that for each $j \in \mathbb{N}$

$$\|\boldsymbol{\sigma}_j - \mathscr{P}^k_{\mathrm{div},h}(\boldsymbol{\sigma}_j)\|_{\mathrm{div},\Omega} \le C h \{|\boldsymbol{\sigma}_j|_{1,\Omega} + |\mathrm{div}\,\boldsymbol{\sigma}_j|_{1,\Omega}\} \tag{4.35}$$

and

$$\|u_j - \mathscr{P}^k_h(u_j)\|_{0,\Omega} \le C h |u_j|_{1,\Omega}. \tag{4.36}$$

Now, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\mathrm{div},\Omega} < \varepsilon/4 \quad \text{and} \quad \|u - u_N\|_{0,\Omega} < \varepsilon/4.$$

In turn, for $j = N$ we deduce from (4.35) and (4.36) that there exists $h_0 > 0$ such that

$$\|\boldsymbol{\sigma}_N - \mathscr{P}^k_{\mathrm{div},h}(\boldsymbol{\sigma}_N)\|_{\mathrm{div},\Omega} < \varepsilon/4 \quad \text{and} \quad \|u_N - \mathscr{P}^k_h(u_N)\|_{0,\Omega} < \varepsilon/4 \qquad \forall\, h \le h_0.$$

Therefore, from the Cea estimate (4.31) we conclude that for each $h \le h_0$

$$\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathrm{div},\Omega} + \|u - u_h\|_{0,\Omega} &\le C_2 \Big\{ \mathrm{dist}(\boldsymbol{\sigma}, H_h) + \mathrm{dist}(u, Q_h) \Big\} \\
&\le C_2 \Big\{ \|\boldsymbol{\sigma} - \mathscr{P}^k_{\mathrm{div},h}(\boldsymbol{\sigma}_N)\|_{\mathrm{div},\Omega} + \|u - \mathscr{P}^k_h(u_N)\|_{0,\Omega} \Big\} \\
&\le C_2 \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_N\|_{\mathrm{div},\Omega} + \|\boldsymbol{\sigma}_N - \mathscr{P}^k_{\mathrm{div},h}(\boldsymbol{\sigma}_N)\|_{\mathrm{div},\Omega} \\
&\quad + \quad \|u - u_N\|_{0,\Omega} + \|u_N - \mathscr{P}^k_h(u_N)\|_{0,\Omega} \Big\} \\
&\le C_2\, \varepsilon,
\end{aligned}$$

which shows the convergence (4.34).

□

4.3 Primal-mixed formulation of the Poisson problem

In this section we analyze a Galerkin scheme for the primal-mixed formulation of the 2D version of the Poisson problem studied in Section 2.4.4. To this end, we recall that, given a bounded domain $\Omega \subseteq \mathbb{R}^2$ with polygonal boundary Γ, and given data $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, the primal-mixed formulation reduces to (cf. (2.70)): Find $(u, \xi) \in H \times Q$ such that

$$\begin{aligned}
a(u, v) + b(v, \xi) &= F(v) \qquad \forall\, v \in H, \\
b(u, \lambda) \qquad\quad &= G(\lambda) \qquad \forall\, \lambda \in Q,
\end{aligned}$$

where $H := H^1(\Omega)$, $Q := H^{-1/2}(\Gamma)$, a and b are the bilinear forms defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \quad \forall (u,v) \in H \times H,$$
$$b(v,\lambda) := \langle \lambda, v \rangle \quad \forall (v,\lambda) \in H \times Q,$$

and the functionals $F \in H'$ and $G \in Q$ are given by

$$F(v) := \int_\Omega f v \quad \forall v \in H, \qquad G(\lambda) = \langle \lambda, g \rangle \quad \forall \lambda \in Q.$$

Then, given a regular family of triangularizations $\{\mathscr{T}_h\}_{h>0}$ of $\overline{\Omega}$, we introduce the subspaces of H (cf. (4.21)) and Q

$$H_h := X_h^1 := \left\{ v \in C(\overline{\Omega}) : \quad v|_K \in \mathbb{P}_1(K) \qquad \forall K \in \mathscr{T}_h \right\},$$
$$Q_{\widetilde{h}} := \left\{ \lambda \in L^2(\Gamma) : \quad \lambda|_{\widetilde{\Gamma}_j} \in \mathbb{P}_0(\widetilde{\Gamma}_j) \qquad \forall j \in \{1, \cdots, m\} \right\},$$

where $\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \cdots, \widetilde{\Gamma}_m\}$ is a partition of Γ (independent of the one inherited from $\mathscr{T}_h$), and $\widetilde{h} := \max \left\{ |\widetilde{\Gamma}_j| : j \in \{1, \cdots, m\} \right\}$. Hence, the associated Galerkin scheme is: Find $(u_h, \xi_{\widetilde{h}}) \in H_h \times Q_{\widetilde{h}}$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \xi_{\widetilde{h}}) &= F(v_h) \qquad \forall v_h \in H_h, \\ b(u_h, \lambda_{\widetilde{h}}) \qquad\qquad &= G(\lambda_{\widetilde{h}}) \qquad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}}. \end{aligned} \tag{4.37}$$

We now let V_h be the discrete kernel of b, that is

$$\begin{aligned} V_h &:= \left\{ v_h \in H_h : \quad b(v_h, \lambda_{\widetilde{h}}) = 0 \qquad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}} \right\} \\ &= \left\{ v_h \in H_h : \quad \langle \lambda_{\widetilde{h}}, v_h \rangle \quad = 0 \qquad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}} \right\}. \end{aligned}$$

Note that, in particular, $\lambda_{\widetilde{h}} \equiv 1$ belongs to $Q_{\widetilde{h}}$ and therefore

$$V_h \subseteq \widehat{V} := \left\{ v \in H : \quad \langle 1, v \rangle = 0 \right\},$$

that is

$$V_h \subseteq \widehat{V} := \left\{ v \in H : \int_\Gamma v = 0 \right\}.$$

Then, utilizing the generalized Poincaré inequality (cf. [46, Theorem 5.11.2]), one can show that $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ are equivalent in $\widehat{V}$ and hence in V_h. It follows that

$$a(v_h, v_h) = |v_h|_{1,\Omega} \geq c\,\|v_h\|_{1,\Omega} \qquad \forall v_h \in V_h,$$

which shows that a is V_h-elliptic.

We prove next that b satisfies the discrete inf-sup condition, that is there exists $\beta > 0$, independent of h, such that

$$\sup_{\substack{v_h\in H_h\\ v_h\neq 0}} \frac{b(v_h,\lambda_{\widetilde{h}})}{\|v_h\|_{1,\Omega}} \geq \beta\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \qquad \forall\,\lambda_{\widetilde{h}} \in Q_{\widetilde{h}},$$

which is

$$\sup_{\substack{v_h\in H_h\\ v_h\neq 0}} \frac{\langle\lambda_{\widetilde{h}},v_h\rangle}{\|v_h\|_{1,\Omega}} \geq \beta\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \qquad \forall\,\lambda_{\widetilde{h}} \in Q_{\widetilde{h}}.$$

For this purpose we need an inverse inequality for $Q_{\widetilde{h}}$, which is shown by the following lemma for a generic space Q_h.

Lemma 4.6. *Let $\{\Gamma_1,\Gamma_2,\cdots,\Gamma_m\}$ be a partition of Γ, denote $h_j := |\Gamma_j| \quad \forall\, j \in \{1,\cdots,m\}$, assume that there exists $c>0$ such that*

$$h_j \geq c\,h := c \max_{i\in\{1,\cdots,m\}} h_i \qquad \forall\, j\in\{1,\cdots,m\},$$

and define

$$Q_h := \Big\{\lambda_h \in L^2(\Gamma): \quad \lambda_h|_{\Gamma_j} \in \mathbb{P}_0(\Gamma_j) \quad \forall\, j\in\{1,\cdots,m\}\Big\}.$$

Then there exists $C>0$ such that

$$\|\lambda_h\|_{r,\Gamma} \leq C\,h^{-1/2-r}\,\|\lambda_h\|_{-1/2,\Gamma} \qquad \forall\,\lambda_h\in Q_h, \quad \forall\, r\in[-1/2,0].$$

Proof. Since clearly

$$\|\lambda_h\|_{-1/2,\Gamma} \leq h^0\,\|\lambda_h\|_{-1/2,\Gamma} \quad \forall\,\lambda_h\in Q_h\,,$$

it suffices to show that

$$\|\lambda_h\|_{0,\Gamma} \leq C\,h^{-1/2}\,\|\lambda_h\|_{-1/2,\Gamma} \quad \forall\,\lambda_h\in Q_h\,, \tag{4.38}$$

and then conclude by interpolation estimates (cf. [49, Appendix B]). In fact, given $\lambda_h\in Q_h$, we let $\lambda_j := \lambda_h|_{\Gamma_j} \in \mathbb{P}_0(\Gamma_j)$ and observe that

$$\|\lambda_h\|_{0,\Gamma}^2 = \sum_{j=1}^m \|\lambda_h\|_{0,\Gamma_j}^2 = \sum_{j=1}^m h_j\,\lambda_j^2 = \sum_{j=1}^m h_j\,\|\lambda_j\|_{0,\widehat{\Gamma}}^2,$$

where $\widehat{\Gamma}$ is a reference segment of measure $|\widehat{\Gamma}|=1$. For instance, we can consider $\widehat{\Gamma} := \{(x,0): \quad x\in]0,1[\}$. Then, using the equivalence of norms in finite dimension, we have that

$$\|\lambda_h\|_{0,\Gamma}^2 \leq \widehat{c}\sum_{j=1}^m h_j\,\|\lambda_j\|_{-1/2,00,\widehat{\Gamma}}^2\,, \tag{4.39}$$

where $\|\cdot\|_{-1/2,00,\widehat{\Gamma}}$ is the norm of $H_{00}^{-1/2}(\widehat{\Gamma})$, the dual of $H_{00}^{1/2}(\widehat{\Gamma})$. On the other hand, applying the inequality (cf. (3.15), Lemma 3.12)

$$|v|_{m,K} \leq c \, \|B_K^{-1}\|^m \, |\det B_K|^{1/2} \, |\widehat{v}|_{m,\widehat{K}}$$

to $K = \Gamma_j$ and $\widehat{K} = \widehat{\Gamma}$, we obtain

$$\|v\|_{0,\Gamma_j} \leq c \left\{ \frac{|\Gamma_j|}{|\widehat{\Gamma}|} \right\}^{1/2} \|\widehat{v}\|_{0,\widehat{\Gamma}} = \widehat{c} \, h_j^{1/2} \, \|\widehat{v}\|_{0,\widehat{\Gamma}} \qquad \forall \, v \in L^2(\Gamma_j)$$

and

$$|v|_{1,\Gamma_j} \leq c \left\{ \frac{\widehat{h}}{\rho_j} \right\}^1 \left\{ \frac{|\Gamma_j|}{|\widehat{\Gamma}|} \right\}^{1/2} |\widehat{v}|_{1,\widehat{\Gamma}} \leq \widehat{c} \, h_j^{-1/2} \, |\widehat{v}|_{1,\widehat{\Gamma}} \qquad \forall \, v \in H^1(\Gamma_j).$$

Then, according to the interpolation estimates for Sobolev spaces (cf. [49, Appendix B]) and using that $H_{00}^{1/2}(\Gamma_j) = (H_0^1(\Gamma_j), L^2(\Gamma_j))_{1/2}$, we find that

$$\|v\|_{1/2,00,\Gamma_j} \leq \widehat{c} \, \|\widehat{v}\|_{1/2,00,\widehat{\Gamma}} \qquad \forall \, v \in H_{00}^{1/2}(\Gamma_j),$$

where $\|\cdot\|_{1/2,00,S}$ denotes the norm of $H_{00}^{1/2}(S)$ for $S \in \{\Gamma_j, \widehat{\Gamma}\}$.

Analogously, applying now (cf. (3.14), Lemma 3.12)

$$|\widehat{v}|_{m,\widehat{K}} \leq c \, \|B_K\|^m \, |\det B_K|^{-1/2} \, |v|_{m,K}$$

to $K = \Gamma_j$ and $\widehat{K} = \widehat{\Gamma}$, using interpolation estimates again, and noting that $H_{00}^{1/2}(\widehat{\Gamma})$ is given by $(H_0^1(\widehat{\Gamma}), L^2(\widehat{\Gamma}))_{1/2}$, we deduce that

$$\|\widehat{v}\|_{1/2,00,\widehat{\Gamma}} \leq \widehat{c} \, \|v\|_{1/2,00,\Gamma_j} \qquad \forall \, \widehat{v} \in H_{00}^{1/2}(\widehat{\Gamma}),$$

and therefore

$$\|v\|_{1/2,00,\Gamma_j} \cong \|\widehat{v}\|_{1/2,00,\widehat{\Gamma}}.$$

Consequently, given $\lambda \in H_{00}^{-1/2}(\Gamma_j)$, we obtain by a duality argument that

$$\begin{aligned}
\|\widehat{\lambda}\|_{-1/2,00,\widehat{\Gamma}} &= \sup_{\substack{\widehat{v} \in H_{00}^{1/2}(\widehat{\Gamma}) \\ \widehat{v} \neq 0}} \frac{\langle \widehat{\lambda}, \widehat{v} \rangle_{\widehat{\Gamma}}}{\|\widehat{v}\|_{1/2,00,\widehat{\Gamma}}} \\
&= \sup_{\substack{\widehat{v} \in H_{00}^{1/2}(\widehat{\Gamma}) \\ \widehat{v} \neq 0}} \frac{h_j^{-1} \, \langle \lambda, v \rangle_{\Gamma_j}}{\|\widehat{v}\|_{1/2,00,\widehat{\Gamma}}} \\
&\leq C \sup_{\substack{\widehat{v} \in H_{00}^{1/2}(\widehat{\Gamma}) \\ \widehat{v} \neq 0}} \frac{h_j^{-1} \, \|\lambda\|_{-1/2,00,\Gamma_j} \, \|v\|_{1/2,00,\Gamma_j}}{\|v\|_{1/2,00,\Gamma_j}} \\
&= C \, h_j^{-1} \, \|\lambda\|_{-1/2,00,\Gamma_j} \,.
\end{aligned}$$

In this way, employing the above estimate in (4.39), we conclude that

$$\begin{aligned}\|\lambda_h\|_{0,\Gamma}^2 &\le \widehat{c}\sum_{j=1}^{m} h_j h_j^{-2}\,\|\lambda_j\|_{-1/2,00,\Gamma_j}^2\\ &\le \widehat{c}h^{-1}\sum_{j=1}^{m}\|\lambda_j\|_{-1/2,00,\Gamma_j}^2 \le \widehat{c}h^{-1}\,\|\lambda_h\|_{-1/2,\Gamma}^2,\end{aligned}$$

which gives (4.38) and completes the proof.

□

We are now in a position to show the discrete inf-sup condition for b.

Lemma 4.7. *There exist $C_0 > 0$ and $\beta > 0$, independent of h and $\widetilde{h}$, such that for each $h \le C_0\widetilde{h}$ there holds*

$$\sup_{\substack{v_h\in H_h\\ v_h\neq 0}} \frac{\langle \lambda_{\widetilde{h}}, v_h\rangle}{\|v_h\|_{1,\Omega}} \ge \beta\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \qquad \forall\,\lambda_{\widetilde{h}} \in Q_{\widetilde{h}}.$$

Proof. Given $\lambda_{\widetilde{h}} \in Q_{\widetilde{h}}$, we let $z \in H^1(\Omega)$ be the unique solution of the problem

$$-\Delta z + z = 0 \quad \text{in} \quad \Omega, \quad \nabla z\cdot\mathbf{n} = \lambda_{\widetilde{h}} \quad \text{in} \quad \Gamma.$$

The continuous dependence result provided by the classical Lax-Milgram Lemma (cf. Theorem 1.1) establishes that

$$\|z\|_{1,\Omega} \le c\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}.$$

In addition, since $Q_{\widetilde{h}} \subseteq H^{\varepsilon}(\Gamma)$ for some $\varepsilon > 0$, it follows by elliptic regularity (cf [42]) that $z \in H^{1+\delta}(\Omega) \quad \forall\,\delta \in [0,\delta_0]$, where $\delta_0 := \min\{\frac{1}{2}+\varepsilon, \frac{\pi}{\omega}\}$ and ω is the largest interior angle of Ω. We then fix $\delta \in (0,\delta_0)$, $\delta < 1/2$, and observe that

$$\|z\|_{1+\delta,\Omega} \le C\,\|\lambda_{\widetilde{h}}\|_{-1/2+\delta,\Gamma}.$$

On the other hand, since (cf. (4.7))

$$\|v - \mathbf{P}_{1,h}^1(v)\|_{1,\Omega} \le C h\,\|v\|_{2,\Omega} \qquad \forall\,v \in H^2(\Omega)$$

and clearly

$$\|v - \mathbf{P}_{1,h}^1(v)\|_{1,\Omega} \le h^0\,\|v\|_{1,\Omega} \qquad \forall\,v \in H^1(\Omega),$$

the estimates for the interpolation of Sobolev spaces (cf. [49, Appendix B]) imply

$$\|v - \mathbf{P}_{1,h}^1(v)\|_{1,\Omega} \le C h^{\delta}\,\|v\|_{1+\delta,\Omega} \qquad \forall\,v \in H^{1+\delta}(\Omega).$$

It follows that

$$\|z - \mathbf{P}_{1,h}^1(z)\|_{1,\Omega} \le C h^{\delta}\,\|z\|_{1+\delta,\Omega} \le C h^{\delta}\,\|\lambda_{\widetilde{h}}\|_{-1/2+\delta,\Gamma},$$

and using the inverse inequality for $Q_{\widetilde{h}}$ (cf. Lemma 4.6), that is

$$\|\lambda_{\widetilde{h}}\|_{-1/2+\delta,\Gamma} \leq C\widetilde{h}^{-\delta}\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma},$$

we arrive at

$$\|z - \mathbf{P}^1_{1,h}(z)\|_{1,\Omega} \leq C\left\{\frac{h}{\widetilde{h}}\right\}^{\delta}\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}. \tag{4.40}$$

In turn, it is clear that

$$\|\mathbf{P}^1_{1,h}(z)\|_{1,\Omega} \leq \|z\|_{1,\Omega} \leq c\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}. \tag{4.41}$$

Next, using the Green identity in $H(\mathrm{div};\Omega)$ (cf. Lemma 1.4), we see that

$$\begin{aligned}\langle\lambda_{\widetilde{h}}, z\rangle &= \langle\nabla z\cdot\mathbf{n}, z\rangle = \langle\gamma_{\mathbf{n}}(\nabla z), \gamma_0(z)\rangle \\ &= \int_\Omega\Big\{z\,\mathrm{div}\,\nabla z + \nabla z\cdot\nabla z\Big\} = \|z\|^2_{1,\Omega},\end{aligned}$$

and, recalling from Theorem 1.7 that $\gamma_{\mathbf{n}} : H(\mathrm{div};\Omega) \to H^{-1/2}(\Gamma)$ is bounded, we obtain that

$$\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} = \|\gamma_{\mathbf{n}}(\nabla z)\|_{-1/2,\Gamma} \leq \|\nabla z\|_{\mathrm{div},\Omega} = \|z\|_{1,\Omega}\,,$$

which yields

$$\langle\lambda_{\widetilde{h}}, z\rangle \geq \|\lambda_{\widetilde{h}}\|^2_{-1/2,\Gamma}\,. \tag{4.42}$$

In this way, employing the estimates (4.40), (4.41), and (4.42), we find that

$$\begin{aligned}\sup_{\substack{v_h\in H_h\\ v_h\neq 0}}\frac{\langle\lambda_{\widetilde{h}}, v_h\rangle}{\|v_h\|_{1,\Omega}} &\geq \frac{|\langle\lambda_{\widetilde{h}}, \mathbf{P}^1_{1,h}(z)\rangle|}{\|\mathbf{P}^1_{1,h}(z)\|_{1,\Omega}} \\ &\geq \frac{|\langle\lambda_{\widetilde{h}}, \mathbf{P}^1_{1,h}(z)\rangle|}{c\,\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}} \\ &\geq \widetilde{C}\left\{\frac{|\langle\lambda_{\widetilde{h}}, z\rangle|}{\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}} - \frac{|\langle\lambda_{\widetilde{h}}, z - \mathbf{P}^1_{1,h}(z)\rangle|}{\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma}}\right\} \\ &\geq \widetilde{C}\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} - C\left\{\frac{h}{\widetilde{h}}\right\}^{\delta}\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \\ &= \left\{\widetilde{C} - C\left(\frac{h}{\widetilde{h}}\right)^{\delta}\right\}\|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma},\end{aligned}$$

where, choosing $h \leq C_0\,\widetilde{h}$, with $C_0 = \left(\dfrac{\widetilde{C}}{2C}\right)^{1/\delta}$, we deduce the existence of $\beta > 0$ such that

$$\sup_{\substack{v_h \in H_h \\ v_h \neq 0}} \frac{\langle \lambda_{\widetilde{h}}, v_h \rangle}{\|v_h\|_{1,\Omega}} \geq \beta \, \|\lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \qquad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}},$$

thus completing the proof

$\square$

Consequently, applying the results from the discrete Babuška-Brezzi theory (cf. Theorems 2.4 and 2.6), we deduce that $\forall h \leq C_0 \widetilde{h}$ there exists a unique pair $(u_h, \xi_{\widetilde{h}}) \in H_h \times Q_{\widetilde{h}}$ solution of the Galerkin scheme (4.37), and there holds the Cea estimate

$$\begin{aligned} &\|u - u_h\|_{1,\Omega} + \|\xi - \xi_{\widetilde{h}}\|_{-1/2,\Gamma} \\ &\qquad \leq c \left\{ \inf_{v_h \in H_h} \|u - v_h\|_{1,\Omega} + \inf_{\lambda_{\widetilde{h}} \in Q_{\widetilde{h}}} \|\xi - \lambda_{\widetilde{h}}\|_{-1/2,\Gamma} \right\}. \end{aligned} \tag{4.43}$$

Note that the first term on the right hand side of the above equation reduces to $\|u - \mathbf{P}^1_{1,h}(u)\|_{1,\Omega}$, which can be bounded by means of (4.7). In order to estimate the second term we need the following previous result.

Lemma 4.8. *Let $\mathscr{P}^0_{\widetilde{h}} : L^2(\Gamma) \to Q_{\widetilde{h}}$ be the orthogonal projector with respect to the $L^2(\Gamma)$-inner product. Then there holds*

$$\|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{-1/2,\Gamma} \leq c\,\widetilde{h}\,\|\lambda\|_{1/2,\Gamma} \qquad \forall \lambda \in H^{1/2}(\Gamma).$$

Proof. Starting from the estimates

$$\|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{0,\Gamma} \leq \widetilde{h}^0 \, \|\lambda\|_{0,\Gamma} \quad \forall \lambda \in L^2(\Gamma)$$

and

$$\|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{0,\Gamma} \leq C\widetilde{h}\,\|\lambda\|_{1,\Gamma} \quad \forall \lambda \in H^1(\Gamma),$$

the latter being consequence of the Deny-Lions and Bramble-Hilbert Lemmas (cf. Theorems 3.4 and 3.5), we find by interpolation that

$$\|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{0,\Gamma} \leq C\widetilde{h}^{1/2} \, \|\lambda\|_{1/2,\Gamma} \qquad \forall \lambda \in H^{1/2}(\Gamma).$$

Next, using a duality argument and the above estimate, we have that for each $\lambda \in H^{1/2}(\Gamma)$ there holds

$$\begin{aligned} \|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{-1/2,\Gamma} &= \sup_{\substack{\eta \in H^{1/2}(\Gamma) \\ \eta \neq \Theta}} \frac{\langle \lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda), \eta \rangle}{\|\eta\|_{1/2,\Gamma}} \\ &= \sup_{\substack{\eta \in H^{1/2}(\Gamma) \\ \eta \neq \Theta}} \frac{\langle \lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda), \eta \rangle_{0,\Gamma}}{\|\eta\|_{1/2,\Gamma}} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{\eta \in H^{1/2}(\Gamma) \\ \eta \neq 0}} \frac{\langle \lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda), \eta - \mathscr{P}^0_{\widetilde{h}}(\eta) \rangle_{0,\Gamma}}{\|\eta\|_{1/2,\Gamma}} \\
&\leq \sup_{\substack{\eta \in H^{1/2}(\Gamma) \\ \eta \neq 0}} \frac{\|\lambda - \mathscr{P}^0_{\widetilde{h}}(\lambda)\|_{0,\Gamma} \, \|\eta - \mathscr{P}^0_{\widetilde{h}}(\eta)\|_{0,\Gamma}}{\|\eta\|_{1/2,\Gamma}} \\
&\leq \sup_{\substack{\eta \in H^{1/2}(\Gamma) \\ \eta \neq 0}} \frac{C \widetilde{h}^{1/2} \|\lambda\|_{1/2,\Gamma} \, C \widetilde{h}^{1/2} \, \|\eta\|_{1/2,\Gamma}}{\|\eta\|_{1/2,\Gamma}} \\
&= \widetilde{C} \widetilde{h} \|\lambda\|_{1/2,\Gamma},
\end{aligned}
$$

which completes the proof.

□

We now let $\mathscr{P}_{-1/2,\widetilde{h}} : H^{-1/2}(\Gamma) \to Q_{\widetilde{h}}$ be the orthogonal projector with respect to the $H^{-1/2}(\Gamma)$-inner product. It is then clear that

$$
\inf_{\lambda_{\widetilde{h}} \in Q_{\widetilde{h}}} \|\xi - \lambda_{\widetilde{h}}\|_{-1/2,\Gamma} = \|\xi - \mathscr{P}_{-1/2,\widetilde{h}}(\xi)\|_{-1/2,\Gamma},
$$

and that

$$
\|\xi - \mathscr{P}_{-1/2,\widetilde{h}}(\xi)\|_{-1/2,\Gamma} \leq \|\xi\|_{-1/2,\Gamma} \qquad \forall \xi \in H^{-1/2}(\Gamma). \tag{4.44}
$$

In addition, utilizing Lemma 4.8 we obtain that

$$
\|\xi - \mathscr{P}_{-1/2,\widetilde{h}}(\xi)\|_{-1/2,\Gamma} \leq \|\xi - \mathscr{P}^0_{\widetilde{h}}(\xi)\|_{-1/2,\Gamma} \leq C \widetilde{h} \|\xi\|_{1/2,\Gamma} \quad \forall \xi \in H^{1/2}(\Gamma),
$$

that is

$$
\|\xi - \mathscr{P}_{-1/2,\widetilde{h}}(\xi)\|_{-1/2,\Gamma} \leq C \widetilde{h} \|\xi\|_{1/2,\Gamma} \qquad \forall \xi \in H^{1/2}(\Gamma),
$$

which, together with (4.44), and thanks to the interpolation estimates for Sobolev spaces, give

$$
\|\xi - \mathscr{P}_{-1/2,\widetilde{h}}(\xi)\|_{-1/2,\Gamma} \leq C \widetilde{h}^{r+\frac{1}{2}} \|\xi\|_{r,\Gamma} \quad \forall \xi \in H^r(\Gamma), \forall r \in [-1/2, 1/2]. \tag{4.45}
$$

Therefore, recalling that

$$
\|v - \mathbf{P}^1_{1,h}(v)\|_{1,\Omega} \leq C h^l \|v\|_{l+1,\Omega} \quad \forall v \in H^{l+1}(\Omega), \, 0 \leq l \leq 1, \tag{4.46}
$$

we conclude from (4.43), (4.45), and (4.46), that

$$
\|u - u_h\|_{1,\Omega} + \|\xi - \xi_{\widetilde{h}}\|_{-1/2,\Gamma} \leq C \left\{ h^l \|u\|_{l+1,\Omega} + \widetilde{h}^{r+1/2} \|\xi\|_{r,\Gamma} \right\}
$$

for each $u \in H^{l+1}(\Omega)$, $0 \leq l \leq 1$, and for each $\xi \in H^r(\Gamma)$, $-1/2 \leq r \leq 1/2$.

4.4 The Poisson problem with Neumann boundary conditions

In this section we analyze a Galerkin scheme for the 2D version of the Poisson problem studied in Section 2.4.2 with Neumann boundary conditions, that is when $\Gamma_N = \Gamma$. In other words, given Ω a bounded domain of $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$, we are interested in the boundary value problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \mathbf{n}} = g \quad \text{in} \quad \Gamma, \quad \int_\Omega u = 0\,,$$

for which we need to assume that the data satisfy the compatibility condition

$$\int_\Omega f + \langle g, 1 \rangle = 0.$$

Then defining the auxiliary unknowns

$$\boldsymbol{\sigma} := \nabla u \quad \text{in} \quad \Omega \quad \text{and} \quad \xi := -\gamma_0(u) \quad \text{in} \quad \Gamma\,,$$

and proceeding as in Sections 2.4.1 and 2.4.2, one arrives at the mixed variational formulation: Find $(\boldsymbol{\sigma}, (u, \xi)) \in H \times Q$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (u, \xi)) &= 0 \qquad \forall \boldsymbol{\tau} \in H, \\ b(\boldsymbol{\sigma}, (v, \lambda)) \qquad\qquad &= -\int_\Omega f v + \langle g, \lambda \rangle \qquad \forall (v, \lambda) \in Q, \end{aligned}$$

where $H := H(\text{div}; \Omega)$, $Q := L_0^2(\Omega) \times H^{1/2}(\Gamma)$, and the bounded bilinear forms $a : H \times H \to \mathbb{R}$ and $b : H \times Q \to \mathbb{R}$ are defined by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in H\,, \tag{4.47}$$

and

$$b(\boldsymbol{\tau}, (v, \lambda)) := \int_\Omega v \,\text{div}\, \boldsymbol{\tau} + \langle \boldsymbol{\tau} \cdot \mathbf{n}, \lambda \rangle \quad \forall \boldsymbol{\tau} \in H, \forall (v, \lambda) \in Q\,, \tag{4.48}$$

As in Chapter 3, from now on we omit the symbol $\gamma_{\mathbf{n}}$ to denote the respective normal traces and simply write $\boldsymbol{\tau} \cdot \mathbf{n}$ instead of $\gamma_{\mathbf{n}}(\boldsymbol{\tau})$.

We now consider finite dimensional subspaces $H_h \subseteq H$, $Q_h^u \subseteq L_0^2(\Omega)$ and $Q_h^\xi \subseteq H^{1/2}(\Gamma)$, and define

$$Q_h := Q_h^u \times Q_h^\xi \subseteq Q\,.$$

Then, the associated Galerkin scheme reduces to: Find $(\boldsymbol{\sigma}_h, (u_h, \xi_h)) \in H_h \times Q_h$ such that

$$
\begin{aligned}
a(\sigma_h, \tau_h) + b(\tau_h, (u_h, \xi_h)) &= 0 \quad \forall\, \tau_h \in H_h, \\
b(\sigma_h, (v_h, \lambda_h)) &= -\int_\Omega f v_h + \langle g, \lambda_h \rangle \quad \forall\, (v_h, \lambda_h) \in Q_h.
\end{aligned}
\tag{4.49}
$$

For the analysis of (4.49), we first focus on the discrete inf-sup condition for b, that is on the eventual existence of $\beta > 0$, independent of h, such that

$$
\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b(\tau_h, (v_h, \lambda_h))}{\|\tau_h\|_H} \geq \beta\, \|(v_h, \lambda_h)\|_Q \quad \forall\, (v_h, \lambda_h) \in Q_h \,. \tag{4.50}
$$

Since b can be decomposed as the sum of two bilinear forms b_1 and b_2, that is (cf. (4.48))

$$
b(\tau, (v, \lambda)) = b_1(\tau, v) + b_2(\tau, \lambda) \quad \forall\, \tau \in H, \forall\, (v, \lambda) \in Q, \tag{4.51}
$$

we could certainly utilize the corresponding characterization result established in [40, Theorem 7] to prove (4.50). Alternatively, and due to the same decomposition, one could also employ the slightly different equivalence given in [45, Theorem 3.1]. However, and in order to provide further points of view to this analysis, in what follows we apply another procedure that can be seen as a combination of the above mentioned approaches. Indeed, we first use the boundedness of the normal trace of vectors in $H(\mathrm{div};\Omega)$ (see eq. (1.44) in the proof of Theorem 1.7) to deduce that

$$
\begin{aligned}
\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b(\tau_h, (v_h, \lambda_h))}{\|\tau_h\|_H} &\geq \sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b_1(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{div},\Omega}} - \|\lambda_h\|_{1/2,\Gamma} \\
&= \sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{\displaystyle\int_\Omega v_h \operatorname{div} \tau_h}{\|\tau_h\|_{\mathrm{div},\Omega}} - \|\lambda_h\|_{1/2,\Gamma} \quad \forall\, (v_h, \lambda_h) \in Q_h \,.
\end{aligned}
$$

Next, considering the particular subspaces (cf. (4.1) and (4.3))

$$
\begin{aligned}
H_h &:= H_h^0 := \Big\{ \tau_h \in H(\mathrm{div};\Omega) : \quad \tau_h|_K \in RT_0(K) \quad \forall\, K \in \mathscr{T}_h \Big\}, \\
Q_h^u &:= Y_h^0 \cap L_0^2(\Omega) := \Big\{ v_h \in L_0^2(\Omega) : \quad v_h|_K \in \mathbb{P}_0(K) \quad \forall\, K \in \mathscr{T}_h \Big\},
\end{aligned}
$$

and employing the analysis from Section 4.2, we obtain that

$$
\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq 0}} \frac{b(\tau_h, (v_h, \lambda_h))}{\|\tau_h\|_H} \geq \widehat{\beta}\, \|v_h\|_{0,\Omega} - \|\lambda_h\|_{1/2,\Gamma} \quad \forall\, (v_h, \lambda_h) \in Q_h \,, \tag{4.52}
$$

with a constant $\widehat{\beta} > 0$, independent of h. In turn, it is straightforward to see that

$$\begin{aligned} \sup_{\substack{\tau_h\in H_h\\ \tau_h\neq 0}} \frac{b(\tau_h,(v_h,\lambda_h))}{\|\tau_h\|_H} &\geq \sup_{\substack{\tau_h\in V_{1,h}\\ \tau_h\neq 0}} \frac{b_2(\tau_h,\lambda_h)}{\|\tau_h\|_{\mathrm{div},\Omega}} \\ &= \sup_{\substack{\tau_h\in V_{1,h}\\ \tau_h\neq 0}} \frac{\langle\tau_h\cdot\mathbf{n},\lambda_h\rangle}{\|\tau_h\|_{\mathrm{div},\Omega}} \qquad \forall\,(v_h,\lambda_h)\in Q_h\,, \end{aligned} \tag{4.53}$$

where $V_{1,h}$ is the discrete kernel of b_1, that is,

$$\begin{aligned} V_{1,h} &= \left\{\tau_h\in H_h: \quad b_1(\tau_h,v_h) := \int_\Omega v_h \operatorname{div}\tau_h = 0 \quad \forall\, v_h\in Q_h^u\right\} \\ &= \left\{\tau_h\in H_h: \quad \operatorname{div}\tau_h\in\mathbb{P}_0(\Omega)\right\}. \end{aligned} \tag{4.54}$$

Hence, it is easy to see from (4.52) and (4.53) that, in order to conclude (4.50), it suffices to show that there exists a constant $\widetilde{\beta}>0$, independent of h, such that

$$\sup_{\substack{\tau_h\in V_{1,h}\\ \tau_h\neq 0}} \frac{\langle\tau_h\cdot\mathbf{n},\lambda_h\rangle}{\|\tau_h\|_{\mathrm{div},\Omega}} \geq \widetilde{\beta}\,\|\lambda_h\|_{1/2,\Gamma} \qquad \forall\,\lambda_h\in Q_h^\xi\,. \tag{4.55}$$

Throughout the rest of the section we aim to show (4.55). To this end we now introduce the following definition.

Definition 4.1. Let $\Phi_h(\Gamma) := \{\tau_h\cdot\mathbf{n}|_\Gamma: \quad \tau_h\in V_{1,h}\}$. We say that a linear operator $\mathscr{L}_h:\Phi_h(\Gamma)\to V_{1,h}$ is a STABLE DISCRETE LIFTING if

i) $\mathscr{L}_h(\phi_h)\cdot\mathbf{n} = \phi_h \quad \text{on} \quad \Gamma \quad \forall\,\phi_h\in\Phi_h(\Gamma)$,
ii) $\exists\, c>0$, independent of h, such that

$$\|\mathscr{L}_h(\phi_h)\|_{\mathrm{div},\Omega} \leq c\,\|\phi_h\|_{-1/2,\Gamma} \qquad \forall\,\phi_h\in\Phi_h(\Gamma).$$

Lemma 4.9. *Assume that there exists a stable discrete lifting $\mathscr{L}_h:\Phi_h(\Gamma)\to V_{1,h}$. Then the discrete inf-sup condition* (4.55) *is equivalent to the existence of $C>0$, independent of h, such that*

$$\sup_{\substack{\phi_h\in\Phi_h(\Gamma)\\ \phi_h\neq 0}} \frac{\langle\phi_h,\lambda_h\rangle}{\|\phi_h\|_{-1/2,\Gamma}} \geq C\,\|\lambda_h\|_{1/2,\Gamma} \qquad \forall\,\lambda_h\in Q_h^\xi\,. \tag{4.56}$$

Proof. It suffices to see, according to ii) in Definition 4.1, that

$$\frac{|\langle\phi_h,\lambda_h\rangle|}{\|\phi_h\|_{-1/2,\Gamma}} \leq \frac{c\,|\langle\phi_h,\lambda_h\rangle|}{\|\mathscr{L}_h(\phi_h)\|_{\mathrm{div},\Omega}} \leq c \sup_{\substack{\tau_h\in V_{1,h}\\ \tau_h\neq 0}} \frac{|\langle\tau_h\cdot\mathbf{n},\lambda_h\rangle|}{\|\tau_h\|_{\mathrm{div},\Omega}}$$

and, according to the bound given by (1.44) (cf. Theorem 1.7), that

$$\frac{|\langle \tau_h \cdot \mathbf{n}, \lambda_h \rangle|}{\|\tau_h\|_{\mathrm{div},\Omega}} \le \frac{|\langle \tau_h \cdot \mathbf{n}, \lambda_h \rangle|}{\|\tau_h \cdot \mathbf{n}\|_{-1/2,\Gamma}} \le \sup_{\substack{\phi_h \in \Phi_h(\Gamma) \\ \phi_h \neq 0}} \frac{|\langle \phi_h, \lambda_h \rangle|}{\|\phi_h\|_{-1/2,\Gamma}},$$

whence (4.55) and (4.56) are equivalent.

□

In what follows we provide sufficient conditions for the existence of a stable discrete lifting $\mathscr{L}_h : \Phi_h(\Gamma) \to V_{1,h}$. For this purpose, we proceed as in [39, Sections 5.2 and 5.3], and assume that $\mathscr{T}_h$ is quasi-uniform around Γ. This means that there exist a neighborhood Ω_Γ of Γ and a constant $c > 0$, independent of h, such that, denoting

$$\mathscr{T}_{h,\Gamma} := \{K \in \mathscr{T}_h : \quad K \cap \Omega_\Gamma \neq \emptyset\}, \tag{4.57}$$

there holds

$$\max_{K \in \mathscr{T}_{h,\Gamma}} h_K \le c \min_{K \in \mathscr{T}_{h,\Gamma}} h_K.$$

It is important to remark here that, while the above requirement of quasi-uniformity was removed recently in [48, Sections 4 and 5] for the 2D case, we prefer to keep it throughout the rest of the present analysis since, up to our knowledge, the approach to be shown below is also the only known one that can be applied to derive the existence of stable discrete liftings in 3D (see, e.g. [33, Lemma 7.5]).

Now, because of the regularity of $\mathscr{T}_h$ (cf. (3.32)), which means that

$$\frac{h_K}{\rho_K} \le c \quad \forall K \in \mathscr{T}_h, \quad \forall h > 0,$$

or equivalently that $\{\mathscr{T}_h\}_{h>0}$ satisfies the minimum angle condition, the quasi-uniformity assumption implies that the partition on Γ inherited from $\mathscr{T}_h$, say Γ_h, is also quasi-uniform, that is there exists $c > 0$, independent of h, such that

$$h_\Gamma := \max\Big\{|e| : \quad e \in \Gamma_h\Big\} \le c \min\Big\{|e| : \quad e \in \Gamma_h\Big\}.$$

We now define

$$\widetilde{\Phi}_h(\Gamma) := \Big\{\phi_h \in L^2(\Gamma) : \quad \phi_h|_e \in \mathbb{P}_0(e) \qquad \forall e \in \Gamma_h\Big\},$$

and notice that $\Phi_h(\Gamma) \subseteq \widetilde{\Phi}_h(\Gamma)$. In addition, the quasi-uniformity of Γ_h implies that $\widetilde{\Phi}_h(\Gamma)$, which coincides with the space $Q_{\tilde{h}}$ given in Section 4.3, satisfies the inverse inequality (cf. Lemma 4.6)

$$\|\phi_h\|_{-1/2+\delta,\Gamma} \le C h_\Gamma^{-\delta} \|\phi_h\|_{-1/2,\Gamma} \qquad \forall \phi_h \in \widetilde{\Phi}_h(\Gamma), \qquad \forall \delta \in [0,1/2]. \tag{4.58}$$

Theorem 4.1. *Under the above stated assumptions, there exists a stable discrete lifting $\mathscr{L}_h : \Phi_h(\Gamma) \to V_{1,h}$.*

Proof. Let $\phi_h \in \widetilde{\Phi}_h(\Gamma)$ and let $v \in H^1(\Omega)$ be the unique solution of the problem

$$\Delta v = \frac{1}{|\Omega|}\int_\Gamma \phi_h \quad \text{in} \quad \Omega, \quad \nabla v\cdot \mathbf{n} = \phi_h \quad \text{on} \quad \Gamma, \quad \int_\Omega v = 0.$$

The respective continuous dependence result says that $\|v\|_{1,\Omega} \le C_1\,\|\phi_h\|_{-1/2,\Gamma}$. In turn, the elliptic regularity result in non-convex polygonal domains (cf. [42]) establishes that there exists $\delta \in (0,1/2)$ such that $v \in H^{1+\delta}(\Omega)$ and

$$\|v\|_{1+\delta,\Omega} \le C\,\|\phi_h\|_{-1/2+\delta,\Gamma}\,.$$

It follows that $\nabla v \in [H^\delta(\Omega)]^2 \cap H(\mathrm{div};\Omega)$ (note that $\mathrm{div}\,(\nabla v) = \Delta v = \frac{1}{|\Omega|}\int_\Gamma \phi_h \in \mathbb{R}$), and hence, we can define (see remark right before Lemma 3.19)

$$\mathscr{L}_h(\phi_h) := \Pi_h^0(\nabla v)\,.$$

According to the above and (3.8) (cf. Lemma 3.7), we obtain

$$\mathrm{div}\,\mathscr{L}_h(\phi_h) = \mathrm{div}\,\Pi_h^0(\nabla v) = \mathscr{P}_h^0(\mathrm{div}\,\nabla v) = \mathscr{P}_h^0(\Delta v) = \frac{1}{|\Omega|}\int_\Gamma \phi_h \in \mathbb{P}_0(\Omega)\,,$$

which confirms that $\mathscr{L}_h(\phi_h) \in V_{1,h}$, and thanks to (3.36) (see proof of Lemma 3.18), we find that

$$\mathscr{L}_h(\phi_h)\cdot\mathbf{n} = \Pi_h^0(\nabla v)\cdot\mathbf{n} = \mathscr{P}_{h,\Gamma}^0(\nabla v\cdot\mathbf{n}) = \mathscr{P}_{h,\Gamma}^0(\phi_h) = \phi_h,$$

where $\mathscr{P}_{h,\Gamma}^0 : L^2(\Gamma) \longrightarrow \widetilde{\Phi}_h(\Gamma)$ is the orthogonal projector. Note that this last identity also shows that $\widetilde{\Phi}_h(\Gamma) \subseteq \Phi_h(\Gamma)$, and therefore we deduce that

$$\begin{aligned}\Phi_h(\Gamma) &:= \Big\{\tau_h\cdot\mathbf{n}|_\Gamma : \quad \tau_h \in V_{1,h}\Big\}\\ &= \widetilde{\Phi}_h(\Gamma) := \Big\{\phi_h \in L^2(\Gamma) : \quad \phi_h|_e \in \mathbb{P}_0(e) \quad \forall\, e \in \Gamma_h\Big\}.\end{aligned}$$

It remains to show that $\mathscr{L}_h : \Phi_h(\Gamma) \to V_{1,h}$ is uniformly bounded. To this end, we first observe that

$$\|\mathscr{L}_h(\phi_h)\|_{\mathrm{div},\Omega}^2 = \|\mathscr{L}_h(\phi_h)\|_{0,\Omega}^2 + \left\|\frac{1}{|\Omega|}\int_\Gamma \phi_h\right\|_{0,\Omega}^2 \le \|\mathscr{L}_h(\phi_h)\|_{0,\Omega}^2 + C\,\|\phi_h\|_{-1/2,\Gamma}^2\,.$$

Next, we recall from (4.57) the definition of $\mathscr{T}_{h,\Gamma}$ and introduce the sets

$$\Omega_h^1 := \cup\Big\{K \in \mathscr{T}_h : \quad K \notin \mathscr{T}_{h,\Gamma}\Big\} \subseteq \Omega\setminus\Omega_\Gamma$$

and

$$\Omega_h^2 := \Omega\backslash\Omega_h^1 = \cup\Big\{K \in \mathscr{T}_{h,\Gamma}\Big\}.$$

Since $\Omega\backslash\Omega_\Gamma$ is strictly contained in Ω, the interior elliptic regularity result (cf. [49, Theorem 4.16]) implies that $v|_{\Omega\backslash\Omega_\Gamma} \in H^2(\Omega\backslash\Omega_\Gamma)$ and

$$\|v\|_{2,\Omega\setminus\Omega_\Gamma} \leq C_2 \|\phi_h\|_{-1/2,\Gamma}\,.$$

It follows that

$$\begin{aligned}
\|\mathscr{L}_h(\phi_h)\|_{0,\Omega} &\leq \|\mathscr{L}_h(\phi_h)\|_{0,\Omega_h^1} + \|\mathscr{L}_h(\phi_h)\|_{0,\Omega_h^2} \\
&= \|\Pi_h^0(\nabla v)\|_{0,\Omega_h^1} + \|\Pi_h^0(\nabla v)\|_{0,\Omega_h^2} \\
&\leq C\,\|\nabla v\|_{1,\Omega_h^1} + \|\nabla v\|_{0,\Omega_h^2} + \|\nabla v - \Pi_h^0(\nabla v)\|_{0,\Omega_h^2} \\
&\leq C\,\|v\|_{2,\Omega_h^1} + \|v\|_{1,\Omega_h^2} + \|\nabla v - \Pi_h^0(\nabla v)\|_{0,\Omega_h^2} \\
&\leq CC_2\,\|\phi_h\|_{-1/2,\Gamma} + C_1\,\|\phi_h\|_{-1/2,\Gamma} + \|\nabla v - \Pi_h^0(\nabla v)\|_{0,\Omega_h^2}.
\end{aligned}$$

On the other hand, applying the estimate (3.37) (cf. Lemma 3.19) and the inverse inequality (4.58), we get

$$\begin{aligned}
\|\nabla v - \Pi_h^0(\nabla v)\|_{0,\Omega_h^2}^2 &= \sum_{K\in\mathscr{T}_{h,\Gamma}} \|\nabla v - \Pi_K^0(\nabla v)\|_{0,K}^2 \\
&\leq C \sum_{K\in\mathscr{T}_{h,\Gamma}} h_K^{2\delta} \left\{ |\nabla v|_{\delta,K}^2 + \left\| \frac{1}{|\Omega|}\int_\Gamma \phi_h \right\|_{0,K}^2 \right\} \\
&\leq \overline{C} \max_{K\in\mathscr{T}_{h,\Gamma}} h_K^{2\delta} \left\{ \|v\|_{1+\delta,\Omega_h^2}^2 + \|\phi_h\|_{-1/2,\Gamma}^2 \right\} \\
&\leq \overline{C} \max_{K\in\mathscr{T}_{h,\Gamma}} h_K^{2\delta} \left\{ \|v\|_{1+\delta,\Omega}^2 + \|\phi_h\|_{-1/2,\Gamma}^2 \right\} \\
&\leq C \max_{K\in\mathscr{T}_{h,\Gamma}} h_K^{2\delta} \left\{ \|\phi_h\|_{-1/2+\delta,\Gamma}^2 + \|\phi_h\|_{-1/2,\Gamma}^2 \right\} \\
&\leq C \max_{K\in\mathscr{T}_{h,\Gamma}} h_K^{2\delta} \left\{ h_\Gamma^{-2\delta}\,\|\phi_h\|_{-1/2,\Gamma}^2 + \|\phi_h\|_{-1/2,\Gamma}^2 \right\} \\
&\leq C\,\|\phi_h\|_{-1/2,\Gamma}^2,
\end{aligned}$$

where the fact that $h_K \leq C\,h_\Gamma \quad \forall K \in \mathscr{T}_{h,\Gamma}$ has been used in the last inequality. Consequently, gathering together the above estimates, we conclude that

$$\|\mathscr{L}_h(\phi_h)\|_{\mathrm{div},\Omega} \leq C\,\|\phi_h\|_{-1/2,\Gamma} \qquad \forall\,\phi_h \in \Phi_h(\Gamma),$$

which finishes the proof.

□

We now aim to prove the following result, which, according to Lemma 4.9, will suffice to conclude the required discrete inf-sup condition for the term on Γ.

Lemma 4.10. *There exists $\beta > 0$, independent of h, such that*

$$\sup_{\substack{\phi_h\in\Phi_h(\Gamma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, \lambda_h\rangle}{\|\phi_h\|_{-1/2,\Gamma}} \geq \beta\,\|\lambda_h\|_{1/2,\Gamma} \qquad \forall\,\lambda_h \in Q_h^\xi.$$

For the above mentioned purpose we proceed as in [39, Section 5.3] and describe in what follows two different procedures under which Lemma 4.10 is proved.

PROCEDURE 1. The analysis below is based on the approach originally proposed in [9]. In fact, we now set

$$\Phi_h(\Gamma) = \widetilde{\Phi}_h(\Gamma) := \left\{ \phi_h \in L^2(\Gamma) : \quad \phi_h|_e \in \mathbb{P}_0(e) \quad \forall\, e \in \Gamma_h \right\},$$

where Γ_h is the partition on Γ inherited from $\mathscr{T}_h$, and let $h_\Gamma := \max \left\{ |e| : e \in \Gamma_h \right\}$. In addition, we introduce the space

$$Q^{\xi}_{\widetilde{h}} := \left\{ \lambda_{\widetilde{h}} \in C(\Gamma) : \quad \lambda_{\widetilde{h}}|_{\widetilde{\Gamma}_j} \in \mathbb{P}_1(\widetilde{\Gamma}_j) \qquad \forall\, j \in \{1, \cdots, m\} \right\},$$

where $\left\{ \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \cdots, \widetilde{\Gamma}_m \right\}$ is another partition of Γ and

$$\widetilde{h} := \max \left\{ |\widetilde{\Gamma}_j| : j \in \{1, \cdots, m\} \right\}.$$

Then we have the following result.

Lemma 4.11. *There exist c_0, $\beta > 0$, independent of h_Γ and $\widetilde{h}$, such that $\forall\, h_\Gamma \le c_0 \widetilde{h}$*

$$\sup_{\substack{\phi_h \in \Phi_h(\Gamma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_{\widetilde{h}} \rangle}{\|\phi_h\|_{-1/2,\Gamma}} \ge \beta\, \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} \qquad \forall\, \lambda_{\widetilde{h}} \in Q^{\xi}_{\widetilde{h}}.$$

Proof. Given $\lambda_{\widetilde{h}} \in Q^{\xi}_{\widetilde{h}}$, we let $z \in H^1(\Omega)$ be the unique solution of the problem

$$-\Delta z + z = 0 \quad \text{in} \quad \Omega, \quad z = \lambda_{\widetilde{h}} \quad \text{on} \quad \Gamma.$$

Since $Q^{\xi}_{\widetilde{h}} \subseteq H^1(\Gamma)$, we obtain by elliptic regularity (cf. [42]) that $z \in H^{1+\delta}(\Omega)$ and $\|z\|_{1+\delta,\Omega} \le C\, \|\lambda_{\widetilde{h}}\|_{1/2+\delta,\Gamma}$ $\forall\, \delta \in [0, \delta_0]$, where $\delta_0 := \min \left\{ \frac{1}{2}, \frac{\pi}{\omega} \right\}$ and ω is the largest interior angle of Ω. In what follows we fix $\delta \in (0, \delta_0]$, $\delta < 1/2$, and observe that $\nabla z \cdot \mathbf{n}\Big|_\Gamma \in H^{-1/2+\delta}(\Gamma)$ and

$$\Big\| \nabla z \cdot \mathbf{n} \Big\|_{-1/2+\delta,\Gamma} \le C\, \|z\|_{1+\delta,\Omega}\,.$$

Then, according to the approximation properties of $\Phi_h(\Gamma)$, whose details are described after this proof, we find that

$$\begin{aligned} \Big\| \nabla z \cdot \mathbf{n} - \mathscr{P}_h^{-1/2}\, (\nabla z \cdot \mathbf{n}) \Big\|_{-1/2,\Gamma} &\le C\, h_\Gamma^\delta\, \Big\| \nabla z \cdot \mathbf{n} \Big\|_{-1/2+\delta,\Gamma} \\ &\le C\, h_\Gamma^\delta\, \|z\|_{1+\delta,\Omega} \le C\, h_\Gamma^\delta\, \|\lambda_{\widetilde{h}}\|_{1/2+\delta,\Gamma}, \end{aligned}$$

where $\mathscr{P}_h^{-1/2} : H^{-1/2}(\Gamma) \to \Phi_h(\Gamma)$ is the orthogonal projector with respect to the $H^{-1/2}(\Gamma)$-inner product. Hence, applying the inverse inequality for $Q_{\widetilde{h}}^{\xi}$ (see remark after the proof), we deduce that

$$\left\| \nabla z \cdot \mathbf{n} - \mathscr{P}_h^{-1/2}(\nabla z \cdot \mathbf{n}) \right\|_{-1/2,\Gamma} \leq C \left(\frac{h_\Gamma}{\widetilde{h}} \right)^{\delta} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} .$$

In turn, using that $\|\nabla z\|_{\mathrm{div},\Omega} = \|z\|_{1,\Omega}$ and that $\Delta z = z$ in Ω, it follows that

$$\left\| \mathscr{P}_h^{-1/2}(\nabla z \cdot \mathbf{n}) \right\|_{-1/2,\Gamma} \leq \left\| \nabla z \cdot \mathbf{n} \right\|_{-1/2,\Gamma} \leq \|z\|_{1,\Omega} \leq \overline{C} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} .$$

On the other hand, it is clear that

$$\left\langle \nabla z \cdot \mathbf{n}, \lambda_{\widetilde{h}} \right\rangle = \left\langle \nabla z \cdot \mathbf{n}, z \right\rangle = \|z\|_{1,\Omega}^2 \geq \widetilde{C} \|z\|_{1/2,\Gamma}^2 = \widehat{C} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma}^2 .$$

In this way, we deduce that

$$\begin{aligned}
\sup_{\substack{\phi_h \in \Phi_h(\Gamma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_{\widetilde{h}} \rangle}{\|\phi_h\|_{-1/2,\Gamma}} &\geq \frac{\left\langle \mathscr{P}_h^{-1/2}(\nabla z \cdot \mathbf{n}), \lambda_{\widetilde{h}} \right\rangle}{\left\| \mathscr{P}_h^{-1/2}(\nabla z \cdot \mathbf{n}) \right\|_{-1/2,\Gamma}} \\
&\geq \overline{C} \frac{1}{\|\lambda_{\widetilde{h}}\|_{1/2,\Gamma}} \left| \left\langle \nabla z \cdot \mathbf{n}, \lambda_{\widetilde{h}} \right\rangle - \left\langle \nabla z \cdot \mathbf{n} - \mathscr{P}_h^{-1/2}(\nabla z \cdot \mathbf{n}), \lambda_{\widetilde{h}} \right\rangle \right| \\
&\geq \overline{C} \frac{1}{\|\lambda_{\widetilde{h}}\|_{1/2,\Gamma}} \left\{ \widehat{C} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma}^2 - C \left(\frac{h_\Gamma}{\widetilde{h}} \right)^{\delta} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma}^2 \right\} \\
&\geq \left\{ C_1 - C_2 \left(\frac{h_\Gamma}{\widetilde{h}} \right)^{\delta} \right\} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} ,
\end{aligned}$$

from which, taking $C_0 = \left(\dfrac{C_1}{2C_2} \right)^{1/\delta}$, the proof is concluded.

□

We notice now that, in order to have the inverse inequality for $Q_{\widetilde{h}}^{\xi}$, one needs to see that

$$\|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} \leq \widetilde{h}^0 \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} \quad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}}^{\xi}$$

and then prove that

$$\|\lambda_{\widetilde{h}}\|_{1,\Gamma} \leq C \widetilde{h}^{-1/2} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} \quad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}}^{\xi} ,$$

for which it suffices to show that $|\lambda_{\widetilde{h}}|_{1,\Gamma} \leq C \widetilde{h}^{-1/2} \|\lambda_{\widetilde{h}}\|_{1/2,\Gamma} \quad \forall \lambda_{\widetilde{h}} \in Q_{\widetilde{h}}^{\xi}$.

PROCEDURE 2. Let $\Phi_h(\Gamma) := \left\{\phi_h \in L^2(\Gamma) : \phi_h|_e \in \mathbb{P}_0(e) \quad \forall\, e \in \Gamma_h\right\}$, define $h_\Gamma := \max\left\{|e| : e \in \Gamma_h\right\}$, and assume that the number of edges e of Γ_h is even. Then we set

$$Q_h^\xi := \left\{\lambda_h \in C(\Gamma) : \quad \lambda_h|_e \in \mathbb{P}_1(e) \quad \forall\, e \in \Gamma_{2h}\right\},$$

where Γ_{2h} is the partition of Γ that arises by joining adjacent edges (certainly lying on the same line). Note that $\dim \Phi_h(\Gamma) = 2 \dim Q_h^\xi$. Similarly as before, the goal here is to show the existence of $\beta > 0$ such that

$$\sup_{\substack{\phi_h \in \Phi_h(\Gamma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h\rangle}{\|\phi_h\|_{-1/2,\Gamma}} \geq \beta\, \|\lambda_h\|_{1/2,\Gamma} \qquad \forall\, \lambda_h \in Q_h^\xi. \tag{4.59}$$

To this end, we assume from now on (see details in [39]) that there exist $\widehat{\Phi}_h(\Gamma) \subseteq \Phi_h(\Gamma)$ and constants $\beta_0, \beta_1 > 0$, such that $\dim \widehat{\Phi}_h(\Gamma) = \dim Q_h^\xi$,

$$\sup_{\substack{\phi_h \in \widehat{\Phi}_h(\Gamma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h\rangle}{\|\phi_h\|_{0,\Gamma}} \geq \beta_0\, \|\lambda_h\|_{0,\Gamma} \qquad \forall\, \lambda_h \in Q_h^\xi\,, \tag{4.60}$$

and

$$\sup_{\substack{\phi_h \in \widehat{\Phi}_h(\Gamma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h\rangle}{\|\phi_h\|_{-1,\Gamma}} \geq \beta_1\, \|\lambda_h\|_{1,\Gamma} \qquad \forall\, \lambda_h \in Q_h^\xi\,, \tag{4.61}$$

so that (4.59) is proved next by "*interpolating*" (4.60) and (4.61). In fact, we have the following result (cf. [54]).

Lemma 4.12. *There exists $\beta > 0$, independent of h, such that* (4.59) *holds.*

Proof. Let $G_h : L^2(\Gamma) \to Q_h^\xi$ be the operator defined by $G_h(\lambda) := \lambda_h$ for each $\lambda \in L^2(\Gamma)$, where λ_h is the unique element in Q_h^ξ such that

$$\langle \phi_h, \lambda_h\rangle = \langle \phi_h, \lambda\rangle \qquad \forall\, \phi_h \in \widehat{\Phi}_h(\Gamma)\,.$$

Note that the inf-sup condition (4.60) and the fact that $\dim \widehat{\Phi}_h(\Gamma) = \dim Q_h^\xi$ guarantee the existence and uniqueness of λ_h. Observe, in addition, that λ_h is a Petrov-Galerkin type approximation of λ. Now, it is easy to see that (4.60) and (4.61) can be put together in the form

$$\sup_{\substack{\phi_h \in \widehat{\Phi}_h(\Gamma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h\rangle}{\|\phi_h\|_{-s,\Gamma}} \geq \beta_s\, \|\lambda_h\|_{s,\Gamma} \qquad \forall\, \lambda_h \in Q_h^\xi, \qquad \forall\, s \in \{0,1\}.$$

Applying the above inequality to $G_h(\lambda) \in Q_h^\xi$, we obtain

$$
\begin{aligned}
\|G_h(\lambda)\|_{s,\Gamma} &\le \frac{1}{\beta_s} \sup_{\substack{\phi_h\in\widehat{\Phi}_h(\Gamma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, G_h(\lambda)\rangle}{\|\phi_h\|_{-s,\Gamma}} \\
&= \frac{1}{\beta_s} \sup_{\substack{\phi_h\in\widehat{\Phi}_h(\Gamma)\\ \phi_h\neq 0}} \frac{\langle \phi_h, \lambda\rangle}{\|\phi_h\|_{-s,\Gamma}} \le \frac{1}{\beta_s}\,\|\lambda\|_{s,\Gamma} \quad \forall\,\lambda\in H^s(\Gamma)\,,
\end{aligned}
$$

that is

$$
\|G_h(\lambda)\|_{s,\Gamma} \,\le\, \beta_s^{-1}\,\|\lambda\|_{s,\Gamma} \qquad \forall\,\lambda\in H^s(\Gamma), \qquad \forall\, s\in\{0,1\},
$$

and then, thanks to the interpolation estimates for Sobolev spaces (cf. [49, Appendix B]), we get

$$
\|G_h(\lambda)\|_{1/2,\Gamma} \,\le\, (\beta_0\,\beta_1)^{-1/2}\,\|\lambda\|_{1/2,\Gamma} \qquad \forall\,\lambda\in H^{1/2}(\Gamma)\,. \tag{4.62}
$$

We now let $\mathscr{P}_h^{-1/2} : H^{-1/2}(\Gamma) \to \widehat{\Phi}_h(\Gamma)$ be the orthogonal projector. Then, given $\lambda\in H^{1/2}(\Gamma)$, we consider the functional $f_\lambda : H^{-1/2}(\Gamma)\to\mathbb{R}$ defined by

$$
f_\lambda(\phi) := \langle \mathscr{P}_h^{-1/2}(\phi), \lambda\rangle \qquad \forall\,\phi\in H^{-1/2}(\Gamma)\,,
$$

and let $v_\lambda := J^{-1}(f_\lambda)\in H^{1/2}(\Gamma)$, where $J : H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)''\equiv H^{-1/2}(\Gamma)'$ is the usual isometry characterizing reflexive spaces, that is $J(v)(F) = F(v) \quad \forall\, F\in H^{-1/2}(\Gamma)\equiv H^{1/2}(\Gamma)', \quad \forall\, v\in H^{1/2}(\Gamma)$. It follows that $\forall\,\phi_h\in\widehat{\Phi}_h(\Gamma)$ there holds

$$
\begin{aligned}
&\langle \phi_h, G_h(v_\lambda)\rangle = \langle \phi_h, v_\lambda\rangle = J(v_\lambda)(\phi_h) = f_\lambda(\phi_h)\\
&= \langle \mathscr{P}_h^{-1/2}(\phi_h), \lambda\rangle = \langle \phi_h, \lambda\rangle = \langle \phi_h, G_h(\lambda)\rangle\,,
\end{aligned}
$$

and therefore $G_h(v_\lambda) = G_h(\lambda)$. Consequently, utilizing (4.62) and the fact that $\|\mathscr{P}_h^{-1/2}(\phi)\|_{-1/2,\Gamma}$ is certainly bounded by $\|\phi\|_{-1/2,\Gamma}$, we find that for each $\lambda\in H^{1/2}(\Gamma)$ there holds

$$
\begin{aligned}
&(\beta_0\,\beta_1)^{1/2}\,\|G_h(\lambda)\|_{1/2,\Gamma} \,=\, (\beta_0\,\beta_1)^{1/2}\,\|G_h(v_\lambda)\|_{1/2,\Gamma}\\
&\le \|v_\lambda\|_{1/2,\Gamma} \,=\, \sup_{\substack{\phi\in H^{-1/2}(\Gamma)\\ \phi\neq 0}} \frac{|\langle \phi, v_\lambda\rangle|}{\|\phi\|_{-1/2,\Gamma}}\\
&= \sup_{\substack{\phi\in H^{-1/2}(\Gamma)\\ \phi\neq 0}} \frac{|\langle \mathscr{P}_h^{-1/2}(\phi), \lambda\rangle|}{\|\phi\|_{-1/2,\Gamma}}\\
&\le \sup_{\substack{\phi\in H^{-1/2}(\Gamma)\\ \phi\neq 0}} \frac{|\langle \mathscr{P}_h^{-1/2}(\phi), \lambda\rangle|}{\|\mathscr{P}_h^{-1/2}(\phi)\|_{-1/2,\Gamma}}\\
&\le \sup_{\substack{\phi_h\in\widehat{\Phi}_h(\Gamma)\\ \phi_h\neq 0}} \frac{\langle \phi_h\,\lambda\rangle}{\|\phi_h\|_{-1/2,\Gamma}}\,.
\end{aligned}
$$

Finally, applying the above to $\lambda_h \in Q_h^\xi$, noting in this case that $G_h(\lambda_h) = \lambda_h$, and recalling that $\widehat{\Phi}_h(\Gamma) \subseteq \Phi_h(\Gamma)$, we conclude that

$$(\beta_0 \beta_1)^{1/2} \|\lambda_h\|_{1/2,\Gamma} = (\beta_0 \beta_1)^{1/2} \|G_h(\lambda_h)\|_{1/2,\Gamma}$$

$$\leq \sup_{\substack{\phi_h \in \widehat{\Phi}_h(\Gamma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h \rangle}{\|\phi_h\|_{-1/2,\Gamma}} \leq \sup_{\substack{\phi_h \in \Phi_h(\Gamma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \lambda_h \rangle}{\|\phi_h\|_{-1/2,\Gamma}} \qquad \forall \lambda_h \in Q_h^\xi ,$$

which constitutes the required discrete inf-sup condition (4.59).

□

We now look at the discrete kernel V_h of b, that is

$$V_h := \Big\{ \tau_h \in H_h : \quad b(\tau_h, (v_h, \lambda_h)) = 0 \quad \forall (v_h, \lambda_h) \in Q_h \Big\},$$

which, according to (4.51) and (4.54), yields

$$V_h := \Big\{ \tau_h \in H_h : \quad \operatorname{div} \tau_h \in \mathbb{P}_0(\Omega) \quad \text{and} \quad \langle \tau_h \cdot \mathbf{n}, \lambda_h \rangle = 0 \quad \forall \lambda_h \in Q_h^\xi \Big\}.$$

Hence, the V_h-ellipticity of a follows straightforwardly from [39, Lemma 3.2] by making use only of the first property characterizing the elements of V_h.

Consequently, by applying again the discrete Babuška-Brezzi theory (cf. Theorems 2.4 and 2.6), we conclude that (4.49) has a unique solution $(\sigma_h, (u_h, \xi_h)) \in H_h \times Q_h$, and there exists a constant $C > 0$, independent of h, such that

$$\begin{aligned} &\|(\sigma, (u, \xi)) - (\sigma_h, (u_h, \xi_h))\|_{H \times Q} \\ &\qquad \leq \quad C \inf_{(\tau_h, (v_h, \lambda_h)) \in H_h \times Q_h} \|(\sigma, (u, \xi)) - (\tau_h, (v_h, \lambda_h))\|_{H \times Q} . \end{aligned}$$

The approximation properties of H_h and Q_h^u are somehow already established by (4.32) and (4.33) (see also (4.4) and (4.21)), whereas the one of Q_h^ξ is given by (cf. [39, **(AP3)**])

$$\|\lambda - \mathscr{P}_{1/2,h}(\lambda)\|_{1/2,\Gamma} \leq C h^\delta \|\lambda\|_{1/2+\delta,\Gamma} \quad \forall \lambda \in H^{1/2+\delta}(\Gamma), \quad \forall \delta \in [0,1],$$

where $\mathscr{P}_{1/2,h} : H^{1/2}(\Gamma) \to Q_h^\xi$ is the orthogonal projector with respect to the inner product of $H^{1/2}(\Gamma)$.

4.5 The linear elasticity problem

In this section we analyze the Galerkin scheme for the 2D version of the linear elasticity problem with Dirichlet boundary conditions studied in Section 2.4.3.1. To