Proof. As announced at the beginning of the present section, the proof follows from a straightforward application of Theorem 2.5, bearing in mind the equivalences between i) and the pairs of assumptions (2.37) - (2.40) and (2.41) - (2.42), and between ii) and (2.36). In turn, (2.43) and (2.44) follow from (2.31) and (2.32), respectively, replacing in both $\|\big(\varPi\mathcal{A}\big)^{-1}\|$ by $\dfrac{1}{\alpha}$, as stipulated in (2.39).

∎

2.2.3 Discrete analysis

2.2.4 Continuous analysis: general case

Here we extend the results from Section 2.2.2 to the general case of variational formulations of the Babuška-Brezzi type, in which three bilinear forms are involved and the unknowns and test functions spaces do not necessarily coincide. While the logical sequence of the analysis is basically the same of the aforementioned section, for sake of completeness in what follows we provide most details.

2.2.4.1 The operator equation

Let $(X_2, \|\cdot\|_{X_2})$, $(M_1, \cdot\|_{M_1})$, $(X_1, \|\cdot\|_{X_1})$, and $(M_2, \cdot\|_{M_2})$ be real Banach spaces, and let $a : X_2 \times X_1 \to R$, $b_i : X_i \times M_i \to R$, $i \in \big\{1, 2\big\}$, be bounded bilinear forms. Then, given $F \in X_1'$ and $G \in M_2'$, we consider the problem: Find $(\sigma, u) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\sigma,\tau) \,+\, b_1(\tau,u) &= F(\tau) \qquad \forall\, \tau \,\in\, X_1 \\ b_2(\sigma,v) \qquad\qquad &= G(v) \qquad \forall\, v \,\in\, M_2\,. \end{aligned} \tag{2.45}$$

Now, similarly as in (2.22) and (2.23), we let $\mathcal{A} \in \mathcal{L}(X_2, X_1')$ and $\mathcal{B}_i \in \mathcal{L}(X_i, M_i')$ be the linear and bounded operators induced by a and b_i, $i \in \big\{1, 2\big\}$, respectively, so that, in particular $\mathcal{B}_1' \in \mathcal{L}(M_1'', X_1')$ is defined as

$$\mathcal{B}_1'(\mathcal{G})(\tau) \,:=\, \mathcal{G}(\mathcal{B}_1(\tau)) \qquad \forall\, \mathcal{G} \in M_1'', \quad \forall\, \tau \in X_1\,. \tag{2.46}$$

Next, if $\mathcal{J}_1 : M_1 \longrightarrow M_1''$ is the injective operator defined as $\mathcal{J}_1(v)(G) \,:=\, G(v) \quad \forall\, G \in M_1'$, $\forall\, v \in M_1$, we have that

$$\mathcal{B}_1'(\mathcal{J}_1(u))(\tau) \,:=\, \mathcal{J}_1(u)(\mathcal{B}_1(\tau)) \,=\, \mathcal{B}_1(\tau)(u) \,=\, b_1(\tau, u)\,,$$

and hence (2.45) can be rewritten, equivalently, as: Find $(\sigma, u) \in X_2 \times M_1$ such that

$$\begin{aligned} \mathcal{A}(\sigma) \,+\, \mathcal{B}_1'(\mathcal{J}_1(u)) &= F \\ \mathcal{B}_2(\sigma) \qquad\qquad &= G\,. \end{aligned} \tag{2.47}$$

Furthermore, assuming from now on that M_1 is reflexive, which means that the operator $\mathcal{J}_1 : M_1 \longrightarrow M_1''$ is bijective, problem (2.47) can be reformulated as: Find $(\sigma, \mathcal{G}) \in X_2 \times M_1''$ such that

$$\begin{aligned} \mathcal{A}(\sigma) \,+\, \mathcal{B}_1'(\mathcal{G}) &= F \\ \mathcal{B}_2(\sigma) \qquad\quad &= G\,, \end{aligned} \tag{2.48}$$

and then define $u \,:=\, \mathcal{J}_1^{-1}(\mathcal{G})$. Similarly as done in Section 2.2.2.2, the solvability of systems of operator equations of the type given by (2.48) is analyzed in the following section.

2.2.4.2 A previous solvability result

We now let $(Z_2, \|\cdot\|_{Z_2})$, $(Z_1, \|\cdot\|_{Z_1})$, $(Y_2, \|\cdot\|_{Y_2})$, and $(Y_1, \|\cdot\|_{Y_1})$ be real Banach spaces, and consider operators $\mathbf{A} \in \mathcal{L}(Z_2, Z_1')$ and $\mathbf{B}_i \in \mathcal{L}(Z_i, Y_i)$, $i \in \{1,2\}$, so that $\mathbf{B}_i' \in \mathcal{L}(Y_i', Z_i')$. Then, given $F \in Z_1'$ and $y \in Y_2$, we are interested in the problem: Find $(\sigma, U) \in Z_2 \times Y_1'$ such that

$$\begin{aligned} \mathbf{A}(\sigma) \, + \, \mathbf{B}_1'(U) &= F \\ \mathbf{B}_2(\sigma) \qquad\quad &= y\,. \end{aligned} \tag{2.49}$$

It is easy to see that (2.48) is a particular case of (2.49) with the spaces

$$Z_2 \,=\, X_2\,, \quad Z_1 \,=\, X_1\,, \quad Y_2 \,=\, M_2'\,, \quad \text{and} \quad Y_1 \,=\, M_1'\,.$$

Next, for each $i \in \{1,2\}$ we introduce the null space of $\mathbf{B}_i$, that is

$$K_i \,:=\, N(\mathbf{B}_i) \,:=\, \Big\{ \tau \in Z_i : \quad \mathbf{B}_i(\tau) \,=\, 0 \Big\}\,.$$

In addition, we let $\Pi\mathbf{A} : K_2 \longrightarrow K_1'$ be the operator defined by

$$\Pi\mathbf{A}(\tau)(\zeta) \,:=\, \mathbf{A}(\tau)(\zeta) \qquad \forall\,\tau \in K_2\,, \quad \forall\,\zeta \in K_1\,.$$

Then, necessary and sufficient conditions for the problem (2.49) to be well-posed, are provided by the following theorem.

Theorem 2.7. *For each pair $(F, y) \in Z_1' \times Y_2$, problem (2.49) has a unique solution that depends continuously on (F, y) if and only if:*

i) $\Pi\mathbf{A} : K_2 \longrightarrow K_1'$ *is an isomorphism, and*
ii) *for each $i \in \{1,2\}$, $\mathbf{B}_i : Z_i \longrightarrow Y_i$ is surjective.*

Proof. Let us first assume i) and ii). Then, given $(F, y) \in Z_1' \times Y_2$, the surjectivity of $\mathbf{B}_2$ (cf. ii)) and Lemma 2.4, part i), guarantee the existence of a positive constant β_2, so that for the given y, there exists $\sigma_y \in Z_2$ such that $\mathbf{B}_2(\sigma_y) = y$ and

$$\|\sigma_y\| \,\leqslant\, \frac{1}{\beta_2}\,\|y\|\,. \tag{2.50}$$

Next, since $\big(F - \mathbf{A}(\sigma_y)\big)|_{K_1} \in K_1'$, we deduce from i) that there exists a unique $\sigma_0 \in K_2$ such that $\Pi\mathbf{A}(\sigma_0) \,=\, \big(F - \mathbf{A}(\sigma_y)\big)|_{K_1}$, that is $\mathbf{A}(\sigma_0)(\tau) \,=\, F(\tau) - \mathbf{A}(\sigma_y)(\tau)$ for all $\tau \in K_1$, or equivalently $F - \mathbf{A}(\sigma) \,\in\, K_1^\circ$, where $\sigma \,:=\, \sigma_0 + \sigma_y$. In addition, thanks to the bounded inverse theorem, we get

$$\|\sigma_0\| \,=\, \|\big(\Pi\mathbf{A}\big)^{-1}\big(F - \mathbf{A}(\sigma_y)\big)|_{K_1}\| \,\leqslant\, \|\big(\Pi\mathbf{A}\big)^{-1}\|\,\Big\{\|F\| + \|\mathbf{A}\|\,\|\sigma_y\|\Big\}\,. \tag{2.51}$$

In turn, knowing that $R(\mathbf{B}_1)$ is closed, $R(\mathbf{B}_1')$ is as well, and hence $R(\mathbf{B}_1') \,=\, K_1^\circ$, whence there exists $U \in Y_1'$ such that $\mathbf{B}_1'(U) \,=\, F - \mathbf{A}(\sigma)$. In this way, noting that there holds $\mathbf{B}_2(\sigma) \,=\, \mathbf{B}_2(\sigma_y) \,=\, y$, it readily follows that $(\sigma, U) \in Z_2 \times Y_1'$ is solution of (2.49). Furthermore, thanks to the surjectivity of $\mathbf{B}_1$ (cf. ii)) and Lemma 2.4, part ii), there exists a positive constant β_1 such that

$$\|U\| \,\leqslant\, \frac{1}{\beta_1}\,\|\mathbf{B}_1'(U)\| \,=\, \frac{1}{\beta_1}\,\|F - \mathbf{A}(\sigma)\| \,\leqslant\, \frac{1}{\beta_1}\,\Big\{\|F\| + \|\mathbf{A}\|\,\|\sigma\|\Big\}\,. \tag{2.52}$$

In this way, according to (2.50), (2.51), and (2.52), we arrive at

$$\|\sigma\| \leqslant \|(\mathit{\Pi}\mathbf{A})^{-1}\| \, \|F\| + \frac{1}{\beta_2}\Big(1 + \|(\mathit{\Pi}\mathbf{A})^{-1}\| \, \|A\|\Big)\|y\| \,, \tag{2.53}$$

and

$$\|U\| \leqslant \frac{1}{\beta_1}\Big(1 + \|(\mathit{\Pi}\mathbf{A})^{-1}\| \, \|A\|\Big)\|F\| + \frac{\|A\|}{\beta_1\,\beta_2}\Big(1 + \|(\mathit{\Pi}\mathbf{A})^{-1}\| \, \|A\|\Big)\|y\| \,, \tag{2.54}$$

which shows that the solution (σ, U) depends continuously on the data (F, y). In order to establish its uniqueness, we now let $(\widetilde{\sigma}, \widetilde{U}) \in Z_2 \times Y_1'$ be a solution of the associated homogeneous system, that is

$$\begin{aligned} \mathbf{A}(\widetilde{\sigma}) + \mathbf{B}_1'(\widetilde{U}) &= 0 \in Z_1' \\ \mathbf{B}_2(\widetilde{\sigma}) \qquad\qquad &= 0 \in Y_2 \,. \end{aligned} \tag{2.55}$$

It is clear from the second row that $\widetilde{\sigma} \in K_2$, and evaluating the first one with $\tau \in K_1$, we get

$$0 = \mathbf{A}(\widetilde{\sigma})(\tau) + \mathbf{B}_1'(\widetilde{U})(\tau) = \mathit{\Pi}\mathbf{A}(\widetilde{\sigma})(\tau) + \widetilde{U}(\mathbf{B}_1(\boldsymbol{\tau})) = (\mathit{\Pi}\mathbf{A})(\widetilde{\sigma})(\tau) \,,$$

which says that $\mathit{\Pi}\mathbf{A}(\widetilde{\sigma}) = 0 \in K_1'$. Then, thanks to the hypothesis i), we deduce that $\widetilde{\sigma} = 0$, and hence the first equation of (2.55) becomes $\mathbf{B}_1'(\widetilde{U}) = 0 \in Z_1'$. Noting now that $N(\mathbf{B}_1') = R(\mathbf{B}_1)^\circ = Y_1^\circ = \{0\} \subseteq Y_1'$, we conclude that $\widetilde{U} = 0 \in Y_1'$, which ends the proof of the sufficiency of i) and ii). Conversely, let us assume that there exists a continuous dependence constant $C_{\mathtt{cd}} > 0$ such that for each $(F, y) \in Z_1' \times Y_2$ there exists a unique $(\sigma, U) \in Z_2 \times Y_1'$ solution to (2.49) and there holds

$$\|\sigma\| + \|U\| \leqslant C_{\mathtt{cd}} \Big\{ \|F\| + \|y\| \Big\} \,. \tag{2.56}$$

Now, letting in particular F be the null functional, we find that for each $y \in Y_2$ there exists a unique $(\widehat{\sigma}, \widehat{U}) \in Z_2 \times Y_1'$ such that

$$\begin{aligned} \mathbf{A}(\widehat{\sigma}) + \mathbf{B}_1'(\widehat{U}) &= 0 \\ \mathbf{B}_2(\widehat{\sigma}) \qquad\qquad &= y \,, \end{aligned}$$

so that from the above second row it follows that $\mathbf{B}_2$ is surjective. Next, given $U \in Y_1'$, we let $F := \mathbf{B}_1'(U) \in Z_1'$ and deduce that there exists a unique $(\bar{\sigma}, \bar{U}) \in Z_2 \times Y_1'$ such that

$$\begin{aligned} \mathbf{A}(\bar{\sigma}) + \mathbf{B}_1'(\bar{U}) &= \mathbf{B}_1'(U) \\ \mathbf{B}_2(\bar{\sigma}) \qquad\qquad &= 0 \,, \end{aligned}$$

and

$$\|\bar{\sigma}\| + \|\bar{U}\| \leqslant C_{\mathtt{cd}} \, \|\mathbf{B}_1'(U)\| \,.$$

However, it is clear from the uniqueness of solution of (2.49) that $\bar{\sigma} = 0$ and $\bar{U} = U$, whence the foregoing inequality becomes

$$\|U\| \leqslant C_{\mathtt{cd}} \, \|\mathbf{B}_1'(U)\| \qquad \forall\, U \in Y_1' \,,$$

which says that $\mathbf{B}_1'$ is injective and of closed range, or equivalently, that $\mathbf{B}_1$ is surjective, thus completing the proof of ii). It remains to prove i), that is that $\Pi\mathbf{A} : K_2 \longrightarrow K_1'$ is bijective. Indeed, for the surjectivity, we let $\widetilde{F}_0 \in K_1'$ and observe first, thanks to the Hahn-Banach theorem, that there exists $\widetilde{F} \in Z_1'$ such that $\widetilde{F}|_{K_1} = \widetilde{F}_0$ and $\|\widetilde{F}\|_{Z_1'} = \|\widetilde{F}_0\|_{K_1'}$. Hence, there exists a unique $(\widetilde{\sigma}, \widetilde{U}) \in Z_2 \times Y_1'$ such that

$$\begin{aligned} \mathbf{A}(\widetilde{\sigma}) + \mathbf{B}_1'(\widetilde{U}) &= \widetilde{F} \\ \mathbf{B}_2(\widetilde{\sigma}) \qquad\qquad &= 0\,, \end{aligned}$$

from which it is clear that $\widetilde{\sigma} \in K_2$. Then, it is easily seen from the first row of the foregoing system that $\widetilde{F}_0(\tau) = \widetilde{F}(\tau) = \mathbf{A}(\widetilde{\sigma})(\tau)$ for all $\tau \in K_1$, that is $\Pi\mathbf{A}(\widetilde{\sigma}) = \widetilde{F}_0$, which proves that $\Pi\mathbf{A}$ is surjective. For the injectivity of $\Pi\mathbf{A}$, we now let $\bar{\sigma} \in K_2$ such that $\Pi\mathbf{A}(\bar{\sigma}) = 0$, that is $\mathbf{A}(\bar{\sigma})(\tau) = 0$ for all $\tau \in K_1$, which means that $\mathbf{A}(\bar{\sigma}) \in K_1^{\circ}$. Knowing that $\mathbf{B}_1$ is surjective, we have that $R(\mathbf{B}_1')$ is closed as well and $R(\mathbf{B}_1') = K_1^{\circ}$, whence there exists $\bar{U} \in Y_1'$ such that $\mathbf{B}_1'(\bar{U}) = -\mathbf{A}(\bar{\sigma})$. In this way, we can write

$$\begin{aligned} \mathbf{A}(\bar{\sigma}) + \mathbf{B}_1'(\bar{U}) &= 0 \\ \mathbf{B}_2(\bar{\sigma}) \qquad\qquad &= 0\,, \end{aligned}$$

from which, in particular, $\bar{\sigma} = 0$, which concludes the proof.

2.2.4.3 The main result

2.2.5 Discrete analysis: general case

2.3 Perturbed saddle-point problems

In this section we basically follow [6] and [12, Theorems 3.2 and 4.1] to analyze the solvability of the continuous and discrete schemes arising from perturbed saddle-point problems in Banach spaces. More precisely, given reflexive Banach spaces H and Q, bounded bilinear forms $a : H \times H \longrightarrow R$, $b : H \times Q \to R$, and $c : Q \times Q \longrightarrow R$, and functionals $F \in H'$ and $G \in Q'$, we are interested in seeking $(\sigma, u) \in H \times Q$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= F(\tau) \quad \forall\, \tau \in H\,, \\ b(\sigma,v) - c(u,v) &= G(v) \quad \forall\, v \in Q\,. \end{aligned} \tag{2.57}$$

In the particular case in which H and Q are Hilbert spaces, the well-posedness of (2.57) and its associated Galerkin scheme, as well as the corresponding error analysis, can be found in [5, Theorem 1.2, Section II.1.2] and [5, Proposition 2.11, Section II.2.4]).

We begin the analysis with some preliminary results.

2.3.1 Preliminaries

Let $\mathbf{B} : H \to Q'$ and $\mathbf{B}^{\mathbf{t}} : Q \to H'$ be the bounded linear operators induced by b, that is

$$\mathbf{B}(\tau)(v) := b(\tau, v) \quad \forall\, \tau \in H\,,\ \forall\, v \in Q \tag{2.58}$$

and