We now look at the discrete kernel V_h of b, that is

$$V_h := \left\{ \boldsymbol{\tau}_h \in H_h : \quad b(\boldsymbol{\tau}_h, (v_h, \lambda_h)) = 0 \quad \forall\, (v_h, \lambda_h) \in Q_h \right\},$$

which, according to (4.51) and (4.54), yields

$$V_h := \left\{ \boldsymbol{\tau}_h \in H_h : \quad \operatorname{div} \boldsymbol{\tau}_h \in \mathbb{P}_0(\Omega) \quad \text{y} \quad \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, \lambda_h \rangle = 0 \quad \forall\, \lambda_h \in Q_h^{\xi} \right\}.$$

Hence, the V_h-ellipticity of a follows straightforwardly from [39, Lemma 3.2] by making use only of the first property characterizing the elements of V_h.

Consequently, by applying again the discrete Babuška-Brezzi theory (cf. Theorems 2.4 and 2.6), we conclude that (4.49) has a unique solution $(\boldsymbol{\sigma}_h, (u_h, \xi_h)) \in H_h \times Q_h$, and there exists a constant $C > 0$, independent of h, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma}, (u, \xi)) \,-\, (\boldsymbol{\sigma}_h, (u_h, \xi_h))\|_{H \times Q} \\ &\qquad \leq \quad C \inf_{(\boldsymbol{\tau}_h, (v_h, \lambda_h)) \in H_h \times Q_h} \|(\boldsymbol{\sigma}, (u, \xi)) \,-\, (\boldsymbol{\tau}_h, (v_h, \lambda_h))\|_{H \times Q}\,. \end{aligned}$$

The approximation properties of H_h and Q_h^u are somehow already established by (4.32) and (4.33) (see also (4.4) and (4.21)), whereas the one of Q_h^{ξ} is given by (cf. [39, **(AP3)**])

$$\|\lambda - \mathcal{P}_{1/2,h}(\lambda)\|_{1/2,\Gamma} \,\leq\, C\, h^{\delta}\, \|\lambda\|_{1/2+\delta,\Gamma} \quad \forall\, \lambda \,\in\, H^{1/2+\delta}(\Gamma)\,, \quad \forall\, \delta \,\in\, [0,1]\,,$$

where $\mathcal{P}_{1/2,h} : H^{1/2}(\Gamma) \to Q_h^{\xi}$ is the orthogonal projector with respect to the inner product of $H^{1/2}(\Gamma)$.

4.5 The linear elasticity problem

In this section we analyze the Galerkin scheme for the 2D version of the linear elasticity problem with Dirichlet boundary conditions studied in Section 2.4.3. To this end, we recall that, given a bounded domain $\Omega \subseteq \mathbb{R}^2$ with Lipschitz-continuous boundary Γ, and given $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the corresponding mixed formulation reduces to (cf. (2.50)): Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\rho})) \in H_0 \times Q$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\rho})) \quad &= \quad F(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \,\in\, H_0\,, \\ b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) \quad &= \quad G(\mathbf{v}, \boldsymbol{\eta}) \qquad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \,\in\, Q\,, \end{aligned} \tag{4.63}$$

where (cf. Section 2.4.3)

$$H_0 \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}, \quad Q \,:=\, \mathbf{L}^2(\Omega) \,\times\, \mathbb{L}^2_{\text{skew}}(\Omega),$$

$a \,:\, H_0 \,\times\, H_0 \,\to$ and $b \,:\, H_0 \,\times\, Q \,\to\, \mathbb{R}$ are the bilinear forms defined by

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \int_\Omega \mathcal{C}^{-1}\,\boldsymbol{\zeta} \,:\, \boldsymbol{\tau} \,=\, \frac{1}{2\mu}\int_\Omega \boldsymbol{\zeta} \,:\, \boldsymbol{\tau} \,-\, \frac{\lambda}{4\mu(\lambda \,+\, \mu)}\int_\Omega \operatorname{tr}(\boldsymbol{\zeta})\operatorname{tr}(\boldsymbol{\tau}) \tag{4.64}$$

for all $(\boldsymbol{\zeta},\boldsymbol{\tau}) \in H_0 \times H_0$, and

$$b(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})) \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} \,+\, \int_\Omega \boldsymbol{\eta} \,:\, \boldsymbol{\tau}, \tag{4.65}$$

for all $(\boldsymbol{\tau},(\mathbf{v},\boldsymbol{\eta})) \in H_0 \times Q$, and the functionals $F \in\, H_0'$ and $G \in\, Q'$ are given by

$$F(\boldsymbol{\tau}) \,:=\, 0 \quad \forall\,\boldsymbol{\tau} \in\, H_0, \qquad G(\mathbf{v},\boldsymbol{\eta}) \,:=\, -\int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\,(\mathbf{v},\boldsymbol{\eta}) \in\, Q. \tag{4.66}$$

We now let $\{H_h\}_{h>0}$, $\{Q_{1,h}\}_{h>0}$ and $\{Q_{2,h}\}_{h>0}$ be families of arbitrary finite element subspaces of H_0, $Q_1 \,:=\, \mathbf{L}^2(\Omega)$, and $Q_2 \,:=\, \mathbb{L}^2_{\text{skew}}(\Omega)$, respectively. Then, denoting $Q_h \,:=\, Q_{1,h} \,\times\, Q_{2,h}$, we consider the associated Galerkin scheme: Find $(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\boldsymbol{\rho}_h)) \in H_h \,\times\, Q_h$ such that

$$\begin{array}{rcll} a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h,(\mathbf{u}_h,\boldsymbol{\rho}_h)) & = & F(\boldsymbol{\tau}_h) & \forall\,\boldsymbol{\tau}_h \,\in\, H_h\,, \\ b(\boldsymbol{\sigma}_h,(\mathbf{v}_h,\boldsymbol{\eta}_h)) & = & G(\mathbf{v}_h,\boldsymbol{\eta}_h) & \forall\,(\mathbf{v}_h,\boldsymbol{\eta}_h) \,\in\, Q_h\,. \end{array} \tag{4.67}$$

Our goal is to apply the theory from Section 2.5 to find specific subspaces H_h, $Q_{1,h}$ and $Q_{2,h}$ ensuring the unique solvability and stability of (4.67). More precisely, assuming in advance that a is going to be elliptic on the discrete kernel V_h of b (which actually will be shown later on), we concentrate in what follows in proving the discrete inf-sup condition for b, that is the existence of $\beta \,>\, 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in H_h \\ \boldsymbol{\tau}_h \neq 0}} \frac{b(\boldsymbol{\tau}_h,(\mathbf{v}_h,\,\boldsymbol{\eta}_h))}{\|\boldsymbol{\tau}_h\|_H} \,\geq\, \beta\|(\mathbf{v}_h,\,\boldsymbol{\eta}_h)\|_Q \qquad \forall\,(\mathbf{v}_h,\,\boldsymbol{\eta}_h) \,\in\, Q_h. \tag{4.68}$$

In order to accomplish this, we know from Fortin's Lemma (cf. Lemma 2.6) that it suffices to build a sequence of uniformly bounded operators $\{\Pi_h\}_{h>0} \,\subseteq\, \mathcal{L}(H,H_h)$ such that

$$b(\boldsymbol{\tau} \,-\, \Pi_h(\boldsymbol{\tau}),\,(\mathbf{v},\boldsymbol{\eta}_h)) \,=\, 0 \qquad \forall\,(\mathbf{v},\boldsymbol{\eta}_h) \in\, Q_h. \tag{4.69}$$

We now let $b_1 : H_0 \times Q_1 \to \mathbb{R}$ and $b_2 : H_0 \times Q_2 \to \mathbb{R}$ be the bounded bilinear forms such that

$$b(\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\eta})) = b_1(\boldsymbol{\tau}, \mathbf{v}) + b_2(\boldsymbol{\tau}, \boldsymbol{\eta}) \quad \forall (\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\eta})) \in H_0 \times Q,$$

that is

$$b_1(\boldsymbol{\tau}, \mathbf{v}) := \int_\Omega \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{\tau} \quad \text{and} \quad b_2(\boldsymbol{\tau}, \boldsymbol{\eta}) := \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}.$$

It follows that (4.69) can be rewritten as

$$b_1(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau}), \mathbf{v}_h) + b_2(\boldsymbol{\tau} - \Pi_h(\boldsymbol{\tau}), \boldsymbol{\eta}_h) = 0 \qquad \forall (\mathbf{v}_h, \boldsymbol{\eta}_h) \in Q_h. \tag{4.70}$$

In addition, if we assume for a moment that we already have a sequence of uniformly bounded operators $\{\Pi_{1,h}\}_{h>0} \subseteq \mathcal{L}(H, H_h)$ such that

$$b_1(\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau}), \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in Q_{1,h}\,, \tag{4.71}$$

then we aim to find a second sequence of uniformly bounded operators $\{\Pi_{2,h}\}_{h>0} \subseteq \mathcal{L}(H, H_h)$ such that

i) $b_1(\Pi_{2,h}(\boldsymbol{\tau}), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in Q_{1,h}$, and

ii) $b_2(\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau}) - \Pi_{2,h}(\boldsymbol{\tau}), \boldsymbol{\eta}_h) = 0 \quad \forall \boldsymbol{\eta}_h \in Q_{2,h}$,

so that defining $\Pi_h := \Pi_{1,h} + \Pi_{2,h}$ the condition (4.70) (equivalently, (4.69)) is satisfied. Indeed, it is easy to see from the analysis in Section 4.2 that, given a triangularization $\mathcal{T}_h$ of $\overline{\Omega}$ and an integer $k \geq 0$, and defining

$$H_h := \{\boldsymbol{\tau}_h \in H_0 : \quad \boldsymbol{\tau}_{h,i}|_K \in RT_k(K) \quad \forall K \in \mathcal{T}_h\} \tag{4.72}$$

and

$$Q_{1,h} := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in [\mathbb{P}_k(K)]^2 \quad \forall K \in \mathcal{T}_h\}\,, \tag{4.73}$$

where $\boldsymbol{\tau}_{h,i}$ stands for the i-th row of $\boldsymbol{\tau}_h$, then one can proceed by rows as in (4.29) to define a uniformly bounded explicit family $\{\Pi_{1,h}\}_{h>0}$ satisfying (4.71).

It remains therefore to build a family $\{\Pi_{2,h}\}_{h>0} \subseteq \mathcal{L}(H, H_h)$ verifying i) and ii). For this purpose, we now follow the approach introduced first in [23] (see also [12] for further extensions). More precisely, we let X_h and Y_h be stable finite element subspaces for the usual primal formulation of the Stokes problem, and, given $\boldsymbol{\tau} \in H_0$, consider the associated Galerkin scheme: Find $(\mathbf{z}_h, p_h) \in X_h \times Y_h$ such that

$$\begin{aligned} \int_\Omega \nabla \mathbf{z}_h : \nabla \omega_h + \int_\Omega p_h \operatorname{div} \omega_h \quad &= 0 \quad \forall\, \omega_h \in X_h, \\ \int_\Omega q_h \operatorname{div} \mathbf{z}_h \quad &= \int_\Omega (\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau})) : S(q_h) \quad \forall\, q_h \in Y_h, \end{aligned} \tag{4.74}$$

where $S(q) := \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \in Q_2 := \mathbb{L}^2_{\text{skew}}(\Omega) \quad \forall\, q \in L^2(\Omega)$. Note that the stability of (4.74) guarantees the existence of a constant $C > 0$, independent of h, such that

$$\|\mathbf{z}_h\|_{1,\Omega} + \|p_h\|_{0,\Omega} \le C\, \|\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau})\|_{0,\Omega} \qquad \forall\, \boldsymbol{\tau} \in H_0\,. \tag{4.75}$$

Now, it is easy to see that

$$\int_\Omega q_h \operatorname{div} \mathbf{z}_h = \int_\Omega \operatorname{curl} \mathbf{z}_h : S(q_h),$$

where, denoting $\mathbf{z}_h := (z_{h,1}, z_{h,2})^{\mathbf{t}} \in X_h$,

$$\operatorname{curl} \mathbf{z}_\mathrm{h} := \begin{pmatrix} -\dfrac{\partial z_{h,1}}{\partial x_2} & \dfrac{\partial z_{h,1}}{\partial x_1} \\ -\dfrac{\partial z_{h,2}}{\partial x_2} & \dfrac{\partial z_{h,2}}{\partial x_1} \end{pmatrix},$$

and hence the second equation in (4.74) can be rewritten as

$$\int_\Omega \Big(\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau}) - \operatorname{curl} \mathbf{z}_h\Big) : S(q_h) = 0 \qquad \forall\, q_h \in Y_h.$$

Thus, the comparison between this identity and the required condition ii) for $\Pi_{2,h}$, suggests to define $\Pi_{2,h}(\boldsymbol{\tau}) := \operatorname{curl} \mathbf{z}_h$ under the assumptions that $\operatorname{curl}(X_h) \subseteq H_h$ and $Q_{2,h} \subseteq S(Y_h)$. In turn, since $\mathbf{div}\, \Pi_{2,h}(\boldsymbol{\tau}) = \mathbf{div} \operatorname{curl} \mathbf{z}_\mathrm{h} = 0$, it follows that

$$b_1\left(\Pi_{2,h}(\boldsymbol{\tau}),\, \mathbf{v}_h\right) = \int_\Omega \mathbf{v}_h \cdot \mathbf{div}\, \Pi_{2,h}(\boldsymbol{\tau}) = 0 \qquad \forall\, \mathbf{v}_h \in Q_{1,h},$$

which shows that i) is also satisfied. In addition, thanks to the uniform boundedness of $\{\Pi_{1,h}\}_{h>0}$ (cf. (4.30)) and the stability result given by (4.75), we find that for each $\boldsymbol{\tau} \in H_0$ there holds

$$\begin{aligned} \|\Pi_{2,h}(\boldsymbol{\tau})\|_{\mathbf{div},\Omega} &= \|\operatorname{curl} \mathbf{z}_\mathrm{h}\|_{0,\Omega} \le \|\mathbf{z}_\mathrm{h}\|_{1,\Omega} \\ &\le C\, \|\boldsymbol{\tau} - \Pi_{1,h}(\boldsymbol{\tau})\|_{0,\Omega} \le C \Big\{ \|\boldsymbol{\tau}\|_{0,\Omega} + \|\Pi_{1,h}(\boldsymbol{\tau})\|_{0,\Omega} \Big\} \\ &\le C \Big\{ \|\boldsymbol{\tau}\|_{0,\Omega} + C_1\, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega} \Big\} \le C_2\, \|\boldsymbol{\tau}\|_{\mathbf{div},\Omega}\,, \end{aligned}$$

which shows that $\{\Pi_{2,h}\}_{h>0}$ is uniformly bounded as well.

As a consequence of the previous analysis, we can say that, given a pair (X_h, Y_h) yielding a well-posed Galerkin scheme for the Stokes problem, the discrete inf-sup condition for b is insured by redefining

$$H_h := \Big\{ \boldsymbol{\tau}_h \in H_0 : \quad \boldsymbol{\tau}_{h,i}|_K \in RT_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\} + \operatorname{curl}(X_h)\,, \tag{4.76}$$

by keeping $Q_{1,h}$ as in (4.73), and by defining

$$Q_{2,h} := S(Y_h). \tag{4.77}$$

In particular, if we consider the "Mini" finite element (cf. [41, Section4.1]) given by

$$X_h := \Big\{ \omega_h \in [C(\Omega)]^2 : \quad \omega_h|_K \in [\mathbb{P}_1(K) \oplus \langle\, b_K \,\rangle]^2 \quad \forall\, K \in \mathcal{T}_h \Big\}$$

and

$$Y_h := \Big\{ q_h \in C(\Omega) : \quad q_h|_K \in \mathbb{P}_1(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

where b_K is the bubble function on the triangle K, then (4.76) (with $k = 0$) and (4.77) become

$$H_h := \Big\{ \boldsymbol{\tau}_h \in H_0 : \quad \boldsymbol{\tau}_{h,i}|_K \in RT_0(K) + \langle\, \operatorname{curl} b_K \,\rangle \quad \forall\, K \in \mathcal{T}_h \Big\}$$

and

$$Q_{2,h} := \left\{ \boldsymbol{\eta}_h := \begin{pmatrix} 0 & q_h \\ -q_h & 0 \end{pmatrix} : \quad q_h \in C(\Omega) \quad \text{y} \quad q_h|_K \in \mathbb{P}_1(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

which, together with $Q_{1,h}$ given by (4.73) (with $k = 0$), constitutes the well-known PEERS finite element subspace of order 0 for linear elasticity (cf. [2]).

Finally, it is easy to see from these definitions of H_h, $Q_{1,h}$, and $Q_{2,h}$ that the discrete kernel of b becomes

$$V_h := \Big\{ \boldsymbol{\tau}_h \in H_h : \quad \mathbf{div}\, \boldsymbol{\tau}_h = 0 \ \text{ in } \ \Omega \quad \text{and} \quad \int_\Omega \boldsymbol{\eta}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, \boldsymbol{\eta}_h \in Q_{2,h} \Big\}.$$

Hence, according to the inequalities given by (2.52) and Lemma 2.3, we conclude that a is V_h-elliptic, which completes the hypotheses required by the discrete Babuška-Brezzi theory (cf. Theorem 2.4) for the well-posedness of (4.67).

We end this monograph by mentioning that certainly many interesting topics have been left aside of it, which include, just to name a few, *nonlinear boundary value problems, time-dependent problems, a posteriori error analysis, and further applications in continuum mechanics and electromagnetism* (see, e.g. the recent book [13] and the extensive list of references therein for a thorough discussion on them). In particular, it is worth mentioning that in the case of the linear elasticity problem, new stable mixed finite element methods in 2D and 3D with either strong symmetry or weakly imposed symmetry for the stresses, have been derived during the last decade using the finite element exterior calculus, a quite abstract framework involving several sophisticated mathematical tools (see, e.g. [3], [4], [5], and [6]). In addition, concerning a posteriori error estimates for mixed finite element methods, we refer to the key contributions in [1] and [17], and also, within the context of the linear elasticity and Stokes problems, to [18], [19] and [47]. Furthermore, and as complementary bibliographic material addressing some of the related contributions by the author together with his main collaborators and former students, we may also refer to [10], [11], [21], [24], [25], [27], [28], [29], [30], [32], [33], [34], [36], [37], and [38], which deal mainly with *augmented mixed methods for linear and nonlinear problems in elasticity and fluid mechanics, twofold saddle point variational formulations, fluid-solid interaction problems, and the corresponding a posteriori error analyses*. We hope to write an extended version of the present book in the near future, which incorporates the contents of most of the above mentioned references.