq\, \mu_1 \|\mathbf{s}_{h,sym}\|_{0,\Omega}^2 \,+\, \mu_1 C_{\mathtt{d}}^2 \Big\{ \|\mathbf{s}_{h,skw}\|_{0,\Omega}^2 \,+\, \|\mathbf{v}_h\|_{0,4;\Omega}^2 \Big\} \\ &\geq\, \mu_1 \, \min\big\{1, C_{\mathtt{d}}^2\big\} \, \|\vec{\mathbf{v}}_h\|^2 \,, \end{aligned}$$

which, together with the fact that $c(\mathbf{w}_h; \vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) = 0$, yields the $\mathbf{V}_h$-ellipticty of both a_{ϕ_h} and $\mathcal{A}_{\mathbf{w}_h,\phi_h}$ with constant $\alpha_{\mathtt{d}} \,:=\, \mu_1 \, \min\big\{1, C_{\mathtt{d}}^2\big\}$. In turn, it is clear from Assumption 4.1 that b satisfies the corresponding inf-sup condition required by Theorem 3.1. In this way, a straightforward application of this theorem implies the unique solvability of (4.10). Moreover, recalling from (3.51) that there hold $\|F_{\phi_h}\| \,\leq\, |\Omega|^{1/2} \, \|\phi_h\|_{0,4;\Omega} \, \|\boldsymbol{g}\|_{\infty,\Omega}$ and $\|G\| \,\leq\, \|\mathbf{u}_D\|_{1/2,\Gamma}$, and applying the a priori estimate given by the first inequality in (3.27), we deduce that

$$\|S_h(\mathbf{w}_h, \phi_h)\| \,:=\, \|\vec{\mathbf{u}}_h\| \,\leq\, \frac{1}{\alpha_{\mathtt{d}}} \, |\Omega|^{1/2} \, \|\phi_h\|_{0,4;\Omega} \, \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \frac{1}{\beta_{\mathtt{d}}} \Big(1 + \frac{2\mu_2 + \|\mathbf{w}_h\|_{0,4;\Omega}}{\alpha_{\mathtt{d}}}\Big) \|\mathbf{u}_D\|_{1/2,\Gamma} \,,$$

which yields (4.9) with $C_{S,\mathtt{d}}$ depending on Ω, μ_2, $\alpha_{\mathtt{d}}$ and $\beta_{\mathtt{d}}$. □

We remark here that, proceeding similarly to the derivation of (3.52), we obtain that

$$\|\boldsymbol{\sigma}_h\| \,\leq\, \left(1 + \frac{2\mu_2 + \|\mathbf{w}_h\|_{0,4;\Omega}}{\alpha_{\mathtt{d}}}\right) \left\{ \frac{|\Omega|^{1/2}}{\beta_{\mathtt{d}}} \|\phi_h\|_{0,4;\Omega} \, \|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \frac{2\mu_2 + \|\mathbf{w}_h\|_{0,4;\Omega}}{\beta_{\mathtt{d}}^2} \, \|\mathbf{u}_D\|_{1/2,\Gamma} \right\} . \tag{4.11}$$

Next, for the well-posedness of problem (4.3), we need the following assumptions:

Assumption **4.3** *There exists a positive constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\vec{\psi}_h \in \widetilde{\mathbf{H}}_h \\ \vec{\psi}_h \neq \mathbf{0}}} \frac{\widetilde{b}(\vec{\psi}_h, \widetilde{\boldsymbol{\tau}}_h)}{\|\vec{\psi}_h\|} \,\geq\, \widetilde{\beta}_{\mathtt{d}} \, \|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \,. \tag{4.12}$$

Assumption **4.4** *Let $\widetilde{\mathbf{V}}_h$ be the discrete kernel of $\widetilde{b}$, that is*

$$\widetilde{\mathbf{V}}_h := \left\{ \vec{\psi}_h := (\psi_h, \widetilde{\mathbf{s}}_h) \in \widetilde{\mathbf{H}}_h : \quad \int_\Omega \widetilde{\boldsymbol{\tau}}_h \cdot \widetilde{\mathbf{s}}_h \,+\, \int_\Omega \psi_h \operatorname{div}(\widetilde{\boldsymbol{\tau}}_h) \,=\, 0 \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \right\}. \tag{4.13}$$

Then, there exists a positive constant $\widetilde{C}_{\mathtt{d}}$, independent of h, such that

$$\|\widetilde{\mathbf{s}}_h\|_{0,\Omega} \,\geq\, \widetilde{C}_{\mathtt{d}}\, \|\psi_h\|_{0,4;\Omega} \qquad \forall\, \vec{\psi}_h := (\psi_h, \widetilde{\mathbf{s}}_h) \in \widetilde{\mathbf{V}}_h\,. \tag{4.14}$$

The following lemma constitutes the discrete analogue of Lemma 3.6.

Lemma 4.3 *For each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, problem (4.3) has a unique solution $(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$. Moreover there exists a positive constant $C_{\widetilde{S},\mathtt{d}}$, independent of h and $\mathbf{w}_h$, such that*

$$\|\widetilde{S}_h(\mathbf{w}_h)\| := \|\vec{\varphi}_h\| \,\leq\, C_{\widetilde{S},\mathtt{d}} \left\{ \left(1 \,+\, \|\mathbb{K}\|_{\infty,\Omega}\right) \|\varphi_D\|_{1/2,\Gamma} \,+\, \|\mathbf{w}_h\|_{0,4;\Omega}\, \|\varphi_D\|_{1/2,\Gamma} \right\}. \tag{4.15}$$

Proof. We proceed as in Lemma 4.2. Indeed, given $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, we let $\widetilde{\mathcal{A}}_{\mathbf{w}_h} : \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{H}}_h \to \mathrm{R}$ be the bilinear form defined as

$$\widetilde{\mathcal{A}}_{\mathbf{w}_h}(\vec{\varphi}_h, \vec{\psi}_h) \,:=\, \widetilde{a}(\vec{\varphi}_h, \vec{\psi}_h) \,+\, \widetilde{c}_{\mathbf{w}_h}(\vec{\varphi}_h, \vec{\psi}_h) \qquad \forall\, \vec{\varphi}_h,\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,,$$

so that problem (4.3) can be reformulated as: Find $(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ such that

$$\begin{aligned} \widetilde{\mathcal{A}}_{\mathbf{w}_h}(\vec{\varphi}_h, \vec{\psi}_h) \,+\, \widetilde{b}(\vec{\psi}_h, \widetilde{\boldsymbol{\sigma}}_h) \quad &= \quad 0 \qquad && \forall\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,, \\ \widetilde{b}(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h) \quad &= \quad \widetilde{G}(\widetilde{\boldsymbol{\tau}}_h) \qquad && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,. \end{aligned} \tag{4.16}$$

As in the proof of Lemma 3.6, we observe from (3.34) and Lemma 3.4 that $\widetilde{\mathcal{A}}_{\mathbf{w}_h}$ is bounded with $\|\widetilde{\mathcal{A}}_{\mathbf{w}_h}\| \leq \|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}_h\|_{0,4;\Omega}$. In turn, denoting by $\kappa > 0$ the smallest eigenvalue of the uniformly positive tensor $\mathbb{K}$, and employing (4.14) (cf. Assumption 4.4), we find that for each $\vec{\psi}_h := (\psi_h, \widetilde{\mathbf{s}}_h) \in \widetilde{\mathbf{V}}_h$ there holds

$$\begin{aligned} \widetilde{a}(\vec{\psi}_h, \vec{\psi}_h) \,&=\, \int_\Omega \mathbb{K}\widetilde{\mathbf{s}}_h \cdot \widetilde{\mathbf{s}}_h \,\geq\, \kappa\, \|\widetilde{\mathbf{s}}_h\|_{0,\Omega}^2 \\ &\geq\, \frac{\kappa}{2}\, \|\widetilde{\mathbf{s}}_h\|_{0,\Omega}^2 \,+\, \frac{\kappa}{2}\, \widetilde{C}_{\mathtt{d}}^2\, \|\psi_h\|_{0,4;\Omega}^2 \\ &\geq\, \frac{\kappa}{2}\, \min\left\{1, \widetilde{C}_{\mathtt{d}}^2\right\} \|\vec{\psi}_h\|^2\,, \end{aligned}$$

which, together with the fact that $\widetilde{c}_{\mathbf{w}_h}(\vec{\psi}_h, \vec{\psi}_h) = 0$, proves the $\widetilde{\mathbf{V}}_h$-ellipticity of both $\widetilde{a}$ and $\widetilde{\mathcal{A}}_{\mathbf{w}_h}$ with constant $\widetilde{\alpha}_{\mathtt{d}} := \frac{\kappa}{2} \min\left\{1, \widetilde{C}_{\mathtt{d}}^2\right\}$. Thus, bearing in mind the discrete inf-sup condition satisfied by $\widetilde{b}$ (cf. (4.12) in Assumption 4.3), another application of Theorem 3.1 confirms the unique solvability of (4.16). In addition, recalling from (3.18) that $\|\widetilde{G}\| \leq \|\varphi_D\|_{1/2,\Gamma}$, and applying the a priori estimate given by the first inequality in (3.27), we find that

$$\|\widetilde{S}_h(\mathbf{w}_h)\| \,:=\, \|\vec{\varphi}_h\| \,\leq\, \frac{1}{\widetilde{\beta}_{\mathtt{d}}}\left(1 + \frac{\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}_h\|_{0,4;\Omega}}{\widetilde{\alpha}_{\mathtt{d}}}\right) \|\varphi_D\|_{1/2,\Gamma}\,,$$

which shows (4.15) with $C_{\widetilde{S},\mathtt{d}}$ depending on $\widetilde{\alpha}_{\mathtt{d}}$ and $\widetilde{\beta}_{\mathtt{d}}$. $\square$

We now notice that, following the same arguments yielding (3.55), we are able to show that

$$\|\widetilde{\boldsymbol{\sigma}_h}\| \,\le\, \left(\frac{\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}_h\|_{0,4;\Omega}}{\widetilde{\beta}_{\mathtt{d}}^2}\right) \left\{1 + \frac{\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{w}_h\|_{0,4;\Omega}}{\widetilde{\alpha}_{\mathtt{d}}}\right\} \|\varphi_D\|_{1/2,\Gamma}\,. \tag{4.17}$$

The discrete analogue of Lemma 3.7 is stated next. Its proof, being a simple adaptation of the arguments proving that lemma, is omitted.

Lemma 4.4 *Given $r > 0$, let W_h be the closed ball in $\mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ with center at the origin and radius r, and assume that the data satisfy*

$$\Big\{\big(1 + \|\varphi_D\|_{1/2,\Gamma}\big)\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma}\Big\} \,\le\, \frac{r}{C_{\mathtt{d}}(r)}\,, \tag{4.18}$$

where $C_{\mathtt{d}}(r) := C_{S,\mathtt{d}}\,\max\big\{1, C_{\widetilde{S},\mathtt{d}}\big\}\,(r+1) \,+\, C_{\widetilde{S},\mathtt{d}}$, and $C_{S,\mathtt{d}}$ and $C_{\widetilde{S},\mathtt{d}}$ are the constants specified in Lemmas 4.2 and 4.3, respectively. Then, there holds $T_h(W_h) \subseteq W_h$.

We now address the Lipschitz continuity of T_h, which, analogously to the continuous case, is consequence of the fact that both S_h and $\widetilde{S}_h$ satisfy this property. Indeed, in what follows we state the discrete analogues of Lemmas 3.8 and 3.9.

Lemma 4.5 *There exists a positive constant $L_{S,\mathtt{d}}$, independent of h, and depending on L_μ and $\alpha_{\mathtt{d}}$, such that*

$$\begin{aligned}&\|S_h(\mathbf{w}_h,\phi_h) - S_h(\mathbf{z}_h,\psi_h)\|\\ &\quad\le\, L_{S,\mathtt{d}}\Big\{\|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}\,\|S(\mathbf{z}_h,\psi_h)\| \,+\, \|\phi_h - \psi_h\|_{0,4;\Omega}\Big(\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|S_{2,h}(\mathbf{z}_h,\psi_h)\|_{0,4;\Omega}\Big)\Big\}\end{aligned} \tag{4.19}$$

for all $(\mathbf{w}_h,\phi_h)$, $(\mathbf{z}_h,\psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$.

Proof. Given $(\mathbf{w}_h,\phi_h)$, $(\mathbf{z}_h,\psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$, we let $\vec{\mathbf{u}}_h = (\mathbf{u}_h,\mathbf{t}_h) := S_h(\mathbf{w}_h,\phi_h)$ and $\vec{\mathbf{u}}_{0,h} = (\mathbf{u}_{0,h},\mathbf{t}_{0,h}) := S_h(\mathbf{z}_h,\psi_h)$ be the respective solutions of (4.2) (equivalently (4.10)). Then, the proof of (4.19), starting now from the $\mathbf{V}_h$-ellipticity of a_{ϕ_h} with constant $\alpha_{\mathtt{d}}$ (cf. proof of Lemma 4.2), is very similar to the one for Lemma 3.8. However, since a regularity assumption such as (3.57) is not available in the present discrete setting, we estimate $a_{\psi_h} - a_{\phi_h}$ by using an $\mathbf{L^4} - \mathbf{L^4} - \mathbf{L^2}$ argument. In this way, instead of proceeding as in (3.62), we simply obtain

$$\big|a_{\psi_h}(\vec{\mathbf{u}}_{0,h},\vec{\mathbf{u}}_h - \vec{\mathbf{u}}_{0,h}) \,-\, a_{\phi_h}(\vec{\mathbf{u}}_{0,h},\vec{\mathbf{u}}_h - \vec{\mathbf{u}}_{0,h})\big| \,\le\, 2L_\mu\,\|\psi_h - \phi_h\|_{0,4;\Omega}\,\|\mathbf{t}_{0,h}\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}_h - \vec{\mathbf{u}}_{0,h}\|\,.$$

The rest of the estimates are similar to those in the proof of Lemma 3.8, and hence further details are omitted. □

In turn, the result for the operator $\widetilde{S}_h$ is established as follows

Lemma 4.6 *There exists a positive constant $L_{\widetilde{S},\mathtt{d}}$, independent of h, and depending on $\widetilde{\alpha}_{\mathtt{d}}$ and $C_{\widetilde{S},\mathtt{d}}$ (cf. Lemma 4.3), such that*

$$\begin{aligned}&\|\widetilde{S}_h(\mathbf{w}_h) - \widetilde{S}_h(\mathbf{z}_h)\|\\ &\quad\le\, L_{\widetilde{S},\mathtt{d}}\,\|\mathbf{z}_h - \mathbf{w}_h\|_{0,4;\Omega}\Big\{\big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma} \,+\, \|\mathbf{z}_h\|_{0,4;\Omega}\,\|\varphi_D\|_{1/2,\Gamma}\Big\}\end{aligned} \tag{4.20}$$

for all $\mathbf{w}_h$, $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$.

Proof. It follows very closely the arguments from the proof of Lemma 3.9. $\square$

As a straightforward consequence of the previous two lemmas, we now establish the continuity of the operator T_h.

Lemma 4.7 *There exists a positive constant $L_{T,\mathtt{d}}$, depending on $L_{S,\mathtt{d}}$, $L_{\widetilde{S},\mathtt{d}}$, and $C_{S,\mathtt{d}}$, such that*

$$\begin{aligned} &\|T_h(\mathbf{w}_h,\phi_h) - T_h(\mathbf{z}_h,\psi_h)\| \\ &\quad\le\, L_{T,\mathtt{d}}\Big\{1 + \Big(1 + \|\mathbb{K}\|_{\infty,\Omega} + \|\psi_h\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} + (1+\|\mathbf{z}_h\|_{0,4;\Omega})\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big)\,\|\varphi_D\|_{1/2,\Gamma}\Big\} \\ &\quad\times\Big(1 + \|(\mathbf{z}_h,\psi_h)\|\Big)\,\Big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|S_{2,h}(\mathbf{z}_h,\psi_h)\|_{0,4;\Omega}\Big)\,\|(\mathbf{w}_h,\phi_h) - (\mathbf{z}_h,\psi_h)\| \end{aligned} \tag{4.21}$$

for all $(\mathbf{w}_h,\phi_h)$, $(\mathbf{z}_h,\psi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$.

Proof. Proceeding as in the proof of Lemma 3.10, but now using the definition of T_h (cf. (4.4)) and the Lipschitz-continuity of $\widetilde{S}_h$ (cf. (4.20)), we readily find that

$$\begin{aligned} \|T_h(\mathbf{w}_h,\phi_h) - T_h(\mathbf{z}_h,\psi_h)\| \,\le\, &\Big\{1 + L_{\widetilde{S},\mathtt{d}}\big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma} \\ &+ L_{\widetilde{S},\mathtt{d}}\|S_{1,h}(\mathbf{z}_h,\psi_h)\|\,\|\varphi_D\|_{1/2,\Gamma}\Big\}\,\|S_{1,h}(\mathbf{w}_h,\phi_h) - S_{1,h}(\mathbf{z}_h,\psi_h)\|\,. \end{aligned} \tag{4.22}$$

Then, the Lipschitz-continuity of S_h (cf. (4.19)) yields

$$\begin{aligned} &\|S_{1,h}(\mathbf{w}_h,\phi_h) - S_{1,h}(\mathbf{z}_h,\psi_h)\| \\ &\quad\le\, L_{S,\mathtt{d}}\Big\{\|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}\,\|S_h(\mathbf{z}_h,\psi_h)\| + \|\phi_h - \psi_h\|_{0,4;\Omega}\,\Big(\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \|S_{2,h}(\mathbf{z}_h,\psi_h)\|_{0,4;\Omega}\Big)\Big\}\,, \end{aligned} \tag{4.23}$$

and the a priori estimate of S_h (cf. (4.9)) establishes

$$\|S_h(\mathbf{z}_h,\psi_h)\| \,\le\, C_{S,\mathtt{d}}\Big\{\|\psi_h\|_{0,4;\Omega}\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 \,+\, \|\mathbf{z}_h\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{4.24}$$

Finally, employing (4.24) in (4.23), replacing the resulting estimate in (4.22), bounding $\|S_{1,h}(\mathbf{z}_h,\psi_h)\|$ in (4.22) by (4.24), and performing some minor algebraic manipulations, we obtain (4.21) with the constant $L_{T,\mathtt{d}} := L_{S,\mathtt{d}}\,\max\big\{1, L_{\widetilde{S},\mathtt{d}}, C_{S,\mathtt{d}}L_{\widetilde{S},\mathtt{d}}\big\}\,\max\big\{2C_{S,\mathtt{d}}, 1\big\}$. $\square$

We are now in position of applying the Brower fixed point theorem to establish a solvability result for the coupled problem (4.1).

Theorem 4.8 *Given $r > 0$, let W_h be the closed ball in $\mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ with center at the origin and radius r, and assume that the data satisfy* (4.18)*, that is*

$$\Big\{\big(1 + \|\varphi_D\|_{1/2,\Gamma}\big)\,\big(\|\boldsymbol{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\big) \,+\, \big(1 + \|\mathbb{K}\|_{\infty,\Omega}\big)\,\|\varphi_D\|_{1/2,\Gamma}\Big\} \,\le\, \frac{r}{C_{\mathtt{d}}(r)}\,, \tag{4.25}$$

where the constant $C_{\mathtt{d}}(r)$ is specified in Lemma 4.4. Then, the operator T_h has a fixed point $(\mathbf{u}_h,\varphi_h) \in W_h$. Equivalently, the coupled problem (4.1) *has at least a solution $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, with $(\mathbf{u}_h,\varphi_h) \in W_h$. Moreover, there hold the following a priori estimates*

$$\|\vec{\mathbf{u}}_h\| \,\le\, C_{S,\mathtt{d}}\Big\{r\,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \big(1 + r\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,, \tag{4.26}$$

$$\|\vec{\varphi}_h\| \,\le\, C_{\widetilde{S},\mathtt{d}}\Big\{1 \,+\, \|\mathbb{K}\|_{\infty,\Omega} \,+\, r\Big\}\,\|\varphi_D\|_{1/2,\Gamma}\,, \tag{4.27}$$

$$\|\boldsymbol{\sigma}_h\| \,\le\, \left(1 + \frac{2\mu_2 + r}{\alpha_{\mathtt{d}}}\right) \left\{ \frac{|\Omega|^{1/2}}{\beta_{\mathtt{d}}} r \,\|\boldsymbol{g}\|_{\infty,\Omega} \,+\, \frac{2\mu_2 + r}{\beta_{\mathtt{d}}^2} \,\|\mathbf{u}_D\|_{1/2,\Gamma} \right\}, \tag{4.28}$$

and

$$\|\widetilde{\boldsymbol{\sigma}}_h\| \,\le\, \left(\frac{\|\mathbb{K}\|_{\infty,\Omega} + r}{\widetilde{\beta}_{\mathtt{d}}^2}\right) \left\{1 + \frac{\|\mathbb{K}\|_{\infty,\Omega} + r}{\widetilde{\alpha}_{\mathtt{d}}}\right\} \|\varphi_D\|_{1/2,\Gamma}\,. \tag{4.29}$$

Proof. It follows similarly to the proof of Theorem 3.11. Indeed, we first notice from Lemma 4.4 that the assumption (4.25) guarantees that $T_h(W_h) \subseteq W_h$. Next, it is easy to see from (4.21) (cf. Lemma 4.7) that $T_h : W_h \to W_h$ is continuous, and hence the Brouwer theorem implies the existence of at least a fixed point $(\mathbf{u}_h, \varphi_h) \in W_h$ of T_h. Then, defining $\vec{\mathbf{u}}_h := S_h(\mathbf{u}_h, \varphi_h)$ and $\vec{\varphi}_h := \widetilde{S}_h(\mathbf{u}_h)$, and letting $\boldsymbol{\sigma}_h$ and $\widetilde{\boldsymbol{\sigma}}_h$ be the second components of the solutions to (4.2) and (4.3) (or (4.10) and (4.16), respectively, with $(\mathbf{w}_h, \phi_h) = (\mathbf{u}_h, \varphi_h)$, we conclude that $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \widetilde{\mathbf{H}}_h \times \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ constitute a solution of (4.1) with $(\mathbf{u}_h, \varphi_h) \in W_h$. Finally, the estimates (4.26), (4.27), (4.28), and (4.29) follow straightforwardly from (4.9), (4.15), (4.11), and (4.17), respectively, by bounding $\|\mathbf{w}_h\| = \|\mathbf{u}_h\|$ and $\|\phi_h\| = \|\varphi_h\|$ by r. $\square$

We end this section by stressing that, in the particular case of a constant viscosity, the estimate (3.64) and the Banach fixed-point theorem can be applied to improve the foregoing result by proving both existence and uniqueness of solution of (4.1) .

5 Specific finite element subspaces

In this section we employ some tools from functional analysis to derive specific finite element subspaces $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{L}^4(\Omega)$, $\mathbb{H}_h^{\mathbf{t}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subseteq \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathrm{H}_h^{\varphi} \subseteq \mathrm{L}^4(\Omega)$, $\mathbf{H}_h^{\widetilde{\mathbf{t}}} \subseteq \mathbf{L}^2(\Omega)$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \subseteq \mathbf{H}(\mathrm{div}_{4/3};\Omega)$, satisfying the crucial discrete inf-sup conditions given by Assumptions 4.1, 4.2, 4.3, and 4.4. In what follows, given a positive integer ℓ and a set $\mathcal{O} \subseteq \mathrm{R}^n$, $\mathrm{P}_\ell(\mathcal{O})$ stands for the space of polynomials of degree $\le \ell$ defined on $\mathcal{O}$, with vector and tensorial versions denoted by $\mathbf{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^n$ and $\mathbb{P}_\ell(\mathcal{O}) := [\mathrm{P}_\ell(\mathcal{O})]^{n\times n}$, respectively. We begin the analysis with a section providing a couple of abstract results on inf-sup conditions.

5.1 Preliminary results on inf-sup conditions

In what follows, given X and Y reflexive Banach spaces and a bounded bilinear form $b : X \times Y \longrightarrow \mathrm{R}$, we let $B : X \longrightarrow Y'$ and $B' : Y \longrightarrow X'$ be the bounded linear operator and its *modified adjoint* induced by b, respectively, which are defined by

$$B(x)(y) \,:=\, b(x,y) \quad \text{and} \quad B'(y)(x) \,:=\, b(x,y) \qquad \forall\,(x,y) \in X \times Y\,.$$

Note that the concept *modified adjoint* employed here refers to the fact that, while the adjoint of B should actually act from Y'' to X', the reflexivity of Y allows to redefine it as stated.

Then, we have the following result.

Lemma 5.1 *Let X, Y, Y_1, Y_2, and Z be reflexive Banach spaces with Y_1 and Y_2 being closed subspaces of Y such that $Y = Y_1 \oplus Y_2$, and assume that the norm of Y can be redefined, equivalently, but with constants independent of Y_1 and Y_2, as $\|y\| := \|y_1\| + \|y_2\| \;\; \forall\, y = y_1 + y_2 \in Y$, with $y_j \in Y_j$ for $j \in \{1,2\}$. In addition, let $b : \big(X \times Y\big) \times Z \longrightarrow \mathrm{R}$ be a bounded bilinear form with boundedness constant denoted by $\|b\|$, and define the following subspaces:*

$$V \,:=\, \Big\{(x,y) \in X \times Y : \quad b\big((x,y),z\big) = 0 \quad \forall\, z \in Z\Big\}, \tag{5.1}$$

and

$$Z_0 \;:=\; \Big\{ z \in Z : \quad b\big((x,y_2),z\big) = 0 \quad \forall\,(x,y_2) \in X \times Y_2 \Big\}\,. \tag{5.2}$$

Then, the following statements are equivalent:

1) *there exist positive constants β_1 and β_2 such that*

$$\sup_{\substack{(x,y)\in X\times Y\\ (x,y)\neq \mathbf{0}}} \frac{b\big((x,y),z\big)}{\|(x,y)\|} \;\geq\; \beta_1\,\|z\| \qquad \forall\, z \in Z\,, \tag{5.3}$$

and

$$\|y_1\| \;\geq\; \beta_2\,\|(x,y_2)\| \quad \forall\,(x,y) \in V, \quad \textit{with} \quad y = y_1 + y_2 \in Y_1 \oplus Y_2 = Y\,. \tag{5.4}$$

2) *there exist positive constants β_3 and β_4 such that*

$$\sup_{\substack{z\in Z\\ z\neq 0}} \frac{b\big((x,y_2),z\big)}{\|z\|} \;\geq\; \beta_3\,\|(x,y_2)\| \qquad \forall\,(x,y_2) \in X \times Y_2\,, \tag{5.5}$$

and

$$\sup_{\substack{y_1\in Y_1\\ y_1\neq 0}} \frac{b\big((0,y_1),z\big)}{\|y_1\|} \;\geq\; \beta_4\,\|z\| \qquad \forall\, z \in Z_0\,. \tag{5.6}$$

Proof. Let us first assume 1) and prove 2). Then, taking in particular $z \in Z_0$ in (5.3), which means that $b\big((x,y_2),z\big) = 0 \;\; \forall\,(x,y_2) \in X \times Y_2$, and using that certainly $\|(x,y)\| \;\geq\; \|y_1\|$, we obtain

$$\beta_1\,\|z\| \;\leq\; \sup_{\substack{(x,y)\in X\times Y\\ (x,y)\neq \mathbf{0}}} \frac{b\big((x,y),z\big)}{\|(x,y)\|} \;=\; \sup_{\substack{(x,y)\in X\times Y\\ (x,y)\neq \mathbf{0}}} \frac{b\big((0,y_1),z\big)}{\|(x,y)\|} \;\leq\; \sup_{\substack{y_1\in Y_1\\ y_1\neq 0}} \frac{b\big((0,y_1),z\big)}{\|y_1\|}\,,$$

which shows (5.6) with $\beta_4 = \beta_1$. Now, denoting by $B : X \times Y \longrightarrow Z'$ the bounded linear operator induced by b with modified adjoint $B' : Z \longrightarrow \big(X \times Y\big)'$, we observe that (5.3) says equivalently that B' is injective and of closed rangle, that is that B is surjective, which in turn is equivalent to the inf-sup condition

$$\sup_{\substack{z\in Z\\ z\neq 0}} \frac{b\big((x,y),z\big)}{\|z\|} \;\geq\; \beta_1\,\|\,[(x,y)]\,\|_{\big(X\times Y\big)/V} \qquad \forall\,(x,y) \in X \times Y\,. \tag{5.7}$$

Thus, taking in particular $(x,y) = (x,y_2) \in X \times Y_2$, using (5.4), and employing the triangle inequality, we get

$$\begin{aligned}
\|\,[(x,y_2)]\,\|_{\big(X\times Y\big)/V} \;&:=\; \inf_{(w,s)\in V} \|(x,y_2) - (w,s)\| \;=\; \inf_{(w,s)\in V} \|(x-w,y_2-s_2-s_1)\| \\
&=\; \inf_{(w,s)\in V} \Big\{ \|x-w\| + \|y_2-s_2\| + \|s_1\| \Big\} \;=\; \inf_{(w,s)\in V} \Big\{ \|(x,y_2)-(w,s_2)\| + \|s_1\| \Big\} \\
&\geq\; \min\big\{1,\beta_2\big\} \inf_{(w,s)\in V} \Big\{ \|(x,y_2)-(w,s_2)\| + \|(w,s_2)\| \Big\} \;\geq\; \min\big\{1,\beta_2\big\}\,\|(x,y_2)\|\,,
\end{aligned}$$

which, together with (5.7), yields (5.5) with $\beta_3 \;=\; \beta_1 \min\big\{1,\beta_2\big\}$. Conversely, in what follows we suppose 2) and demonstrate 1). In fact, denoting by $B_1 : Y_1 \longrightarrow Z_0'$ the bounded linear operator induced by $b\big((0,\cdot),\cdot\big) : Y_1 \times Z_0 \longrightarrow \mathrm{R}$ with modified adjoint $B_1' : Z_0 \longrightarrow Y_1'$, we first realize that (5.6) says that B_1' is injective and of closed range, that is that B_1 is surjective. In this way, given $G \in Z'$, we obviously have that $G|_{Z_0} \in Z_0'$, and hence there exists $y_1 \in Y_1$ such that $B_1(y_1) = G|_{Z_0}$, that is $b\big((0,y_1),z\big) = G(z) \quad \forall\, z \in Z_0$, and also $\beta_4\, \|y_1\| \,\le\, \|G|_{Z_0}\| \,\le\, \|G\|$. On the other hand, denoting by $B_2 : X \times Y_2 \longrightarrow Z'$ the bounded linear operator induced by $b|_{\big(X\times Y_2\big)\times Z}$, we notice that (5.5) establishes that B_2 is injective and of closed range. It follows that $R(B_2) = N(B_2')^\circ$, with

$$N(B_2') \,:=\, \Big\{ z \in Z : \quad b\big((x,y_2),z\big) = 0 \quad \forall\, (x,y_2) \in X \times Y_2 \Big\} \,=\, Z_0\,,$$

and therefore

$$R(B_2) = \Big\{ F \in Z' : \quad F(z) = 0 \quad \forall\, z \in Z_0 \Big\} \equiv \Big(Z/Z_0 \Big)'\,.$$

According to the above, and since $G - b\big((0,y_1),\cdot\big) \in R(B_2)$, there exists a unique $(x_G, y_2) \in X \times Y_2$ such that $B_2(x_G,y_2) = G - b\big((0,y_1),\cdot\big)$, that is $b\big((x_G,y_2),z\big) = G(z) - b\big((0,y_1),z\big) \;\; \forall\, z \in Z$, and

$$\beta_3\, \|(x_G,y_2)\| \le \|G - b\big((0,y_1),\cdot\big)\| \le \|G\| + \|B_1\|\,\|y_1\| \le \big(1 + \beta_4^{-1}\|B_1\|\big)\, \|G\|\,.$$

Thus, defining $y_G := y_1 + y_2 \in Y$, we readily see that $b\big((x_G,y_G),z\big) = G(z) \;\; \forall\, z \in Z$, and

$$\|(x_G,y_G)\| \,\le\, \Big\{ \beta_3^{-1}\big(1 + \beta_4^{-1}\|B_1\|\big) + \beta_4^{-1} \Big\}\, \|G\|\,.$$

Next, given arbitrary $z \in Z$ and $G \in Z'$, and defining $\beta_1 := \Big\{ \beta_3^{-1}\big(1 + \beta_4^{-1}\|B_1\|\big) + \beta_4^{-1} \Big\}^{-1}$, we find that

$$\sup_{\substack{(x,y)\in X\times Y\\ (x,y)\neq \mathbf{0}}} \frac{b\big((x,y),z\big)}{\|(x,y)\|} \,\ge\, \frac{\big|b\big((x_G,y_G),z\big)\big|}{\|(x_G,y_G)\|} \,\ge\, \beta_1\, \frac{|G(z)|}{\|G\|}\,,$$

from which, taking supremum on all $G \in Z'$, we conclude (5.3). It remains to prove (5.4). To this end, given $(x,y) \in V$, with $y = y_1 + y_2 \in Y_1 \oplus Y_2 = Y$, we first recall that for each $z \in Z$ there holds $0 = b\big((x,y),z\big) = b\big((x,y_2),z\big) + b\big((0,y_1),z\big)$, that is $b\big((x,y_2),z\big) \,=\, -b\big((0,y_1),z\big)$. Hence, employing (5.5) we deduce that

$$\beta_3\, \|(x,y_2)\| \,\le\, \sup_{\substack{z\in Z\\ z\neq 0}} \frac{\big|b\big((x,y_2),z\big)\big|}{\|z\|} \,=\, \sup_{\substack{z\in Z\\ z\neq 0}} \frac{\big|b\big((0,y_1),z\big)\big|}{\|z\|} \,\le\, \|b\|\, \|y_1\|\,,$$

which yields (5.4) with $\beta_2 = \beta_3\, \|b\|^{-1}$. □

Having established the equivalence given by Lemma 5.1, we now provide sufficient conditions for the inf-sup condition (5.5).

Lemma 5.2 *In addition to the notations and assumptions from Lemma* 5.1*, we now introduce the subspace*

$$Z_1 \,:=\, \Big\{ z \in Z : \quad b\big((x,0),z\big) = 0 \qquad \forall\, x \in X \Big\}\,, \tag{5.8}$$

and assume that there exist positive constants β_5 and β_6 such that

$$\sup_{\substack{z\in Z\\ z\neq 0}} \frac{b\big((x,0),z\big)}{\|z\|} \,\ge\, \beta_5\, \|x\| \qquad \forall\, x \in X\,, \tag{5.9}$$

and

$$\sup_{\substack{z\in Z_1\\ z\neq 0}} \frac{b\big((0,y_2),z\big)}{\|z\|} \;\geq\; \beta_6\,\|y_2\| \qquad \forall\, y_2 \in Y_2\,. \tag{5.10}$$

Then, the inf-sup condition (5.5) *is satisfied.*

Proof. Given $(x, y_2) \in X \times Y_2$, we begin by noticing that (5.9) and (5.10) guarantee the existence of $\widetilde{z} \in Z$ and $\widehat{z} \in Z_1$, respectively, satisfying $\|\widetilde{z}\| = \|\widehat{z}\| = 1$, and the inequalities

$$b\big((x,0),\widetilde{z}\big) \;\geq\; \frac{\beta_5}{2}\,\|x\| \quad \text{and} \quad b\big((0,y_2),\widehat{z}\big) \;\geq\; \frac{\beta_6}{2}\,\|y_2\|\,.$$

Then, defining $\bar{z} := C_1\,\widetilde{z} + C_2\,\widehat{z}$, with positive constants C_1 and C_2 to be chosen later on, which yields $\|\bar{z}\| \leq C_1 + C_2$, and using that $b\big((x,0),\widehat{z}\big) = 0$, we obtain

$$\begin{aligned}
\sup_{\substack{z\in Z\\ z\neq 0}} \frac{b\big((x,y_2),z\big)}{\|z\|} \;&\geq\; \frac{\big|b\big((x,y_2),\bar{z}\big)\big|}{\|\bar{z}\|} \;=\; \frac{\big|b\big((x,0),\bar{z}\big) + b\big((0,y_2),\bar{z}\big)\big|}{\|\bar{z}\|} \\
&=\; \frac{\big|C_1\, b\big((x,0),\widetilde{z}\big) + C_2\, b\big((0,y_2),\widehat{z}\big) + C_1\, b\big((0,y_2),\widetilde{z}\big)\big|}{\|\bar{z}\|} \\
&\geq\; \frac{1}{C_1 + C_2}\left\{ \frac{C_1\,\beta_5}{2}\,\|x\| \;+\; \Big(\frac{C_2\,\beta_6}{2} - C_1\,\|b\|\Big)\,\|y_2\| \right\},
\end{aligned}$$

from which, choosing C_1 and C_2 such that $C_1 > 0$ and $\dfrac{C_2\,\beta_6}{2} > C_1\,\|b\|$, we arrive at (5.5) with β_3 depending on C_1, C_2, β_5, β_6, and $\|b\|$. For instance, taking $C_1 = 1$ and $C_2 = 4\|b\|\beta_6^{-1}$, we get $\beta_3 = \frac{1}{\left(1+4\|b\|\beta_6^{-1}\right)}\,\min\left\{\frac{\beta_5}{2}, \|b\|\right\}$. □

At this point we remark that a particular case of the equivalence between the statements (1) and (3) in [30, Theorem 3.1] would imply that actually (5.5) and the pair (5.9) - (5.10) are equivalent. However, we believe that the necessity of (5.9) - (5.10), and particularly that of (5.10), requires additionally that the kernel of the bilinear form $b\big((0,\cdot),\cdot\big) : Y_2 \times Z_1 \longrightarrow \mathrm{R}$ be the null space, that is that

$$\Big\{ y_2 \in Y_2 : \quad b\big((0,y_2),z\big) = 0 \quad \forall\, z \in Z_1 \Big\} \;=\; \Big\{0\Big\}\,, \tag{5.11}$$

which is not included in the statement of [30, Theorem 3.1]. In any case, and though (5.11) clearly follows from (5.10), for our analysis below we do not need neither such equivalence nor (5.11) as such, but only the sufficiency provided by Lemma 5.2.

5.2 The subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, and $\mathbb{H}_h^{\boldsymbol{\sigma}}$

We now aim to derive specific finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, and $\mathbb{H}_h^{\boldsymbol{\sigma}}$ satisfying the Assumptions 4.1 and 4.2. To this end, we first split $\mathbb{H}_h^{\mathbf{t}}$ as $\mathbb{H}_h^{\mathbf{t}} = \mathbb{H}_{h,sym}^{\mathbf{t}} \oplus \mathbb{H}_{h,skw}^{\mathbf{t}}$, where

$$\mathbb{H}_{h,sym}^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \mathbf{s}_h^{\mathtt{t}} \;=\; \mathbf{s}_h \Big\}\,, \tag{5.12}$$

and

$$\mathbb{H}_{h,skw}^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \mathbf{s}_h^{\mathtt{t}} \;=\; -\,\mathbf{s}_h \Big\}\,, \tag{5.13}$$

and observe, due to the orthogonality between $\mathbb{H}^{\mathbf{t}}_{h,sym}$ and $\mathbb{H}^{\mathbf{t}}_{h,skw}$, that for each $\mathbf{s}_h = \mathbf{s}_{h,sym} + \mathbf{s}_{h,skw} \in \mathbb{H}^{\mathbf{t}}_{h,sym} \oplus \mathbb{H}^{\mathbf{t}}_{h,skw} = \mathbb{H}^{\mathbf{t}}_h$ there holds

$$\frac{1}{\sqrt{2}}\Big\{\|\mathbf{s}_{h,sym}\|_{0,\Omega} + \|\mathbf{s}_{h,skw}\|_{0,\Omega}\Big\} \le \|\mathbf{s}_h\|_{0,\Omega} \le \|\mathbf{s}_{h,sym}\|_{0,\Omega} + \|\mathbf{s}_{h,skw}\|_{0,\Omega}\,.$$

Then, applying Lemmas 5.1 and 5.2 (particularly the fact that (5.3) and (5.4) follow from (5.6), (5.9), and (5.10)) to the setting given by the spaces

$$X = \mathbf{H}^{\mathbf{u}}_h\,,\quad Y_1 = \mathbb{H}^{\mathbf{t}}_{h,sym}\,,\quad Y_2 = \mathbb{H}^{\mathbf{t}}_{h,skw}\,,\quad Y = \mathbb{H}^{\mathbf{t}}_h\,,\quad Z = \mathbb{H}^{\boldsymbol{\sigma}}_h\,,$$

and our bilinear form b (cf. (3.14)), we conclude that, in order to verify ASSUMPTIONS 4.1 and 4.2, we just need to show the corresponding inf-sup conditions given by (5.6), (5.9), and (5.10). In other words, we need to prove that there exist positive constants β_4, β_5, and β_6, independent of h, such that

$$\sup_{\substack{\mathbf{s}_{h,sym}\in\mathbb{H}^{\mathbf{t}}_{h,sym}\\ \mathbf{s}_{h,sym}\neq 0}} \frac{b\big((0,\mathbf{s}_{h,sym}),\boldsymbol{\tau}_h\big)}{\|\mathbf{s}_{h,sym}\|_{0,\Omega}} = \sup_{\substack{\mathbf{s}_{h,sym}\in\mathbb{H}^{\mathbf{t}}_{h,sym}\\ \mathbf{s}_{h,sym}\neq 0}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,sym}}{\|\mathbf{s}_{h,sym}\|_{0,\Omega}} \ge \beta_4\,\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\,\boldsymbol{\tau}_h \in Z_{0,h}\,, \tag{5.14}$$

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\\ \boldsymbol{\tau}_h\neq 0}} \frac{b\big((\mathbf{v}_h,0),\boldsymbol{\tau}_h\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} = \sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}^{\boldsymbol{\sigma}}_h\\ \boldsymbol{\tau}_h\neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \beta_5\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h\,, \tag{5.15}$$

and

$$\sup_{\substack{\boldsymbol{\tau}_h\in Z_{1,h}\\ \boldsymbol{\tau}_h\neq 0}} \frac{b\big((0,\mathbf{s}_{h,skw}),\boldsymbol{\tau}_h\big)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} = \sup_{\substack{\boldsymbol{\tau}_h\in Z_{1,h}\\ \boldsymbol{\tau}_h\neq 0}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,skw}}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \ge \beta_6\,\|\mathbf{s}_{h,skw}\|_{0,\Omega} \qquad \forall\,\mathbf{s}_{h,skw} \in \mathbb{H}^{\mathbf{t}}_{h,skw}\,, \tag{5.16}$$

where, according to (5.2) and (5.8), we have

$$\begin{aligned} Z_{0,h} &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h : \quad b\big((\mathbf{v}_h,\mathbf{s}_{h,skw}),\boldsymbol{\tau}_h\big) = 0 \quad \forall\,(\mathbf{v}_h,\mathbf{s}_{h,skw}) \in \mathbf{H}^{\mathbf{u}}_h \times \mathbb{H}^{\mathbf{t}}_{h,skw}\Big\} \\ &= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h : \quad \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\,\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h \\ &\qquad\qquad \text{and} \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,skw} = 0 \qquad \forall\,\mathbf{s}_{h,skw} \in \mathbb{H}^{\mathbf{t}}_{h,skw}\Big\}\,, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} Z_{1,h} &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h : \quad b\big((\mathbf{v}_h,0),\boldsymbol{\tau}_h\big) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h\Big\} \\ &= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h : \quad \int_\Omega \mathbf{v}_h\cdot\mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\,\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h\Big\}\,. \end{aligned} \tag{5.18}$$

Throughout the rest of this section we address the verification of (5.16), for which we concentrate on the 2D case. As a result of this analysis we will be able to propose specific finite element subspaces $\mathbf{H}^{\mathbf{u}}_h$, $\mathbb{H}^{\mathbf{t}}_h$, and $\mathbb{H}^{\boldsymbol{\sigma}}_h$, which will then be considered in Section 5.3 to prove the remaining inf-sup conditions (5.14) and (5.15).

In order to deal with (5.16), we now proceed as in [9] and [24] (see also [10, Proposition 9.3.2] and [28, Section 4.5]), and let U_h and $\widehat{Q}_h$ be arbitrary finite element subspaces of $\mathbf{H}_0^1(\Omega)$ and $\mathrm{L}^2(\Omega)$, respectively, such that $\mathrm{P}_0(\Omega) \subseteq \widehat{Q}_h$, and so that U_h and $Q_h := \widehat{Q}_h \cap \mathrm{L}_0^2(\Omega)$ yield stability of the Galerkin scheme associated with the primal formulation of the Stokes problem. This means that, for each pair $(f,g) \in \mathbf{H}_0^1(\Omega)' \times \mathrm{L}_0^2(\Omega)'$, there exists a unique $(\mathbf{z}_h, p_h) \in U_h \times Q_h$, with $\mathbf{z}_h := (z_{h,1}, z_{h,2})^{\mathbf{t}}$, such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{z}_h : \nabla \mathbf{w}_h + \int_\Omega p_h \operatorname{div}(\mathbf{w}_h) &= f(\mathbf{w}_h) \qquad \forall\, \mathbf{w}_h \in U_h\,, \\ \int_\Omega q_h \operatorname{div}(\mathbf{z}_h) &= g(q_h) \qquad \forall\, q_h \in Q_h\,, \end{aligned} \tag{5.19}$$

and

$$\|\mathbf{z}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} \le C_0 \Big\{ \|f\| + \|g\| \Big\}, \tag{5.20}$$

with a positive constant C_0 independent of h and the subspaces U_h and $\widehat{Q}_h$. In particular, from now on we consider f as the null functional and g as the functional induced by a given $\widehat{q}_h \in \widehat{Q}_h$, that is $g(q_h) := \displaystyle\int_\Omega \widehat{q}_h\, q_h \;\forall\, q_h \in Q_h$. In this way, assuming that $\mathbf{P}_1(\Omega) \subseteq U_h$, and taking $\mathbf{w}_h(\mathbf{x}) := (-x_2, x_1)^{\mathbf{t}}$ $\forall\, \mathbf{x} := (x_1, x_2)^{\mathbf{t}} \in \Omega$, the first equation of (5.19) gives

$$\int_\Omega \Big\{ -\frac{\partial z_{h,1}}{\partial x_2} + \frac{\partial z_{h,2}}{\partial x_1} \Big\} = 0\,. \tag{5.21}$$

In turn, we let $\widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ be a finite element subspace of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ such that $\mathbb{P}_0(\Omega) \subseteq \widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, and set $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Then we assume that $\mathbf{H}_h^{\mathbf{u}}$ and $\widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ are chosen such that $\mathbf{div}(\widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{H}_h^{\mathbf{u}}$, whence (5.18) yields

$$Z_{1,h} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{5.22}$$

Next, we set

$$c_h := \frac{1}{|\Omega|} \int_\Omega \widehat{q}_h\,, \qquad \mathbf{curl}(\mathbf{z}_h) := \begin{pmatrix} -\dfrac{\partial z_{h,1}}{\partial x_2} & \dfrac{\partial z_{h,1}}{\partial x_1} \\ -\dfrac{\partial z_{h,2}}{\partial x_2} & \dfrac{\partial z_{h,2}}{\partial x_1} \end{pmatrix},$$

and define the tensor

$$\widehat{\boldsymbol{\tau}}_h = \mathbf{curl}(\mathbf{z}_h) + c_h \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which is obviously divergence free. In addition, we see from (5.21) that $\int_\Omega \operatorname{tr}(\widehat{\boldsymbol{\tau}}_h) = 0$, and assuming that $\mathbf{curl}\big(U_h\big) + \mathbb{P}_0(\Omega) \subseteq \widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, we realize that $\widehat{\boldsymbol{\tau}}_h \in Z_{1,h}$. Then, we notice that $\widehat{q}_h - c_h \in Q_h$, and observe, thanks to the divergence theorem, the fact that $\mathbf{z}_h$ vanishes on Γ, and the second equation of (5.19), that

$$\int_\Omega \widehat{q}_h \operatorname{div}(\mathbf{z}_h) = \int_\Omega \big(\widehat{q}_h - c_h\big) \operatorname{div}(\mathbf{z}_h) = \int_\Omega \widehat{q}_h \big(\widehat{q}_h - c_h\big) = \|\widehat{q}_h\|_{0,\Omega}^2 - |\Omega|\, c_h^2\,.$$

In this way, considering the particular choice $\mathbf{s}_{h,skw} = \begin{pmatrix} 0 & \widehat{q}_h \\ -\widehat{q}_h & 0 \end{pmatrix}$, we find that

$$\int_\Omega \widehat{\boldsymbol{\tau}}_h : \mathbf{s}_{h,skw} = \int_\Omega \widehat{q}_h \operatorname{div}(\mathbf{z}_h) + |\Omega|\, c_h^2 = \|\widehat{q}_h\|_{0,\Omega}^2 = \frac{1}{2} \|\mathbf{s}_{h,skw}\|_{0,\Omega}^2\,, \tag{5.23}$$

whereas the stability estimate (5.20) and the definition of c_h give

$$\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{4/3};\Omega} \,=\, \|\widehat{\boldsymbol{\tau}}_h\|_{0,\Omega} \,\leq\, |\mathbf{z}_h|_{1,\Omega} \,+\, \|c_h\|_{0,\Omega} \,\leq\, \widehat{C}_0\,\|\widehat{q}_h\|_{0,\Omega} \,=\, \frac{\widehat{C}_0}{\sqrt{2}}\,\|\mathbf{s}_{h,skw}\|_{0,\Omega}\,, \tag{5.24}$$

with a constant $\widehat{C}_0$ depending on C_0 and $|\Omega|$, and hence we conclude from (5.23) and (5.24) that

$$\sup_{\substack{\boldsymbol{\tau}_h\in Z_{1,h}\\ \boldsymbol{\tau}_h\neq 0}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,skw}}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\geq\, \frac{\displaystyle\int_\Omega \widehat{\boldsymbol{\tau}}_h : \mathbf{s}_{h,skw}}{\|\widehat{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\geq\, \beta_6\,\|\mathbf{s}_{h,skw}\|_{0,\Omega}\,,$$

with $\beta_6 = \sqrt{2}/(2\widehat{C}_0)$.

Summarizing, our previous analysis has shown the inf-sup condition (5.16) under the hypotheses

$$\begin{gathered} \mathrm{P}_0(\Omega) \,\subseteq\, \widehat{Q}_h\,, \qquad \mathbf{P}_1(\Omega) \,\subseteq\, U_h\,, \qquad \mathbb{P}_0(\Omega) \,\subseteq\, \widehat{\mathbb{H}}^{\boldsymbol{\sigma}}_h\,, \\ \mathbf{div}(\widehat{\mathbb{H}}^{\boldsymbol{\sigma}}_h) \,\subseteq\, \mathbf{H}^{\mathbf{u}}_h\,, \qquad \mathbf{curl}\big(U_h\big) + \mathbb{P}_0(\Omega) \,\subseteq\, \widehat{\mathbb{H}}^{\boldsymbol{\sigma}}_h\,, \end{gathered} \tag{5.25}$$

and assuming that $\mathbb{H}^{\mathbf{t}}_{h,skw}$ is defined as

$$\mathbb{H}^{\mathbf{t}}_{h,skw} \,:=\, \left\{ \mathbf{s}_{h,skw} \,:=\, \begin{pmatrix} 0 & \widehat{q}_h \\ -\,\widehat{q}_h & 0 \end{pmatrix} : \qquad \widehat{q}_h \in \widehat{Q}_h \right\}. \tag{5.26}$$

In addition, it is not difficult to see that the three-dimensional case follows analogously, by suitably modifying the definition of **curl** and the right-hand side of the second equation of (5.19), thus concluding (5.25) and the 3D version of (5.26) as well. We omit further details and refer to [10, Proposition 9.3.2].

In what follows, we consider the particular example of spaces U_h and Q_h given by the Scott-Vogelius pair, which, being usually employed to approximate the solutions of the Navier-Stokes equations, has also been shown to be stable for the Stokes problem with optimal approximation properties (see, e.g [38], [39], [41], [42], [43], and [44]). More precisely, given a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles (in R^2) or tetrahedra (in R^3) , we denote by $\mathcal{T}_h^{\mathsf{b}}$ the corresponding barycentric refinement of $\mathcal{T}_h$. In addition, letting h_K be the diameter of each $K \in \mathcal{T}_h^{\mathsf{b}}$, we also denote by h the meshsize of $\mathcal{T}_h^{\mathsf{b}}$, that is $h := \max\left\{ h_K : \quad K \in \mathcal{T}_h^{\mathsf{b}} \right\}$. In turn, for each $K \in \mathcal{T}_h^{\mathsf{b}}$ we let ρ_K be the diameter of the largest ball contained in K. Then, for each integer k such that $k+1 \geq n$, we define the Scott-Vogelius spaces as

$$U_h \,:=\, \left\{ \mathbf{w}_h \in \mathbf{C}(\bar{\Omega}) : \quad \mathbf{w}_h|_K \,\in\, \mathbf{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\,, \quad \mathbf{w}_h = 0 \quad \text{on} \quad \Gamma \right\}, \tag{5.27}$$

$$\widehat{Q}_h \,:=\, \left\{ \widehat{q}_h \in \mathrm{L}^2(\Omega) : \quad \widehat{q}_h|_K \,\in\, \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}} \right\}, \quad \text{and} \quad Q_h \,:=\, \widehat{Q}_h \,\cap\, \mathrm{L}^2_0(\Omega)\,. \tag{5.28}$$

According to the above, we observe that the first two inclusions in (5.25) are clearly satisfied. Next, it is straightforward to see that

$$\mathbf{curl}\big(U_h\big) + \mathbb{P}_0(\Omega) \,\subseteq\, \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \,\in\, \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}} \right\},$$

and therefore, letting $\mathbf{RT}_k(K) \,:=\, \mathbf{P}_k(K) \,\oplus\, \mathrm{P}_k(K)\,\mathbf{x}$ be the local Raviart-Thomas space of order k for each $K \in \mathcal{T}_h^{\mathsf{b}}$, where $\mathbf{x}$ denotes a generic vector in Ω, we deduce that, in order to satisfy the third and fifth inclusions of (5.25), it suffices to define

$$\widehat{\mathbb{H}}^{\boldsymbol{\sigma}}_h \,:=\, \left\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{c}^{\mathtt{t}}\,\boldsymbol{\tau}_h|_K \,\in\, \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n\,, \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}} \right\}, \tag{5.29}$$

and thus

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{c}^{\mathtt{t}}\, \boldsymbol{\tau}_h|_K \,\in\, \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n\,, \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,. \tag{5.30}$$

Morever, it is straightforward to see that, setting

$$\mathbf{H}_h^{\mathbf{u}} := \Big\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \,\in\, \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,, \tag{5.31}$$

the fourth inclusion in (5.25) is also verified, whereas (5.26) and (5.28) suggest to introduce

$$\mathbb{H}_h^{\mathbf{t}} := \Big\{ \mathbf{s}_h \,\in\, \mathbb{L}^2_{\mathrm{tr}}(\Omega) : \qquad \mathbf{s}_h|_K \,\in\, \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,. \tag{5.32}$$

In the next section we recall and provide several useful results on Raviart-Thomas spaces within the Banach framework given by the Sobolev spaces $\mathrm{W}^{m,p}$, and then in Section 5.4 we employ the specific finite element subspaces given by (5.30), (5.31), and (5.32), and the aforementioned results, to prove the remaining inf-sup conditions (5.14) and (5.15).

5.3 Some useful results on Raviart-Thomas spaces

We begin by defining for each $p > \dfrac{2n}{n+2}$:

$$\mathbf{H}_p := \Big\{ \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_p;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,p}(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,, \tag{5.33}$$

and

$$\widehat{\mathbf{H}}_h^{\boldsymbol{\sigma}} := \Big\{ \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_p;\Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,. \tag{5.34}$$

In addition, we let $\Pi_h^k : \mathbf{H}_p \to \widehat{\mathbf{H}}_h^{\boldsymbol{\sigma}}$ be the Raviart-Thomas interpolation operator, which is characterized for each $\boldsymbol{\tau} \in \mathbf{H}_p$ by the identities (see e.g. [23, Section 1.2.7]):

$$\int_e (\Pi_h^k(\boldsymbol{\tau}) \cdot \boldsymbol{\nu})\, \xi \,=\, \int_e (\boldsymbol{\tau} \cdot \boldsymbol{\nu})\, \xi \qquad \forall\, \xi \in \mathrm{P}_k(e), \quad \forall\, \text{edge or face } e \text{ of } \mathcal{T}_h^{\mathtt{b}}\,,$$

and

$$\int_K \Pi_h^k(\boldsymbol{\tau}) \cdot \psi \,=\, \int_K \boldsymbol{\tau} \cdot \psi \qquad \forall\, \psi \in \mathbf{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \quad (\text{if } k \geq 1)\,.$$

In turn, given $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we let

$$\mathrm{H}_h^{\mathbf{u}} := \Big\{ v \in \mathrm{L}^q(\Omega) : \quad v|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathtt{b}} \Big\}\,, \tag{5.35}$$

and recall from [23, Lemma 1.41] that there holds

$$\mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big) \,=\, \mathcal{P}_h^k\big(\mathrm{div}(\boldsymbol{\tau})\big) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}_p\,, \tag{5.36}$$

where $\mathcal{P}_h^k : \mathrm{L}^p(\Omega) \to \mathrm{H}_h^{\mathbf{u}}$ is the usual orthogonal projector with respect to the $\mathrm{L}^2(\Omega)$-inner product, which satisfies the following error estimate (see [23, Proposition 1.135]): there exists a positive constant C_0, independent of h, such that for $0 \leq l \leq k+1$ and $1 \leq p \leq \infty$ there holds

$$\|w - \mathcal{P}_h^k(w)\|_{0,p;\Omega} \,\leq\, C_0\, h^l\, \|w\|_{l,p;\Omega} \qquad \forall\, w \in \mathrm{W}^{l,p}(\Omega)\,. \tag{5.37}$$

In addition, we stress that $\mathcal{P}_h^k(w)|_K = \mathcal{P}_K^k(w|_K) \ \ \forall\, w \in \mathrm{L}^p(\Omega)$, where $\mathcal{P}_K^k : \mathrm{L}^p(K) \to \mathrm{P}_k(K)$ is the corresponding local orthogonal projector. Moreover, using the $\mathrm{W}^{m,p}$ version of the Deny-Lions Lemma (cf. [23, B.67]) and the associated scaling estimates (cf. [23, Lemma 1.101]), one can show the following approximation property of the projectors $\mathcal{P}_K^k$: there exists a positive constant C_1, independent of h, such that for $0 \le l \le k$, $0 \le m \le l+1$ and $p > 1$, and for each $K \in \mathcal{T}_h^{\mathrm{b}}$, there hold

$$|w - \mathcal{P}_K^k(w)|_{m,p;K} \,\le\, C_1\, \frac{h_K^{l+1}}{\rho_K^m}\, |w|_{l+1,p;K} \qquad \forall\, w \in \mathrm{W}^{l+1,p}(K)\,. \tag{5.38}$$

In turn, the local approximation properties of Π_h^k are established as in [28, Section 3.4.4], by using again [23, B.67] and [23, Lemma 1.101], but employing also (5.36) and (5.38). The corresponding statement is as follows.

Lemma 5.3 *Given $p > \dfrac{2n}{n+2}$, there exist positive constants C_2 and C_3, independent of h, such that for $0 \le l \le k$ and $0 \le m \le l+1$, and for each $K \in \mathcal{T}_h^{\mathrm{b}}$, there hold*

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{m,p;K} \,\le\, C_2\, \frac{h_K^{l+2}}{\rho_K^{m+1}}\, |\boldsymbol{\tau}|_{l+1,p;K} \tag{5.39}$$

for all $\boldsymbol{\tau} \in \mathbf{W}^{l+1,p}(K)$, and

$$|\mathrm{div}(\boldsymbol{\tau}) - \mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big)|_{m,p;K} \,\le\, C_3\, \frac{h_K^{l+1}}{\rho_K^m}\, |\mathrm{div}(\boldsymbol{\tau})|_{l+1,p;K} \tag{5.40}$$

for all $\boldsymbol{\tau} \in \mathbf{W}^{1,p}(K)$ with $\mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{l+1,p}(K)$.

Next, applying the regularity of the meshes together with the estimates (5.39) (for $m = 0$ and $p = 2$) and (5.40) (for $m = 0$ and $p > \dfrac{2n}{n+2}$) to each $K \in \mathcal{T}_h^{\mathrm{b}}$, we deduce the existence of positive constants $\widetilde{C}_2$ and $\widetilde{C}_3$, independent of h, such that for $0 \le l \le k$ there hold

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \,\le\, \widetilde{C}_2\, h^{l+1}\, |\boldsymbol{\tau}|_{l+1,\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}^{l+1}(\Omega)\,,$$

and

$$\|\mathrm{div}(\boldsymbol{\tau}) - \mathrm{div}\big(\Pi_h^k(\boldsymbol{\tau})\big)\|_{0,p;\Omega} \,\le\, \widetilde{C}_3\, h^{l+1}\, |\mathrm{div}(\boldsymbol{\tau})|_{l+1,p;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{1,p}(\Omega) \text{ with } \ \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{l+1,p}(\Omega)\,,$$

respectively, which yield the existence of a positive constant $\widetilde{C}_4$, independent of h, such that for $0 \le l \le k$ there hold

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{\mathrm{div}_p;\Omega} \,\le\, \widetilde{C}_4\, h^{l+1} \Big\{ |\boldsymbol{\tau}|_{l+1,\Omega} \,+\, |\mathrm{div}(\boldsymbol{\tau})|_{l+1,p;\Omega} \Big\}\,, \tag{5.41}$$

for all $\boldsymbol{\tau} \in \mathbf{H}^{l+1}(\Omega)$ with $\mathrm{div}(\boldsymbol{\tau}) \in \mathrm{W}^{l+1,p}(\Omega)$.

Furthermore, we have the following additional estimate concerning Π_h^k, which will be employed below in the proof of Lemma 5.5 for the particular case $p = 4/3$.

Lemma 5.4 *Assume that $1 \le p < n$ and $p \le 2 \le \frac{np}{n-p}$. Then, there exists a positive constant C_5, independent of h, such that for $0 \le l \le k$ there holds*

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,\Omega} \,\le\, C_5\, h^{l+1-n(2-p)/2p}\, |\boldsymbol{\tau}|_{l+1,p;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{l+1,p}(\Omega)\,. \tag{5.42}$$

Proof. We first observe that the assumptions on p and the Sobolev embedding Theorem (cf. [1, Theorem 4.12], [23, Corollary B.43], [37, Theorem 1.3.4]) guarantee the continuous injection of $\mathrm{W}^{1,p}(\mathcal{O})$ into $\mathrm{L}^2(\mathcal{O})$ for each open set $\mathcal{O}$ with Lipschitz-continuous boundary (cf. [23, Theorem B.37]). In particular, and denoting by $\widehat{K}$ the reference triangle (or tetrahedron in R^3) for $\mathcal{T}_h^{\mathrm{b}}$, the above implies the existence of a positive constant $\widehat{c}$, depending only on $\widehat{K}$, such that

$$\|w\|_{0,\widehat{K}} \le \widehat{c}\,\|w\|_{1,p;\widehat{K}} \qquad \forall\, w \in \mathrm{W}^{1,p}(\widehat{K})\,. \tag{5.43}$$

Next, given $K \in \mathcal{T}_h^{\mathrm{b}}$, we let $F_K : \widehat{K} \to K$ be the bijective affine mapping defined by $F_K(\widehat{\mathbf{x}}) := B_K\mathbf{x} + b_K$ $\forall \mathbf{x} \in \widehat{K}$, with $B_K \in \mathrm{R}^{n\times n}$ invertible and $b_K \in \mathrm{R}^n$. Then, using $\widehat{\ }$ to denote composition with F_K, we obtain from the usual scaling estimates (cf. [23, Lemma 1.101]) and (5.43) that

$$\begin{aligned}
\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,K} \,&\le\, |\det B_K|^{1/2}\,\|\widehat{\boldsymbol{\tau}} - \widehat{\Pi_h^k(\boldsymbol{\tau})}\|_{0,\widehat{K}} \,\le\, \widehat{c}\,|\det B_K|^{1/2}\,\|\widehat{\boldsymbol{\tau}} - \widehat{\Pi_h^k(\boldsymbol{\tau})}\|_{1,p;\widehat{K}} \\
&\le\, \widehat{c}\,|\det B_K|^{1/2}\left\{|\widehat{\boldsymbol{\tau}} - \widehat{\Pi_h^k(\boldsymbol{\tau})}|_{0,p;\widehat{K}} \,+\, |\widehat{\boldsymbol{\tau}} - \widehat{\Pi_h^k(\boldsymbol{\tau})}|_{1,p;\widehat{K}}\right\} \\
&\le\, \widehat{c}\,|\det B_K|^{-(2-p)/2p}\left\{|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{0,p;K} \,+\, \|B_K\|\,|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{1,p;K}\right\}.
\end{aligned} \tag{5.44}$$

Now, employing again the regularity of the meshes together with the estimate (5.39) for $m = 0$ and $m = 1$, we find a positive constant $\bar{C}_2$, independent of h, such that

$$|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{0,p;K} \,\le\, \bar{C}_2\, h_K^{l+1}\,|\boldsymbol{\tau}|_{l+1,p;K} \qquad \text{and} \qquad |\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})|_{1,p;K} \,\le\, \bar{C}_2\, h_K^{l}\,|\boldsymbol{\tau}|_{l+1,p;K}\,, \tag{5.45}$$

for all $\boldsymbol{\tau} \in \mathbf{W}^{l+1,p}(K)$. In this way, replacing (5.45) back into (5.44), and recalling that $|\det B_K| = O(h_K^n)$ and $\|B_K\| = O(h_K)$, we readily deduce that

$$\|\boldsymbol{\tau} - \Pi_h^k(\boldsymbol{\tau})\|_{0,K} \,\le\, 2\,\widehat{c}\,\bar{C}_2\, h_K^{l+1-n(2-p)/2p}\,|\boldsymbol{\tau}|_{l+1,p;K} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{W}^{l+1,p}(K)\,,$$

from which, taking square, and then summing up over all $K \in \mathcal{T}_h^{\mathrm{b}}$, we arrive at (5.42) and conclude the proof. □

We now let $\mathbb{H}_p$ be the tensorial version of $\mathbf{H}_p$ (cf. (5.33)) and observe that $\widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ (cf. (5.29)) and $\mathbf{H}_h^{\mathbf{u}}$ (cf. (5.31)) are the tensorial and vector versions of $\widehat{\mathbf{H}}_h^{\boldsymbol{\sigma}}$ (cf. (5.34)) and $\mathrm{H}_h^{\mathbf{u}}$ (cf. (5.35)), respectively, for $p = 4/3$. Then, we let $\boldsymbol{\Pi}_h^k : \mathbb{H}_p \to \widehat{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ be the corresponding Raviart-Thomas interpolation operator, which is defined row-wise by Π_h^k, and let $\boldsymbol{\mathcal{P}}_h^k : \mathbf{L}^p(\Omega) \to \mathbf{H}_h^{\mathbf{u}}$ be the corresponding orthogonal projector with respect to the $\mathbf{L}^2(\Omega)$-inner product, which is defined component-wise by $\mathcal{P}_h^k$. We end this section by highlighting that $\boldsymbol{\Pi}_h^k$ and $\boldsymbol{\mathcal{P}}_h^k$ satisfy the analogue of all the properties described above for Π_h^k and $\mathcal{P}_h^k$.

5.4 The remaining inf-sup conditions for $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, and $\mathbb{H}_h^{\boldsymbol{\sigma}}$

We first establish the discrete inf-sup condition (5.15).

Lemma 5.5 *There exists a positive constant β_5, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\ge\, \beta_5\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,. \tag{5.46}$$

Proof. The proof begins similarly to [13, Lemma 4.4] (see also [14, Lemma 3.3]). Indeed, given $\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}$, we let $\mathcal{O}$ be a convex bounded domain containing $\bar{\Omega}$, and define

$$\boldsymbol{g} := \begin{cases} |\mathbf{v}_h|^2 \, \mathbf{v}_h & \text{in} \quad \Omega\,, \\ \mathbf{0} & \text{in} \quad \mathcal{O} \setminus \bar{\Omega}\,. \end{cases} \tag{5.47}$$

It is easy to see that $\boldsymbol{g} \in \mathbf{L}^{4/3}(\mathcal{O})$ with

$$\|\boldsymbol{g}\|_{0,4/3;\mathcal{O}} \,=\, \|\boldsymbol{g}\|_{0,4/3;\Omega} \,=\, \|\, |\mathbf{v}_h|^2 \, \mathbf{v}_h \,\|_{0,4/3;\Omega} \,=\, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \tag{5.48}$$

It follows that there exists a unique $\mathbf{z} \in \mathbf{W}^{2,4/3}(\mathcal{O}) \cap \mathbf{W}_0^{1,4/3}(\mathcal{O})$ solution to the Dirichlet boundary value problem

$$\Delta \mathbf{z} \,=\, \boldsymbol{g} \quad \text{in} \quad \mathcal{O}\,, \qquad \mathbf{z} \,=\, \mathbf{0} \quad \text{on} \quad \partial\mathcal{O}\,, \tag{5.49}$$

and the corresponding regularity estimate (see e.g. [27]) guarantees the existence of a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|\mathbf{z}\|_{2,4/3;\mathcal{O}} \,\leq\, C_{\mathtt{reg}} \, \|\boldsymbol{g}\|_{0,4/3;\Omega} \,=\, C_{\mathtt{reg}} \, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \tag{5.50}$$

Next, we set $\boldsymbol{\zeta} := \nabla \mathbf{z}|_\Omega \in \mathbb{W}^{1,4/3}(\Omega)$, and observe from (5.49) and (5.50) that

$$\mathbf{div}(\boldsymbol{\zeta}) \,=\, \boldsymbol{g} \,=\, |\mathbf{v}_h|^2 \, \mathbf{v}_h \quad \text{in} \quad \Omega\,, \tag{5.51}$$

and

$$\|\boldsymbol{\zeta}\|_{1,4/3;\Omega} \,\leq\, \|\mathbf{z}\|_{2,4/3;\mathcal{O}} \,\leq\, C_{\mathtt{reg}} \, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \tag{5.52}$$

Furthermore, applying (5.42) to $\boldsymbol{\zeta}$, with $l = 0$ and $p = 4/3$ (which satisfy the assumptions required by Lemma 5.4), we find that

$$\|\boldsymbol{\zeta} - \mathbf{\Pi}_h^k(\boldsymbol{\zeta})\|_{0,\Omega} \,\leq\, C_5 \, h^{1-n/4} \, |\boldsymbol{\zeta}|_{1,4/3;\Omega} \leq \widetilde{C}_5 \, |\boldsymbol{\zeta}|_{1,4/3;\Omega} \,\leq\, \widetilde{C}_5 \, C_{\mathtt{reg}} \, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,, \tag{5.53}$$

with a positive constant $\widetilde{C}_5$, independent of h. Thus, defining $\boldsymbol{\zeta}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\boldsymbol{\zeta}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ as the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-components of $\mathbf{\Pi}_h^k(\boldsymbol{\zeta})$ and $\boldsymbol{\zeta}$, respectively, that is

$$\boldsymbol{\zeta}_h \,:=\, \mathbf{\Pi}_h^k(\boldsymbol{\zeta}) \,-\, \frac{1}{n|\Omega|} \int_\Omega \operatorname{tr}\big(\mathbf{\Pi}_h^k(\boldsymbol{\zeta})\big) \, \mathbb{I} \quad \text{and} \quad \boldsymbol{\zeta}_0 \,:=\, \boldsymbol{\zeta} \,-\, \frac{1}{n|\Omega|} \int_\Omega \operatorname{tr}\big(\boldsymbol{\zeta}\big) \, \mathbb{I}\,,$$

and using (5.52), (5.53), and the continuous injection of $\mathbb{W}^{1,4/3}(\Omega)$ into $\mathbb{L}^2(\Omega)$ with boundedness constant c_0, we obtain

$$\begin{aligned} \|\boldsymbol{\zeta}_h\|_{0,\Omega} \,&\leq\, \|\boldsymbol{\zeta}_h - \boldsymbol{\zeta}_0\|_{0,\Omega} \,+\, \|\boldsymbol{\zeta}_0\|_{0,\Omega} \,\leq\, \|\mathbf{\Pi}_h^k(\boldsymbol{\zeta}) - \boldsymbol{\zeta}\|_{0,\Omega} \,+\, \|\boldsymbol{\zeta}\|_{0,\Omega} \\ &\leq\, \|\mathbf{\Pi}_h^k(\boldsymbol{\zeta}) - \boldsymbol{\zeta}\|_{0,\Omega} \,+\, c_0 \, \|\boldsymbol{\zeta}\|_{1,4/3;\Omega} \,\leq\, \big(\widetilde{C}_5 + c_0\big) \, C_{\mathtt{reg}} \, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \end{aligned} \tag{5.54}$$

In addition, it is clear from (5.36) and (5.51) that

$$\mathbf{div}(\boldsymbol{\zeta}_h) \,=\, \mathbf{div}\big(\mathbf{\Pi}_h^k(\boldsymbol{\zeta})\big) \,=\, \boldsymbol{\mathcal{P}}_h^k\big(\mathbf{div}(\boldsymbol{\zeta})\big) \,=\, \boldsymbol{\mathcal{P}}_h^k\big(|\mathbf{v}_h|^2 \, \mathbf{v}_h\big)\,, \tag{5.55}$$

and hence, utilizing the triangle inequality, (5.51), and (5.48), we get

$$\begin{aligned} \|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,4/3;\Omega} \,&\leq\, \|\mathbf{div}(\boldsymbol{\zeta}) - \mathbf{div}\big(\mathbf{\Pi}_h^k(\boldsymbol{\zeta})\big)\|_{0,4/3;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,4/3;\Omega} \\ &=\, \|\mathbf{div}(\boldsymbol{\zeta}) - \mathbf{div}\big(\mathbf{\Pi}_h^k(\boldsymbol{\zeta})\big)\|_{0,4/3;\Omega} \,+\, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \end{aligned} \tag{5.56}$$

In turn, applying (5.40) with $m = l = 0$ to each $K \in \mathcal{T}_h^{\mathtt{b}}$, and then employing a local inverse inequality for the polynomial $\mathbf{div}(\boldsymbol{\zeta})|_K = |\mathbf{v}_h|^2\,\mathbf{v}_h|_K$, which follows from the usual scaling estimates and the fact that all the norms in any polynomial space defined on $\widehat{K}$ are equivalent, we deduce that

$$\begin{aligned} \|\mathbf{div}(\boldsymbol{\zeta}) - \mathbf{div}\big(\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\big)\|_{0,4/3;K} \,&\leq\, C_3\, h_K\, |\mathbf{div}(\boldsymbol{\zeta})|_{1,4/3;K} \\ &\leq\, \bar{C}_3\, |\mathbf{div}(\boldsymbol{\zeta})|_{0,4/3;K} \,=\, \bar{C}_3\, \|\mathbf{v}_h\|_{0,4;K}^3 \,, \end{aligned} \tag{5.57}$$

with a positive constant $\bar{C}_3$, independent of h. In this way, taking the above inequality to the power $4/3$, and then summing up over all $K \in \mathcal{T}_h^{\mathtt{b}}$, we easily arrive at

$$\|\mathbf{div}(\boldsymbol{\zeta}) - \mathbf{div}\big(\boldsymbol{\Pi}_h^k(\boldsymbol{\zeta})\big)\|_{0,4/3;\Omega} \,\leq\, \bar{C}_3\, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,,$$

which, replaced back into (5.56), yields

$$\|\mathbf{div}(\boldsymbol{\zeta}_h)\|_{0,4/3;\Omega} \,\leq\, \big(1 + \bar{C}_3\big)\, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,, \tag{5.58}$$

and hence, (5.54) and (5.58) imply

$$\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, \Big\{1 + \bar{C}_3 + \big(\widetilde{C}_5 + c_0\big)\, C_{\mathtt{reg}}\Big\}\, \|\mathbf{v}_h\|_{0,4;\Omega}^3 \,. \tag{5.59}$$

Finally, using (5.55) and the orthogonality property of $\boldsymbol{\mathcal{P}}_h^k$, we obtain

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq 0}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\geq\, \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\zeta}_h)}{\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \boldsymbol{\mathcal{P}}_h^k\big(|\mathbf{v}_h|^2\,\mathbf{v}_h\big)}{\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}} \,=\, \frac{\|\mathbf{v}_h\|_{0,4;\Omega}^4}{\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}} \,,$$

which, combined with the estimate (5.59), gives (5.46) with $\beta_5 = \Big\{1 + \bar{C}_3 + \big(\widetilde{C}_5 + c_0\big)\, C_{\mathtt{reg}}\Big\}^{-1}$, thus concluding the proof. □

We now aim to prove the discrete inf-sup condition (5.14), that is the existence of a positive constant β_4, independent of h, such that

$$\sup_{\substack{\mathbf{s}_{h,sym} \in \mathbb{H}_{h,sym}^{\mathbf{t}} \\ \mathbf{s}_{h,sym} \neq 0}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,sym}}{\|\mathbf{s}_{h,sym}\|_{0,\Omega}} \,\geq\, \beta_4\, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in Z_{0,h} \,, \tag{5.60}$$

where (cf. (5.12), (5.13), (5.17))

$$\mathbb{H}_{h,sym}^{\mathbf{t}} \,:=\, \Big\{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \mathbf{s}_h^{\mathtt{t}} \,=\, \mathbf{s}_h\Big\}\,, \quad \mathbb{H}_{h,skw}^{\mathbf{t}} \,:=\, \Big\{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \mathbf{s}_h^{\mathtt{t}} \,=\, -\,\mathbf{s}_h\Big\}\,, \tag{5.61}$$

$$\begin{aligned} Z_{0,h} \,:=\, \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad &\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \\ \text{and} \quad &\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,skw} = 0 \qquad \forall\, \mathbf{s}_{h,skw} \in \mathbb{H}_{h,skw}^{\mathbf{t}}\Big\}\,, \end{aligned} \tag{5.62}$$

and the specific finite element subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\mathbf{t}}$, are defined by (5.30), (5.31), and (5.32), respectively. In particular, according to (5.30) and (5.31), and as observed before, we get

$$Z_{0,h} \,:=\, \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \text{ in } \Omega\,, \quad \text{and} \quad \int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,skw} = 0 \quad \forall\, \mathbf{s}_{h,skw} \in \mathbb{H}_{h,skw}^{\mathbf{t}}\Big\}\,. \tag{5.63}$$

In turn, proceeding exactly as in part of the proof of [28, Theorem 3.3, Section 3.3], it is easy to show that if $\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}$ is such that $\mathbf{div}(\boldsymbol{\tau}_h) = 0$ in Ω, then necessarily $\boldsymbol{\tau}_h|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}_h^{\mathtt{b}}$. Moreover, once knowing the above for a given $\boldsymbol{\tau}_h \in Z_{0,h}$, we realize that the second identity in (5.63) together with the definition of $\mathbb{H}_h^{\mathbf{t}}$ and $\mathbb{H}_{h,skw}^{\mathbf{t}}$, imply that $\boldsymbol{\tau}_h = \boldsymbol{\tau}_h^{\mathbf{t}}$, which yields $\boldsymbol{\tau}_h^{\mathtt{d}} \in \mathbb{H}_{h,sym}^{\mathbf{t}}$. On the other hand, we also recall from [28, Lemma 2.3] that there exists a positive constant c_1, depending only on Ω, such that

$$c_1 \, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \, \leq \, \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega}^2 \, + \, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}^2 \qquad \forall \, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) \,. \tag{5.64}$$

According to the previous discussion, we conclude that for each $\boldsymbol{\tau}_h \in Z_{0,h}$, there holds

$$\sup_{\substack{\mathbf{s}_{h,sym} \in \mathbb{H}_{h,sym}^{\mathbf{t}} \\ \mathbf{s}_{h,sym} \neq 0}} \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \mathbf{s}_{h,sym}}{\|\mathbf{s}_{h,sym}\|_{0,\Omega}} \, \geq \, \frac{\displaystyle\int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\tau}_h^{\mathtt{d}}}{\|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega}} \, = \, \|\boldsymbol{\tau}_h^{\mathtt{d}}\|_{0,\Omega} \, \geq \, c_1^{1/2} \, \|\boldsymbol{\tau}_h\|_{0,\Omega} \, = \, c_1^{1/2} \, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,,$$

which proves (5.60), that is (5.14), with $\beta_4 = c_1^{1/2}$.

5.5 The finite element subspaces H_h^{φ}, $\mathbf{H}_h^{\widetilde{\mathbf{t}}}$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$

In this section we specify finite element subspaces H_h^{φ}, $\mathbf{H}_h^{\widetilde{\mathbf{t}}}$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ satisfying the Assumptions 4.3 and 4.4. To this end, we now apply Lemma 5.1 to the setting given by the spaces

$$X \, = \, \mathrm{H}_h^{\varphi} \,, \quad Y \, = \, Y_1 \, = \, \mathbf{H}_h^{\widetilde{\mathbf{t}}} \,, \quad Y_2 \, = \, \Big\{ \mathbf{0} \Big\} \,, \quad Z \, = \, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \,,$$

and our bilinear form $\widetilde{b}$ (cf. (3.16)). In this way, we conclude that verifying the aforementioned assumptions is equivalent to showing the corresponding inf-sup conditions given by (5.5) and (5.6). This means that we just need to prove that there exist positive constants $\widetilde{\beta}_3$ and $\widetilde{\beta}_4$, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \\ \widetilde{\boldsymbol{\tau}}_h \neq 0}} \frac{\widetilde{b}\big((\psi_h, 0), \widetilde{\boldsymbol{\tau}}_h\big)}{\|\widetilde{\boldsymbol{\tau}}_h\|} \, = \, \sup_{\substack{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \\ \widetilde{\boldsymbol{\tau}}_h \neq 0}} \frac{\displaystyle\int_\Omega \psi_h \, \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_{4/3};\Omega}} \, \geq \, \widetilde{\beta}_3 \, \|\psi_h\|_{0,4;\Omega} \qquad \forall \, \psi_h \in \mathrm{H}_h^{\varphi} \,, \tag{5.65}$$

and

$$\sup_{\substack{\widetilde{\mathbf{s}}_h \in \mathbf{H}_h^{\widetilde{\mathbf{t}}} \\ \widetilde{\mathbf{s}}_h \neq 0}} \frac{\widetilde{b}\big((0, \widetilde{\mathbf{s}}_h), \widetilde{\boldsymbol{\tau}}_h\big)}{\|\widetilde{\mathbf{s}}_h\|} \, = \, \sup_{\substack{\widetilde{\mathbf{s}}_h \in \mathbf{H}_h^{\widetilde{\mathbf{t}}} \\ \widetilde{\mathbf{s}}_h \neq 0}} \frac{\displaystyle\int_\Omega \widetilde{\mathbf{s}}_h \cdot \widetilde{\boldsymbol{\tau}}_h}{\|\widetilde{\mathbf{s}}_h\|_{0,\Omega}} \, \geq \, \widetilde{\beta}_4 \, \|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_{4/3};\Omega} \qquad \forall \, \widetilde{\boldsymbol{\tau}}_h \in \widetilde{Z}_{0,h} \,, \tag{5.66}$$

where, according to (5.2), we have

$$\begin{aligned} \widetilde{Z}_{0,h} \, &:= \, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} : \quad \widetilde{b}\big((\psi_h, 0), \widetilde{\boldsymbol{\tau}}_h\big) = 0 \quad \forall \, \psi_h \in \mathrm{H}_h^{\varphi} \Big\} \\ &= \, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} : \quad \int_\Omega \psi_h \, \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h) = 0 \quad \forall \, \psi_h \in \mathrm{H}_h^{\varphi} \Big\} \,. \end{aligned} \tag{5.67}$$

In virtue of the above discussion, and bearing in mind the analysis already developed in Sections 5.2, 5.3 and 5.4, in particular realizing the similarities between the pairs of discrete inf-sup conditions given by (5.15) - (5.14) and (5.65) - (5.66), we propose now to define H_h^{φ}, $\mathbf{H}_h^{\widetilde{\mathbf{t}}}$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ as follows

$$\mathrm{H}_h^{\varphi} \, := \, \Big\{ \psi_h \in \mathrm{L}^4(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K) \qquad \forall \, K \in \mathcal{T}_h^{\mathtt{b}} \Big\} \,, \tag{5.68}$$

$$\mathbf{H}_h^{\widetilde{\mathbf{t}}} \,:=\, \Big\{ \widetilde{\mathbf{s}}_h \in \mathbf{L}^2(\Omega) : \quad \widetilde{\mathbf{s}}_h|_K \in \mathbf{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathsf{b}} \Big\}\,, \tag{5.69}$$

and

$$\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega) : \quad \widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{RT}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathsf{b}} \Big\}\,. \tag{5.70}$$

It is clear from (5.68) and (5.70) that $\mathrm{div}\big(\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big) \subseteq \mathrm{H}_h^{\varphi}$, and hence (5.67) becomes

$$\widetilde{Z}_{0,h} \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h) = 0 \quad \text{in} \quad \Omega \Big\}\,.$$

Moreover, proceeding again as in part of the proof of [28, Theorem 3.3, Section 3.3], we can show that if $\widetilde{\boldsymbol{\tau}}_h \in \widetilde{Z}_{0,h}$, then necessarily $\widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}$, and hence $\widetilde{Z}_{0,h} \subseteq \mathbf{H}_h^{\widetilde{\mathbf{t}}}$. It follows that for each $\widetilde{\boldsymbol{\tau}}_h \in \widetilde{Z}_{0,h}$ there holds

$$\sup_{\substack{\widetilde{\mathbf{s}}_h \in \mathbf{H}_h^{\widetilde{\mathbf{t}}} \\ \widetilde{\mathbf{s}}_h \neq 0}} \frac{\displaystyle\int_\Omega \widetilde{\mathbf{s}}_h \cdot \widetilde{\boldsymbol{\tau}}_h}{\|\widetilde{\mathbf{s}}_h\|_{0,\Omega}} \,\geq\, \frac{\displaystyle\int_\Omega \widetilde{\boldsymbol{\tau}}_h \cdot \widetilde{\boldsymbol{\tau}}_h}{\|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}} \,=\, \|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \,=\, \|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_{4/3};\Omega)}\,,$$

which shows (5.66) with $\widetilde{\beta}_4 = 1$.

In turn, due to the definitions of H_h^{φ} and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$ (cf. (5.68) and (5.70)), the inf-sup condition (5.65) corresponds essentially to the vector version of (5.46), and hence its proof is almost verbatim to the one of Lemma 5.5. The only difference lies on the fact that in this case the supremum on the left hand side of (5.65) is bounded below by choosing simply $\widetilde{\boldsymbol{\zeta}}_h \,=\, \Pi_h^k(\nabla z|_\Omega) \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, where, taking $\mathcal{O}$ as before, $z \in \mathrm{W}^{2,4/3}(\mathcal{O}) \cap \mathrm{W}_0^{1,4/3}(\mathcal{O})$ is the unique solution of the scalar version of (5.49), that is, given $\psi_h \in \mathrm{H}_h^{\varphi}$, z solves:

$$\Delta z \,=\, g \,:=\, \begin{cases} |\psi_h|^2\, \psi_h & \text{in} \quad \Omega \\ \quad 0 & \text{in} \quad \mathcal{O} \setminus \bar{\Omega} \end{cases} \,, \qquad z \,=\, 0 \quad \text{on} \quad \partial\mathcal{O}\,.$$

We omit further details and refer to the proof of Lemma 5.5.

We end this section by collecting next the approximation properties of the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, H_h^{φ}, $\mathbf{H}_h^{\widetilde{\mathbf{t}}}$, and $\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, which basically follow from interpolation estimates of Sobolev spaces and the approximation properties provided by the projectors $\mathcal{P}_h^k$ and $\boldsymbol{\mathcal{P}}_h^k$ (cf. (5.37)), and the interpolation operators Π_h^k and $\boldsymbol{\Pi}_h^k$ (cf. (5.41)) (see, also [10], [12], [14], [28]):

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \,:=\, \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \,\leq\, C\, h^l\, \|\mathbf{v}\|_{l,4;\Omega}\,. \tag{5.71}$$

($\mathbf{AP}_h^{\mathbf{t}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$ there holds

$$\mathrm{dist}(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}) \,:=\, \inf_{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \,\leq\, C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,. \tag{5.72}$$

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, C\, h^l \Big\{ \|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \Big\}\,. \tag{5.73}$$

($\mathbf{AP}_h^{\varphi}$) there exists $C>0$, independent of h, such that for each $l\in[0,k+1]$, and for each $\psi\in \mathrm{W}^{l,4}(\Omega)$ there holds

$$\mathrm{dist}(\psi,\mathrm{H}_h^{\varphi}) \,:=\, \inf_{\psi_h\in \mathrm{H}_h^{\varphi}} \|\psi-\psi_h\|_{0,4;\Omega} \,\le\, C\,h^l\,\|\psi\|_{l,4;\Omega}\,. \tag{5.74}$$

($\mathbf{AP}_h^{\widetilde{\mathbf{t}}}$) there exists $C>0$, independent of h, such that for each $l\in[0,k+1]$, and for each $\widetilde{\mathbf{s}}\in \mathbf{H}^l(\Omega)$ there holds

$$\mathrm{dist}(\widetilde{\mathbf{s}},\mathbf{H}_h^{\widetilde{\mathbf{t}}}) \,:=\, \inf_{\widetilde{\mathbf{s}}_h\in \mathbf{H}_h^{\widetilde{\mathbf{t}}}} \|\widetilde{\mathbf{s}}-\widetilde{\mathbf{s}}_h\|_{0,\Omega} \,\le\, C\,h^l\,\|\widetilde{\mathbf{s}}\|_{l,\Omega}\,. \tag{5.75}$$

($\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}}$) there exists $C>0$, independent of h, such that for each $l\in[0,k+1]$, and for each $\widetilde{\boldsymbol{\tau}}\in \mathbf{H}^l(\Omega)\cap \mathbf{H}(\mathrm{div}_{4/3};\Omega)$ with $\mathrm{div}(\widetilde{\boldsymbol{\tau}})\in \mathrm{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}(\widetilde{\boldsymbol{\tau}},\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}) \,:=\, \inf_{\widetilde{\boldsymbol{\tau}}_h\in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}} \|\widetilde{\boldsymbol{\tau}}-\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_{4/3};\Omega} \,\le\, C\,h^l\,\Big\{\|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} \,+\, \|\mathrm{div}(\widetilde{\boldsymbol{\tau}})\|_{l,4/3;\Omega}\Big\}\,. \tag{5.76}$$

6 A priori error analysis

In this section we derive an a priori error estimate for our Galerkin scheme with arbitrary finite element subspaces satisfying the hypothesis stated in Section 4.2. More precisely, according to what was established by Theorems 3.11 and 4.8, we let $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in \mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})\in \widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, with $(\mathbf{u},\varphi)\in W$, be the unique solution of the coupled problem (3.13), and let $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\in \mathbf{H}_h\times\mathbb{H}_h^{\boldsymbol{\sigma}}$ and $(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\in \widetilde{\mathbf{H}}_h\times\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, with $(\mathbf{u}_h,\varphi_h)\in W_h$, be a solution of the discrete coupled problem (4.1), respectively. Then, we are interested in obtaining a Cea estimate for the error

$$\|(\vec{\mathbf{u}},\boldsymbol{\sigma})-(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\| \,+\, \|(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})-(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|\,.$$

To this end, we establish next an ad-hoc Strang-type estimate for saddle point problems. In what follows, given a subspace X_h of a generic Banach space $\big(X,\|\cdot\|_X\big)$, we set for each $x\in X$

$$\mathrm{dist}(x,X_h) \,:=\, \inf_{x_h\in X_h}\|x-x_h\|_X\,.$$

Lemma 6.1 *Let* H *and* Q *be reflexive Banach spaces, and let* $a:\mathrm{H}\times\mathrm{H}\longrightarrow \mathrm{R}$ *and* $b:\mathrm{H}\times\mathrm{Q}\longrightarrow\mathrm{R}$ *be bounded bilinear forms with induced operators* $A\in\mathcal{L}(\mathrm{H},\mathrm{H}')$ *and* $B\in\mathcal{L}(\mathrm{H},\mathrm{Q}')$*, respectively, such that* a *and* b *satisfy the hypotheses of Theorem* 3.1*. Furthermore, let* $\big\{H_h\big\}_{h>0}$ *and* $\big\{Q_h\big\}_{h>0}$ *be sequences of finite dimensional subspaces of* H *and* Q*, respectively, and for each* $h>0$ *consider a bounded bilinear form* $a_h: H\times H\longrightarrow R$ *with induced operator* $A_h\in\mathcal{L}(\mathrm{H},\mathrm{H}')$*, such that* $a_h|_{H_h\times H_h}$ *and* $b|_{H_h\times Q_h}$ *satisfy the hypotheses of Theorem* 3.1 *as well, with constants* $\widetilde{\alpha}$ *and* $\widetilde{\beta}$*, both independent of* h*. In turn, given* $F\in H'$*,* $G\in Q'$*, and a sequence of functionals* $\big\{F_h\big\}_{h>0}$*, with* $F_h\in H_h'$ *for each* $h>0$*, we let* $(u,\sigma)\in H\times Q$ *and* $(u_h,\sigma_h)\in H_h\times Q_h$ *be the unique solutions, respectively, to the problems*

$$\begin{aligned} a(u,v)\,+\,b(v,\sigma) &= F(v) \qquad \forall\, v\in \mathrm{H}\,,\\ b(v,\tau) &= G(\tau) \qquad \forall\,\tau\in\mathrm{Q}\,, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} a_h(u_h,v_h)\,+\,b(v_h,\sigma_h) &= F_h(v_h) \qquad \forall\, v_h\in \mathrm{H}_h\,,\\ b(v_h,\tau_h) &= G(\tau_h) \qquad \forall\,\tau_h\in\mathrm{Q}_h\,. \end{aligned} \tag{6.2}$$

Then, there holds

$$\begin{aligned} \|u - u_h\| + \|\sigma - \sigma_h\| \,&\le\, C_{S,1}\,\mathrm{dist}\big(u, H_h\big) \,+\, C_{S,2}\,\mathrm{dist}\big(\sigma, Q_h\big) \\ &+\, C_{S,3}\left\{ \|F - F_h\|_{H_h'} \,+\, \|a(u,\cdot) - a_h(u,\cdot)\|_{H_h'} \right\}, \end{aligned} \tag{6.3}$$

where $C_{S,i}$, $i \in \{1,2,3\}$, are positive constants depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, $\|A\|$, $\|A_h\|$, and $\|B\|$, whose explicit expressions are as follows

$$\begin{aligned} C_{S,1} \,&:=\, \left(1 + \frac{\|A_h\|}{\widetilde{\beta}}\right)\left(1 + \frac{\|B\|}{\widetilde{\beta}}\right)\left(1 + \frac{2\|A\|}{\widetilde{\alpha}} + \frac{\|A_h\|}{\widetilde{\alpha}}\right), \\ C_{S,2} \,&:=\, 1 + \frac{\|B\|}{\widetilde{\beta}} \,+\, \frac{\|B\|}{\widetilde{\alpha}} \,+\, \frac{\|A_h\|\,\|B\|}{\widetilde{\alpha}\widetilde{\beta}}, \\ C_{S,3} \,&:=\, \frac{1}{\widetilde{\alpha}} \,+\, \frac{1}{\widetilde{\beta}} \,+\, \frac{\|A_h\|}{\widetilde{\alpha}\widetilde{\beta}}. \end{aligned} \tag{6.4}$$

Proof. It is basically a simple modification of the proof of [28, Theorem 2.6]. We omit further details and just highlight the fact that the consistency term in (6.3) involving the bilinear forms a and a_h does not appear within any infimum, as for the classical Strang-type estimates, but it is placed free, together with the consistency term involving F and F_h. Indeed, thanks to the boundedness of A and A_h, we find that for each w_h, $v_h \in H_h$ there holds

$$\begin{aligned} \frac{a(w_h,v_h) - a_h(w_h,v_h)}{\|v_h\|_H} \,&=\, \frac{a(w_h,v_h) - a(u,v_h) + a(u,v_h) - a_h(u,v_h) + a_h(u,v_h) - a_h(w_h,v_h)}{\|v_h\|_H} \\ &\le\, \big(\|A\| + \|A_h\|\big)\,\|u - w_h\|_H \,+\, \frac{a(u,v_h) - a_h(u,v_h)}{\|v_h\|_H}, \end{aligned}$$

and hence the usual expression given by

$$\inf_{w_h \in H_h} \left\{ \|u - w_h\|_H \,+\, \|a(w_h,\cdot) - a_h(w_h,\cdot)\|_{H_h'} \right\},$$

can be replaced by

$$\big(1 + \|A\| + \|A_h\|\big)\,\mathrm{dist}\big(u, H_h\big) \,+\, \|a(u,\cdot) - a_h(u,\cdot)\|_{H_h'},$$

where

$$\|a(u,\cdot) - a_h(u,\cdot)\|_{H_h'} \,:=\, \sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{a(u,v_h) - a_h(u,v_h)}{\|v_h\|_H}.$$

□

In order to apply Lemma 6.1, we rewrite (3.13) and (4.1) as suggested in the proofs of Lemmas 3.5, 3.6, 4.2, and 4.3, that is

$$\begin{aligned} \mathcal{A}_{\mathbf{u},\varphi}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) \,+\, b(\vec{\mathbf{v}}, \boldsymbol{\sigma}) \,&=\, F_\varphi(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}, \\ b(\vec{\mathbf{u}}, \boldsymbol{\tau}) \,&=\, G(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \end{aligned} \tag{6.5}$$

$$\begin{aligned} \widetilde{\mathcal{A}}_{\mathbf{u}}(\vec{\varphi}, \vec{\psi}) \,+\, \widetilde{b}(\vec{\psi}, \widetilde{\boldsymbol{\sigma}}) \,&=\, 0 \qquad \forall\, \vec{\psi} \in \widetilde{\mathbf{H}}, \\ \widetilde{b}(\vec{\varphi}, \widetilde{\boldsymbol{\tau}}) \,&=\, \widetilde{G}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_{4/3};\Omega), \end{aligned} \tag{6.6}$$

$$
\begin{aligned}
\mathcal{A}_{\mathbf{u}_h,\varphi_h}(\vec{\mathbf{u}}_h,\vec{\mathbf{v}}_h) + b(\vec{\mathbf{v}}_h,\boldsymbol{\sigma}_h) &= F_{\varphi_h}(\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h\,,\\
b(\vec{\mathbf{u}}_h,\boldsymbol{\tau}_h) &= G(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,
\end{aligned} \tag{6.7}
$$

and

$$
\begin{aligned}
\widetilde{\mathcal{A}}_{\mathbf{u}_h}(\vec{\varphi}_h,\vec{\psi}_h) + \widetilde{b}(\vec{\psi}_h,\widetilde{\boldsymbol{\sigma}}_h) &= 0 \qquad \forall\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,,\\
\widetilde{b}(\vec{\varphi}_h,\widetilde{\boldsymbol{\tau}}_h) &= \widetilde{G}(\widetilde{\boldsymbol{\tau}}_h) \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\,.
\end{aligned} \tag{6.8}
$$

where

$$
\mathcal{A}_{\mathbf{u},\varphi}(\vec{\mathbf{w}},\vec{\mathbf{v}}) := a_{\varphi}(\vec{\mathbf{w}},\vec{\mathbf{v}}) + c(\mathbf{u};\vec{\mathbf{w}},\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{6.9}
$$

$$
\widetilde{\mathcal{A}}_{\mathbf{u}}(\vec{\phi},\vec{\psi}) := \widetilde{a}(\vec{\phi},\vec{\psi}) + \widetilde{c}_{\mathbf{u}}(\vec{\phi},\vec{\psi}) \qquad \forall\, \vec{\phi},\, \vec{\psi} \in \widetilde{\mathbf{H}}\,, \tag{6.10}
$$

$$
\mathcal{A}_{\mathbf{u}_h,\varphi_h}(\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) := a_{\varphi_h}(\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) + c(\mathbf{u}_h;\vec{\mathbf{w}}_h,\vec{\mathbf{v}}_h) \qquad \forall\, \vec{\mathbf{w}}_h,\, \vec{\mathbf{v}}_h \in \mathbf{H}_h \times \mathbf{H}_h\,, \tag{6.11}
$$

and

$$
\widetilde{\mathcal{A}}_{\mathbf{u}_h}(\vec{\phi}_h,\vec{\psi}_h) := \widetilde{a}(\vec{\phi}_h,\vec{\psi}_h) + \widetilde{c}_{\mathbf{u}_h}(\vec{\phi}_h,\vec{\psi}_h) \qquad \forall\, \vec{\phi}_h,\, \vec{\psi}_h \in \widetilde{\mathbf{H}}_h\,. \tag{6.12}
$$

We begin by collecting several useful properties of the foregoing bilinear forms to be employed in what follows. First we recall from the proofs of Lemmas 3.5, 3.6, 4.2, and 4.3, and the estimates (3.33) and (3.35), that they are all bounded with

$$
\begin{aligned}
&\|\mathcal{A}_{\mathbf{u},\varphi}\| \le \big(2\mu_2 + \|\mathbf{u}\|_{0,4;\Omega}\big)\,, \qquad \|\mathcal{A}_{\mathbf{u}_h,\varphi_h}\| \le \big(2\mu_2 + \|\mathbf{u}_h\|_{0,4;\Omega}\big)\,,\\
&\|\widetilde{\mathcal{A}}_{\mathbf{u}}\| \le \big(\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{u}\|_{0,4;\Omega}\big)\,, \qquad \|\widetilde{\mathcal{A}}_{\mathbf{u}_h}\| \le \big(\|\mathbb{K}\|_{\infty,\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega}\big)\,,\\
&\|b\| \le 1\,, \quad \text{and} \quad \|\widetilde{b}\| \le 1\,.
\end{aligned} \tag{6.13}
$$

Next, proceeding as for the derivation of (3.63), and then employing the regularity estimate (3.57) and the fact that the norms of both $\mathbf{u}$ and φ are bounded by the radius r of the ball W (cf. Theorem 3.11), we readily obtain for each $\vec{\mathbf{v}}_h \in \mathbf{H}_h$

$$
\begin{aligned}
&\big|a_{\varphi}(\vec{\mathbf{u}},\vec{\mathbf{v}}_h) - a_{\varphi_h}(\vec{\mathbf{u}},\vec{\mathbf{v}}_h)\big|\\
&\quad\le 2L_{\mu}\,\|i_{\epsilon}\|\,c(\epsilon,n,|\Omega|)\,\|\varphi-\varphi_h\|_{0,4;\Omega}\,\|\mathbf{t}\|_{\epsilon,\Omega}\,\|\vec{\mathbf{v}}_h\|\\
&\quad\le c_1(\boldsymbol{g},\mathbf{u}_D)\,\|\varphi-\varphi_h\|_{0,4;\Omega}\,\|\vec{\mathbf{v}}_h\|\,,
\end{aligned} \tag{6.14}
$$

with

$$
c_1(\boldsymbol{g},\mathbf{u}_D) := 2L_{\mu}\,\|i_{\epsilon}\|\,c(\epsilon,n,|\Omega|)\,c_S\Big\{r\,\|\boldsymbol{g}\|_{\infty,\Omega} + \big(1+r\big)\,\|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma}\Big\}\,,
$$

whereas (3.44) and the a priori estimate (3.74) (cf. Theorem 3.11) guarantee that

$$
\big|c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}_h) - c(\mathbf{u}_h;\vec{\mathbf{u}},\vec{\mathbf{v}}_h)\big| \le \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}\|\,\|\vec{\mathbf{v}}_h\| \le c_2(\boldsymbol{g},\mathbf{u}_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\mathbf{v}}_h\|\,, \tag{6.15}
$$

with

$$
c_2(\boldsymbol{g},\mathbf{u}_D) := C_S\Big\{r\,\|\boldsymbol{g}\|_{\infty,\Omega} + \big(1+r\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.
$$

In this way, the definitions of $\mathcal{A}_{\mathbf{u},\varphi}$ (cf. (6.9)) and $\mathcal{A}_{\mathbf{u}_h,\varphi_h}$ (cf. (6.11)), together with (6.14) and (6.15), imply that for each $\vec{\mathbf{v}}_h \in \mathbf{H}_h$ there holds

$$
\big|\mathcal{A}_{\mathbf{u},\varphi}(\vec{\mathbf{u}},\vec{\mathbf{v}}_h) - \mathcal{A}_{\mathbf{u}_h,\varphi_h}(\vec{\mathbf{u}},\vec{\mathbf{v}}_h)\big| \le \Big\{c_1(\boldsymbol{g},\mathbf{u}_D)\,\|\varphi-\varphi_h\|_{0,4;\Omega} + c_2(\boldsymbol{g},\mathbf{u}_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,\|\vec{\mathbf{v}}_h\|\,,
$$

which yields

$$\big\|\mathcal{A}_{\mathbf{u},\varphi}(\vec{\mathbf{u}},\cdot) - \mathcal{A}_{\mathbf{u}_h,\varphi_h}(\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,\le\, \Big\{c_1(\boldsymbol{g},\mathbf{u}_D)\,\|\varphi-\varphi_h\|_{0,4;\Omega} + c_2(\boldsymbol{g},\mathbf{u}_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,. \tag{6.16}$$

Similarly, according to the definitions of $\widetilde{\mathcal{A}}_{\mathbf{u}}$ (cf. (6.10)) and $\widetilde{\mathcal{A}}_{\mathbf{u}_h}$ (cf. (6.12)), the inequality (3.46), and the a priori estimate (3.75) (cf. Theorem 3.11), we find that for each $\vec{\psi}_h \in \widetilde{\mathbf{H}}_h$ there holds

$$\begin{aligned}\big|\widetilde{\mathcal{A}}_{\mathbf{u}}(\vec{\varphi},\vec{\psi}_h) - \widetilde{\mathcal{A}}_{\mathbf{u}_h}(\vec{\varphi},\vec{\psi}_h)\big| \,&=\, \big|\widetilde{c}_{\mathbf{u}}(\vec{\varphi},\vec{\psi}_h) \,-\, \widetilde{c}_{\mathbf{u}_h}(\vec{\varphi},\vec{\psi}_h)\big| \\ &\le\, \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\varphi}\|\,\|\vec{\psi}_h\| \,\le\, c_3(\varphi_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\vec{\psi}_h\|\,,\end{aligned}$$

with

$$c_3(\varphi_D) \,:=\, C_{\widetilde{S}}\Big\{1 \,+\, \|\mathbb{K}\|_{\infty,\Omega} \,+\, r\Big\}\,\|\varphi_D\|_{1/2,\Gamma}\,,$$

which yields

$$\big\|\widetilde{\mathcal{A}}_{\mathbf{u}}(\vec{\varphi},\cdot) - \widetilde{\mathcal{A}}_{\mathbf{u}_h}(\vec{\varphi},\cdot)\|_{\widetilde{\mathbf{H}}_h'} \,\le\, c_3(\varphi_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,. \tag{6.17}$$

Furthermore, it readily follows from (3.51) (see also (3.60)) that

$$\|F_\varphi - F_{\varphi_h}\|_{\mathbf{H}_h'} \,\le\, c_4(\boldsymbol{g})\,\|\varphi-\varphi_h\|_{0,4;\Omega}\,, \tag{6.18}$$

with

$$c_4(\boldsymbol{g}) \,:=\, |\Omega|^{1/2}\,\|\boldsymbol{g}\|_{\infty,\Omega}\,.$$

Having established the above, we now recall from Sections 3.3 and 4.2 that the pairs of bilinear forms $\big(\mathcal{A}_{\mathbf{u},\varphi},b\big)$ and $\big(\mathcal{A}_{\mathbf{u}_h,\varphi_h},b\big)$ do satisfy the hypotheses of Lemma 6.1 on $\mathbf{H}\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ and $\mathbf{H}_h\times\mathbb{H}_h^{\boldsymbol{\sigma}}$, respectively, the latter with constants $\alpha_{\mathtt{d}}$ (cf. proof of Lemma 4.2) and $\beta_{\mathtt{d}}$ (cf. Assumption 4.1). Hence, applying the aforementioned lemma to the context given by problems (6.5) and (6.7), and bearing in mind the consistency estimates (6.16) and (6.18), we deduce that

$$\begin{aligned}\|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\| \,&\le\, \bar{C}_{S,1}\,\mathrm{dist}\big(\vec{\mathbf{u}},\mathbf{H}_h\big) \,+\, \bar{C}_{S,2}\,\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \\ &\quad+\, \bar{C}_{S,3}\Big\{\big(c_1(\boldsymbol{g},\mathbf{u}_D)+c_4(\boldsymbol{g})\big)\,\|\varphi-\varphi_h\|_{0,4;\Omega} \,+\, c_2(\boldsymbol{g},\mathbf{u}_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{6.19}$$

where the constants $\bar{C}_{S,1}$, $\bar{C}_{S,2}$, and $\bar{C}_{S,3}$, depending on μ_2, r, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, are computed according to (6.4), after using (6.13) to bound both $\|\mathcal{A}_{\mathbf{u},\varphi}\|$ and $\|\mathcal{A}_{\mathbf{u}_h,\varphi_h}\|$ by $\big(2\mu_2 + r\big)$.

In turn, we also recall from Sections 3.3 and 4.2 that the pairs of bilinear forms $\big(\widetilde{\mathcal{A}}_{\mathbf{u}},\widetilde{b}\big)$ and $\big(\widetilde{\mathcal{A}}_{\mathbf{u}_h},\widetilde{b}\big)$ satisfy the hypotheses of Lemma 6.1 as well on $\widetilde{\mathbf{H}}\times\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\widetilde{\mathbf{H}}_h\times\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}$, respectively, the latter with constants $\widetilde{\alpha}_{\mathtt{d}}$ (cf. proof of Lemma 4.3) and $\widetilde{\beta}_{\mathtt{d}}$ (cf. Assumption 4.3). Therefore, applying again the aforementioned lemma to the context given now by problems (6.6) and (6.8), and bearing in mind the consistency estimate (6.17), we arrive at

$$\|(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) - (\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\| \,\le\, \widehat{C}_{S,1}\,\mathrm{dist}\big(\vec{\varphi},\widetilde{\mathbf{H}}_h\big) \,+\, \widehat{C}_{S,2}\,\mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}},\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big) \,+\, \widehat{C}_{S,3}\,c_3(\varphi_D)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,, \tag{6.20}$$

where, similarly as before, the constants $\widehat{C}_{S,1}$, $\widehat{C}_{S,2}$, and $\widehat{C}_{S,3}$, depending on $\|\mathbb{K}\|_{\infty,\Omega}$, r, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, are computed according to (6.4), after using (6.13) to bound both $\|\widetilde{\mathcal{A}}_{\mathbf{u}}\|$ and $\|\widetilde{\mathcal{A}}_{\mathbf{u}_h}\|$ by $\big(\|\mathbb{K}\|_{\infty,\Omega} + r\big)$.

The required Cea estimate will now follow from (6.19) and (6.20). In fact, bounding $\|\varphi-\varphi_h\|_{0,4;\Omega}$ in (6.19) by the right hand side of (6.20), we obtain

$$
\begin{aligned}
\|(\vec{\mathbf{u}},\boldsymbol{\sigma})-(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\| \,\le\, & \bar{C}_{S,1}\,\mathrm{dist}\big(\vec{\mathbf{u}},\mathbf{H}_h\big) \,+\, \bar{C}_{S,2}\,\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big)\\
&+\,\bar{C}_{S,3}\,\widehat{C}_{S,1}\,\big(c_1(\boldsymbol{g},\mathbf{u}_D)+c_4(\boldsymbol{g})\big)\,\mathrm{dist}\big(\vec{\varphi},\widetilde{\mathbf{H}}_h\big)\\
&+\,\bar{C}_{S,3}\,\widehat{C}_{S,2}\,\big(c_1(\boldsymbol{g},\mathbf{u}_D)+c_4(\boldsymbol{g})\big)\,\mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}},\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big)\\
&+\,\bar{C}_{S,3}\,\Big\{\widehat{C}_{S,3}\,\big(c_1(\boldsymbol{g},\mathbf{u}_D)+c_4(\boldsymbol{g})\big)\,c_3(\varphi_D)\,+\,c_2(\boldsymbol{g},\mathbf{u}_D)\Big\}\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,
\end{aligned}
\tag{6.21}
$$

from which, imposing the constant multiplying $\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}$ in (6.21) to be sufficiently small, say $\le 1/2$, we derive the a priori error estimate for $\|(\vec{\mathbf{u}},\boldsymbol{\sigma})-(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|$, which, employed then to bound the third term on the right hand side of (6.20), provides the corresponding upper bound for $\|(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})-(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|$. More precisely, we have thus demostrated the following result.

Theorem 6.2 *Assume that the data $\boldsymbol{g}$, $\mathbf{u}_D$, and φ_D satisfy*

$$
\bar{C}_{S,3}\,\Big\{\widehat{C}_{S,3}\,\big(c_1(\boldsymbol{g},\mathbf{u}_D)+c_4(\boldsymbol{g})\big)\,c_3(\varphi_D)\,+\,c_2(\boldsymbol{g},\mathbf{u}_D)\Big\}\,\le\,\frac{1}{2}\,. \tag{6.22}
$$

Then, there exists a positive constant C, independent of h, but depending on μ_2, $\|\mathbb{K}\|_{\infty,\Omega}$, r, $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, and the data $\boldsymbol{g}$, $\mathbf{u}_D$, and φ_D, such that

$$
\begin{aligned}
&\|(\vec{\mathbf{u}},\boldsymbol{\sigma})-(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\| \,+\, \|(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})-(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|\\
&\quad\le\, C\,\Big\{\mathrm{dist}\big(\vec{\mathbf{u}},\mathbf{H}_h\big)\,+\,\mathrm{dist}\big(\boldsymbol{\sigma},\mathbb{H}_h^{\boldsymbol{\sigma}}\big)\,+\,\mathrm{dist}\big(\vec{\varphi},\widetilde{\mathbf{H}}_h\big)\,+\,\mathrm{dist}\big(\widetilde{\boldsymbol{\sigma}},\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}}\big)\Big\}\,.
\end{aligned}
\tag{6.23}
$$

We are now able to provide the rates of convergence of the Galerkin Scheme (4.1) when the finite element subspaces specified in Sections 5.2 and 5.5 are employed.

Theorem 6.3 *Assume that there exists $l\in[0,k+1]$ such that $\mathbf{u}\in\mathbf{W}^{l,4}(\Omega)$, $\mathbf{t}\in\mathbb{H}^l(\Omega)\cap\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}^l(\Omega)\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{l,4/3}(\Omega)$, $\varphi\in\mathrm{W}^{l,4}(\Omega)$, $\widetilde{\mathbf{t}}\in\mathbf{H}^l(\Omega)$, $\widetilde{\boldsymbol{\sigma}}\in\mathbf{H}^l(\Omega)\cap\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and $\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\in\mathbf{W}^{l,4/3}(\Omega)$. Then, there exists $C>0$, independent of h, such that*

$$
\begin{aligned}
\|(\vec{\mathbf{u}},\boldsymbol{\sigma})-(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\| \,+\, \|(\vec{\varphi},\widetilde{\boldsymbol{\sigma}})-(\vec{\varphi}_h,\widetilde{\boldsymbol{\sigma}}_h)\| \,\le\, & C\,h^l\,\Big\{\|\mathbf{u}\|_{l,4;\Omega}+\|\mathbf{t}\|_{l,\Omega}+\|\boldsymbol{\sigma}\|_{l,\Omega}\\
&+\,\|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega}+\|\varphi\|_{l,4;\Omega}+\|\widetilde{\mathbf{t}}\|_{l,\Omega}+\|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega}+\|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,4/3;\Omega}\Big\}\,.
\end{aligned}
\tag{6.24}
$$

Proof. It follows straightforwardly from (6.23) and the approximation properties described at the end of Section 5.5. □

We end this section with the postprocessing of the pressure. Indeed, the identity (2.5) and the orthogonal decomposition for the pseudostress tensor provided by (3.10) (recall that $\boldsymbol{\sigma}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\subseteq\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$), suggest to define the discrete pressure as

$$
p_h \,:=\, -\,\frac{1}{2n}\mathrm{tr}\big(2\boldsymbol{\sigma}_h+2c_h\mathbb{I}+\mathbf{u}_h\otimes\mathbf{u}_h\big)\,,
$$

with

$$
c_h \,:=\, -\,\frac{1}{2n|\Omega|}\int_\Omega \mathrm{tr}\big(\mathbf{u}_h\otimes\mathbf{u}_h\big)\,.
$$

In turn, since $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, the modified equation for the continuous pressure becomes

$$p \,=\, -\,\frac{1}{2n}\mathrm{tr}\big(2\boldsymbol{\sigma} + 2c\mathbb{I} + \mathbf{u}\otimes\mathbf{u}\big)\,,$$

with

$$c \,:=\, -\,\frac{1}{2n|\Omega|}\int_{\Omega}\mathrm{tr}\big(\mathbf{u}\otimes\mathbf{u}\big)\,.$$

Then, it is easy to prove that there exists a positive constant C, independent of h, such that

$$\|p - p_h\|_{0,\Omega} \,\le\, C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\right\},$$

whence the rate of convergence of p_h is the same one provided by the rest of the variables (cf. (6.24)).

7 Numerical results

This section presents a few numerical examples in 2D to illustrate the performance of our fully-mixed formulation (4.1) and to support the respective convergence theoretical results with the specific finite element subspaces derived in Section 5. Accordingly, as required for the stability of the Scott-Vogelius pair, the computations are performed on barycenter refined meshes $\mathcal{T}_h^{\mathsf{b}}$ created from regular triangulations $\mathcal{T}_h$ of the domain Ω (see Figure 7.1 for an example of it). So, for $k \ge n-1 = 1$, the discrete spaces approximating $\mathbf{u}$, $\mathbf{t}$, $\boldsymbol{\sigma}$, φ, $\widetilde{\mathbf{t}}$, and $\widetilde{\boldsymbol{\sigma}}$ are then given, respectively, as (cf. (5.30)-(5.32), (5.68)-(5.70))

$$\mathbf{H}_h^{\mathbf{u}} \,:=\, \Big\{\mathbf{v}_h \in \mathbf{L}^4(\Omega): \quad \mathbf{v}_h|_K \,\in\, \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,,$$

$$\mathbb{H}_h^{\mathbf{t}} \,:=\, \Big\{\mathbf{s}_h \,\in\, \mathbb{L}^2_{\mathrm{tr}}(\Omega): \qquad \mathbf{s}_h|_K \,\in\, \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,,$$

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \,:=\, \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega): \quad \mathbf{c}^{\mathtt{t}}\,\boldsymbol{\tau}_h|_K \,\in\, \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n\,, \quad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,,$$

$$\mathrm{H}_h^{\varphi} \,:=\, \Big\{\psi_h \in \mathrm{L}^4(\Omega): \quad \psi_h|_K \in \mathrm{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,,$$

$$\mathbf{H}_h^{\widetilde{\mathbf{t}}} \,:=\, \Big\{\widetilde{\mathbf{s}}_h \in \mathbf{L}^2(\Omega): \quad \widetilde{\mathbf{s}}_h|_K \in \mathbf{P}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,,$$

and

$$\mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}} \,:=\, \Big\{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}(\mathrm{div}_{4/3};\Omega): \quad \widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{RT}_k(K) \qquad \forall\, K \in \mathcal{T}_h^{\mathsf{b}}\Big\}\,.$$

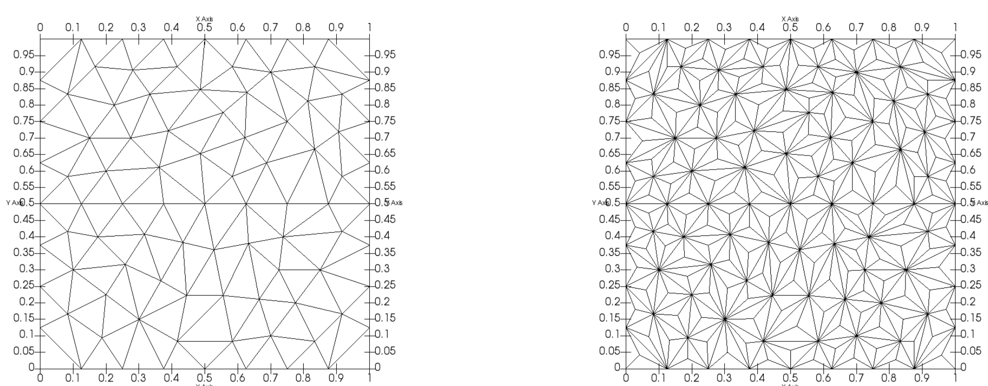

Figure 7.1: Example of a regular triangulation $\mathcal{T}_h$ and its barycentric refinement $\mathcal{T}_h^{\mathsf{b}}$ in $\Omega \,:=\, [0,1]^2$

The computational implementation is based on a **FreeFem++** code (cf. [29]). A Newton-Raphson algorithm was used for the resolution of the nonlinear problem (4.1), with initial guess $(\mathbf{u},\varphi) = (\mathbf{0},0)$, and the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^{m+1}$ and $\mathbf{coeff}^{m}$, is sufficiently small, that is,

$$\frac{||\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}||_{\ell^2}}{||\mathbf{coeff}^{m+1}||_{\ell^2}} < \mathtt{tol}\,,$$

where $\mathtt{tol}$ is a specified tolerance and $||\cdot||_{\ell^2}$ is the standard ℓ^2-norm in R^N with N denoting the total number of degrees of freedom defined by the finite element family $(\mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\mathbf{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathrm{H}_h^{\varphi}, \mathbf{H}_h^{\widetilde{\mathbf{t}}}, \mathbf{H}_h^{\widetilde{\boldsymbol{\sigma}}})$. At each iteration, the resulting linear systems were solved by means of the direct linear solver UMFPACK (cf. [21]) and the trace condition on the stress $\boldsymbol{\sigma}$ is enforced through a penalization strategy. As usual, the individual errors associated to the main unknowns are computed as

$$e(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,, \quad e(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega} \quad e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,,$$

$$e(\varphi) := \|\varphi-\varphi_h\|_{0,4;\Omega}\,, \quad e(\widetilde{\mathbf{t}}) := \|\widetilde{\mathbf{t}}-\widetilde{\mathbf{t}}_h\|_{0,4;\Omega}\,, \quad e(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}}-\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_{4/3};\Omega}\,,$$

and the error associated to the postprocessed pressure as

$$e(p) := \|p-p_h\|_{0,\Omega}\,.$$

In turn, for all $\star \in \{\mathbf{u},\mathbf{t},\boldsymbol{\sigma},\varphi,\widetilde{\mathbf{t}},\widetilde{\boldsymbol{\sigma}},p\}$, we let $r(\star)$ be the experimental convergence rate given by

$$r(\star) := \frac{\log(\mathsf{e}(\star)/\mathsf{e}'(\star))}{\log(h/h')}\,,$$

where h and h' denote two consecutive mesh sizes with errors $\mathsf{e}(\star)$ and $\mathsf{e}'(\star)$, respectively.

7.1 Example 1: accuracy assessment

In our first example, we study the accuracy of the approximations by manufacturing an exact solution of the nonlinear problem (2.1) defined in the square $\Omega := (-1,1)^2$. We then consider the data defined for each $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$ as

$$\mu = 1, \quad \mathbb{K}(\mathbf{x}) = \begin{bmatrix} e^{-x_1} & x_1/10 \\ x_2/10 & e^{-x_2} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{g}(\mathbf{x}) = (0,-1)^t\,,$$

and the terms at the right-hand sides are adjusted in such a way that the exact solutions are given by the smooth functions

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} 4x_2(x_1^2-1)^2(x_2^2-1) \\ -4x_1(x_2^2-1)^2(x_1^2-1) \end{pmatrix}, \quad p(\mathbf{x}) = (x_1-0.5)(x_2-0.5)-0.25\,,$$

and

$$\varphi(\mathbf{x}) = e^{-x_1^2-x_2^2} - \frac{1}{2},$$

whereas the Dirichlet data $\mathbf{u}_D$ and φ_D are imposed according to the exact solutions.

Values of errors and corresponding convergence rates associated to the approximations with the finite element family $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ are summarized in Table 1. There, we observe that the convergence rates are quadratic with respect to h for all the unknowns in their respective norms. These findings are in agreement with the theoretical error bounds from Section 6 (cf. estimate (6.24)). We mention that 4 Newton steps were required to reach a tolerance $\mathtt{tol}$ = 1E-08. The exact velocity magnitude, the exact pressure and the exact temperature as well as the corresponding approximation computed with our fully-mixed method on a barycenter refined mesh with $N = 1917696$ degrees of freedom are depicted in Figure 7.2.

Finite Element Family: $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$								
N	h	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	
7536	0.5000	1.0046e-01	-	5.8517e-01	-	1.9043e00	-	
30048	0.2500	2.7087e-02	1.8953	1.5853e-01	1.8884	4.8726e-01	1.9710	
120000	0.1250	6.9415e-03	1.9665	3.9956e-02	1.9906	1.2253e-01	1.9938	
479616	0.06250	1.7467e-03	1.9917	1.0027e-02	1.9956	3.0724e-02	1.9969	
1917696	0.03125	4.3739e-04	1.9982	2.5141e-03	1.9963	7.6952e-03	1.9979	
$e(\varphi)$	$r(\varphi)$	$e(\widetilde{\mathbf{t}})$	$r(\widetilde{\mathbf{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(p)$	$r(p)$	It.
7.8148e-03	-	3.2988e-02	-	1.0277e-01	-	4.6875e-01	-	4
1.9960e-03	1.9736	9.5172e-03	1.7974	2.7264e-02	1.9188	1.1722e-01	2.0041	4
4.9931e-04	2.0014	2.5139e-03	1.9228	6.9473e-03	1.9747	2.8878e-02	2.0235	4
1.2481e-04	2.0013	6.4399e-04	1.9659	1.7496e-03	1.9905	7.1529e-03	2.0145	4
3.1202e-05	2.0006	1.6283e-04	1.9841	4.3876e-04	1.9961	1.7796e-03	2.0075	4

Table 1: Example 1: Convergence history and Newton iteration count for the fully-mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation. Here, N stands for the number of degrees of freedom associated to each barycenter refined mesh $\mathcal{T}_h^{\mathrm{b}}$.

7.2 Example 2: non-convex domain and temperature-dependent viscosity

In this example, we set the problem (2.1) on an "U" shaped non-convex domain, that is, we set $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\begin{aligned}
\Omega_1 &:= \Big\{ \mathbf{x} := (x_1, x_2)^{\mathrm{t}} : \quad -1 < x_1 < 0.5, \quad -\sin(x_1) < x_2 < 0 \Big\}, \\
\Omega_2 &:= \Big\{ \mathbf{x} := (x_1, x_2)^{\mathrm{t}} : \quad -0.5 < x_1 < 0.5, \quad -\sin(x_1) < x_2 < -\frac{1}{2}\sin(x_1) \Big\}, \\
\Omega_3 &:= \Big\{ \mathbf{x} := (x_1, x_2)^{\mathrm{t}} : \quad 0.5 < x_1 < 1, \quad -\sin(x_1) < x_2 < 0 \Big\},
\end{aligned}$$

and test the performance of our fully-mixed technique considering the temperature-dependent viscosity, thermal conductivity and body force given by

$$\mu(\varphi) = e^{-\varphi}, \quad \mathbb{K}(\mathbf{x}) = e^{x_1+x_2}\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{g}(\mathbf{x}) = (1,0)^t .$$

Again, the right-hand sides and the boundary Dirichlet data are adjusted in terms of the manufactured exact solutions

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} 4x_2(x_1^2-1)^2(x_2^2-1) \\ -4x_1(x_2^2-1)^2(x_1^2-1) \end{pmatrix}, \quad p(\mathbf{x}) = \sin(x_1x_2), \quad \text{and} \quad \varphi(\mathbf{x}) = \cos(x_1x_2) + 1 .$$

In Table 2 we present the errors and the convergence rates associated to the approximations with the finite element family $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbf{P}_2 - \mathbf{RT}_2$ ($k = 2$). Once again, in concordance with the theoretical error bounds predicted in Section 6, we find that the errors associated to all the unknowns are of order $O(h^3)$, as expected. A total of 4 Newton iterations, in average, were required to reach a tolerance `tol` = 1E-08. In Figure 7.3, we display the velocity magnitude, the pressure and the temperature versus the corresponding approximations driven by our fully-mixed technique on a barycenter refined mesh with $N = 600885$ degrees of freedom.

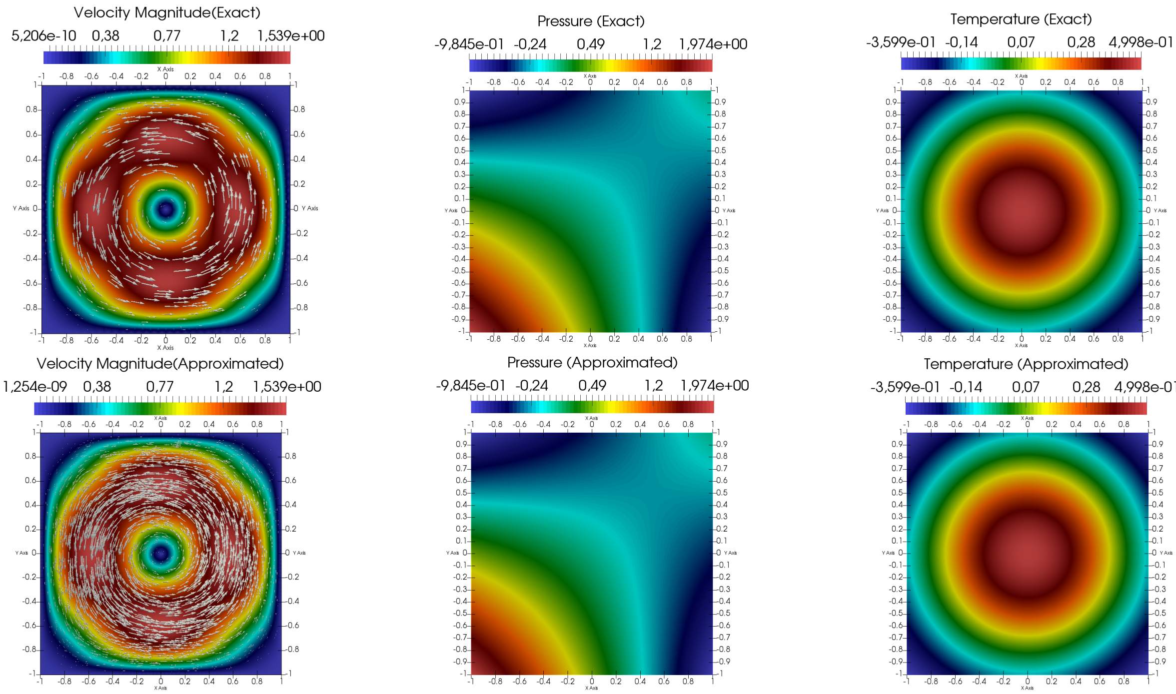

Figure 7.2: Example 1: Exact (first panel) and approximated (second panel) velocity magnitude, pressure and temperature, with $k = 1$ and number of degrees of freedom $N = 1917696$.

7.3 Example 3: natural convection in a square cavity

In this last example, we consider the natural convection of a fluid in a square cavity with different heat walls. This phenomenon has been widely studied with different types of boundary conditions (see [7, 20, 22], for instance). Such as in [4], we consider the problem (2.1) with dimensionless numbers: Find $(\mathbf{u}, p, \varphi)$ such that

$$\begin{aligned} -Pr\,\mathbf{div}(2\mu(\varphi)\boldsymbol{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p &= Ra\ \ \varphi\boldsymbol{g} && \text{in} \quad \Omega\,, \\ \mathrm{div}\mathbf{u} &= 0 && \text{in} \quad \Omega\,, \\ -\mathrm{div}(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi &= 0 && \text{in} \quad \Omega\,, \end{aligned} \tag{7.1}$$

where Pr and Ra are the Prandtl and Rayleight numbers, defined respectively as the ratio of momentum diffusivity to thermal diffusivity, and the ratio of buoyancy forces to viscosity forces times the Prandtl number. Hence, we model the cavity as $\Omega = (0,1)^2$ and consider Prandtl and Rayleight numbers, viscosity, thermal conductivity and body force given by

$$Pr = 0.5, \quad Ra = 2000, \quad \mu(\varphi) = \exp(-\varphi), \quad \mathbb{K}(\mathbf{x}) = \mathbb{I}, \quad \text{and} \quad \boldsymbol{g}(\mathbf{x}) = (0,1)^t.$$

In addition, as in [4], the system (7.1) is completed with the boundary conditions

$$\mathbf{u}_D(\mathbf{x}) = \mathbf{0}, \quad \text{and} \quad \varphi_D(\mathbf{x}) = \frac{1}{2}\Big(1 - \cos(2\pi x_1)\Big)\Big(1 - x_2\Big) \quad \text{on} \quad \Gamma\,.$$

The last condition results in the left, top and right walls with zero-temperature, and describes a sinusoidal profile in the bottom wall, with a peak of temperature $\varphi = 1$ at $x = 0.5$. In Figure 7.4, we display the approximation of the temperature, its gradient, some components of the vorticity tensor

Finite Element Family: $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbf{P}_2 - \mathbf{RT}_2$								
DOF	h	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	
8208	0.3943	1.4610e-03	-	2.8051e-02	-	1.6080e-02	-	
36216	0.1957	2.0810e-04	2.6258	6.2386e-03	2.0254	2.8834e-03	2.3155	
159966	0.10299	1.6248e-05	3.4333	7.7387e-04	2.8100	3.5436e-04	2.8225	
600885	0.04973	1.4079e-06	3.6962	9.0125e-05	3.2494	4.0883e-05	3.2636	
2524257	0.02682	1.8288e-07	3.4512	1.2541e-05	2.7480	5.4697e-06	2.8028	
$e(\varphi)$	$r(\varphi)$	$e(\widetilde{\mathbf{t}})$	$r(\widetilde{\mathbf{t}})$	$e(\widetilde{\boldsymbol{\sigma}})$	$r(\widetilde{\boldsymbol{\sigma}})$	$e(p)$	$r(p)$	It.
1.3278e-04	-	3.9332e-03	-	3.1475e-03	-	5.5932e-03	-	5
9.5052e-06	3.5528	5.1895e-04	2.7289	3.8306e-04	2.8358	8.5282e-04	2.5340	4
6.8654e-07	3.5382	6.8708e-05	2.7223	4.4652e-05	2.8956	9.5312e-05	2.9504	4
5.0659e-08	3.9390	8.0110e-06	3.2476	5.5112e-06	3.1616	1.1564e-05	3.1875	4
4.7396e-09	3.3012	1.0029e-06	2.8953	6.8055e-07	2.9145	1.4522e-06	2.8911	4

Table 2: Example 2: Convergence history and Newton iteration count for the fully-mixed $\mathbf{P}_2 - \mathbb{P}_2 - \mathbb{RT}_2 - \mathrm{P}_2 - \mathbb{P}_2 - \mathbf{RT}_2$ approximation on a non-convex domain and with temperature-dependent viscosity. Here, N stands for the number of degrees of freedom associated to each barycenter refined mesh $\mathcal{T}_h^{\mathtt{b}}$.

of the fluid (which is computed as a direct postprocessing of the velocity gradient, that is $\frac{1}{2}\big(\mathbf{t}_h - \mathbf{t}_h^{\mathtt{t}}\big)$), the pressure and the velocity magnitude. Our results are in concordance with those obtained in [4] and what is expected to be observed from the physical point of view, in accordance to [20].

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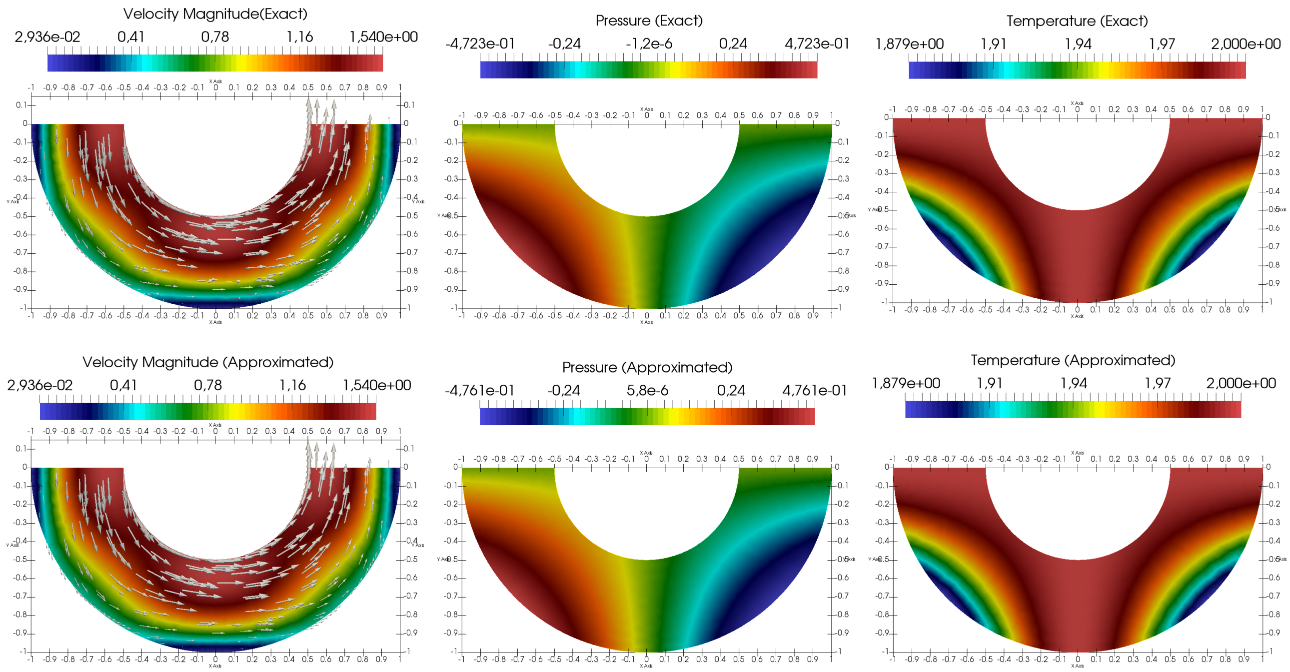

Figure 7.3: Example 2: Exact (first panel) and approximated (second panel) velocity magnitude, pressure and temperature, with $k = 2$ and number of degrees of freedom $N = 600885$.

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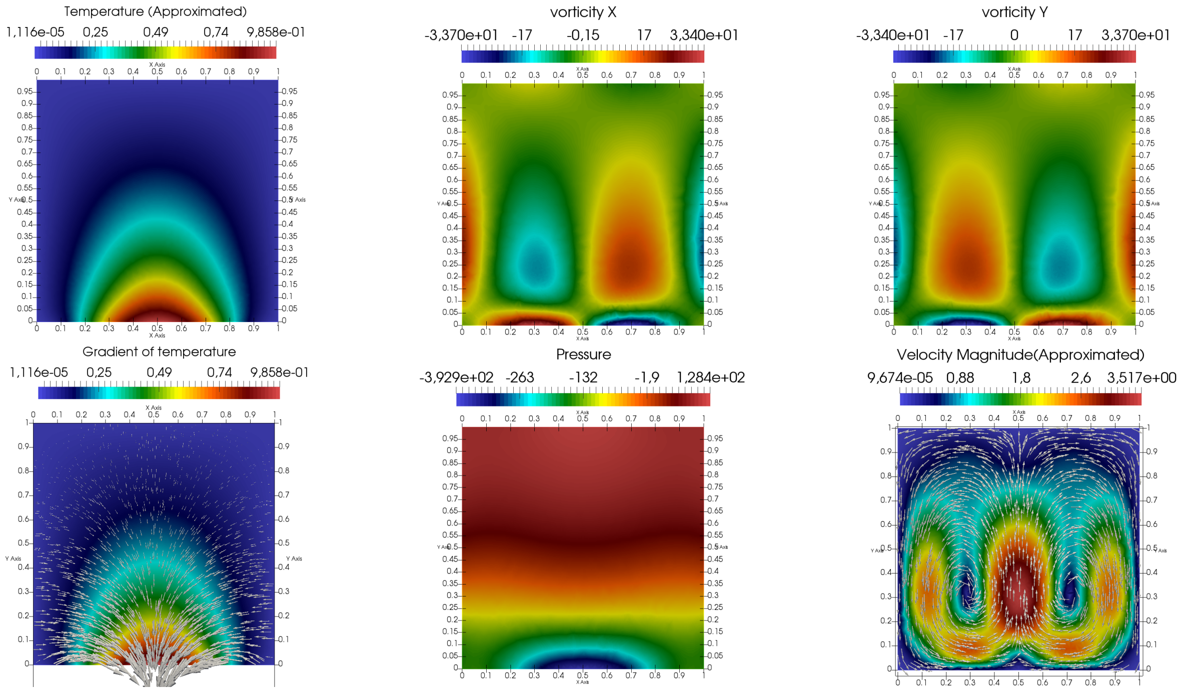

Figure 7.4: Example 3: Natural Convection in a Square Cavity, with $k = 1$, DOF=1132626

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