On continuous and discrete solubility of perturbed Babuška-Brezzi type problems

Alonso J. Bustos - Benjamín N. Venegas

University of Concepción

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Outline

Let H and Q be reflexive Banach spaces, $a : H \times H \to \mathbb{R}$, $b : H \times Q \to \mathbb{R}$ and $c : Q \times Q \to \mathbb{R}$ bounded bilinear forms, and functionals $f \in H'$ and $g \in Q'$.
Our purpose is to analyze the following problem: find $(\sigma, u) \in H \times Q$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \quad \forall \tau \in H \\ b(\sigma,v) - c(u,v) &= g(v) \quad \forall v \in Q \end{aligned} \tag{1.1}$$

We denote by $\|a\|$, $\|b\|$ and $\|c\|$ the smallest positive constants such that

$$\begin{aligned} |a(\zeta,\tau)| &\leq \|a\| \, \|\zeta\|_H \, \|\tau\|_H \quad \forall (\zeta,\tau) \in H \times H \\ |b(\tau,v)| &\leq \|b\| \, \|\tau\|_H \, \|v\|_Q \quad \forall (\tau,v) \in H \times Q \\ |c(w,v)| &\leq \|c\| \, \|w\|_Q \, \|v\|_Q \quad \forall (w,v) \in Q \times Q \end{aligned}$$

Later, we shall consider a and c symmetric and positive semi-definite. In this case, can be proved that

$$c(w,v) \leq c(w,w)^{1/2} c(v,v)^{1/2} \quad \forall (w,v) \in Q \times Q$$

and we define $|\cdot|_c : Q \to \mathbb{R}$ such that $|w|_c := c(w,w)^{1/2}$ for all $w \in Q$, which is a semi-norm in Q.

Since H and Q are reflexive Banach spaces, we let $\mathcal{J}_H : H \to H''$ and $\mathcal{J}_Q : Q \to Q''$ be the isometric and bijective linear mappings given by

$$\mathcal{J}_H(\tau)(F) := F(\tau) \quad \forall \tau \in H, \quad \forall F \in H' \tag{1.2}$$

$$\mathcal{J}_Q(v)(G) := G(v) \quad \forall v \in Q, \quad \forall G \in Q' \tag{1.3}$$

In this context, we will show results in the case that the null space of one of the operators induced by b is $\{\theta\}$. In particular, we have existence and uniqueness in the continuous and discrete case, and we know the respective Cea estimate. Then, we shall compare with the general case in which the null space is not trivial.

Now, we introduce some notation and results which will be helpful later.
Let $\boldsymbol{B} : H \to Q'$ and $\boldsymbol{B}^t : Q \to H'$ be the bounded linear operators induced by the bilinear form b, that is

$$\boldsymbol{B}(\tau)(v) := b(\tau, v), \quad \forall v \in Q$$

and

$$\boldsymbol{B}^t(v)(\tau) := b(\tau, v), \quad \forall \tau \in H,$$

and their respective null spaces

$$V := N(\boldsymbol{B}) \quad \text{and} \quad W := N(\boldsymbol{B}^t).$$

Also, assuming that V and W admit topological complements, we denote them by $V^\perp$ and $W^\perp$ and we denote by

$$i : V^\perp \to H \quad \text{and} \quad j : W^\perp \to Q$$

the respective canonical injections. Also, we recall a direct consequence of the open mapping theorem, which asures the existence of positive constants C_H, C_Q, depending only on H and Q, respectively, such that

$$\|\tau_0\|_H + \|\bar{\tau}\|_H \leq C_H \|\tau\|_H \quad \text{and} \quad \|v_0\|_Q + \|\bar{v}\|_Q \leq C_Q \|v\|_Q$$

for all $\tau := \tau_0 + \bar{\tau} \in H$, where $\tau_0 \in V$, $\bar{\tau} \in V^\perp$ and for all $v = v_0 + \bar{v} \in Q$, where $v_0 \in W$, $\bar{v} \in W^\perp$.

Lemma 2.1.

There hold

$$\frac{1}{C_H}\|\tau\|_H \leq \operatorname{dist}(\tau, V) \leq \|\tau\|_H \quad \forall \tau \in V^{\perp}, \tag{2.1}$$

and

$$\frac{1}{C_Q}\|v\|_Q \leq \operatorname{dist}(v, W) \leq \|v\|_Q \quad \forall v \in W^{\perp}. \tag{2.2}$$

Lemma 2.2.

The following statements are equivalent:

i) $\boldsymbol{B}^t \circ j : W^\perp \to H'$ *is injective and of closed range, that is, there exists a constant* $\tilde{\beta} > 0$ *such that*

$$\left\| \boldsymbol{B}^t(v) \right\|_{H'} := \sup_{\substack{\tau \in H \\ \tau \neq \theta}} \frac{b(\tau, v)}{\|\tau\|_H} \geq \tilde{\beta} \, \|v\|_Q \quad \forall v \in W^\perp \tag{2.3}$$

ii) $j' \circ \boldsymbol{B} : H \to (W^\perp)'$ *is surjective*

iii) $\boldsymbol{B} \circ i : V^\perp \to Q'$ *is injective and of closed range, that is, there exists a constant* $\hat{\beta} > 0$ *such that*

$$\|\boldsymbol{B}(\tau)\|_{Q'} := \sup_{\substack{v \in Q \\ v \neq \theta}} \frac{b(\tau, v)}{\|v\|_Q} \geq \hat{\beta} \, \|\tau\|_H \quad \forall \tau \in V^\perp \tag{2.4}$$

iv) $i' \circ \boldsymbol{B}^t : Q \to (V^\perp)'$ *is surjective.*

Lemma 2.3.

Let X and Y be two Banach spaces and let $A \in \mathcal{L}(X, Y)$ be a surjective operator. Let $\alpha > 0$. The property

$$\forall y \in Y : \exists x \in X : A(x) = y \quad \textit{and} \quad \alpha \|x\|_X \leq \|y\|_Y ,$$

implies

$$\inf_{\substack{F \in Y' \\ F \neq \theta}} \sup_{\substack{x \in X \\ x \neq \theta}} \frac{A'(F)(x)}{\|F\|_{Y'} \|x\|_X} \geq \alpha$$

The converse is true if X is reflexive.

Theorem 3.1.

Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathbb{R}$, $b : H \times Q \to \mathbb{R}$ and $c : Q \times Q \to \mathbb{R}$ be given bounded bilinear forms. Assume that:

i) *a and c are symmetric and positive semi-definite,*

ii) *there exists a constant $\tilde{\alpha} > 0$ such that*

$$\sup_{\substack{\tau \in V \\ \tau \neq \theta}} \frac{a(\vartheta, \tau)}{\|\tau\|_H} \geq \tilde{\alpha} \|\vartheta\|_H \quad \forall \vartheta \in V \tag{3.1}$$

iii) *and there exists a constant $\tilde{\beta} > 0$ such that*

$$\sup_{\substack{\tau \in H \\ \tau \neq \theta}} \frac{b(\tau, v)}{\|\tau\|_H} \geq \tilde{\beta} \|v\|_Q \quad \forall v \in Q \tag{3.2}$$

Then, for each pair $(f, g) \in H' \times Q'$ *there exists a unique* $(\sigma, u) \in H \times Q$ *solution to* (1.1). *Moreover, there exists a constant* $\tilde{C} > 0$, *depending only on* $\|a\|, \|c\|, \tilde{\alpha}$, *and* $\tilde{\beta}$, *such that*

$$\|(\sigma, u)\|_{H \times Q} \leq \tilde{C} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\} \tag{3.3}$$

Proof Scheme.

We fix a pair $(\zeta, w) \in H \times Q$ and define the functionals $F_{(\zeta,w)} : H \to \mathbb{R}$ and $G_{(\zeta,w)} : Q \to \mathbb{R}$ by

$$F_{(\zeta,w)}(\tau) := a(\zeta, \tau) + b(\tau, w) \quad \forall \tau \in H$$

and

$$G_{(\zeta,w)}(v) := b(\zeta, v) - c(w, v) \quad \forall v \in Q.$$

One can easily note that they are actually elements of H' and Q', respectively. Moreover, we define the bilinear form $\boldsymbol{A} : (H \times Q) \times (H \times Q) \to \mathbb{R}$ by

$$\boldsymbol{A}((\zeta, w), (\tau, v)) := F_{(\zeta,w)}(\tau) + G_{(\zeta,w)}(v) \quad \forall (\zeta, w), (\tau, v) \in (H \times Q)$$

and the functional $\boldsymbol{F} : H \times Q \to \mathbb{R}$ by

$$\boldsymbol{F}(\tau, v) := f(\tau) + g(v) \quad \forall (\tau, v) \in H \times Q$$

Which allows us to write the original problem (see (1.1)) equivalently, as:

Find $(\sigma, u) \in H \times Q$ such that

$$\boldsymbol{A}((\sigma, u), (\tau, v)) = \boldsymbol{F}(\tau, v) \quad \forall (\tau, v) \in H \times Q \tag{3.4}$$

In this way, recalling the Generalized Lax-Milgram lemma, the problem (1.1)(cf. (3.4)) is well-posed if and only if the following hypotheses are satisfied:

1. there exists a constant $\alpha > 0$ such that

$$S(\zeta, w) := \sup_{\substack{(\tau,v)\in H\times Q\\(\tau,v)\neq\theta}} \frac{\boldsymbol{A}((\zeta, w), (\tau, v))}{\|(\tau, v)\|_{H\times Q}} \geq \alpha \|(\zeta, w)\|_{H\times Q}$$

2. for each $(\tau, v) \in H \times Q$, $(\tau, v) \neq \theta$:

$$\sup_{(\zeta,w)\in H\times Q} \boldsymbol{A}((\zeta, w)(\tau, v)) > 0$$

Certainly, the fact that a and c are symmetric (see i)) guarantees the symmetry of $\boldsymbol{A}$, which results in the second hypothesis being redundant.
On the other hand, we notice that there is an equivalency between $S(\zeta, w)$ and the expression $\left\|F_{(\zeta,w)}\right\|_{H'} + \left\|G_{(\zeta,w)}\right\|_{Q'}$. More precisely, one has

$$\frac{1}{2}\left\{\left\|F_{(\zeta,w)}\right\|_{H'} + \left\|G_{(\zeta,w)}\right\|_{Q'}\right\} \leq S(\zeta, w) \leq \left\|F_{(\zeta,w)}\right\|_{H'} + \left\|G_{(\zeta,w)}\right\|_{Q'}, \tag{3.5}$$

for each $(\zeta, w) \in H \times Q$.
The lower bound implies that it suffices to prove the existence of $\tilde{C} > 0$ such that

$$\|(\zeta, w)\|_{H\times Q} \leq \tilde{C}\left\{\left\|F_{(\zeta,w)}\right\|_{H'} + \left\|G_{(\zeta,w)}\right\|_{Q'}\right\}, \tag{3.6}$$

for each $(\zeta, w) \in H \times Q$, to reach the desired continous inf-sup condition.
To do so, we first observe that $\boldsymbol{B}^t = \boldsymbol{B}' \circ \mathcal{J}_Q$. Indeed, given $v \in Q$ and $\tau \in H$, we have

$$(\boldsymbol{B}'\circ\mathcal{J}_Q)(v)(\tau) = \boldsymbol{B}'(\mathcal{J}_Q(v))(\tau) = (\mathcal{J}_Q(v)\circ\boldsymbol{B})(\tau) = \mathcal{J}_Q(v)(\boldsymbol{B}(\tau)) = \boldsymbol{B}(\tau)(v) = \boldsymbol{B}^t(v)(\tau)$$

Using this and the inf-sup condition for b (see (3.2)), we conclude that $\boldsymbol{B}$ is surjective.
Now, using the converse of Lemma 2.3, we know that exists $\bar{\zeta} \in H$ such that

$$\boldsymbol{B}(\bar{\zeta}) = \boldsymbol{B}(\zeta) \quad \text{and} \quad \left\|\bar{\zeta}\right\|_H \leq \frac{1}{\tilde{\beta}}\|\boldsymbol{B}(\zeta)\|_{Q'} = \frac{1}{\tilde{\beta}}\left\|\boldsymbol{B}(\bar{\zeta})\right\|_{Q'} \tag{3.7}$$

It is easy to see that $\zeta_0 := \zeta - \bar{\zeta} \in V$ and therefore we can apply the continuous inf-sup for a (see (3.1)) to $\vartheta = \zeta_0$, that is

$$\sup_{\substack{\tau \in V \\ \tau \neq \theta}} \frac{a(\zeta_0, \tau)}{\|\tau\|_H} \geq \tilde{\alpha} \, \|\zeta_0\|_H$$

Observing that $F_{(\zeta,w)}(\tau) = a(\zeta, \tau)$ for all $\tau \in V$, the above inequality implies that

$$\|\zeta_0\|_H \leq \frac{1}{\tilde{\alpha}} \sup_{\substack{\tau \in V \\ \tau \neq \theta}} \frac{a(\zeta_0, \tau)}{\|\tau\|_H} = \frac{1}{\tilde{\alpha}} \sup_{\substack{\tau \in V \\ \tau \neq \theta}} \frac{a(\zeta, \tau) - a(\bar{\zeta}, \tau)}{\|\tau\|_H} \leq \frac{1}{\tilde{\alpha}} \left\|F_{(\zeta,w)}\right\|_{H'} + \frac{1}{\tilde{\alpha}} \|a\| \left\|\bar{\zeta}\right\|_H \tag{3.8}$$

Next, noting from the definition of $G_{(\zeta,w)}$ that $b(\bar{\zeta}, v) = G_{(\zeta,w)}(v) + c(w, v)$ for all $v \in Q$, it follows from the inequality in (3.7) that

$$\left\|\bar{\zeta}\right\|_H \leq \frac{1}{\tilde{\beta}} \left\|\boldsymbol{B}(\bar{\zeta})\right\|_{Q'} = \frac{1}{\tilde{\beta}} \sup_{\substack{v \in Q \\ v \neq \theta}} \frac{b(\bar{\zeta}, v)}{\|v\|_Q} = \frac{1}{\tilde{\beta}} \sup_{\substack{v \in Q \\ v \neq \theta}} \frac{G_{(\zeta,w)}(v) + c(w, v)}{\|v\|_Q},$$

and since c is symmetric and positive semi-definite,
$c(w, v) \leq c(w, w)^{1/2} c(v, v)^{1/2} \leq |w|_c \, \|c\|^{1/2} \, \|v\|_Q$, we obtain that

$$\left\|\bar{\zeta}\right\|_H \leq \frac{1}{\tilde{\beta}} \left\|G_{(\zeta,w)}\right\|_{Q'} + \frac{\|c\|^{1/2}}{\tilde{\beta}} |w|_c \tag{3.9}$$

Thanks to (3.8) and (3.9), we have the following upper bound

$$\|\zeta\|_H \leq \frac{1}{\tilde{\alpha}} \left\|F_{(\zeta,w)}\right\|_{H'} + \left(1 + \frac{\|a\|}{\tilde{\alpha}}\right) \frac{1}{\tilde{\beta}} \left\|G_{(\zeta,w)}\right\|_{Q'} + \left(1 + \frac{\|a\|}{\tilde{\alpha}}\right) \frac{\|c\|^{1/2}}{\tilde{\beta}} |w|_c \quad (3.10)$$

Now, we shall focus on bounding $|w|_c$ in terms of $\left\|F_{(\zeta,w)}\right\|_{H'}$ and $\left\|G_{(\zeta,w)}\right\|_{Q'}$. In this way, remembering the definition of $F_{(\zeta,w)}$, we know that $b(\tau, w) = F_{(\zeta,w)}(\tau) - a(\zeta, \tau)$ for all $\tau \in H$, so that applying the third hypothesis (see (3.2)) to $w \in Q$, we deduce that

$$\|w\|_Q \leq \frac{1}{\tilde{\beta}} \left\|F_{(\zeta,w)}\right\|_{H'} + \frac{\|a\|}{\tilde{\beta}} \|\zeta\|_H \quad (3.11)$$

We notice that $F_{(\zeta,w)}(\zeta) - G_{(\zeta,w)}(w) = a(\zeta,\zeta) + c(w,w)$ and since the first hypothesis we know that $a(\zeta,\zeta) \geq 0$, from which it follows that

$$|w|_c^2 = c(w,w) \leq \left\|F_{(\zeta,w)}\right\|_{H'} \|\zeta\|_H + \left\|G_{(\zeta,w)}\right\|_{Q'} \|w\|_Q$$

and then, employing the bounds for $\|\zeta\|_H$ and $\|w\|_Q$ provided by (3.10) and (3.11), and applying Young's inequality, we arrive at

$$|w|_c \leq \left(2\,\text{máx}\{\tilde{C}_1, \tilde{C}_2\}\right)^{1/2} \left\{\left\|F_{(\zeta,w)}\right\|_{H'} + \left\|G_{(\zeta,w)}\right\|_{Q'}\right\} \quad (3.12)$$

where

$$\tilde{C}_1 := \frac{1}{\tilde{\alpha}} + \frac{1}{\tilde{\beta}} \left(1 + \frac{\|a\|}{\tilde{\alpha}}\right) + \frac{\|c\|}{\tilde{\beta}^2} \left(1 + \frac{\|a\|}{\tilde{\alpha}}\right)^2$$

and

$$\tilde{C}_2 := \frac{1}{\tilde{\beta}}\left(1 + \frac{\|a\|}{\tilde{\alpha}}\right)\left\{1 + \frac{\|a\|}{\tilde{\beta}} + \frac{\|a\|^2\,\|c\|}{\tilde{\beta}^3}\left(1 + \frac{\|a\|}{\tilde{\alpha}}\right)\right\}$$

Finally, (3.10), (3.11) and (3.12) complete the proof.

□

We remark that (3.5) and the conclusion of the previous theorem, give us that the global inf-sup condition for $\boldsymbol{A}$ holds, namely

$$\sup_{\substack{(\tau,v)\in H\times Q\\(\tau,v)\neq\theta}} \frac{\boldsymbol{A}((\zeta,w),(\tau,v))}{\|(\tau,v)\|_{H\times Q}} \geq \frac{1}{2\tilde{C}}\,\|(\zeta,w)\|_{H\times Q} \qquad \forall(\zeta,w)\in H\times Q \tag{3.13}$$

Let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be families of finite dimensional subspaces of H and Q, respectively. We introduce the Galerkin scheme associated with (1.1): Find $(\sigma_h, u_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= f(\tau_h) \quad \forall \tau_h \in H_h \\ b(\sigma_h, v_H) - c(u_h, v_h) &= g(v_h) \quad \forall v_h \in Q_h \end{aligned} \tag{3.14}$$

We let $\boldsymbol{B}_h : H_h \to Q'_h$ and $\boldsymbol{B}_h^t : Q_h \to H'_h$ be the discrete versions of the bounded linear operator induced by b, and define the respective null spaces

$$V_h := N(\boldsymbol{B}_h) := \{\tau_h \in H_h : \quad b(\tau_h, v_h) = 0 \quad \forall v_h \in Q_h\}$$

and

$$W_h := N(\boldsymbol{B}_h^t) := \{v_h \in Q_h : \quad b(\tau_h, v_h) = 0 \quad \forall \tau_h \in H_h\}$$

Now, we consider the case in which $W_h = \{\theta\}$.

Theorem 3.2.

Assume that:

I) a *and* c *are symmetric and positive semi-definite,*

II) *there exists a constant* $\tilde{\alpha}_d > 0$*, independent of* h*, such that*

$$\sup_{\substack{\tau_h \in V_h \\ \tau_h \neq \theta}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_H} \geq \tilde{\alpha}_d \|\vartheta_h\|_H \quad \forall \vartheta_h \in V_h \tag{3.15}$$

III) *and there exists a constant* $\tilde{\beta}_d > 0$*, independent of* h*, such that*

$$\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq \theta}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \geq \tilde{\beta}_d \|v_h\|_Q \quad \forall v_h \in Q_h \tag{3.16}$$

Then, for each pair $(f, g) \in H' \times Q'$ *there exists a unique* $(\sigma_h, u_h) \in H_h \times Q_h$ *solution to* (3.14)*. Moreover, there exists a constant* $\tilde{C}_{\mathsf{d}} > 0$*, depending only on* $\|a\|$*,* $\|c\|$*,* $\tilde{\alpha}_{\mathsf{d}}$ *and* $\tilde{\beta}_{\mathsf{d}}$*, such that*

$$\|\sigma_h\|_H + \|u_h\|_Q \leq \tilde{C}_{\mathsf{d}} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\} \tag{3.17}$$

Proof Scheme.

First, notice that the third hypothesis is equivalent to

$$\left\| \boldsymbol{B}_h^t(v_h) \right\|_{H_h'} \geq \tilde{\beta}_{\mathsf{d}} \|v_h\|_Q \quad \forall v_h \in Q_h \tag{3.18}$$

Similarly to the continuous case, we have $\boldsymbol{B}_h^t = \boldsymbol{B}_h' \circ \mathcal{J}_{Q_h}$ and (3.18) is equivalent to stating

$$\left\| \boldsymbol{B}_h'(\mathcal{G}_h) \right\|_{H_h'} \geq \tilde{\beta}_{\mathsf{d}} \|\mathcal{G}_h\|_{Q_h''} \quad \forall \mathcal{G}_h \in Q_h''$$

Applying the converse of Lemma (2.3), we conclude that for each $G_h \in Q_h'$ there exists $\vartheta_h \in H_h$ such that

$$\boldsymbol{B}_h(\vartheta_h) = G_h \quad \text{and} \quad \|\vartheta_h\|_H \leq \frac{1}{\tilde{\beta}_{\mathsf{d}}} \|G_h\|_{Q_h'} \tag{3.19}$$

In this way, given $(\zeta_h, w_h) \in H_h \times Q_h$, we deduce the existence of $\bar{\zeta}_h \in H_h$ such that

$$\boldsymbol{B}_h(\bar{\zeta}_h) = \boldsymbol{B}_h(\zeta_h) \quad \text{and} \quad \left\|\bar{\zeta}_h\right\|_H \leq \frac{1}{\tilde{\beta}_{\mathsf{d}}} \|\boldsymbol{B}_h(\zeta_h)\|_{Q'} = \frac{1}{\tilde{\beta}_{\mathsf{d}}} \left\|\boldsymbol{B}_h(\bar{\zeta}_h)\right\|_{Q'}, \tag{3.20}$$

so that ζ_h can be decomposed as

$$\zeta_h = \zeta_{0,h} + \bar{\zeta}_h,$$

with $\zeta_{0,h} := \zeta_h - \bar{\zeta}_h \in V_h$.
The rest of the proof follows as the one of Theorem (3.1).

□

Similarly as the continuous case, we have the global discrete inf-sup condition for $\boldsymbol{A}$, that is

$$\sup_{\substack{(\tau_h,v_h)\in H_h\times Q_h \\ (\tau_h,v_h)\neq\theta}} \frac{\boldsymbol{A}((\zeta_h,w_h),(\tau_h,v_h))}{\|(\tau_h,v_h)\|_{H\times Q}} \geq \frac{1}{2\tilde{C}_{\mathsf{d}}} \|(\zeta_h,w_h)\|_{H\times Q} \qquad \forall(\zeta_h,w_h)\in H_h\times Q_h \tag{3.21}$$

Theorem 3.3.

Assume the hypotheses of (3.1) *and* (3.2), *and let* $(\sigma,u)\in H\times Q$ *and* $(\sigma_h,u_h)\in H_h\times Q_h$ *be the unique solutions of* (1.1) *and* (3.14), *respectively. Then, there exists a constant* $\hat{C}_d > 0$, *depending only on* $\|a\|$, $\|b\|$, $\|c\|$, $\tilde{\alpha}_d$ *and* $\tilde{\beta}_d$, *such that*

$$\|\sigma-\sigma_h\|_H + \|u-u_h\|_Q \leq \hat{C}_d\left\{\mathit{dist}(\sigma,H_h) + \mathit{dist}(u,Q_h)\right\} \tag{3.22}$$

Proof Scheme.

Due to the equivalence between (1.1) and (3.4), it is clear that (3.14) can be, equivalently, rewritten as: Find $(\sigma_h, u_h) \in H_h \times Q_h$ such that

$$\boldsymbol{A}((\sigma_h, u_h), (\tau_h, v_h)) = \boldsymbol{F}(\tau_h, v_h) \quad \forall (\tau_h, v_h) \in H_h \times Q_h \tag{3.23}$$

We first apply the triangle inequality to obtain

$$\|(\sigma, u) - (\sigma_h, u_h)\|_{H\times Q} \leq \|(\sigma, u) - (\zeta_h, w_h)\|_{H\times Q} + \|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{H\times Q}$$

for each $(\zeta_h, w_h) \in H_h \times Q_h$. Then we employ the global discrete inf-sup condition (3.21), which gives

$$\|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{H\times Q} \leq 2\, \tilde{C}_{\mathsf{d}} \sup_{\substack{(\tau_h, v_h) \in H_h \times Q_h \\ (\tau_h, v_h) \neq \theta}} \frac{\boldsymbol{A}((\sigma_h, u_h) - (\zeta_h, w_h), (\tau_h, v_h))}{\|(\tau_h, v_h)\|_{H\times Q}}$$

and finally we use that $\boldsymbol{A}((\sigma_h, u_h), (\tau_h, v_h)) = \boldsymbol{A}((\sigma, u), (\tau_h, v_h))$ for each $(\tau_h, v_h) \in H_h \times Q_h$, along with the boundedness of $\boldsymbol{A}$, that is

$$\begin{aligned}\|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{H\times Q} &\leq 2\, \tilde{C}_{\mathsf{d}} \sup_{\substack{(\tau_h, v_h) \in H_h \times Q_h \\ (\tau_h, v_h) \neq \theta}} \frac{\boldsymbol{A}((\sigma, u) - (\zeta_h, w_h), (\tau_h, v_h))}{\|(\tau_h, v_h)\|_{H\times Q}} \\ &\leq 2\, \tilde{C}_{\mathsf{d}}\, \|\boldsymbol{A}\|\, \|(\sigma, u) - (\zeta_h, w_h)\|_{H\times Q}\end{aligned}$$

and then

$$\|(\sigma, u) - (\sigma_h, u_h)\|_{H\times Q} \leq (1 + 2\tilde{C}_{\mathsf{d}}\,\|\boldsymbol{A}\|)\,\|(\sigma, u) - (\zeta_h, w_h)\|_{H\times Q}$$

We define $\hat{C}_{\mathsf{d}} := 1 + 2\tilde{C}_{\mathsf{d}}\,\|\boldsymbol{A}\|$ and we arrive at

$$\|\sigma - \sigma_h\|_H + \|u - u_h\|_Q \leq \hat{C}_{\mathsf{d}}\left\{\|\sigma - \zeta_h\|_H + \|u - w_h\|_Q\right\}$$

for all $(\zeta_h, w_h) \in H \times Q$. Which allows us to conclude (3.22). $\square$

Theorem 4.1.

Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathbb{R}$, $b : H \times Q \to \mathbb{R}$, and $c : Q \times Q \to \mathbb{R}$ be bounded bilinear forms. In addition, let $\boldsymbol{B} : H \to Q'$ and $\boldsymbol{B}^t : Q \to H'$ be the bounded linear operators induced by b, and let $V := N(\boldsymbol{B})$ and $W := N(\boldsymbol{B}^t)$ be the respective null spaces. Assume that:

i) *there exist closed subspaces $V^\perp$ and $W^\perp$ of H and Q, respectively, such that $H = V \oplus V^\perp$ and $Q = W \oplus W^\perp$,*

ii) *a and c are symmetric and positive semi-definite, the latter meaning that*

$$a(\tau,\tau) \geq 0 \quad \forall \tau \in H \quad \textit{and} \quad c(v,v) \geq 0 \quad \forall v \in Q \tag{4.1}$$

iii) *there exists a constant $\tilde{\alpha} > 0$ such that*

$$\sup_{\substack{\tau \in V \\ \tau \neq \theta}} \frac{a(\vartheta,\tau)}{\|\tau\|_H} \geq \tilde{\alpha}\,\|\vartheta\|_H \quad \forall \vartheta \in V, \tag{4.2}$$

iv) *there exists a constant $\tilde{\beta} > 0$ such that*

$$\sup_{\substack{\tau \in H \\ \tau \neq \theta}} \frac{b(\tau,v)}{\|\tau\|_H} \geq \tilde{\beta}\,\|v\|_Q \quad \forall v \in W^\perp, \tag{4.3}$$

v) *and there exists a constant* $\tilde{\gamma} > 0$ *such that*

$$\sup_{\substack{v \in W \\ v \neq \theta}} \frac{c(z,v)}{\|v\|_Q} \geq \tilde{\gamma}\, \|z\|_Q \quad \forall z \in W. \tag{4.4}$$

Then, for each pair $(f,g) \in H' \times Q'$ *there exists a unique* $(\sigma, u) \in H \times Q$ *solution to* (1.1) *(equivalently* (3.4)*). Moreover, there exists a constant* $\tilde{C} > 0$*, depending only on* $\|a\|$, $\|c\|$, $\tilde{\alpha}$, $\tilde{\beta}$, C_H*, and* $\tilde{\gamma}$*, such that*

$$\|(\sigma,u)\|_{H \times Q} \leq \tilde{C} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\} \tag{4.5}$$

Theorem 4.2.

In addition to the previous notations and definitions, assume that:

i) a *and* c *are symmetric and positive-definite,*

ii) *there exists a constant* $\tilde{\alpha}_d > 0$, *independent of* h, *such that*

$$\sup_{\substack{\tau_h \in V_h \\ \tau_h \neq \theta}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_H} \geq \tilde{\alpha}_d \|\vartheta_h\|_H \quad \forall \vartheta_h \in V_h, \tag{4.6}$$

iii) *there exists a constant* $\tilde{\beta}_d > 0$, *independent of* h, *such that*

$$\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq \theta}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \geq \tilde{\beta}_d \|v_h\|_Q \quad \forall v_h \in W_h^{\perp}, \tag{4.7}$$

iv) *there exists a constant* $\hat{\beta}_d > 0$, *independent of* h, *such that*

$$\sup_{\substack{v_h \in Q_h \\ v_h \neq \theta}} \frac{b(\tau_h, v_h)}{\|v_h\|_Q} \geq \hat{\beta}_d \|\tau_h\|_H \quad \forall \tau_h \in V_h^{\perp}, \tag{4.8}$$

v) *and there exists a constant* $\tilde{\gamma}_d > 0$, *independent of* h, *such that*

$$\sup_{\substack{v_h \in W_h \\ v \neq \theta}} \frac{c(z_h, v_h)}{\|v_h\|_Q} \geq \tilde{\gamma}_d \|z_h\|_Q \quad \forall z_h \in W_h. \tag{4.9}$$

Then, for each pair $(f,g) \in H' \times Q'$ *there exists a unique* $(\sigma_h, u_h) \in H_h \times Q_h$ *solution to* (3.14)*. Moreover, there exists a constant* $\tilde{C}_d > 0$*, depending only on* $\|a\|$*,* $\|c\|$*,* $\tilde{\alpha}_d$*,* $\tilde{\beta}_d$*,* $\hat{\beta}_d$*, and* $\tilde{\gamma}_d$*, such that*

$$\|\sigma_h\|_H + \|u_h\|_Q \leq \tilde{C}_d \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\} \tag{4.10}$$

Theorem 4.3.

Assume the hypotheses of Theorems (4.1) *and* (4.2), *and let* $(\sigma, u) \in H \times Q$ *and* $(\sigma_h, u_h) \in H_h \times Q_h$ *be the unique solutions of* (1.1) *and* (3.14), *respectively. Then, there exists a constant* $\hat{C}_d > 0$, *depending only on* $\|a\|$, $\|b\|$, $\|c\|$, $\tilde{\alpha}_d$, $\tilde{\beta}_d$, $\hat{\beta}_d$, *and* $\tilde{\gamma}_d$, *such that*

$$\|\sigma - \sigma_h\|_H + \|u - u_h\|_Q \leq \hat{C}_d \left\{ \mathit{dist}(\sigma, H_h) + \mathit{dist}(u, Q_h) \right\} \tag{4.11}$$