Chapter 3

Raviart-Thomas Spaces

In this chapter we introduce the Raviart-Thomas spaces, which constitute the most classical finite element subspaces of $H(\mathrm{div};\Omega)$, and prove their main interpolation and approximation properties . Several aspects of our analysis follow the approaches from [16], [50], and [52].

3.1 Preliminary results

In what follows, Ω is a bounded and connected domain of $\mathbb{R}^n$, $n \in \{2,3\}$, with polyhedral boundary Γ, and $\mathcal{T}_h$ is a triangularization of $\overline{\Omega}$. More precisely, $\mathcal{T}_h$ is a finite family of triangles (in $\mathbb{R}^2$) or tetrahedra (in $\mathbb{R}^3$), such that:

i) $\overline{\Omega} = \bigcup\limits_{K\in\mathcal{T}_h} K$.

ii) $\overset{\circ}{K} \neq 0 \quad \forall\, K \in \mathcal{T}_h$.

iii) $\overset{\circ}{K}_i \cap \overset{\circ}{K}_j = \phi \quad \forall\, K_i, K_j \in \mathcal{T}_h,\ K_i \neq K_j$.

iv) If $F = K_i \cap K_j,\ \ K_i, K_j \in \mathcal{T}_h,\ K_i \neq K_j$, then F is a common face, a common side, or a common vertex of K_i and K_j.

v) $\mathrm{diam}(K) =: h_K \leq h \qquad \forall\, K \in \mathcal{T}_h$.

In addition, to each $\mathcal{T}_h$ we associate a fixed reference polyhedron $\widehat{K}$, which can or can't belong to $\mathcal{T}_h$, and a family of affine mappings $\{T_K\}_{K\in\mathcal{T}_h}$ such that

a) $T_K : \mathbb{R}^n \to \mathbb{R}^n, T_K(\widehat{x}) = B_K \widehat{x} + b_K \quad \forall \widehat{x} \in \mathbb{R}^n$, with $B_K \in \mathbb{R}^{n\times n}$ invertible, and $b_K \in \mathbb{R}^n$.

b) $K = T_K(\widehat{K}) \quad \forall K \in \mathcal{T}_h$.

One usually considers $\widehat{K}$ as the unit simplex, that is the triangle with vertices (1,0), (0,1), and (0,0) in $\mathbb{R}^2$, or the tetrahedron with vertices (1,0,0), (0,1,0), (0,0,1) and (0,0,0) in $\mathbb{R}^3$.

Throughout the rest of this section we demonstrate a sequence of results characterizing the spaces $H^1(\Omega)$ and $H(\mathrm{div};\Omega)$ in terms of their local behaviours on the elements of the triangularization $\mathcal{T}_h$. In what follows, $\langle \cdot,\cdot\rangle_{\partial K}$ denotes the duality between $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$ for each $K \in \mathcal{T}_h$. In turn, we omit the symbol $\gamma_{\boldsymbol{\nu}}$ to denote the respective normal traces, and simply write, when no confusion arises, $\boldsymbol{\tau}\cdot\boldsymbol{\nu} \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega)$ and $\boldsymbol{\tau}\cdot\boldsymbol{\nu}_K \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};K)$, where $\boldsymbol{\nu}_K$ is the normal vector to ∂K. Similarly, we omit the symbol γ_0 and just write $v|_\Gamma$ (or only v) for $v \in H^1(\Omega)$, and $v|_{\partial K}$ (or only v) for $v \in H^1(K)$.

LEMA 3.1 *Define the spaces* $X := \left\{ v \in L^2(\Omega) : \quad v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h \right\}$ *and* $H_0(\mathrm{div};\Omega) := \left\{ \boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \quad \boldsymbol{\tau}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma \right\}$. *Then*

$$H^1(\Omega) = \left\{ v \in X : \quad \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K} = 0 \quad \forall \boldsymbol{\tau} \in H_0(\mathrm{div};\Omega) \right\}.$$

DEMOSTRACIÓN. We proceed by double inclusion. Let $v \in X$ such that

$$\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K} = 0 \quad \forall \boldsymbol{\tau} \in H_0(\mathrm{div};\Omega).$$

Since $v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h$, we have for each $\boldsymbol{\tau} \in H_0(\mathrm{div};\Omega)$ that

$$\int_K \boldsymbol{\tau}\cdot\nabla v = -\int_K v\,\mathrm{div}\,\boldsymbol{\tau} + \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K},$$

which yields

$$\sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\tau}\cdot\nabla v = -\int_\Omega v\,\mathrm{div}\,\boldsymbol{\tau}.$$

In particular, for $\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n \subseteq H_0(\mathrm{div};\Omega)$ the above identity becomes

$$\langle \nabla v, \boldsymbol{\tau}\rangle_{[\mathcal{D}'(\Omega)]^n\times[\mathcal{D}(\Omega)]^n} = \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\tau}\cdot\nabla v = \int_\Omega \boldsymbol{\tau}\cdot w,$$

where $\langle \cdot, \cdot \rangle_{[\mathcal{D}'(\Omega)]^n \times [\mathcal{D}(\Omega)]^n}$ stands for the distributional pairing of $[\mathcal{D}'(\Omega)]^n$ and $[\mathcal{D}(\Omega)]^n$, and $w \in [L^2(\Omega)]^n$ is given by $w|_K = \nabla(v|_K) \quad \forall K \in \mathcal{T}_h$. This proves that $\nabla v = w$ in $[\mathcal{D}'(\Omega)]^n$, and hence $v \in H^1(\Omega)$.

Conversely, let $v \in H^1(\Omega)$. It is clear that $v \in X$ since obviously $v \in L^2(\Omega)$ and $v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h$. Now, given $\boldsymbol{\tau} \in H_0(\mathrm{div};\Omega)$, we utilize the Green identity (1.50) (cf. Lemma 1.4) in $H(\mathrm{div};\Omega)$ and $H(\mathrm{div};K) \quad \forall K \in \mathcal{T}_h$, to deduce that

$$\begin{aligned} 0 &= \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, v \rangle_\Gamma = \int_\Omega \boldsymbol{\tau} \cdot \nabla v + \int_\Omega v \operatorname{div} \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\tau} \cdot \nabla v + \int_\Omega v \operatorname{div} \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \Big\{ - \int_K v \operatorname{div} \boldsymbol{\tau} + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} \Big\} + \int_\Omega v \operatorname{div} \boldsymbol{\tau} \\ &= \sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K}, \end{aligned}$$

which completes the proof.

□

An immediate consequence of the above theorem is given by the following result.

Lema 3.2 *Let* $X := \Big\{ v \in L^2(\Omega) : \quad v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h \Big\}$. *Then:*

$$H^1(\Omega) = \Big\{ v \in X : \quad \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K \, v = 0 \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n \Big\}.$$

Demostración. The proof follows by employing Lemma 3.1, the inclusion $[C_0^\infty(\Omega)]^n \subseteq H_0(\mathrm{div};\Omega)$, the fact that $\boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall \boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n, \quad \forall K \in \mathcal{T}_h$, and the identity (cf. (1.45))

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v \rangle_{\partial K} = \int_{\partial K} \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K \, v \quad \forall v \in H^1(K), \quad \forall \boldsymbol{\tau} \in [H^1(K)]^n.$$

We omit further details.

□

In order to further simplify the characterization of $H^1(\Omega)$ given by the previous lemmas, we need the following technical result.

Lema 3.3 *Let* K_i, $K_j \in \mathcal{T}_h$ *be adjacent polyhedra with common face/side* F *and let* $z \in L^2(F)$ *such that* $\int_F z \, \rho = 0 \quad \forall \rho \in C_0^\infty(K_i \cup K_j)$. *Then* $z = 0$ *on* F.

DEMOSTRACIÓN. Using that $C_0^\infty(F)$ is dense in $L^2(F)$, it suffices to show that $\displaystyle\int_F z\varphi = 0 \quad \forall\,\varphi \in C_0^\infty(F)$. To this end, let G be a perpendicular line to F, and let $x = (x_1, x_2, \cdots, x_n)$ be the representation of a coordinate system with $(x_1, x_2, \cdots, x_{n-1}) \in F$, $x_n \in G$, and the origin given by the intersection point of F and G (which can be assumed to be the barycenter of F). Then, given $\varphi \in C_0^\infty(F)$, we can construct, via regularization techniques, a function $\psi \in C_0^\infty(G)$ such that $\psi(0) = 1$ and so that $\operatorname{sop}\varphi \times \operatorname{sop}\psi$ is contained in the interior of $K_i \cup K_j$. Hence, defining the function $\rho(x) := \varphi(x_1, x_2, \cdots, x_{n-1})\,\psi(x_n)$, we have that $\rho \in C_0^\infty(K_i \cup K_j)$ and $\rho|_F = \varphi$, which implies that $0 = \displaystyle\int_F z\,\rho = \int_F z\,\varphi$, thus finishing the proof.

□

We are able now to prove the following theorem.

TEOREMA 3.1 *Let* $X := \Big\{ v \in L^2(\Omega) : \quad v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h \Big\}$. *Then:*

$$H^1(\Omega) = \Big\{ v \in X : \quad v|_{K_i} - v|_{K_j} = 0 \quad \textit{in} \quad L^2(F)$$
$$\forall\, K_i, K_j \in \mathcal{T}_h \quad \textit{that are adjacent with common face/side } F \Big\}.$$

DEMOSTRACIÓN. Let $v \in X$ such that $v|_{K_i} - v|_{K_j} = 0$ in $L^2(F)$ $\forall\, K_i, K_j \in \mathcal{T}_h$ that are adjacent with common face/side F. Then, given $\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n$ we have $\boldsymbol{\tau}\cdot\boldsymbol{\nu} = 0$ in Γ and hence

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v = \sum_{F\in I_h(\Omega)} \int_F \Big(v|_{K_{i,F}} - v|_{K_{j,F}} \Big) \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_{i,F}}\,,$$

where $I_h(\Omega)$ is the set of interior faces/sides of $\mathcal{T}_h$, and $K_{i,F}$ and $K_{j,F}$ are the adjacent polyhedra with common face/side F. Note here that $\boldsymbol{\nu}_{K_{i,F}} = -\,\boldsymbol{\nu}_{K_{j,F}}$. It follows that

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v = 0 \qquad \forall\,\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n\,,$$

which, thanks to Lemma 3.2, implies that $v \in H^1(\Omega)$.

Conversely, let $v \in H^1(\Omega)$. It is clear from Lemma 3.2 that

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v = 0 \qquad \forall\,\boldsymbol{\tau} \in [C_0^\infty(\Omega)]^n\,.$$

In particular, given $\boldsymbol{\tau} \in [C_0^\infty(K_i \cup K_j)]^n$, with $K_i, K_j \in \mathcal{T}_h$ adjacent with common face/side F, we obtain

$$
\begin{aligned}
0 &= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v = \int_F \Big(v|_{K_i} - v|_{K_j}\Big)\boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_i} \\
&= \int_F \Big(v|_{K_i} - v|_{K_j}\Big)\boldsymbol{\nu}_{K_i}\cdot\boldsymbol{\tau},
\end{aligned}
$$

from which, applying Lemma 3.3 to a non-null component of $\boldsymbol{\nu}_{K_i}$, we deduce that $v|_{K_i} - v|_{K_j} = 0$ in $L^2(F)$.

$\square$

Our next goal is to characterize the space $H(\mathrm{div};\Omega)$ in terms of the local behaviours. We begin with the following lemma, which constitutes a kind of dual result to Lemma 3.1.

Lema 3.4 *Let* $Y := \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^n : \quad \boldsymbol{\tau}|_K \in H(\mathrm{div};K) \quad \forall\, K\in\mathcal{T}_h\Big\}$. *Then*

$$
H(\mathrm{div};\Omega) = \Big\{\boldsymbol{\tau} \in Y : \quad \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K,\, v\rangle_{\partial K} = 0 \quad \forall\, v \in H_0^1(\Omega)\Big\}.
$$

Demostración. We proceed by double inclusion. Let $\boldsymbol{\tau} \in Y$ such that

$$
\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K,\, v\rangle_{\partial K} = 0 \qquad \forall\, v \in H_0^1(\Omega).
$$

Since $\boldsymbol{\tau}|_K \in H(\mathrm{div};K) \quad \forall\, K\in\mathcal{T}_h$, we have for each $v \in H_0^1(\Omega)$

$$
\int_K v\,\mathrm{div}\,\boldsymbol{\tau} = -\int_K \boldsymbol{\tau}\cdot\nabla v + \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K}\,,
$$

which gives

$$
\sum_{K\in\mathcal{T}_h} \int_K v\,\mathrm{div}\,\boldsymbol{\tau} = -\int_\Omega \boldsymbol{\tau}\cdot\nabla v\,.
$$

In particular, for $v \in C_0^\infty(\Omega) \subseteq H_0^1(\Omega)$, the above identity reduces to

$$
\langle \mathrm{div}\,\boldsymbol{\tau}, v\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \sum_{K\in\mathcal{T}_h} \int_K v\,\mathrm{div}\,\boldsymbol{\tau} = \int_\Omega v\, z,
$$

where $\langle\cdot,\cdot\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)}$ is the distributional pairing of $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$, and $z \in L^2(\Omega)$ is given by $z|_K = \mathrm{div}\,(\boldsymbol{\tau}|_K) \quad \forall\, K\in\mathcal{T}_h$. This shows that $\mathrm{div}\,\boldsymbol{\tau} = z$ in $\mathcal{D}'(\Omega)$, and hence $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$.

Conversely, let $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$. It is clear that $\boldsymbol{\tau} \in Y$ since obviously $\boldsymbol{\tau} \in [L^2(\Omega)]^n$ and $\boldsymbol{\tau}|_K \in H(\mathrm{div};K) \quad \forall\, K \in \mathcal{T}_h$. Thus, given $v \in H_0^1(\Omega)$, we first utilize the Green identity (1.50) (cf. Lemma 1.4) in $H(\mathrm{div};\Omega)$ and $H(\mathrm{div};K) \quad \forall\, K \in \mathcal{T}_h$, and proceed next as in the second part of the proof of Lemma 3.1, to conclude that

$$0 \,=\, \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu},\, v\rangle_\Gamma = \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}_K, v\rangle_{\partial K},$$

which completes the proof.

$\square$

The following theorem is consequence of the above lemma and the technical result given by Lemma 3.3.

Teorema 3.2 *Let* $Z := \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^n : \quad \boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall\, K \in \mathcal{T}_h\Big\}$. *Then*

$$H(\mathrm{div};\Omega) \cap Z \,=\, \Big\{\boldsymbol{\tau} \in Z : \quad \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_i} + \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_j} \,=\, 0 \quad \textit{in} \quad L^2(F) \\ \forall\, K_i, K_j \in \mathcal{T}_h \quad \textit{that are adjacent with common face/side } F\Big\}.$$

Demostración. Let $\boldsymbol{\tau} \in Z$ such that $\boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_i} + \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_j} = 0$ in $L^2(F) \quad \forall\, K_i, K_j \in \mathcal{T}_h$ that are adjacent with common face/side F. Then, given $v \in H_0^1(\Omega)$, we use that $\boldsymbol{\tau}\cdot\boldsymbol{\nu}_K \in L^2(\partial K)$ since $\boldsymbol{\tau}|_K \in [H^1(K)]^n \quad \forall\, K \in \mathcal{T}_h$, and employ the same notation of Theorem 3.1, to deduce that

$$\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K} \,=\, \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v \,=\, \sum_{F\in I_h(\Omega)} \int_F \Big(\boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_{i,F}} + \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_{j,F}}\Big)\, v \,=\, 0\,,$$

which, thanks to Lemma 3.4, yields $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$.

Conversely, let $\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \cap Z$. It follows again from Lemma 3.4, that

$$\sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K} \,=\, 0 \qquad \forall\, v \,\in\, H_0^1(\Omega)\,.$$

In particular, for $v \in C_0^\infty(K_i \cup K_j)$, where $K_i, K_j \in \mathcal{T}_h$ are adjacent polyhedra with common face/side F, we find that

$$0 \,=\, \sum_{K\in\mathcal{T}_h} \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K, v\rangle_{\partial K} \,=\, \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\tau}\cdot\boldsymbol{\nu}_K\, v \,=\, \int_F \Big(\boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_i} + \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_j}\Big)\, v\,,$$

and hence, in virtue of Lemma 3.3, we conclude that $\boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_i} + \boldsymbol{\tau}\cdot\boldsymbol{\nu}_{K_j} = 0$ in $L^2(F)$.

$\square$