New mixed finite element methods for the coupled convective Brinkman-Forchheimer and double-diffusion equations*

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Abstract

In this paper we introduce and analyze new Banach spaces-based mixed finite element methods for the stationary nonlinear problem arising from the coupling of the convective Brinkman-Forchheimer equations with a double diffusion phenomenon. Besides the velocity and pressure variables, the symmetric stress and the skew-symmetric vorticity tensors are introduced as auxiliary unknowns of the fluid. Thus, the incompressibility condition allows to eliminate the pressure, which, along with the velocity gradient and the shear stress, can be computed afterwards via postprocessing formulae depending on the velocity and the aforementioned new tensors. Regarding the diffusive part of the coupled model, and additionally to the temperature and concentration of the solute, their gradients and pseudoheat/pseudodiffusion vectors are incorporated as further unknowns as well. The resulting mixed variational formulation, settled within a Banach spaces framework, consists of a nonlinear perturbation of, in turn, a nonlinearly perturbed saddle-point scheme, coupled with a usual saddle-point system. A fixed-point strategy, combined with classical and recent solvability results for suitable linearizations of the decoupled problems, including in particular, the Banach-Nečas-Babuška theorem and the Babuška-Brezzi theory, are employed to prove, jointly with the Banach fixed-point theorem, the well-posedness of the continuous and discrete formulations. Both PEERS and AFW elements of order $\ell \geqslant 0$ for the fluid variables, and piecewise polynomials of degree $\leqslant \ell$ together with Raviart-Thomas elements of order ℓ for the unknowns of the diffusion equations, constitute feasible choices for the Galerkin scheme. In turn, optimal a priori error estimates, including those for the postprocessed unknowns, are derived, and corresponding rates of convergence are established. Finally, several numerical experiments confirming the latter and illustrating the good performance of the proposed methods, are reported.

Key words: convective Brinkman–Forchheimer, stress-vorticity tensor-velocity formulation, double diffusion, fixed point theory, mixed finite element methods, *a priori* error analysis

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1 Introduction

The multiphysics problem of double-diffusive convection in which two scalar fields, such as heat and concentration of a solute, affect the density distribution in a fluid-saturated highly porous medium, has been intensively studied in recent years (see, e.g., [33], [36], [37], [17], [15], and references therein). Applications include predicting and controlling processes arising in geophysics, oceanography, chemical engineering, and energy technology, to name a few. In particular, some of them includes groundwater system in karst aquifers, fast flows in fractured or vuggy aquifers or reservoirs, chemical processing, convective flow of carbon nanotubes, and propagation of biological fluids (see, for instance, [3], [8], [24], and [39]). In this regard, we remark that much of the research in porous medium has been focused on the use of Darcy's law. However, this fundamental equation may be inaccurate for modeling fluid flow through porous media with high Reynolds numbers or through media with high porosity. To overcome this limitation, it is possible to consider the convective Brinkman–Forchheimer equations (see, e.g., [19], [38], [34], [13], and [14]), where terms are added to Darcy's equation in order to take into account the above described physical aspects. Moreover, this fact has motivated the introduction of the corresponding coupling with a system of advection-diffusion equations (also called double-diffusion equations), through convective terms and the body force.

Concerning literature devoted to studying the coupling of the Brinkman–Forchheimer and double-diffusion equations, we first highlight that, up to the authors' knowledge, [33] constitutes one of the first works in analyzing the well-posedness and regularity of solution for a velocity-pressure-temperature-concentration variational formulation. Later on, a finite volume method to solve the coupling of the time-dependent Brinkman–Forchheimer and double-diffusion equations was adopted in [37]. The focus of this work was on the validity of the Brinkman–Forchheimer model when various combinations of the thermal Rayleigh number, permeability ratio, inclination angle, thermal conductivity and buoyancy ratio are considered. More recently, an augmented fully-mixed formulation based on the introduction of the fluid pseudostress tensor, and the pseudoheat and pseudodiffusive vectors (besides the velocity, temperature and concentration fields) was analyzed in [17]. Meanwhile, a non-augmented Banach spaces-based fully-mixed formulation was proposed and analyzed in [15]. In particular, this latter scheme is written equivalently as a fixed-point equation, so that the well-known Banach theorem, combined with classical results on nonlinear monotone operators and Babuška-Brezzi's theory in Banach spaces, are applied to prove the unique solvability of the continuous and discrete systems.

Regarding the literature focused on the analysis of the convective Brinkman–Forchheimer (CBF) equations, we start referring to [19], where the authors analyze the continuous dependence of solutions of the CBF equations written in velocity-pressure formulation on the Forchheimer coefficient in H^1 norm. In turn, an approximation of solutions for the incompressible CBF equations via the artificial compressibility method was proposed and developed in [38], where a family of perturbed compressible CBF equations that approximate the incompressible CBF equations is introduced. Furthermore, the well-posedness of the corresponding velocity-pressure variational formulation of the two-dimensional stationary CBF equations was analyzed in [34]. In addition, error estimates for a mixed finite element approximation were obtained, and a one-step Newton iteration algorithm initialized using a fixed-point iteration, was proposed. Recently, an augmented mixed pseudostress-velocity formulation was analyzed in [13]. In there, the well-posedness of the problem is achieved by combining a fixed-point strategy, the Lax–Milgram theorem, and the well-known Schauder and Banach fixed-point theorems. We also mention [14], where a Banach spaces-based mixed formulation was proposed and analyzed for the CBF problem, but differently from the techniques previously developed in [13], no augmentation procedure was needed for the formulation nor for the solvability analysis. The resulting non-augmented scheme is then written equivalently as a fixed-point equation, so that results recently established in [21] for perturbed saddle-point problems in Banach spaces, along with the well-known Banach–Nečas–Babuška and Banach theorems, are applied to prove the well-posedness of the continuous and discrete systems.

We point out that the motivation of employing an augmented approach, as in [17] and [13], is originated by the wish of performing the respective solvability analysis of the equations within a Hilbertian framework. However, it is well known that the introduction of additional terms into the formulation, while having some advantages, also leads to much more expensive schemes in terms of complexity and computational implementation. In order to overcome this, in recent years there has arisen an increasing development on Banach spaces-based mixed finite element methods to solve a wide family of single and coupled nonlinear problems in continuum mechanics. In particular, we refer to [11], [10], [18], [20], [7], [27], and [12], for the analysis of mixed formulations within a Banach framework of the Poisson, Navier–Stokes, Brinkman–Forchheimer, Boussinesq, coupled flow-transport, Navier–Stokes–Brinkman, and chemotaxis-Navier–Stokes equations. This kind of procedures shows two advantages at least: no augmentation is required, and the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy–Schwarz and Hölder inequalities to the terms resulting from the testing and integration by parts of the equations of the model. As a consequence, simpler and closer to the original physical model formulations are obtained.

According to the previous discussion, and aiming to continue extending the applicability of the aforementioned framework, the goal of the present paper is to develop and analyze a new Banach spaces-based fully-mixed formulation, augmentation free, for the coupling of the convective Brinkman–Forchheimer and double-difusion equations, and study its numerical approximation by the associated mixed finite element method. To this end, and unlike [13] and [14], where the model is based on the pseudostress tensor, and hence on the velocity gradient, here we assume the constitutive equation to be given with respect to the symmetric stress, so that it depends on the strain rate tensor. The reason for proceeding in this way lies on the subsequent purpose, to be discussed elsewhere, of being able to extend our approach to the case of variable viscosity, for which we expect dependence on the strain rate tensor rather than on the velocity gradient. This is the case, for instance, of the rheology models for granular flows, usually represented by the Navier–Stokes equations, in which, as suggested by the evidence available in the literature (see, e.g. [30]), the friction coefficient, which can be understood as an equivalent viscosity, depends on the inertial number, which, in turn, is defined in terms of the symmetric part of the velocity gradient. As a consequence of the utilization of the symmetric stress, and in order to be able to apply the usual integration by parts formula to the velocity gradient multiplied by a suitable test tensor, we decompose the strain rate as the difference between the aforementioned gradient and the skew-symmetric vorticity tensor, so that the latter becomes a natural auxiliary unknown. In addition to the above, and similarly as done in previous related papers that use either stress or pseudostress (see, e.g. [13], [14], and [29]), the pressure unknown is then eliminated thanks to the incompressibility condition, but computed afterwards in terms of the stress (or the pseudostress) and the velocity.

As regards the double-difussion equations, we follow [15, 20] and adopt a dual-mixed formulation making use of the temperature/concentration gradients and the pseudoheat/pseudodiffusion vectors as further unknowns. The resulting mixed formulation is written as a nonlinear perturbation of, in turn, a nonlinearly perturbed saddle-point scheme, coupled with a usual saddle-point system. Then, similarly to [15], [22], [27], and [12], we combine a fixed-point argument, the abstract results provided in [21], the Banach–Nečas–Babuška theorem, Babuška-Brezzi's theory in Banach spaces, sufficiently small data assumptions, and the Banach theorem, to establish existence and uniqueness of solution of both the continuous and discrete formulations. In this regard, and since the formulation is similar to the ones considered in [15], [21], and [22], our present analysis certainly makes use of the corresponding results available there. In addition, applying an ad-hoc Strang-type lemma in Banach spaces established in [17], we are able to derive the corresponding *a priori* error estimates for arbitrary discrete subspaces. Next, employing PEERS and AFW elements of order $\ell \geqslant 0$ for approximating the fluid variables, and piecewise polynomials of degree $\leqslant \ell$ together with Raviart–Thomas elements of order ℓ for the unknowns of the double-diffusion equations, we prove that the corresponding discrete methods are convergent with optimal rates.

The paper is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. The model problem is introduced in Section 2, and all the auxiliary variables to be employed in the setting of the formulation are defined there. Next, in Section 3 we derive the corresponding fully-mixed variational formulation in Banach spaces, whereas, the well-posedness of this continuous scheme is established in Section 4. The corresponding Galerkin system is introduced and analyzed in Section 5, where the discrete analogue of the theory used in the continuous case is employed to prove existence and uniqueness of solution. *A priori* error estimates for arbitrary finite element subspaces are also obtained there. In Section 6 we establish the corresponding rates of convergence for specific discrete subspaces. Finally, the performance of the method is illustrated in Section 7 throughout several numerical examples in 2D and 3D, with and without manufactured solutions, which confirm the accuracy and flexibility of our fully-mixed finite element method.

Preliminary notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\boldsymbol{\nu}$ be the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n},$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\,\zeta_{ij}\,.$$

Furthermore, for each $t \in [1, +\infty)$ we introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{ \boldsymbol{\eta} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\eta}) \in \mathrm{L}^t(\Omega) \Big\}, \quad \text{and}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\},$$

equipped with the natural norms

$$\|\boldsymbol{\eta}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\eta}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\eta})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\eta} \in \mathbf{H}(\mathrm{div}_t;\Omega)\,, \quad \text{and}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,.$$

Additionally, we recall that, proceeding as in [25, eq. (1.43), Section 1.3.4] (see also [11, Section 4.1] and [20, Section 3.1]), one can prove that for $t \in \begin{cases} (1,+\infty] \text{ in } \mathrm{R}^2\,, \\ [\frac{6}{5},+\infty] \text{ in } \mathrm{R}^3\,, \end{cases}$ there holds

$$\langle \boldsymbol{\eta}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{ \boldsymbol{\eta}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\eta}) \Big\} \quad \forall\,(\boldsymbol{\eta},v) \in \mathbf{H}(\mathrm{div}_t;\Omega) \times \mathrm{H}^1(\Omega)\,, \tag{1.1}$$

and

$$\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{v}\rangle = \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega) \times \mathbf{H}^1(\Omega)\,, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.1) (resp. (1.2)) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) and $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$).

2 The model problem

In what follows we consider the steady convective Brinkman–Forchheimer equations introduced in [34] (see also [38, 13, 14]) coupled with double-diffusion equations, similarly as done in [15]. More precisely, we focus on finding a velocity field $\mathbf{u}$, a pressure field p, a temperature field ϕ_1, and a concentration field ϕ_2, the latter two defining a vector unknown $\boldsymbol{\phi} := (\phi_1,\phi_2)$, such that

$$\begin{aligned}
-\,\mathbf{div}\big(\mu\,\mathbf{e}(\mathbf{u})\big) + (\nabla\mathbf{u})\mathbf{u} + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} + \nabla p &= \mathbf{f}(\boldsymbol{\phi}) && \text{in} \quad \Omega\,,\\
\mathrm{div}(\mathbf{u}) &= 0 && \text{in} \quad \Omega\,,\\
-\,\mathrm{div}(\mathbf{Q}_1\,\nabla\phi_1) + \mathtt{R}_1\,\mathbf{u}\cdot\nabla\phi_1 &= g_1 && \text{in} \quad \Omega\,,\\
-\,\mathrm{div}(\mathbf{Q}_2\,\nabla\phi_2) + \mathtt{R}_2\,\mathbf{u}\cdot\nabla\phi_2 &= g_2 && \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_{\mathrm{D}}\,, \quad \phi_1 = \phi_{1,\mathrm{D}}\,, \quad \text{and} \quad \phi_2 &= \phi_{2,\mathrm{D}} && \text{on} \quad \Gamma\,,
\end{aligned} \tag{2.1}$$

where μ is the Brinkman coefficient (or effective viscosity), which is assumed to be eventually variable, and bounded, that is there exist constants μ_0, $\mu_1 > 0$, such that

$$\mu_0 \leqslant \mu(\mathbf{x}) \leqslant \mu_1 \qquad \forall\,\mathbf{x}\in\Omega\,. \tag{2.2}$$

In addition, $\mathbf{e}(\mathbf{u}) := \frac{1}{2}\big(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\big)$ is the symmetric part of $\nabla\mathbf{u}$, also named strain rate tensor, $\mathtt{D} > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient, ρ is a given number in $[3,4]$, and $\mathbf{f}(\boldsymbol{\phi})$ is an external force defined by

$$\mathbf{f}(\boldsymbol{\phi}) := -\,(\phi_1 - \phi_{1,\mathtt{r}})\,\mathbf{g} + \frac{1}{\varrho}\,(\phi_2 - \phi_{2,\mathtt{r}})\,\mathbf{g}, \tag{2.3}$$

where $\mathbf{g}$ represents the potential type gravitational acceleration, $\phi_{1,\mathtt{r}}$ and $\phi_{2,\mathtt{r}}$ are the reference temperature and concentration of a solute, respectively, and ϱ is a parameter experimentally valued that can be assumed to be $\geqslant 1$ (see [33, Section 2] for details). The spaces to which $\phi_{1,\mathtt{r}}$ and $\phi_{2,\mathtt{r}}$ belong will be specified later on. In turn, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ denote the thermal and concentration diffusion tensors, respectively, which are assumed to belong to $\mathbb{L}^\infty(\Omega)$, whereas $\mathtt{R}_1$ is the thermal Rayleigh number and

$\mathtt{R}_2$ is the solute Rayleigh number. In addition, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are assumed to be uniformly positive definite tensors, which means that there exist positive constants C_1 and C_2, such that

$$\mathbf{v} \cdot \mathbf{Q}_j(\mathbf{x})\mathbf{v} \,\geqslant\, C_j\,|\mathbf{v}|^2 \quad \forall\,\mathbf{v} \in \mathrm{R}^n, \quad \forall\,\mathbf{x} \in \Omega, \quad j \in \{1,2\}\,, \tag{2.4}$$

and g_1 and g_2 are given source terms in suitable spaces to be specified later on. Finally, $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$ and $\phi_{i,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$, $i \in \big\{1,2\big\}$, are given Dirichlet data.

Owing to the incompressibility of the fluid and the Dirichlet boundary condition for $\mathbf{u}$, the datum $\mathbf{u}_{\mathrm{D}}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_{\mathrm{D}} \cdot \boldsymbol{\nu} \,=\, 0\,. \tag{2.5}$$

In addition, due to the pressure gradient in (2.1), and in order to guarantee uniqueness of this unknown, p will be sought in the space

$$\mathrm{L}^2_0(\Omega) \,:=\, \Big\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \Big\}\,.$$

Now, in order to derive a fully mixed formulation for (2.1), thus yielding the Dirichlet boundary conditions to become natural, we proceed similarly to [27] (see also [10] for related approaches), and introduce as a further unknown the symmetric tensor $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma} \,:=\, \mu\,\mathbf{e}(\mathbf{u}) \,-\, (\mathbf{u} \otimes \mathbf{u}) \,-\, p\,\mathbb{I}\,. \tag{2.6}$$

In this way, the first equation of (2.1) can be rewritten as

$$\mathtt{D}\,\mathbf{u} \,+\, \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \,-\, \mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f}(\boldsymbol{\phi}) \quad \text{in} \quad \Omega\,, \tag{2.7}$$

whereas applying the trace operator to $\boldsymbol{\sigma}$ and utilizing the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in Ω, we obtain

$$p \,=\, -\,\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma} \,+\, (\mathbf{u} \otimes \mathbf{u})\big)\,. \tag{2.8}$$

Moreover, applying the deviatoric operator to (2.6) and dividing by μ, we find that

$$\frac{1}{\mu}\,\boldsymbol{\sigma}^{\mathtt{d}} \,+\, \frac{1}{\mu}\,(\mathbf{u} \otimes \mathbf{u})^{\mathtt{d}} \,=\, \mathbf{e}(\mathbf{u}) \,=\, \nabla\mathbf{u} \,-\, \boldsymbol{\gamma}\,, \tag{2.9}$$

where the skew-symmetric vorticity tensor

$$\boldsymbol{\gamma} \,:=\, \frac{1}{2}\Big(\nabla\mathbf{u} \,-\, \big(\nabla\mathbf{u}\big)^{\mathtt{t}}\Big) \tag{2.10}$$

is introduced as a further unknown. Note that the diagonal entries of $\boldsymbol{\gamma}$ are all null, and that, modulus the sign and the factor $\frac{1}{2}$ involved, the off diagonal ones are given by the components of the vorticity vector

$$\underline{\mathrm{curl}}(\mathbf{u}) = \begin{cases} \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} & , \quad \text{for } n = 2\,, \\ \qquad \nabla \times \mathbf{u} & , \quad \text{for } n = 3\,. \end{cases}$$

This is the reason why we employ the adjective "vorticity" to refer to $\boldsymbol{\gamma}$.

Next, for the double-diffusion equations we consider for each $j \in \{1,2\}$ the temperature (or concentration) gradient $\mathbf{t}_j$, and the corresponding pseudoheat (or pseudodiffusion) $\boldsymbol{\vartheta}_j$, as auxiliary unknowns, which are defined, respectively, by

$$\mathbf{t}_j \,:=\, \nabla\phi_j\,, \quad \boldsymbol{\vartheta}_j \,:=\, \mathbf{Q}_j\,\mathbf{t}_j \,-\, \frac{1}{2}\,\mathtt{R}_j\,\phi_j\,\mathbf{u}, \quad \forall\, j \in \{1,2\}, \quad \text{in} \quad \Omega\,, \tag{2.11}$$

whence the third and fourth equations of (2.1) can be rewritten as

$$\frac{1}{2}\,\mathtt{R}_j\,\mathbf{u}\cdot\mathbf{t}_j \,-\, \mathrm{div}(\boldsymbol{\vartheta}_j) \,=\, g_j \quad \text{in} \quad \Omega\,, \quad j \in \{1,2\}\,. \tag{2.12}$$

Consequently, gathering (2.7), (2.9), (2.11), and (2.12), and incorporating the Dirichlet boundary conditions, we find that (2.1) can be rewritten, equivalently, as follows: Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})$ and $(\phi_j, \mathbf{t}_j, \boldsymbol{\vartheta}_j)$, $j \in \{1,2\}$, in suitable spaces to be indicated below, such that

$$\begin{array}{rcll}
\dfrac{1}{\mu}\,\boldsymbol{\sigma}^{\mathrm{d}} + \dfrac{1}{\mu}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \,+\, \boldsymbol{\gamma} & = & \nabla\mathbf{u} & \text{in} \quad \Omega\,, \\
\mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) & = & \mathbf{f}(\boldsymbol{\phi}) & \text{in} \quad \Omega\,, \\
\mathbf{t}_j & = & \nabla\phi_j & \text{in} \quad \Omega\,, \quad j \in \{1,2\}\,, \\
\mathbf{Q}_j\,\mathbf{t}_j \,-\, \dfrac{1}{2}\,\mathtt{R}_j\,\phi_j\,\mathbf{u} & = & \boldsymbol{\vartheta}_j & \text{in} \quad \Omega\,, \quad j \in \{1,2\}\,, \\
\dfrac{1}{2}\,\mathtt{R}_j\,\mathbf{u}\cdot\mathbf{t}_j \,-\, \mathrm{div}(\boldsymbol{\vartheta}_j) & = & g_j & \text{in} \quad \Omega\,, \quad j \in \{1,2\}\,, \\
\mathbf{u} \,=\, \mathbf{u}_{\mathrm{D}}\,, \quad \phi_1 \,=\, \phi_{1,\mathrm{D}}\,, \quad \text{and} \quad \phi_2 & = & \phi_{2,\mathrm{D}} & \text{on} \quad \Gamma\,, \\
\displaystyle\int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} \,+\, (\mathbf{u}\otimes\mathbf{u})\big) & = & 0 & \text{in} \quad \Omega\,.
\end{array} \tag{2.13}$$

We remark here that, as suggested by (2.8), p is eliminated from the present formulation and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ by using that identity. This fact justifies the last equation in (2.13), which aims to ensure that the resulting p does belong to $\mathrm{L}^2_0(\Omega)$. In addition, we observe that (2.9) and (2.10) follow from the first equation of (2.13), so that there is no need to include them explicitly here. Indeed, knowing that $\boldsymbol{\sigma}$ must be sought symmetric and that $(\mathbf{u}\otimes\mathbf{u})$ is symmetric, it follows that $\boldsymbol{\sigma}^{\mathbf{d}}$ and $(\mathbf{u}\otimes\mathbf{u})^{\mathbf{d}}$ share this property as well. In this way, using also that $\boldsymbol{\gamma}$ must be sought skew-symmetric, we realize that (2.9) arises after taking the symmetric part of the first equation of (2.13). In turn, it is readily seen that taking the skew-symmetric part of the same equation, one obtains (2.10) .

On the other hand, we notice also that further variables of interest, such as the velocity gradient $\nabla\mathbf{u}$, and the shear stress tensor $\widetilde{\boldsymbol{\sigma}} \,:=\, \mu\,\mathbf{e}(\mathbf{u}) \,-\, p\,\mathbb{I}$, can be easily computed, respectively, as follows

$$\nabla\mathbf{u} \,=\, \frac{1}{\mu}\,\boldsymbol{\sigma}^{\mathrm{d}} + \frac{1}{\mu}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \,+\, \boldsymbol{\gamma} \quad \text{and} \quad \widetilde{\boldsymbol{\sigma}} \,=\, \boldsymbol{\sigma} \,+\, (\mathbf{u}\otimes\mathbf{u})\,. \tag{2.14}$$

We highlight here that, precisely, having defined $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ as auxiliary unknowns allows the derivation of the post-processing formulas provided by (2.14), which constitutes a clear advantage of incorporating those tensors into our formulation. We reinforce this remark at the end of Section 5 when referring to the discrete versions of (2.8) and (2.14), and the associated rates of convergence.

3 The variational formulation

In this section we follow [10] and [20] (see also [14, 15, 27, 28, 29]) to derive a mixed formulation for (2.13) within a Banach spaces framework.

As compared with just a convective Brinkman-Forchheimer problem, in which the velocity $\mathbf{u}$ appears nonlinearly in the respective convective and Forchheimer terms, the fact that we are considering here the coupling with double-diffusion equations, which also involve $\mathbf{u}$ nonlinearly in two other distinct terms (cf. fourth and fifth rows of (2.13)), makes us, first of all, to face the derivation of the right space in which this unknown must be sought. Thus, regarding the spaces to which the different unknowns involved must belong, this initial difficulty requires now the correct matching of four terms. Moreover, the present Forchheimer term, involving a power of $|\mathbf{u}|$ ranging between 1 and 2, is more general than the one from [15] and [17], which simply reduces to $\mathtt{F}\,|\mathbf{u}|\mathbf{u}$, so that this aspect must be carefully considered as well. Furthermore, the fixed-point strategy to be employed later on for the solvability of our coupled problem, will require, besides the one taking care of the convective and Forchheimer terms, the introduction of two additional operators, one for each diffusion equation. As a consequence, the corresponding continuous and discrete analyses become a bit more cumbersome.

We begin by testing the first equation of (2.13) against a tensor $\boldsymbol{\tau}$ associated with the unknown $\boldsymbol{\sigma}$, so that, using the identity $\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau} = \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}$, we formally get

$$\int_\Omega \frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \frac{1}{\mu}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} \;=\; \int_\Omega \nabla\mathbf{u} : \boldsymbol{\tau}\,. \tag{3.1}$$

We observe that the first and third expressions on the left hand side of (3.1) make sense for $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\boldsymbol{\gamma} \in \mathbb{L}^2(\Omega)$. In turn, seeking originally $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which is in line with the condition that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, and assuming that $\boldsymbol{\tau}$ is taken in $\mathbb{H}(\mathbf{div}_t;\Omega)$, with t fitting the ranges for the validity of (1.1) and (1.2), we can apply the latter, and employ the Dirichlet boundary condition on $\mathbf{u}$, to obtain

$$\int_\Omega \nabla\mathbf{u} : \boldsymbol{\tau} \;=\; -\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \;+\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma\,. \tag{3.2}$$

In this way, replacing (3.2) back into (3.1), we arrive at

$$\int_\Omega \frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \frac{1}{\mu}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} \;=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D\rangle_\Gamma \tag{3.3}$$

for all $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$. Now, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, and using Hölder's inequality, we conclude from the second term in (3.3) that it suffices to look for $\mathbf{u}$ in $\mathbf{L}^{t'}(\Omega)$ instead of $\mathbf{H}^1(\Omega)$, where t, $t' \in (1,+\infty)$ are conjugate to each other. In addition, employing the Cauchy–Schwarz and Hölder inequalities, we readily deduce that the convective nonlinear term is well defined if $\mathbf{u} \in \mathbf{L}^4(\Omega)$, which yields to choose $t' = 4$, and thus $t = 4/3$, whence the test space for $\boldsymbol{\tau}$ becomes $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$.

On the other hand, linking the spaces to which the unknown $\boldsymbol{\sigma}$ and its test functions $\boldsymbol{\tau}$ belong, we impose to look for $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. Hence, testing the second equation of (2.13) against $\mathbf{v} \in \mathbf{L}^4(\Omega)$, formally yields

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \mathtt{D}\int_\Omega \mathbf{u}\cdot\mathbf{v} - \mathtt{F}\int_\Omega |\mathbf{u}|^{\rho-2}\mathbf{u}\cdot\mathbf{v} \;=\; -\int_\Omega \mathbf{f}(\boldsymbol{\phi})\cdot\mathbf{v} \tag{3.4}$$

for all $\mathbf{v} \in \mathbf{L}^4(\Omega)$, from which the first term is bounded thanks to the fact that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$. Next, noting that for $\rho \in [3,4]$ there holds $2(\rho-2) \leqslant 4$, we consider the continuous injection $i_{2(\rho-2)} : \mathbf{L}^4(\Omega) \to \mathbf{L}^{2(\rho-2)}(\Omega)$ and observe that $\|i_{2(\rho-2)}\| \leqslant |\Omega|^{(4-\rho)/4(\rho-2)}$. In this way, applying the Cauchy–Schwarz and Hölder inequalities to the third term on the left-hand side of (3.4), we find that

$$\left|\int_\Omega |\mathbf{w}|^{\rho-2}\mathbf{u}\cdot\mathbf{v}\right| \;\leqslant\; \|\mathbf{w}\|_{0,2(\rho-2);\Omega}^{\rho-2}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega} \;\leqslant\; |\Omega|^{(4-\rho)/4}\,\|\mathbf{w}\|_{0,4;\Omega}^{\rho-2}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,,$$

which proves that the aforementioned term is well-defined for $\mathbf{u}$, $\mathbf{w}$, $\mathbf{v} \in \mathbf{L}^4(\Omega)$. In turn, being $\mathbf{L}^4(\Omega)$ certainly contained in $\mathbf{L}^2(\Omega)$ guarantees that the second term in (3.4) is bounded as well, whereas the right hand side of (3.4) becomes well defined if $\mathbf{f}(\phi)$ (cf. (2.3)) belongs to $\mathbf{L}^{4/3}(\Omega)$, which is assumed from now on. We will refer again to this issue later on.

Finally, the symmetry of $\boldsymbol{\sigma}$ (cf. (2.6)) is imposed weakly as

$$\int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma} = 0 \quad \forall\, \boldsymbol{\delta} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,, \tag{3.5}$$

where

$$\mathbb{L}^2_{\mathrm{skew}}(\Omega) \,:=\, \Big\{ \boldsymbol{\delta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\delta}^{\mathrm{t}} = -\boldsymbol{\delta} \Big\}\,.$$

We stress here that the above weak imposition, which is possible thanks to the availability of the space to which $\boldsymbol{\gamma}$ must belong, constitutes another advantage of having included $\boldsymbol{\sigma}$ along with $\boldsymbol{\gamma}$ as additional unknowns. Indeed, if only $\boldsymbol{\sigma}$ were introduced, then the symmetry of this tensor would need to be incorporated into the definition of the space in which $\boldsymbol{\sigma}$ is sought. While this fact does not yield any difficulty at the continuous level, the situation would turn out to be much less comfortable for the associated Galerkin scheme since it is well-known that the finite element subspaces yielding symmetry are much more expensive in terms of the number of degrees of freedom involved. We provide further details on this issue later on in Section 6.2.

According to the previous analysis, the weak formulation of the convective Brinkman–Forchheimer problem (2.13) reduces at first instance to: Find $(\boldsymbol{\sigma}, \boldsymbol{\gamma}, \mathbf{u}) \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathbf{L}^4(\Omega)$ such that (3.3), (3.4) and (3.5) hold for all $(\boldsymbol{\tau}, \boldsymbol{\delta}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega) \times \mathbf{L}^4(\Omega)$. However, similarly as in [10] (see also [14], [20]), we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{3.6}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \Big\}\,,$$

which means that each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ can be uniquely decomposed as

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 + d_0\,\mathbb{I} \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad d_0 := \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,\in\, \mathrm{R}\,.$$

In particular, using the last equation of (2.13), we obtain

$$\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 + c_0\,\mathbb{I} \quad \text{with} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 := -\frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,, \tag{3.7}$$

which says that c_0 is know explicitly in terms of $\mathbf{u}$. Therefore, in order to fully determine $\boldsymbol{\sigma}$, it only remains to find its $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-component $\boldsymbol{\sigma}_0$, which is renamed from now on simply as $\boldsymbol{\sigma}$.

Next, using the compatibility condition (2.5), we observe that both sides of (3.3) vanish when $\boldsymbol{\tau} = \mathbb{I}$, and hence testing against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Therefore, bearing in mind the foregoing discussion, denoting

$$\mathbf{H} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \quad \mathbf{Q} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{skew}}(\Omega)\,,$$

and setting

$$\vec{\mathbf{u}} = (\mathbf{u}, \boldsymbol{\gamma})\,, \quad \vec{\mathbf{v}} = (\mathbf{v}, \boldsymbol{\delta})\,, \quad \vec{\mathbf{z}} = (\mathbf{z}, \boldsymbol{\chi}) \,\in\, \mathbf{Q}\,,$$

we arrive at the following mixed formulation for the convective Brinkman–Forchheimer equations: Find $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$
\begin{aligned}
a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\vec{\mathbf{u}}) + \mathbf{b}(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) \quad &= \quad \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}\,, \\
b(\boldsymbol{\sigma},\vec{\mathbf{v}}) - c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) \quad &= \quad \mathbf{F}_{\boldsymbol{\phi}}(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{Q}\,,
\end{aligned}
\tag{3.8}
$$

where the bilinear forms $a : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ and $b : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$ are defined as

$$
a(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \int_\Omega \frac{1}{\mu}\,\boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \,\in\, \mathbf{H} \times \mathbf{H}\,,
\tag{3.9}
$$

and

$$
b(\boldsymbol{\tau},\vec{\mathbf{v}}) \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}}) \,\in\, \mathbf{H} \times \mathbf{Q}\,,
\tag{3.10}
$$

whereas, for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, the bilinear forms $\mathbf{b}(\mathbf{w};\cdot,\cdot) : \mathbf{L}^4(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \to \mathrm{R}$ and $c(\mathbf{w};\cdot,\cdot) : \mathbf{Q} \times \mathbf{Q} \to \mathrm{R}$ are given by

$$
\mathbf{b}(\mathbf{w};\mathbf{v},\boldsymbol{\tau}) \,:=\, \int_\Omega \frac{1}{\mu}\,(\mathbf{w}\otimes\mathbf{v}) : \boldsymbol{\tau} \qquad \forall\,(\mathbf{v},\boldsymbol{\tau}) \,\in\, \mathbf{L}^4(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,
\tag{3.11}
$$

and

$$
c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,:=\, \mathtt{D}\int_\Omega \mathbf{u}\cdot\mathbf{v} + \mathtt{F}\int_\Omega |\mathbf{w}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v} \qquad \forall\,(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,\in\, \mathbf{Q}\times\mathbf{Q}\,.
\tag{3.12}
$$

Finally, the linear and bounded functionals $\mathbf{G} : \mathbf{H} \to \mathrm{R}$ and $\mathbf{F}_{\boldsymbol{\phi}} : \mathbf{Q} \to \mathrm{R}$ reduce to

$$
\mathbf{G}(\boldsymbol{\tau}) \,:=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\,\boldsymbol{\tau} \,\in\, \mathbf{H} \quad \text{and} \quad \mathbf{F}_{\boldsymbol{\phi}}(\vec{\mathbf{v}}) \,:=\, \int_\Omega \mathbf{f}(\boldsymbol{\phi})\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{v}} \,\in\, \mathbf{Q}\,.
\tag{3.13}
$$

On the other hand, for the double diffusion equations, which are described by the third up to the fifth rows of (2.13), we proceed similarly as for the convective Brinkman-Forchheimer equations, and look originally for $\phi_j \in \mathrm{H}^1(\Omega)$, which, besides yielding $\mathbf{t}_j \in \mathbf{L}^2(\Omega)$, is in line as well with the fact that the data $\phi_{i,\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$, $i \in \{1,2\}$. Then, testing the aforementioned third equation against $\boldsymbol{\eta}_j \in \mathbf{H}(\mathrm{div}_t;\Omega)$, with t as before, applying now (1.1), and using the Dirichlet boundary condition on ϕ_j, we get

$$
\int_\Omega \mathbf{t}_j\cdot\boldsymbol{\eta}_j + \int_\Omega \phi_j\,\mathrm{div}(\boldsymbol{\eta}_j) \,=\, \big\langle \boldsymbol{\eta}_j\cdot\boldsymbol{\nu}, \phi_{j,D} \big\rangle \quad j \,\in\, \{1,2\}\,.
\tag{3.14}
$$

In this way, knowing that $\mathrm{div}(\boldsymbol{\eta}_j) \in \mathrm{L}^t(\Omega)$, we realize from the second term of (3.14) and Hölder's inequality that it suffices to look for ϕ_j in $\mathrm{L}^{t'}(\Omega)$. Needless to say, it is clear that the first term makes sense since both $\mathbf{t}_j$ and $\boldsymbol{\eta}_j$ belong to $\mathbf{L}^2(\Omega)$. Next, letting $\mathbf{L}^2(\Omega)$ be as well the space of test functions associated with the unknown $\mathbf{t}_j$, the corresponding testing of the fourth row of (2.13) formally gives

$$
\int_\Omega \mathbf{Q}_j\,\mathbf{t}_j\cdot\mathbf{r}_j \,-\, \frac{1}{2}\mathtt{R}_j\int_\Omega \phi_j\,(\mathbf{u}\cdot\mathbf{r}_j) - \int_\Omega \boldsymbol{\vartheta}_j\cdot\mathbf{r}_j = 0
\tag{3.15}
$$

for all $\mathbf{r}_j \in \mathbf{L}^2(\Omega)$, so that the third term of (3.15) is well-defined if $\boldsymbol{\vartheta}_j \in \mathbf{L}^2(\Omega)$. In turn, regarding the second term, and bearing in mind that from the analysis of the Brinkman–Forchheimer equations we know that $\mathbf{u}$ must be sought in $\mathbf{L}^4(\Omega)$, direct applications of the Cauchy–Schwarz and Hölder inequalities imply

$$
\Big|\int_\Omega \phi_j\,(\mathbf{u}\cdot\mathbf{r}_j)\Big| \,\leqslant\, \|\phi_j\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{r}_j\|_{0,\Omega}\,,
\tag{3.16}
$$

from which it is natural to fix the seeking space for ϕ_j as $\mathrm{L}^4(\Omega)$, that is $t' = 4$, which yields $t = 4/3$. In this way, letting $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\mathrm{L}^4(\Omega)$ be as well the spaces where $\boldsymbol{\vartheta}_j$ is sought and where the test functions associated with ϕ_j belong to, respectively, we can test the fifth row of (2.13) against $\psi_j \in \mathrm{L}^4(\Omega)$ to obtain

$$\frac{1}{2}\mathtt{R}_j \int_\Omega \psi_j\,(\mathbf{u}\cdot\mathbf{t}_j) - \int_\Omega \psi_j\,\mathrm{div}(\boldsymbol{\vartheta}_j) \;=\; \int_\Omega g_j\,\psi_j\,. \tag{3.17}$$

Note that the first and second terms of (3.17) are well-defined thanks to the analogue estimate (3.16) and the fact that $\mathrm{div}(\boldsymbol{\vartheta}_j) \in \mathrm{L}^{4/3}(\Omega)$, whereas the expression on the right-hand side makes sense if $g_j \in \mathrm{L}^{4/3}(\Omega)$, which we assume from now on. Therefore, introducing the spaces

$$\widetilde{\mathbf{H}} \;:=\; \mathrm{L}^4(\Omega)\times\mathbf{L}^2(\Omega) \quad\text{and}\quad \widetilde{\mathbf{Q}} \;:=\; \mathbf{H}(\mathrm{div}_{4/3};\Omega)\,,$$

setting the variables

$$\vec{\phi}_j \;=\; (\phi_j,\mathbf{t}_j)\,,\quad \vec{\psi}_j \;=\; (\psi_j,\mathbf{r}_j)\,,\quad \vec{\xi}_j \;=\; (\xi_j,\mathbf{s}_j) \;\in\; \widetilde{\mathbf{H}}\,,$$

and grouping conveniently (3.14), (3.15), and (3.17), we arrive at the weak formulation: Find $(\vec{\phi}_j,\boldsymbol{\vartheta}_j) \in \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}$, $j\in\{1,2\}$, such that

$$\begin{array}{rcll} \widetilde{a}_j(\vec{\phi}_j,\vec{\psi}_j) \,+\, \widetilde{c}_j(\mathbf{u};\vec{\phi}_j,\vec{\psi}_j) \,+\, \widetilde{b}(\vec{\psi}_j,\boldsymbol{\vartheta}_j) & = & \widetilde{F}_j(\vec{\psi}_j) & \forall\,\vec{\psi}_j \,\in\, \widetilde{\mathbf{H}}\,,\\ \widetilde{b}(\vec{\phi}_j,\boldsymbol{\eta}_j) & = & \widetilde{G}_j(\boldsymbol{\eta}_j) & \forall\,\boldsymbol{\eta}_j \,\in\, \widetilde{\mathbf{Q}}\,, \end{array} \tag{3.18}$$

where, for $j\in\{1,2\}$, the bilinear forms $\widetilde{a}_j : \widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}} \to \mathrm{R}$, $\widetilde{b} : \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}} \to \mathrm{R}$, and the linear and bounded functionals $\widetilde{F}_j : \widetilde{\mathbf{H}} \to \mathrm{R}$ and $\widetilde{G}_j : \widetilde{\mathbf{Q}} \to \mathrm{R}$ are defined, respectively, as:

$$\widetilde{a}_j(\vec{\xi}_j,\vec{\psi}_j) \;:=\; \int_\Omega \mathbf{Q}_j\,\mathbf{s}_j\cdot\mathbf{r}_j \qquad \forall\,(\vec{\xi}_j,\vec{\psi}_j) \,\in\, \widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}}\,, \tag{3.19}$$

$$\widetilde{b}(\vec{\xi}_j,\boldsymbol{\eta}_j) \;:=\; -\int_\Omega \mathbf{s}_j\cdot\boldsymbol{\eta}_j - \int_\Omega \xi_j\,\mathrm{div}(\boldsymbol{\eta}_j) \qquad \forall\,(\vec{\xi}_j,\boldsymbol{\eta}_j) \,\in\, \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}\,, \tag{3.20}$$

$$\widetilde{F}_j(\vec{\psi}_j) \;:=\; \int_\Omega g_j\,\psi_j \qquad \forall\,\vec{\psi}_j \in \widetilde{\mathbf{H}}\,,\quad\text{and} \tag{3.21}$$

$$\widetilde{G}_j(\boldsymbol{\eta}_j) \;:=\; -\big\langle \boldsymbol{\eta}_j\cdot\boldsymbol{\nu}, \phi_{j,D}\big\rangle \qquad \forall\,\boldsymbol{\eta}_j \,\in\, \widetilde{\mathbf{Q}}\,, \tag{3.22}$$

whereas, given $\mathbf{w}\in\mathbf{L}^4(\Omega)$, the bilinear form $\widetilde{c}_j(\mathbf{w};\cdot,\cdot) : \widetilde{\mathbf{H}}\times\widetilde{\mathbf{H}} \to \mathrm{R}$ is given by

$$\widetilde{c}_j(\mathbf{w};\vec{\xi}_j,\vec{\psi}_j) \;:=\; \frac{1}{2}\mathtt{R}_j\Big\{\int_\Omega \psi_j\,(\mathbf{w}\cdot\mathbf{s}_j) - \int_\Omega \xi_j\,(\mathbf{w}\cdot\mathbf{r}_j)\Big\} \qquad \forall\,\vec{\xi}_j\,,\,\vec{\psi}_j \,\in\, \widetilde{\mathbf{H}}\,. \tag{3.23}$$

Summarizing, the fully mixed formulation of the Brinkman–Forchheimer equations coupled with double diffusion equations (cf. (2.13)) reads: Find $(\boldsymbol{\sigma},\vec{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\phi}_j,\boldsymbol{\vartheta}_j) \in \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}$, $j\in\{1,2\}$, such that

$$\begin{array}{rcll} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\vec{\mathbf{u}}) + \mathbf{b}(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) & = & \mathbf{G}(\boldsymbol{\tau}) & \forall\,\boldsymbol{\tau} \,\in\, \mathbf{H}\,,\\ b(\boldsymbol{\sigma},\vec{\mathbf{v}}) - c(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) & = & \mathbf{F}_{\boldsymbol{\phi}}(\vec{\mathbf{v}}) & \forall\,\vec{\mathbf{v}} \,\in\, \mathbf{Q}\,,\\ \widetilde{a}_j(\vec{\phi}_j,\vec{\psi}_j) \,+\, \widetilde{c}_j(\mathbf{u};\vec{\phi}_j,\vec{\psi}_j) \,+\, \widetilde{b}(\vec{\psi}_j,\boldsymbol{\vartheta}_j) & = & \widetilde{F}_j(\vec{\psi}_j) & \forall\,\vec{\psi}_j \,\in\, \widetilde{\mathbf{H}}\,,\\ \widetilde{b}(\vec{\phi}_j,\boldsymbol{\eta}_j) & = & \widetilde{G}_j(\boldsymbol{\eta}_j) & \forall\,\boldsymbol{\eta}_j \,\in\, \widetilde{\mathbf{Q}}\,. \end{array} \tag{3.24}$$

4 Analysis of the coupled problem

In this section we combine classical and new results on the solvability of variational formulations in Banach spaces to establish the well-posedness of (3.24).

4.1 Preliminaries

The stability properties of the operators and functionals involved in (3.24) are provided first. In fact, direct applications of the Cauchy-Schwarz and Hölder inequalities, along with the upper bounds of μ (cf. (2.2)), the continuity of the normal trace operator in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and the continuity of the injection $i_4 : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ and its vectorial version $\boldsymbol{i}_4$, yield the existence of positive constants, denoted and given as:

$$
\begin{gathered}
\|a\| := \frac{1}{\mu_0}, \quad \|b\| := 1, \quad \|\widetilde{a}_j\| := \|\mathbf{Q}_j\|_{0,\infty;\Omega}, \quad \|\widetilde{b}\| := 2, \\
\|\mathbf{G}\| := \|\mathbf{u}_D\|_{1/2,\Gamma}\,\|\boldsymbol{i}_4\|, \quad \|\mathbf{F}_{\boldsymbol{\varphi}}\| := \|\mathbf{g}\|_{0,\Omega}\Big\{\|\boldsymbol{\varphi}\|_{0,4;\Omega} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega}\Big\}, \\
\|\widetilde{F}_j\| := \|g_j\|_{0,4/3;\Omega}, \quad \text{and} \quad \|\widetilde{G}_j\| := \|\phi_{j,D}\|_{1/2,\Gamma}\,\|i_4\|,
\end{gathered}
\tag{4.1}
$$

where $j \in \{1,2\}$ and $\boldsymbol{\varphi} = (\varphi_1,\varphi_2) \in \mathrm{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, such that there hold

$$
\begin{array}{rcll}
|a(\boldsymbol{\zeta},\boldsymbol{\tau})| & \leqslant & \|a\|\,\|\boldsymbol{\zeta}\|_{\mathbf{H}}\,\|\boldsymbol{\tau}\|_{\mathbf{H}} & \forall\,\boldsymbol{\zeta},\boldsymbol{\tau} \in \mathbf{H}, \\
|b(\boldsymbol{\tau},\vec{\mathbf{v}})| & \leqslant & \|b\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} & \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}, \\
|\widetilde{a}_j(\vec{\phi}_j,\vec{\psi}_j)| & \leqslant & \|\widetilde{a}_j\|\,\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}\,\|\vec{\psi}_j\|_{\widetilde{\mathbf{H}}} & \forall\,\vec{\phi}_j,\,\vec{\psi}_j \in \widetilde{\mathbf{H}}, \\
|\widetilde{b}(\vec{\psi}_j,\boldsymbol{\eta}_j)| & \leqslant & \|\widetilde{b}\|\,\|\vec{\psi}_j\|_{\widetilde{\mathbf{H}}}\,\|\boldsymbol{\eta}_j\|_{\widetilde{\mathbf{Q}}} & \forall\,(\vec{\psi}_j,\boldsymbol{\eta}_j) \in \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}, \\
|\mathbf{G}(\boldsymbol{\tau})| & \leqslant & \|\mathbf{G}\|\,\|\boldsymbol{\tau}\|_{\mathbf{H}} & \forall\,\boldsymbol{\tau} \in \mathbf{H}, \\
|\mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}})| & \leqslant & \|\mathbf{F}_{\boldsymbol{\varphi}}\|\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} & \forall\,\vec{\mathbf{v}} \in \mathbf{Q}, \\
|\widetilde{F}_j(\vec{\psi}_j)| & \leqslant & \|\widetilde{F}_j\|\,\|\vec{\psi}_j\|_{\widetilde{\mathbf{H}}} & \forall\,\vec{\psi}_j \in \widetilde{\mathbf{H}}, \quad \text{and} \\
|\widetilde{G}_j(\boldsymbol{\eta}_j)| & \leqslant & \|\widetilde{G}_j\|\,\|\boldsymbol{\eta}_j\|_{\widetilde{\mathbf{Q}}} & \forall\,\boldsymbol{\eta}_j \in \widetilde{\mathbf{Q}}.
\end{array}
\tag{4.2}
$$

In turn, given $\mathbf{w} \in \mathbf{L}^4(\Omega)$, we apply the Cauchy-Schwarz and Hölder inequalities, similarly as we did in (4.1) - (4.2), and previously in (3.16), to derive the following bounds for $\mathbf{b}$ (cf. (3.11)), c (cf. (3.12)), and $\widetilde{c}_j$ (cf. (3.23))

$$
\begin{array}{rcll}
|\mathbf{b}(\mathbf{w};\mathbf{v},\boldsymbol{\tau})| & \leqslant & \dfrac{1}{\mu_0}\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} & \forall\,(\mathbf{v},\boldsymbol{\tau}) \in \mathbf{L}^4(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \\
|c(\mathbf{w};\vec{\mathbf{v}},\vec{\mathbf{z}})| & \leqslant & |\Omega|^{1/2}\left(\mathtt{D} + \mathtt{F}\,\|\mathbf{w}\|^{\rho-2}_{0,4;\Omega}\right)\|\vec{\mathbf{v}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{z}}\|_{\mathbf{Q}} & \forall\,\vec{\mathbf{v}},\vec{\mathbf{z}} \in \mathbf{Q}, \quad \text{and} \\
|\widetilde{c}_j(\mathbf{w};\vec{\phi}_j,\vec{\psi}_j)| & \leqslant & \mathtt{R}_j\,\|\mathbf{w}\|_{0,4;\Omega}\,\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}\,\|\vec{\psi}_j\|_{\widetilde{\mathbf{H}}} & \forall\,\vec{\phi}_j,\,\vec{\psi}_j \in \widetilde{\mathbf{H}}.
\end{array}
\tag{4.3}
$$

Moreover, noting from the definition of $\widetilde{c}_j$ (cf. (3.23)) that $\widetilde{c}_j(\cdot;\vec{\phi}_j,\vec{\psi}_j)$ is linear, we readily deduce from the third row of (4.3) that

$$
|\widetilde{c}_j(\mathbf{w};\vec{\phi}_j,\vec{\psi}_j) - \widetilde{c}_j(\mathbf{z};\vec{\phi}_j,\vec{\psi}_j)| \leqslant \mathtt{R}_j\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}\,\|\vec{\psi}_j\|_{\widetilde{\mathbf{H}}}
\tag{4.4}
$$

for all $\mathbf{w},\mathbf{z} \in \mathbf{L}^4(\Omega)$ and for all $\vec{\phi}_j,\vec{\psi}_j \in \widetilde{\mathbf{H}}$, and it is also clear from (3.23) that there holds

$$
\widetilde{c}_j(\mathbf{w};\vec{\psi}_j,\vec{\psi}_j) = 0
\tag{4.5}
$$

for all $\mathbf{w} \in \mathbf{L}^4(\Omega)$ and for all $\vec{\psi}_j \in \widetilde{\mathbf{H}}$.

4.2 A fixed point strategy

In what follows, we proceed similarly to [15] (see also [14]) and adopt a fixed-point strategy to address the well–posedness of (3.24). We begin by letting $\mathbf{S} : \mathbf{L}^4(\Omega) \times (\mathrm{L}^4(\Omega) \times \mathrm{L}^4(\Omega)) \to \mathbf{L}^4(\Omega)$ be the operator defined as

$$\mathbf{S}(\mathbf{w}, \boldsymbol{\varphi}) := \mathbf{u} \qquad \forall\, (\mathbf{w}, \boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega) \times (\mathrm{L}^4(\Omega) \times \mathrm{L}^4(\Omega)), \tag{4.6}$$

where $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) := (\boldsymbol{\sigma}, (\mathbf{u}, \gamma)) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of the problem arising from the first two rows of (3.24) after replacing $\mathbf{b}(\mathbf{u}; \cdot, \cdot)$, $c(\mathbf{u}; \cdot, \cdot)$, and $\mathbf{F}_{\boldsymbol{\phi}}$ by $\mathbf{b}(\mathbf{w}; \cdot, \cdot)$, $c(\mathbf{w}; \cdot, \cdot)$, and $\mathbf{F}_{\boldsymbol{\varphi}}$, respectively, that is

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\mathbf{u}}) + \mathbf{b}(\mathbf{w}; \mathbf{u}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}, \\ b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) &= \mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{Q}. \end{aligned} \tag{4.7}$$

Equivalently, introducing the bilinear form $\mathbf{A}_{\mathbf{w}} : (\mathbf{H} \times \mathbf{Q}) \times (\mathbf{H} \times \mathbf{Q}) \to \mathrm{R}$ given by

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) := a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \vec{\mathbf{u}}) + b(\boldsymbol{\sigma}, \vec{\mathbf{v}}) - c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \tag{4.8}$$

for all $(\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$, the uncoupled problem (4.7) can be rewritten as

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}, \vec{\mathbf{u}}), (\boldsymbol{\tau}, \vec{\mathbf{v}})) + \mathbf{b}(\mathbf{w}; \mathbf{u}, \boldsymbol{\tau}) = \mathbf{G}(\boldsymbol{\tau}) + \mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \qquad \forall\, (\boldsymbol{\tau}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}. \tag{4.9}$$

In turn, for each $j \in \{1, 2\}$ we define the operator $\widetilde{\mathbf{S}}_j : \mathbf{L}^4(\Omega) \to \mathrm{L}^4(\Omega)$ given by

$$\widetilde{\mathbf{S}}_j(\mathbf{w}) := \phi_j \qquad \forall\, \mathbf{w} \in \mathbf{L}^4(\Omega),$$

where $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := ((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j) \in \widetilde{\mathbf{H}} \times \widetilde{\mathbf{Q}}$ is the unique solution (to be confirmed below) of the problem arising from the third and fourth rows of (3.24) after replacing $\widetilde{c}_j(\mathbf{u}; \vec{\phi}_j, \vec{\psi}_j)$ by $\widetilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j)$, that is

$$\begin{aligned} \widetilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \widetilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j) + \widetilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \widetilde{F}_j(\vec{\psi}_j) \qquad \forall\, \vec{\psi}_j \in \widetilde{\mathbf{H}}, \\ \widetilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \widetilde{G}_j(\boldsymbol{\eta}_j) \qquad \forall\, \boldsymbol{\eta}_j \in \widetilde{\mathbf{Q}}. \end{aligned} \tag{4.10}$$

Similarly as for (4.9), for each $j \in \{1, 2\}$ we define the bilinear form $\widetilde{a}_{j,\mathbf{w}} : \widetilde{\mathbf{H}} \times \widetilde{\mathbf{H}} \to \mathrm{R}$ as

$$\widetilde{a}_{j,\mathbf{w}}(\vec{\phi}_j, \vec{\psi}_j) := \widetilde{a}_j(\vec{\phi}_j, \vec{\psi}_j) + \widetilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\psi}_j) \qquad \forall\, \vec{\phi}_j,\, \vec{\psi}_j \in \widetilde{\mathbf{H}}, \tag{4.11}$$

which allows us to restate (4.10) as

$$\begin{aligned} \widetilde{a}_{j,\mathbf{w}}(\vec{\phi}_j, \vec{\psi}_j) + \widetilde{b}(\vec{\psi}_j, \boldsymbol{\vartheta}_j) &= \widetilde{F}_j(\vec{\psi}_j) \qquad \forall\, \vec{\psi}_j \in \widetilde{\mathbf{H}}, \\ \widetilde{b}(\vec{\phi}_j, \boldsymbol{\eta}_j) &= \widetilde{G}_j(\boldsymbol{\eta}_j) \qquad \forall\, \boldsymbol{\eta}_j \in \widetilde{\mathbf{Q}}, \end{aligned} \tag{4.12}$$

Hence, defining $\widetilde{\mathbf{S}} : \mathbf{L}^4(\Omega) \to \mathrm{L}^4(\Omega) \times \mathrm{L}^4(\Omega)$ as

$$\widetilde{\mathbf{S}}(\mathbf{w}) := \big(\widetilde{\mathbf{S}}_1(\mathbf{w}), \widetilde{\mathbf{S}}_2(\mathbf{w})\big) \qquad \forall\, \mathbf{w} \in \mathbf{L}^4(\Omega), \tag{4.13}$$

and letting $\mathbf{T} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ be the operator defined by

$$\mathbf{T}(\mathbf{w}) := \mathbf{S}\big(\mathbf{w}, \widetilde{\mathbf{S}}(\mathbf{w})\big) \qquad \forall\, \mathbf{w} \in \mathbf{L}^4(\Omega), \tag{4.14}$$

we see that solving (3.24) is equivalent to seeking a fixed-point of $\mathbf{T}$, that is $\mathbf{u} \in \mathbf{L}^4(\Omega)$ such that

$$\mathbf{T}(\mathbf{u}) = \mathbf{u}. \tag{4.15}$$

4.3 Well posedness of the uncoupled problems

In this section we utilize the Banach–Nečas–Babuška Theorem (cf. [23, Theorem 2.6]), along with recent solvability results for perturbed saddle-point problems in Banach spaces (cf. [21], [22]), and the Banach version of the Babuška-Brezzi theory (cf. [23, Theorem 2.34]), to show that the uncoupled problems (4.7) (or (4.9)) and (4.12) are well–posed, which means, equivalently, that the operators $\mathbf{S}$ (cf. (4.6)) and $\widetilde{\mathbf{S}}$ (cf. (4.13)) are well–defined. We begin by remarking that, being $\mathrm{L}^p(\Omega)$ reflexive for each $p \in (1,+\infty)$, all the spaces involved in the formulations (4.9) and (4.12), namely $\mathbf{L}^2(\Omega)$, $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\text{skew}}(\Omega)$, $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$, and $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, are easily shown to be reflexive as well.

In what follows we address the solvability of (4.7), for which we first show that the bilinear forms a (cf. (3.9)), b (cf. (3.10)), and $c(\mathbf{w};\cdot,\cdot)$ (cf. (3.12)), for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, which define the bilinear form $\mathbf{A}_{\mathbf{w}}$ (cf. (4.8)), satisfy the hypotheses of [21, Theorem 3.4]. In fact, it is clear from their respective definitions that a and $c(\mathbf{w};\cdot,\cdot)$ are symmetric and positive semi-definite, which confirms the hypothesis **i)** of [21, Theorem 3.4]. Now, letting $\mathbf{V}$ be the null space of the linear and bounded operator induced by b, we readily see (cf. (3.10)) that

$$\mathbf{V} \,=\, \Big\{ \boldsymbol{\zeta} \in \mathbf{H} : \quad \boldsymbol{\zeta} \,=\, \boldsymbol{\zeta}^{\mathrm{t}} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\zeta}) \,=\, 0 \Big\} . \tag{4.16}$$

In addition, it is already well-known that a slight modification of [25, Lemma 2.3] (see also [9, Proposition IV.3.1], [26, Lemma 3.3], and [10, Lemma 3.1]) allows to prove the existence of a positive constant c_1, depending on Ω and the norm of the continuous injection $i_4 : \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$, such that

$$c_1\,\|\boldsymbol{\zeta}\|_{0,\Omega} \,\leqslant\, \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,4/3;\Omega} \qquad \forall\, \boldsymbol{\zeta} \,\in\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) . \tag{4.17}$$

Thus, thanks to the boundedness of μ (cf. (2.2)) and the inequality (4.17), we deduce that

$$a(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\geqslant\, \frac{1}{\mu_1}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|^2_{0,\Omega} \,\geqslant\, \alpha\,\|\boldsymbol{\zeta}\|^2_{\mathbf{H}} \qquad \forall\, \boldsymbol{\zeta} \,\in\, \mathbf{V} , \tag{4.18}$$

with $\alpha := \frac{c_1^2}{\mu_1}$, which easily implies the verification of the continuous inf-sup condition for a required by the hypothesis **ii)** of [21, Theorem 3.4]. On the other hand, letting c_P be the positive constant yielding Poincaré's inequality, that is such that $\|\mathbf{v}\|^2_{1,\Omega} \leqslant c_P\,|\mathbf{v}|^2_{1,\Omega} \quad \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega)$, and recalling that $\boldsymbol{i}_4$ is the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, it can be proved (cf. [29, Lemma 3.5]) that there exists a positive constant β, depending only on c_P and $\|\boldsymbol{i}_4\|$, such that

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbf{V}\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau},\vec{\mathbf{v}})}{\|\boldsymbol{\tau}\|_{\mathbf{H}}} \,\geqslant\, \beta\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \qquad \forall\, \vec{\mathbf{v}} \,\in\, \mathbf{Q} ,$$

which accomplishes the hypothesis **iii)** of [21, Theorem 3.4]. Furthermore, letting $\delta > 0$ be an arbitrary radius, we introduce the ball

$$\mathbf{W}(\delta) \,:=\, \Big\{ \mathbf{w} \,\in\, \mathbf{L}^4(\Omega) : \quad \|\mathbf{w}\|_{0,4;\Omega} \,\leqslant\, \delta \Big\} , \tag{4.19}$$

so that for each $\mathbf{w} \in \mathbf{W}(\delta)$ the boundedness estimate for $c(\mathbf{w};\cdot,\cdot)$ becomes (cf. (4.3))

$$|c(\mathbf{w};\vec{\mathbf{v}},\vec{\mathbf{z}})| \,\leqslant\, |\Omega|^{1/2}\left(\mathtt{D} \,+\, \mathtt{F}\,\delta^{\rho-2}\right)\|\vec{\mathbf{v}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{z}}\|_{\mathbf{Q}} \quad \forall\, \vec{\mathbf{v}}\,,\vec{\mathbf{z}} \,\in\, \mathbf{Q} . \tag{4.20}$$

Hence, bearing also in mind the expression for $\|a\|$ (cf. (4.1)), a straightforward application of [21, Theorem 3.4] ensures the existence of a positive constant $\alpha_{\mathbf{A}}$, depending only on μ_0, $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, δ, ρ, α, and β, such that for each $\mathbf{w} \in \mathbf{W}(\delta)$ there holds

$$\sup_{\substack{(\boldsymbol{\zeta},\vec{\mathbf{z}})\in\mathbf{H}\times\mathbf{Q}\\ (\boldsymbol{\zeta},\vec{\mathbf{z}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\vec{\mathbf{v}}),(\boldsymbol{\zeta},\vec{\mathbf{z}}))}{\|(\boldsymbol{\zeta},\vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}}} \,\geqslant\, \alpha_{\mathbf{A}}\,\|(\boldsymbol{\tau},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}} \qquad \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}}) \,\in\, \mathbf{H}\times\mathbf{Q} . \tag{4.21}$$

Then, combining (4.21) with the boundedness estimate for $\mathbf{b}(\mathbf{w};\cdot,\cdot)$ (cf. (4.3)), we arrive at

$$\sup_{\substack{(\boldsymbol{\zeta},\vec{\mathbf{z}})\in\mathbf{H}\times\mathbf{Q}\\(\boldsymbol{\zeta},\vec{\mathbf{z}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\vec{\mathbf{v}}),(\boldsymbol{\zeta},\vec{\mathbf{z}})) \,+\, \mathbf{b}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})}{\|(\boldsymbol{\zeta},\vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}}} \,\geqslant\, \left\{\alpha_{\mathbf{A}} - \frac{1}{\mu_0}\,\|\mathbf{w}\|_{0,4;\Omega}\right\} \|(\boldsymbol{\tau},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}$$

for all $(\boldsymbol{\tau},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}$, from which, under the additional assumption that $\|\mathbf{w}\|_{0,4;\Omega} \leqslant \frac{\mu_0\,\alpha_{\mathbf{A}}}{2}$, we conclude that

$$\sup_{\substack{(\boldsymbol{\zeta},\vec{\mathbf{z}})\in\mathbf{H}\times\mathbf{Q}\\(\boldsymbol{\zeta},\vec{\mathbf{z}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\vec{\mathbf{v}}),(\boldsymbol{\zeta},\vec{\mathbf{z}})) \,+\, \mathbf{b}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})}{\|(\boldsymbol{\zeta},\vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}}} \,\geqslant\, \frac{\alpha_{\mathbf{A}}}{2}\,\|(\boldsymbol{\tau},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}} \tag{4.22}$$

for all $(\boldsymbol{\tau},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}$. Similarly, noting that $\mathbf{A}_{\mathbf{w}}$ is symmetric (since a and c are), using again the boundedness estimate for $\mathbf{b}(\mathbf{w};\cdot,\cdot)$ (cf. (4.3)), and under the same assumption on $\mathbf{w}$, we obtain

$$\sup_{\substack{(\boldsymbol{\tau},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\(\boldsymbol{\tau},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\tau},\vec{\mathbf{v}}),(\boldsymbol{\zeta},\vec{\mathbf{z}})) \,+\, \mathbf{b}(\mathbf{w};\mathbf{v},\boldsymbol{\zeta})}{\|(\boldsymbol{\tau},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}} \,\geqslant\, \frac{\alpha_{\mathbf{A}}}{2}\,\|(\boldsymbol{\zeta},\vec{\mathbf{z}})\|_{\mathbf{H}\times\mathbf{Q}} \tag{4.23}$$

for all $(\boldsymbol{\zeta},\vec{\mathbf{z}}) \in \mathbf{H}\times\mathbf{Q}$.

We are now in position of establishing next the well–posedness of (4.9), thanks to which the operator $\mathbf{S}$ is well-defined.

Theorem 4.1 *Given $\delta > 0$, let $r \in (0, r_0]$, with*

$$r_0 \,:=\, \min\left\{\delta, \frac{\mu_0\,\alpha_{\mathbf{A}}}{2}\right\}. \tag{4.24}$$

Then, for each $(\mathbf{w},\boldsymbol{\varphi}) \in \mathbf{L}^4(\Omega)\times\big(\mathrm{L}^4(\Omega)\times\mathrm{L}^4(\Omega)\big)$ such that $\|\mathbf{w}\|_{0,4;\Omega} \leqslant r$, (4.9) (equivalently, (4.7)) has a unique solution $(\boldsymbol{\sigma},\vec{\mathbf{u}}) := \big(\boldsymbol{\sigma},(\mathbf{u},\gamma)\big) \in \mathbf{H}\times\mathbf{Q}$, and hence one can define $\mathbf{S}(\mathbf{w},\boldsymbol{\varphi}) := \mathbf{u}$. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on $\|\boldsymbol{i}_4\|$, $\|\mathbf{g}\|_{0,\Omega}$, and $\alpha_{\mathbf{A}}$, such that

$$\|\mathbf{S}(\mathbf{w},\boldsymbol{\varphi})\|_{0,4;\Omega} \,=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \|(\boldsymbol{\sigma},\vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, C_{\mathbf{S}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \|\boldsymbol{\varphi}\|_{0,4;\Omega}\right\}. \tag{4.25}$$

Proof. It is clear from (4.22) and (4.23) that the bilinear form $\mathbf{A}_{\mathbf{w}} + \mathbf{b}(\mathbf{w};\cdot,\cdot)$ satisfies the assumptions of the Banach–Nečas–Babuška Theorem (cf. [23, Theorem 2.6]), and hence, knowing from (4.1) and (4.2) that $\mathbf{G} \in \mathbf{H}'$ and $\mathbf{F}_{\boldsymbol{\varphi}} \in \mathbf{Q}'$, the proof reduces to a straightforward application of that theorem. In particular, the a priori estimate (4.25) follows from [23, Theorem 2.6, eq. (2.5)] and the upper bounds for $\|\mathbf{G}\|_{\mathbf{H}'}$ and $\|\mathbf{F}_{\boldsymbol{\varphi}}\|_{\mathbf{Q}'}$ (cf. (4.1)). □

On the other hand, in order to derive the well–posedness of (4.12), equivalently (4.10), we aim to prove that the bilinear forms $\widetilde{a}_{j,\mathbf{w}}$ (cf. (4.11)) and $\widetilde{b}$ (cf. (3.20)) satisfy the hypotheses of [23, Theorem 2.34]. In this way, letting $\widetilde{\mathbf{V}}$ be the null space of the linear and bounded operator induced by the bilinear form $\widetilde{b}$, we first observe that (cf. [15, eq. (3.35)])

$$\widetilde{\mathbf{V}} \,=\, \left\{\vec{\psi} \,:=\, (\psi,\mathbf{r}) \,\in\, \widetilde{\mathbf{H}}: \quad \psi \,\in\, \mathrm{H}_0^1(\Omega) \quad \text{and} \quad \mathbf{r} \,=\, \nabla\,\psi\right\}.$$

Next, according to the definition of $\widetilde{a}_{j,\mathbf{w}}$ (cf. (4.11)), with a given $\mathbf{w} \in \mathbf{L}^4(\Omega)$, and employing (4.5) and (2.4), we obtain, similarly as in the proof of [15, Lemma 3.2], that for each $\vec{\psi}_j := (\psi_j,\mathbf{r}_j) \in \widetilde{\mathbf{V}}$ there holds

$$\widetilde{a}_{j,\mathbf{w}}(\vec{\psi}_j,\vec{\psi}_j) \,=\, \widetilde{a}_j(\vec{\psi}_j,\vec{\psi}_j) \,=\, \int_\Omega \mathbf{Q}_j\,|\mathbf{r}_j|^2 \,\geqslant\, \widetilde{\alpha}_j\,\|\vec{\psi}_j\|^2_{\widetilde{\mathbf{H}}}, \tag{4.26}$$

where $\widetilde{\alpha}_j$ is a positive constant depending only on C_j (cf. (2.4)), $\|i_4\|$, and c_P. Then, it is easily seen that (4.26) implies the hypotheses on $\widetilde{a}_{j,\mathbf{w}}$ required in [23, Theorem 2.34, eq. (2.28)]. Furthermore, we recall from [20, Lemma 3.3] that $\widetilde{b}$ satisfies the continuous inf-sup condition required in [23, Theorem 2.34, eq. (2.29)], that is, there exists a positive constant $\widetilde{\beta}$, depending only on $|\Omega|$, such that

$$\sup_{\substack{\vec{\psi}\in\widetilde{\mathbf{H}}\\ \vec{\psi}\neq\mathbf{0}}} \frac{\widetilde{b}(\vec{\psi},\boldsymbol{\eta})}{\|\vec{\psi}\|_{\widetilde{\mathbf{H}}}} \geqslant \widetilde{\beta}\, \|\boldsymbol{\eta}\|_{\widetilde{\mathbf{Q}}} \qquad \forall\, \boldsymbol{\eta} \in \widetilde{\mathbf{Q}}\,.$$

Consequently, the well-posedness of (4.12), and thus the well-definedness of the operator $\widetilde{\mathbf{S}}$ (cf. (4.13)), is stated as follows.

Theorem 4.2 *For each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, and for each $j \in \{1,2\}$, there exists a unique $(\vec{\phi}_j,\boldsymbol{\vartheta}_j) := \big((\phi_j,\mathbf{t}_j),\boldsymbol{\vartheta}_j\big) \in \widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}$ solution to (4.12) (equivalently, (4.10)), and hence one can define $\widetilde{\mathbf{S}}_j(\mathbf{w}) := \phi_j$. Moreover, there exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending only on $\widetilde{\alpha}_j$, $\widetilde{\beta}$, $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|i_4\|$, and $\mathtt{R}_j$, $j \in \big\{1,2\big\}$, such that*

$$\begin{aligned} \|\widetilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega} &:= \|\big(\widetilde{\mathbf{S}}_1(\mathbf{w}),\widetilde{\mathbf{S}}_2(\mathbf{w})\big)\|_{0,4;\Omega} \\ &\leqslant C_{\widetilde{\mathbf{S}}} \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{w}\|_{0,4;\Omega}\big)\, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\}\,. \end{aligned} \tag{4.27}$$

Proof. Bearing in mind the previous discussion on $\widetilde{a}_{j,\mathbf{w}}$ and $\widetilde{b}$, for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$ and for each $j \in \{1,2\}$, and recalling from (4.1) and (4.2) that $\widetilde{F}_j \in \widetilde{\mathbf{H}}'$ and $\widetilde{G}_j \in \widetilde{\mathbf{Q}}'$, the proof follows from a direct application of [23, Theorem 2.34]. In this way, the corresponding a priori estimate (cf. [23, Theorem 2.34, eq. (2.30)]) yields

$$\|\widetilde{\mathbf{S}}_j(\mathbf{w})\|_{0,4;\Omega} \,=\, \|\phi_j\|_{0,4;\Omega} \,\leqslant\, \|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}} \,\leqslant\, \frac{1}{\widetilde{\alpha}_j}\,\|\widetilde{F}_j\| \,+\, \frac{1}{\widetilde{\beta}}\left(1 \,+\, \frac{\|\widetilde{a}_{j,\mathbf{w}}\|}{\widetilde{\alpha}_j}\right)\|\widetilde{G}_j\|\,,$$

so that, noting from (4.1) and (4.3) that $\|\widetilde{a}_{j,\mathbf{w}}\| \leqslant \|\mathbf{Q}_j\|_{0,\infty;\Omega} + \mathtt{R}_j\,\|\mathbf{w}\|_{0,4;\Omega}$, and employing the expressions for $\|\widetilde{F}_j\|$ and $\|\widetilde{G}_j\|$ provided in (4.1), the foregoing estimate becomes

$$\|\widetilde{\mathbf{S}}_j(\mathbf{w})\|_{0,4;\Omega} \,\leqslant\, \|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}} \,\leqslant\, \widetilde{C}_j \Big\{ \|g_j\|_{0,4/3;\Omega} \,+\, \big(1 \,+\, \|\mathbf{w}\|_{0,4;\Omega}\big)\, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\}\,, \tag{4.28}$$

where $\widetilde{C}_j$ is a positive constant depending on $\widetilde{\alpha}_j$, $\widetilde{\beta}$, $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|i_4\|$, and $\mathtt{R}_j$. Finally, summing up in (4.28) over $j \in \big\{1,2\big\}$, we arrive at (4.27) with $C_{\widetilde{\mathbf{S}}} \,=\, \widetilde{C}_1 \,+\, \widetilde{C}_2$. □

For sake of completeness, we provide next the upper bound for the component $\boldsymbol{\vartheta}_j$ of the solution of (4.12). In fact, according now to the second inequality in [23, Theorem 2.34, eq. (2.30)], we find that

$$\|\boldsymbol{\vartheta}_j\|_{\widetilde{\mathbf{Q}}} \,\leqslant\, \frac{1}{\widetilde{\beta}}\left(1 \,+\, \frac{\|\widetilde{a}_{j,\mathbf{w}}\|}{\widetilde{\alpha}_j}\right)\|\widetilde{F}_j\| \,+\, \frac{\|\widetilde{a}_{j,\mathbf{w}}\|}{\widetilde{\beta}^2}\left(1 \,+\, \frac{\|\widetilde{a}_{j,\mathbf{w}}\|}{\widetilde{\alpha}_j}\right)\|\widetilde{G}_j\|\,,$$

which yields

$$\|\boldsymbol{\vartheta}_j\|_{\widetilde{\mathbf{Q}}} \,\leqslant\, \widetilde{M}_j\,\big(1 \,+\, \|\mathbf{w}\|_{0,4;\Omega}\big) \Big\{ \|g_j\|_{0,4/3;\Omega} \,+\, \big(1 \,+\, \|\mathbf{w}\|_{0,4;\Omega}\big)\, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\}\,, \tag{4.29}$$

with a positive constant $\widetilde{M}_j$ depending as well on $\widetilde{\alpha}_j$, $\widetilde{\beta}$, $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|i_4\|$, and $\mathtt{R}_j$.

4.4 Solvability analysis of the fixed–point equation

Having proved the well–posedness of the uncoupled problems (4.7) and (4.10), in particular the former under the assumption on $\mathbf{w}$ specified in Theorem 4.1, thus ensuring that the operators $\mathbf{S}$ (cf. (4.6)), $\widetilde{\mathbf{S}}$ (cf. (4.13)), and hence $\mathbf{T}$ (cf. (4.14)), are well–defined, our next goal is to establish the existence of a unique fixed–point of $\mathbf{T}$. For this purpose, in what follows we aim to verify the hypotheses of the Banach theorem, starting by providing a suitable condition guaranteeing that $\mathbf{T}$ maps a ball into itself. Indeed, given $r \in (0, r_0]$, with r_0 as in (4.24), we let, as in (4.19),

$$\mathbf{W}(r) := \Big\{ \mathbf{w} \in \mathbf{L}^4(\Omega) : \quad \|\mathbf{w}\|_{0,4;\Omega} \leqslant r \Big\}, \tag{4.30}$$

and observe, thanks to the a priori estimates (4.25) and (4.27), that for each $\mathbf{w} \in \mathbf{W}(r)$ there holds

$$\begin{aligned} \|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} \,&=\, \|\mathbf{S}\big(\mathbf{w}, \widetilde{\mathbf{S}}(\mathbf{w})\big)\|_{0,4;\Omega} \,\leqslant\, C_{\mathbf{S}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \|\widetilde{\mathbf{S}}(\mathbf{w})\|_{0,4;\Omega} \Big\} \\ &\leqslant\, C_{\mathbf{T}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big)\, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \Big\}, \end{aligned} \tag{4.31}$$

where $C_{\mathbf{T}} := C_{\mathbf{S}} \max\big\{1, C_{\widetilde{\mathbf{S}}}\big\}$. Then, we have the following result.

Lemma 4.3 *Given $r \in (0, r_0]$, with r_0 as in (4.24), assume that the data satisfy*

$$C_{\mathbf{T}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big)\, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \Big\} \,\leqslant\, r\,. \tag{4.32}$$

Then, there holds $\mathbf{T}\big(\mathbf{W}(r)\big) \,\subseteq\, \mathbf{W}(r)$.

Proof. It is a direct consequence of the estimate (4.31). □

Next, we aim to show that the operator $\mathbf{T}$ is Lipschitz continuous, for which, according to (4.14), it suffices to prove suitable continuity properties for $\mathbf{S}$ and $\widetilde{\mathbf{S}}$. In order to derive the corresponding result for $\mathbf{S}$, we need the technical estimate for c provided by the following lemma.

Lemma 4.4 *For each $\rho \in [3, 4]$ there exists a positive constant L_c, depending only on $\mathtt{F}$, $|\Omega|$, and ρ, such that*

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \,-\, c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \,\leqslant\, L_c \Big\{ \|\mathbf{w}\|_{0,4;\Omega} \,+\, \|\mathbf{z}\|_{0,4;\Omega} \Big\}^{\rho-3} \|\mathbf{w} \,-\, \mathbf{z}\|_{0,4;\Omega}\, \|\mathbf{u}\|_{0,4;\Omega}\, \|\mathbf{v}\|_{0,4;\Omega} \tag{4.33}$$

for all $\mathbf{w}$, $\mathbf{z} \in \mathbf{L}^4(\Omega)$, and for all $\vec{\mathbf{u}}$, $\vec{\mathbf{v}} \in \mathbf{Q}$.

Proof. We begin by noticing from the definition of c (cf. (3.12)) that, given $\mathbf{w}$, $\mathbf{z} \in \mathbf{L}^4(\Omega)$, and $\vec{\mathbf{u}} := (\mathbf{u}, \gamma)$, $\vec{\mathbf{v}} := (\mathbf{v}, \boldsymbol{\delta}) \in \mathbf{Q}$, there holds

$$|c(\mathbf{w}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \,-\, c(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}})| \,\leqslant\, \mathtt{F} \int_{\Omega} \big||\mathbf{w}|^{\rho-2} \,-\, |\mathbf{z}|^{\rho-2}\big|\, |\mathbf{u} \cdot \mathbf{v}|\,. \tag{4.34}$$

Next, defining $\widetilde{\mathbf{w}} := \big(|\mathbf{w}|, \mathbf{0}\big)$, $\widetilde{\mathbf{z}} := \big(|\mathbf{z}|, \mathbf{0}\big) \in \mathrm{R}^n$, with $\mathbf{0} \in \mathrm{R}^{n-1}$, we observe that

$$\big||\mathbf{w}|^{\rho-2} \,-\, |\mathbf{z}|^{\rho-2}\big| = \big||\widetilde{\mathbf{w}}|^{\rho-3}\, \widetilde{\mathbf{w}} \,-\, |\widetilde{\mathbf{z}}|^{\rho-3}\, \widetilde{\mathbf{z}}\big|\,, \tag{4.35}$$

and recall from [31, Lemma 5.3] that for each $t \geqslant 2$ there exists a positive constant C_t such that

$$\big||\mathbf{x}|^{t-2}\,\mathbf{x} - |\mathbf{y}|^{t-2}\,\mathbf{y}\big| \,\leqslant\, C_t\,\big(|\mathbf{x}| + |\mathbf{y}|\big)^{t-2}\,|\mathbf{x} - \mathbf{y}| \qquad \forall\,\mathbf{x},\,\mathbf{y} \in \mathrm{R}^n\,,$$

so that applying the foregoing inequality with $t = \rho - 1$, and denoting $C(\rho) := C_{\rho-1}$, we deduce that

$$\big||\widetilde{\mathbf{w}}|^{\rho-3}\,\widetilde{\mathbf{w}} \,-\, |\widetilde{\mathbf{z}}|^{\rho-3}\,\widetilde{\mathbf{z}}\big| \,\leqslant\, C(\rho)\,\big(|\widetilde{\mathbf{w}}| \,+\, |\widetilde{\mathbf{z}}|\big)^{\rho-3}\,|\widetilde{\mathbf{w}} \,-\, \widetilde{\mathbf{z}}|\,. \tag{4.36}$$

Thus, replacing (4.36) back into (4.35), and then the resulting estimate back into (4.34), returning to the original variables, and using, in particular, that $|\widetilde{\mathbf{w}} \,-\, \widetilde{\mathbf{z}}| \,=\, \big||\mathbf{w}| - |\mathbf{z}|\big| \,\leqslant\, |\mathbf{w} - \mathbf{z}|$, we arrive at

$$|c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,-\, c(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}})| \,\leqslant\, C(\rho)\,\mathtt{F}\int_\Omega \big(|\mathbf{w}| \,+\, |\mathbf{z}|\big)^{\rho-3}\,|\mathbf{w} \,-\, \mathbf{z}|\,|\mathbf{u}\cdot\mathbf{v}|\,,$$

from which, applying Cauchy-Schwarz's inequality, we deduce that

$$|c(\mathbf{w};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,-\, c(\mathbf{z};\vec{\mathbf{u}},\vec{\mathbf{v}})| \,\leqslant\, C(\rho)\,\mathtt{F}\,\big\|\big(|\mathbf{w}| \,+\, |\mathbf{z}|\big)^{\rho-3}\big\|_{0,4;\Omega}\,\|\mathbf{w} \,-\, \mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,. \tag{4.37}$$

It remains to estimate the expression $\big\|\big(|\mathbf{w}| \,+\, |\mathbf{z}|\big)^{\rho-3}\big\|_{0,4;\Omega}$. The case $\rho = 3$ is straightforward since $\big\|\big(|\mathbf{w}| + |\mathbf{z}|\big)^{\rho-3}\big\|_{0,4;\Omega}$ becomes $|\Omega|^{1/4}$, which yields (4.33) with $L_c := C(\rho)\,\mathtt{F}\,|\Omega|^{1/4}$. In turn, when $\rho = 4$, we get by triangle inequality that $\big\|\big(|\mathbf{w}| \,+\, |\mathbf{z}|\big)^{\rho-3}\big\|_{0,4;\Omega} \,=\, \|\,|\mathbf{w}| \,+\, |\mathbf{z}|\,\|_{0,4;\Omega} \,\leqslant\, \|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega}$, which implies (4.33) with $L_c := C(\rho)\,\mathtt{F}$. Finally, if $\rho \in (3,4)$, we apply Hölder's inequality with $r := \frac{1}{4-\rho} \in (1,+\infty)$ and its conjugate $r' := \frac{1}{\rho-3}$, to obtain

$$\big\|\big(|\mathbf{w}| \,+\, |\mathbf{z}|\big)^{\rho-3}\big\|_{0,4;\Omega} \,\leqslant\, |\Omega|^{(4-\rho)/4}\,\|\,|\mathbf{w}| + |\mathbf{z}|\,\|_{0,4;\Omega}^{\rho-3} \,\leqslant\, |\Omega|^{(4-\rho)/4}\,\Big(\|\mathbf{w}\|_{0,4;\Omega} + \|\mathbf{z}\|_{0,4;\Omega}\Big)^{\rho-3}\,,$$

which, along with (4.37), gives (4.33) with $L_c := C(\rho)\,\mathtt{F}\,|\Omega|^{(4-\rho)/4}$. Summarizing, (4.33) holds with this latter value of L_c for all $\rho \in [3,4]$. $\square$

The announced property for $\mathbf{S}$ is established next.

Lemma 4.5 *Let $r \in (0, r_0]$, with r_0 as in (4.24). Then, there exists a positive constant $L_{\mathbf{S}}$, depending only on $\alpha_{\mathbf{A}}$, $\|\mathbf{g}\|_{0,\Omega}$, μ_0, L_c, and r, such that*

$$\|\mathbf{S}(\mathbf{w},\boldsymbol{\varphi}) \,-\, \mathbf{S}(\mathbf{z},\boldsymbol{\xi})\|_{0,4;\Omega} \,\leqslant\, L_{\mathbf{S}}\Big\{\|\boldsymbol{\varphi} \,-\, \boldsymbol{\xi}\|_{0,4;\Omega} \,+\, \|\mathbf{S}(\mathbf{z},\boldsymbol{\xi})\|_{0,4;\Omega}\,\|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\Big\} \tag{4.38}$$

for all $(\mathbf{w},\boldsymbol{\varphi})$, $(\mathbf{z},\boldsymbol{\xi}) \in \mathbf{W}(r) \times \big(\mathrm{L}^4(\Omega) \times \mathrm{L}^4(\Omega)\big)$.

Proof. Let $(\mathbf{w},\boldsymbol{\varphi})$, $(\mathbf{z},\boldsymbol{\xi}) \in \mathbf{W}(r) \times \big(\mathrm{L}^4(\Omega) \times \mathrm{L}^4(\Omega)\big)$ such that $\mathbf{S}(\mathbf{w},\boldsymbol{\varphi}) \,=\, \mathbf{u}_1$ and $\mathbf{S}(\mathbf{z},\boldsymbol{\xi}) \,=\, \mathbf{u}_2$, where, for each $i \in \big\{1,2\big\}$, $(\boldsymbol{\sigma}_i,\vec{\mathbf{u}}_i) \,:=\, \big(\boldsymbol{\sigma}_i,(\mathbf{u}_i,\gamma_i)\big) \in \mathbf{H} \times \mathbf{Q}$ is the corresponding unique solution of (4.9) (equivalently, (4.7)), that is

$$\begin{aligned}
\mathbf{A}_{\mathbf{w}}\big((\boldsymbol{\sigma}_1,\vec{\mathbf{u}}_1),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \,+\, \mathbf{b}(\mathbf{w};\mathbf{u}_1,\boldsymbol{\tau}) \;&=\; \mathbf{F}_{\boldsymbol{\varphi}}(\vec{\mathbf{v}}) \,+\, \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}}) \,\in\, \mathbf{H} \times \mathbf{Q}\,,\\
\mathbf{A}_{\mathbf{z}}\big((\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \,+\, \mathbf{b}(\mathbf{z};\mathbf{u}_2,\boldsymbol{\tau}) \;&=\; \mathbf{F}_{\boldsymbol{\xi}}(\vec{\mathbf{v}}) \,+\, \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}}) \,\in\, \mathbf{H} \times \mathbf{Q}\,.
\end{aligned} \tag{4.39}$$

Then, applying (4.22) to $(\boldsymbol{\sigma}_1,\vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2) \in \mathbf{H} \times \mathbf{Q}$, we obtain

$$\begin{aligned}
\|\mathbf{S}(\mathbf{w},\boldsymbol{\varphi}) \,-\, \mathbf{S}(\mathbf{z},\boldsymbol{\xi})\|_{0,4;\Omega} \,&=\, \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4;\Omega} \,\leqslant\, \|(\boldsymbol{\sigma}_1,\vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2)\|_{\mathbf{H}\times\mathbf{Q}}\\
&\leqslant\, \frac{2}{\alpha_{\mathbf{A}}}\sup_{\substack{(\boldsymbol{\tau},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\(\boldsymbol{\tau},\vec{\mathbf{v}})\neq\mathbf{0}}}\frac{\mathbf{A}_{\mathbf{w}}\big((\boldsymbol{\sigma}_1,\vec{\mathbf{u}}_1) - (\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \,+\, \mathbf{b}(\mathbf{w};\mathbf{u}_1 \,-\, \mathbf{u}_2,\boldsymbol{\tau})}{\|(\boldsymbol{\tau},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}\,.
\end{aligned} \tag{4.40}$$

Now, adding and subtracting $\mathbf{A}_{\mathbf{z}}\big((\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2),(\boldsymbol{\tau},\vec{\mathbf{v}})\big)$, and using (4.39) and the fact that there holds $\big(\mathbf{A}_{\mathbf{z}}-\mathbf{A}_{\mathbf{w}}\big)\big((\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \,=\, c(\mathbf{w};\vec{\mathbf{u}}_2,\vec{\mathbf{v}})-c(\mathbf{z};\vec{\mathbf{u}}_2,\vec{\mathbf{v}})$, we get after some algebraic manipulations

$$\begin{aligned}&\mathbf{A}_{\mathbf{w}}\big((\boldsymbol{\sigma}_1,\vec{\mathbf{u}}_1)-(\boldsymbol{\sigma}_2,\vec{\mathbf{u}}_2),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \,+\, \mathbf{b}(\mathbf{w};\mathbf{u}_1\,-\,\mathbf{u}_2,\boldsymbol{\tau})\\ &=\, \big(\mathbf{F}_{\boldsymbol{\varphi}}-\mathbf{F}_{\boldsymbol{\xi}}\big)(\vec{\mathbf{v}})+\mathbf{b}(\mathbf{z}-\mathbf{w};\mathbf{u}_2,\boldsymbol{\tau})+c(\mathbf{w};\vec{\mathbf{u}}_2,\vec{\mathbf{v}})-c(\mathbf{z};\vec{\mathbf{u}}_2,\vec{\mathbf{v}})\,.\end{aligned} \tag{4.41}$$

Next, it readily follows from the definitions of the function $\mathbf{f}$ (cf. (2.3)) and the functional $\mathbf{F}_{\boldsymbol{\phi}}$ (cf. (3.13)) that

$$\big(\mathbf{F}_{\boldsymbol{\varphi}}-\mathbf{F}_{\boldsymbol{\xi}}\big)(\vec{\mathbf{v}}) \,\leqslant\, \|\mathbf{g}\|_{0,\Omega}\,\|\boldsymbol{\varphi}-\boldsymbol{\xi}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{4.42}$$

whereas it is clear from the first row of (4.3) that

$$\mathbf{b}(\mathbf{z}\,-\,\mathbf{w};\mathbf{u}_2,\boldsymbol{\tau}) \,\leqslant\, \frac{1}{\mu_0}\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{u}_2\|_{0,4;\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}\,. \tag{4.43}$$

In turn, applying Lemma 4.4, and using that both $\|\mathbf{w}\|_{0,4;\Omega}$ and $\|\mathbf{z}\|_{0,4;\Omega}$ are bounded by r, we find that

$$\begin{aligned}c(\mathbf{w};\vec{\mathbf{u}}_2,\vec{\mathbf{v}})-c(\mathbf{z};\vec{\mathbf{u}}_2,\vec{\mathbf{v}}) \,&\leqslant\, L_c\Big\{\|\mathbf{w}\|_{0,4;\Omega}\,+\,\|\mathbf{z}\|_{0,4;\Omega}\Big\}^{\rho-3}\,\|\mathbf{w}\,-\,\mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{u}_2\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\\ &\leqslant\, L_c\,(2r)^{\rho-3}\,\|\mathbf{w}\,-\,\mathbf{z}\|_{0,4;\Omega}\,\|\mathbf{u}_2\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,.\end{aligned} \tag{4.44}$$

Finally, replacing (4.41) back into (4.40), employing the upper bounds provided by (4.42), (4.43), and (4.44), and recalling that $\mathbf{u}_2=\mathbf{S}(\mathbf{z},\boldsymbol{\xi})$, we arrive a the required inequality (4.38) with a positive constant $L_{\mathbf{S}}$ as indicated. □

The following lemma proves the Lipschitz continuity of the operator $\widetilde{\mathbf{S}}$.

Lemma 4.6 *Let $r\in(0,r_0]$, with r_0 as in (4.24). Then, there exists a positive constant $L_{\widetilde{\mathbf{S}}}$, depending only on $\widetilde{C}_j$ (cf. (4.28)), $\mathtt{R}_j$, and $\widetilde{\alpha}_j$, $j\in\{1,2\}$, such that*

$$\|\widetilde{\mathbf{S}}(\mathbf{w})-\widetilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} \,\leqslant\, L_{\widetilde{\mathbf{S}}}\sum_{j=1}^{2}\Big\{\|g_j\|_{0,4/3;\Omega}\,+\,\big(1+\|\mathbf{w}\|_{0,4;\Omega}\big)\,\|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\,\|\mathbf{w}-\mathbf{z}\|_{0,4;\Omega} \tag{4.45}$$

for all $\mathbf{w},\,\mathbf{z}\in\mathbf{W}(r)$.

Proof. We proceed similarly to the proof of [15, Lemma 3.11]. Indeed, given $r\in(0,r_0]$ and $\mathbf{w},\mathbf{z}\in\mathbf{W}(r)$, we let $\widetilde{\mathbf{S}}(\mathbf{w}) := (\phi_1,\phi_2)\in \mathrm{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$ and $\widetilde{\mathbf{S}}(\mathbf{z}) := (\xi_1,\xi_2)\in \mathrm{L}^4(\Omega)\times\mathrm{L}^4(\Omega)$, where, for each $j\in\{1,2\}$, $(\vec{\phi}_j,\boldsymbol{\vartheta}_j) := \big((\phi_j,\mathbf{t}_j),\boldsymbol{\vartheta}_j\big)\in\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}$ and $(\vec{\xi}_j,\boldsymbol{\zeta}_j) := \big((\xi_j,\mathbf{s}_j),\boldsymbol{\zeta}_j\big)\in\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}$ are the unique solutions of (4.10) (equivalently, (4.12)) with $\widetilde{c}_j(\mathbf{w};\cdot,\cdot)$ and $\widetilde{c}_j(\mathbf{z};\cdot,\cdot)$ (equivalently, with $\widetilde{a}_{j,\mathbf{w}}$ and $\widetilde{a}_{j,\mathbf{z}}$), respectively. It follows from the subtraction of the corresponding second equations of (4.10) that $\vec{\phi}_j-\vec{\xi}_j\in\widetilde{\mathbf{V}}$. In addition, testing the first equations of (4.10) against $\vec{\psi}_j \,=\, \vec{\phi}_j-\vec{\xi}_j$, and then subtracting them, we deduce that

$$\widetilde{a}_j(\vec{\phi}_j-\vec{\xi}_j,\vec{\phi}_j-\vec{\xi}_j) \,=\, \widetilde{c}_j(\mathbf{z};\vec{\xi}_j,\vec{\phi}_j-\vec{\xi}_j)-\widetilde{c}_j(\mathbf{w};\vec{\phi}_j,\vec{\phi}_j-\vec{\xi}_j)\,,$$

from which, subtracting and adding $\vec{\phi}_j$ in the second component of the first term on the right hand-side, and using the identity (4.5), we get

$$\widetilde{a}_j(\vec{\phi}_j-\vec{\xi}_j,\vec{\phi}_j-\vec{\xi}_j) \,=\, \widetilde{c}_j(\mathbf{z};\vec{\phi}_j,\vec{\phi}_j-\vec{\xi}_j)-\widetilde{c}_j(\mathbf{w};\vec{\phi}_j,\vec{\phi}_j-\vec{\xi}_j)\,.$$

In this way, employing now the ellipticity of $\widetilde{a}_j$ (cf. (4.26)), the foregoing identity, and the continuity property for $\widetilde{c}_j$ provided by (4.4), we find that

$$\widetilde{\alpha}_j \, \|\vec{\phi}_j - \vec{\xi}_j\|^2_{\widetilde{\mathbf{H}}} \,\leqslant\, \widetilde{a}_j(\vec{\phi}_j - \vec{\xi}_j, \vec{\phi}_j - \vec{\xi}_j) \,=\, \widetilde{c}_j(\mathbf{z}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j) - \widetilde{c}_j(\mathbf{w}; \vec{\phi}_j, \vec{\phi}_j - \vec{\xi}_j)$$
$$\leqslant\, \mathtt{R}_j \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \, \|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}} \, \|\vec{\phi}_j - \vec{\xi}_j\|_{\widetilde{\mathbf{H}}} \,,$$

which, along with the a priori estimate for $\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}$ given by (4.28), yields

$$\begin{aligned}\|\widetilde{\mathbf{S}}_j(\mathbf{w}) - \widetilde{\mathbf{S}}_j(\mathbf{z})\|_{0,4;\Omega} \,&=\, \|\phi_j - \xi_j\|_{0,4;\Omega} \,\leqslant\, \|\vec{\phi}_j - \vec{\xi}_j\|_{\widetilde{\mathbf{H}}} \,\leqslant\, \widetilde{\alpha}_j^{-1} \, \mathtt{R}_j \, \|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \\ &\leqslant\, \widetilde{\alpha}_j^{-1} \, \mathtt{R}_j \, \widetilde{C}_j \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{w}\|_{0,4;\Omega}\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \,.\end{aligned} \tag{4.46}$$

Finally, summing up in (4.46) over $j \in \{1, 2\}$, we get (4.45) with $L_{\widetilde{\mathbf{S}}} := \max\limits_{j \in \{1,2\}} \big\{ \widetilde{\alpha}_j^{-1} \, \mathtt{R}_j \, \widetilde{C}_j \big\}$. □

As a consequence of Lemmas 4.5 and 4.6, we are able now to prove the Lipschitz continuity of $\mathbf{T}$.

Lemma 4.7 *Let $r \in (0, r_0]$, with r_0 as in (4.24). Then, there exists a positive constant $L_{\mathbf{T}}$, depending only on $L_{\mathbf{S}}$, $L_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{S}}$, and $C_{\widetilde{\mathbf{S}}}$, such that*

$$\begin{aligned}&\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\mathbf{z})\|_{0,4;\Omega} \\ &\quad \leqslant\, L_{\mathbf{T}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \Big\} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega}\end{aligned} \tag{4.47}$$

for all $\mathbf{w}$, $\mathbf{z} \in \mathbf{W}(r)$.

Proof. Given $\mathbf{w}$, $\mathbf{z} \in \mathbf{W}(r)$, we first deduce from the definition of $\mathbf{T}$ (cf. (4.14)) and the continuity property of $\mathbf{S}$ (cf. Lemma 4.5) that

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\mathbf{z})\|_{0,4;\Omega} \,&=\, \|\mathbf{S}\big(\mathbf{w}, \widetilde{\mathbf{S}}(\mathbf{w})\big) - \mathbf{S}\big(\mathbf{z}, \widetilde{\mathbf{S}}(\mathbf{z})\big)\|_{0,4;\Omega} \\ &\leqslant\, L_{\mathbf{S}} \Big\{ \|\widetilde{\mathbf{S}}(\mathbf{w}) - \widetilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} + \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathbf{S}}(\mathbf{z})\big)\|_{0,4;\Omega} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \Big\} \,.\end{aligned} \tag{4.48}$$

In turn, the Lipschitz-continuity of $\widetilde{\mathbf{S}}$ (cf. Lemma 4.6) yields

$$\|\widetilde{\mathbf{S}}(\mathbf{w}) - \widetilde{\mathbf{S}}(\mathbf{z})\|_{0,4;\Omega} \,\leqslant\, L_{\widetilde{\mathbf{S}}} \, \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} \,+\, \big(1 + r\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \, \|\mathbf{w} - \mathbf{z}\|_{0,4;\Omega} \,, \tag{4.49}$$

whereas the a priori estimates for $\mathbf{S}$ (cf. (4.25)) and $\widetilde{\mathbf{S}}$ (cf. (4.27)) imply

$$\begin{aligned}&\|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathbf{S}}(\mathbf{z})\big)\|_{0,4;\Omega} \\ &\quad \leqslant\, C_{\mathbf{S}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + C_{\widetilde{\mathbf{S}}} \, \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \Big\} \,,\end{aligned} \tag{4.50}$$

where the fact that both $\|\mathbf{w}\|_{0,4;\Omega}$ and $\|\mathbf{z}\|_{0,4;\Omega}$ are bounded by r has been utilized in (4.49) and (4.50), respectively. Finally, replacing the latter estimates back into (4.48), and performing simple algebraic manipulations, we arrive at (4.47) and end the proof. □

The main result of this section, which refers to the solvability of (4.15) (equivalently, (3.24)), is stated as follows.

Theorem 4.8 *Given $r \in (0, r_0]$, with r_0 as in (4.24), assume that, in addition to the hypothesis of Lemma 4.3 (cf. (4.32)), the data satisfy*

$$L_{\mathbf{T}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \left\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big) \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} < 1 . \tag{4.51}$$

Then, there exists a unique $\mathbf{u} \in \mathbf{W}(r)$ (cf. (4.30)) fixed point of $\mathbf{T}$ (cf. (4.15)). Equivalently, (3.24) has a unique solution $(\boldsymbol{\sigma}, \vec{\mathbf{u}}) := \big(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{H} \times \mathbf{Q}$ and $(\vec{\phi}_j, \boldsymbol{\vartheta}_j) := \big((\phi_j, \mathbf{t}_j), \boldsymbol{\vartheta}_j\big) \in \widetilde{\mathbf{H}} \times \widetilde{\mathbf{Q}}$, $j \in \{1, 2\}$, with $\mathbf{u} \in \mathbf{W}(r)$. Moreover, there exist positive constants $\mathcal{C}$, $\mathcal{C}_1$, and $\mathcal{C}_2$, depending on $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}$, $\|\mathbf{Q}_1\|_{0,\infty;\Omega}$, $\|\mathbf{Q}_2\|_{0,\infty;\Omega}$, $\|i_4\|$, $\mathtt{R}_1$, $\mathtt{R}_2$, r, $\|\boldsymbol{i}_4\|$, $\|\mathbf{g}\|_{0,\Omega}$, and $\alpha_{\mathbf{A}}$, such that there hold the following a priori bounds

$$\|(\boldsymbol{\sigma}, \vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \mathcal{C} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} \right\} , \tag{4.52}$$

and for each $j \in \{1, 2\}$

$$\|(\vec{\phi}_j, \boldsymbol{\vartheta}_j)\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} \leqslant \mathcal{C}_j \left\{ \|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma} \right\} . \tag{4.53}$$

Proof. It is clear from Lemma 4.3 and the assumption (4.51) that $\mathbf{T}$ is a contraction that maps the ball $\mathbf{W}(r)$ into itself, and hence a straightforward application of the classical Banach fixed–point theorem implies the indicated solvabilities of (4.15) and (3.24). Furthermore, since $\mathbf{u} = \mathbf{T}(\mathbf{u}) = \mathbf{S}\big(\mathbf{u}, \widetilde{\mathbf{S}}(\mathbf{u})\big)$, we deduce that $\boldsymbol{\phi} := (\phi_1, \phi_2) = \widetilde{\mathbf{S}}(\mathbf{u})$, whence (4.53) follows from (4.28) and (4.29), whereas (4.52) is consequence of (4.25) and (4.53). We omit further details. $\square$

5 The Galerkin scheme

The Galerkin scheme of the fully–mixed formulation (3.24) is introduced and analyzed in this section. In particular, regarding the solvability analyses of the discrete versions of the decoupled problems studied in Section 4.3, we now apply [21, Theorem 3.5], [23, Theorem 2.22], and [23, Proposition 2.42], which correspond to the discrete analogues of [21, Theorem 3.4], [23, Theorem 2.6], and [23, Theorem 2.34], respectively.

5.1 Preliminaries

We begin by letting $\left\{\mathcal{T}_h\right\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and set $h := \max\left\{h_K : \quad K \in \mathcal{T}_h\right\}$. Then, we let $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, H_h^{ϕ}, $\mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\vartheta}}$ be arbitrary finite element subpaces of $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$, $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\mathsf{skew}}(\Omega)$, $\mathrm{L}^4(\Omega)$, $\mathbf{L}^2(\Omega)$, and $\mathbf{H}(\mathrm{div}_{4/3}; \Omega)$, respectively. Specific choices of them, satisfying suitable hypotheses to be introduced along the discussion, will be described later on in Section 6. Note that h stands for both, the size of the triangulation $\mathcal{T}_h$ and the sub-index of each subspace. Then, defining

$$\mathbf{H}_h := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) , \quad \mathbf{Q}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma} , \quad \widetilde{\mathbf{H}}_h := \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{t}} , \quad \widetilde{\mathbf{Q}}_h := \mathbf{H}_h^{\boldsymbol{\vartheta}} , \tag{5.1}$$

and setting the notations

$$\vec{\mathbf{u}}_h = (\mathbf{u}_h, \boldsymbol{\gamma}_h) , \quad \vec{\mathbf{v}}_h = (\mathbf{v}_h, \boldsymbol{\delta}_h) , \quad \vec{\mathbf{z}}_h = (\mathbf{z}_h, \boldsymbol{\zeta}_h) \in \mathbf{Q}_h ,$$

and for $j \in \{1,2\}$

$$\vec{\phi}_{j,h} \,=\, (\phi_{j,h}, \mathbf{t}_{j,h})\,, \quad \vec{\psi}_{j,h} \,=\, (\psi_{j,h}, \mathbf{r}_{j,h})\,, \quad \vec{\xi}_{j,h} \,=\, (\xi_{j,h}, \mathbf{s}_{j,h}) \,\in\, \widetilde{\mathbf{H}}_h\,,$$

the Galerkin scheme associated with (3.24) reads: Find $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) \in \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{Q}}_h$, $j \in \{1,2\}$, such that

$$\begin{array}{rcll} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \vec{\mathbf{u}}_h) + \mathbf{b}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) & = & \mathbf{G}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \,\in\, \mathbf{H}_h\,, \\ b(\boldsymbol{\sigma}_h, \vec{\mathbf{v}}_h) - c(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) & = & \mathbf{F}_{\boldsymbol{\phi}_h}(\vec{\mathbf{v}}_h) & \forall\, \vec{\mathbf{v}}_h \,\in\, \mathbf{Q}_h\,, \\ \widetilde{a}_j(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) \,+\, \widetilde{c}_j(\mathbf{u}_h; \vec{\phi}_{j,h}, \vec{\psi}_{j,h}) \,+\, \widetilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) & = & \widetilde{F}_j(\vec{\psi}_{j,h}) & \forall\, \vec{\psi}_{j,h} \,\in\, \widetilde{\mathbf{H}}_h\,, \\ \widetilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) & = & \widetilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall\, \boldsymbol{\eta}_{j,h} \,\in\, \widetilde{\mathbf{Q}}_h\,, \end{array} \tag{5.2}$$

where $\boldsymbol{\phi}_h \,:=\, (\phi_{1,h}, \phi_{2,h}) \in \mathrm{H}_h^\phi \times \mathrm{H}_h^\phi$.

5.2 Discrete fixed point strategy

In order to address the solvability of (5.2), we adopt the discrete analogue of the fixed point strategy employed in Section 4.2. Indeed, we start by introducing the operator $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \big(\mathrm{H}_h^\phi \times \mathrm{H}_h^\phi\big) \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\mathbf{S}_h(\mathbf{w}_h, \boldsymbol{\varphi}_h) \,:=\, \mathbf{u}_h \quad \forall\, (\mathbf{w}_h, \boldsymbol{\varphi}_h) \,\in\, \mathbf{H}_h^{\mathbf{u}} \times \big(\mathrm{H}_h^\phi \times \mathrm{H}_h^\phi\big)\,, \tag{5.3}$$

where $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) \,:=\, \big(\boldsymbol{\sigma}_h, (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the problem arising from the first two rows of (5.2) when $\mathbf{b}(\mathbf{u}_h; \cdot, \cdot)$, $c(\mathbf{u}_h; \cdot, \cdot)$, and $\mathbf{F}_{\boldsymbol{\phi}_h}$, are replaced by $\mathbf{b}(\mathbf{w}_h; \cdot, \cdot)$, $c(\mathbf{w}_h; \cdot, \cdot)$, and $\mathbf{F}_{\boldsymbol{\varphi}_h}$, respectively, that is

$$\begin{array}{rcll} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \vec{\mathbf{u}}_h) + \mathbf{b}(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) & = & \mathbf{G}(\boldsymbol{\tau}_h) & \forall\, \boldsymbol{\tau}_h \,\in\, \mathbf{H}_h\,, \\ b(\boldsymbol{\sigma}_h, \vec{\mathbf{v}}_h) - c(\mathbf{w}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) & = & \mathbf{F}_{\boldsymbol{\varphi}_h}(\vec{\mathbf{v}}_h) & \forall\, \vec{\mathbf{v}}_h \,\in\, \mathbf{Q}_h\,, \end{array} \tag{5.4}$$

or, equivalently, as the discrete analogue of (4.9)

$$\mathbf{A}_{\mathbf{w}_h}\big((\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h), (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h)\big) \,+\, \mathbf{b}(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) \,=\, \mathbf{G}(\boldsymbol{\tau}_h) \,+\, \mathbf{F}_{\boldsymbol{\varphi}_h}(\vec{\mathbf{v}}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \vec{\mathbf{v}}_h) \,\in\, \mathbf{H}_h \times \mathbf{Q}_h\,, \tag{5.5}$$

where, given $\mathbf{w}_h \in \mathbf{H}_h^u$, $\mathbf{A}_{\mathbf{w}_h} : \big(\mathbf{H}_h \times \mathbf{Q}_h\big) \times \big(\mathbf{H}_h \times \mathbf{Q}_h\big) \to \mathrm{R}$ is defined according to (4.8).

On the other hand, for each $j \in \big\{1,2\big\}$ we introduce the operator $\widetilde{\mathbf{S}}_{j,h} : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^\phi$ defined by

$$\widetilde{\mathbf{S}}_{j,h}(\mathbf{w}_h) \,:=\, \phi_{j,h} \qquad \forall\, \mathbf{w}_h \,\in\, \mathbf{H}_h^{\mathbf{u}}\,,$$

where $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) \,:=\, \big((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{Q}}_h$ is the unique solution (to be confirmed below) of the problem that arises from the third and fourth rows of (5.2) when $\widetilde{c}_j(\mathbf{u}_h; \cdot, \cdot)$ is replaced by $\widetilde{c}_j(\mathbf{w}_h; \cdot, \cdot)$, that is

$$\begin{array}{rcll} \widetilde{a}_j(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) \,+\, \widetilde{c}_j(\mathbf{w}_h; \vec{\phi}_{j,h}, \vec{\psi}_{j,h}) \,+\, \widetilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) & = & \widetilde{F}_j(\vec{\psi}_{j,h}) & \forall\, \vec{\psi}_{j,h} \,\in\, \widetilde{\mathbf{H}}_h\,, \\ \widetilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) & = & \widetilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall\, \boldsymbol{\eta}_{j,h} \,\in\, \widetilde{\mathbf{Q}}_h\,. \end{array} \tag{5.6}$$

Equivalently, defining $\widetilde{a}_{j,\mathbf{w}_h} : \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{H}}_h \to \mathrm{R}$, for each $\mathbf{w}_h \in \mathbf{H}_h$, as in (4.11), we can restate (5.6) as

$$\begin{array}{rcll} \widetilde{a}_{j,\mathbf{w}_h}(\vec{\phi}_{j,h}, \vec{\psi}_{j,h}) \,+\, \widetilde{b}(\vec{\psi}_{j,h}, \boldsymbol{\vartheta}_{j.h}) & = & \widetilde{F}_j(\vec{\psi}_{j,h}) & \forall\, \vec{\psi}_{j,h} \,\in\, \widetilde{\mathbf{H}}_h\,, \\ \widetilde{b}(\vec{\phi}_{j,h}, \boldsymbol{\eta}_{j,h}) & = & \widetilde{G}_j(\boldsymbol{\eta}_{j,h}) & \forall\, \boldsymbol{\eta}_{j,h} \,\in\, \widetilde{\mathbf{Q}}_h\,. \end{array} \tag{5.7}$$

In this way, defining $\widetilde{\mathbf{S}}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\phi}$ as

$$\widetilde{\mathbf{S}}_h(\mathbf{w}_h) := \big(\widetilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \widetilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)\big) \qquad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{5.8}$$

and letting $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator given by

$$\mathbf{T}_h(\mathbf{w}_h) := \mathbf{S}_h\big(\mathbf{w}_h, \widetilde{\mathbf{S}}_h(\mathbf{w}_h)\big) \qquad \forall\, \mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{5.9}$$

we realize that solving (5.2) is equivalent to seeking a fixed-point of $\mathbf{T}_h$, that is $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h . \tag{5.10}$$

5.3 Discrete solvability analysis

In this section we address the solvability of (5.2) via the corresponding analysis of the equivalent fixed–point equation (5.10), which previously requires to prove that the operators $\mathbf{S}_h$ (cf. (5.3)) and $\widetilde{\mathbf{S}}_h$ (cf. (5.8)), and hence $\mathbf{T}_h$, are well-defined. Equivalently, that the uncoupled problems (5.5) (or (5.4)) and (5.7) (or (5.6)) are well-posed.

We begin with the analysis of (5.5), for which we aim to prove that the bilinear forms a, b, and $c(\mathbf{w}_h; \cdot, \cdot)$, for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, when restricted to the corresponding finite element subspaces, satisfy the assumptions of [21, Theorem 3.5]. In fact, being the hypothesis **i)** of [21, Theorem 3.5] basically the same as the one of [21, Theorem 3.4], namely the symmetry and positive semi-definedness of a and $c(\mathbf{w}_h; \cdot, \cdot)$, which was already clarified in Section 4.3, we only need to concentrate here on **ii)** and **iii)** of [21, Theorem 3.5]. To this end, we first consider the following hypotheses on $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$:

(**H.1**) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiplies of the identity tensor $\mathbb{I}$, and

(**H.2**) $\mathbf{div}\big(\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}\big) \subseteq \mathbf{H}_h^{\mathbf{u}}$.

It follows from (**H.1**) and the decomposition (3.6) that $\mathbf{H}_h$ (cf. (5.1)) can be redefined as

$$\mathbf{H}_h := \left\{ \boldsymbol{\zeta}_h - \Big(\frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\Big)\, \mathbb{I} : \quad \boldsymbol{\zeta}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \right\}.$$

In turn, letting $\mathbf{V}_h$ be the kernel of $b|_{\mathbf{H}_h \times \mathbf{Q}_h}$, we readily deduce, thanks to the definition of b (cf. (3.10)) and (**H.2**), that

$$\mathbf{V}_h := \left\{ \boldsymbol{\zeta}_h \in \mathbf{H}_h : \quad \mathbf{div}(\boldsymbol{\zeta}_h) = 0 \quad \text{and} \quad \int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\zeta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}_h^{\gamma} \right\}.$$

Consequently, while $\mathbf{V}_h$ is not necessarily contained in $\mathbf{V}$ (cf. (4.16)), the fact that the elements of $\mathbf{V}_h$ are still divergence–free, along with the inequality (4.17), suffice to conclude the discrete analogue of (4.18), and with the same constant, namely

$$a(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \geqslant \alpha_{\mathtt{d}}\, \|\boldsymbol{\zeta}_h\|_{\mathbf{H}}^2 \qquad \forall\, \boldsymbol{\zeta}_h \in \mathbf{V}_h, \tag{5.11}$$

with $\alpha_{\mathtt{d}} := \frac{c_1^2}{\mu_1}$. Similarly as for the continuous case, it is easily seen that (5.11) yields the discrete inf-sup condition for a required by the hypothesis **ii)** of [21, Theorem 3.5].

Furthermore, in order to continue the analysis, we introduce the discrete inf-sup condition for b as a third hypothesis, that is:

(**H.3**) there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbf{H}_h\\ \boldsymbol{\tau}_h\neq\mathbf{0}}} \frac{b(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{H}}} \,\geqslant\, \beta_{\mathtt{d}}\,\|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \qquad \forall\,\vec{\mathbf{v}}_h\,\in\,\mathbf{Q}_h\,.$$

Next, proceeding analogously to the continuous case, we consider the same radius δ employed in (4.19), and introduce now the discrete ball

$$\mathbf{W}_h(\delta)\,:=\,\Big\{\mathbf{w}_h\,\in\,\mathbf{H}_h^{\mathbf{u}}:\quad \|\mathbf{w}_h\|_{0,4;\Omega}\,\leqslant\,\delta\Big\}\,, \tag{5.12}$$

so that the boundedness for $c(\mathbf{w}_h;\cdot,\cdot)|_{\mathbf{Q}_h\times\mathbf{Q}_h}$ becomes exactly as in (4.20), that is

$$|c(\mathbf{w}_h;\vec{\mathbf{v}}_h,\vec{\mathbf{z}}_h)|\,\leqslant\,|\Omega|^{1/2}\left(\mathtt{D}\,+\,\mathtt{F}\,\delta^{\rho-2}\right)\|\vec{\mathbf{v}}_h\|_{\mathbf{Q}}\,\|\vec{\mathbf{z}}_h\|_{\mathbf{Q}}\quad\forall\,\vec{\mathbf{v}}_h\,,\vec{\mathbf{z}}_h\,\in\,\mathbf{Q}_h\,.$$

Hence, having satisfied all the hypotheses of [21, Theorem 3.5], a straightforward application of this result implies the existence of a positive constant $\alpha_{\mathbf{A},\mathtt{d}}$, depending only on μ_0, $|\Omega|$, $\mathtt{D}$, $\mathtt{F}$, δ, ρ, $\alpha_{\mathtt{d}}$, and $\beta_{\mathtt{d}}$, such that for each $\mathbf{w}_h\in\mathbf{W}_h(\delta)$ there holds the discrete analogue of (4.21), that is

$$\sup_{\substack{(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h\\ (\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}_h}\big((\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h),(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\big)}{\|(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\|_{\mathbf{H}\times\mathbf{Q}}}\,\geqslant\,\alpha_{\mathbf{A},\mathtt{d}}\,\|(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\qquad\forall\,(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\,\in\,\mathbf{H}_h\,\times\,\mathbf{Q}_h\,. \tag{5.13}$$

Thus, using (5.13) and the boundedness property of $\mathbf{b}(\mathbf{w};\cdot,\cdot)$ (cf. (4.3)), similarly as for the derivation of (4.22), we deduce that for each $\mathbf{w}_h\in\mathbf{W}_h(\delta)$ such that $\|\mathbf{w}_h\|_{0,4;\Omega}\,\leqslant\,\frac{\mu_0\,\alpha_{\mathbf{A},\mathtt{d}}}{2}$, there holds

$$\sup_{\substack{(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h\\ (\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{w}_h}((\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h),(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h))\,+\,\mathbf{b}(\mathbf{w}_h;\mathbf{v}_h,\boldsymbol{\zeta}_h)}{\|(\boldsymbol{\zeta}_h,\vec{\mathbf{z}}_h)\|_{\mathbf{H}\times\mathbf{Q}}}\,\geqslant\,\frac{\alpha_{\mathbf{A},\mathtt{d}}}{2}\,\|(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}$$

for all $(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\,\in\,\mathbf{H}_h\,\times\,\mathbf{Q}_h$.

Therefore, the well–posedness of (5.5) is established as follows.

Theorem 5.1 *Given $\delta>0$, let $r\in(0,r_{0,\mathtt{d}}]$, with*

$$r_{0,\mathtt{d}}\,:=\,\min\left\{\delta,\frac{\mu_0\,\alpha_{\mathbf{A},\mathtt{d}}}{2}\right\}. \tag{5.14}$$

Then, for each $(\mathbf{w}_h,\boldsymbol{\varphi}_h)\in\mathbf{H}_h^{\mathbf{u}}\times\left(\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\phi}\right)$ such that $\|\mathbf{w}_h\|_{0,4;\Omega}\leqslant r$, (5.5) (equivalently, (5.4)) has a unique solution $(\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h):=\big(\boldsymbol{\sigma}_h,(\mathbf{u}_h,\gamma_h)\big)\in\mathbf{H}_h\times\mathbf{Q}_h$, and hence one can define $\mathbf{S}_h(\mathbf{w}_h,\boldsymbol{\varphi}_h):=\mathbf{u}_h$. Moreover, there exists a positive constant $C_{\mathbf{S},\mathtt{d}}$, depending only on $\|\boldsymbol{i}_4\|$, $\|\mathbf{g}\|_{0,\Omega}$, and $\alpha_{\mathbf{A},\mathtt{d}}$, such that

$$\begin{aligned}\|\mathbf{S}_h(\mathbf{w}_h,\boldsymbol{\varphi}_h)\|_{0,4;\Omega}\,&=\,\|\mathbf{u}_h\|_{0,4;\Omega}\,\leqslant\,\|(\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\ &\leqslant\,C_{\mathbf{S},\mathtt{d}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega}+\|\boldsymbol{\varphi}_h\|_{0,4;\Omega}\Big\}\,.\end{aligned} \tag{5.15}$$

Proof. Similarly as for the proof of Theorem 4.1, we observe now that the bilinear form $\mathbf{A}_{\mathbf{w}_h}+\mathbf{b}(\mathbf{w}_h;\cdot,\cdot)$ satisfies the hypotheses of [23, Theorem 2.22], so that, noting in this case that $\mathbf{G}|_{\mathbf{H}_h}\,\in\,\mathbf{H}_h'$ and $\mathbf{F}_{\boldsymbol{\varphi}_h}|_{\mathbf{Q}_h}\in\mathbf{Q}_h'$, an application of that theorem proves the present result. □

On the other hand, in order to establish the well-posedness of (5.7) (equivalently, (5.6)), in what follows we show that the bilinear forms $\widetilde{a}_{j,\mathbf{w}_h}|_{\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{H}}_h}$ and $\widetilde{b}|_{\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{Q}}_h}$ satisfy the hypotheses of [23, Proposition 2.42]. To this end, we proceed as in [20, Section 5.5] (see also [7, Section 4.3, Lemma 4.2] and [15, Section 4.2, Lemmas 4.1 and 4.5]), and introduce first the kernel of $\widetilde{b}|_{\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{Q}}_h}$, that is

$$\widetilde{\mathbf{V}}_h\,:=\,\Big\{\vec{\psi}_h\,=\,(\psi_h,\mathbf{r}_h)\,\in\,\widetilde{\mathbf{H}}_h:\quad\widetilde{b}(\vec{\psi}_h,\boldsymbol{\eta}_h)\,=\,0\quad\forall\,\boldsymbol{\eta}_h\,\in\,\widetilde{\mathbf{Q}}_h\Big\}\,,$$

and

$$\mathrm{Z}_{0,h} := \Big\{ \boldsymbol{\eta}_h \in \widetilde{\mathbf{Q}}_h : \quad \widetilde{b}(\vec{\psi}_h, \boldsymbol{\eta}_h) = 0 \quad \forall\, \vec{\psi}_h = (\psi_h, 0) \in \widetilde{\mathbf{H}}_h \Big\},$$

which become, respectively,

$$\widetilde{\mathbf{V}}_h := \Big\{ \vec{\psi}_h = (\psi_h, \mathbf{r}_h) \in \widetilde{\mathbf{H}}_h : \quad \int_\Omega \mathbf{r}_h \cdot \boldsymbol{\eta}_h + \int_\Omega \psi_h \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \forall\, \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \Big\},$$

and

$$\mathrm{Z}_{0,h} := \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}} : \quad \int_\Omega \psi_h \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \forall\, \psi_h \in \mathrm{H}_h^{\phi} \Big\}. \tag{5.16}$$

Next, we consider the following assumptions on the subspaces H_h^{ϕ}, $\mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\vartheta}}$:

(**H.4**) $\operatorname{div}(\mathbf{H}_h^{\boldsymbol{\vartheta}}) \subseteq \mathrm{H}_h^{\phi}$,

(**H.5**) $\mathrm{Z}_{0,h} \subseteq \mathbf{H}_h^{\mathbf{t}}$, and

(**H.6**) there exists a positive constant $\beta_{1,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \\ \boldsymbol{\eta}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \psi_h \operatorname{div}(\boldsymbol{\eta}_h)}{\|\boldsymbol{\eta}_h\|_{\widetilde{\mathbf{Q}}}} \geqslant \beta_{1,\mathtt{d}} \|\psi_h\|_{0,4;\Omega} \qquad \forall\, \psi_h \in \mathrm{H}_h^{\phi}.$$

As a consequence of (**H.4**) we easily deduce from (5.16) that

$$\mathrm{Z}_{0,h} := \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}} : \quad \operatorname{div}(\boldsymbol{\eta}_h) = 0 \quad \text{in} \quad \Omega \Big\}, \tag{5.17}$$

and thus, given $\boldsymbol{\eta}_h \in \mathrm{Z}_{0,h}$, and using (**H.5**), we bound the supremum by below with $\mathbf{r}_h := \boldsymbol{\eta}_h \in \mathbf{H}_h^{\mathbf{t}}$, to deduce that

$$\sup_{\substack{\mathbf{r}_h \in \mathbf{H}_h^{\mathbf{t}} \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{r}_h \cdot \boldsymbol{\eta}_h}{\|\mathbf{r}_h\|_{0,\Omega}} \geqslant \|\boldsymbol{\eta}_h\|_{0,\Omega} = \beta_{2,\mathtt{d}} \|\boldsymbol{\eta}_h\|_{\widetilde{\mathbf{Q}}} \quad \forall\, \boldsymbol{\eta}_h \in \mathrm{Z}_{0,h}, \tag{5.18}$$

with $\beta_{2,\mathtt{d}} = 1$. Consequently, invoking [20, Lemma 5.1] with local notation there given by $X = \mathrm{H}_h^{\phi}$, $Y = Y_1 = \mathbf{H}_h^{\mathbf{t}}$, $Y_2 = \{\mathbf{0}\}$, $V = \widetilde{\mathbf{V}}_h$, $Z = \mathbf{H}_h^{\boldsymbol{\vartheta}}$, and $Z_0 = \mathrm{Z}_{0,h}$, we conclude that (**H.6**) and (5.18) are equivalent to the existence of positive constants $\widetilde{\beta}_{\mathtt{d}}$ and $\widetilde{C}_{\mathtt{d}}$ such that

$$\sup_{\substack{\vec{\psi}_h \in \widetilde{\mathbf{H}}_h \\ \vec{\psi}_h \neq \mathbf{0}}} \frac{\widetilde{b}(\vec{\psi}_h, \boldsymbol{\eta}_h)}{\|\vec{\psi}_h\|_{\widetilde{\mathbf{H}}}} \geqslant \widetilde{\beta}_{\mathtt{d}} \|\boldsymbol{\eta}_h\|_{\widetilde{\mathbf{Q}}} \qquad \forall\, \boldsymbol{\eta}_h \in \widetilde{\mathbf{Q}}_h, \tag{5.19}$$

and

$$\|\mathbf{r}_h\|_{0,\Omega} \geqslant \widetilde{C}_{\mathtt{d}} \|\psi_h\|_{0,4;\Omega} \qquad \forall\, \vec{\psi}_h = (\psi_h, \mathbf{r}_h) \in \widetilde{\mathbf{V}}_h. \tag{5.20}$$

Note that (5.19) constitutes the discrete inf-sup condition for $\widetilde{b}$ required in [23, Proposition 2.42, eq. (2.36)]. In turn, given $\vec{\psi}_{j,h} = (\psi_{j,h}, \mathbf{r}_{j,h}) \in \widetilde{\mathbf{V}}_h$, we use (4.5) and (2.4), similarly to the first part of the derivation of (4.26), but then, differently from there, employ (5.20) to conclude that

$$\begin{aligned} \widetilde{a}_{j,\mathbf{w}_h}(\vec{\psi}_{j,h}, \vec{\psi}_{j,h}) &= \widetilde{a}_j(\vec{\psi}_{j,h}, \vec{\psi}_{j,h}) = \int_\Omega \mathbf{Q}_j\, |\mathbf{r}_{j,h}|^2 \geqslant C_j \|\mathbf{r}_{j,h}\|_{0,\Omega}^2, \\ &\geqslant \frac{C_j}{2} \Big\{ \widetilde{C}_{\mathtt{d}}^2 \|\psi_{j,h}\|_{0,4;\Omega}^2 + \|\mathbf{r}_{j,h}\|_{0,\Omega}^2 \Big\} \geqslant \widetilde{\alpha}_{j,\mathtt{d}} \|\vec{\psi}_{j,h}\|_{\widetilde{\mathbf{H}}}^2, \end{aligned} \tag{5.21}$$

with $\widetilde{\alpha}_{j,\mathtt{d}} := \frac{C_j}{2}\min\left\{\widetilde{C}_{\mathtt{d}}^2,\,1\right\}$. Then, analogously to the continuous case, it is readily seen that (5.21) yields the discrete inf-sup condition for $\widetilde{a}_{j,\mathbf{w}_h}$ required in [23, Proposition 2.42, eq. (2.35)].

We are now in position to state the discrete analogue of Theorem 4.2.

Theorem 5.2 *For each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, and for each $j \in \{1,2\}$, (5.7) (equivalently, (5.6)) has a unique solution $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) := \big((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{Q}}_h$, and hence one can define $\widetilde{\mathbf{S}}_{j,h}(\mathbf{w}_h) := \phi_{j,h}$. Moreover, there exists a positive constant $C_{\widetilde{\mathbf{S}},\mathtt{d}}$, depending only on $\widetilde{\alpha}_{j,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|i_4\|$, and $\mathtt{R}_j$, $j \in \big\{1,2\big\}$, such that*

$$
\begin{aligned}
\|\widetilde{\mathbf{S}}_h(\mathbf{w})\|_{0,4;\Omega} &:= \|\big(\widetilde{\mathbf{S}}_{1,h}(\mathbf{w}_h), \widetilde{\mathbf{S}}_{2,h}(\mathbf{w}_h)\big)\|_{0,4;\Omega} \\
&\leqslant C_{\widetilde{\mathbf{S}},\mathtt{d}} \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big) \|\phi_{j,D}\|_{1/2,\Gamma} \Big\}\,.
\end{aligned}
\tag{5.22}
$$

Proof. According to the previous discussion on $\widetilde{b}$ and $\widetilde{a}_{j,\mathbf{w}_h}$, for each $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$, and bearing in mind that $\widetilde{a}_{j,\mathbf{w}_h}|_{\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{H}}_h}$, $\widetilde{b}|_{\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{Q}}_h}$, $\widetilde{F}_j|_{\widetilde{\mathbf{H}}_h}$, and $\widetilde{G}_j|_{\widetilde{\mathbf{Q}}_h}$ are all bounded, the existence of a unique solution of (5.7), for each $j \in \big\{1,2\big\}$, follows from a direct application of [23, Proposition 2.42]. In turn, employing the discrete version of the first inequality in [23, Theorem 2.34, eq. (2.30)], we get the a priori estimate for $\|\widetilde{\mathbf{S}}_{j,h}(\mathbf{w}_h)\|_{0,4;\Omega}$, from which, summing up over $j \in \big\{1,2\big\}$, we arrive at (5.22). □

At this point we remark that, similarly as for the continuous case, the component $\boldsymbol{\vartheta}_{j,h}$ of the solution of (5.7) can be bounded employing the discrete version of the second inequality in [23, Theorem 2.34, eq. (2.30)], which yields

$$
\|\boldsymbol{\vartheta}_{j,h}\|_{\widetilde{\mathbf{Q}}} \leqslant \widetilde{M}_{j,\mathtt{d}}\big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big)\Big\{\|g_j\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big)\|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\,, \tag{5.23}
$$

where $\widetilde{M}_{j,\mathtt{d}}$ is a positive constant depending on $\widetilde{\alpha}_{j,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|i_4\|$, and $\mathtt{R}_j$.

Having established, thanks to Theorems 5.1 and 5.2, that $\mathbf{S}_h$ (cf. (5.3)), $\widetilde{\mathbf{S}}_h$ (cf. (5.8)), and hence $\mathbf{T}_h$ (cf. (5.9)), are well-defined, we now aim to show that $\mathbf{T}_h$ has a unique fixed-point. More precisely, analogously to the continuous case, in what follows we prove that $\mathbf{T}_h$ verifies the hypotheses of the Banach theorem. For this purpose, given $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in (5.14), we first follow (5.12) and define

$$
\mathbf{W}_h(r) := \Big\{\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}: \quad \|\mathbf{w}_h\|_{0,4;\Omega} \leqslant r\Big\}\,. \tag{5.24}
$$

Then, using now the a priori estimates (5.15) and (5.22), we easily deduce the existence of a positive constant $C_{\mathbf{T},\mathtt{d}}$, depending only on $C_{\mathbf{S},\mathtt{d}}$ and $C_{\widetilde{\mathbf{S}},\mathtt{d}}$, such that for each $\mathbf{w}_h \in \mathbf{W}_h(r)$ there holds

$$
\|\mathbf{T}_h(\mathbf{w}_h)\|_{0,4;\Omega} \leqslant C_{\mathbf{T},\mathtt{d}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathtt{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2}\Big\{\|g_j\|_{0,4/3;\Omega} + \big(1+r\big)\|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\right\},
$$

which constitutes the discrete version of (4.31). Hence, we are able to state next the discrete analogue of Lemma 4.3

Lemma 5.3 *Given $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in (5.14), assume that the data satisfy*

$$
C_{\mathbf{T},\mathtt{d}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathtt{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2}\Big\{\|g_j\|_{0,4/3;\Omega} + \big(1+r\big)\|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\right\} \leqslant r\,. \tag{5.25}
$$

Then, there holds $\mathbf{T}_h\big(\mathbf{W}_h(r)\big) \subseteq \mathbf{W}_h(r)$.

In turn, employing similar arguments to those yielding Lemmas 4.5, 4.6, and 4.7, we are able to show their discrete counterparts, that is the continuity properties of $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, and $\mathbf{T}_h$. However, being the respective proofs almost verbatim to the continuous ones, we omit the details and just state the corresponding results as follows.

Lemma 5.4 *Let $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in* (5.14)*. Then, there exists a positive constant $L_{\mathbf{S},\mathtt{d}}$, depending only on $\alpha_{\mathbf{A},\mathtt{d}}$, $\|\mathbf{g}\|_{0,\Omega}$, μ_0, L_c (cf. Lemma 4.4), and r, such that*

$$\|\mathbf{S}_h(\mathbf{w}_h, \boldsymbol{\varphi}_h) - \mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\xi}_h)\|_{0,4;\Omega} \leqslant L_{\mathbf{S},\mathtt{d}} \Big\{ \|\boldsymbol{\varphi}_h - \boldsymbol{\xi}_h\|_{0,4;\Omega} + \|\mathbf{S}_h(\mathbf{z}_h, \boldsymbol{\xi}_h)\|_{0,4;\Omega} \, \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega} \Big\}$$

for all $(\mathbf{w}_h, \boldsymbol{\varphi}_h)$, $(\mathbf{z}_h, \boldsymbol{\xi}_h) \in \mathbf{W}_h(r) \times \big(\mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\phi}\big)$.

Lemma 5.5 *Let $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in* (5.14)*. Then, there exists a positive constant $L_{\widetilde{\mathbf{S}},\mathtt{d}}$, depending only on $\widetilde{C}_j$ (cf.* (4.28)*), $\mathtt{R}_j$, and $\widetilde{\alpha}_{j,\mathtt{d}}$, $j \in \{1,2\}$, such that*

$$\begin{aligned}
&\|\widetilde{\mathbf{S}}_h(\mathbf{w}_h) - \widetilde{\mathbf{S}}_h(\mathbf{z}_h)\|_{0,4;\Omega} \\
&\qquad \leqslant L_{\widetilde{\mathbf{S}},\mathtt{d}} \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{w}_h\|_{0,4;\Omega}\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \, \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}
\end{aligned}$$

for all $\mathbf{w}_h$, $\mathbf{z}_h \in \mathbf{W}_h(r)$.

Thanks to Lemmas 5.4 and 5.5, the Lipschitz continuity of $\mathbf{T}_h$ (cf. (5.9)) is stated as follows.

Lemma 5.6 *Let $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in* (5.14)*. Then, there exists a positive constant $L_{\mathbf{T},\mathtt{d}}$, depending only on $L_{\mathbf{S},\mathtt{d}}$, $L_{\widetilde{\mathbf{S}},\mathtt{d}}$, $C_{\mathbf{S},\mathtt{d}}$, and $C_{\widetilde{\mathbf{S}},\mathtt{d}}$, such that*

$$\begin{aligned}
&\|\mathbf{T}_h(\mathbf{w}_h) - \mathbf{T}_h(\mathbf{z}_h)\|_{0,4;\Omega} \\
&\quad \leqslant L_{\mathbf{T},\mathtt{d}} \bigg\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathtt{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \bigg\} \|\mathbf{w}_h - \mathbf{z}_h\|_{0,4;\Omega}
\end{aligned}$$

for all $\mathbf{w}_h$, $\mathbf{z}_h \in \mathbf{W}_h(r)$.

We end this section with the solvability result for (5.10) (and hence for (5.2)).

Theorem 5.7 *Given $r \in (0, r_{0,\mathtt{d}}]$, with $r_{0,\mathtt{d}}$ as in* (5.14)*, assume that, in addition to the hypothesis of Lemma* 5.3 *(cf.* (5.25)*), the data satisfy*

$$L_{\mathbf{T},\mathtt{d}} \bigg\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathtt{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2} \Big\{ \|g_j\|_{0,4/3;\Omega} + \big(1 + r\big) \, \|\phi_{j,D}\|_{1/2,\Gamma} \Big\} \bigg\} < 1 \,. \tag{5.26}$$

Then, there exists a unique $\mathbf{u}_h \in \mathbf{W}_h(r)$ (cf. (5.24)*) fixed point of $\mathbf{T}_h$ (cf.* (5.10)*). Equivalently,* (5.2) *has a unique solution $(\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h) := \big(\boldsymbol{\sigma}_h, (\mathbf{u}_h, \gamma_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $(\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h}) := \big((\phi_{j,h}, \mathbf{t}_{j,h}), \boldsymbol{\vartheta}_{j,h}\big) \in \widetilde{\mathbf{H}}_h \times \widetilde{\mathbf{Q}}_h$, $j \in \{1,2\}$, with $\mathbf{u}_h \in \mathbf{W}_h(r)$. Moreover, there exist positive constants $\mathcal{C}_{\mathtt{d}}$, $\mathcal{C}_{1,\mathtt{d}}$, and $\mathcal{C}_{2,\mathtt{d}}$,* *depending on* $\widetilde{\alpha}_{1,\mathtt{d}}$, $\widetilde{\alpha}_{2,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\mathbf{Q}_1\|_{0,\infty;\Omega}$, $\|\mathbf{Q}_2\|_{0,\infty;\Omega}$, $\|i_4\|$, $\mathtt{R}_1$, $\mathtt{R}_2$, r, $\|\boldsymbol{i}_4\|$, $\|\mathbf{g}\|_{0,\Omega}$, *and* $\alpha_{\mathbf{A},\mathtt{d}}$, *such that there hold the following a priori bounds*

$$\|(\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \mathcal{C}_{\mathtt{d}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2}\left\{\|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma}\right\}\right\}, \tag{5.27}$$

and for each $j \in \{1,2\}$

$$\|(\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} \,\leqslant\, \mathcal{C}_{j,\mathtt{d}}\left\{\|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma}\right\}. \tag{5.28}$$

Proof. It proceeds analogously to the proof of Theorem 4.8. Indeed, since $\mathbf{T}_h$ is a contraction that maps the ball $\mathbf{W}_h(r)$ into itself, which is consequence of Lemma 5.3 and assumption (5.26), a direct application of the Banach fixed-point theorem confirms the solvabilities of (5.10) and (5.2). In turn, noting that $\mathbf{u}_h = \mathbf{T}_h(\mathbf{u}_h) = \mathbf{S}_h\big(\mathbf{u}_h,\widetilde{\mathbf{S}}_h(\mathbf{u}_h)\big)$ and $\boldsymbol{\phi}_h := (\phi_{1,h},\phi_{2,h}) = \widetilde{\mathbf{S}}_h(\mathbf{u}_h)$, the a priori estimates (5.27) and (5.28) follow from (5.15), (5.22), and (5.23). $\square$

5.4 A priori error analysis

In this section we consider arbitrary finite element subspaces satisfying the hypotheses (**H.1**) up to (**H.6**) introduced in Section 5.3, and derive the Céa estimate for the global error

$$\|(\boldsymbol{\sigma},\vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sum_{j=1}^{2}\,\|(\vec{\phi}_j,\boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}}\,,$$

where $\big((\boldsymbol{\sigma},\vec{\mathbf{u}}),(\vec{\phi}_j,\boldsymbol{\vartheta}_j)\big) := \big((\boldsymbol{\sigma},(\mathbf{u},\gamma)),((\phi_j,\mathbf{t}_j),\boldsymbol{\vartheta}_j)\big) \in \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}\big)$, $j\in\{1,2\}$, with $\mathbf{u}\in\mathbf{W}(r)$, is the unique solution of (3.24), and $\big((\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h),(\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\big) := \big((\boldsymbol{\sigma}_h,(\mathbf{u}_h,\gamma_h)),((\phi_{j,h},\mathbf{t}_{j,h}),\boldsymbol{\vartheta}_{j,h})\big) \in \big(\mathbf{H}_h\times\mathbf{Q}_h\big)\times\big(\widetilde{\mathbf{H}}_h\times\widetilde{\mathbf{Q}}_h\big)$, $j\in\{1,2\}$, with $\mathbf{u}_h\in\mathbf{W}_h(r)$, is the unique solution of (5.2). To this end, in what follows we apply known Strang-type estimates to the pairs of associated continuous and discrete schemes arising from (3.24) and (5.2), once they are split according to the two decoupled problems. Hereafter, given a subspace X_h of an arbitrary Banach space $(X,\|\cdot\|_X)$, we set

$$\mathrm{dist}(x,X_h) \,:=\, \inf_{x_h\in X_h}\|x-x_h\|_X\,.$$

We begin the analysis with the first two equations of (3.24) and (5.2), which can be rewritten as

$$\begin{aligned}
\mathcal{A}\big((\boldsymbol{\sigma},\vec{\mathbf{u}}),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \quad &= \quad \mathcal{F}\big((\boldsymbol{\tau},\vec{\mathbf{v}})\big) \qquad\quad \forall\,(\boldsymbol{\tau},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\,,\quad \text{and}\\
\mathcal{A}_h\big((\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h),(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\big) \quad &= \quad \mathcal{F}_h\big((\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\big) \qquad \forall\,(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}\big((\boldsymbol{\zeta},\vec{\mathbf{w}}),(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \quad &:= \quad \mathbf{A}_{\mathbf{u}}((\boldsymbol{\zeta},\vec{\mathbf{w}}),(\boldsymbol{\tau},\vec{\mathbf{v}})) \,+\, \mathbf{b}(\mathbf{u};\mathbf{w},\boldsymbol{\tau})\,,\\
\mathcal{A}_h\big((\boldsymbol{\zeta}_h,\vec{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\big) \quad &:= \quad \mathbf{A}_{\mathbf{u}_h}((\boldsymbol{\zeta}_h,\vec{\mathbf{w}}_h),(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)) \,+\, \mathbf{b}(\mathbf{u}_h;\mathbf{w}_h,\boldsymbol{\tau}_h)\,,\\
\mathcal{F}\big(\boldsymbol{\tau},\vec{\mathbf{v}})\big) \quad &:= \quad \mathbf{G}(\boldsymbol{\tau}) \,+\, \mathbf{F}_{\boldsymbol{\phi}}(\vec{\mathbf{v}})\,,\quad \text{and}\\
\mathcal{F}_h\big(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\big) \quad &:= \quad \mathbf{G}(\boldsymbol{\tau}_h) \,+\, \mathbf{F}_{\boldsymbol{\phi}_h}(\vec{\mathbf{v}}_h)\,,
\end{aligned}$$

for all $(\boldsymbol{\zeta},\vec{\mathbf{w}})$, $(\boldsymbol{\tau},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}$, for all $(\boldsymbol{\zeta}_h,\vec{\mathbf{w}}_h)$, $(\boldsymbol{\tau}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$. Then, applying the a priori error bound provided by [17, Lemma 5.1], and then suitably bounding the resulting consistency estimate, which is given by $\|\mathcal{A}\big((\boldsymbol{\sigma},\vec{\mathbf{u}}),(\cdot,\cdot)\big) - \mathcal{A}_h\big((\boldsymbol{\sigma},\vec{\mathbf{u}}),(\cdot,\cdot)\big)\|_{\left(\mathbf{H}_h\times\mathbf{Q}_h\right)'}$, we deduce the existence of a positive constant $\mathcal{C}_{\mathtt{ST}}$, depending only on $\|\mathcal{A}\|$, $\|\mathcal{A}_h\|$, and $\alpha_{\mathbf{A},\mathtt{d}}$, and thus, easily shown to be independent of h, such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \mathcal{C}_{\mathtt{ST}}\Big\{&\mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{H}_h) \,+\, \mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{Q}_h) \,+\, \|\mathbf{F}_{\boldsymbol{\phi}} - \mathbf{F}_{\boldsymbol{\phi}_h}\|_{\mathbf{Q}_h'} \\ &+\, \|\mathbf{b}(\mathbf{u};\mathbf{u},\cdot) - \mathbf{b}(\mathbf{u}_h;\mathbf{u},\cdot)\|_{\mathbf{H}_h'} \,+\, \|c(\mathbf{u};\vec{\mathbf{u}},\cdot) - c(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{Q}_h'}\Big\}\,.\end{aligned} \tag{5.29}$$

Note, in particular, that $\|\mathcal{A}\|$ depends proportionally on $\|a\| = \frac{1}{\mu_0}$, $\|b\| = 1$, $\|\mathbf{b}(\mathbf{u},\cdot,\cdot)\| = \frac{1}{\mu_0}\,\|\mathbf{u}\|_{0,4;\Omega}$, and $\|c(\mathbf{u};\cdot,\cdot)\| = |\Omega|^{1/2}\left(\mathtt{D} + \mathtt{F}\|\mathbf{u}\|_{0,4;\Omega}^{\rho-2}\right)$, with $\|\mathbf{u}\|_{0,4;\Omega}$ bounded by r. An analogue remark is valid for $\|\mathcal{A}_h\|$. Next, proceeding as for the derivation of (4.42), we readily obtain

$$\|\mathbf{F}_{\boldsymbol{\phi}} - \mathbf{F}_{\boldsymbol{\phi}_h}\|_{\mathbf{Q}_h'} \,\leqslant\, \|\mathbf{g}\|_{0,\Omega}\,\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,4;\Omega}\,. \tag{5.30}$$

In turn, bearing in mind the definition of $\mathbf{b}$ (cf. (3.11)), we find that for each $\boldsymbol{\tau}_h \in \mathbf{H}_h$ there holds

$$\mathbf{b}(\mathbf{u};\mathbf{u},\boldsymbol{\tau}_h) - \mathbf{b}(\mathbf{u}_h;\mathbf{u},\boldsymbol{\tau}_h) \,=\, \mathbf{b}(\mathbf{u}-\mathbf{u}_h;\mathbf{u},\boldsymbol{\tau}_h)\,,$$

from which, employing the boundedness property of $\mathbf{b}$ (cf. first row of (4.3)), we conclude that

$$\|\mathbf{b}(\mathbf{u};\mathbf{u},\cdot) - \mathbf{b}(\mathbf{u}_h;\mathbf{u},\cdot)\|_{\mathbf{H}_h'} \,\leqslant\, \frac{1}{\mu_0}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,. \tag{5.31}$$

Similarly, using the continuity property of c provided by (4.33) (cf. Lemma 4.4), we get

$$\begin{aligned}\|c(\mathbf{u};\vec{\mathbf{u}},\cdot) - c(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{Q}_h'} \,&\leqslant\, L_c\Big\{\|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{u}_h\|_{0,4;\Omega}\Big\}^{\rho-3}\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} \\ &\leqslant\, L_c\,\big(2r\big)^{\rho-3}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,.\end{aligned} \tag{5.32}$$

In this way, replacing the bounds given by (5.30), (5.31), and (5.32), back into (5.29), we arrive at

$$\begin{aligned}\|(\boldsymbol{\sigma},\vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \widehat{\mathcal{C}}_{\mathtt{ST}}\Big\{&\mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{H}_h) \,+\, \mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{Q}_h) \\ &+\, \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{0,4;\Omega} \,+\, \|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,,\end{aligned} \tag{5.33}$$

where $\widehat{\mathcal{C}}_{\mathtt{ST}}$ is a positive constant depending only on $\mathcal{C}_{\mathtt{ST}}$, $\|\mathbf{g}\|_{0,\Omega}$, μ_0, L_c, r, and ρ.

On the other hand, proceeding analogously with the third and fourth equations of (3.24) and (5.2), but using now the particular Strang-type estimate provided by [20, Lemma 6.1] (see also [7, Lemma 5.1] for a slightly more general result), we deduce, for each $j \in \{1,2\}$, the existence of a positive constant $\mathcal{C}_{j,\mathtt{ST}}$ depending only on $\widetilde{\alpha}_{j,\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\|\widetilde{a}_j\| = \|\mathbf{Q}_j\|_{0,\infty;\Omega}$, $\|\widetilde{c}_j(\mathbf{u};\cdot,\cdot)\| = \mathtt{R}_j\,\|\mathbf{u}\|_{0,4;\Omega}$, $\|\widetilde{c}_j(\mathbf{u}_h;\cdot,\cdot)\| = \mathtt{R}_j\,\|\mathbf{u}_h\|_{0,4;\Omega}$, and $\|\widetilde{b}\| = 2$, and hence, easily shown to be independent of h, such that

$$\begin{aligned}\|(\vec{\phi}_j,\boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} \,\leqslant\, \mathcal{C}_{j,\mathtt{ST}}\Big\{&\mathrm{dist}\,(\vec{\phi}_j,\widetilde{\mathbf{H}}_h) \,+\, \mathrm{dist}\,(\boldsymbol{\vartheta}_j,\widetilde{\mathbf{Q}}_h) \\ &+\, \|\widetilde{c}_j(\mathbf{u};\vec{\phi}_j,\cdot) - \widetilde{c}_j(\mathbf{u}_h;\vec{\phi}_j,\cdot)\|_{\widetilde{\mathbf{H}}_h'}\Big\}\,.\end{aligned} \tag{5.34}$$

Now, bearing in mind the definition of $\widetilde{c}_j$ (cf. (3.23)), we obtain for each $\vec{\psi}_{j,h} \in \widetilde{\mathbf{H}}_h$

$$\widetilde{c}_j(\mathbf{u};\vec{\phi}_j,\vec{\psi}_{j,h}) - \widetilde{c}_j(\mathbf{u}_h;\vec{\phi}_j,\vec{\psi}_{j,h}) \,=\, \widetilde{c}_j(\mathbf{u}-\mathbf{u}_h;\vec{\phi}_j,\vec{\psi}_{j,h})\,,$$

from which, using the boundedness property of $\widetilde{c}_j$ (cf. third row of (4.3)), we deduce that

$$\|\widetilde{c}_j(\mathbf{u};\vec{\phi}_j,\cdot) - \widetilde{c}_j(\mathbf{u}_h;\vec{\phi}_j,\cdot)\|_{\widetilde{\mathbf{H}}'_h} \leqslant \mathtt{R}_j\,\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,$$

so that (5.34) becomes

$$\|(\vec{\phi}_j,\boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} \leqslant \widehat{\mathcal{C}}_{j,\mathtt{ST}}\Big\{\mathrm{dist}\,(\vec{\phi}_j,\widetilde{\mathbf{H}}_h) + \mathrm{dist}\,(\boldsymbol{\vartheta}_j,\widetilde{\mathbf{Q}}_h) + \|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,, \tag{5.35}$$

where $\widehat{\mathcal{C}}_{j,\mathtt{ST}}$ is a positive constant depending only on $\mathcal{C}_{j,\mathtt{ST}}$ and $\mathtt{R}_j$.

We now proceed to suitably combine (5.33) and (5.35) to derive the final Céa estimate. Indeed, multiplying (5.33) by $\frac{1}{2\,\widehat{\mathcal{C}}_{\mathtt{ST}}}$, summing up in (5.35) over $j\in\{1,2\}$, adding the resulting inequalities, bounding $\|\vec{\mathbf{u}}\|_{0,4;\Omega}$ and $\|\vec{\phi}_j\|_{\widetilde{\mathbf{H}}}$ by the right hand sides of (4.52) and (4.53), respectively, and then performing some algebraic manipulations, we find that

$$\begin{aligned}
\frac{1}{2\,\widehat{\mathcal{C}}_{\mathtt{ST}}}\,\|(\boldsymbol{\sigma},\vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \sum_{j=1}^{2}\|(\vec{\phi}_j,\boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} &\leqslant \frac{1}{2}\,\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{0,4;\Omega}\\
+\frac{1}{2}\Big\{\mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{H}_h) + \mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{Q}_h)\Big\} + \sum_{j=1}^{2}\widehat{\mathcal{C}}_{j,\mathtt{ST}}\Big\{\mathrm{dist}\,(\vec{\phi}_j,\widetilde{\mathbf{H}}_h) + \mathrm{dist}\,(\boldsymbol{\vartheta}_j,\widetilde{\mathbf{Q}}_h)\Big\}&\\
+\widehat{\mathcal{C}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2}\Big\{\|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,&
\end{aligned} \tag{5.36}$$

where $\widehat{\mathcal{C}}$ is a positive constant depending only on $\mathcal{C}$ (cf. (4.52)), $\mathcal{C}_j$ (cf. (4.53)), and $\widehat{\mathcal{C}}_{j,\mathtt{ST}}$, $j\in\{1,2\}$.

Having established (5.36), the announced Céa estimate is stated as follows.

Theorem 5.8 *Assume that the data satisfy*

$$\widehat{\mathcal{C}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{\phi}_{\mathbf{r}}\|_{0,4;\Omega} + \sum_{j=1}^{2}\Big\{\|g_j\|_{0,4/3;\Omega} + \|\phi_{j,D}\|_{1/2,\Gamma}\Big\}\Big\} \leqslant \frac{1}{4\,\widehat{\mathcal{C}}_{\mathtt{ST}}}\,. \tag{5.37}$$

Then, there exists a positive constant $\widetilde{\mathcal{C}}$, depending only on $\widehat{\mathcal{C}}_{\mathtt{ST}}$ and $\widehat{\mathcal{C}}_{j,\mathtt{ST}}$, $j\in\{1,2\}$, and hence, independent of h, such that

$$\begin{aligned}
&\|(\boldsymbol{\sigma},\vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \sum_{j=1}^{2}\|(\vec{\phi}_j,\boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h},\boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}}\\
&\quad\leqslant \widetilde{\mathcal{C}}\Big\{\mathrm{dist}\,(\boldsymbol{\sigma},\mathbf{H}_h) + \mathrm{dist}\,(\vec{\mathbf{u}},\mathbf{Q}_h) + \sum_{j=1}^{2}\Big\{\mathrm{dist}\,(\vec{\phi}_j,\widetilde{\mathbf{H}}_h) + \mathrm{dist}\,(\boldsymbol{\vartheta}_j,\widetilde{\mathbf{Q}}_h)\Big\}\Big\}\,.
\end{aligned} \tag{5.38}$$

Proof. It follows directly from (5.36) after realizing that the first term on its right hand side can be subtracted from the second one on the left hand side, whereas, under (5.37), a similar procedure applies to the corresponding last and first terms. □

Furthermore, as suggested by (2.8), (2.14), and (3.7), we can approximate the pressure p, the velocity gradient $\nabla\mathbf{u}$, and the shear stress tensor $\widetilde{\boldsymbol{\sigma}}$, by the following postprocessing formulae:

$$p_h := -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma}_h + (\mathbf{u}_h\otimes\mathbf{u}_h)\big) + \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\,, \tag{5.39}$$

$$(\nabla \mathbf{u})_h \,:=\, \frac{1}{\mu}\,\boldsymbol{\sigma}_h^{\mathrm{d}} + \frac{1}{\mu}\,(\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} \,+\, \boldsymbol{\gamma}_h\,, \quad \text{and} \tag{5.40}$$

$$\widetilde{\boldsymbol{\sigma}}_h \,:=\, \boldsymbol{\sigma}_h \,-\, \left(\frac{1}{n\,|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}_h \otimes \mathbf{u}_h)\right)\mathbb{I} \,+\, (\mathbf{u}_h \otimes \mathbf{u}_h)\,. \tag{5.41}$$

Thus, is is not difficult to show that there exists a positive constant c, independent of h, though depending either on r or the data providing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, such that

$$\begin{aligned} &\|p - p_h\|_{0,\Omega} \,+\, \|\nabla\mathbf{u} - (\nabla\mathbf{u})_h\|_{0,\Omega} \,+\, \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\ &\quad \leqslant\, c\,\Big\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \,+\, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}\Big\}\,, \end{aligned} \tag{5.42}$$

which, certainly, is bounded by the right hand side of (5.38) as well.

We highlight here in advance that, as a consequence of (5.42), and as it will be established at the end of Section 6 by Theorem 6.1, the approximations of p, $\nabla\mathbf{u}$, and $\widetilde{\boldsymbol{\sigma}}$ provided by (5.39), (5.40), and (5.41), share the same rates of convergence of $\boldsymbol{\sigma}_h$, $\mathbf{u}_h$, and $\boldsymbol{\gamma}_h$, which certainly emphasizes further the advantageous of having introduced $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ as auxiliary unknowns of our mixed formulation.

6 Specific finite element subspaces

In this section we resort to [29, Section 4.4] and [28, Section 4.5] to specify two examples of finite element subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, H_h^{ϕ}, $\mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\vartheta}}$ satisfying the hypotheses (**H.1**) up to (**H.6**) stated in Section 5.3, and then establish the associated rates of convergence for the Galerkin scheme (5.2). Although it will become clear below, we remark in advance that the two aforementioned examples are actually determined by two possible choices for the first three subspaces since the remaining three are kept the same in both cases.

6.1 Preliminaries

Given an integer $k \geqslant 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of degree at most k defined on K with vector and tensorial counterparts $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^n$ and $\mathbb{P}_k(K) := [\mathrm{P}_k(K)]^{n\times n}$, respectively. In addition, we let $\mathbf{RT}_k(K) := \mathbf{P}_k(K) + \mathrm{P}_k(K)\mathbf{x}$ be the local Raviart–Thomas space of order k defined on K, where $\mathbf{x}$ stands for a generic vector in R^n. Furthermore, denoting by b_K the bubble function on K, which is given by the product of its $n+1$ barycentric coordinates, we set the local bubble space of order k as

$$\begin{aligned} \mathbf{B}_k(K) \,&:=\, \operatorname{curl}(b_K\,\mathrm{P}_k(K)) \quad \text{if} \quad n \,=\, 2\,, \quad \text{and} \\ \mathbf{B}_k(K) \,&:=\, \operatorname{curl}(b_K\,\mathbf{P}_k(K)) \quad \text{if} \quad n \,=\, 3\,, \end{aligned} \tag{6.1}$$

where $\operatorname{curl}(v) := (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$ if $n = 2$ and $v : K \to \mathrm{R}$, and $\operatorname{curl}(\mathbf{v}) = \nabla \times \mathbf{v}$ if $n = 3$ and $\mathbf{v} : K \to \mathrm{R}^3$. Next, we introduce the global spaces

$$
\begin{aligned}
\mathrm{P}_k(\Omega) &:= \Big\{ v_h \in \mathrm{L}^2(\Omega): \quad v_h|_K \in \mathrm{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbf{P}_k(\Omega) &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbb{P}_k(\Omega) &:= \Big\{ \boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega): \quad \boldsymbol{\delta}_h|_K \in \mathbb{P}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbf{RT}_k(\Omega) &:= \Big\{ \boldsymbol{\eta}_h \in \mathbf{H}(\mathbf{div};\Omega): \quad \boldsymbol{\eta}_h|_K \in \mathbf{RT}_k(K)\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbb{RT}_k(\Omega) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_k(K)\,, \quad \forall\, i \in \{1,\dots,n\}\,, \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbb{B}_k(\Omega) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{B}_k(K)\,, \quad \forall\, i \in \{1,\dots,n\}\,, \forall\, K \in \mathcal{T}_h \Big\}\,,
\end{aligned}
$$

where $\boldsymbol{\tau}_{h,i}$ denotes the ith-row of $\boldsymbol{\tau}_h$. It is clear that $\mathrm{P}_k(\Omega)$ and $\mathbf{P}_k(\Omega)$ are also subspaces of $\mathrm{L}^4(\Omega)$ and $\mathbf{L}^4(\Omega)$, respectively. In addition, being $\mathbf{H}(\mathrm{div};\Omega)$ and $\mathbb{H}(\mathbf{div};\Omega)$ contained in $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$ and $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, respectively, we notice that the spaces $\mathbb{RT}_k(\Omega)$ and $\mathbb{B}_k(\Omega)$ are both subspaces of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well, whereas $\mathbf{RT}_k(\Omega)$ is contained in $\mathbf{H}(\mathrm{div}_{4/3};\Omega)$.

6.2 Two examples

To begin with, we proceed as in [27, Section 4.4] and [28, Section 4.5], and employ the stable triplets for linear elasticity derived in [29, Section 4.4], to define two examples of finite element subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, satisfying (**H.1**) up to (**H.3**). In what follows, ℓ is a non-negative integer.

The first example for $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, is PEERS$_\ell$, the plane elasticity element with reduced symmetry of order $\ell \geqslant 0$, whose stability for the mixed finite element formulation of the linear elasticity problem, within the classical Hilbertian framework, was originally established in [4] for $\ell = 0$ and $n = 2$, and later on proved for $\ell \geqslant 0$ and $n \in \{2,3\}$ in [35]. The corresponding subspaces are defined as follows:

$$
\begin{gathered}
\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega)\,, \qquad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and}\\
\mathbb{H}_h^{\gamma} := [C(\bar{\Omega})]^{n\times n} \cap \mathbb{L}^2_{\text{skew}}(\Omega) \cap \mathbb{P}_{\ell+1}(\Omega)\,.
\end{gathered}
\tag{6.2}
$$

The second example for $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, is AFW$_\ell$, the Arnold-Falk-Winther element of order $\ell \geqslant 0$, whose corresponding aforementioned stability can be found in [5]. In this case, the subspaces are given by:

$$
\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)\,, \qquad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and} \quad \mathbb{H}_h^{\gamma} := \mathbb{L}^2_{\text{skew}}(\Omega) \cap \mathbb{P}_\ell(\Omega)\,. \tag{6.3}
$$

On purpose of the above, and in order to complement a corresponding remark from Section 3, we stress here that, in the case of a symmetric stress-displacement formulation, the first finite elements that are known to be stable in 2D for linear elasticity, that is that satisfy the discrete inf-sup condition for the resulting bilinear form b, are those provided in [6]. The one of lowest order there consists of piecewise cubic polynomials for the stress, with 24 degrees of freedom per triangle, and piecewise linear functions for the displacement. In turn, an analogue of the above in 3D was proposed in [1], which employs piecewise quartic stresses with 162 degrees of freedom per tetrahedron, and piecewise linear displacements. Thus, it is easy to see that the lowest order ones from (6.2) and (6.3), which are obtained for $\ell = 0$, involve much less local degrees of freedom.

Regarding the verification of the hypotheses by the subspaces specified in (6.2) and (6.3), we first observe that (**H.1**) is clearly satisfied in both cases. The same holds with (**H.2**) since $\mathbf{div}\big(\mathbb{RT}_\ell(\Omega)\big)$ and $\mathbf{div}\big(\mathbb{P}_{\ell+1}(\Omega)\big)$ are contained in $\mathbf{P}_\ell(\Omega)$, which coincides with $\mathbf{H}_h^{\mathbf{u}}$ in the two examples, whereas, according to (6.1), the tensors in $\mathbb{B}_\ell(\Omega)$ are divergence-free. In turn, we recall that the discrete inf-sup condition for b required in the assumption (**H.3**), was proved in [29, Lemma 4.8] for PEERS_ℓ as well as for AFW_ℓ. We omit further details and refer to the analysis developed in [29, Section 4.4.2].

On the other hand, specific finite element subspaces H_h^ϕ, $\mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\vartheta}}$, are set as follows:

$$\mathrm{H}_h^\phi := \mathrm{P}_\ell(\Omega)\,, \quad \mathbf{H}_h^{\mathbf{t}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and} \quad \mathbf{H}_h^{\boldsymbol{\vartheta}} := \mathbf{RT}_\ell(\Omega)\,. \tag{6.4}$$

Similarly as a previous remark, the fact that $\mathrm{div}\big(\mathbf{RT}_\ell(\Omega)\big)$ is contained in $\mathrm{P}_\ell(\Omega) = \mathrm{H}_h^\phi$, guarantees that (**H.4**) is satisfied. In addition, knowing from (5.17) that, besides being contained in $\mathbf{H}_h^{\boldsymbol{\vartheta}} = \mathbf{RT}_\ell(\Omega)$, the vector functions of $\mathrm{Z}_{0,h}$ are divergence-free, we deduce, from a particular argument provided in the proof of [25, Theorem 3.3], that $\mathrm{Z}_{0,h} \subseteq \mathbf{P}_\ell(\Omega)$, which confirms (**H.5**). Finally, the discrete inf-sup condition required by (**H.6**), which coincides with [20, eq. (5.64)], is basically proved in the last part of [20, Section 5.5] by realizing that it reduces to the vector version of [20, Lemma 5.5, eq. (5.45)].

6.3 The rates of convergence

In this section we first collect the approximation properties of the finite element spaces defined in Section 6.2, and then establish the associated rates of convergence of the Galerkin scheme (5.2).

We begin with the approximation properties of PEERS_ℓ (cf. (6.2)) and AFW_ℓ (cf. (6.3)), which basically follow from the analogue properties of the Raviart–Thomas and AFW interpolation operators, and of the orthogonal projectors $\boldsymbol{\mathcal{P}}_h^\ell : \mathbf{L}^1(\Omega) \to \mathbf{P}_\ell(\Omega)$ and $\mathbb{P}_h^\ell : \mathbb{L}^1(\Omega) \to \mathbb{P}_\ell(\Omega)$ (cf. [23, Proposition 1.135]), along with the use of the commuting diagram properties and of the interpolation estimates of Sobolev spaces. They read as follows (cf. [29, Section 4.4.3], [27, Section 4.4.4]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists a positive constant C, independent of h, such that for each $s \in (0, \ell+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{s,4/3}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\tau}, \mathbf{H}_h) \,\leqslant\, C\,h^s \left\{ \|\boldsymbol{\tau}\|_{s,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{s,4/3;\Omega} \right\},$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists a positive constant C, independent of h, such that for each $s \in [0, \ell+1]$, and for each $\mathbf{v} \in \mathbf{W}^{s,4}(\Omega)$, there holds

$$\mathrm{dist}\,(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \,\leqslant\, C\,h^s\,\|\mathbf{v}\|_{s,4;\Omega}\,, \quad \text{and}$$

($\mathbf{AP}_h^{\boldsymbol{\gamma}}$) there exists a positive constant C, independent of h, such that for each $s \in [0, \ell+1]$, and for each $\boldsymbol{\delta} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\text{skew}}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\delta}, \mathbb{H}_h^{\boldsymbol{\gamma}}) \,\leqslant\, C\,h^s\,\|\boldsymbol{\delta}\|_{s,\Omega}\,.$$

Furthermore, regarding the approximation properties of the subspaces defined in (6.4), they are given as indicated next:

($\mathbf{AP}_h^\phi$) there exists a positive constant C, independent of h, such that for each $s \in [0, \ell+1]$, and for each $\psi \in \mathrm{W}^{s,4}(\Omega)$, there holds

$$\mathrm{dist}\,(\psi, \mathrm{H}_h^\phi) \,\leqslant\, C\,h^s\,\|\psi\|_{s,4;\Omega}\,,$$

($\mathbf{AP}_h^{\mathbf{t}}$) there exists a positive constant C, independent of h, such that for each $s \in [0, \ell+1]$, and for each $\mathbf{r} \in \mathbf{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{r}, \mathbf{H}_h^{\mathbf{t}}) \leqslant C\, h^s \, \|\mathbf{r}\|_{s;\Omega}\,, \quad \text{and}$$

($\mathbf{AP}_h^{\boldsymbol{\vartheta}}$) there exists a positive constant C, independent of h, such that for each $s \in (0, \ell+1]$, and for each $\boldsymbol{\eta} \in \mathbf{H}^s(\Omega) \cap \mathbf{H}(\operatorname{div}_{4/3};\Omega)$, with $\operatorname{div}(\boldsymbol{\eta}) \in \mathrm{W}^{s,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\eta}, \mathbf{H}_h^{\boldsymbol{\vartheta}}) \leqslant C\, h^s \Big\{ \|\boldsymbol{\eta}\|_{s,\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\eta})\|_{s,4/3;\Omega} \Big\}.$$

In this way, as a consequence of Theorem 5.8, (5.42), and the approximation properties ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^{\boldsymbol{\gamma}}$), ($\mathbf{AP}_h^{\phi}$), ($\mathbf{AP}_h^{\mathbf{t}}$), and ($\mathbf{AP}_h^{\boldsymbol{\vartheta}}$), we conclude the rates of convergence of the Galerkin Scheme (5.2) with the finite element subspaces defined in Section 6.2. More precisely, we have the following result.

Theorem 6.1 *In addition to the hypotheses of Theorems* 4.8, 5.7, *and* 5.8, *assume that there exists* $s \in (0, \ell+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{s,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{s,4}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathrm{skew}}(\Omega)$, $\phi_j \in \mathrm{W}^{s,4}(\Omega)$, $\mathbf{t}_j \in \mathbf{H}^s(\Omega)$, $\boldsymbol{\vartheta}_j \in \mathbf{H}^s(\Omega) \cap \mathbf{H}(\operatorname{div}_{4/3};\Omega)$, *and* $\operatorname{div}(\boldsymbol{\vartheta}_j) \in \mathrm{W}^{s,4/3}(\Omega)$, $j \in \{1,2\}$. *Then, there exists a positive constant* C, *independent of* h *such that*

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \vec{\mathbf{u}}) - (\boldsymbol{\sigma}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sum_{j=1}^{2} \|(\vec{\phi}_j, \boldsymbol{\vartheta}_j) - (\vec{\phi}_{j,h}, \boldsymbol{\vartheta}_{j,h})\|_{\widetilde{\mathbf{H}}\times\widetilde{\mathbf{Q}}} \\
&+\, \|p - p_h\|_{0,\Omega} \,+\, \|\nabla\mathbf{u} - \big(\nabla\mathbf{u}\big)_h\|_{0,\Omega} \,+\, \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \\
&\quad \leqslant C\, h^s \Big\{ \|\mathbf{u}\|_{s,4;\Omega} \,+\, \|\boldsymbol{\sigma}\|_{s,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{s,4/3;\Omega} \,+\, \|\boldsymbol{\gamma}\|_{s,\Omega} \\
&\quad + \sum_{j=1}^{2} \Big(\|\phi_j\|_{s,4;\Omega} \,+\, \|\mathbf{t}_j\|_{s,\Omega} \,+\, \|\boldsymbol{\vartheta}_j\|_{s,\Omega} \,+\, \|\operatorname{div}(\boldsymbol{\vartheta}_j)\|_{s,4/3;\Omega} \Big) \Big\}.
\end{aligned} \tag{6.5}$$

7 Numerical results

In this section we consider the two pairs of finite element subspaces detailed in Section 6 to present three examples illustrating the performance of the mixed finite element method (5.2) on a set of quasi-uniform triangulations of the respective domains. In order to compare (5.2) with a standard finite element method for the velocity-pressure-temperature-concentration formulation of (2.1), we first summarize below the advantages of the former with respect to the latter:

i) regarding the fluid equations, we highlight that (5.2) along with (5.39), (5.40), and (5.41), provide approximations of the symmetric stress tensor, the velocity, the skew-symmetric vorticity tensor (and hence of the vorticity vector), the pressure, the velocity gradient, and the shear stress tensor, and all them with the same orders of convergence. On the contrary, for obtaining approximations of the aforementioned tensor variables by means of a standard finite element formulation, whose only fluid unknowns are the velocity and the pressure, one would need to perform corresponding numerical differentiation processes with the consequent loss of accuracy that they produce;

ii) regarding the double diffusion equations, we highlight that (5.2) provides approximations of the temperature, the concentration, the gradient of both, as well as of the pseudoheat and pseudodiffusion vectors, and all them with the same orders of convergence. An analogue remark to the one given in the second part of i) is valid here for the aforementioned vector variables with respect to a standard finite element formulation, whose only unknowns are the temperature and the concentration; and

iii) regarding the Dirichlet boundary conditions, we highlight that they are all natural for (5.2), so that they arise automatically in the respective linear functionals after performing the usual integration by parts procedures. On the contrary, being essential for a velocity-pressure-temperature-concentration formulation, they need to be incorporated either by continuous and discrete trace liftings or by means of the introduction of suitable Lagrange multipliers. In both cases, the solvability analyses and the corresponding derivation of the a priori error estimates and rates of convergence, while feasible, become a bit more involved.

Secondly, proceeding similarly as in [14], we observe that the global degrees of freedom for (5.2) with $\ell = 0$ are a bit higher than those obtained with a standard finite element formulation employing, for instance, Bernardi-Raugel or MINI-element in the fluid, and continuous piecewise linear elements in the diffusion equations. Nevertheless, we believe that this minor disadvantage of (5.2) is clearly compensated by the advantages described in i) and ii).

In what follows, we refer to the sets of finite element subspaces generated by $\ell = \{0, 1\}$ in Section 6 as simply $\mathrm{PEERS}_\ell - \mathrm{P}_\ell - \mathbf{P}_\ell - \mathbf{RT}_\ell$ and $\mathrm{AFW}_\ell - \mathrm{P}_\ell - \mathbf{P}_\ell - \mathbf{RT}_\ell$. The corresponding numerical methods have been implemented using open source finite element libraries: `FEniCS` [2] and `FreeFem++` [32]. We have used `FEniCS` for Examples 1 and 2, and `FreeFem++` for the Example 3. A Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-06$ is used for the resolution of the nonlinear problem (5.2). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DOF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DOF}}} \leqslant \mathsf{tol},$$

where $\|\cdot\|_{\mathtt{DOF}}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$ with $\mathtt{DOF}$ denoting the total number of degrees of freedom defining the finite element subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}, \mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\gamma}, \mathrm{H}_h^{\phi}, \mathbf{H}_h^{\mathbf{t}}$, and $\mathbf{H}_h^{\boldsymbol{\vartheta}}$ (cf. (6.2)–(6.4)).

We now introduce some additional notation. The individual errors are denoted by

$$\begin{gathered}
\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{0,\Omega}, \\
\mathsf{e}(p) := \|p - p_h\|_{0,\Omega}, \quad \mathsf{e}(\nabla\mathbf{u}) := \|\nabla\mathbf{u} - (\nabla\mathbf{u})_h\|_{0,\Omega}, \\
\mathsf{e}(\phi_j) := \|\phi_j - \phi_{j,h}\|_{0,4;\Omega}, \quad \mathsf{e}(\mathbf{t}_j) := \|\mathbf{t}_j - \mathbf{t}_{j,h}\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\vartheta}_j) := \|\boldsymbol{\vartheta}_j - \boldsymbol{\vartheta}_{j,h}\|_{\mathrm{div}_{4/3};\Omega}, \quad j \in \{1, 2\},
\end{gathered}$$

where p_h and $(\nabla\mathbf{u})_h$ stand for the post-processed pressure and velocity gradient suggested by (5.39) and (5.40), respectively. We stress here that we are also able to recover the shear stress tensor $\widetilde{\boldsymbol{\sigma}}$ by the post-processing formula (5.41). However, for the sake of simplicity, in the numerical essays below we will focus only on the pressure field and velocity gradient tensor. Moreover, for each $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, p, \nabla\mathbf{u}, \phi_j, \mathbf{t}_j, \boldsymbol{\vartheta}_j\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by $\mathsf{r}(\star) := \log\big(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star)\big)/\log(h/\widehat{h})$, where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{\mathsf{e}}$, respectively.

The examples to be considered in this section are described next. In all of them, we take $\varrho = 1, \mathtt{R}_1 = 1, \mathtt{R}_2 = 1$, and $\boldsymbol{\phi}_{\mathtt{r}} = (0,0)$. In turn, in the first two examples the tensors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are taken as the identity matrix $\mathbb{I}$, which satisfy (2.4). In addition, the null mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a real Lagrange multiplier strategy.

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test we corroborate the rates of convergence in a two-dimensional domain. The domain is the square $\Omega = (0,1)^2$. We consider the inertial power $\rho = 3$, the potential type gravitational acceleration $\mathbf{g} = (0,-1)^{\mathrm{t}}$, the effective viscosity $\mu(x_1,x_2) = \exp(-x_1x_2)$, and adjust the data $\mathbf{f}, g_1$, and g_2 in (2.13) such that the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1,x_2) = \cos(\pi x_1)\sin(0.5\,\pi x_2)\,,$$

$$\phi_1(x_1,x_2) = 0.5 + 0.5\cos(x_1x_2)\,, \quad \text{and} \quad \phi_2(x_1,x_2) = 0.1 + 0.3\exp(x_1x_2)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Tables 7.1 and 7.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations when $\mathtt{D} = 1$ and $\mathtt{F} = 10$. As already announced, we stress that we are able not only to approximate the original unknowns but also the pressure field and the velocity gradient through the formulae (5.39)–(5.40). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{\ell+1})$ predicted by Theorem 6.1 are attained for $\ell = \{0,1\}$ for both PEERS_ℓ and AFW_ℓ based schemes. The Newton method exhibits a behavior independent of the meshsize, converging in five iterations in almost all cases. In Figure 7.1 we display some solutions obtained with the mixed $\mathrm{PEERS}_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation with meshsize $h = 0.013$ and $24,200$ triangle elements (actually representing $1,260,602$ DOF).

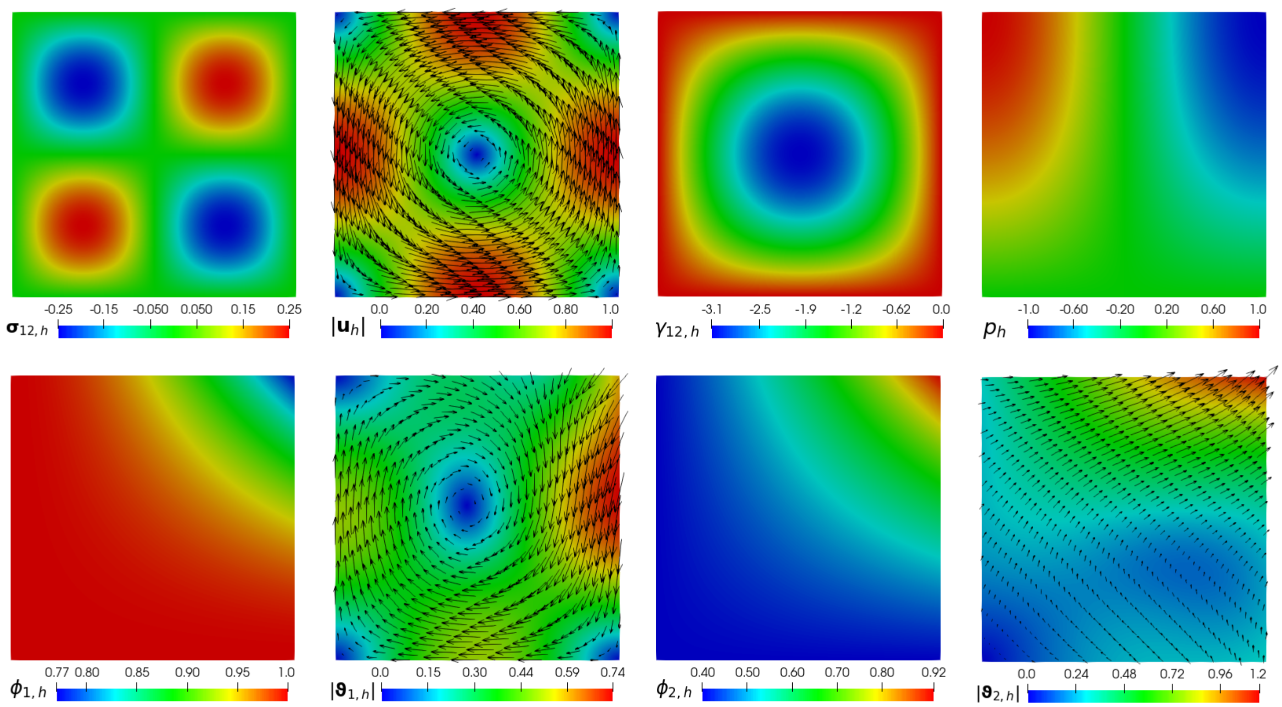

Figure 7.1: [Example 1, $\ell = 1$] Computed pseudostress tensor component, magnitude of the velocity, vorticity tensor component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots).

PEERS$_0$ − P$_0$ − $\mathbf{P}_0$ − $\mathbf{RT}_0$ approximation											
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(γ)	r(γ)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
570	0.354	2.2E+00	–	2.3E-01	–	1.6E-01	–	2.8E-01	–	8.6E-01	–
2194	0.177	1.1E+00	0.988	1.2E-01	1.003	5.1E-02	1.666	1.4E-01	0.948	4.5E-01	0.927
8610	0.088	5.5E-01	1.003	5.8E-02	1.063	2.1E-02	1.304	6.9E-02	1.039	2.3E-01	0.973
30002	0.047	2.9E-01	1.004	3.1E-02	1.057	9.2E-03	1.301	3.6E-02	1.037	1.2E-01	0.993
119402	0.024	1.5E-01	1.002	1.6E-02	1.111	3.4E-03	1.422	1.8E-02	1.015	6.2E-02	1.000
400402	0.013	8.0E-02	1.001	8.5E-03	1.059	1.4E-03	1.479	9.7E-03	1.005	3.4E-02	1.001

e(ϕ_1)	r(ϕ_1)	e($\mathbf{t}_1$)	r($\mathbf{t}_1$)	e($\boldsymbol{\vartheta}_1$)	r($\boldsymbol{\vartheta}_1$)	e(ϕ_2)	r(ϕ_2)	e($\mathbf{t}_2$)	r($\mathbf{t}_2$)	e($\boldsymbol{\vartheta}_2$)	r($\boldsymbol{\vartheta}_2$)	it
2.2E-02	–	8.4E-02	–	1.6E-01	–	4.2E-02	–	7.6E-02	–	1.5E-01	–	6
1.1E-02	0.979	5.0E-02	0.736	8.9E-02	0.883	2.1E-02	0.981	4.1E-02	0.904	7.8E-02	0.946	6
5.5E-03	0.997	2.7E-02	0.916	4.6E-02	0.961	1.1E-02	0.995	2.1E-02	0.962	3.9E-02	0.980	5
2.9E-03	1.000	1.4E-02	0.973	2.5E-02	0.988	5.7E-03	0.999	1.1E-02	0.987	2.1E-02	0.993	5
1.5E-03	1.000	7.3E-03	0.991	1.2E-02	0.996	2.8E-03	1.000	5.6E-03	0.996	1.1E-02	0.998	5
8.0E-04	1.000	4.0E-03	0.997	6.7E-03	0.999	1.6E-03	1.000	3.1E-03	0.999	5.8E-03	0.999	5

AFW$_0$ − P$_0$ − $\mathbf{P}_0$ − $\mathbf{RT}_0$ approximation											
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(γ)	r(γ)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
625	0.354	2.0E+00	–	2.3E-01	–	4.0E-01	–	1.5E-01	–	8.2E-01	–
2401	0.177	9.8E-01	1.027	1.2E-01	0.965	2.1E-01	0.944	6.7E-02	1.164	4.2E-01	0.943
9409	0.088	4.9E-01	1.010	5.8E-02	0.991	1.0E-01	0.986	3.2E-02	1.078	2.1E-01	0.987
32761	0.047	2.6E-01	1.003	3.1E-02	0.998	5.6E-02	0.996	1.7E-02	1.024	1.1E-01	0.997
130321	0.024	1.3E-01	1.001	1.6E-02	0.999	2.8E-02	0.999	8.2E-03	1.007	5.7E-02	0.999
436921	0.013	7.0E-02	1.000	8.5E-03	1.000	1.5E-02	1.000	4.5E-03	1.002	3.1E-02	1.000

e(ϕ_1)	r(ϕ_1)	e($\mathbf{t}_1$)	r($\mathbf{t}_1$)	e($\boldsymbol{\vartheta}_1$)	r($\boldsymbol{\vartheta}_1$)	e(ϕ_2)	r(ϕ_2)	e($\mathbf{t}_2$)	r($\mathbf{t}_2$)	e($\boldsymbol{\vartheta}_2$)	r($\boldsymbol{\vartheta}_2$)	it
2.2E-02	–	8.3E-02	–	1.6E-01	–	4.2E-02	–	7.6E-02	–	1.5E-01	–	5
1.1E-02	0.978	5.0E-02	0.731	8.9E-02	0.883	2.1E-02	0.981	4.1E-02	0.905	7.8E-02	0.947	5
5.5E-03	0.997	2.7E-02	0.915	4.6E-02	0.960	1.1E-02	0.996	2.1E-02	0.963	3.9E-02	0.980	5
2.9E-03	1.000	1.4E-02	0.973	2.5E-02	0.987	5.7E-03	0.999	1.1E-02	0.987	2.1E-02	0.993	5
1.5E-03	1.000	7.3E-03	0.991	1.2E-02	0.996	2.8E-03	1.000	5.6E-03	0.996	1.1E-02	0.998	5
8.0E-04	1.000	4.0E-03	0.997	6.7E-03	0.999	1.6E-03	1.000	3.1E-03	0.999	5.8E-03	0.999	5

Table 7.1: [Example 1, $\ell = 0$] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with $\rho = 3$, $\mathtt{D} = 1$, and $\mathtt{F} = 10$.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example we consider the cube domain $\Omega = (0,1)^3$, the model parameter $\rho = 3.5, \mathtt{D} = 1$, $\mathtt{F} = 10$, $\mu(x_1,x_2,x_3) = \exp(-x_1x_2x_3)$, and $\mathbf{g} = (0,0,-1)$. The manufactured solution is given by

$$\mathbf{u}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \quad p(x_1,x_2,x_3) = \cos(\pi x_1)\exp(x_2+x_3)\,,$$

$$\phi_1(x_1,x_2,x_3) = 0.5 + 0.5\cos(x_1x_2x_3)\,, \quad \text{and} \quad \phi_2(x_1,x_2,x_3) = 0.1 + 0.3\exp(x_1x_2x_3)\,.$$

Similarly to the first example, the data $\mathbf{f}, g_1, g_2$ and $\mathbf{u}_{\mathrm{D}}, \phi_{1,\mathrm{D}}, \phi_{2,\mathrm{D}}$ are computed from (2.13) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using $\ell = 0$ is shown in Table 7.3. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 6.1. In addition, some components of the numerical solution are displayed in Figure 7.2, which were built using the mixed PEERS$_0$−P$_0$−$\mathbf{P}_0$−$\mathbf{RT}_0$ approximation with meshsize $h = 0.087$ and $48,000$ tetrahedral elements (actually representing $1,479,784$ DOF).

PEERS$_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation											
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
1746	0.354	3.4E-01	–	4.1E-02	–	5.7E-02	–	3.4E-02	–	1.3E-01	–
6818	0.177	8.8E-02	1.959	1.1E-02	1.969	1.7E-02	1.733	9.4E-03	1.836	3.6E-02	1.867
26946	0.088	2.3E-02	1.964	2.6E-03	1.992	5.4E-03	1.674	2.6E-03	1.851	1.0E-02	1.861
94202	0.047	6.5E-03	1.979	7.5E-04	1.998	1.7E-03	1.817	7.9E-04	1.912	3.0E-03	1.905
375602	0.024	1.6E-03	1.991	1.9E-04	1.999	4.5E-04	1.926	2.0E-04	1.960	7.8E-04	1.954
1260602	0.013	4.9E-04	1.996	5.6E-05	2.000	1.4E-04	1.973	6.1E-05	1.983	2.3E-04	1.980

e(ϕ_1)	r(ϕ_1)	e($\mathbf{t}_1$)	r($\mathbf{t}_1$)	e($\boldsymbol{\vartheta}_1$)	r($\boldsymbol{\vartheta}_1$)	e(ϕ_2)	r(ϕ_2)	e($\mathbf{t}_2$)	r($\mathbf{t}_2$)	e($\boldsymbol{\vartheta}_2$)	r($\boldsymbol{\vartheta}_2$)	it
2.0E-03	–	1.3E-02	–	2.2E-02	–	3.4E-03	–	7.9E-03	–	1.7E-02	–	6
5.1E-04	2.006	3.7E-03	1.792	6.1E-03	1.878	8.9E-04	1.971	2.2E-03	1.827	4.5E-03	1.926	6
1.3E-04	2.002	9.8E-04	1.919	1.6E-03	1.950	2.2E-04	1.994	5.9E-04	1.929	1.2E-03	1.969	5
3.6E-05	2.001	2.9E-04	1.965	4.5E-04	1.978	6.1E-05	1.999	1.7E-04	1.967	3.3E-04	1.985	5
9.0E-06	2.000	7.2E-05	1.984	1.1E-04	1.990	1.5E-05	2.000	4.3E-05	1.984	8.3E-05	1.993	5
2.7E-06	2.000	2.2E-05	1.992	3.4E-05	1.995	4.5E-06	2.000	1.3E-05	1.992	2.5E-05	1.996	5

AFW$_1 - \mathrm{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ approximation											
DOF	h	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\gamma}$)	r($\boldsymbol{\gamma}$)	e(p)	r(p)	e($\nabla\mathbf{u}$)	r($\nabla\mathbf{u}$)
1745	0.354	3.1E-01	–	4.1E-02	–	6.4E-02	–	1.8E-02	–	1.3E-01	–
6817	0.177	7.6E-02	2.010	1.1E-02	1.959	1.7E-02	1.949	4.3E-03	2.042	3.3E-02	1.958
26945	0.088	1.9E-02	2.006	2.6E-03	1.989	4.2E-03	1.980	1.1E-03	1.963	8.4E-03	1.986
94201	0.047	5.3E-03	2.003	7.5E-04	1.997	1.2E-03	1.992	3.2E-04	1.972	2.4E-03	1.995
375601	0.024	1.3E-03	2.001	1.9E-04	1.999	3.0E-04	1.997	8.1E-05	1.984	6.0E-04	1.998
1260601	0.013	4.0E-04	2.001	5.6E-05	2.000	8.9E-03	1.998	2.4E-05	1.992	1.8E-04	1.999

e(ϕ_1)	r(ϕ_1)	e($\mathbf{t}_1$)	r($\mathbf{t}_1$)	e($\boldsymbol{\vartheta}_1$)	r($\boldsymbol{\vartheta}_1$)	e(ϕ_2)	r(ϕ_2)	e($\mathbf{t}_2$)	r($\mathbf{t}_2$)	e($\boldsymbol{\vartheta}_2$)	r($\boldsymbol{\vartheta}_2$)	it
2.0E-03	–	1.3E-02	–	2.2E-02	–	3.4E-03	–	7.9E-03	–	1.7E-02	–	6
5.1E-04	2.006	3.7E-03	1.783	6.1E-03	1.877	8.6E-04	1.971	2.2E-03	1.820	4.5E-03	1.926	5
1.3E-04	2.002	9.8E-04	1.918	1.6E-03	1.950	2.2E-04	1.994	5.9E-04	1.928	1.2E-03	1.969	5
3.6E-05	2.001	2.9E-04	1.965	4.5E-04	1.978	6.1E-05	1.999	1.7E-04	1.968	3.3E-04	1.985	5
9.0E-06	2.000	7.2E-05	1.984	1.1E-04	1.990	1.5E-05	2.000	4.3E-05	1.984	8.3E-05	1.993	5
2.7E-06	2.000	2.2E-05	1.992	3.4E-05	1.995	4.5E-06	2.000	1.3E-05	1.992	2.5E-05	1.996	5

Table 7.2: [Example 1, $\ell = 1$] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with $\rho = 3$, $\mathtt{D} = 1$, and $\mathtt{F} = 10$.

Example 3: Flow through a 2D porous media with fracture network.

Inspired by [18, Example 4, Section 6], we finally focus on a flow through a porous medium with a fracture network considering strong jump discontinuities of the parameters $\mathtt{D}$ and $\mathtt{F}$ across the two regions. We consider the square domain $\Omega = (-1,1)^2$ with an internal fracture network denoted as Ω_{f} (see the first plot in Figure 7.3), and boundary Γ, whose left, right, upper and lower parts are given by $\Gamma_{\mathrm{left}} = \{-1\} \times (-1,1)$, $\Gamma_{\mathrm{right}} = \{1\} \times (-1,1)$, $\Gamma_{\mathrm{top}} = (-1,1) \times \{1\}$, and $\Gamma_{\mathrm{bottom}} = (-1,1) \times \{-1\}$, respectively. Note that the boundary of the internal fracture network is defined as a union of segments. The prescribed mesh file is available in `https://github.com/scaucao/Fracture_network-mesh_CBF-DD`. We consider the coupling of the convective Brinkman–Forchheimer and double-diffusion equations (2.13) in the whole domain Ω with inertial power $\rho = 4$, $\mu = 1$, $\mathbf{Q}_1 = 0.1\,\mathbb{I}$ and $\mathbf{Q}_2 = 0.2\,\mathbb{I}$, but with different values of the parameters $\mathtt{D}$ and $\mathtt{F}$ for the interior and the exterior of the fracture, namely

$$\mathtt{F} = \left\{ \begin{array}{rl} 10 & \text{in} \quad \Omega_{\mathrm{f}} \\ 1 & \text{in} \quad \overline{\Omega}\backslash\Omega_{\mathrm{f}} \end{array} \right. \quad \text{and} \quad \mathtt{D} = \left\{ \begin{array}{rl} 1 & \text{in} \quad \Omega_{\mathrm{f}} \\ 1000 & \text{in} \quad \overline{\Omega}\backslash\Omega_{\mathrm{f}} \end{array} \right. . \tag{7.1}$$

$\mathtt{DOF}$	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\gamma)$	$\mathtt{r}(\gamma)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\nabla\mathbf{u})$	$\mathtt{r}(\nabla\mathbf{u})$
PEERS$_0$ $-$ P$_0$ $-$ $\mathbf{P}_0$ $-$ $\mathbf{RT}_0$ approximation											
1642	0.866	9.2E+00	–	5.7E-01	–	1.2E+00	–	1.2E+00	–	2.4E+00	–
12376	0.433	4.8E+00	0.953	3.1E-01	0.898	4.3E-01	1.477	6.3E-01	0.881	1.4E+00	0.721
96268	0.217	2.4E+00	1.002	1.6E-01	0.971	1.2E-01	1.793	3.1E-01	1.054	7.6E-01	0.910
509926	0.124	1.4E+00	1.015	8.9E-02	0.994	5.0E-02	1.604	1.6E-01	1.104	4.4E-01	0.968
1479784	0.087	9.4E-01	1.012	6.3E-02	0.998	2.9E-02	1.577	1.1E-01	1.084	3.1E-01	0.987

$\mathtt{e}(\phi_1)$	$\mathtt{r}(\phi_1)$	$\mathtt{e}(\mathbf{t}_1)$	$\mathtt{r}(\mathbf{t}_1)$	$\mathtt{e}(\boldsymbol{\vartheta}_1)$	$\mathtt{r}(\boldsymbol{\vartheta}_1)$	$\mathtt{e}(\phi_2)$	$\mathtt{r}(\phi_2)$	$\mathtt{e}(\mathbf{t}_2)$	$\mathtt{r}(\mathbf{t}_2)$	$\mathtt{e}(\boldsymbol{\vartheta}_2)$	$\mathtt{r}(\boldsymbol{\vartheta}_2)$	$\mathtt{it}$
2.9E-02	–	1.0E-01	–	3.2E-01	–	6.0E-02	–	1.1E-01	–	2.3E-01	–	6
1.6E-02	0.868	8.2E-02	0.295	1.9E-01	0.760	3.1E-02	0.931	6.3E-02	0.849	1.3E-01	0.864	6
8.0E-03	0.971	4.8E-02	0.788	9.9E-02	0.926	1.6E-02	0.982	3.3E-02	0.918	6.6E-02	0.957	6
4.6E-03	0.994	2.8E-02	0.925	5.7E-02	0.975	9.1E-03	0.995	1.9E-02	0.966	3.8E-02	0.986	6
3.2E-03	0.998	2.0E-02	0.965	4.0E-02	0.989	6.4E-03	0.998	1.4E-02	0.983	2.7E-02	0.994	6

$\mathtt{DOF}$	h	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(\gamma)$	$\mathtt{r}(\gamma)$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\nabla\mathbf{u})$	$\mathtt{r}(\nabla\mathbf{u})$
AFW$_0$ $-$ P$_0$ $-$ $\mathbf{P}_0$ $-$ $\mathbf{RT}_0$ approximation											
1993	0.866	8.7E+00	–	5.6E-01	–	1.1E+00	–	1.0E+00	–	2.1E+00	–
14881	0.433	4.3E+00	1.032	3.0E-01	0.903	6.4E-01	0.727	4.9E-01	1.093	1.3E+00	0.712
114817	0.217	2.1E+00	1.045	1.6E-01	0.959	3.4E-01	0.928	2.2E-01	1.119	6.8E-01	0.922
605641	0.124	1.2E+00	1.019	8.9E-02	0.988	1.9E-01	0.980	1.3E-01	1.050	3.9E-01	0.978
1754401	0.087	8.1E-01	1.008	6.2E-02	0.995	1.4E-01	0.992	8.7E-02	1.021	2.7E-01	0.992

$\mathtt{e}(\phi_1)$	$\mathtt{r}(\phi_1)$	$\mathtt{e}(\mathbf{t}_1)$	$\mathtt{r}(\mathbf{t}_1)$	$\mathtt{e}(\boldsymbol{\vartheta}_1)$	$\mathtt{r}(\boldsymbol{\vartheta}_1)$	$\mathtt{e}(\phi_2)$	$\mathtt{r}(\phi_2)$	$\mathtt{e}(\mathbf{t}_2)$	$\mathtt{r}(\mathbf{t}_2)$	$\mathtt{e}(\boldsymbol{\vartheta}_2)$	$\mathtt{r}(\boldsymbol{\vartheta}_2)$	$\mathtt{it}$
2.9E-02	–	9.9E-02	–	3.1E-01	–	6.0E-02	–	1.1E-01	–	2.3E-01	–	5
1.6E-02	0.869	8.1E-02	0.275	1.9E-01	0.747	3.1E-02	0.934	6.3E-02	0.846	1.3E-02	0.863	5
8.0E-03	0.972	4.8E-02	0.775	9.8E-02	0.917	1.6E-02	0.984	3.3E-02	0.915	6.6E-02	0.954	5
4.6E-03	0.995	2.8E-02	0.922	5.7E-02	0.970	9.1E-03	0.996	1.9E-02	0.965	3.8E-02	0.983	5
3.2E-03	0.998	2.0E-02	0.964	4.0E-02	0.986	6.4E-03	0.998	1.4E-02	0.983	2.7E-02	0.992	5

Table 7.3: [Example 2] Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed approximations with $\rho = 3.5$, $\mathtt{D} = 1$, and $\mathtt{F} = 10$.

The parameter choice corresponds to increased inertial effect ($\mathtt{F} = 10$) in the fracture and a high permeability ($\mathtt{D} = 1$), compared to reduced inertial effect ($\mathtt{F} = 1$) in the porous medium and low permeability ($\mathtt{D} = 1000$). In addition, $\mathbf{g} = (0, -1)$, the source terms are $g_1 = 0$ and $g_2 = 0$, and the boundaries conditions are

$$
\begin{gathered}
\boldsymbol{\sigma}\boldsymbol{\nu} = \begin{cases} (-100\,(x_2 - 1),\, 0)^{\mathrm{t}} & \text{on} \quad \Gamma_{\mathrm{left}}\,, \\ (0,\, -100\,(x_1 - 1))^{\mathrm{t}} & \text{on} \quad \Gamma_{\mathrm{top}}\,, \end{cases} \qquad \boldsymbol{\sigma}\boldsymbol{\nu} = (0,\,0)^{\mathrm{t}} \quad \text{on} \quad \Gamma_{\mathrm{right}} \cup \Gamma_{\mathrm{bottom}}\,, \\
\phi_1 = 0.3 \quad \text{on} \quad \Gamma_{\mathrm{bottom}}\,, \qquad \phi_1 = 0 \quad \text{on} \quad \Gamma_{\mathrm{top}}\,, \qquad \boldsymbol{\vartheta}_1 \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_{\mathrm{left}} \cup \Gamma_{\mathrm{right}}\,, \\
\phi_2 = 0.2 \quad \text{on} \quad \Gamma_{\mathrm{bottom}}\,, \qquad \phi_2 = 0 \quad \text{on} \quad \Gamma_{\mathrm{top}}\,, \qquad \boldsymbol{\vartheta}_2 \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma_{\mathrm{left}} \cup \Gamma_{\mathrm{right}}\,,
\end{gathered}
\tag{7.2}
$$

which drives the flow in a diagonal direction from the left-top corner to the right-bottom corner of the square domain Ω. We remark that the analysis developed in the previous sections can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example.

In Figure 7.3, we display the computed magnitude of the pseudostress tensor, velocity, velocity gradient, and gradients of the temperature and concentration, and the temperature and concentration fields, which were built using the fully-mixed $\mathrm{AFW}_0 - \mathrm{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$ scheme on a mesh with $h = 0.029$ and $31,932$ triangle elements (actually representing $576,216\,\mathtt{DOF}$). As we expected, the velocity in the fractures is higher than the velocity in the porous medium, due to smaller fractures thickness and the parameter setting (7.1). In addition, the velocity is higher in branches of the network where the

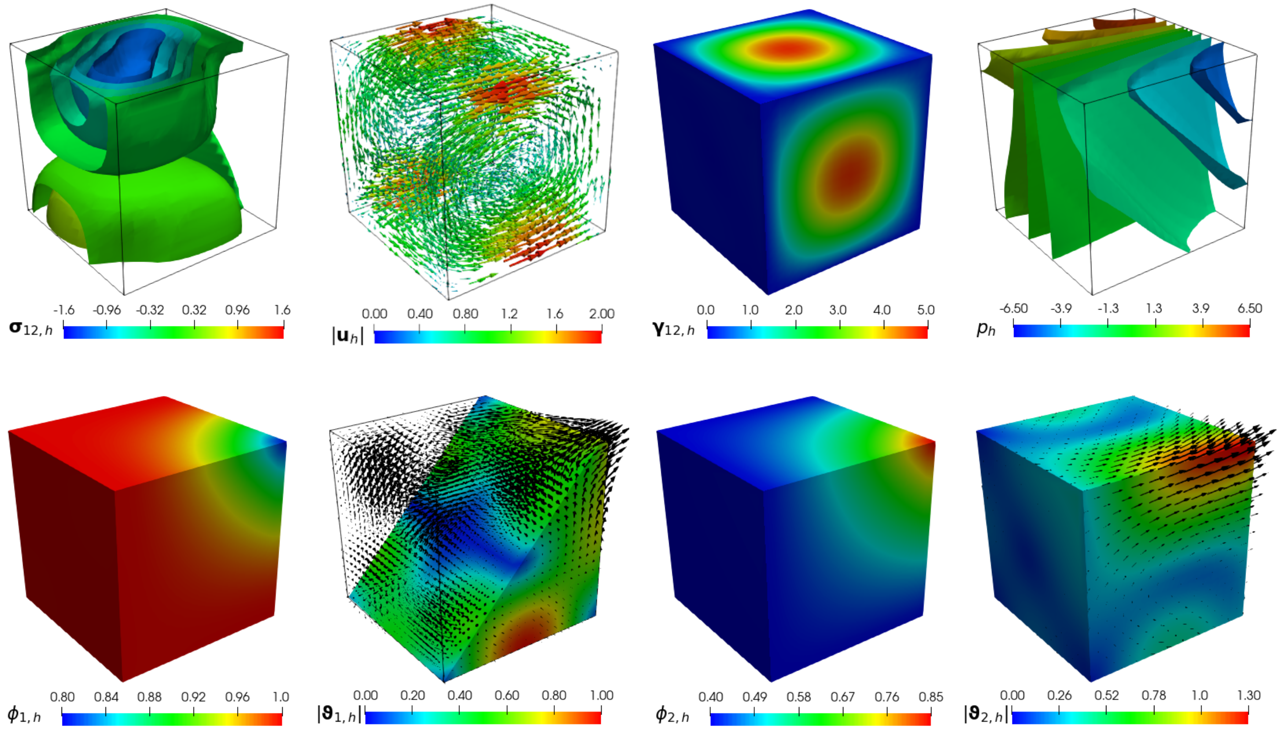

Figure 7.2: [Example 2] Computed pseudostress tensor component, magnitude of the velocity, vorticity tensor component, and pressure field (top plots); temperature field, magnitude of the pseudoheat vector, concentration field, and magnitude of the pseudodiffusion vector (bottom plots).

fluid enters from the left-top corner and decreases toward the right-bottom corner of the domain. In turn, we observe a sharp velocity gradient across the interfaces between the fractures and the porous medium. The pseudostress is consistent with the boundary conditions (7.2) and it is more diffused since it includes the pressure field. In turn, the temperature and concentration are zero on the top of the domain and go increasing towards the bottom of it, which is consistent with the behavior observed in the magnitude of the temperature and concentration gradients. This example illustrates the ability of the method to provide accurate resolution and numerically stable results for heterogeneous inclusions with high aspect ratio and complex geometry, as presented in the network of thin fractures. We notice that the mesh used in this example was built by considering a quasi-uniform refinement. Nevertheless, this refinement can be improved and automatized by employing a suitable *a posteriori* error indicator, as in [16] and [13], that captures the aforementioned discontinuity of the parameters and localize the refinement where it is needed. The corresponding *a posteriori* error analysis and numerical implementation will be addressed in a future work.

Data Availability. Enquiries about data availability should be directed to the authors.

Conflict of interest. The authors have not disclosed any competing interests.

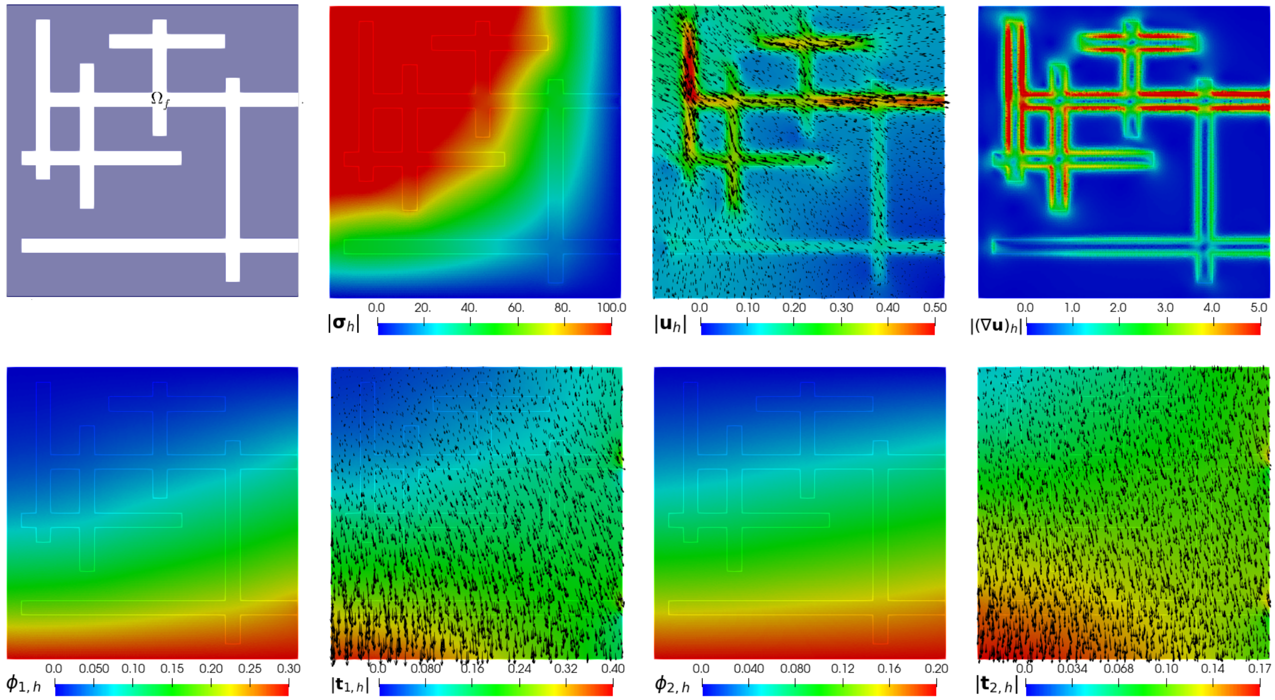

Figure 7.3: [Example 3] Domain configuration, computed magnitude of the pseudostress tensor, velocity, and velocity gradient tensor (top plots); concentration field, magnitude of the temperature gradient, concentration field, and magnitude of the concentration gradient (bottom plots).

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