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A Banach spaces-based analysis of a new mixed-primal finite element method for a coupled flow-transport problem[☆]

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Abstract

In this paper we introduce and analyze a new finite element method for a strongly coupled flow and transport problem in R^n, $n \in \{2, 3\}$, whose governing equations are given by a scalar nonlinear convection–diffusion equation coupled with the Stokes equations. The variational formulation for this model is obtained by applying a suitable dual-mixed method for the Stokes system and the usual primal procedure for the transport equation. In this way, and differently from the techniques previously developed for this and related coupled problems, no augmentation procedure needs to be incorporated now into the solvability analysis, which constitutes the main advantage of the present approach. The resulting continuous and discrete schemes, which involve the Cauchy fluid stress, the velocity of the fluid, and the concentration as the only unknowns, are then equivalently reformulated as fixed point operator equations. Consequently, the well-known Schauder, Banach, and Brouwer theorems, combined with Babuška–Brezzi's theory in Banach spaces, monotone operator theory, regularity assumptions, and Sobolev imbedding theorems, allow to establish the corresponding well-posedness of them. In particular, Raviart–Thomas approximations of order $k \geq 0$ for the stress, discontinuous piecewise polynomials of degree $\leq k$ for the velocity, and continuous piecewise polynomials of degree $\leq k + 1$ for the concentration, becomes a feasible choice for the Galerkin scheme. Next, suitable Strang-type lemmas are employed to derive optimal a priori error estimates. Finally, several numerical results illustrating the performance of the mixed-primal scheme and confirming the theoretical rates of convergence, are provided.

MSC: 65N30; 65N12; 76R05; 76D07; 65N15

Keywords: Stokes equations; Nonlinear transport problem; Fixed point theory; Sedimentation–consolidation process; Finite element methods; A priori error analysis

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1. Introduction

Several practical applications in engineering, including natural and thermal convection, chemical distillation processes, fluidized beds, solid–liquid separation, and sedimentation–consolidation processes, among others, deal with the transport of a species density in an immiscible fluid. Regarding the understanding and prediction of the behavior of such problems, the main difficulties involved have to do with the highly nonlinear character of the advection and diffusion terms, the strong interaction of velocity and solids volume fraction, the complexity of the resulting coupled problem, the saddle-point structure of either the flow or transport problem, or both, and the eventual non-homogeneous boundary conditions, whose handling could imply the introduction of Lagrange multipliers as further unknowns. In any case, most of these difficulties influence the solvability analysis of the continuous and discrete schemes, and particularly the construction of appropriate finite element subspaces yielding stability of the latter, in addition to the derivation of a priori error bounds and corresponding rates of convergence.

Now, concerning the unknowns of interest, we stress that until certain time ago, the main sought physical quantities of most of the above mentioned models were the velocity and the pressure of the fluid, and the local solids concentration (see, e.g. [1–3] for the particular phenomenon given by the sedimentation–consolidation of particles). More recently, the development of new numerical methods that directly approximate other variables of physical relevance, such as the principal components of the fluid or solids stress tensors, the velocity gradient and the vorticity of the fluid, the concentration gradient, and even some boundary traces, has gained considerable attention by the community of numerical analysis of partial differential equations. In turn, the need of computing accurate approximations of additional fields has also arised in related problems in continuum mechanics, thus motivating, for instance, the derivation of new mixed variational formulations and associated Galerkin schemes for linear and nonlinear elasticity, Navier–Stokes, Boussinesq, and other equations (see, e.g. [4–11] and the references therein).

In the present paper we are interested in the coupled flow and transport problem determined by a scalar nonlinear convection–diffusion equation interacting with the Stokes equations, which serves as a prototype for certain sedimentation–consolidation processes, and also models the transport of species concentration within a viscous fluid. Indeed, diverse combinations of primal and mixed finite element methods have been proposed lately in the literature for the numerical solution of this and related models, whose most distinctive feature is the fact that, not only the viscosity of the fluid, but also the diffusion coefficient and the function describing hindered settling, depend on the solution to the transport problem. In addition, the first order term of the latter includes the velocity of the fluid as a factor. In particular, we first refer to [12], where the solvability of our model of interest was analyzed by means of an augmented dual-mixed method in the fluid and the usual primal scheme in the transport equation, thus yielding a three-field augmented mixed-primal variational formulation, whose unknowns, given by the Cauchy stress, the velocity of the fluid, and the concentration, are sought in $\mathbb{H}(\mathbf{div};\Omega)$ (the space of tensors in $[\mathrm{L}^2(\Omega)]^{n\times n}$ with divergence in $[\mathrm{L}^2(\Omega)]^n$), $[\mathrm{H}^1(\Omega)]^n$, and $\mathrm{H}^1(\Omega)$, respectively. The well-posedness of the continuous and discrete formulations, rewritten as fixed point operator equations, is established by using the classical Schauder and Brouwer theorems, respectively. In addition, suitable regularity assumptions and the Sobolev embedding and Rellich–Kondrachov compactness theorems, are also employed in the continuous analysis. In turn, the stability of the associated Galerkin scheme is guaranteed with Raviart–Thomas spaces of order k for the stress, and continuous piecewise polynomials of degree $\leq k+1$ for both the velocity and the concentration. Optimal a priori error estimates and consequent rates of convergence are also derived there.

On the other hand, the approach from [12] was extended in [13] to the case of a strongly coupled flow and transport system typically encountered in sedimentation–consolidation processes. The solvability of this model had been previously discussed in [14] for the case of large fluid viscosity, using the technique of parabolic regularization. Additionally, the existence of solutions to a related model for chemically reacting non-Newtonian fluid had been established in [15] as well. Regarding the governing equations in [13], they consist in the Brinkman problem with variable viscosity, written in terms of Cauchy pseudo-stresses and bulk velocity of the mixture, coupled with a nonlinear advection–nonlinear diffusion equation describing the transport of the solids volume fraction. Moreover, as in [12], the viscosity also depends on the concentration, but differently from there, where an explicit dependence on the gradient of the concentration was assumed for the effective diffusivity, this coefficient is supposed in [13] to depend only on the scalar value of the concentration. An augmented mixed approach for the Brinkman problem and the usual primal weak form for the transport equation are then employed to derive the variational formulation of the coupled problem. In this way, similarly as in [12], the corresponding continuous and discrete solvability analyses are performed by combining fixed point arguments, elliptic regularity estimates, and some classical results. The techniques from [12] and [13] have also been applied in [16] to a model describing the flow-transport interaction in a porous-fluidic domain. In this case, the medium consists of a highly permeable material, where the flow of an incompressible viscous fluid is governed by Brinkman equations (written in terms of vorticity, velocity and pressure, as in [17]), and a porous medium where Darcy's law describes fluid motion using filtration velocity and pressure. In turn, an augmented fully-mixed variational formulation for the model from [12], in which both the dual-mixed method and the augmentation procedure are applied to each one of the equations, was recently introduced and analyzed in [18]. Furthermore, reliable and efficient residual-based *a posteriori* error estimators for the models and corresponding methods studied in [12] and [13] are derived in [19] and [20], respectively.

While the augmentation procedure has played a crucial role in all the aforementioned references, particularly to make possible the solvability analyses in suitable Hilbert space frameworks, and also to allow more flexibility in the choice of the finite elements subspaces yielding the stability of the associated Galerkin schemes, it is no less true that the introduction of additional terms into the formulation certainly leads to much more expensive schemes because of the extra computations that need to be performed in order to set up the stiffness matrix and load vector of the resulting discrete system. As a consequence of this fact, several efforts have been made in recent years aiming to avoid the introduction of augmented terms and appealing to a Banach space framework for analyzing the continuous and discrete formulations of diverse problems in continuum mechanics. The list of works in this direction includes, for instance, [21–24], and [25], all of which, irrespective of dealing with different models, namely Poisson, Navier–Stokes, and Boussinesq equations, share a Banach saddle-point structure for the resulting variational formulations.

According to all the previous discussion, our long-term goal is to extend the applicability of the Banach spaces-based analysis, and together with it, to avoid the use of any augmentation procedure, to address the solvability of a large family of coupled flow-transport problems, which includes those studied in [12,13], and [16], as the most representative ones. In this way, in order to begin contributing in this direction, in the present work we employ some of the theoretical tools from [21,22], and even [12], to propose a new mixed-primal finite element method for the model analyzed in [12]. In particular, since the formulation for the transport equation is the same one employed in [12], our present analysis certainly makes use of the corresponding results available there. The contents of this work are organized as follows. The remainder of this section describes some useful notation to be utilized along the paper. The model problem is introduced in Section 2, and all the auxiliary variables to be employed in the setting of the formulation are defined there. As in [12], the pressure unknown is eliminated, which, however, can be recovered later on via a postprocessing formula. In Section 3.1 we derive the variational formulation of the coupled problem by using a non-augmented dual-mixed approach for Stokes, which constitutes the main advantage with respect to [12], and the classical primal method for the transport. In this way, the resulting mixed-primal scheme yields the Cauchy fluid stress and the velocity of the fluid living in suitable Banach spaces, whereas the concentration lies in the usual Hilbert space $\mathrm{H}^1(\Omega)$. Then, a global fixed-point strategy combined with Babuška–Brezzi's theory in the fluid and classical results on monotone operators in the transport equation, allow to establish the well-posedness of the continuous formulation. Next, in Section 4 we introduce the associated Galerkin scheme and address its solvability by employing the discrete analogue tools of Section 3.1. Thus, the stability of the mixed-primal finite element method is guaranteed with Raviart–Thomas spaces of order $k \geq 0$ for the stress, discontinuous piecewise polynomials of degree $\leq k$ for the velocity, and continuous piecewise polynomials of degree $\leq k+1$ for the concentration. The a priori error estimates and the associated rates of convergence are deduced in Section 5 by using suitable Strang-type lemmas and the approximation properties of the finite element subspaces involved. Finally, the performance of the method is illustrated in Section 6 with several numerical examples in 2D and 3D, which also confirm the aforementioned rates.

Preliminary notations

In what follows $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, is a given bounded domain with polyhedral boundary Γ, whose outward unit normal vector is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual, and $\langle\cdot,\cdot\rangle$

stands for the corresponding duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Furthermore, as usual $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in R^n. Also, for any vector field $\mathbf{v}=(v_i)_{i=1,n}$ we let $\nabla\mathbf{v}$ and $\mathrm{div}(\mathbf{v})$ be its gradient and divergence, respectively. In addition, for any tensor $\boldsymbol{\tau}=(\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta}=(\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n}\tau_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n}\tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau}-\frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

Finally, for any pair of normed spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$, we provide the product space $X\times Y$ with the natural norm $\|(x,y)\|_{X\times Y} := \|x\|_X+\|y\|_Y \quad \forall\,(x,y)\in X\times Y$.

2. The model problem

The following system of partial differential equations describes the stationary state of the transport of species in an immiscible fluid occupying the domain Ω:

$$\begin{gathered}\boldsymbol{\sigma}=\mu(\phi)\,\nabla\mathbf{u}-p\,\mathbb{I}, \quad -\mathbf{div}(\boldsymbol{\sigma})=\boldsymbol{f}\phi, \quad \mathrm{div}(\mathbf{u})=0,\\ \mathbf{p}=\vartheta\,(|\nabla\phi|)\,\nabla\phi-\phi\mathbf{u}-\gamma(\phi)\boldsymbol{k}, \quad -\mathrm{div}(\mathbf{p})=g,\\ \int_\Omega p=0,\end{gathered} \tag{2.1}$$

where the sought quantities are the Cauchy fluid stress $\boldsymbol{\sigma}$, the local volume-average velocity of the fluid $\mathbf{u}$, the pressure p, and the local concentration of species ϕ. In turn, $\boldsymbol{f}\in\mathbf{L}^2(\Omega)$ and $g\in\mathrm{L}^2(\Omega)$ are given functions, and, as observed from the second equation in (2.1), the driving force of the mixture depends linearly on ϕ. In addition, the kinematic effective viscosity, μ; the diffusion coefficient, ϑ; and the one-dimensional flux function describing hindered settling, γ; depend nonlinearly on ϕ, whereas $\boldsymbol{k}$ is a vector pointing in the direction of gravity. Furthermore, ϑ is assumed of class C^1 and we suppose that there exist positive constants μ_1, μ_2, γ_1, γ_2, ϑ_1, and ϑ_2, such that

$$\mu_1\le\mu(s)\le\mu_2 \quad \text{and} \quad \gamma_1\le\gamma(s)\le\gamma_2 \qquad \forall\,s\in\mathrm{R}, \tag{2.2}$$

$$\vartheta_1\le\vartheta(s)\le\vartheta_2 \quad \text{and} \quad \vartheta_1\le\vartheta(s)+s\,\vartheta'(s)\le\vartheta_2 \qquad \forall\,s\ge 0. \tag{2.3}$$

Examples of nonlinear functions ϑ that satisfy the hypothesis (2.3) are $\vartheta(s) := 2+\frac{1}{1+s}\ \forall\,s\ge 0$, which is basically of academic character, and the well-known Carreau law in fluid mechanics given by $\vartheta(s) := m_1+m_2(1+s^2)^{(m_3-2)/2}\ \forall\,s\ge 0$, with $m_1,\,m_2>0$ and $m_3\in(0,2)$. The latter is indeed considered in the numerical results reported below in Section 6. Other examples of diffusion coefficients, which are typically found in the applied literature, replace the $|\nabla\phi|$-dependence of ϑ by simply $\vartheta(\phi)$. In particular, analogously as it can be observed for the augmented formulations introduced and analyzed in [12] and [13], the structure of (2.1) may also serve as a prototype model for generalized Boussinesq or related problems (see, e.g. [26–28]), where ϕ represents the adimensional temperature of a fluid, and $\vartheta(\phi) := \vartheta_0\exp(\phi)$, with a positive parameter ϑ_0. In turn, the above remark is also valid for sedimentation–consolidation systems (see, e.g. [1,3,29]), where ϕ is the volumetric fraction of the solids (or simply concentration), and $\vartheta(\phi) := \vartheta_0\left(1-\dfrac{\phi}{\widetilde{\phi}}\right)^{-m}$, with positive parameters ϑ_0, $\widetilde{\phi}$, and m. In these cases in which ϑ depends only on ϕ, the corresponding assumption (2.3) reduces just to its first inequality, that is $\vartheta_1\le\vartheta(\phi)\le\vartheta_2$, which, thinking for instance of the aforementioned exponential law, is easily satisfied if the values of ϕ are assumed to lie in a suitable range.

Additionally, we assume that μ and γ are Lipschitz continuous, that is that there exist positive constants L_μ and L_γ such that

$$|\mu(s)-\mu(t)| \le L_\mu\,|s-t| \qquad \forall\,s,t\in\mathrm{R}, \tag{2.4}$$

and

$$|\gamma(s)-\gamma(t)| \le L_\gamma\,|s-t| \qquad \forall\,s,t\in\mathrm{R}. \tag{2.5}$$

Finally, given $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, the following Dirichlet boundary conditions complement (2.1):

$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \phi = 0 \quad \text{on} \quad \Gamma\,, \tag{2.6}$$

where, due to the incompressibility of the fluid, the datum $\mathbf{u}_D$ must satisfy the compatibility constraint $\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} = 0$. On the other hand, it is easy to see that the first and third equations in (2.1) are equivalent to

$$\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} \quad \text{and} \quad p = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \tag{2.7}$$

which allows us to eliminate the pressure p, thus arriving at the following equivalent coupled system

$$\begin{gathered}
\frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \qquad -\mathbf{div}(\boldsymbol{\sigma}) = \boldsymbol{f}\phi \quad \text{in} \quad \Omega\,,\\
\mathbf{p} = \vartheta(|\nabla\phi|)\,\nabla\phi - \phi\mathbf{u} - \gamma(\phi)\boldsymbol{k} \quad \text{in} \quad \Omega\,, \qquad -\operatorname{div}(\mathbf{p}) = g \quad \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \qquad \phi = 0 \quad \text{on} \quad \Gamma\,, \qquad \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0\,.
\end{gathered} \tag{2.8}$$

We stress here that the incompressibility condition is implicitly present in the first equation of (2.8), that is in the constitutive equation relating $\boldsymbol{\sigma}$ and $\mathbf{u}$. In addition, the uniqueness condition for p, originally given by $\int_\Omega p = 0$, is now stated as $\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = 0$, which certainly follows from the postprocessed formula for p provided by the second expression in (2.7).

3. The continuous formulation

In this section we introduce and analyze a mixed-primal formulation of the coupled problem (2.8).

3.1. A mixed-primal approach

We first observe that the Dirichlet condition for ϕ motivates the introduction of the space

$$\mathrm{H}^1_0(\Omega) := \Big\{\psi \in \mathrm{H}^1(\Omega): \quad \psi = 0 \quad \text{on} \quad \Gamma\Big\},$$

for which, thanks to the Poincaré inequality, there exists a positive constant c_p, depending only on Ω, such that

$$\|\psi\|_{1,\Omega} \le c_p\,|\psi|_{1,\Omega} \qquad \forall\,\psi \in \mathrm{H}^1_0(\Omega)\,. \tag{3.1}$$

Moreover, the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^4(\Omega)$, which is valid in R^n, $n \in \{2,3\}$, as proved in [30, Theorem 4.12] (see also [31, Theorem 1.3.4]), yields the existence of a positive constant $c(\Omega) > 0$, depending only on Ω, such that

$$\|\psi\|_{0,4;\Omega} \le c(\Omega)\,\|\psi\|_{1,\Omega} \qquad \forall\,\psi \in \mathrm{H}^1(\Omega)\,. \tag{3.2}$$

Next, in order to derive the variational formulation of (2.8), we begin with the transport equation. In fact, multiplying the respective equilibrium equation $\operatorname{div}(\mathbf{p}) = -g$ by $\psi \in \mathrm{H}^1_0(\Omega)$, integrating by parts, taking into account the third equation of (2.8), and bearing in mind the Dirichlet boundary condition $\phi = 0$ on Γ, we deduce that the primal approach for the concentration becomes: Find $\phi \in \mathrm{H}^1_0(\Omega)$ such that

$$A_{\mathbf{u}}(\phi,\psi) = G_\phi(\psi)\,, \tag{3.3}$$

where

$$A_{\mathbf{u}}(\varphi,\psi) := \int_\Omega \vartheta(|\nabla\varphi|)\,\nabla\varphi\cdot\nabla\psi - \int_\Omega \varphi\mathbf{u}\cdot\nabla\psi \qquad \forall\,\varphi,\ \psi \in \mathrm{H}^1_0(\Omega)\,, \tag{3.4}$$

and

$$G_\phi(\psi) := \int_\Omega \gamma(\phi)\boldsymbol{k}\cdot\nabla\psi + \int_\Omega g\psi \qquad \forall\,\psi \in \mathrm{H}^1_0(\Omega)\,. \tag{3.5}$$

Concerning $A_{\mathbf{u}}$, we first recall from [32, Theorem 3.8] (see also [12, Lemma 3.5]) that, thanks to the assumptions on ϑ (cf. (2.3)), the nonlinear operator induced by its first term is strongly monotone in $\mathrm{H}_0^1(\Omega)$ and Lipschitz-continuous in $\mathrm{H}^1(\Omega)$ with constants ϑ_1 and $\widetilde{\vartheta}_2 := \max\big\{\vartheta_2, 2\vartheta_2 - \vartheta_1\big\}$, respectively, that is there hold

$$\int_\Omega \Big\{\vartheta(|\nabla\varphi|)\nabla\varphi - \vartheta(|\nabla\psi|)\nabla\psi\Big\}\cdot\nabla(\varphi-\psi) \,\geq\, \vartheta_1\,|\varphi-\psi|_{1,\Omega}^2 \qquad \forall\,\varphi,\ \psi\in \mathrm{H}^1(\Omega)\,, \tag{3.6}$$

and

$$\left|\int_\Omega \Big\{\vartheta(|\nabla\phi|)\nabla\phi - \vartheta(|\nabla\varphi|)\nabla\varphi\Big\}\cdot\nabla\psi\right| \,\leq\, \widetilde{\vartheta}_2\,|\phi-\varphi|_{1,\Omega}\,|\psi|_{1,\Omega} \qquad \forall\,\phi,\ \varphi,\ \psi\in \mathrm{H}^1(\Omega)\,. \tag{3.7}$$

Furthermore, we notice, using (2.3), Cauchy–Schwarz's inequality, and (3.2), that there hold

$$\left|\int_\Omega \vartheta(|\nabla\varphi|)\,\nabla\varphi\cdot\nabla\psi\right| \,\leq\, \vartheta_2\,\|\varphi\|_{1,\Omega}\|\psi\|_{1,\Omega} \qquad \forall\,\varphi,\psi\in \mathrm{H}^1(\Omega)\,, \tag{3.8}$$

and

$$\left|\int_\Omega \varphi\,\mathbf{v}\cdot\nabla\psi\right| \,\leq\, c(\Omega)\,\|\varphi\|_{1,\Omega}\|\mathbf{v}\|_{0,4;\Omega}\,|\psi|_{1,\Omega} \qquad \forall\,\varphi,\ \psi\in \mathrm{H}^1(\Omega),\quad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,, \tag{3.9}$$

which shows that $A_{\mathbf{u}}$ is well-defined if the given $\mathbf{u}$ lies in $\mathbf{L}^4(\Omega)$, and hence from now on we look for this unknown in the latter space. Throughout the rest of the paper, we make no notational distinction between the semilinear form $A_{\mathbf{u}} : \mathrm{H}_0^1(\Omega)\times\mathrm{H}_0^1(\Omega)\to \mathrm{R}$ (as defined by (3.4)) and its induced nonlinear operator $A_{\mathbf{u}} : \mathrm{H}_0^1(\Omega)\to\mathrm{H}_0^1(\Omega)'$, which maps each $\varphi\in\mathrm{H}_0^1(\Omega)$ into a functional $A_{\mathbf{u}}(\varphi)\in\mathrm{H}_0^1(\Omega)'$, whose evaluation in an arbitrary $\psi\in\mathrm{H}_0^1(\Omega)$ is precisely the right hand side of (3.4). In turn, regarding the functional G_ϕ, we readily observe from (3.5) and (2.2) that it is bounded, independently of ϕ, with

$$\|G_\phi\| \,\leq\, \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}\,. \tag{3.10}$$

Furthermore, multiplying the first equation of (2.8) by a sufficiently smooth arbitrary tensor $\boldsymbol{\tau}$ (living in a suitable space to be described later), integrating by parts, taking into account the Dirichlet boundary condition $\mathbf{u}=\mathbf{u}_D$ on Γ and the identity $\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}=\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}$, we formally obtain

$$\int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\sigma}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle\,. \tag{3.11}$$

We observe here, thanks to the boundedness of μ (cf. (2.2)), that the first expression on the left-hand side of (3.11) makes sense if both $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ belong to $\mathbb{L}^2(\Omega)$, whereas the second one requires that $\mathbf{div}(\boldsymbol{\tau})$ lies in $\mathbf{L}^{4/3}(\Omega)$, which follows from Hölder's inequality and the fact that $\mathbf{u}$ is already sought in $\mathbf{L}^4(\Omega)$. The above suggests to introduce now the Banach space

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,:=\, \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega):\quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^{4/3}(\Omega)\Big\},$$

endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \,:=\, \|\boldsymbol{\tau}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega).$$

In this way, the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$ (cf. [30, Theorem 4.12], [31, Theorem 1.3.4]) guarantees that $\boldsymbol{\tau}\boldsymbol{\nu}$ (as defined in [33, eq. (1.43)]) belongs to $\mathbf{H}^{-1/2}(\Gamma)$ not only for $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega)$ but also when $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, which proves that (3.11) makes full sense for all $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Moreover, it is easy to show (see, e.g. [22, Section 3.1] or [34, Lemma 3.5]) that there exists a positive constant $\widetilde{c}(\Omega)$, depending only on $c(\Omega)$ (cf. (3.2)), such that

$$\|\boldsymbol{\tau}\boldsymbol{\nu}\|_{-1/2,\Gamma} \,\leq\, \widetilde{c}(\Omega)\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,. \tag{3.12}$$

Thus, looking for the unknown $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well, we realize that the equilibrium equation $-\mathbf{div}(\boldsymbol{\sigma})=\boldsymbol{f}\phi$ in Ω can be imposed, equivalently, as

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,=\, -\int_\Omega \boldsymbol{f}\phi\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,. \tag{3.13}$$

In addition, the null mean value of $\mathrm{tr}(\boldsymbol{\sigma})$ stated as the last equation of (2.8) says that $\boldsymbol{\sigma}$ must be actually sought in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega): \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\}.$$

Therefore, given $\phi \in \mathrm{H}^1_0(\Omega)$, we collect (3.11) and (3.13) to arrive at first instance to the following mixed formulation for the flow: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,, \end{aligned} \tag{3.14}$$

where $\mathbf{a}_\phi : \mathbb{H}(\mathbf{div}_{4/3};\Omega)\times\mathbb{H}(\mathbf{div}_{4/3};\Omega)\to\mathrm{R}$, $\mathbf{b}:\mathbb{H}(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)\to\mathrm{R}$, $\mathbf{F}:\mathbb{H}(\mathbf{div}_{4/3};\Omega)\to\mathrm{R}$, and $\mathbf{G}_\phi:\mathbf{L}^4(\Omega)\to\mathrm{R}$ are the bounded bilinear and linear forms, respectively, defined as

$$\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \frac{1}{\mu(\phi)}\boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}\,, \qquad \mathbf{b}(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \tag{3.15}$$

$$\mathbf{F}(\boldsymbol{\tau}) := \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle\,, \quad \text{and} \quad \mathbf{G}_\phi(\mathbf{v}) := -\int_\Omega \boldsymbol{f}\phi\cdot\mathbf{v} \tag{3.16}$$

for all $\boldsymbol{\zeta},\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and for all $\mathbf{v}\in\mathbf{L}^4(\Omega)$. In fact, note that there hold

$$|\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau})| \le \frac{1}{\mu_1}\|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \text{and} \quad |\mathbf{b}(\boldsymbol{\zeta},\mathbf{v})| \le \|\boldsymbol{\zeta}\|_{\mathbf{div}_{4/3};\Omega}\|\mathbf{v}\|_{0,4;\Omega}\,,$$

which confirms boundedness constants given by

$$\|\mathbf{a}_\phi\| = \frac{1}{\mu_1} \quad \text{and} \quad \|\mathbf{b}\| = 1\,. \tag{3.17}$$

In turn, employing the duality between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ together with the estimate (3.12), and then applying Cauchy–Schwarz's inequality and the continuous injection given by (3.2), we readily show that

$$|\mathbf{F}(\boldsymbol{\tau})| \le \widetilde{c}(\Omega)\|\mathbf{u}_D\|_{1/2,\Gamma}\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \text{and} \quad |\mathbf{G}_\phi(\mathbf{v})| \le c(\Omega)\|\boldsymbol{f}\|_{0,\Omega}\|\phi\|_{1,\Omega}\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{3.18}$$

which yields

$$\|\mathbf{F}\| \le \widetilde{c}(\Omega)\|\mathbf{u}_D\|_{1/2,\Gamma} \quad \text{and} \quad \|\mathbf{G}_\phi\| \le c(\Omega)\|\boldsymbol{f}\|_{0,\Omega}\|\phi\|_{1,\Omega}\,. \tag{3.19}$$

Furthermore, thanks to the compatibility condition for $\mathbf{u}_D$ and the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\oplus\mathrm{R}\,\mathbb{I},$$

it is easily shown that imposing the first equation of (3.14) against $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and therefore (3.14) reduces to: Find $(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,. \end{aligned} \tag{3.20}$$

Finally, combining (3.20) and (3.3), we arrive at the following mixed-primal formulation of our coupled problem (2.8): Find $(\boldsymbol{\sigma},\mathbf{u},\phi)\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)\times\mathrm{H}^1_0(\Omega)$ such that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) \qquad \forall\mathbf{v}\in\mathbf{L}^4(\Omega)\,,\\ A_{\mathbf{u}}(\phi,\psi) &= G_\phi(\psi) \qquad \forall\psi\in\mathrm{H}^1_0(\Omega)\,. \end{aligned} \tag{3.21}$$

3.2. A fixed point strategy

In what follows we proceed similarly as in [12] (see also [13,16]) and utilize a fixed point strategy to analyze the solvability of (3.21). For this purpose, we first let $\mathbf{S}:\mathrm{H}^1_0(\Omega)\to\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ be the operator defined by:

$$\mathbf{S}(\phi) = (\mathbf{S}_1(\phi),\mathbf{S}_2(\phi)) := (\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega) \qquad \forall\,\phi\in\mathrm{H}^1_0(\Omega),$$

where $(\boldsymbol{\sigma}, \mathbf{u})$ is the unique solution (to be confirmed below) of (3.20) with the given ϕ. In turn, we let $\widetilde{\mathbf{S}} : \mathrm{H}_0^1(\Omega) \times \mathbf{L}^4(\Omega) \to \mathrm{H}_0^1(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\phi, \mathbf{u}) := \widetilde{\phi} \qquad \forall (\phi, \mathbf{u}) \in \mathrm{H}_0^1(\Omega) \times \mathbf{L}^4(\Omega),$$

with $\widetilde{\phi} \in \mathrm{H}_0^1(\Omega)$ being the unique solution (to be confirmed below) of the problem:

$$A_{\mathbf{u}}(\widetilde{\phi}, \widetilde{\psi}) = G_\phi(\widetilde{\psi}) \qquad \forall\, \widetilde{\psi} \in \mathrm{H}_0^1(\Omega) \tag{3.22}$$

with the given $(\phi, \mathbf{u})$. Then, we set the operator $\mathbf{T} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ as

$$\mathbf{T}(\phi) := \widetilde{\mathbf{S}}(\phi, \mathbf{S}_2(\phi)) \qquad \forall \phi \in \mathrm{H}_0^1(\Omega), \tag{3.23}$$

and realize that solving (3.21) is equivalent to seeking a fixed point of $\mathbf{T}$, that is: Find $\phi \in \mathrm{H}_0^1(\Omega)$ such that

$$\mathbf{T}(\phi) = \phi\,. \tag{3.24}$$

3.3. Well-posedness of the uncoupled problems

In this section we show that the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are indeed well-defined, which means, equivalently, that the uncoupled problems (3.20) and (3.22) are well-posed. We begin with (3.20), for which we employ the classical Babuška–Brezzi theory in Banach spaces (see e.g. [35, Theorem 2.34]), which, given $\phi \in \mathrm{H}_0^1(\Omega)$, requires two inf–sup conditions of $\mathbf{a}_\phi$ on the kernel of $\mathbf{b}$, and an inf–sup condition of $\mathbf{b}$. To this end, we now recall from [34, Lemma 3.2] that a simple modification of the proof of [33, Lemma 2.3] (see also [36, Proposition 3.1, Chapter IV]) allows to show that there exists $c_1 > 0$, depending only on Ω, such that

$$c_1\, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \;\le\; \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{3.25}$$

Next, we let $\mathbb{V}$ be the kernel of $\mathbf{b}$, that is

$$\mathbb{V} := \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega) \Big\},$$

which clearly reduces to

$$\mathbb{V} = \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in} \quad \Omega \Big\}\,. \tag{3.26}$$

Then, we have the following result.

Lemma 3.1. *There exists a positive constant α such that for each $\phi \in \mathrm{H}_0^1(\Omega)$ there holds*

$$\mathbf{a}_\phi(\boldsymbol{\tau}, \boldsymbol{\tau}) \;\ge\; \alpha\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}^2 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V}\,. \tag{3.27}$$

Proof. Given $\phi \in \mathrm{H}_0^1(\Omega)$ and $\boldsymbol{\tau} \in \mathbb{V}$, we easily deduce, according to the definition of $\mathbf{a}_\phi$ (cf. (3.15)), the boundedness of μ (cf. (2.2)), and the inequality (3.25), that

$$\mathbf{a}_\phi(\boldsymbol{\tau}, \boldsymbol{\tau}) \;=\; \int_\Omega \frac{1}{\mu(\phi)} \boldsymbol{\tau}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \;\ge\; \frac{1}{\mu_2}\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \;\ge\; \frac{c_1}{\mu_2}\, \|\boldsymbol{\tau}\|_{0,\Omega}^2 \;=\; \frac{c_1}{\mu_2}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}^2,$$

which proves (3.27) with $\alpha = c_1/\mu_2$. $\square$

As a straightforward consequence of (3.27) it follows that

$$\sup_{\boldsymbol{\zeta} \in \mathbb{V}} \mathbf{a}_\phi(\boldsymbol{\zeta}, \boldsymbol{\tau}) \;>\; 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{V} \setminus \{\mathbf{0}\}\,, \quad \forall \phi \in \mathrm{H}_0^1(\Omega)\,. \tag{3.28}$$

In turn, the aforementioned inf–sup condition of the bilinear form $\mathbf{b}$ is stated as follows.

Lemma 3.2. *There exists a positive constant β, depending on n, c_p (cf. (3.1)) and $c(\Omega)$ (cf. (3.2)), such that*

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega}} \;\ge\; \beta\, \|\mathbf{v}\|_{0,4;\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,. \tag{3.29}$$

Proof. See [34, Lemma 3.4] for details. $\square$

According to the previous results, we are now able to prove the well-definedness of the operator $\mathbf{S}$.

Lemma 3.3. *For each $\phi \in \mathrm{H}^1_0(\Omega)$ there exists a unique $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ solution to the problem (3.20). Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on μ_1, α, β, and $c(\Omega)$, and hence independent of ϕ, such that*

$$\|\mathbf{S}(\phi)\| \,=\, \|(\boldsymbol{\sigma},\mathbf{u})\| \,\le\, C_{\mathbf{S}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi\|_{1,\Omega}\right\} \qquad \forall\,\phi \in \mathrm{H}^1_0(\Omega)\,. \tag{3.30}$$

Proof. Given $\phi \in \mathrm{H}^1_0(\Omega)$, we first recall from (3.17) and (3.19) that $\mathbf{a}_\phi$, $\mathbf{b}$, $\mathbf{F}$, and $\mathbf{G}_\phi$ are all bounded. Then, thanks to Lemmas 3.1 and 3.2, and inequality (3.28), the proof follows from a straightforward application of the Babuška–Brezzi theory in Banach spaces (see, e.g. [35, Theorem 2.34]) to problem (3.20). In particular, the corresponding a priori estimate reads

$$\|(\boldsymbol{\sigma},\mathbf{u})\| \,\le\, C\left\{\|\mathbf{F}\| \,+\, \|\mathbf{G}_\phi\|\right\}, \tag{3.31}$$

with a positive constant C depending only on $\|\mathbf{a}_\phi\| = \frac{1}{\mu_1}$, α, and β. In this way, the foregoing inequality together with (3.17) and (3.19) imply (3.30) and complete the proof. $\square$

We now establish the unique solvability of the nonlinear problem (3.22), which confirms the well-definedness of the operator $\widetilde{\mathbf{S}}$.

Lemma 3.4. *Let $\phi \in \mathrm{H}^1_0(\Omega)$, $\delta \in\,]0,1[$, and $\mathbf{u} \in \mathbf{L}^4(\Omega)$ such that $\|\mathbf{u}\|_{0,4;\Omega} < \frac{\delta\vartheta_1}{c(\Omega)\,c_p}$ (cf. (2.3), (3.1), (3.9)). Then, there exists a unique $\widetilde{\phi} := \widetilde{\mathbf{S}}(\phi,\mathbf{u}) \in \mathrm{H}^1_0(\Omega)$ solution to (3.22), and there holds*

$$\|\widetilde{\mathbf{S}}(\phi,\mathbf{u})\| \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\le\, \frac{c_p^2}{(1-\delta)\,\vartheta_1}\left\{\gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}\right\}. \tag{3.32}$$

Proof. We proceed similarly as in [12, Lemma 3.5] by observing first, thanks to (3.6), (3.1), and (3.9), that for each $\widetilde{\phi}$, $\widetilde{\varphi} \in \mathrm{H}^1_0(\Omega)$ there holds

$$\begin{aligned} A_{\mathbf{u}}(\widetilde{\phi},\widetilde{\phi}-\widetilde{\varphi}) - A_{\mathbf{u}}(\widetilde{\varphi},\widetilde{\phi}-\widetilde{\varphi}) &\ge \vartheta_1\,|\widetilde{\phi}-\widetilde{\varphi}|^2_{1,\Omega} \,-\, c(\Omega)\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,|\widetilde{\phi}-\widetilde{\varphi}|_{1,\Omega} \\ &\ge\, \left\{\vartheta_1 \,-\, c(\Omega)c_p\|\mathbf{u}\|_{0,4;\Omega}\right\}|\widetilde{\phi}-\widetilde{\varphi}|^2_{1,\Omega} \,\ge\, c_p^{-2}\left\{\vartheta_1 \,-\, c(\Omega)c_p\|\mathbf{u}\|_{0,4;\Omega}\right\}\|\widetilde{\phi}-\widetilde{\varphi}\|^2_{1,\Omega}\,, \end{aligned} \tag{3.33}$$

which, using the hypothesis on $\mathbf{u}$, implies the strong monotonicity of the operator $A_{\mathbf{u}}$ with constant $\widetilde{\alpha} := c_p^{-2}(1-\delta)\vartheta_1$. In turn, employing now (3.7), and again (3.9) and the aforementioned bound on $\mathbf{u}$, we obtain that for each $\widetilde{\phi}$, $\widetilde{\varphi}$, $\widetilde{\psi} \in \mathrm{H}^1_0(\Omega)$ there holds

$$\begin{aligned} \left|A_{\mathbf{u}}(\widetilde{\phi},\widetilde{\psi}) - A_{\mathbf{u}}(\widetilde{\varphi},\widetilde{\psi})\right| &\le\, \left\{\widetilde{\vartheta}_2\,|\widetilde{\phi}-\widetilde{\varphi}|_{1,\Omega} \,+\, c(\Omega)\,\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\right\}|\widetilde{\psi}|_{1,\Omega} \\ \le\, \left\{\widetilde{\vartheta}_2 \,+\, c(\Omega)\,\|\mathbf{u}\|_{0,4;\Omega}\right\}\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,|\widetilde{\psi}|_{1,\Omega} &\le\, \left\{\widetilde{\vartheta}_2 \,+\, \frac{\delta\vartheta_1}{c_p}\right\}\|\widetilde{\phi}-\widetilde{\varphi}\|_{1,\Omega}\,|\widetilde{\psi}|_{1,\Omega}\,, \end{aligned} \tag{3.34}$$

which implies the Lipschitz-continuity of $A_{\mathbf{u}}$ with constant $\widetilde{L} = \widetilde{\vartheta}_2 + \frac{\delta\vartheta_1}{c_p}$. In this way, given $\phi \in \mathrm{H}^1_0(\Omega)$, a classical result on the bijectivity of monotone operators (cf. [37, Theorem 3.3.23]) guarantees the existence of a unique $\widetilde{\phi} \in \mathrm{H}^1_0(\Omega)$ solution to (3.22). Moreover, exactly as derived in [12, Lemma 3.5], we find that $\|\widetilde{\phi}\|_{1,\Omega} \le \widetilde{\alpha}^{-1}\,\|G_\phi\|$, which, together with the upper bound for $\|G_\phi\|$ (cf. (3.10)), gives (3.32) and ends the proof. $\square$

We remark here that the assumption on $\mathbf{u}$ in Lemma 3.4 depends on a particular value of $\delta \in\,]0,1[$. In this regard, we notice that the closer to 1, the larger the range for choosing $\mathbf{u}$, but then the constant in the a priori estimate (3.32) tends to blow up. Conversely, the closer to 0, the smaller the range for $\mathbf{u}$, but then the constant in (3.32) remains bounded. According to the above, in what follows we simply consider the midpoint $\delta = 1/2$, which yields

$$\widetilde{\alpha} = \frac{\vartheta_1}{2\,c_p^2} \qquad \text{and} \qquad \widetilde{L} = \widetilde{\vartheta}_2 + \frac{\vartheta_1}{2c_p}\,, \tag{3.35}$$

and with which the assumption on $\mathbf{u}$ and the a priori estimate for $\widetilde{\mathbf{S}}(\phi, \mathbf{u}) = \widetilde{\phi}$ in Lemma 3.4, become

$$\|\mathbf{u}\|_{0,4;\Omega} \,<\, \frac{\vartheta_1}{2\,c(\Omega)\,c_p} \tag{3.36}$$

and

$$\|\widetilde{\mathbf{S}}(\phi, \mathbf{u})\| \,=\, \|\widetilde{\phi}\|_{1,\Omega} \,\leq\, C_{\widetilde{\mathbf{S}}} \left\{ \gamma_2 |\Omega|^{1/2} \|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \right\}, \tag{3.37}$$

respectively, where $C_{\widetilde{\mathbf{S}}} := \frac{1}{\widetilde{\alpha}}$.

3.4. Solvability analysis of the fixed point equation

Having established in the previous section that the uncoupled problems (3.20) and (3.22) are well-posed, equivalently that the operators $\mathbf{S}$, $\widetilde{\mathbf{S}}$, and hence $\mathbf{T}$ are well defined, we now address the solvability analysis of the fixed point equation (3.24). To this end, in this section we apply the Schauder fixed point theorem, whose statement is as follows (see, e.g. [38, Theorem 9.12-1(b)]).

Theorem 3.5. *Let W be a closed and convex subset of a Banach space X and let $T : W \to W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

We now proceed to verify that, under suitable assumptions on the data, the operator $\mathbf{T}$ satisfies the hypotheses of Theorem 3.5. We begin with the following result.

Lemma 3.6. *Given $r > 0$, we let W be the closed and convex subset of $\mathrm{H}^1_0(\Omega)$ defined by*

$$W \,:=\, \left\{ \phi \in \mathrm{H}^1_0(\Omega) : \quad \|\phi\|_{1,\Omega} \,\leq\, r \right\},$$

and assume that the data satisfy

$$\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, r\|\boldsymbol{f}\|_{0,\Omega} \,<\, \frac{\vartheta_1}{2\,C_{\mathbf{S}}\,c(\Omega)\,c_p} \quad \textit{and} \quad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega} \,\leq\, \frac{r}{C_{\widetilde{\mathbf{S}}}}. \tag{3.38}$$

Then $\mathbf{T}(W) \subseteq W$.

Proof. Given $\phi \in W$, we get from (3.30) that

$$\|\mathbf{S}(\phi)\| = \|(\boldsymbol{\sigma}, \mathbf{u})\| \,\leq\, C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, r\|\boldsymbol{f}\|_{0,\Omega} \right\},$$

and hence, thanks to the first restriction in (3.38), we observe that $\mathbf{u} = \mathbf{S}_1(\phi)$ satisfies the hypothesis of Lemma 3.4 given by (3.36). Moreover, the corresponding a priori estimate given by (3.37) yields

$$\|\mathbf{T}(\phi)\| = \|\widetilde{\mathbf{S}}(\phi, \mathbf{u})\| \,\leq\, C_{\widetilde{\mathbf{S}}} \left\{ \gamma_2 |\Omega|^{1/2} \|\boldsymbol{k}\| + \|g\|_{0,\Omega} \right\},$$

which, due to the second hypothesis in (3.38), proves that $\mathbf{T}(\phi) \in W$, thus finishing the proof. □

Throughout the rest of the paper we assume further regularity for the problem defining the operator $\mathbf{S}$. More precisely, we suppose that $\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}(\Gamma)$ for some $\varepsilon \in (0, 1)$ (when $n = 2$) or $\varepsilon \in (\frac{1}{2}, 1)$ (when $n = 3$), and that for each $\psi \in \mathrm{H}^1_0(\Omega)$ with $\|\psi\|_{1,\Omega} \,\leq\, r$, $r > 0$ given, there hold $(\boldsymbol{\zeta}, \mathbf{w}) := \mathbf{S}(\psi) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \cap \mathbb{H}^\varepsilon(\Omega) \times \mathbf{L}^4(\Omega) \cap \mathbf{W}^{\varepsilon,4}(\Omega)$ and

$$\|\boldsymbol{\zeta}\|_{\varepsilon,\Omega} + \|\mathbf{w}\|_{\varepsilon,4;\Omega} \,\leq\, \widetilde{C}_{\mathbf{S}}(r) \left\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega} \|\psi\|_{1,\Omega} \right\}, \tag{3.39}$$

with a positive constant $\widetilde{C}_{\mathbf{S}}(r)$ independent of the given ψ but depending on the upper bound r of its H^1-norm.

We now aim to prove the continuity and compactness properties of $\mathbf{T}$, which will be straightforward consequences of the following two lemmas providing the compactness of $\mathbf{S}$ and the continuity of $\widetilde{\mathbf{S}}$, respectively. In this regard, we remark in advance that the further regularity assumption specified by (3.39) plays a key role in the proof of the first result, which is established as follows.

Lemma 3.7. *There exists a positive constant $L_{\mathbf{S}}$, depending on μ_1, L_μ, α, β and ε, such that*

$$\begin{aligned} &\|\mathbf{S}(\phi)-\mathbf{S}(\psi)\| \\ &\quad\leq L_{\mathbf{S}}\left\{\|f\|_{0,\Omega}\,\|\phi-\psi\|_{0,4;\Omega} + \|\mathbf{S}_1(\psi)\|_{\varepsilon,\Omega}\,\|\phi-\psi\|_{0,n/\varepsilon;\Omega}\right\} \qquad \forall\,\phi,\,\psi\in \mathrm{H}^1_0(\Omega)\,. \end{aligned} \tag{3.40}$$

Proof. We base our arguments on the proof of [12, Lemma 3.9]. Indeed, letting $H := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$, we first notice that the a priori estimate (3.31) of problem (3.20), with a given $\varphi\in\mathrm{H}^1_0(\Omega)$, is equivalent to stating that

$$\|(\boldsymbol{\rho},\mathbf{z})\| \leq C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\varphi(\boldsymbol{\rho},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\tau},\mathbf{z})+\mathbf{b}(\boldsymbol{\rho},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|} \qquad \forall\,(\boldsymbol{\rho},\mathbf{z})\in H\,, \tag{3.41}$$

with a positive constant C depending only on μ_1, α and β, and hence independent of φ. Next, given $\phi,\,\psi\in\mathrm{H}^1_0(\Omega)$, we let $(\boldsymbol{\sigma},\mathbf{u})=\mathbf{S}(\phi)$ and $(\boldsymbol{\zeta},\mathbf{w})=\mathbf{S}(\psi)$, which means, according to the definition of $\mathbf{S}$ provided in Section 3.2, that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle_\Gamma \qquad &&\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\ \mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= -\int_\Omega f\phi\cdot\mathbf{v} &&\forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,, \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} \mathbf{a}_\psi(\boldsymbol{\zeta},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\tau},\mathbf{w}) &= \langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle_\Gamma \qquad &&\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega),\\ \mathbf{b}(\boldsymbol{\zeta},\mathbf{v}) &= -\int_\Omega f\psi\cdot\mathbf{v} &&\forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,. \end{aligned} \tag{3.43}$$

Then, applying (3.41) with $(\boldsymbol{\rho},\mathbf{z})=\mathbf{S}(\phi)-\mathbf{S}(\psi)=(\boldsymbol{\sigma}-\boldsymbol{\zeta},\mathbf{u}-\mathbf{w})$ and $\varphi=\phi$, and then using (3.42), (3.43), and the definitions of $\mathbf{a}_\phi$ and $\mathbf{a}_\psi$ (cf. (3.15)), we first arrive at

$$\begin{aligned} \|\mathbf{S}(\phi)-\mathbf{S}(\psi)\| &\leq C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\phi(\boldsymbol{\sigma}-\boldsymbol{\zeta},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\tau},\mathbf{u}-\mathbf{w})+\mathbf{b}(\boldsymbol{\sigma}-\boldsymbol{\zeta},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &= C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\mathbf{a}_\psi(\boldsymbol{\zeta},\boldsymbol{\tau})-\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau})+\mathbf{b}(\boldsymbol{\sigma}-\boldsymbol{\zeta},\mathbf{v})}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &= C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{\displaystyle\int_\Omega\left(\frac{\mu(\phi)-\mu(\psi)}{\mu(\psi)\mu(\phi)}\right)\boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} - \int_\Omega f(\phi-\psi)\cdot\mathbf{v}}{\|(\boldsymbol{\tau},\mathbf{v})\|}\,. \end{aligned} \tag{3.44}$$

Next, employing the boundedness and Lipschitz-continuity of μ (cf. (2.2), (2.4)), and the Cauchy–Schwarz inequality, it follows from (3.44) that

$$\begin{aligned} \|\mathbf{S}(\phi)-\mathbf{S}(\psi)\| &\leq C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{L_\mu\,\mu_1^{-2}\displaystyle\int_\Omega|(\phi-\psi)\boldsymbol{\zeta}^{\mathrm{d}}|\,|\boldsymbol{\tau}^{\mathrm{d}}| + \|f\|_{0,\Omega}\,\|\phi-\psi\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}}{\|(\boldsymbol{\tau},\mathbf{v})\|}\\ &\leq C \sup_{\substack{(\boldsymbol{\tau},\mathbf{v})\in H\\(\boldsymbol{\tau},\mathbf{v})\neq\mathbf{0}}} \frac{L_\mu\,\mu_1^{-2}\,\|(\phi-\psi)\boldsymbol{\zeta}\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{0,\Omega} + \|f\|_{0,\Omega}\,\|\phi-\psi\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}}{\|(\boldsymbol{\tau},\mathbf{v})\|}\,, \end{aligned}$$

which, applying now the Hölder inequality to the expression $\|(\phi-\psi)\boldsymbol{\zeta}\|_{0,\Omega}$, yields

$$\|\mathbf{S}(\phi)-\mathbf{S}(\psi)\| \leq C\left\{L_\mu\,\mu_1^{-2}\,\|\boldsymbol{\zeta}\|_{0,2p;\Omega}\,\|\phi-\psi\|_{0,2q;\Omega} + \|f\|_{0,\Omega}\|\phi-\psi\|_{0,4;\Omega}\right\}, \tag{3.45}$$

where $p,q\in\,]1,+\infty)$ are such that $\frac{1}{p}+\frac{1}{q}=1$. Next, bearing in mind the further regularity in (3.39), we notice that the Sobolev embedding Theorem (cf. [30, Theorem 4.12] and [31, Theorem 1.3.4]) establishes the continuous injection $i_\varepsilon:\mathrm{H}^\varepsilon(\Omega)\to\mathrm{L}^{\varepsilon^*}(\Omega)$ with boundedness constant C_ε, where

$$\varepsilon^* := \begin{cases} \frac{2}{1-\varepsilon} & \text{if} \quad n=2,\\ \frac{6}{3-2\varepsilon} & \text{if} \quad n=3. \end{cases}$$

Thus, choosing p such that $2p = \varepsilon^*$, we deduce that $\boldsymbol{\zeta} := \mathbf{S}_1(\psi)$ does in fact belong to $\mathbb{L}^{2p}(\Omega)$, and hence, thanks to the aforementioned continuity, there holds

$$\|\boldsymbol{\zeta}\|_{0,2p;\Omega} \,\le\, C_\varepsilon \, \|\boldsymbol{\zeta}\|_{\varepsilon,\Omega} \,, \tag{3.46}$$

which, when needed, can be bounded by (3.39), yielding for each ψ with $\|\psi\|_{1,\Omega} \le r$

$$\|\boldsymbol{\zeta}\|_{0,2p;\Omega} \,\le\, C_\varepsilon \, \widetilde{C}_{\mathbf{S}}(r) \Big\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega} \|\psi\|_{1,\Omega} \Big\}.$$

In addition, according to the above choice of p, we readily find that

$$2q := \frac{2p}{p-1} = \begin{cases} \frac{2}{\varepsilon} & \text{if} \quad n = 2, \\ \frac{3}{\varepsilon} & \text{if} \quad n = 3, \end{cases} \;= \frac{n}{\varepsilon}\,. \tag{3.47}$$

In this way, (3.45) and (3.46), together with (3.47), imply (3.40) and complete the proof. □

On the other hand, the continuity of $\widetilde{\mathbf{S}}$ is proved next.

Lemma 3.8. *There exists a positive constant $L_{\widetilde{\mathbf{S}}}$, depending on ϑ_1, L_γ, c_p, and $c(\Omega)$, such that for all $(\phi, \mathbf{u})$, $(\varphi, \mathbf{w}) \in \mathrm{H}^1_0(\Omega) \times \mathbf{L}^4(\Omega)$, with $\|\mathbf{u}\|_{0,4;\Omega}$, $\|\mathbf{w}\|_{0,4;\Omega} \le \frac{\vartheta_1}{2\,c(\Omega)\,c_p}$, there holds*

$$\|\widetilde{\mathbf{S}}(\phi, \mathbf{u}) - \widetilde{\mathbf{S}}(\varphi, \mathbf{w})\| \,\le\, L_{\widetilde{\mathbf{S}}} \Big\{ \|\boldsymbol{k}\| \|\phi - \varphi\|_{0,\Omega} + \|\widetilde{\mathbf{S}}(\varphi, \mathbf{w})\| \, \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} \Big\}.$$

Proof. This proof, being essentially the same as the one of [12, Lemma 3.10], is based on the strong monotonicity of $A_{\mathbf{u}}$ with respective constant $\widetilde{\alpha} = \frac{\vartheta_1}{2c_p^2}$ (cf. (3.35)), the Lipschitz continuity of γ (cf. (2.5)), the Cauchy–Schwarz inequality, and estimate (3.9) (which is the analogue of [12, eq. (3.5)]). Further details are omitted and we refer the interested reader to [12, Lemma 3.10]. □

We are now in position to establish the announced properties of $\mathbf{T}$ as straightforward consequences of Lemmas 3.7 and 3.8. More precisely, we have the following result.

Lemma 3.9. *Given $r > 0$, we let $W := \Big\{ \phi \in \mathrm{H}^1_0(\Omega) : \quad \|\phi\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy (3.38). In turn, let $L_{\mathbf{S}}$ and $L_{\widetilde{\mathbf{S}}}$ be the constants provided by Lemmas 3.7 and 3.8. Then, there holds*

$$\begin{aligned} \|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{1,\Omega} \,&\le\, L_{\widetilde{\mathbf{S}}} \, \|\boldsymbol{k}\| \, \|\phi - \varphi\|_{0,\Omega} \,+\, L_{\mathbf{S}} \, L_{\widetilde{\mathbf{S}}} \, \|\mathbf{T}(\varphi)\|_{1,\Omega} \, \|\boldsymbol{f}\|_{0,\Omega} \, \|\phi - \varphi\|_{0,4;\Omega} \\ &+\, L_{\mathbf{S}} \, L_{\widetilde{\mathbf{S}}} \, \|\mathbf{T}(\varphi)\|_{1,\Omega} \, \|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega} \, \|\phi - \varphi\|_{0,n/\varepsilon;\Omega} \qquad \forall\, \phi,\, \varphi \in W\,, \end{aligned} \tag{3.48}$$

and hence $\mathbf{T} : W \to W$ is continuous and $\overline{\mathbf{T}(W)}$ is compact.

Proof. We proceed basically as in the proofs of [12, Lemmmas 3.11 and 3.12]. In fact, we begin by recalling from Lemma 3.6 that the hypothesis (3.38) on the data guarantees that $\mathbf{T}(W) \subseteq W$. Next, in order to deduce (3.48) it suffices to recall from (3.23) that $\mathbf{T}(\phi) = \widetilde{\mathbf{S}}(\phi, \mathbf{S}_2(\phi))$ for all $\phi \in \mathrm{H}^1_0(\Omega)$, and then apply Lemmas 3.7 and 3.8, in addition to perform some algebraic manipulations. In turn, thanks to the Rellich–Kondrachov compactness Theorem (cf. [30, Theorem 6.3], [31, Theorem 1.3.5]) and the ranges for ε specified by the regularity assumption (3.39), we know that $\mathrm{H}^1(\Omega)$ is compactly embedded in $\mathrm{L}^4(\Omega)$, $\mathrm{L}^2(\Omega)$, and $\mathrm{L}^{n/\varepsilon}(\Omega)$. In this way, these compact (and hence continuous) injections together with (3.48), allow to prove, exactly as done for [12, Lemma 3.12], the remaining properties of $\mathbf{T}$. □

According to the above results, the main result concerning the solvability of (3.21) reads as follows.

Theorem 3.10. *Given $r > 0$, we let $W := \Big\{ \phi \in \mathrm{H}^1_0(\Omega) : \quad \|\phi\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy (3.38). Then the mixed-primal formulation (3.21) has at least one solution $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}^1_0(\Omega)$ with $\phi \in W$, and there holds*

$$\|\phi\|_{1,\Omega} \,\le\, C_{\widetilde{\mathbf{S}}} \Big\{ \gamma_2 |\Omega|^{1/2} \|\boldsymbol{k}\| + \|g\|_{0,\Omega} \Big\} \tag{3.49}$$

and

$$\|(\boldsymbol{\sigma},\mathbf{u})\| \;\le\; C_{\mathbf{S}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\|\phi\|_{1,\Omega}\right\}. \tag{3.50}$$

In turn, if the data $\boldsymbol{k}$, $\boldsymbol{f}$, *and* $\mathbf{u}_D$ *are sufficiently small so that*

$$L_{\mathbf{T}} \;:=\; L_{\widetilde{\mathbf{S}}}\,\|\boldsymbol{k}\| + L_{\mathbf{S}}\,L_{\widetilde{\mathbf{S}}}\,r\left\{\Big(c(\Omega) + r\,\widetilde{C}_\varepsilon\,\widetilde{C}_{\mathbf{S}}(r)\Big)\,\|\boldsymbol{f}\|_{0,\Omega} \;+\; \widetilde{C}_\varepsilon\,\widetilde{C}_{\mathbf{S}}(r)\,\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma}\right\} \;<\; 1, \tag{3.51}$$

where $\widetilde{C}_\varepsilon$ *is the boundedness constant of the continuous injection of* $\mathrm{H}^1(\Omega)$ *into* $\mathrm{L}^{n/\varepsilon}(\Omega)$, *and* $c(\Omega)$, $\widetilde{C}_{\mathbf{S}}(r)$, $L_{\mathbf{S}}$, *and* $L_{\widetilde{\mathbf{S}}}$ *are given by* (3.2), (3.39), *and Lemmas* 3.7 *and* 3.8, *respectively, then* (3.21) *has a unique solution* $(\boldsymbol{\sigma},\mathbf{u},\phi) \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)\times\mathrm{H}^1_0(\Omega)$ *with* $\phi \in W$.

Proof. The proof is just a recontextualization of the one of [12, Theorem 3.13]. Indeed, according to the equivalence between (3.21) and (3.24), and thanks to Lemmas 3.6 and 3.9, the existence of solution is a straightforward application of the Schauder fixed point Theorem (cf. Theorem 3.5). Then, the estimates (3.49) and (3.50) follow from (3.37) and (3.30), respectively. In turn, employing the estimates $\|\mathbf{T}(\varphi)\|_{1,\Omega} = \|\varphi\|_{1,\Omega} \le r$, $\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega} \le \widetilde{C}_{\mathbf{S}}(r)\left\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|\boldsymbol{f}\|_{0,\Omega}\|\varphi\|_{1,\Omega}\right\}$ (cf. (3.39)), $\|\psi\|_{0,n/\varepsilon;\Omega} \le \widetilde{C}_\varepsilon\,\|\psi\|_{1,\Omega}$, and $\|\psi\|_{0,4;\Omega} \le c(\Omega)\,\|\psi\|_{1,\Omega}$ (cf. (3.2)) for $\psi \in \mathrm{H}^1(\Omega)$, it follows straightforwardly from (3.48) that $\|\mathbf{T}(\phi)-\mathbf{T}(\varphi)\|_{1,\Omega} \le L_{\mathbf{T}}\,\|\phi-\varphi\|_{1,\Omega} \quad \forall\,\phi,\,\varphi \in W$. Therefore, the Banach fixed-point theorem and the assumption (3.51) complete the proof. □

We end this section by stressing that the further regularity assumption specified by (3.39) is not needed if the viscosity μ is constant. Indeed, in that case the term $\mathbf{a}_\psi(\boldsymbol{\zeta},\boldsymbol{\tau})-\mathbf{a}_\phi(\boldsymbol{\zeta},\boldsymbol{\tau})$ vanishes in (3.44), and hence (3.40) reduces to

$$\|\mathbf{S}(\phi) - \mathbf{S}(\psi)\| \;\le\; L_{\mathbf{S}}\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi-\psi\|_{0,4;\Omega} \qquad \forall\,\phi,\;\psi \in \mathrm{H}^1_0(\Omega). \tag{3.52}$$

As a consequence, Lemma 3.9 and Theorem 3.10 are also improved accordingly. More precisely, thanks to the use of (3.52) instead of (3.40), the resulting estimates in the new statements of these results will not include neither one of the expressions or constants arising from that extra regularity or from a related continuous Sobolev injection involving ε. In particular, the term containing $\|\mathbf{S}_1(\varphi)\|_{\varepsilon,\Omega}$ and $\|\phi-\varphi\|_{0,n/\varepsilon;\Omega}$ disappears from (3.48), whereas in (3.51) the constant $\widetilde{C}_\varepsilon$ and $\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma}$ are not present anymore and should be replaced by 0. We omit further details.

4. The discrete formulation

In this section we introduce the Galerkin scheme associated with (3.21) and study its solvability and convergence.

4.1. A mixed-primal finite element method

We first let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω by triangles K (resp. tetrahedra K in R^3) and set $h := \max\{h_K :\; K \in \mathcal{T}_h\}$. In turn, given an integer $l \ge 0$ and a subset S of R^n, we denote by $\mathrm{P}_l(S)$ the space of polynomials of total degree at most l defined on S. Hence, for each integer $k \ge 0$ and for each $K \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as $\mathrm{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\,\mathbf{x}$, where $\mathbf{x} := (x_1,\ldots,x_n)^{\mathrm{t}}$ is a generic vector of R^n and $\widetilde{\mathrm{P}}_k(K)$ is the space of polynomials of total degree equal to k defined on T. In this way, introducing the finite element subspaces:

$$\mathbb{H}^{\boldsymbol{\sigma}}_h := \left\{\boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \boldsymbol{c}^{\mathrm{t}}\,\boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\,\boldsymbol{c} \in \mathrm{R}^n\,, \quad \forall\,K \in \mathcal{T}_h\right\}, \tag{4.1}$$

$$\mathbf{H}^{\mathbf{u}}_h := \left\{\mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\,K \in \mathcal{T}_h\right\}, \tag{4.2}$$

$$\mathrm{H}^{\phi}_h := \left\{\psi_h \in C(\Omega)\cap\mathrm{H}^1_0(\Omega) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\,K \in \mathcal{T}_h\right\}, \tag{4.3}$$

the aforementioned Galerkin scheme reads: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h,\phi_h) \in \mathbb{H}^{\boldsymbol{\sigma}}_h \times \mathbf{H}^{\mathbf{u}}_h \times \mathrm{H}^{\phi}_h$ such that

$$\begin{aligned}
\mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h,\mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\boldsymbol{\tau}_h \in \mathbb{H}^{\boldsymbol{\sigma}}_h,\\
\mathbf{b}(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) && \forall\mathbf{v}_h \in \mathbf{H}^{\mathbf{u}}_h,\\
A_{\mathbf{u}_h}(\phi_h,\psi_h) &= G_{\phi_h}(\psi_h) && \forall\psi_h \in \mathrm{H}^{\phi}_h\,.
\end{aligned} \tag{4.4}$$

where the corresponding definitions of the bilinear forms $\mathbf{a}_{\phi_h}$, $\mathbf{b}$, and $A_{\mathbf{u}_h}$, and the linear functionals $\mathbf{F}$, $\mathbf{G}_{\phi_h}$, and G_{ϕ_h}, are given in (3.4), (3.5), (3.15), and (3.16), with $\phi = \phi_h$ and $\mathbf{u} = \mathbf{u}_h$.

Before proceeding with the solvability analysis of (4.4), we find it important to remark here that this discrete scheme differs from the non-augmented one arising from [12] when the stabilization constants there are set to zero. In fact, while the resulting equations and the finite element subspaces used to approximate $\boldsymbol{\sigma}$ and ϕ are the same, the difference lies in $\mathbf{H}_h^{\mathbf{u}}$, the one employed to approximate $\mathbf{u}$, which consists here of the piecewise polynomial vectors of degree $\leq k$ (cf. (4.2)), whereas in [12] it is given by the continuous piecewise polynomial vectors of degree $\leq k+1$. We recall that the latter is imposed by the fact that in [12] the unknown $\mathbf{u}$ belongs to $\mathbf{H}^1(\Omega)$. Then, this apparently minor difference in the continuity and the polynomial degrees involved makes a big difference in the respective analyses since, on the contrary to the present subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$, which, as stated below in Lemma 4.1, do satisfy the discrete inf–sup condition for $\mathbf{b}$, the corresponding subspaces in [12] do not. As a consequence, the resulting linearized systems at each Picard step in the fluid equations of [12] would not be neither stable nor invertible. Precisely, in the present paper we introduce the right continuous and discrete spaces for analyzing that non-augmented formulation.

4.2. A discrete fixed-point strategy

In what follows we reformulate (4.4) by adopting the discrete analogue of the fixed point strategy introduced in Section 3.2. Hence, we now let $\mathbf{S}_h : \mathrm{H}_h^{\phi} \longrightarrow \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\mathbf{S}_h(\phi_h) = (\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h)) := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \quad \forall\, \phi_h \in \mathrm{H}_h^{\phi},$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution of

$$\begin{aligned} \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) & \forall \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}, \end{aligned} \tag{4.5}$$

with the given $\phi_h \in \mathrm{H}_h^{\phi}$. In addition, we let $\tilde{\mathbf{S}}_h : \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}} \longrightarrow \mathrm{H}_h^{\phi}$ be the operator defined by

$$\tilde{\mathbf{S}}_h(\phi_h, \mathbf{u}_h) := \tilde{\phi}_h \qquad \forall\, (\phi_h, \mathbf{u}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}},$$

where $\tilde{\phi}_h \in \mathrm{H}_h^{\phi}$ is the unique solution of

$$A_{\mathbf{u}_h}(\tilde{\phi}_h, \tilde{\psi}_h) = G_{\phi_h}(\tilde{\psi}_h) \qquad \forall\, \tilde{\psi}_h \in \mathrm{H}_h^{\phi}, \tag{4.6}$$

with the given $(\phi_h, \mathbf{u}_h) \in \mathrm{H}_h^{\phi} \times \mathbf{H}_h^{\mathbf{u}}$. Finally, we define the operator $\mathbf{T}_h : \mathrm{H}_h^{\phi} \longrightarrow \mathrm{H}_h^{\phi}$ by

$$\mathbf{T}_h(\phi_h) := \tilde{\mathbf{S}}_h(\phi_h, \mathbf{S}_{2,h}(\phi_h)) \qquad \forall\, \phi_h \in \mathrm{H}_h^{\phi}, \tag{4.7}$$

and realize that (4.4) can be rewritten, equivalently, as: Find $\phi_h \in \mathrm{H}_h^{\phi}$ such that

$$\mathbf{T}_h(\phi_h) = \phi_h\,. \tag{4.8}$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.5) and (4.6) are well-posed, which is addressed in the first part of the following section.

4.3. Solvability analysis of the discrete fixed-point equation

We begin by showing that the discrete operator $\mathbf{S}_h$ is well-defined. To this end, we now let $\mathbb{V}_h$ be the discrete kernel of $\mathbf{b}$, that is

$$\mathbb{V}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \Big\},$$

which, using from (4.1)–(4.2) that $\mathbf{div}\big(\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \subseteq \mathbf{H}_h^{\mathbf{u}}$, becomes

$$\mathbb{V}_h = \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0} \quad \text{in} \quad \Omega \Big\}.$$

It follows that $\mathbb{V}_h \subseteq \mathbb{V}$ (cf. (3.26)), and hence, thanks to Lemma 3.1, we readily deduce that, with the same constant α from Lemma 3.1, there holds

$$\mathbf{a}_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha\, \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{V}_h\,, \quad \forall \phi_h \in \mathrm{H}_h^{\phi}\,, \tag{4.9}$$

which certainly implies that the bilinear form $\mathbf{a}_{\phi_h}$ satisfies the corresponding hypothesis required by the discrete Babuška–Brezzi theory in Banach spaces (cf. [35, Proposition 2.42]). Besides the already proved boundedness of the linear functionals involved (cf. (3.19)), the requirements of this abstract result are completed with the discrete inf–sup condition for the bilinear form $\mathbf{b}$, which we recall next from [22] (see also [21]).

Lemma 4.1. *There exists a positive constant β_{d}, independent of h, such that*

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h\neq 0}} \frac{\mathbf{b}(\boldsymbol{\tau}_h,\mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \,\geq\, \beta_{\mathrm{d}}\,\|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\,\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}. \tag{4.10}$$

Proof. It relies on several useful results concerning the Raviart–Thomas spaces and their interpolation and projection operators involved, within the framework of suitable Banach spaces, and also on an elliptic regularity estimate. For further details, we refer to [22, Section 5.3] and [22, Lemma 5.5, Section 5.4]. $\square$

It is important to remark here that the choice of the finite element subspaces for approximating $\boldsymbol{\sigma}$ and $\mathbf{u}$ is not limited to the Raviart–Thomas ones introduced in (4.1) and (4.2), but to any pair $\big(\mathbb{H}_h^{\boldsymbol{\sigma}},\mathbf{H}_h^{\mathbf{u}}\big)$ satisfying $\mathbf{div}\big(\mathbb{H}_h^{\boldsymbol{\sigma}}\big)\subseteq\mathbf{H}_h^{\mathbf{u}}$ and the discrete inf–sup condition for $\mathbf{b}$ (cf. (4.10)). Indeed, as shown by (4.9), the former yields the $\mathbb{V}_h$-ellipticity of $\mathbf{a}_{\phi_h}$, whereas, according to the analysis provided in [22], the latter requires the existence of corresponding interpolation and orthogonal projection operators verifying the commuting diagram property given by [22, eq. (5.35)], and the local approximation properties stated in [22, eq. (5.37)] and [22, Lemma 5.3]. In this regard, we stress for instance that, besides Raviart–Thomas, the BDM spaces also verify the above (cf. [39]). In this way, the aforementioned properties allow to show the additional interpolation estimate provided by [22, Lemma 5.4], which finally plays a key role in the proof of (4.10) (cf. [22, Lemma 5.5, Section 5.4]). Needless to say, under the same regularity assumptions on the exact solutions, Raviart–Thomas and BDM share the same approximation properties (cf. [39]), and hence they lead to the same rates of convergence for our discrete scheme.

We are now in position to establish next the discrete analogue of Lemma 3.3.

Lemma 4.2. *For each $\phi_h\in\mathrm{H}_h^{\phi}$ there exists a unique $\mathbf{S}_h(\phi_h):=(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}$ solution to the problem (4.5). Moreover, there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, depending only on μ_1, α, β_{d}, $\widetilde{c}(\Omega)$, and $c(\Omega)$, and hence independent of ϕ_h, such that*

$$\|\mathbf{S}_h(\phi_h)\| \,=\, \|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\leq\, C_{\mathbf{S},\mathrm{d}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi_h\|_{1,\Omega}\right\} \qquad \forall\,\phi_h\in\mathrm{H}_h^{\phi}. \tag{4.11}$$

Proof. It follows from (4.9), Lemma 4.1, and the discrete Babuška–Brezzi theory in Banach spaces (cf. [35, Proposition 2.42]). In particular, the corresponding a priori estimate reduces to

$$\|\mathbf{S}_h(\phi_h)\| \,:=\, \|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\leq\, \bar{C}\left\{\|\mathbf{F}|_{\mathbb{H}_h^{\boldsymbol{\sigma}}}\| \,+\, \|\mathbf{G}_{\phi_h}|_{\mathbf{H}_h^{\mathbf{u}}}\|\right\}, \tag{4.12}$$

with a positive constant $\bar{C}$ depending only on μ_1, α, and β_{d}. In this way, (4.12), the fact that the discrete norms $\|\mathbf{F}|_{\mathbb{H}_h^{\boldsymbol{\sigma}}}\| \,:=\, \sup\limits_{\substack{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}}\dfrac{|\mathbf{F}(\boldsymbol{\tau}_h)|}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}}$ and $\|\mathbf{G}_{\phi_h}|_{\mathbf{H}_h^{\mathbf{u}}}\| \,:=\, \sup\limits_{\substack{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}\\ \mathbf{v}_h\neq\mathbf{0}}}\dfrac{|\mathbf{G}_{\phi_h}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{0,4;\Omega}}$ are obviously bounded by $\|\mathbf{F}\|$ and $\|\mathbf{G}_{\phi_h}\|$, respectively, and the bounds for the latter provided in (3.19), imply (4.11). $\square$

In turn, the discrete analogue of Lemma 3.4 and the corresponding estimates (3.36) and (3.37), reads as follows.

Lemma 4.3. *Let $\phi\in\mathrm{H}_h^{\phi}$ and $\mathbf{u}_h\in\mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{u}_h\|_{0,4;\Omega}<\frac{\vartheta_1}{2\,c(\Omega)\,c_p}$ (cf. (2.3), (3.1), (3.9)). Then, there exists a unique $\widetilde{\phi}_h:=\widetilde{\mathbf{S}}_h(\phi_h,\mathbf{u}_h)\in\mathrm{H}_h^{\phi}$ solution to (4.6), and there holds*

$$\|\widetilde{\mathbf{S}}_h(\phi_h,\mathbf{u}_h)\| = \|\widetilde{\phi}_h\|_{1,\Omega} \,\leq\, C_{\widetilde{\mathbf{S}}}\left\{\gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| \,+\, \|g\|_{0,\Omega}\right\}. \tag{4.13}$$

Proof. It suffices to observe that the strong monotonicity and Lipschitz-continuity of $A_{\mathbf{u}_h}$ on $\mathrm{H}_0^1(\Omega)\times\mathrm{H}_0^1(\Omega)$ is certainly transferred to $\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\phi}$ with the same constants $\widetilde{\alpha}$ and $\widetilde{L}$, respectively, provided in (3.35). Therefore, under the stipulated hypotheses, another straightforward application of [37, Theorem 3.3.23] yields the unique solvability of (4.6) and the a priori estimate (4.13). $\square$

We now aim to show the solvability of (4.4) by analyzing the equivalent fixed point equation (4.8). To this end, in what follows we verify the hypotheses of the Brouwer fixed point theorem, which is recalled next (cf. [38, Theorem 9.9-2]).

Theorem 4.4. *Let W be a compact and convex subset of a finite dimensional Banach space X, and let $T : W \to W$ be a continuous mapping. Then T has at least one fixed point.*

We begin with the discrete version of Lemma 3.6.

Lemma 4.5. *Given $r > 0$, we let $W_h := \Big\{\phi_h \in \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \le r\Big\}$, and assume that*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + r\,\|\boldsymbol{f}\|_{0,\Omega} < \frac{\vartheta_1}{2\,C_{\mathbf{S},\mathrm{d}}\,c(\Omega)\,c_p} \quad \text{and} \quad \gamma_2\,|\Omega|^{1/2}\,\|\boldsymbol{k}\| + \|g\|_{0,\Omega} \le \frac{r}{C_{\widetilde{\mathbf{S}}}}. \tag{4.14}$$

Then $\mathbf{T}_h(W_h) \subseteq W_h$.

Proof. Similarly to the proof of Lemma 3.6, it is a direct consequence of Lemmas 4.2 and 4.3. $\square$

In turn, the discrete analogue of Lemma 3.7 is provided next. We notice in advance that, instead of the regularity assumption employed in the proof of that result, which is not needed nor could be applied in the present discrete case, we simple utilize an $\mathrm{L}^4 - \mathrm{L}^4 - \mathrm{L}^2$ argument.

Lemma 4.6. *There exists a positive constant $L_{\mathbf{S},\mathrm{d}}$, depending on μ_1, L_μ, α, and β_{d}, such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\psi_h)\| \le L_{\mathbf{S},\mathrm{d}}\Big\{\|\boldsymbol{f}\|_{0,\Omega} + \|\mathbf{S}_{1,h}(\psi_h)\|_{0,4;\Omega}\Big\}\,\|\phi_h - \psi_h\|_{0,4;\Omega} \qquad \forall\,\phi_h, \psi_h \in \mathrm{H}_h^\phi. \tag{4.15}$$

Proof. We begin by observing that the discrete stability estimate for $\mathbf{S}_h$ provided by (4.12), with a given $\phi_h \in \mathrm{H}_h^\phi$, is equivalent to stating that (cf. [35, Proposition 2.36])

$$\|(\boldsymbol{\rho}_h, \mathbf{z}_h)\| \le \bar{C} \sup_{\substack{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\\(\boldsymbol{\tau}_h,\mathbf{v}_h)\neq\mathbf{0}}} \frac{\mathbf{a}_\varphi(\boldsymbol{\rho}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{z}_h) + \mathbf{b}(\boldsymbol{\rho}_h, \mathbf{v}_h)}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|} \tag{4.16}$$

for all $(\boldsymbol{\rho}_h, \mathbf{z}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$. Then, according to the foregoing inequality, the rest of the proof proceeds exactly as done for Lemma 3.7, except for the last part of the derivation of the discrete analogue of (3.45). In fact, given $\phi_h,\ \psi_h \in \mathrm{H}_h^\phi$, we now let $(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = \mathbf{S}_h(\phi_h)$ and $(\boldsymbol{\zeta}_h, \mathbf{w}_h) = \mathbf{S}_h(\psi_h)$, and instead of applying Hölder's inequality with the values of p and q determined by the regularity parameter ε, we simply employ Cauchy–Schwarz's inequality to obtain

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\psi_h)\| \le \bar{C}\Big\{L_\mu\,\mu_1^{-2}\,\|\boldsymbol{\zeta}_h\|_{0,4;\Omega}\,\|\phi_h - \psi_h\|_{0,4;\Omega} + \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi_h - \psi_h\|_{0,4;\Omega}\Big\}, \tag{4.17}$$

which readily yields (4.15) and finishes the proof. $\square$

The continuity of $\widetilde{\mathbf{S}}_h$, that is the discrete analogue of Lemma 3.8, is shown now.

Lemma 4.7. *There exists a positive constant $L_{\widetilde{\mathbf{S}},\mathrm{d}}$, depending on ϑ_1, L_γ, c_p, and $c(\Omega)$, such that for all $(\phi_h, \mathbf{u}_h)$, $(\varphi_h, \mathbf{w}_h) \in \mathrm{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$, with $\|\mathbf{u}_h\|_{0,4;\Omega}$, $\|\mathbf{w}_h\|_{0,4;\Omega} \le \frac{\vartheta_1}{2\,c(\Omega)\,c_p}$, there holds*

$$\|\widetilde{\mathbf{S}}_h(\phi_h, \mathbf{u}_h) - \widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h)\| \le L_{\widetilde{\mathbf{S}},\mathrm{d}}\Big\{\|\boldsymbol{k}\|\,\|\phi_h - \varphi_h\|_{0,\Omega} + \|\widetilde{\mathbf{S}}_h(\varphi_h, \mathbf{w}_h)\|\,\|\mathbf{u}_h - \mathbf{w}_h\|_{0,4;\Omega}\Big\}.$$

Proof. It proceeds as in the proof of Lemma 3.8 by using now the strong monotonicity of $A_{\mathbf{u}_h}$, the Lipschitz-continuity of γ, Cauchy–Schwarz's inequality, and again estimate (3.9). We omit further details. $\square$

The continuity of the discrete fixed-point operator $\mathbf{T}_h$ is proved next.

Lemma 4.8. *Given $r > 0$, we let $W_h := \Big\{ \phi_h \in \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy (4.14). In turn, let $L_{\mathbf{S},\mathrm{d}}$ and $L_{\widetilde{\mathbf{S}},\mathrm{d}}$ be the constants provided by Lemmas 4.6 and 4.7. Then, there holds*

$$\begin{aligned} &\|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{1,\Omega} \\ &\quad \le L_{\widetilde{\mathbf{S}},\mathrm{d}} \, \|\boldsymbol{k}\| \, \|\phi_h - \varphi_h\|_{0,\Omega} \, + \, L_{\mathbf{S},\mathrm{d}} \, L_{\widetilde{\mathbf{S}},\mathrm{d}} \, \|\mathbf{T}_h(\varphi_h)\|_{1,\Omega} \, \|\boldsymbol{f}\|_{0,\Omega} \, \|\phi_h - \varphi_h\|_{0,4;\Omega} \\ &\quad + L_{\mathbf{S},\mathrm{d}} \, L_{\widetilde{\mathbf{S}},\mathrm{d}} \, \|\mathbf{T}_h(\varphi_h)\|_{1,\Omega} \, \|\mathbf{S}_{1,h}(\varphi_h)\|_{0,4;\Omega} \, \|\phi_h - \varphi_h\|_{0,4;\Omega} \qquad \forall \, \phi_h, \, \varphi_h \in W_h \,, \end{aligned} \tag{4.18}$$

and hence $\mathbf{T}_h : W_h \to W_h$ is continuous.

Proof. In order to obtain (4.18), it suffices to recall from (4.7) that $\mathbf{T}_h(\phi_h) := \tilde{\mathbf{S}}_h(\phi_h, \mathbf{S}_{2,h}(\phi_h))$ for all $\phi_h \in \mathrm{H}_h^\phi$, and then apply the estimates provided by Lemmas 4.6 and 4.7. Thus, the compact (and therefore continuous) injections of $\mathrm{H}^1(\Omega)$ into both $\mathrm{L}^4(\Omega)$ and $\mathrm{L}^2(\Omega)$ confirm the continuity of $\mathbf{T}_h$. $\square$

We stress here that, while $\|\mathbf{T}_h(\varphi_h)\|_{1,\Omega}$ can be certainly bounded by r in (4.18), the lack of a bound independent of h for the expression $\|\mathbf{S}_{1,h}(\varphi_h)\|_{0,4;\Omega}$ that also appears there, stops us of deriving a controllable Lipschitz-continuity constant for $\mathbf{T}_h$. This is the reason why we are not able to apply the Banach fixed-point theorem to $\mathbf{T}_h$, but only the Brouwer one (cf. Theorem 4.4) as we state next.

Theorem 4.9. *Given $r > 0$, we let $W_h := \Big\{ \phi_h \in \mathrm{H}_h^\phi : \quad \|\phi_h\|_{1,\Omega} \le r \Big\}$, and assume that the data satisfy (4.14). Then the Galerkin scheme (4.4) has at least one solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^\phi$ with $\phi_h \in W_h$, and there holds*

$$\|\phi_h\|_{1,\Omega} \, \le \, C_{\widetilde{\mathbf{S}}} \Big\{ \gamma_2 \, |\Omega|^{1/2} \, \|\boldsymbol{k}\| \, + \, \|g\|_{0,\Omega} \Big\} \tag{4.19}$$

and

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \, \le \, C_{\mathbf{S},\mathrm{d}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \, + \, \|\boldsymbol{f}\|_{0,\Omega} \, \|\phi_h\|_{1,\Omega} \Big\} . \tag{4.20}$$

Proof. Thanks to the equivalence between (4.4) and (4.8), the existence of solution follows from Lemmas 4.5 and 4.8, and a direct application of Theorem 4.4. In addition, the a priori estimates (4.19) and (4.20) are consequences of (4.13) and (4.11), respectively. $\square$

We end this section by stressing that our Galerkin scheme does not provide exact conservation of momentum, but only in an approximate sense. In fact, using again that $\mathbf{div}\big(\mathbb{H}_h^{\boldsymbol{\sigma}}\big) \subseteq \mathbf{H}_h^{\mathbf{u}}$, we deduce from the second equation of (4.4) that $\mathbf{div}(\boldsymbol{\sigma}_h) + \mathcal{P}_h(\boldsymbol{f}\phi_h) = \mathbf{0}$ in Ω, where $\mathcal{P}_h$ is the $\mathbf{L}^2(\Omega)$-orthogonal projection onto $\mathbf{H}_h^{\mathbf{u}}$. In this way, even if $\boldsymbol{f}$ were piecewise constant, the fact that ϕ_h is a continuous piecewise polynomial of degree $\le k+1$ (cf. (4.3)), which is certainly not contained in the scalar version of $\mathbf{H}_h^{\mathbf{u}}$, prevents us of establishing the aforementioned property exactly. A way of circumventing this is employing a mixed formulation for the transport equation as well, as done for instance in [34] and [40] for related problems, which we plan to address in a forthcoming work. The verification of the present approximate conservation of momentum is illustrated below in Section 6.

5. A priori error analysis

Given $(\boldsymbol{\sigma}, \mathbf{u}, \phi) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega) \times \mathrm{H}_0^1(\Omega)$ with $\phi \in W$, and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^\phi$ with $\phi_h \in W_h$, solutions of (3.21) and (4.4), respectively, we now aim to derive a corresponding a priori error estimate. For this purpose, we now recall from those equations, that the above means that

$$\begin{aligned} \mathbf{a}_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) \;&=\; \mathbf{F}(\boldsymbol{\tau}) && \forall \, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \,, \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) \;&=\; \mathbf{G}_\phi(\mathbf{v}) && \forall \, \mathbf{v} \in \mathbf{L}^4(\Omega) \,, \\ \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) \;&=\; \mathbf{F}(\boldsymbol{\tau}_h) && \forall \, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) \;&=\; \mathbf{G}_{\phi_h}(\mathbf{v}_h) && \forall \, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \,, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} A_{\mathbf{u}}(\phi, \psi) \;&=\; G_\phi(\psi) && \forall \, \psi \in \mathrm{H}_0^1(\Omega) \,, \\ A_{\mathbf{u}_h}(\phi_h, \psi_h) \;&=\; G_{\phi_h}(\psi_h) && \forall \, \psi_h \in \mathrm{H}_h^\phi \,. \end{aligned} \tag{5.2}$$

We begin by denoting as usual

$$\operatorname{dist}(\phi, \mathrm{H}_h^\phi) := \inf_{\varphi_h \in \mathrm{H}_h^\phi} \|\phi - \varphi_h\|_{1,\Omega}$$

and

$$\operatorname{dist}\big((\boldsymbol{\sigma}, \mathbf{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\big) := \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}} \|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\tau}_h, \mathbf{v}_h)\|.$$

Next, we recall two Strang-type lemmas that will be utilized in our subsequent analysis. The first one deals with strongly monotone and Lipschitz-continuous nonlinear operators.

Lemma 5.1. *Let H be a Hilbert space, $F \in H'$ and $\mathbf{A} : H \longrightarrow H'$ a nonlinear operator. In addition, let $\{H_n\}_{n\in N}$ be a sequence of finite dimensional subspaces of H, and for each $n \in N$ consider a nonlinear operator $\mathbf{A}_n : H_n \longrightarrow H_n'$ and a functional $F_n \in H_n'$. Assume that the family $\{\mathbf{A}\} \cup \{\mathbf{A}_n\}_{n\in N}$ is uniformly Lipschitz continuous and strongly monotone with constants Λ_{LC} and Λ_{SM}, respectively. In turn, let $u \in H$ and $u_n \in H_n$ such that*

$$[\mathbf{A}(u), v] = [F, v] \quad \forall\, v \in H \quad \text{and} \quad [\mathbf{A}_n(u_n), v_n] = [F_n, v_n] \quad \forall\, v_n \in H_n,$$

where $[\cdot,\cdot]$ denotes the duality pairing between H' (resp. H_n') and H (resp. H_n). Then for each $n \in N$ there holds

$$\|u - u_n\| \le \Lambda_{\mathrm{ST}} \left\{ \sup_{\substack{w_n \in H_n \\ w_n \neq \mathbf{0}}} \frac{[F, w_n] - [F_n, w_n]}{\|w_n\|} + \inf_{\substack{v_n \in H_n \\ v_n \neq \mathbf{0}}} \left(\|u - v_n\| + \sup_{\substack{w_n \in H_n \\ w_n \neq \mathbf{0}}} \frac{[\mathbf{A}(v_n), w_n] - [\mathbf{A}_n(v_n), w_n]}{\|w_n\|} \right) \right\},$$

with $\Lambda_{\mathrm{ST}} := \Lambda_{\mathrm{SM}}^{-1} \max\Big\{1, \Lambda_{\mathrm{SM}} + \Lambda_{\mathrm{LC}}\Big\}$.

Proof. It is a particular case of [41, Theorem 6.4]. □

The second theorem concerns the Babuška–Brezzi theory in Banach spaces.

Lemma 5.2. *Let H and Q be reflexive Banach spaces, $F \in H'$, $G \in Q'$, and $a : H \times H \to \mathrm{R}$ and $b : H \times Q \to \mathrm{R}$ bounded bilinear forms. In addition, let $\{H_n\}_{n\in\mathrm{N}}$ and $\{Q_n\}_{n\in\mathrm{N}}$ be sequences of finite dimensional subspaces of H and Q, respectively, and for each $n \in \mathrm{N}$ consider bilinear forms $a_n : H_n \times H_n \to \mathrm{R}$ and $b_n : H_n \times Q_n \to \mathrm{R}$, and functionals $F_n \in H_n'$ and $G_n \in Q_n'$. Assume that the families $\{a\} \cup \{a_n\}_{n\in\mathrm{N}}$ and $\{b\} \cup \{b_n\}_{n\in\mathrm{N}}$ uniformly verify the hypotheses of the continuous and discrete Babuška–Brezzi theories in Banach spaces (cf. [35, Theorem 2.34, Proposition 2.42]) with constants α and β, independent of $n \in N$, and that they are uniformly bounded with constants Λ_a and Λ_b, respectively. In turn, let $(\sigma, u) \in H \times Q$ and $(\sigma_n, u_n) \in H_n \times Q_n$ such that*

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= F(\tau) \qquad \forall\, \tau \in H\,, \\ b(\sigma, v) &= G(v) \qquad \forall\, v \in Q\,, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} a_n(\sigma_n, \tau_n) + b_n(\tau_n, u_n) &= F_n(\tau_n) \qquad \forall\, \tau_n \in H_n\,, \\ b_n(\sigma_n, v_n) &= G_n(v_n) \qquad \forall\, v_n \in Q_n\,. \end{aligned} \tag{5.4}$$

Then, there exists a positive constant Λ_{ST}, depending only on α, β, Λ_a, and Λ_b, such that for each $n \in N$ there holds

$$\begin{aligned} \|(\sigma, u) - (\sigma_n, u_n)\|_{H\times Q} \le \Lambda_{\mathrm{ST}} \Bigg\{ & \sup_{\substack{\tau_n \in H_n \\ \tau_n \neq \mathbf{0}}} \frac{F(\tau_n) - F_n(\tau_n)}{\|\tau_n\|_H} + \sup_{\substack{v_n \in Q_n \\ v_n \neq \mathbf{0}}} \frac{G(v_n) - G_n(v_n)}{\|v_n\|_Q} \\ & + \inf_{\substack{\zeta_n \in H_n \\ \zeta_n \neq \mathbf{0}}} \left(\|\sigma - \zeta_n\|_H + \sup_{\substack{\tau_n \in H_n \\ \tau_n \neq \mathbf{0}}} \frac{a_n(\zeta_n, \tau_n) - a(\zeta_n, \tau_n)}{\|\tau_n\|_H} + \sup_{\substack{v_n \in Q_n \\ v_n \neq \mathbf{0}}} \frac{b_n(\zeta_n, v_n) - b(\zeta_n, v_n)}{\|v_n\|_Q} \right) \\ & + \inf_{\substack{w_n \in Q_n \\ w_n \neq \mathbf{0}}} \left(\|u - w_n\|_Q + \sup_{\substack{\tau_n \in H_n \\ \tau_n \neq \mathbf{0}}} \frac{b_n(\tau_n, w_n) - b(\tau_n, w_n)}{\|\tau_n\|_H} \right) \Bigg\}\,. \end{aligned}$$

Proof. We begin by applying the triangular inequality to obtain

$$\|(\sigma,u)-(\sigma_n,u_n)\|_{H\times Q} \;\leq\; \|(\sigma,u)-(\zeta_n,w_n)\|_{H\times Q} \;+\; \|(\sigma_n,u_n)-(\zeta_n,w_n)\|_{H\times Q} \tag{5.5}$$

for all $(\zeta_n,w_n)\in H_n\times Q_n$. On the other hand, similarly as we did for (4.16), we observe that the fact that a_n and b_n satisfy the hypotheses of the discrete Babuška–Brezzi theory (cf. [35, Proposition 2.42]), guarantees, according to [35, Proposition 2.36], the global discrete inf–sup condition

$$\|(\rho_n,z_n)\|_{H\times Q} \;\leq\; \Lambda \sup_{\substack{(\tau_n,v_n)\in H_n\times Q_n\\ (\tau_n,v_n)\neq \mathbf{0}}} \frac{a_n(\rho_n,\tau_n)+b_n(\tau_n,z_n)+b_n(\rho_n,v_n)}{\|(\tau_n,v_n)\|_{H\times Q}} \quad \forall\,(\rho_n,z_n)\in H_n\times Q_n\,, \tag{5.6}$$

with $\Lambda := \frac{1}{\beta}+\left(1+\frac{\Lambda_a}{\beta}\right)\left\{\frac{1}{\beta}+\frac{1}{\alpha}\left(1+\frac{\Lambda_a}{\beta}\right)\right\}$. In this way, applying the inequality (5.6) to the pair $(\rho_n,z_n) := (\sigma_n,u_n)-(\zeta_n,w_n)$, and using the equations from (5.4), we arrive at

$$\begin{aligned} &\|(\sigma_n,u_n)-(\zeta_n,w_n)\|_{H\times Q} \\ &\quad\leq\; \Lambda \sup_{\substack{(\tau_n,v_n)\in H_n\times Q_n\\ (\tau_n,v_n)\neq \mathbf{0}}} \frac{a_n(\sigma_n-\zeta_n,\tau_n)+b_n(\tau_n,u_n-w_n)+b_n(\sigma_n-\zeta_n,v_n)}{\|(\tau_n,v_n)\|_{H\times Q}} \\ &\quad=\; \Lambda \sup_{\substack{(\tau_n,v_n)\in H_n\times Q_n\\ (\tau_n,v_n)\neq \mathbf{0}}} \frac{F_n(\tau_n)-a_n(\zeta_n,\tau_n)-b_n(\tau_n,w_n)+G_n(v_n)-b_n(\zeta_n,v_n)}{\|(\tau_n,v_n)\|_{H\times Q}}\,. \end{aligned} \tag{5.7}$$

The rest of the proof proceeds by adding and subtracting the equations from (5.3) to the upper term on the right hand side of (5.7), by performing suitable algebraic manipulations, and then by plugging the resulting estimate back into (5.5). Finally, we take there infimum with respect to $\zeta_n\in H_n$ and $w_n\in Q_n$. Further details are omitted. □

Now, we have the following preliminary result concerning $\|\phi-\phi_h\|_{1,\Omega}$.

Lemma 5.3. *There exists a positive constant $\widetilde{\Lambda}_{\mathrm{ST}}$, depending only on $\widetilde{\alpha}$ and $\widetilde{L}$ (cf. (3.35)), such that*

$$\begin{aligned} \|\phi-\phi_h\|_{1,\Omega} \;\leq\; \widetilde{\Lambda}_{\mathrm{ST}} \Bigg\{ & L_\gamma\,\|\boldsymbol{k}\|\,\|\phi-\phi_h\|_{0,\Omega} \;+\; c(\Omega)\,\|\phi\|_{1,\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega} \\ & + \Big(1+c(\Omega)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big)\,\mathrm{dist}\,(\phi,\mathrm{H}_h^{\phi}) \Bigg\}\,. \end{aligned} \tag{5.8}$$

Proof. The proof follows very similarly to the one of [12, Lemma 5.2]. In fact, we first recall, from Lemma 3.4 and the remark right after it, that $A_{\mathbf{u}}$ and $A_{\mathbf{u}_h}$ are both strongly monotone and Lipschitz-continuous with the respective constants $\widetilde{\alpha}$ and $\widetilde{L}$ given in (3.35). Hence, applying Lemma 5.1 to the corresponding context given by (5.2), which means taking in that lemma $H=\mathrm{H}_0^1(\Omega)$, $\mathbf{A}=A_{\mathbf{u}}$, $F=G_\phi$, $H_n=\mathrm{H}_h^{\phi}$, $\mathbf{A}_n=A_{\mathbf{u}_h}$, and $F_n=G_{\phi_h}$, we deduce that

$$\begin{aligned} \|\phi-\phi_h\|_{1,\Omega} \;\leq\; \widetilde{\Lambda}_{\mathrm{ST}} \Bigg\{ & \sup_{\substack{\psi_h\in \mathrm{H}_h^{\phi}\\ \psi_h\neq\mathbf{0}}} \frac{G_\phi(\psi_h)-G_{\phi_h}(\psi_h)}{\|\psi_h\|_{1,\Omega}} \\ & + \inf_{\substack{\varphi_h\in \mathrm{H}_h^{\phi}\\ \varphi_h\neq\mathbf{0}}} \left(\|\phi-\varphi_h\|_{1,\Omega} + \sup_{\substack{\psi_h\in \mathrm{H}_h^{\phi}\\ \psi_h\neq\mathbf{0}}} \frac{A_{\mathbf{u}}(\varphi_h,\psi_h)-A_{\mathbf{u}_h}(\varphi_h,\psi_h)}{\|\psi_h\|_{1,\Omega}} \right) \Bigg\}, \end{aligned} \tag{5.9}$$

with $\widetilde{\Lambda}_{\mathrm{ST}} := \widetilde{\alpha}^{-1}\max\left\{1,\widetilde{\alpha}+\widetilde{L}\right\}$. Next, from the definitions of G_ϕ and G_{ϕ_h} (cf. (3.5)), and the Lipschitz-continuity of γ (cf. (2.5)), we obtain

$$|\,G_\phi(\psi_h)-G_{\phi_h}(\psi_h)| \;\leq\; L_\gamma\,\|\boldsymbol{k}\|\,\|\phi-\phi_h\|_{0,\Omega}\,|\psi_h|_{1,\Omega},$$

whereas the definitions of $A_{\mathbf{u}}$ and $A_{\mathbf{u}_h}$ (cf. (3.4)), and the estimate (3.9) give

$$|\,A_{\mathbf{u}}(\varphi_h,\psi_h)-A_{\mathbf{u}_h}(\varphi_h,\psi_h)\,| \;\leq\; c(\Omega)\,\|\varphi_h\|_{1,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,|\psi_h|_{1,\Omega}.$$

In this way, replacing the foregoing estimates back into (5.9), bounding $\|\varphi_h\|_{1,\Omega}$ in the latter one by $\|\phi-\varphi_h\|_{1,\Omega}+\|\phi\|_{1,\Omega}$, and then taking infimum with respect to $\varphi_h\in\mathrm{H}_h^{\phi}$ in the resulting inequality in (5.9), we easily arrive to (5.8) and end the proof. □

Furthermore, the following lemma provides a preliminary estimate for $\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|$.

Lemma 5.4. *There exists a positive constant $\widehat{\Lambda}_{\mathrm{ST}}>0$, depending only on μ_1, α, β, and β_{d}, such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\le\, \widehat{\Lambda}_{\mathrm{ST}}\,\Big\{ & \operatorname{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\big) \\ & + L_\mu\,C_\varepsilon\,\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega}\,\|\phi-\phi_h\|_{0,n/\varepsilon;\Omega} \,+\, c(\Omega)\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\Big\}\,.\end{aligned} \tag{5.10}$$

Proof. We begin by recalling, thanks to Lemmas 3.1, 3.2 and 4.1, and the estimate (4.9), that the bilinear forms $\mathbf{a}_\phi$, $\mathbf{a}_{\phi_h}$, and $\mathbf{b}$ do satisfy the corresponding hypotheses of Lemma 5.2. Thus, applying this result to the context given by (5.1), which means taking in that lemma $H=\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $Q=\mathbf{L}^4(\Omega)$, $a=\mathbf{a}_\phi$, $b=\mathbf{b}$, $F=\mathbf{F}$, $G=\mathbf{G}_\phi$, $H_n=\mathbb{H}_h^{\boldsymbol{\sigma}}$, $Q_n=\mathbf{H}_h^{\mathbf{u}}$, $a_n=\mathbf{a}_{\phi_h}$, $b_n=\mathbf{b}$, $F_n=\mathbf{F}$, and $G_n=\mathbf{G}_{\phi_h}$, we can write

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,\le\, \widehat{C}_{\mathrm{ST}}\,\Bigg\{ & \sup_{\substack{\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}\\ \mathbf{v}_h\neq\mathbf{0}}}\frac{\mathbf{G}_\phi(\mathbf{v}_h)-\mathbf{G}_{\phi_h}(\mathbf{v}_h)}{\|\mathbf{v}_h\|_{0,4;\Omega}} + \operatorname{dist}(\mathbf{u},\mathbf{H}_h^{\mathbf{u}}) \\ & + \inf_{\boldsymbol{\tau}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}}\Bigg(\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} + \sup_{\substack{\boldsymbol{\zeta}_h\in\mathbb{H}_h^{\boldsymbol{\sigma}}\\ \boldsymbol{\zeta}_h\neq\mathbf{0}}}\frac{\mathbf{a}_\phi(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h)-\mathbf{a}_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h)}{\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}}\Bigg)\Bigg\},\end{aligned} \tag{5.11}$$

where $\widehat{C}_{\mathrm{ST}}$ is a positive constant depending only on μ_1, α, β, and β_{d}. Next, using the definitions of $\mathbf{G}_\phi$ and $\mathbf{G}_{\phi_h}$ (cf. (3.5)) and the second estimate in (3.18), we obtain

$$|\mathbf{G}_\phi(\mathbf{v}_h)-\mathbf{G}_{\phi_h}(\mathbf{v}_h)| \,=\, \left|\int_\Omega \boldsymbol{f}\,(\phi-\phi_h)\cdot\mathbf{v}_h\right| \,\le\, c(\Omega)\,\|\boldsymbol{f}\|_{0,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\,\|\mathbf{v}_h\|_{0,4;\Omega}\,. \tag{5.12}$$

In turn, adding and subtracting $\boldsymbol{\sigma}$ in the first component of each expression, we find that

$$\mathbf{a}_\phi(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h) \,-\, \mathbf{a}_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h) \,=\, \mathbf{a}_\phi(\boldsymbol{\tau}_h-\boldsymbol{\sigma},\boldsymbol{\zeta}_h) \,+\, \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}-\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h) \,+\, (\mathbf{a}_\phi-\mathbf{a}_{\phi_h})(\boldsymbol{\sigma},\boldsymbol{\zeta}_h)\,, \tag{5.13}$$

from which, applying the boundedness of $\mathbf{a}_\phi$ to the first and second terms on the right hand side of (5.13), proceeding with the third one as we did for deriving (3.45), in particular employing the Lipschitz-continuity of μ (cf. (2.4)), and making use of (3.46), we arrive at

$$\begin{aligned}&|\mathbf{a}_\phi(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h)-\mathbf{a}_{\phi_h}(\boldsymbol{\tau}_h,\boldsymbol{\zeta}_h)| \\ &\quad\le\, \left\{\frac{2}{\mu_1}\,\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} + \frac{L_\mu}{\mu_1^2}\,C_\varepsilon\,\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega}\,\|\phi-\phi_h\|_{0,n/\varepsilon,\Omega}\right\}\,\|\boldsymbol{\zeta}_h\|_{\mathbf{div}_{4/3};\Omega}\,.\end{aligned} \tag{5.14}$$

Finally, replacing (5.12) and (5.14) back into (5.11), we get (5.10), which ends the proof. □

We now combine the inequalities provided by Lemmas 5.3 and 5.4 to derive the Céa estimate for the total error $\|\phi-\phi_h\|_{1,\Omega} + \|(\boldsymbol{\sigma},\mathbf{u})-(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|$. More precisely, we replace the bound for $\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}$ given by (5.10) into the second term on the right hand side of (5.8). In this way, employing from (3.39) that

$$\|\boldsymbol{\sigma}\|_{\varepsilon,\Omega} \,\le\, \widetilde{C}_{\mathbf{S}}(r)\Big\{\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,+\, \|\boldsymbol{f}\|_{0,\Omega}\,\|\phi\|_{1,\Omega}\Big\}\,, \tag{5.15}$$

recalling that $\|\phi\|_{1,\Omega}$ is bounded by r, denoting by $\widetilde{C}_\varepsilon$ the boundedness constant of the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{n/\varepsilon}(\Omega)$, and performing several algebraic manipulations, we are led to

$$\begin{aligned}\|\phi-\phi_h\|_{1,\Omega} \,\le\, & \Big\{C_1\,\|\boldsymbol{k}\| \,+\, (C_2+r\,C_3)\,\|\boldsymbol{f}\|_{0,\Omega} \,+\, C_3\,\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma}\Big\}\,\|\phi-\phi_h\|_{1,\Omega} \\ & + C_4\operatorname{dist}\big((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\big) \,+\, \widetilde{\Lambda}_{\mathrm{ST}}\,\big(1+c(\Omega)\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\big)\operatorname{dist}(\phi,\mathrm{H}_h^{\phi})\,,\end{aligned} \tag{5.16}$$

where C_i, $i\in\{1,2,3,4\}$, are the positive constants defined by

$$\begin{aligned}C_1 \,:=\, \widetilde{\Lambda}_{\mathrm{ST}}\,L_\gamma\,,\qquad C_2 \,:=\, \widetilde{\Lambda}_{\mathrm{ST}}\,\big(c(\Omega)\big)^2\,r\,\widehat{\Lambda}_{\mathrm{ST}}\,,\\ C_3 \,:=\, \widetilde{\Lambda}_{\mathrm{ST}}\,c(\Omega)\,r\,\widehat{\Lambda}_{\mathrm{ST}}\,L_\mu\,C_\varepsilon\,\widetilde{C}_\varepsilon\,\widetilde{C}_{\mathbf{S}}(r)\,,\quad C_4 \,:=\, \widetilde{\Lambda}_{\mathrm{ST}}\,c(\Omega)\,r\,\widehat{\Lambda}_{\mathrm{ST}}\,.\end{aligned}$$

We highlight here that $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{u}_h\|_{1,\Omega}$ are estimated according to (3.30), and hence the expression in (5.16) multiplying $\operatorname{dist}(\phi, \mathrm{H}_h^\phi)$ can be easily controlled by constants, parameters, and data only. As a consequence of the foregoing discussion, we are able to prove the requested Céa estimate as follows.

Theorem 5.5. *Assume that the data $\boldsymbol{k}$, $\boldsymbol{f}$ and $\mathbf{u}_D$ are sufficiently small so that*

$$C_1\,\|\boldsymbol{k}\| \,+\, (C_2 + r\,C_3)\,\|\boldsymbol{f}\|_{0,\Omega} \,+\, C_3\,\|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \,\leq\, \frac{1}{2}\,. \tag{5.17}$$

Then, there exist a positive constant C, depending only on parameters, data, and other constants, all them independent of h, such that

$$\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,+\, \|\phi - \phi_h\|_{1,\Omega} \,\leq\, C\,\Big\{\operatorname{dist}(\phi, \mathrm{H}_h^\phi) \,+\, \operatorname{dist}\big((\boldsymbol{\sigma},\mathbf{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\big)\Big\}\,. \tag{5.18}$$

Proof. The estimate for $\|\phi - \phi_h\|_{1,\Omega}$ follows straightforwardly from (5.16) and (5.17), and then, the replacement of it back into (5.10), using again that $\|\phi - \phi_h\|_{0,n/\varepsilon;\Omega} \,\leq\, \widetilde{C}_\varepsilon\,\|\phi - \phi_h\|_{1,\Omega}$, completes the proof. □

We recall here that $\boldsymbol{k}$ is just a vector pointing in the direction of gravity, which is not assumed in advance to be of any particular length, and hence the feasibility of (5.17) is guaranteed by choosing it sufficiently small. Alternatively, since this vector appears always multiplied by $\gamma(\phi)$, one could transfer eventual restrictions to this latter function if needed.

We now establish the approximation properties of the subspaces defined by (4.1), (4.2), and (4.3), which follow from interpolation estimates of Sobolev spaces and the approximation properties provided by the orthogonal projectors and the interpolation operators involved in their definitions (see, e.g. [21,22,33,35,36,39]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, C\,h^l\,\Big\{\|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega}\Big\}\,. \tag{5.19}$$

($\mathbf{AP}_h^{\mathbf{u}}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$ there holds

$$\operatorname{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \,:=\, \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \,\leq\, C\,h^l\,\|\mathbf{v}\|_{l,4;\Omega}\,. \tag{5.20}$$

($\mathbf{AP}_h^{\phi}$) there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{H}^{l+1}(\Omega)$ there holds

$$\operatorname{dist}(\psi, \mathrm{H}_h^{\phi}) \,:=\, \inf_{\psi_h \in \mathrm{H}_h^{\phi}} \|\psi - \psi_h\|_{1,\Omega} \,\leq\, C\,h^l\,\|\psi\|_{l+1,\Omega}\,. \tag{5.21}$$

Finally, we conclude this section with the rates of convergence of our Galerkin scheme (4.4).

Theorem 5.6. *In addition to the hypotheses of Theorems 3.10, 4.9, and 5.5, assume that there exists $l \in [0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, and $\phi \in \mathrm{H}^{l+1}(\Omega)$. Then, there exists a positive constant $C_{\mathtt{rc}}$, independent of h, such that*

$$\begin{aligned} &\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\| \,+\, \|\phi - \phi_h\|_{1,\Omega} \\ &\quad\leq\, C_{\mathtt{rc}}\,h^l\,\Big\{\|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} + \|\mathbf{u}\|_{l,4;\Omega} + \|\phi\|_{l+1,\Omega}\Big\}\,. \end{aligned} \tag{5.22}$$

Proof. The result is a straightforward application of Theorem 5.5, ($\mathbf{AP}_h^{\boldsymbol{\sigma}}$), ($\mathbf{AP}_h^{\mathbf{u}}$), and ($\mathbf{AP}_h^{\phi}$). □

6. Numerical results

In this section we present some examples illustrating the performance of our mixed-primal finite element method (4.4), on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a FreeFem++ code (see [42]), in conjunction with the direct linear solver UMFPACK (see [43]). A Picard algorithm with a fixed given tolerance *tol* has been used for the corresponding fixed-point problems (4.8), and the iterations

Table 6.1
Convergence history for Example 1, with a quasi-uniform refinement and a tolerance of 10^{-6}.

$\mathbb{RT}_0$–$\mathbf{P}_0$–P_1										
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\phi)$	$r(\phi)$	$e(p)$	$r(p)$	iter
223	0.3727	55.2038	–	0.4301	–	0.9076	–	2.4540	–	10
885	0.1964	25.2330	1.2224	0.2083	1.1317	0.3814	1.3538	1.0408	1.3393	10
3419	0.0970	12.4148	1.0055	0.1071	0.9431	0.1839	1.0342	0.5211	0.9808	10
13 481	0.0478	6.0144	1.0241	0.0524	1.0109	0.0870	1.0569	0.2526	1.0229	10
53 592	0.0245	3.0233	1.0294	0.0265	1.0170	0.0445	1.0046	0.1318	0.9742	9
216 072	0.0128	1.5024	1.0719	0.0132	1.0696	0.0222	1.0675	0.0642	1.1030	9
$\mathbb{RT}_1$–$\mathbf{P}_1$–P_2										
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\phi)$	$r(\phi)$	$e(p)$	$r(p)$	iter
697	0.3727	18.6311	–	0.1363	–	0.0828	–	0.6266	–	10
2833	0.1964	4.0332	2.3893	0.0274	2.5037	0.0205	2.1790	0.1363	2.3820	10
11 065	0.0970	1.0277	1.9384	0.0069	1.9589	0.0053	1.9073	0.0351	1.9213	10
43 873	0.0478	0.2460	2.0201	0.0017	1.9623	0.0014	1.8540	0.0084	2.0304	10
174 901	0.0245	0.0633	2.0308	0.0004	2.0552	0.0004	2.1022	0.0022	1.9950	9
706 165	0.0128	0.0154	2.1701	0.0001	2.1910	0.0001	2.1887	0.0005	2.1698	9

Table 6.2
Example 1, ℓ^∞-norm of $\mathbf{div}(\boldsymbol{\sigma}_h) + \boldsymbol{\mathcal{P}}_h(\mathbf{f}\phi_h)$ for the mixed-primal $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_1$ approximation.

h	0.3727	0.1964	0.0970	0.0478	0.0245	0.0128
$\Vert\mathbf{div}(\boldsymbol{\sigma}_h) + \boldsymbol{\mathcal{P}}_h(\mathbf{f}\phi_h)\Vert_{\ell^\infty}$	8.22E−06	1.31E−05	1.50E−05	1.54E−05	9.88E−05	9.90E−05

are terminated once the relative error of the entire coefficient vectors between two consecutive iterates, say $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq tol,$$

where $\|\cdot\|$ stands for the usual euclidean norm in R^N, with N denoting the total number of degrees of freedom defining the finite element method.

We now introduce some additional notation. The individual errors are denoted by:

$$e(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega}, \quad e(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega},$$
$$e(\phi) := \|\phi - \phi_h\|_{1,\Omega}, \quad e(p) := \|p - p_h\|_{0,\Omega},$$

where p_h corresponds to the post-processed pressure p_h obtained via the expression (cf. (2.7))

$$p_h = -\frac{1}{n}\mathrm{tr}\,(\boldsymbol{\sigma}_h).$$

Next, as usual, for $\star \in \{\boldsymbol{\sigma}, \mathbf{u}, \phi, p\}$ we let $r(\star)$ be the experimental rate of convergence given by

$$r(\star) := \frac{\log(e(\star)/\widehat{e}(\star))}{\log(h/\widehat{h})},$$

where h and $\widehat{h}$ denote two consecutive meshsizes with associated errors e and $\widehat{e}$, respectively.

6.1. Example 1

In our first example we illustrate the accuracy of our method in 2D by considering a manufactured exact solution defined on $\Omega := (0,1)^2$. We introduce the coefficients $\mu(\phi) = (1 - c\phi)^{-2}$, $\gamma(\phi) = c\phi(1 - c\phi)^2$,

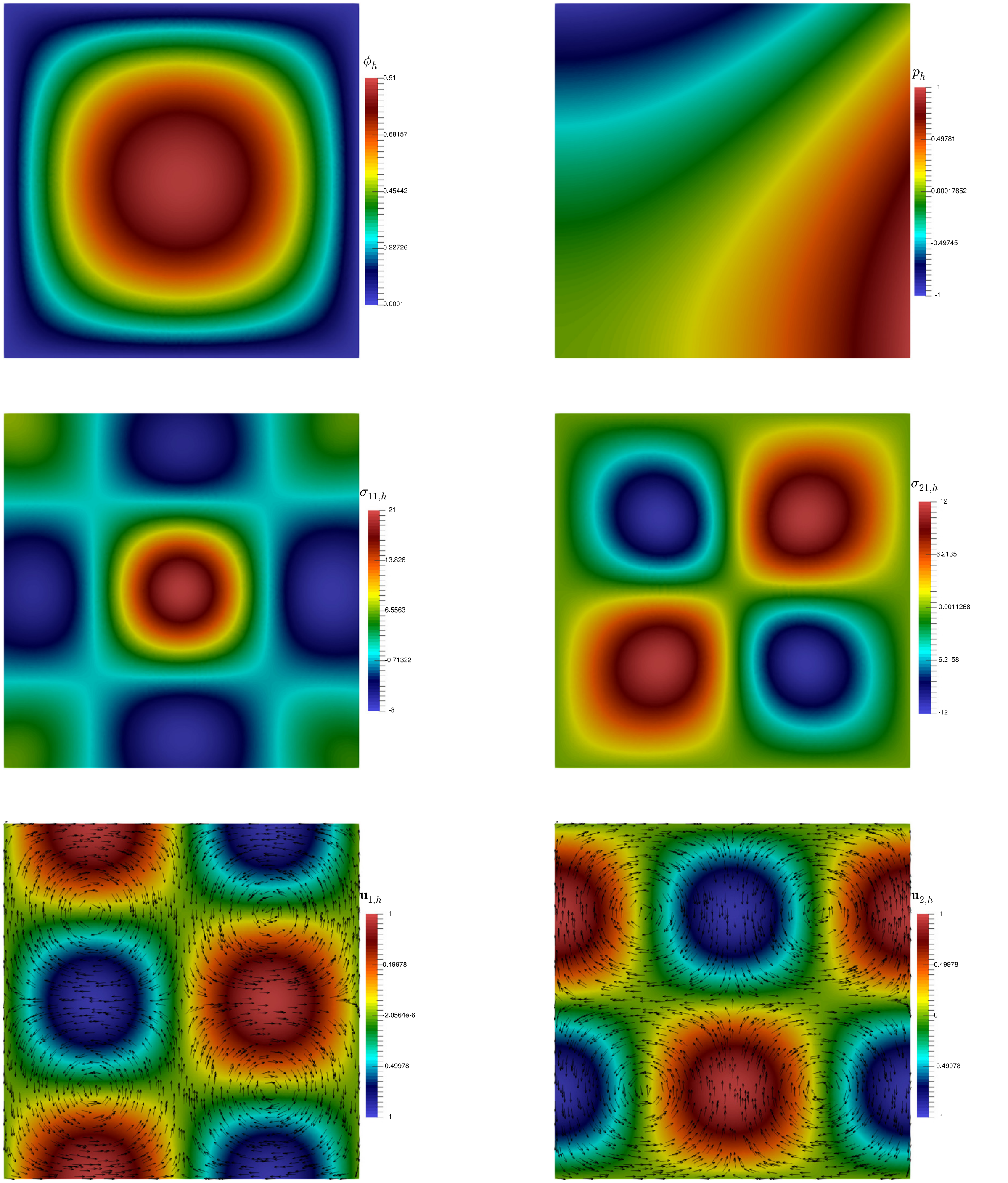

Fig. 6.1. Numerical results for Example 1: From left to right and from up to down: approximation of scalar field concentration ϕ_h, post-processed pressure p_h, stress and velocity components. Snapshot obtained from a simulation with 706 165 DOF.

$\vartheta(|\nabla\phi|) = m_1 + m_2(1 + |\nabla\phi|^2)^{m_3/2-1}$, and the source terms on the right hand sides are adjusted in such a way that the exact solutions are given by the smooth functions

$$\phi(x_1, x_2) = b - b\exp(-x_1(x_1 - 1)x_2(x_2 - 1)),$$

Table 6.3
Convergence history for Example 2, with a quasi-uniform refinement and a tolerance of 10^{-6}.

$\mathbb{RT}_0$–$\mathbf{P}_0$–P_1										
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\phi)$	$r(\phi)$	$e(p)$	$r(p)$	iter
157	0.3750	28.5342	–	0.4421	–	0.1523	–	0.8985	–	7
621	0.2001	12.4799	1.3166	0.2132	1.1614	0.0684	1.2746	0.5605	0.7513	7
2429	0.0950	5.8120	1.0265	0.1058	0.9413	0.0328	0.9883	0.2709	0.9764	7
9697	0.0504	2.7530	1.1780	0.0508	1.1559	0.0159	1.1432	0.1325	1.1274	7
38 280	0.0265	1.3567	1.0986	0.0253	1.0804	0.0079	1.0762	0.0651	1.1025	6
152 767	0.0133	0.6752	1.0166	0.0127	1.0046	0.0040	1.0095	0.0322	1.0274	6
$\mathbb{RT}_1$–$\mathbf{P}_1$–P_2										
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\phi)$	$r(\phi)$	$e(p)$	$r(p)$	iter
481	0.3750	5.4370	–	0.0987	–	0.0186	–	0.3049	–	7
1969	0.2001	1.6640	1.8850	0.0285	1.9760	0.0042	2.3839	0.0675	2.4013	7
7825	0.0950	0.4025	1.9065	0.0068	1.9288	0.0011	1.8341	0.0161	1.9244	7
31 489	0.0504	0.0995	2.2029	0.0017	2.1659	0.0003	2.2435	0.0038	2.2927	7
124 789	0.0265	0.0250	2.1474	0.0004	2.1418	0.0001	2.1410	0.0009	2.1578	7
498 985	0.0133	0.0062	2.0274	0.0001	2.0259	0.0000	2.0370	0.0002	2.0281	6

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(2\pi x_1)\cos(2\pi x_2) \\ -\cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix}, \qquad \boldsymbol{\sigma} = \mu(\phi)\nabla\mathbf{u} - (x_1^2 - x_2^2)\,\mathbb{I}\,,$$

for $(x_1,x_2) \in \overline{\Omega}$. We take $b = 15$, $c = m_1 = m_2 = 1/2$, $m_3 = 3/2$, where ϕ vanishes at Γ and $\mathbf{u}_D$ is imposed accordingly to the exact solution. The mean value of $\operatorname{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a Lagrange multiplier strategy. The domain is partitioned into quasi-uniform meshes with 2^{n+2}, $n \in \{0,\ldots,5\}$ vertices on each side of the domain. Values of errors and corresponding rates associated to $\mathbb{RT}_k$–$\mathbf{P}_k$–P_{k+1} for (4.4) and $k \in \{0,1\}$ are summarized in Table 6.1, whereas plots of some components of the approximate solution are displayed in Fig. 6.1. These findings are in agreement with the theoretical error bounds of (5.22). In addition, and since, as explained at the end of Section 4, our Galerkin scheme provides conservation of momentum in an approximate sense, we illustrate this fact in Table 6.2, where the computed ℓ^∞-norm for $\mathbf{div}(\boldsymbol{\sigma}_h) + \mathcal{P}_h(\boldsymbol{f}\phi_h)$ are displayed. As expected, these values are certainly close to zero.

6.2. *Example 2*

Our second test focuses on the non-convex domain $\Omega = (0,1)^2 \setminus [0.5,1]^2$ under quasi-uniform refinement. The coefficients μ, ϑ and γ, and the exact solution are given as in the previous test. In this case, $b = 3$, $c = m_1 = m_2 = 1/2$, $m_3 = 3/2$. Since ϕ does not identically vanish on Γ, the right hand side must be modified properly. Values of errors and corresponding rates associated to $\mathbb{RT}_k$–$\mathbf{P}_k$–P_{k+1} for (4.4) and $k \in \{0,1\}$ are summarized in Table 6.3, whereas plots of some components of the approximate solution are displayed in Fig. 6.2. Despite the non-convexity of the domain, the experimental rates of convergence are in agreement with the theoretical error bounds of (5.22), which is due to the fact that the exact solution is sufficiently smooth.

6.3. *Example 3*

In this example we illustrate the accuracy of our method in 3D. We consider $\Omega = (0,1)^3$, and the functions μ, ϑ and γ are established as in Example 1. The source terms on the right hand side are adjusted such that the exact solutions are given by

$$\phi(x_1,x_2,x_3) = b - b\exp(x_1(x_1-1)x_2(x_2-1)x_3(x_3-1)),$$

$$\mathbf{u}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3), \end{pmatrix} \qquad \boldsymbol{\sigma} = \mu(\phi)\nabla\mathbf{u} - (x_1 - 0.5)^3\sin(x_3 + x_2)\,\mathbb{I}\,,$$

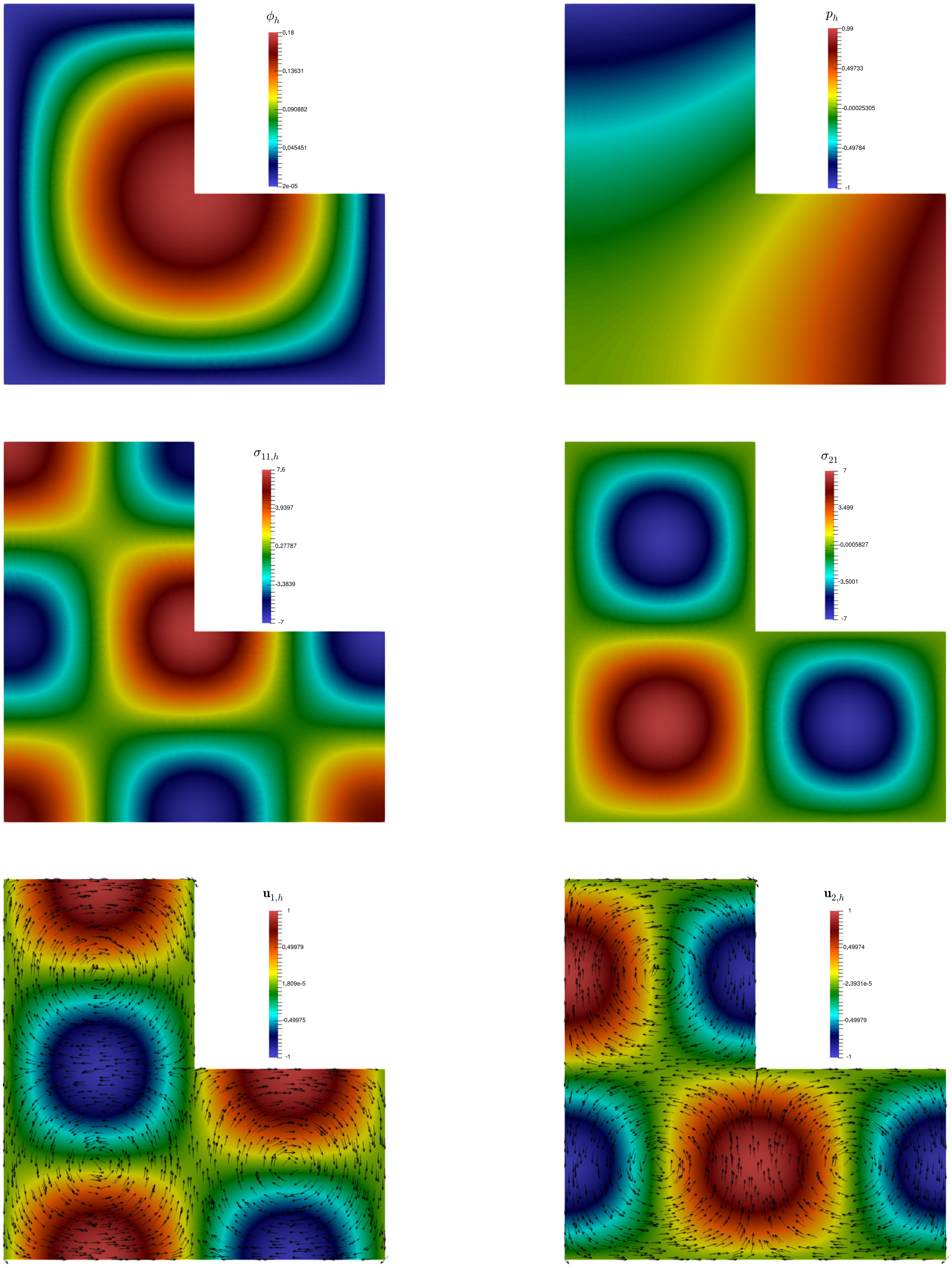

Fig. 6.2. Numerical results for Example 2: From left to right and from up to down: approximation of scalar field concentration ϕ_h, post-processed pressure p_h, stress components and velocity components. Snapshot obtained from a simulation with 498 985 DOF.

for $(x_1, x_2, x_3) \in \overline{\Omega}$. Values of errors for $\mathbb{RT}_0$–$\mathbf{P}_0$–P_1 are summarized in Table 6.4, whereas plots of some components of the approximate solution are displayed in Fig. 6.3. For the most refined meshes, the optimal rate of convergence $O(h)$ is recovered.

Table 6.4
Convergence history for Example 3, with a quasi-uniform refinement and a tolerance of 10^{-6}.

$\mathbb{RT}_0$–$\mathbf{P}_0$–P_1										
DOF	h	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(\phi)$	$r(\phi)$	$e(p)$	$r(p)$	iter
531	0.7071	23.6177	–	0.6122	–	0.4147	–	2.3258	–	6
3869	0.3536	16.0150	0.5604	0.3120	0.9726	0.2444	0.7626	1.4891	0.6432	6
29 529	0.1768	8.3325	0.9426	0.1572	0.9883	0.1278	0.9358	0.5618	1.4063	7
230 705	0.0884	3.8228	1.1241	0.0783	1.0057	0.0647	0.9828	0.1897	1.5664	7
1 823 841	0.0442	1.6982	1.1706	0.0391	1.0015	0.0325	0.9926	0.0743	1.3528	7

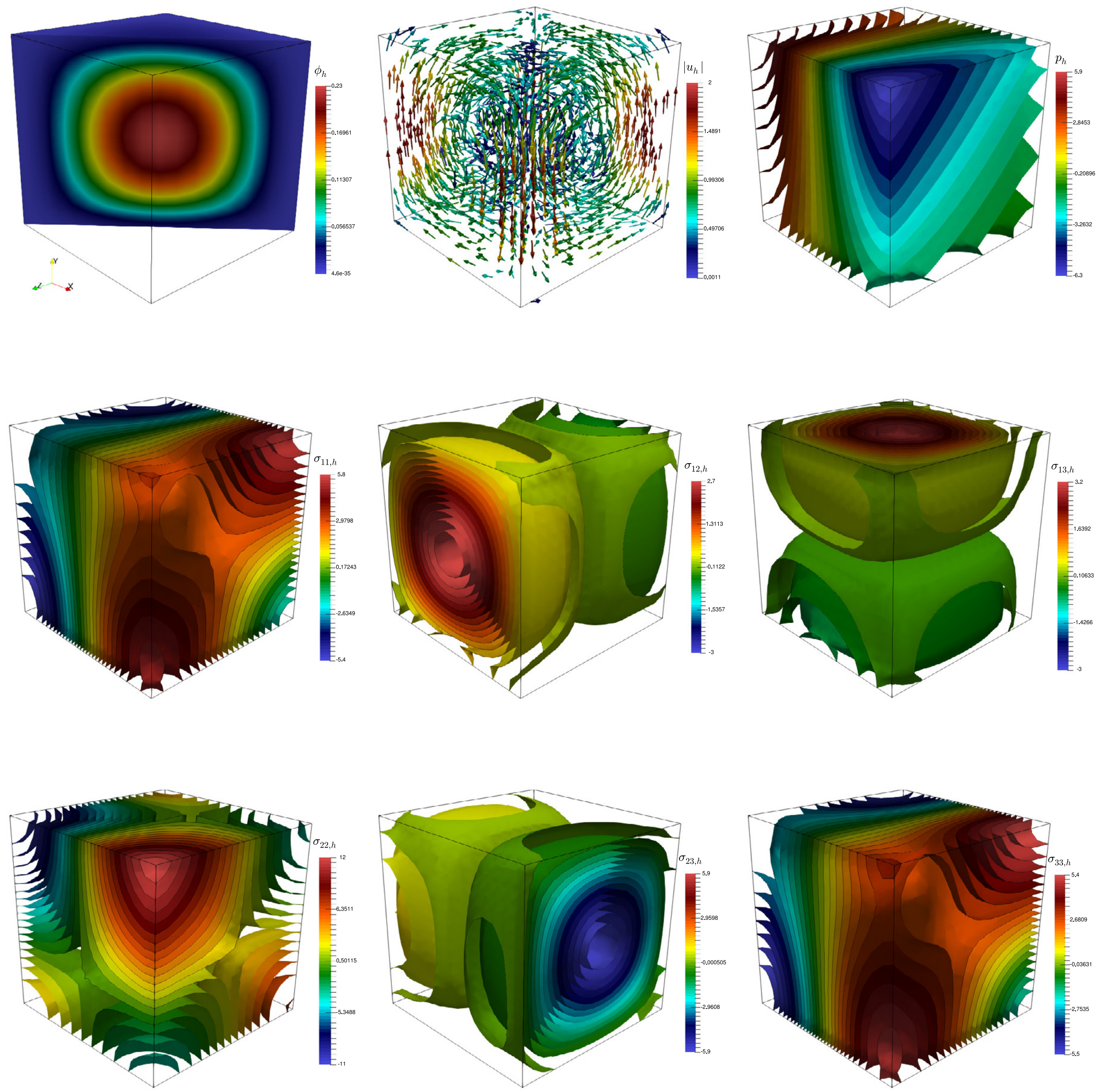

Fig. 6.3. Numerical results for Example 3: From left to right and from up to down: approximation of scalar field concentration ϕ_h, magnitude of the velocity $\mathbf{u}_h$, post-processed pressure p_h, and stress components. Snapshot obtained with 230 705 DOF.

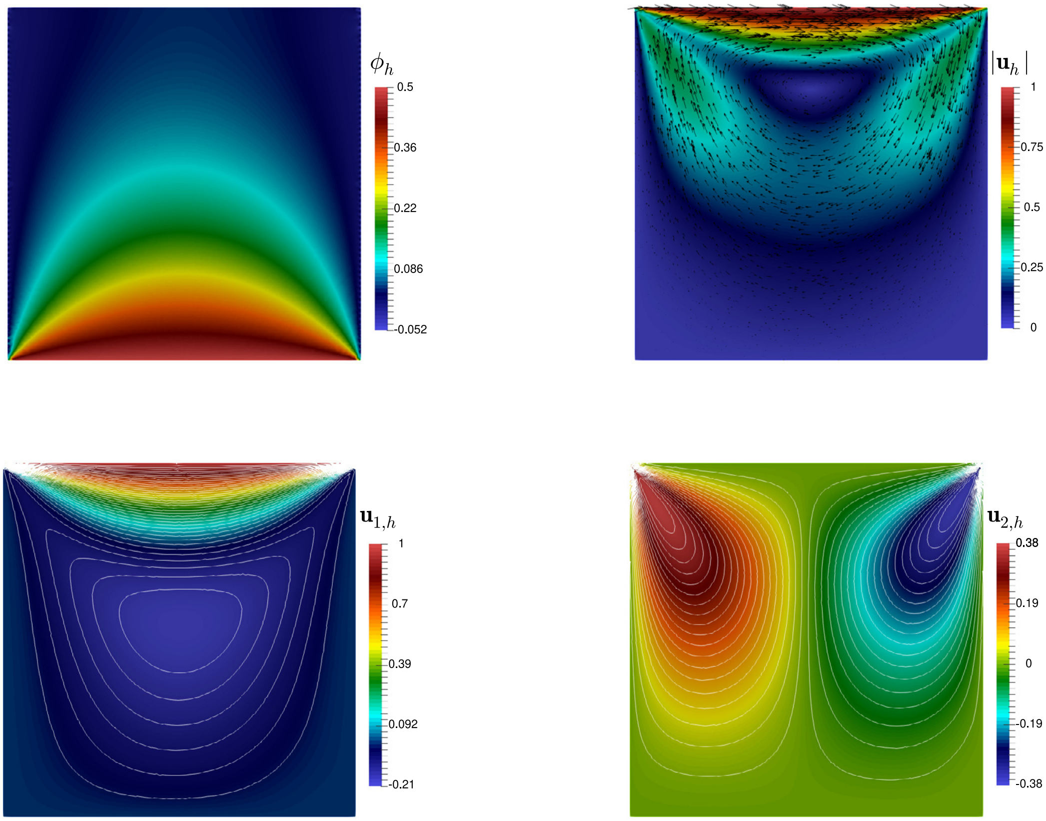

Fig. 6.4. Numerical results for Example 4: From left to right and from up to-down: approximation of scalar field concentration ϕ_h, magnitude of the velocity $\mathbf{u}_h$, and velocity components. Snapshot obtained from a simulation with 216 072 DOF.

6.4. Example 4

In our last example we study the behavior of the model for the well-known lid-driven cavity flow in the square domain $\Omega = (0,1)^2$ with boundary Γ, and whose upper and lower parts are denoted Γ_{top} and Γ_{bottom}, respectively. The functions μ, ϑ and γ are given as in Example 1 with $c = m_1 = m_2 = 1/2$ and $m_3 = 3/2$, whereas the right-hand side data are chosen as $\boldsymbol{f} = (0, -9.81)$ and $g = 0$, and the boundary conditions are

$$\mathbf{u} = (1,0) \quad \text{on} \quad \Gamma_{\mathrm{top}}, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma \setminus \Gamma_{\mathrm{top}},$$

$$\phi = 0.5 \quad \text{on} \quad \Gamma_{\mathrm{bottom}}, \quad \mathbf{p} \cdot \boldsymbol{\nu} = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_{\mathrm{bottom}}.$$

We stress that slight modifications allow to extend our analysis to these new boundary conditions. In Fig. 6.4 we display the approximated scalar concentration field, magnitude of the velocity, and velocity components, which were built using the mixed-primal $\mathbb{RT}_0 - \mathbf{P}_0 - \mathrm{P}_1$ approximation on a quasi-uniform mesh with meshsize $h = 0.0128$ and 39,146 triangle elements (actually representing 216,072 degrees of freedom). As expected, for the magnitude of the velocity we observe a vortex near Γ_{top}. In turn, the concentration is low on the top of the domain and goes increasing towards the bottom of the cavity. In addition, similarly to Example 1, we compute the ℓ^∞-norm of $\mathbf{div}(\boldsymbol{\sigma}_h) + \mathcal{P}_h(\boldsymbol{f}\phi_h)$ obtaining 3.37E−06, which illustrates again that this method conserves momentum in an approximate sense.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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