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On the well-posedness of Banach spaces-based mixed formulations for the nearly incompressible Navier-Lamé and Stokes equations ☆

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ABSTRACT

In this paper we introduce and analyze, up to our knowledge for the first time, Banach spaces-based mixed variational formulations for nearly incompressible linear elasticity and Stokes models. Our interest in this subject is motivated by the respective need that arises from the solvability studies of nonlinear coupled problems in continuum mechanics that involve these equations. We consider pseudostress-based approaches in both cases and apply a suitable integration by parts formula for ad-hoc Sobolev spaces to derive the corresponding continuous schemes. We utilize known and new preliminary results, along with the Babuška-Brezzi theory in Banach spaces, to establish the well-posedness of the formulations for a particular range of the indexes of the Lebesgue spaces involved. Among the aforementioned new results from us, we highlight the construction of a particular operator mapping a tensor Lebesgue space into itself, and the generalization of a classical estimate in L^2 for deviatoric tensors, which plays a key role in the Hilbertian analysis of linear elasticity, to arbitrary Lebesgue spaces. No discrete analysis is performed in this work.

1. Introduction

In many nonlinear models in continuum mechanics, specially in coupled ones, the coefficients, source terms, or arbitrary terms of each equation depend on the unknowns from the other equations involved, which certainly makes the corresponding analyses much more cumbersome than for simple linear problems. Indeed, one of the main challenges that one often encounters there refers to the fact that the natural spaces to which the unknowns belong force the respective variational formulations to be posed in terms of Banach spaces instead of Hilbert ones. In order to overcome this, in some cases one may resort to the incorporation of augmented terms, as done for instance in [3] and [4] for coupled flow-transport problems, in [1] and [10] for the Boussinesq equations, in [8] and [9] for the Navier-Stokes equations, or in [15] and [16] for stress-assisted diffusion, thanks to which one recovers Hilbertian frameworks for the models, which are much easier to analyze. Nevertheless, while showing this and other advantages as well, the augmentation procedure adds further complexity to the problems, mainly affecting the associated discrete schemes and the respective computational implementations, which could be avoided if proper analyses are developed for the original non-augmented variational formulations. Needless to mention, in some models the augmentation is not even possible, as for the coupled Darcy and heat equations, and hence a Banach framework becomes unavoidable in these cases (see, e.g. [18]).

As a matter of illustration of the above, let us briefly recall that the model from [15] and [16] consists of a system of partial differential equations governing the diffusion of a solute interacting with the motion of an elastic solid occupying a bounded domain Ω with boundary Γ. In particular, the respective diffusion coefficient ϑ depends on the Cauchy stress tensor $\boldsymbol{\sigma}$ of the solid, so that the diffusive flux $\mathbf{p}$ and the diffusion equation become

$$\mathbf{p} := \vartheta(\boldsymbol{\sigma})\nabla\phi \quad \text{and} \quad -\mathrm{div}(\mathbf{p}) = g(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{1.1}$$

respectively, where ϕ is the solute concentration, ∇ and div are the usual gradient and divergence operators, respectively, and g is a source term depending on the displacement $\boldsymbol{u}$ of the solid. Then, dividing the first equation of (1.1) by $\vartheta(\boldsymbol{\sigma})$, which is assumed to be strictly positive, multiplying by a test vector $\mathbf{q}$ associated with the unknown $\mathbf{p}$, formally integrating by parts, and assuming for simplicity that ϕ vanishes on Γ, one obtains

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$$\int_\Omega \frac{1}{\vartheta(\boldsymbol{\sigma})}\,\mathbf{p}\cdot\mathbf{q} + \int_\Omega \phi\,\mathrm{div}(\mathbf{q}) = 0\,. \tag{1.2}$$

In turn, denoting by ψ a test function associated with ϕ, the second equation of (1.1) yields

$$\int_\Omega \psi\,\mathrm{div}(\mathbf{p}) = -\int_\Omega \psi\,g(\boldsymbol{u})\,. \tag{1.3}$$

Thus, because of the terms $\vartheta(\boldsymbol{\sigma})$ and $g(\boldsymbol{u})$, with $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ coming from the elasticity model, one can employ fixed point arguments to analyze the solvability of (1.2) - (1.3). A similar procedure is applied to the linear elasticity equation, whose source term depends on ϕ. As a consequence, and in order to derive, in particular, a continuity property of the fixed-point operator for the stress-assisted diffusion problem, most likely one will have to deal, among others, with the following expression arising from the first term of (1.2)

$$\int_\Omega \left\{\frac{\vartheta(\boldsymbol{\tau})-\vartheta(\boldsymbol{\zeta})}{\vartheta(\boldsymbol{\tau})\vartheta(\boldsymbol{\zeta})}\right\}\mathbf{p}\cdot\mathbf{q}\,, \tag{1.4}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$ are generic tensors belonging to the same space where $\boldsymbol{\sigma}$ lives. In this case, if ϑ is assumed to be bounded from below and satisfy a Lipschitz-continuity property, the Cauchy-Schwarz and Hölder inequalities allow to conclude that the above expression can be controlled only if $\boldsymbol{\tau}-\boldsymbol{\zeta}$, $\mathbf{p}$, and $\mathbf{q}$, belong to particular Lebesgue spaces. This simple example illustrates that, even if $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are solutions of a linear elasticity problem, for which the solvability via Hilbert spaces is already well-established, when this equation is coupled with (1.1), the fixed-point argumentation requires that the analysis of the former be performed within a suitable Banach spaces framework. Same conclusions arise if linear elasticity is coupled with other equations, if other model, as Stokes in [3], is employed, or if similar coupled problems are considered.

According to the above discussion, the initial purpose of this work is to introduce and analyze a Banach spaces-based mixed variational formulation for linear elasticity, particularly for the nearly incompressible case, which is of much more interest in applications. Additionally, and because of the similarities between the resulting continuous formulations, we also include the Stokes system in our discussion. In this way, the rest of the paper is organized as follows. In Section 2 we introduce both models of interest and use a suitable integration by parts formula to derive their mixed variational formulations. Some preliminary results, namely the well-posedness of Banach spaces-based primal formulations for the Stokes and Poisson equations, a suitable operator mapping a tensor Lebesgue space into itself, and a generalization to arbitrary Lebesgue spaces of a key inequality for the Hilbertian analysis of linear elasticity, are stated in Section 3. Finally, the well-posedness of the formulations from Section 2 are established in Section 4.

We end this section by mentioning that throughout the rest of the paper we adopt the standard notations for the Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$ and $\mathrm{W}^{\ell,t}_0(\Omega)$, with $\ell\ge 0$ and $t\in[1,+\infty)$. In particular, the corresponding norms and seminorms, either for the scalar, vector, or tensor versions of them, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$ and $|\cdot|_{\ell,t;\Omega}$, respectively. In addition, if $t, t'\in(1,+\infty)$ are conjugate to each other, that is $\frac{1}{t}+\frac{1}{t'}=1$, we let $\mathrm{W}^{-1,t'}(\Omega)$ be the dual of $\mathrm{W}^{1,t}_0(\Omega)$. Also, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. Furthermore, given a generic scalar functional space S, we denote by $\mathbf{S}$ and $\mathbb{S}$ its vector and tensor versions, respectively, examples of which are $\mathbf{W}^{\ell,t}(\Omega):=[\mathrm{W}^{\ell,t}(\Omega)]^n$, $\mathbf{W}^{-1,t'}(\Omega)$, the dual of $\mathbf{W}^{1,t}_0(\Omega):=[\mathrm{W}^{1,t}_0(\Omega)]^n$, and $\mathbb{L}^t(\Omega):=[\mathrm{L}^t(\Omega)]^{n\times n}$. Finally, we let $\mathbb{I}$ be the identity matrix of $\mathbb{R}:=\mathrm{R}^{n\times n}$, and for any $\boldsymbol{\tau}:=(\tau_{ij})$, $\boldsymbol{\zeta}:=(\zeta_{ij})\in\mathbb{R}$, we write as usual

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji}),\qquad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii},\qquad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau}-\frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I},\quad\text{and}\quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij}\zeta_{ij}\,,$$

which corresponds, respectively, to the transpose, the trace, the deviatoric of a tensor $\boldsymbol{\tau}$, and the tensor product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$.

2. The models and their mixed formulations

In this section we define our models of interest and derive their corresponding Banach spaces-based mixed formulations. In what follows, Ω is a bounded Lipschitz-continuous domain of R^n, $n\in\{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at Γ is denoted by $\boldsymbol{\nu}$.

2.1. Nearly incompressible linear elasticity

The aim of the linear elasticity model is to determine the displacement $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\rho}$ of a linear elastic material occupying the region Ω, under the action of external forces. More precisely, given a volume force $\boldsymbol{f}$ and a Dirichlet datum $\boldsymbol{u}_D$, we seek a symmetric tensor field $\boldsymbol{\rho}$ and a vector field $\boldsymbol{u}$ satisfying the constitutive relation given by Hooke's law, the corresponding momentum balance, and a Dirichlet boundary condition on Γ, that is

$$\begin{aligned} \boldsymbol{\rho} &= 2\mu\,\boldsymbol{e}(\boldsymbol{u}) + \lambda\,\mathrm{tr}(\boldsymbol{e}(\boldsymbol{u}))\,\mathbb{I} \quad\text{in}\quad \Omega,\\ \mathbf{div}(\boldsymbol{\rho}) &= -\boldsymbol{f} \quad\text{in}\quad \Omega,\quad\text{and}\quad \boldsymbol{u}=\boldsymbol{u}_D \quad\text{on}\quad \Gamma, \end{aligned} \tag{2.1}$$

where $\boldsymbol{e}(\boldsymbol{u}) := \frac{1}{2}\big(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{t}}\big)$ is the strain tensor of small deformations, $\lambda,\mu>0$ denote the corresponding Lamé constants, and $\mathbf{div}$ stands for the operator div acting along the rows of each tensor. We are particularly interested in the nearly incompressible case, which reduces to assume from now on that λ is sufficiently large. In addition, in order to avoid the symmetry of $\boldsymbol{\rho}$, we reformulate (2.1) in terms of the non-symmetric pseudostress tensor $\boldsymbol{\sigma}$ introduced in [14]. More precisely, according to the analysis provided in [14, Section 2.1], we know that (2.1) is equivalent to the Navier-Lamé equations, which are given by

$$\begin{aligned} \boldsymbol{\sigma} &= \mu\,\nabla\boldsymbol{u} + \big(\lambda+\mu\big)\,\mathrm{tr}(\nabla\boldsymbol{u})\,\mathbb{I} \quad\text{in}\quad \Omega,\\ \mathbf{div}(\boldsymbol{\sigma}) &= -\boldsymbol{f} \quad\text{in}\quad \Omega,\quad\text{and}\quad \boldsymbol{u}=\boldsymbol{u}_D \quad\text{on}\quad \Gamma. \end{aligned} \tag{2.2}$$

Hence, applying matrix trace to the first equation of (2.2), we can express $\mathrm{tr}(\nabla\boldsymbol{u})$ in terms of $\mathrm{tr}(\boldsymbol{\sigma})$ (cf. [14, eq. (2.3)]), so that the former is eliminated and (2.2) is rewritten, equivalently, as

$$\begin{aligned} \frac{1}{\mu}\boldsymbol{\sigma}^{\mathrm{d}} + \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathbb{I} &= \nabla\boldsymbol{u} \quad\text{in}\quad \Omega,\\ \mathbf{div}(\boldsymbol{\sigma}) &= -\boldsymbol{f} \quad\text{in}\quad \Omega,\quad\text{and}\quad \boldsymbol{u}=\boldsymbol{u}_D \quad\text{on}\quad \Gamma. \end{aligned} \tag{2.3}$$

Note that the original Cauchy stress tensor $\boldsymbol{\rho}$ can be recovered in terms of the pseudostress $\boldsymbol{\sigma}$ through the postprocessing formula (cf. [14, eq. (2.14)])

$$\boldsymbol{\rho} = \boldsymbol{\sigma}+\boldsymbol{\sigma}^{\mathrm{t}} - \frac{(\lambda+2\mu)}{\big(n\lambda+(n+1)\mu\big)}\,\mathrm{tr}(\boldsymbol{\sigma})\,\mathbb{I}\,. \tag{2.4}$$

Next, in order to set the Banach spaces-based variational formulation of (2.3), we need a couple of further concepts and tools. Indeed, we first introduce for each $t\in(1,+\infty)$ the Banach space

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^t(\Omega):\quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^t(\Omega)\Big\}\,, \tag{2.5}$$

which is endowed with the natural norm defined as

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{2.6}$$

Note that $\mathbb{H}^2(\mathbf{div}_2;\Omega)$ is the usual Hilbert space $\mathbb{H}(\mathbf{div};\Omega)$. Then, given $t, t'\in(1,+\infty)$ conjugate to each other, we invoke the integration by parts formula (cf. [12, Corollary B. 57])

$$\langle \tau\,\nu, v\rangle_\Gamma = \int_\Omega \Big\{ \tau : \nabla v + v\cdot \mathbf{div}(\tau) \Big\} \qquad \forall\, (\tau, v) \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\times \mathbf{W}^{1,t'}(\Omega)\,, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathbf{W}^{-1/t,t}(\Gamma)$ and $\mathbf{W}^{1/t,t'}(\Gamma)$. Finally, we observe that for each $t\in(1,+\infty)$ there holds

$$\mathbb{H}^t(\mathbf{div}_t;\Omega) = \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{2.8}$$

where

$$\mathbb{H}^t_0(\mathbf{div}_t;\Omega) := \Big\{ \tau\in \mathbb{H}^t(\mathbf{div}_t;\Omega): \quad \int_\Omega \mathrm{tr}(\tau) = 0 \Big\}. \tag{2.9}$$

Equivalently, each $\tau\in\mathbb{H}^t(\mathbf{div}_t;\Omega)$ can be decomposed, uniquely, as

$$\begin{aligned} &\tau = \tau_0 + d\,\mathbb{I}\,, \quad \text{with} \quad \tau_0\in \mathbb{H}^t_0(\mathbf{div}_t;\Omega) \\ &\quad \text{and} \qquad d := \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\tau)\in \mathrm{R}\,. \end{aligned} \tag{2.10}$$

Now, given $r, s\in(1,+\infty)$ conjugate to each other, we assume that $f\in\mathbf{L}^r(\Omega)$ and $u_D\in\mathbf{W}^{1/s,r}(\Gamma)$, and initially look for $(\sigma,u)\in\mathbb{H}^r(\mathbf{div}_r;\Omega)\times\mathbf{W}^{1,r}(\Omega)$ as the solution of (2.3). In this way, multiplying the first equation of (2.3) by a test tensor $\tau\in\mathbb{H}^s(\mathbf{div}_s;\Omega)$, applying (2.7) with $t=s$ and $t'=r$, and using the Dirichlet boundary condition for u, we find that

$$\frac{1}{\mu}\int_\Omega \sigma^{\mathrm d} : \tau^{\mathrm d} + \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\int_\Omega \mathrm{tr}(\sigma)\,\mathrm{tr}(\tau) + \int_\Omega u\cdot\mathbf{div}(\tau) = \langle \tau\,\nu, u_D\rangle_\Gamma\,, \tag{2.11}$$

whereas the second equation of (2.3) tested against $v\in\mathbf{L}^s(\Omega)$ becomes

$$\int_\Omega v\cdot\mathbf{div}(\sigma) = -\int_\Omega f\cdot v\,. \tag{2.12}$$

In turn, taking $\tau=\mathbb{I}$ in (2.11), it follows that

$$\frac{1}{\big(n\lambda+(n+1)\mu\big)}\int_\Omega \mathrm{tr}(\sigma) = \int_\Gamma u_D\cdot\nu\,,$$

from which, along with (2.10), we deduce that

$$\begin{aligned} &\sigma = \sigma_0 + c\,\mathbb{I}\,, \quad \text{with} \quad \sigma_0\in\mathbb{H}^r_0(\mathbf{div}_r;\Omega) \\ &\text{and} \qquad c := \frac{\big(n\lambda+(n+1)\mu\big)}{n|\Omega|}\int_\Gamma u_D\cdot\nu\in\mathrm{R}\,. \end{aligned} \tag{2.13}$$

Regarding the explicit knowledge of the unknown σ, the foregoing equation shows that it only remains to find its $\mathbb{H}^r_0(\mathbf{div}_r;\Omega)$-component σ_0. Hence, replacing $\sigma = \sigma_0 + c\,\mathbb{I}$ back into (2.11), redenoting σ_0 simply by σ, noting that the testing of the resulting (2.11) against $\tau\in\mathbb{H}^s(\mathbf{div}_s;\Omega)$ is equivalent to doing it against $\tau\in\mathbb{H}^s_0(\mathbf{div}_s;\Omega)$, and placing this new equation jointly with (2.12), we arrive at the following mixed variational formulation of (2.3): Find $(\sigma,u)\in X_2\times M_1$ such that

$$\begin{aligned} a(\sigma,\tau) + b_1(\tau,u) &= F(\tau) \qquad \forall\,\tau\in X_1\,, \\ b_2(\sigma,v) &= G(v) \qquad \forall\, v\in M_2\,, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} X_2 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega)\,, \quad M_1 := \mathbf{L}^r(\Omega)\,, \quad X_1 := \mathbb{H}^s_0(\mathbf{div}_s;\Omega)\,, \\ \text{and} \quad M_2 := \mathbf{L}^s(\Omega)\,, \end{aligned} \tag{2.15}$$

and the bilinear forms $a: X_2\times X_1\to\mathrm{R}$ and $b_i : X_i\times M_i\to\mathrm{R}$, $i\in\{1,2\}$, and the functionals $F\in X_1'$ and $G\in M_2'$, are defined, respectively, as

$$\begin{aligned} &a(\zeta,\tau) := \frac{1}{\mu}\int_\Omega \zeta^{\mathrm d}:\tau^{\mathrm d} + \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\int_\Omega \mathrm{tr}(\zeta)\,\mathrm{tr}(\tau) \\ &\forall\,(\zeta,\tau)\in X_2\times X_1\,, \end{aligned} \tag{2.16}$$

$$b_i(\tau,v) := \int_\Omega v\cdot\mathbf{div}(\tau) \qquad \forall\,(\tau,v)\in X_i\times M_i\,, \tag{2.17}$$

$$F(\tau) := \langle\tau\,\nu, u_D\rangle_\Gamma \qquad \forall\,\tau\in X_1\,, \tag{2.18}$$

and

$$G(v) := -\int_\Omega f\cdot v \qquad \forall\, v\in M_2\,. \tag{2.19}$$

We remark here that the above notations for the spaces involved have been chosen for convenience of the definitions of b_1 and b_2.

2.2. The Stokes system

The goal of this model is to determine the pseudostress tensor σ, the velocity u, and the pressure p of a steady flow occupying the region Ω, under the action of external forces. More precisely, given a volume force f and a Dirichlet datum u_D, we now seek a tensor field σ, a vector field u, and a scalar field p such that

$$\begin{aligned} &\sigma = 2\mu\,\nabla u - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\sigma) = -f \quad \text{in} \quad \Omega\,, \\ &\mathrm{div}(u) = 0 \quad \text{in} \quad \Omega\,, \quad \int_\Omega p = 0\,, \quad \text{and} \quad u = u_D \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.20}$$

where μ is the kinematic viscosity, and, as required by the incompressibility equation $\mathrm{div}(u)=0$, the datum u_D satisfies the compatibility condition $\int_\Gamma u_D\cdot\nu = 0$. Then, proceeding exactly as in [17, Section 2.1], we can show that (2.20) can be rewritten as

$$\begin{aligned} &\sigma = 2\mu\,\nabla u - p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\sigma) = -f \quad \text{in} \quad \Omega\,, \\ &p + \frac{1}{n}\mathrm{tr}(\sigma) = 0 \quad \text{in} \quad \Omega\,, \quad \int_\Omega p = 0\,, \quad \text{and} \quad u = u_D \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{2.21}$$

from which, eliminating the pressure p, which can be calculated later on by the postprocessing formula $p = -\frac{1}{n}\mathrm{tr}(\sigma)$, we arrive at the equivalent system

$$\begin{aligned} &\frac{1}{2\mu}\sigma^{\mathrm d} = \nabla u \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\sigma) = -f \quad \text{in} \quad \Omega\,, \\ &\int_\Omega \mathrm{tr}(\sigma) = 0\,, \quad \text{and} \quad u = u_D \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.22}$$

In this way, assuming that $f\in\mathbf{L}^r(\Omega)$ and $u_D\in\mathbf{W}^{1/s,r}(\Gamma)$, and proceeding analogously to the derivation of (2.14), we obtain the following mixed variational formulation of (2.22): Find $(\sigma,u)\in X_2\times M_1$ such that

$$\begin{aligned} \widetilde{a}(\sigma,\tau) + b_1(\tau,u) &= F(\tau) \qquad \forall\,\tau\in X_1\,, \\ b_2(\sigma,v) &= G(v) \qquad \forall\, v\in M_2\,, \end{aligned} \tag{2.23}$$

where the spaces X_2, M_1, X_1 and M_2, the bilinear forms $b_i : X_i\times M_i\to\mathrm{R}$, $i\in\{1,2\}$, and the functionals F and G are those given by (2.15), (2.17), (2.18), and (2.19), whereas the bilinear form $\widetilde{a} : X_2\times X_1\to\mathrm{R}$ is defined as

$$\widetilde{a}(\zeta,\tau) := \frac{1}{2\mu}\int_\Omega \zeta^{\mathrm d}:\tau^{\mathrm d} \qquad \forall\,(\zeta,\tau)\in X_2\times X_1\,. \tag{2.24}$$

Later on in Section 4 we prove the well-posedness of the mixed variational formulations (2.14) and (2.23), for which we establish below some results that will be employed in the respective proofs.

3. Some preliminary results

We begin by considering a Banach spaces-based primal formulation for a slight generalization of the Stokes system (2.20) with viscosity $\mu = 1/2$ and null Dirichlet boundary condition, which, given $r, s\in(1,+\infty)$ conjugate to each other, $g\in\mathbb{L}^r(\Omega)$, and $f\in\mathbf{L}^r(\Omega)$, consists of seeking a pair $(u,p)\in\mathbf{W}^{1,r}(\Omega)\times\mathrm{L}^r(\Omega)$ such that

$$\begin{aligned} \mathbf{div}(\nabla \boldsymbol{u} - p\mathbb{I} - \boldsymbol{g}) &= -\boldsymbol{f} \quad \text{in} \quad \Omega, \\ \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \quad \int_\Omega p &= 0, \quad \text{and} \quad \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma. \end{aligned} \tag{3.1}$$

Note that the above mentioned generalization refers to the incorporation of a further datum $\boldsymbol{g}$ within the divergence operator, whose purpose, rather mathematical than physical, has to do with the introduction in Lemma 3.1 of a key operator for our analysis, and particularly with the verification of its divergence free property (cf. (3.7)).

Then, applying (2.7) with $\boldsymbol{\tau} := \nabla \boldsymbol{u} - p\mathbb{I} - \boldsymbol{g} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$ and $\boldsymbol{v} \in \mathbf{W}_0^{1,s}(\Omega)$, and performing some minor algebraic rearrangements, the testing of the first equation of (3.1) becomes

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_\Omega p \operatorname{div}(\boldsymbol{v}) = \int_\Omega \boldsymbol{g} : \nabla \boldsymbol{v} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in \mathbf{W}_0^{1,s}(\Omega). \tag{3.2}$$

In turn, it is easy to see, thanks to the homogeneous Dirichlet boundary condition satisfied by $\boldsymbol{u}$, that testing the incompressibility equation $\operatorname{div}(\boldsymbol{u}) = 0$ in Ω against $q \in \mathrm{L}^s(\Omega)$ is equivalent to doing it against $q \in \mathrm{L}_0^s(\Omega)$. Consequently, the weak formulation of (3.1) reduces to: Find $(\boldsymbol{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_\Omega p \operatorname{div}(\boldsymbol{v}) &= F(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in \mathbf{W}_0^{1,s}(\Omega), \\ \int_\Omega q \operatorname{div}(\boldsymbol{u}) &= 0 \qquad \forall\, q \in \mathrm{L}_0^s(\Omega), \end{aligned} \tag{3.3}$$

where the functional $F \in \mathbf{W}^{-1,r}(\Omega) := \mathbf{W}_0^{1,s}(\Omega)'$ is defined as

$$F(\boldsymbol{v}) := \int_\Omega \boldsymbol{g} : \nabla \boldsymbol{v} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{W}_0^{1,s}(\Omega). \tag{3.4}$$

We now establish, as a consequence of a more general result from [20], the well-posedness of (3.3), even irrespective of the particular form of F given by (3.4).

Theorem 3.1. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $r, s \in (1,+\infty)$ conjugate to each other. Then, there exists $\delta > 0$ such that for each $r \in \left(\frac{2n}{n+1} - \delta, \frac{2n}{n-1} + \delta\right)$, and for each $F \in \mathbf{W}^{-1,r}(\Omega)$, there exists a unique pair $(\boldsymbol{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ solution to (3.3). Moreover, there exists a positive constant c_r, such that*

$$\|\boldsymbol{u}\|_{1,r;\Omega} + \|p\|_{0,r;\Omega} \le c_r \|F\|_{-1,r;\Omega}. \tag{3.5}$$

Proof. We first assume that $\Omega \subset \mathrm{R}^2$. Then, taking the local parameters $\alpha = -1$ and $q = 2$ in [20, Corollary 1.7], we deduce, according to [20, eq. (1.47)], that there exists $\epsilon \in (0, \frac{1}{2}]$ such that for each $F \in \mathbf{W}^{-1,r}(\Omega)$ the problem (3.3) has a unique solution $(\boldsymbol{u}, p) \in \mathbf{W}_0^{1,r}(\Omega) \times \mathrm{L}_0^r(\Omega)$ satisfying (3.5) whenever the point $\left(\alpha - \frac{1}{r} + 2, \frac{1}{r}\right) = \left(1 - \frac{1}{r}, \frac{1}{r}\right)$ belongs to the two-dimensional region specified by [20, Figure 1]. More precisely, the latter means either

i) $0 < 1 - \dfrac{1}{r} < \dfrac{1}{2} + \epsilon$ and $0 < \dfrac{1}{r} < \dfrac{3}{2} - \dfrac{1}{r} + \epsilon$, or

ii) $\dfrac{1}{2} + \epsilon \le 1 - \dfrac{1}{r} < 1$ and $\dfrac{1}{2} - \dfrac{1}{r} - \epsilon < \dfrac{1}{r} < \dfrac{3}{2} - \dfrac{1}{r} + \epsilon$.

Then, solving these inequalities, one obtains $r \in \left(\frac{4}{3} - \epsilon_1, \frac{2}{1-2\epsilon}\right)$, with $\epsilon_1 := \frac{8\epsilon}{9+6\epsilon}$, and $r \in \left[\frac{2}{1-2\epsilon}, 4 + \epsilon_2\right)$, with $\epsilon_2 := \frac{8\epsilon}{1-2\epsilon}$, as solutions of i) and ii), respectively, so that the final feasible range for r is the interval $\left(\frac{4}{3} - \epsilon_1, 4 + \epsilon_2\right)$. In this way, observing now that $\epsilon_1 < \epsilon < \epsilon_2$, we arrive at the indicated range for r (cf. [20, eq. (1.52)]) with $\delta = \epsilon_1$. In turn, the case $\Omega \subset \mathrm{R}^3$ proceeds analogously by imposing now the point $\left(1 - \frac{1}{r}, \frac{1}{r}\right)$ to belong to the two-dimensional region specified by [20, Figure 2]. We omit further details. □

We stress here that when F is given by (3.4), the a priori estimate (3.5) becomes

$$\|u\|_{1,r;\Omega} + \|p\|_{0,r;\Omega} \le c_r \left\{ \|\boldsymbol{g}\|_{0,r;\Omega} + \|\boldsymbol{f}\|_{0,r;\Omega} \right\}. \tag{3.6}$$

The following result, which constitutes an extension of [18, Lemma 2.3] to the present tensor context, makes use of Theorem 3.1 to introduce a suitable operator mapping $\mathbb{L}^t(\Omega)$ into itself for each t in the range specified by this theorem.

Lemma 3.1. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $t, t' \in (1,+\infty)$ conjugate to each other with t satisfying the range given by Theorem 3.1. Then, there exists a linear and bounded operator $D_t : \mathbb{L}^t(\Omega) \to \mathbb{L}^t(\Omega)$ such that*

$$\mathbf{div}\big(D_t(\boldsymbol{\tau})\big) = \mathbf{0} \quad \textit{in} \quad \Omega, \tag{3.7}$$

and

$$\int_\Omega \operatorname{tr}\big(D_t(\boldsymbol{\tau})\big) = \int_\Omega \operatorname{tr}(\boldsymbol{\tau}), \tag{3.8}$$

for all $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$. In addition, for each $\boldsymbol{\zeta} \in \mathbb{L}^{t'}(\Omega)$ such that $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$ in Ω, there holds

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \big(D_t(\boldsymbol{\tau})\big)^{\mathrm{d}} = \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{L}^t(\Omega). \tag{3.9}$$

Proof. Given $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$, we let $(\boldsymbol{u}, p) \in \mathbf{W}^{1,t}(\Omega) \times \mathrm{L}^t(\Omega)$ be the unique solution, guaranteed by Theorem 3.1, of the Stokes problem (3.1) with $r = t$, $\boldsymbol{g} = \boldsymbol{\tau}$ and $\boldsymbol{f} = \mathbf{0}$, that is

$$\begin{aligned} \mathbf{div}(\nabla \boldsymbol{u} - p\mathbb{I} - \boldsymbol{\tau}) &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega, \quad \int_\Omega p &= 0, \quad \text{and} \quad \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \end{aligned} \tag{3.10}$$

whose weak formulation is given by (3.3) and (3.4). Note that the functional $F \in \mathbf{W}^{-1,t}(\Omega) = \mathbf{W}_0^{1,t'}(\Omega)'$ (cf. (3.4)) reduces in this case to $F(\boldsymbol{v}) := \int_\Omega \boldsymbol{\tau} : \nabla \boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbf{W}_0^{1,t'}(\Omega)$. It follows, in virtue of the continuous dependence result (3.6), that $\|\boldsymbol{u}\|_{1,t;\Omega} + \|p\|_{0,t;\Omega} \le c_t \|\boldsymbol{\tau}\|_{0,t;\Omega}$, so that, defining

$$D_t(\boldsymbol{\tau}) := \boldsymbol{\tau} - (\nabla \boldsymbol{u} - p\mathbb{I}) \in \mathbb{L}^t(\Omega), \tag{3.11}$$

we see that D_t is linear and bounded, namely

$$\|D_t(\boldsymbol{\tau})\|_{0,t;\Omega} \le (1 + n^{1/t} c_t) \|\boldsymbol{\tau}\|_{0,t;\Omega}, \tag{3.12}$$

which implies $\|D_t\| \le (1 + n^{1/t} c_t)$, and clearly $D_t(\boldsymbol{\tau})$ is divergence free in Ω. In addition, since $\operatorname{tr}(\nabla \boldsymbol{u}) = \operatorname{div}(\boldsymbol{u}) = 0$ and $\int_\Omega p = 0$, we readily deduce from (3.11) that for each $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$ there holds

$$\int_\Omega \operatorname{tr}\big(D_t(\boldsymbol{\tau})\big) = \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) + n \int_\Omega p = \int_\Omega \operatorname{tr}(\boldsymbol{\tau}),$$

which proves (3.8). Furthermore, using again that $\operatorname{tr}(\nabla \boldsymbol{u}) = 0$, we have that $\big(D_t(\boldsymbol{\tau})\big)^{\mathrm{d}} = \boldsymbol{\tau}^{\mathrm{d}} - \nabla \boldsymbol{u}$, and hence, given $\boldsymbol{\zeta} \in \mathbb{L}^{t'}(\Omega)$ such that $\mathbf{div}(\boldsymbol{\zeta}) = 0$ in Ω, and applying (2.7) to $\boldsymbol{\zeta} \in \mathbb{H}^{t'}(\mathbf{div}_{t'};\Omega)$ and $\boldsymbol{u} \in \mathbf{W}_0^{1,t}(\Omega)$, we deduce that

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \nabla \boldsymbol{u} = \int_\Omega \boldsymbol{\zeta} : \nabla \boldsymbol{u} = 0,$$

which yields (3.9) and ends the proof. □

On the other hand, for each $t \in (1,+\infty)$ we introduce the subspace of $\mathrm{L}^t(\Omega)$ given by

$$\mathrm{L}^t_0(\Omega) := \left\{ v \in \mathrm{L}^t(\Omega) : \quad \int_\Omega v = 0 \right\}. \tag{3.13}$$

Then, we have from [12, Lemma B.69] (see [6] for the original reference, or [11]) the following result.

Lemma 3.2. *Let Ω be a bounded domain of R^n, $n \geq 2$, which is star-shaped with respect to a ball. Then, for each $t \in (1,+\infty)$ the operator $\mathrm{div} : \mathbf{W}^{1,t}(\Omega) \to \mathrm{L}^t_0(\Omega)$ is surjective.*

Thanks to Lemma 3.2 and the open mapping theorem (cf. [2, Theorem 7.7]), we readily deduce that, given $t \in (1,+\infty)$, there exists a constant $C_t > 0$, such that for every $v \in \mathrm{L}^t_0(\Omega)$ there exists $\boldsymbol{z}_v \in \mathbf{W}^{1,t}_0(\Omega)$ satisfying

$$\mathrm{div}(\boldsymbol{z}_v) = v \quad \text{and} \quad \|\boldsymbol{z}_v\|_{1,t;\Omega} \leq C_t \, \|v\|_{0,t;\Omega}. \tag{3.14}$$

We now employ Lemma 3.2, and particularly (3.14), to provide a generalization from $r = 2$ to any $r \in (1,+\infty)$ of the inequality stated in [7, Chapter IV, Proposition 3.1] (see also [13, Lemma 2.3]), namely

$$\|\boldsymbol{\tau}\|_{0,\Omega} \leq C \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega} \right\} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^2_0(\mathbf{div}_2;\Omega),$$

which plays a key role in the solvability analysis of the classical Hilbertian dual-mixed variational formulation of linear elasticity (cf. [7, Chapter IV, Section IV.3], [13, Section 2.4.3]). More precisely, we have the following result.

Lemma 3.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star-shaped with respect to a ball, and let $r \in (1,+\infty)$. Then, there exist positive constants $\widetilde{C}_r$ and $\widehat{C}_r$ such that*

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,r;\Omega} \leq \widetilde{C}_r \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \tag{3.15}$$

and

$$\|\boldsymbol{\tau}\|_{0,r;\Omega} \leq \widehat{C}_r \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \tag{3.16}$$

for all $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$.

Proof. Given $r, s \in (1,+\infty)$ conjugate to each other, we first recall that the dual of $\mathrm{L}^s(\Omega)$ is identified with $\mathrm{L}^r(\Omega)$. Then, given $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$, which yields $\mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{L}^r_0(\Omega)$, we apply the associated duality argument and the fact that $\mathrm{L}^s(\Omega) = \mathrm{L}^s_0(\Omega) \oplus \mathrm{R}$, to observe that

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,r;\Omega} = \sup_{\substack{v \in \mathrm{L}^s(\Omega) \\ v \neq 0}} \frac{\displaystyle\int_\Omega v\,\mathrm{tr}(\boldsymbol{\tau})}{\|v\|_{0,s;\Omega}} = \sup_{\substack{v \in \mathrm{L}^s_0(\Omega) \\ v \neq 0}} \frac{\displaystyle\int_\Omega v\,\mathrm{tr}(\boldsymbol{\tau})}{\|v\|_{0,s;\Omega}}. \tag{3.17}$$

Next, given $v \in \mathrm{L}^s_0(\Omega)$, $v \neq 0$, we make use of (3.14) (with $t = s$) and proceed analogously to the proof of [7, Chapter IV, Proposition 3.1] to estimate $\int_\Omega v\,\mathrm{tr}(\boldsymbol{\tau})$. Indeed, recalling that $\mathrm{div}(\boldsymbol{z}_v) = \mathrm{tr}(\nabla \boldsymbol{z}_v)$, utilizing the definition and properties of the deviatoric tensors, and then integrating by parts according to (2.7) with $\boldsymbol{\tau} \in \mathbb{H}^r_0(\mathbf{div}_r;\Omega)$ and $\boldsymbol{z}_v \in \mathbf{W}^{1,s}_0(\Omega)$, we find that

$$\begin{aligned}
\int_\Omega v\,\mathrm{tr}(\boldsymbol{\tau}) &= \int_\Omega \mathrm{div}(\boldsymbol{z}_v)\,\mathrm{tr}(\boldsymbol{\tau}) = \int_\Omega \mathrm{tr}(\nabla \boldsymbol{z}_v)\,\boldsymbol{\tau} : \mathbb{I} \\
&= \int_\Omega \boldsymbol{\tau} : \mathrm{tr}(\nabla \boldsymbol{z}_v)\,\mathbb{I} = n \int_\Omega \boldsymbol{\tau} : \left(\nabla \boldsymbol{z}_v - (\nabla \boldsymbol{z}_v)^{\mathrm{d}} \right) \\
&= n \int_\Omega \boldsymbol{\tau} : \nabla \boldsymbol{z}_v - n \int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla \boldsymbol{z}_v \\
&= -n \int_\Omega \boldsymbol{z}_v \cdot \mathbf{div}(\boldsymbol{\tau}) - n \int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla \boldsymbol{z}_v,
\end{aligned}$$

from which, employing Hölder's inequality and (3.14), we obtain

$$\begin{aligned}
\left| \int_\Omega v\,\mathrm{tr}(\boldsymbol{\tau}) \right| &\leq n \, \|\boldsymbol{z}_v\|_{1,s;\Omega} \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\} \\
&\leq n\, C_s \, \|v\|_{0,s;\Omega} \left\{ \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \right\}.
\end{aligned} \tag{3.18}$$

In this way, replacing (3.18) back into (3.17) we arrive at (3.15) with $\widetilde{C}_r := n\, C_s$. Furthermore, using the triangle inequality and the fact that $\|\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\|^r_{0,r;\Omega} = n \|\mathrm{tr}(\boldsymbol{\tau})\|^r_{0,r;\Omega}$, we get

$$\|\boldsymbol{\tau}\|_{0,r;\Omega} \leq \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + \frac{1}{n} \|\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}\|_{0,r;\Omega} = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,r;\Omega} + n^{1/r-1} \, \|\mathrm{tr}(\boldsymbol{\tau})\|_{0,r;\Omega},$$

which, along with (3.15), implies (3.16) with $\widehat{C}_r := 1 + n^{1/r}\, C_s$. $\square$

We end this section with a Banach spaces-based primal formulation for the vector Poisson equation, which, given $r, s \in (1,+\infty)$ conjugate to each other, $\boldsymbol{g} \in \mathbb{L}^r(\Omega)$, and $\boldsymbol{f} \in \mathbf{L}^r(\Omega)$, consists of seeking $\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)$ such that

$$\mathbf{div}(\nabla \boldsymbol{u} - \boldsymbol{g}) = -\boldsymbol{f} \quad \text{in} \quad \Omega, \qquad \boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{3.19}$$

Then, proceeding similarly as for (3.1), that is applying (2.7) with $\boldsymbol{\tau} := \nabla \boldsymbol{u} - \boldsymbol{g} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)$ and $\boldsymbol{w} \in \mathbf{W}^{1,s}_0(\Omega)$, we arrive at the following weak formulation of (3.19): Find $\boldsymbol{u} \in \mathbf{W}^{1,r}_0(\Omega)$ such that

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{w} = F(\boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in \mathbf{W}^{1,s}_0(\Omega), \tag{3.20}$$

where $F \in \mathbf{W}^{-1,r}(\Omega) := \mathbf{W}^{1,s}_0(\Omega)'$ is defined as in (3.4), that is

$$F(\boldsymbol{w}) := \int_\Omega \boldsymbol{g} : \nabla \boldsymbol{w} + \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} \qquad \forall\, \boldsymbol{w} \in \mathbf{W}^{1,s}_0(\Omega). \tag{3.21}$$

We establish next the analogue of Theorem 3.1 for the vector Poisson equation, which arises in this case as a straightforward consequence of more general results provided in [19]. We remark in advance that the arguments of the proof are very similar to those from Theorem 3.1, whereas the resulting ranges for r are exactly the same. In addition, we stress that while [19] addresses the scalar Poisson equation, the analysis and results certainly applies to the present version as well. Actually, there is no intrinsic difference between both versions, so that we provide below some details just for sake of clearness.

Theorem 3.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $r, s \in (1,+\infty)$ conjugate to each other. Then, there exists $\delta > 0$ such that for each $r \in \left(\frac{2n}{n+1} - \delta, \frac{2n}{n-1} + \delta \right)$, and for each $F \in \mathbf{W}^{-1,r}(\Omega)$, there exists a unique $\boldsymbol{u} \in \mathbf{W}^{1,r}_0(\Omega)$ solution to (3.20). Moreover, there exists a positive constant $\bar{c}_r$, such that*

$$\|\boldsymbol{u}\|_{1,r;\Omega} \leq \bar{c}_r \, \|F\|_{-1,r;\Omega}. \tag{3.22}$$

Proof. We first assume that $\Omega \subset \mathrm{R}^3$. Then, taking the local parameter $\alpha = 1$ in [19, Theorem 1.1], we deduce that there exists $\epsilon \in (0,1]$ such that for each $F \in \mathbf{W}^{-1,r}(\Omega)$ the problem (3.20) has a unique solution $\boldsymbol{u} \in \mathbf{W}^{1,r}_0(\Omega)$ satisfying (3.22) whenever:

i) $1 < r \leq \kappa$ and $\dfrac{3}{r} - 1 - \epsilon < 1 < 1 + \dfrac{1}{r}$, or

ii) $\kappa < r < \kappa'$ and $\dfrac{1}{r} < 1 < 1 + \dfrac{1}{r}$, or

iii) $\kappa' \leq r < +\infty$ and $\dfrac{1}{r} < 1 < \dfrac{3}{r} + \epsilon$,

where $\kappa = \frac{2}{1+\epsilon}$ and $\kappa' = \frac{2}{1-\epsilon}$. Then, in order to guarantee that at least one of the above is accomplished, one simply solves the three inequalities on the right hand-side, which gives

$$\frac{3}{2} - \epsilon_1 < r < 3 + \epsilon_2 \quad \text{with} \quad \epsilon_1 := \frac{3\epsilon}{2(2+\epsilon)} \quad \text{and} \quad \epsilon_2 := \frac{3\epsilon}{1-\epsilon}.$$

Hence, noticing that $\epsilon_1 < \epsilon < \epsilon_2$, we obtain the indicated range for r with $\delta = \epsilon_1$. The case $\Omega \subset \mathrm{R}^2$ proceeds analogously by taking now $\alpha = 1$ in [19, Theorem 1.3]. Further details are omitted. $\square$

4. The main results

In this section we apply the Babuška-Brezzi theory in Banach spaces and the results from Section 3 to prove the unique solvability and continuous dependence result for each one of the mixed variational formulations (2.14) and (2.23). For sake of completeness and clearness, we follow [5, Theorem 2.1, Corollary 2.1, Section 2.1] to state below the main theorem concerning the aforementioned theory.

Theorem 4.1. *Let X_1, X_2, M_1, and M_2 be real reflexive Banach spaces, and let $a : X_2 \times X_1 \to \mathrm{R}$ and $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1,2\}$, respectively. In addition, for each $i \in \{1,2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i := \Big\{ \tau \in X_i : \quad b_i(\tau, v) = 0 \quad \forall\, v \in M_i \Big\}.$$

Assume that

i) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{\tau\in\mathcal{K}_1\\ \tau\neq 0}} \frac{a(\zeta,\tau)}{\|\tau\|_{X_1}} \geq \alpha\,\|\zeta\|_{X_2} \qquad \forall\,\zeta\in\mathcal{K}_2\,,$$

ii) *there holds*

$$\sup_{\zeta\in\mathcal{K}_2} a(\zeta,\tau) > 0 \qquad \forall\,\tau\in\mathcal{K}_1\,,\ \tau\neq 0\,,$$

iii) *for each $i \in \{1,2\}$ there exists $\beta_i > 0$ such that*

$$\sup_{\substack{\zeta\in X_i\\ \zeta\neq 0}} \frac{b_i(\zeta,v)}{\|\zeta\|_{X_i}} \geq \beta_i\,\|v\|_{M_i} \qquad \forall\,v\in M_i\,.$$

Then, for each $(F,G) \in X_1' \times M_2'$ there exists a unique $(\sigma,u) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\sigma,\tau) + b_1(\tau,u) &= F(\tau) \qquad \forall\,\tau\in X_1\,,\\ b_2(\sigma,v) &= G(v) \qquad \forall\,v\in M_2\,, \end{aligned} \tag{4.1}$$

and the following a priori estimates hold:

$$\begin{aligned} \|\sigma\|_{X_2} &\leq \frac{1}{\alpha}\,\|F\|_{X_1'} + \frac{1}{\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{M_2'}\,,\\ \|u\|_{M_1} &\leq \frac{1}{\beta_1}\left(1+\frac{\|a\|}{\alpha}\right)\|F\|_{X_1'} + \frac{\|a\|}{\beta_1\,\beta_2}\left(1+\frac{\|a\|}{\alpha}\right)\|G\|_{M_2'}\,. \end{aligned} \tag{4.2}$$

Moreover, i), ii), *and* iii) *are also necessary conditions for the well-posedness of* (4.1).

We begin by providing a tensor version of [18, Lemma 2.2]. Indeed, given $t, t' \in (1,+\infty)$ conjugate to each other, we define for each $\boldsymbol{\tau} \in \mathbb{L}^t(\Omega)$

$$\mathcal{J}_t(\boldsymbol{\tau}) := \begin{cases} \|\boldsymbol{\tau}\|^{t-2}\,\boldsymbol{\tau} & \text{if } \boldsymbol{\tau}\neq\mathbf{0}\,,\\ \mathbf{0} & \text{otherwise}\,, \end{cases} \tag{4.3}$$

and observe, after simple algebraic computations, that

$$\boldsymbol{\tau}_{t'} := \mathcal{J}_t(\boldsymbol{\tau}) \in \mathbb{L}^{t'}(\Omega) \quad \text{if and only if} \quad \boldsymbol{\tau} = \mathcal{J}_{t'}(\boldsymbol{\tau}_{t'}), \quad \text{and} \tag{4.4}$$

$$\int_\Omega \boldsymbol{\tau} : \boldsymbol{\tau}_{t'} = \|\boldsymbol{\tau}\|_{0,t;\Omega}^t = \|\boldsymbol{\tau}_{t'}\|_{0,t';\Omega}^{t'} = \|\boldsymbol{\tau}\|_{0,t;\Omega}\,\|\boldsymbol{\tau}_{t'}\|_{0,t';\Omega}\,. \tag{4.5}$$

Next, for each $i \in \{1,2\}$ we let $K_i \subset X_i$ be the kernel of the bilinear form $\boldsymbol{b}_i$, which, according to the definition of the spaces involved (cf. (2.15)), and b_i (cf. (2.17)), yields

$$K_i := \Big\{ \boldsymbol{\tau} \in X_i : \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \Big\}. \tag{4.6}$$

Then, the inf-sup conditions required for the bilinear form $\boldsymbol{a}$ (cf. (2.16)) are established as follows.

Lemma 4.1. *Assume that r and s satisfy the range specified by Theorem 3.1. Then, there exist positive constants M and α such that for each $\lambda > M$ there hold*

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\boldsymbol{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \geq \alpha\,\|\boldsymbol{\zeta}\|_{X_2} \qquad \forall\,\boldsymbol{\zeta}\in K_2\,, \tag{4.7}$$

and

$$\sup_{\boldsymbol{\zeta}\in K_2} \boldsymbol{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) > 0 \qquad \forall\,\boldsymbol{\tau}\in K_1\,,\ \boldsymbol{\tau}\neq\mathbf{0}\,. \tag{4.8}$$

Proof. We first observe that for each pair $(\boldsymbol{\zeta},\boldsymbol{\tau}) \in X_2 \times X_1 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \times \mathbb{H}_0^s(\mathbf{div}_s;\Omega)$ there holds

$$\left|\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau})\right| \leq n^{1/r}\,\|\mathrm{tr}(\boldsymbol{\zeta})\|_{0,r;\Omega}\,\|\boldsymbol{\tau}\|_{0,s;\Omega}\,, \tag{4.9}$$

which follows from simple applications of the Hölder and triangle inequalities, the latter in $\mathrm{L}^s(\Omega)$ and the former in $\mathrm{L}^r(\Omega)\times\mathrm{L}^s(\Omega)$ and $\mathbf{R}\times\mathbf{R}$. Now, let $\boldsymbol{\zeta} \in K_2$, that is $\boldsymbol{\zeta} \in X_2 := \mathbb{H}_0^r(\mathbf{div}_r;\Omega)$ and $\mathbf{div}(\boldsymbol{\zeta}) = \mathbf{0}$, and assume that $\boldsymbol{\zeta} \neq \mathbf{0}$. Then, bearing in mind the definition of $\boldsymbol{a}$ (cf. (2.16)), and employing (4.9) and (3.15) (cf. Lemma 3.3), we readily find that

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\boldsymbol{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \geq \frac{1}{\mu}\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} - \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \tag{4.10}$$

In turn, letting $\boldsymbol{\zeta}_s := \mathcal{J}_r(\boldsymbol{\zeta}^{\mathrm{d}}) \in \mathbb{L}^s(\Omega)$ as defined in (4.3), we clearly have $\mathrm{tr}(\boldsymbol{\zeta}_s) = 0$, and thus, thanks to Lemma 3.1, it follows that $D_s(\boldsymbol{\zeta}_s)$ belongs to K_1. Next, using (3.9) and (4.5), we get

$$\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}} = \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\zeta}_s^{\mathrm{d}} = \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\zeta}_s = \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}\,,$$

and hence, noting that $\|D_s(\boldsymbol{\zeta}_s)\|_{X_1} = \|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}$, and employing the boundedness of D_s (cf. (3.12)), we deduce that

$$\begin{aligned} \sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{X_1}} &\geq \frac{\displaystyle\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \big(D_s(\boldsymbol{\zeta}_s)\big)^{\mathrm{d}}}{\|D_s(\boldsymbol{\zeta}_s)\|_{X_1}} = \frac{\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,\|\boldsymbol{\zeta}_s\|_{0,s;\Omega}}{\|D_s(\boldsymbol{\zeta}_s)\|_{0,s;\Omega}}\\ &\geq \frac{1}{\|D_s\|}\,\|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,. \end{aligned} \tag{4.11}$$

In this way, replacing the foregoing estimate back into (4.10), we arrive at

$$\sup_{\substack{\boldsymbol{\tau}\in K_1\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{\boldsymbol{a}(\boldsymbol{\zeta},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{X_1}} \geq \left\{ \frac{1}{\mu\|D_s\|} - \frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)} \right\} \|\boldsymbol{\zeta}^{\mathrm{d}}\|_{0,r;\Omega}\,, \tag{4.12}$$

from which, choosing λ sufficiently large such that

$$\frac{\widetilde{C}_r}{n^{1/s}\big(n\lambda+(n+1)\mu\big)} < \frac{1}{2\mu\|D_s\|}\,,$$

that is

$$\lambda > M_s := \frac{\mu}{n^{1+1/s}}\max\Big\{2\|D_s\|\widetilde{C}_r - n^{1/s}(n+1), 0\Big\},$$

and applying (3.16), we conclude (4.7) with $\alpha := \frac{1}{2\mu\|D_s\|\widehat{C}_r}$. On the other hand, given now $\boldsymbol{\tau} \in K_1$, $\boldsymbol{\tau} \neq \mathbf{0}$, we proceed analogously as above, but exchanging the roles of $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$, and obtain

$$\sup_{\zeta\in K_2} \boldsymbol{a}(\zeta,\tau) \geq \sup_{\substack{\zeta\in K_2\\ \zeta\neq 0}} \frac{\boldsymbol{a}(\zeta,\tau)}{\|\zeta\|_{X_2}} \geq \frac{1}{2\mu\|D_r\|\widehat{C}_s}\|\tau\|_{X_1} > 0 \tag{4.13}$$

for $\lambda > M_r := \frac{\mu}{n^{1+1/r}} \max\left\{2\|D_r\|\widetilde{C}_s - n^{1/r}(n+1), 0\right\}$, which proves (4.8). Finally, the proof is completed by choosing $M := \max\left\{M_s, M_r\right\}$. $\square$

We stress here that, constituting the bilinear form $\widetilde{\boldsymbol{a}}$ a key part of $\boldsymbol{a}$, some of the arguments employed in the proof of Lemma 4.1 allow us to establish next the inf-sup conditions required for the former.

Lemma 4.2. *Assume that r and s satisfy the range specified by Theorem 3.1. Then, there exists a positive constant $\widetilde{\alpha}$ such that*

$$\sup_{\substack{\tau\in K_1\\ \tau\neq 0}} \frac{\widetilde{\boldsymbol{a}}(\zeta,\tau)}{\|\tau\|_{X_1}} \geq \widetilde{\alpha}\,\|\zeta\|_{X_2} \qquad \forall\,\zeta\in K_2. \tag{4.14}$$

In addition, there holds

$$\sup_{\zeta\in K_2} \widetilde{\boldsymbol{a}}(\zeta,\tau) > 0 \qquad \forall\,\tau\in K_1,\ \tau\neq \mathbf{0}. \tag{4.15}$$

Proof. It follows straightforwardly from the definition of $\widetilde{\boldsymbol{a}}$ (cf. (2.24)), and the inequalities (4.11), and (3.16), that for each $\zeta\in K_2$, $\zeta\neq\mathbf{0}$, there holds

$$\sup_{\substack{\tau\in K_1\\ \tau\neq 0}} \frac{\widetilde{\boldsymbol{a}}(\zeta,\tau)}{\|\tau\|_{X_1}} = \frac{1}{2\mu}\sup_{\substack{\tau\in K_1\\ \tau\neq 0}} \frac{\displaystyle\int_\Omega \zeta^{\mathrm{d}} : \tau^{\mathrm{d}}}{\|\tau\|_{X_1}} \geq \frac{1}{2\mu\|D_s\|\widehat{C}_r}\|\zeta\|_{X_2}, \tag{4.16}$$

which yields (4.14) with $\widetilde{\alpha} := \frac{1}{2\mu\|D_s\|\widehat{C}_r}$. In addition, given $\tau\in K_1$, $\tau\neq\mathbf{0}$, and proceeding analogously to the derivation of (4.13), that is exchanging the roles of ζ and τ and using (4.16), we easily find that

$$\sup_{\zeta\in K_2} \widetilde{\boldsymbol{a}}(\zeta,\tau) \geq \sup_{\substack{\zeta\in K_2\\ \zeta\neq 0}} \frac{\widetilde{\boldsymbol{a}}(\zeta,\tau)}{\|\zeta\|_{X_2}} \geq \frac{1}{2\mu\|D_r\|\widehat{C}_s}\|\tau\|_{X_1} > 0, \tag{4.17}$$

which shows (4.15) and ends the proof. $\square$

It only remains to verify the inf-sup conditions for the bilinear forms $\boldsymbol{b}_i$, $i\in\{1,2\}$, which we address in what follows.

Lemma 4.3. *Assume that r and s satisfy the range specified by Theorem 3.2. Then, there exist positive constants β_1, β_2 such that for each $i\in\{1,2\}$ there hold*

$$\sup_{\substack{\zeta\in X_i\\ \zeta\neq 0}} \frac{\boldsymbol{b}_i(\zeta,\boldsymbol{v})}{\|\zeta\|_{X_i}} \geq \beta_i\,\|\boldsymbol{v}\|_{M_i} \qquad \forall\,\boldsymbol{v}\in M_i. \tag{4.18}$$

Proof. Having $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ the same algebraic structure (cf. (2.17)), and being the pairs (X_1,M_1) and (X_2,M_2) one obtained from the other by exchanging r and s, we now proceed to show (4.18) only for $i=2$ since the proof for $i=1$ is completely analogous. In this way, given $\boldsymbol{v}\in M_2 := \mathbf{L}^s(\Omega)$, we let $\mathcal{J}_s$ be the vector version of $\mathcal{J}_s$ (cf. (4.3)), and set $\boldsymbol{v}_r := \mathcal{J}_s(\boldsymbol{v})\in\mathbf{L}^r(\Omega)$, for which, similarly to (4.4) and (4.5), there hold

$$\boldsymbol{v} = \mathcal{J}_r(\boldsymbol{v}_r), \quad\text{and}\quad \int_\Omega \boldsymbol{v}\cdot\boldsymbol{v}_r = \|\boldsymbol{v}\|^s_{0,s;\Omega} = \|\boldsymbol{v}_r\|^r_{0,r;\Omega} = \|\boldsymbol{v}\|_{0,s;\Omega}\,\|\boldsymbol{v}_r\|_{0,r;\Omega}. \tag{4.19}$$

Then, we let $\boldsymbol{z}\in\mathbf{W}^{1,r}_0(\Omega)$ be the unique solution, guaranteed by Theorem 3.2, of the vector Poisson equation (3.19) with $\boldsymbol{g}=\mathbf{0}$ and $\boldsymbol{f}=-\boldsymbol{v}_r$, that is

$$\Delta\boldsymbol{z} = \boldsymbol{v}_r \quad\text{in}\quad \Omega, \qquad \boldsymbol{z}=0 \quad\text{on}\quad \Gamma,$$

whose weak formulation is given by (3.20) and (3.21) with $F(\boldsymbol{w}) := -\int_\Omega \boldsymbol{v}_r\cdot\boldsymbol{w}$ for all $\boldsymbol{w}\in\mathbf{W}^{1,s}_0(\Omega)$. It follows that $\|F\|_{-1,r}\leq\|\boldsymbol{v}_r\|_{0,r;\Omega}$, and thus the continuous dependence result (3.22) yields

$$\|\boldsymbol{z}\|_{1,r;\Omega} \leq \bar{c}_r\,\|\boldsymbol{v}_r\|_{0,r;\Omega}. \tag{4.20}$$

Next, we observe that $\mathbf{div}(\nabla\boldsymbol{z}) = \boldsymbol{v}_r$ in Ω, which proves that $\nabla\boldsymbol{z}\in\mathbb{H}^r(\mathbf{div}_r;\Omega)$, and let $\widehat{\zeta}$ be the $\mathbb{H}^r_0(\mathbf{div}_r;\Omega)$-component (cf. (2.8)) of $\nabla\boldsymbol{z}$. In this way, utilizing (4.20) and noting that $\mathbf{div}(\widehat{\zeta}) = \boldsymbol{v}_r$, we deduce that

$$\|\widehat{\zeta}\|_{X_2} = \|\widehat{\zeta}\|_{0,r;\Omega} + \|\mathbf{div}(\widehat{\zeta})\|_{0,r;\Omega} \leq |\boldsymbol{z}|_{1,r;\Omega} + \|\boldsymbol{v}_r\|_{0,r;\Omega} \leq \left(1+\bar{c}_r\right)\|\boldsymbol{v}_r\|_{0,r;\Omega}.$$

Finally, bearing in mind the definition of $\boldsymbol{b}_2$ (cf. (2.17)), and employing (4.19) and the foregoing inequality, we conclude that

$$\sup_{\substack{\zeta\in X_2\\ \zeta\neq 0}} \frac{\boldsymbol{b}_2(\zeta,\boldsymbol{v})}{\|\zeta\|_{X_2}} \geq \frac{\boldsymbol{b}_2(\widehat{\zeta},\boldsymbol{v})}{\|\widehat{\zeta}\|_{X_2}} = \frac{\displaystyle\int_\Omega \boldsymbol{v}\cdot\boldsymbol{v}_r}{\|\widehat{\zeta}\|_{X_2}} \geq \frac{1}{\left(1+\bar{c}_r\right)}\|\boldsymbol{v}\|_{0,s;\Omega}, \tag{4.21}$$

which gives (4.18) for $i=2$ with $\beta_2 := \left(1+\bar{c}_r\right)^{-1}$. $\square$

Regarding the assumptions on r and its conjugate s, we remark here that $\left[\frac{2n}{n+1},\frac{2n}{n-1}\right]$ constitutes the largest subset of $\left(\frac{2n}{n+1}-\delta, \frac{2n}{n-1}+\delta\right)$ guaranteeing that both indexes lie simultaneously within it.

We are now in position to establish below the announced well-posedness of (2.14) and (2.23).

Theorem 4.2. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n\in\{2,3\}$, which is star shaped with respect to a ball, and let r, $s\in(1,+\infty)$ conjugate to each other such that they satisfy the range specified by Theorem 3.1 (which coincides with that of Theorem 3.2). Then, there exists a positive constant M such that for each $\lambda>M$ and for each pair $(\boldsymbol{f},\boldsymbol{u}_D)\in\mathbf{L}^r(\Omega)\times\mathbf{W}^{1/s,r}(\Gamma)$, there exists a unique solution $(\sigma,\boldsymbol{u})\in X_2\times M_1 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega)\times\mathrm{L}^r(\Omega)$ to (2.14). Moreover, there exists a positive constant C, independent of the data and the solution, such that*

$$\|\sigma\|_{r,\mathbf{div}_r;\Omega} + \|\boldsymbol{u}\|_{0,r;\Omega} \leq C\left\{\|\boldsymbol{f}\|_{0r;\Omega} + \|\boldsymbol{u}_D\|_{1/s,r;\Gamma}\right\}.$$

Proof. It follows from Lemmas 4.1 and 4.3, along with a straightforward application of Theorem 4.1. $\square$

Theorem 4.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n\in\{2,3\}$, which is star shaped with respect to a ball, and let r, $s\in(1,+\infty)$ conjugate to each other such that they satisfy the range specified by Theorem 3.2 (which coincides with that of Theorem 3.1). Then, for each pair $(\boldsymbol{f},\boldsymbol{u}_D)\in\mathbf{L}^r(\Omega)\times\mathbf{W}^{1/s,r}(\Gamma)$, there exists a unique solution $(\sigma,\boldsymbol{u})\in X_2\times M_1 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega)\times\mathrm{L}^r(\Omega)$ to (2.23). Moreover, there exists a positive constant C, independent of the data and the solution, such that*

$$\|\sigma\|_{r,\mathbf{div}_r;\Omega} + \|\boldsymbol{u}\|_{0,r;\Omega} \leq C\left\{\|\boldsymbol{f}\|_{0r;\Omega} + \|\boldsymbol{u}_D\|_{1/s,r;\Gamma}\right\}.$$

Proof. It follows from Lemmas 4.2 and 4.3, along with a straightforward application of Theorem 4.1. $\square$

We end the paper by announcing that the extension of the present analysis to the discrete setting of a Banach spaces-based mixed formulation for the stress-assisted diffusion problem studied in [15] and [16], will be reported in a separate work. In particular, it will be shown in this case that the feasible ranges for r and its conjugate s are given by the intervals $(2,\frac{2n}{n-1}]$ and $[\frac{2n}{n+1},2)$, respectively. Several numerical experiments illustrating the performance of the resulting method will also be included there.

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