Chapter 1

Introduction

In this chapter we base most of the presentation on the classical references [8], [20], [41], and [51], and describe the main introductory aspects of the finite and mixed finite element methods. We first recall the particular and general versions of the Lax-Milgram Lemma, and then we introduce two examples illustrating the use of mixed variational formulations to solve boundary value problems. Finally, we present several basic results on traces, integration by parts formulae, and Green's identities for some Sobolev spaces, and in particular for $H(\mathrm{div};\Omega)$.

1.1 The Lax-Milgram Lemma

In order to state and prove this result, the most classical one in the analysis of variational problems, we need some preliminary concepts.

1.1.1 Preliminaries

Definición 1.1 *Let* $(H_1, \langle\cdot,\cdot\rangle_1)$ *and* $(H_2, \langle\cdot,\cdot\rangle_2)$ *be real Hilbert spaces. We say that* $B: H_1 \times H_2 \to \mathbb{R}$ *is a bilinear form if it is linear in each of its components, that is*

i) $B(\alpha\, x + \beta\, y, z) = \alpha\, B(x,z) + \beta\, B(y,z) \quad \forall\, x, y \in H_1, \quad \forall\, z \in H_2, \quad \forall\, \alpha, \beta \in \mathbb{R}.$

ii) $B(x, \alpha\, y + \beta\, z) = \alpha\, B(x,y) + \beta\, B(x,z) \quad \forall\, x \in H_1, \quad \forall\, y, z \in H_2, \quad \forall\, \alpha, \beta \in \mathbb{R}.$

Definición 1.2 *Let* $(H_1, \langle\cdot,\cdot\rangle_1)$ *and* $(H_2, \langle\cdot,\cdot\rangle_2)$ *be real Hilbert spaces with induced norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$*, respectively. We say that a bilinear form* $B: H_1 \times H_2 \to \mathbb{R}$ *is* bounded *if there exists a constant $M > 0$ such that:*

$$|B(x,y)| \leq M \, \|x\|_1 \, \|y\|_2 \quad \forall (x,y) \in H_1 \times H_2.$$

DEFINICIÓN 1.3 *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with induced norm $\| \cdot \|$, and let $B : H \times H \to \mathbb{R}$ be a bilinear form. We say that B is* STRONGLY COERCIVE *(or H-*ELLIPTIC*) if there exists a constant $\alpha > 0$ such that*

$$B(x,x) \ \geq \ \alpha \, \|x\|^2 \qquad \forall \, x \in H.$$

Now, given $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ real Hilbert spaces and $B : H_1 \times H_2 \to \mathbb{R}$ a bounded bilinear form, we are interested in defining the operator $\mathbb{B} : H_1 \to H_2$ induced by B and viceversa. To this end, we consider $v \in H_1$ and define the functional $F_v : H_2 \to \mathbb{R}$ by

$$F_v(w) := B(v,w) \quad \forall \, w \in H_2.$$

Since B is bilinear, it is clear that F_v is linear. In addition, the fact that B is bounded (with constant M) implies that

$$|F_v(w)| \leq M \|v\|_1 \, \|w\|_2 \quad \forall \, w \in H_2,$$

which shows that $F_v \in H_2'$ and

$$\|F_v\| \, \leq \, M \|v\|_1 \quad \forall \, v \in H_1. \tag{1.1}$$

The above induces the definition of the operator $\mathcal{B} : H_1 \to H_2'$ as

$$\mathcal{B}(v) := F_v \quad \forall \, v \in H_1,$$

which, in virtue of the linearity of B in its first component and the inequality (1.1), is linear and bounded with

$$\|\mathcal{B}\|_{\mathcal{L}(H_1, H_2')} \, \leq \, M.$$

Recall here that, given Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y. Finally, if $\mathcal{R}_2 : H_2' \to H_2$ denotes the Riesz mapping, we let $\mathbb{B} : H_1 \to H_2$ be the operator induced by B, that is

$$\mathbb{B} := \mathcal{R}_2 \circ \mathcal{B}, \tag{1.2}$$

or, graphically

$$\begin{array}{ccc} H_1 & \overset{\mathcal{B}}{\longrightarrow} & H_2' \\ & \underset{\mathbb{B}}{\searrow} & \downarrow \mathcal{R}_2 \\ & & H_2 \end{array}$$

Note that the linearity and boundedness of $\mathcal{R}_2$ and $\mathcal{B}$ yield the same properties for $\mathbb{B}$, and there holds

$$\langle \mathbb{B}(v), w\rangle_2 = \langle\, \mathcal{R}_2(\mathcal{B}(v)), w\rangle_2 = \mathcal{B}(v)(w) = B(v,w) \quad \forall\, (v,w) \in H_1 \times H_2\,. \tag{1.3}$$

Conversely, given $\mathbb{B} \in \mathcal{L}(H_1, H_2)$ we define the bilinear form $B : H_1 \times H_2 \to \mathbb{R}$ induced by $\mathbb{B}$ as

$$B(v,w) := \langle \mathbb{B}(v), w\rangle_2 \quad \forall\, (v,w) \in H_1 \times H_2\,. \tag{1.4}$$

1.1.2 The classical version

The following result constitutes the most known version of the Lax-Milgram Lemma.

TEOREMA 1.1 (**The Lax-Milgram Lemma**) *Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $B : H \times H \to \mathbb{R}$ be a bounded and H-elliptic bilinear form with constants M and α, respectively. Then, for each $F \in H'$ there exists a unique $u \in H$ such that*

$$B(u,v) = F(v) \quad \forall\, v \in H \tag{1.5}$$

and

$$\|u\| \le \frac{1}{\alpha}\, \|F\|. \tag{1.6}$$

DEMOSTRACIÓN. Let $\mathbb{B} : H \to H$ be the linear and bounded operator induced by B, that is

$$\langle \mathbb{B}(v), w\rangle = B(v,w) \quad \forall\, (v,w) \in H \times H,$$

and let $\mathcal{R} : H' \to H$ be the corresponding Riesz operator. Then, finding a unique $u \in H$ such that (1.5) holds is equivalent to look for $u \in H$ such that

$$\langle \mathbb{B}(u), v\rangle = \langle \mathcal{R}(F), v\rangle \quad \forall\, v \in H,$$

that is, such that

$$\mathbb{B}(u) = \mathcal{R}(F). \tag{1.7}$$

Moreover, since the above is required for each $F \in H'$, we deduce that the present proof reduces to show that $\mathbb{B} : H \to H$ is bijective. To this end, let us notice from the H-ellipticity of B that for each $v \in H$ there holds

$$\alpha \|v\|^2 \leq B(v,v) = \langle \mathbb{B}(v), v\rangle \leq \|\mathbb{B}(v)\| \, \|v\|,$$

from where

$$\alpha \|v\| \leq \|\mathbb{B}(v)\| \quad \forall v \in H . \tag{1.8}$$

It follows, because of the result characterizing the operators with closed range, that $\mathbb{B}$ is injective and $R(\mathbb{B})$ is a closed subspace of H. Hence, according to the Orthogonal Decomposition Theorem, we have that $H = R(\mathbb{B}) \oplus R(\mathbb{B})^\perp$, and therefore, in order to conclude that $\mathbb{B}$ is surjective, it only remains to prove that $R(\mathbb{B})^\perp = \{0\}$. Indeed, given $w \in R(\mathbb{B})^\perp$ there holds $\langle z, w \rangle = 0 \quad \forall z \in R(\mathbb{B})$, or, equivalently, $\langle \mathbb{B}(v), w\rangle = 0 \quad \forall v \in H$. In particular, taking $v = w$ and utilizing again the H-ellipticity of B, we get

$$0 = \langle \mathbb{B}(w), w\rangle = B(w,w) \geq \alpha \|w\|^2 ,$$

from which $w = 0$, thus completing the proof of the bijectivity of $\mathbb{B}$. Consequently, given $F \in H'$ there exists a unique $u \in H$ such that $\mathbb{B}(u) = \mathcal{R}(F)$, that is

$$B(u,v) = F(v) \quad \forall v \in H .$$

Finally, taking $v = u$ in (1.8) and using that $\|\mathbb{B}(u)\| = \|\mathcal{R}(F)\| = \|F\|$, we obtain

$$\|u\| \leq \frac{1}{\alpha} \|F\| ,$$

which finishes the proof.

□

The Lax-Milgram Lemma and its proof yield several remarks. First of all, we observe that the inequality (1.6) represents a continuous dependence result for the problem (1.5). In fact, given $F_1, \ F_2 \in H'$, let us denote by $u_1, u_2 \in H$ the unique solutions, guaranteed by this lemma, of the problems

$$B(u_1, v) = F_1(v) \qquad \forall v \in H$$

and

$$B(u_2, v) = F_2(v) \qquad \forall v \in H.$$

It follows that $u := u_1 - u_2 \in H$ is in turn the unique solution of

$$B(u,v) = (F_1 - F_2)(v) \qquad \forall\, v \in H\,,$$

whence (1.6) implies that

$$\|u_1 - u_2\| \,\leq\, \frac{1}{\alpha}\,\|F_1 - F_2\|\,.$$

The above inequality shows that the stability of the solution of (1.5) depends strongly on the inverse of the ellipticity constant α. In other words, the larger α the better stability of (1.5).

On the other hand, let us recall from (1.7) that proving the Lax-Milgram Lemma reduces, given $F \in H'$, to show the existence of a unique $u \in H$ such that $\mathbb{B}(u) = \mathcal{R}(F)$. Then, considering a parameter $\rho > 0$, the above is equivalent to find $u \in H$ such that

$$-\rho\,\big\{\mathbb{B}(u) - \mathcal{R}(F)\big\} \,=\, 0,$$

that is, to seek $u \in H$ such that

$$T(u) = u,$$

where $T : H \to H$ is the nonlinear operator defined by

$$T(v) \,:=\, v - \rho\,\big\{\mathbb{B}(v) - \mathcal{R}(F)\big\} \qquad \forall\, v \in H.$$

In this way, an alternative proof of the Lax-Milgram Lemma consists of proving that T has a unique fixed point, which is achieved, in virtue of the corresponding Banach Theorem, by showing that T is a contraction for some $\rho > 0$. In fact, using the ellipticity and boundedness of $\mathbb{B}$, we obtain that

$$\begin{aligned}
&\|T(v) - T(w)\|^2 = \langle\, T(v) - T(w), T(v) - T(w)\,\rangle \\
&= \langle\,(v-w) - \rho\,\mathbb{B}(v-w), (v-w) - \rho\,\mathbb{B}(v-w)\,\rangle \\
&= \|v-w\|^2 - 2\rho\,\langle\,\mathbb{B}(v-w), v-w\,\rangle \,+\, \rho^2\,\|\,\mathbb{B}(v-w)\|^2 \\
&\leq\, (1 - 2\rho\,\alpha \,+\, \rho^2\,M^2)\|v-w\|^2 \qquad \forall\, v, w \in H,
\end{aligned}$$

from which it follows that a sufficient condition for the contractivity of T is that $1 - 2\,\rho\,\alpha \,+\, \rho^2\,M^2 \,<\, 1$, that is

$$\rho \,\in\, \left(0,\, \frac{2\alpha}{M^2}\right)\,.$$

Another interesting aspect of problem (1.5) is the resulting analysis for the case in which the bilinear form B is SYMMETRIC. Indeed, under this additional hypothesis, B becomes a scalar product on H whose induced norm, denoted by $\|\cdot\|_B$, is given by

$$\|v\|_B := B(v,v)^{1/2} \qquad \forall\, v \in H.$$

Hence, thanks to the H-ellipticity and boundedness of B, there holds

$$\alpha\,\|v\|^2 \le B(v,v) = \|v\|_B^2 \le M\,\|v\|^2 \qquad \forall\, v \in H,$$

which proves that $\|\cdot\|$ and $\|\cdot\|_B$ are equivalent in H, and therefore $(H, B(\cdot,\cdot))$ is a Hilbert space. Consequently, given $F \in H'$ (with respect to any of these norms), a straightforward application of the Riesz Representation Theorem (R.R.T.) to $(H, B(\cdot,\cdot))$, yields the existence of a unique $u \in H$ such that

$$F(v) = B(u,v) \qquad \forall\, v \in H.$$

According to the above analysis, the proof of the Lax-Milgram Lemma in the case of a symmetric bilinear form B reduces simply to an application of the R.R.T.. In other words, this classical lemma is nothing but an extension of the R.R.T. to the case of a bounded and H-elliptic bilinear form B.

In what follows we illustrate the applicability of the Lax-Milgram Lemma with a one-dimensional example. To this end and for later use, we recall that, given an interval $\Omega :=]a,b[\subseteq \mathbb{R}$, the corresponding Sobolev space of order 1 is given by

$$H^1(\Omega) := \left\{ v \in L^2(\Omega) : \quad v' \in L^2(\Omega) \right\},$$

where the derivative v' is in the distributional sense. It is easy to prove, using that $L^2(\Omega)$ (with its usual norm $\|\cdot\|_{0,\Omega}$) is Hilbert, that $H^1(\Omega)$ endowed with the inner product $\langle v,w\rangle_{1,\Omega} := \int_\Omega \left\{ v'w' + vw \right\} \quad \forall\, v,\, w \in H^1(\Omega)$ and induced norm $\|\cdot\|_{1,\Omega}$ is also Hilbert. Furthermore, letting $|\cdot|_{1,\Omega}$ to be the associated semi-norm, that is $|v|_{1,\Omega} := \|v'\|_{0,\Omega}$ $\forall\, v \in H^1(\Omega)$, we have the following result.

LEMA 1.1 (**Friedrichs-Poincaré's Inequality**) *Let* $\Omega :=]a,b[\subseteq \mathbb{R}$ *and define* $H_0^1(\Omega) := \left\{ v \in H^1(\Omega) : \quad v(a) = v(b) = 0 \right\}$. *Then, there holds*

$$\|v\|_{1,\Omega}^2 \le \left\{ 1 + \frac{(b-a)^2}{2} \right\} |v|_{1,\Omega}^2 \qquad \forall\, v \in H_0^1(\Omega)\,. \tag{1.9}$$

DEMOSTRACIÓN. We first prove the inequality (1.9) in the space $C_0^\infty(\Omega)$, and then use that $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ with respect to $\|\cdot\|_{1,\Omega}$. In fact, let $\varphi \in C_0^\infty(\Omega)$. Then, for each $x \in \Omega$ there holds

$$\varphi(x) = \int_a^x \varphi'(t)\, dt,$$

which, applying Cauchy-Schwarz's inequality, implies that

$$\begin{aligned} |\varphi(x)|^2 &\le \int_a^x 1^2\, dt \int_a^x (\varphi'(t))^2\, dt = (x-a) \int_a^x (\varphi'(t))^2\, dt \\ &\le (x-a) \int_a^b (\varphi'(t))^2\, dt = (x-a)\, |\varphi|_{1,\Omega}^2. \end{aligned}$$

Then, integrating by parts with respect to $x \in \Omega$ in the above estimate, we find that

$$\|\varphi\|_{0,\Omega}^2 \le |\varphi|_{1,\Omega}^2 \int_a^b (x-a)\, dx = \frac{(b-a)^2}{2}\, |\varphi|_{1,\Omega}^2,$$

and therefore

$$\|\varphi\|_{1,\Omega}^2 = \|\varphi\|_{0,\Omega}^2 + |\varphi|_{1,\Omega}^2 \le \left\{ 1 + \frac{(b-a)^2}{2} \right\} |\varphi|_{1,\Omega}^2. \tag{1.10}$$

Now, given $v \in H_0^1(\Omega)$ we let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ be such that

$$\|v - \varphi_n\|_{1,\Omega} \overset{n\to\infty}{\to} 0. \tag{1.11}$$

It follows from (1.10) that

$$\|\varphi_n\|_{1,\Omega}^2 \le \left\{ 1 + \frac{(b-a)^2}{2} \right\} |\varphi_n|_{1,\Omega}^2 \quad \forall\, n \in \mathbb{N},$$

from which, taking $\lim\limits_{n\to\infty}$ and using (1.11), we conclude (1.9).

□

EJEMPLO 1.1 Given $\Omega =]0,1[$ and $f \in L^2(\Omega)$, we consider the boundary value problem:

$$-u'' = f \quad \text{in} \quad \Omega, \quad u(0) = u(1) = 0.$$

It is easy to see that the corresponding variational formulation is given by: Find $u \in H := H_0^1(\Omega)$ such that

$$B(u,v) := \int_0^1 u'v' = F(v) := \int_0^1 fv \quad \forall\, v \in H. \tag{1.12}$$

It is clear that F is linear and bounded since, applying Cauchy-Schwarz's inequality, we obtain

$$|F(v)| = \left| \int_0^1 fv \right| \le \|f\|_{0,\Omega}\, \|v\|_{0,\Omega} \le \|f\|_{0,\Omega}\, \|v\|_{1,\Omega} \quad \forall\, v \in H,$$

which says that $\|F\| \le \|f\|_{0,\Omega}$. In turn, $B : H \times H \to \mathbb{R}$ is a bounded bilinear form since, using again Cauchy-Schwarz's inequality, we get

$$|B(w,v)| = \left| \int_0^1 w'v' \right| \le |w|_{1,\Omega}\, |v|_{1,\Omega} \le \|w\|_{1,\Omega}\, \|v\|_{1,\Omega} \quad \forall\, w, v \in H.$$

On the other hand, employing the Friedrichs-Poincaré's inequality from Lemma 1.1 with $a = 0$ and $b = 1$, we find that

$$B(v,v) = \int_0^1 (v')^2 = |v|^2_{1,\Omega} \ge \frac{2}{3}\, \|v\|^2_{1,\Omega} \quad \forall\, v \in H,$$

which shows that B is H-elliptic with constant $\alpha = 2/3$. Hence, a direct application of the Lax-Milgram Lemma implies that (1.12) has a unique solution $u \in H_0^1(\Omega)$, which satisfies

$$\|u\|_{1,\Omega} \le \frac{3}{2}\, \|F\| \le \frac{3}{2}\, \|f\|_{0,\Omega}.$$

1.1.3 The general version

The next goal is to derive a more general version of the Lax-Milgram Lemma (see Theorem 1.1). To this end, we now consider real Hilbert spaces $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$, a functional $F \in H_2'$, and a bounded bilinear form $B : H_1 \times H_2 \to \mathbb{R}$, and look for $u \in H_1$ such that

$$B(u,v) = F(v) \quad \forall\, v \in H_2. \tag{1.13}$$

Equivalently, if $\mathbb{B} : H_1 \to H_2$ is the linear and bounded operator induced by B, and $\mathcal{R}_2 : H_2' \to H_2$ is the corresponding Riesz mapping, we are interested in finding $u \in H_1$ such that

$$\mathbb{B}(u) = \mathcal{R}_2(F).$$

Hence, a necessary and sufficient condition for (1.13) to have a unique solution for each $F \in H_2'$ is that $\mathbb{B}$ be bijective. In turn, the bijectivity of $\mathbb{B}$ can be reformulated according to the equivalences provided by the following lemma.

Lema 1.2 *Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be Hilbert spaces with induced norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, and let $\mathbb{B} \in \mathcal{L}(H_1, H_2)$. Then:*

a) *$\mathbb{B}$ is surjective if and only if $\mathbb{B}^*$ is injective and has closed range, that is, if there exists $\alpha > 0$ such that*

$$\|\mathbb{B}^*(v)\|_1 \ge \alpha\, \|v\|_2 \qquad \forall\, v \in H_2. \tag{1.14}$$

b) $\mathbb{B}$ *is injective if and only if*

$$\sup_{v\in H_2} \langle \mathbb{B}(u), v\rangle_2 > 0 \qquad \forall\, u \in H_1,\, u \neq 0.$$

c) $\mathbb{B}^*$ *is surjective if and only if* $\mathbb{B}$ *is injective and has closed range, that is, if there exists* $\alpha > 0$ *such that*

$$\|\mathbb{B}(u)\|_2 \,\geq\, \alpha\,\|u\|_1 \qquad \forall\, u \in H_1. \tag{1.15}$$

d) $\mathbb{B}^*$ *is injective if and only if*

$$\sup_{u\in H_1} \langle \mathbb{B}(u), v\rangle_2 > 0 \quad \forall\, v \in H_2,\, v \neq 0.$$

e) $\mathbb{B}$ *is bijective if and only if* $\mathbb{B}^*$ *is bijective.*

Demostración.

a) Suppose that $R(\mathbb{B}) = H_2$. It follows that $R(\mathbb{B})$ and hence $R(\mathbb{B}^*)$ are closed. In addition, it is clear that $N(\mathbb{B}^*) = R(\mathbb{B})^\perp = H_2^\perp = \{0\}$. Conversely, if $\mathbb{B}^*$ is injective and has closed range, the range of $\mathbb{B}$ is closed as well, and therefore $R(\mathbb{B}) = N(\mathbb{B}^*)^\perp = \{0\}^\perp = H_2$. The equivalence with (1.14) is precisely the characterization result for injective operators with closed range.

b) It suffices to see that $\mathbb{B}$ is injective if and only if $\mathbb{B}(u) \neq 0 \quad \forall\, u \in H_1,\, u \neq 0$.

c) and d) These equivalences follow directly from a) and b) by applying them to $\mathbb{B}^*$.

e) Assume that $\mathbb{B}$ is bijective. It follows from a) that $\mathbb{B}^*$ is injective and has closed range, which yields $R(\mathbb{B}^*) = N(\mathbb{B})^\perp = \{0\}^\perp = H_1$, and thus $\mathbb{B}^*$ is bijective. For the converse it suffices to apply the above implication to $\mathbb{B}^*$ instead of $\mathbb{B}$.

□

It is important to observe here, according to e), that the pairs of conditions a), b) and c), d) are equivalent. To this respect, let us also note that (1.14) and (1.15) can be rewritten as:

$$\|\mathbb{B}^*(v)\|_1 := \sup_{\substack{u\in H_1\\ u\neq 0}} \frac{\langle \mathbb{B}(u), v\rangle_2}{\|u\|_1} = \sup_{\substack{u\in H_1\\ u\neq 0}} \frac{B(u,v)}{\|u\|_1} \geq \alpha\,\|v\|_2 \quad \forall\, v \,\in\, H_2, \tag{1.16}$$

and

$$\|\mathbb{B}(u)\|_2 := \sup_{\substack{v\in H_2\\ v\neq 0}} \frac{\langle \mathbb{B}(u), v\rangle_2}{\|v\|_2} = \sup_{\substack{v\in H_2\\ v\neq 0}} \frac{B(u,v)}{\|v\|_2} \geq \alpha\,\|u\|_1 \quad \forall\, u \in H_1, \tag{1.17}$$

or, respectively,

$$\inf_{\substack{v\in H_2\\ v\neq 0}} \sup_{\substack{u\in H_1\\ u\neq 0}} \frac{B(u,v)}{\|u\|_1\,\|v\|_2} \geq \alpha, \tag{1.18}$$

and

$$\inf_{\substack{u\in H_1\\ u\neq 0}} \sup_{\substack{v\in H_2\\ v\neq 0}} \frac{B(u,v)}{\|u\|_1\,\|v\|_2} \geq \alpha, \tag{1.19}$$

which explains the name of INF-SUP CONDITIONS given to (1.16) and (1.17) (equivalently (1.14) and (1.15)).

In virtue of the above analysis, we can establish next a more general version of the Lax-Milgram Lemma.

TEOREMA 1.2 (**The Generalized Lax-Milgram Lemma**) *Let* $(H_1, \langle\cdot,\cdot\rangle_1)$ *and* $(H_2, \langle\cdot,\cdot\rangle_2)$ *be Hilbert spaces with induced norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$, *respectively, and let* $B: H_1 \times H_2 \to \mathbb{R}$ *be a bounded bilinear form. Assume that:*

i) *there exists* $\alpha > 0$ *such that*

$$\sup_{\substack{v\in H_2\\ v\neq 0}} \frac{B(u,v)}{\|v\|_2} \geq \alpha\,\|u\|_1 \quad \forall\, u \in H_1,$$

ii)

$$\sup_{u\in H_1} B(u,v) > 0 \quad \forall\, v \in H_2,\, v \neq 0.$$

Then, for each $F \in H_2'$ *there exists a unique* $u \in H_1$ *such that*

$$B(u,v) = F(v) \qquad \forall\, v \in H_2$$

and

$$\|u\|_1 \leq \frac{1}{\alpha}\,\|F\|_{H_2'}. \tag{1.20}$$

Moreover, the assumptions i) *and* ii) *are also necessary.*

DEMOSTRACIÓN. It suffices to notice that i) and ii) correspond to the conditions c) and d) from Lemma 1.2, which (both together) are equivalent to the bijectivity of $\mathbb{B}^*$ and consequently equivalent to the bijectivity of $\mathbb{B}$ as well. The estimate (1.20) follows from (1.15) by noting that $\mathbb{B}(u) = R_2(F)$.

□

Certainly, the above theorem can be stated, equivalently, with the conditions a) and b) instead of c) and d) from Lemma 1.2. In this case, one can assume the same constant α in (1.14) and (1.15) since $\|(\mathbb{B}^*)^{-1}\|$ (bounded by $1/\alpha$ in (1.14)) is equal to $\|\mathbb{B}^{-1}\|$ (bounded by $1/\alpha$ in (1.15)), and therefore the estimate (1.20) is also obtained from a) and b), but making use of the identity $\|(\mathbb{B}^*)^{-1}\| = \|\mathbb{B}^{-1}\|$.

Now, in the particular case in which $H_1 = H_2 = H$, the Lax-Milgram Lemma (cf. Theorem 1.1) follows obviously from Theorem 1.2. Indeed, if $B : H \times H \to \mathbb{R}$ is a bounded and H-elliptic bilinear form with constants M and α, respectively, there clearly hold

$$\sup_{\substack{v\in H\\ v\neq 0}} \frac{B(u,v)}{\|v\|} \geq \frac{B(u,u)}{\|u\|} \geq \alpha\,\|u\| \qquad \forall\, u \in H,\, u \neq 0,$$

and

$$\sup_{u\in H} B(u,v) \geq B(v,v) \geq \alpha\,\|v\|^2 > 0 \qquad \forall\, v \in H,\, v \neq 0,$$

which show, respectively, the hypotheses i) and ii) from Theorem 1.2. In turn, if instead of being H-elliptic, it is assumed that $B : H \times H \to \mathbb{R}$ is symmetric, then the operator $\mathbb{B} \in \mathcal{L}(H)$ induced by B becomes self-adjoint, and hence the hypothesis ii) from Theorem 1.2 is redundant. The above suggests the following symmetric version of the generalized Lax-Milgram Lemma.

TEOREMA 1.3 *Let H be a real Hilbert space and let $B : H \times H \to \mathbb{R}$ be a bounded bilinear form. Assume that:*

i) $B(w,v) = B(v,w) \qquad \forall\, w, v \in H.$

ii) *there exists $\alpha > 0$ such that*

$$\sup_{\substack{v\in H\\ v\neq 0}} \frac{B(u,v)}{\|v\|} \geq \alpha\,\|u\| \qquad \forall\, u \in H.$$

Then, for each $F \in H'$ there exists a unique $u \in H$ such that

$$B(u,v) = F(v) \qquad \forall\, v \in H$$

and

$$\|u\| \,\leq\, \frac{1}{\alpha}\,\|F\|\,.$$

Demostración. It is a straightforward corollary of Theorem 1.2.

□

1.2 Examples of mixed formulations

1.2.1 A one-dimensional model

Let a, b, $\kappa \in \mathbb{R}$, $\kappa > 0$, $\Omega :=]0,1[$, $f \in L^2(\Omega)$, and let us consider the boundary value problem:

$$-\,u'' \,+\, \kappa\, u \,=\, f \quad \text{in} \quad \Omega\,, \quad u'(0) \,=\, a\,, \quad u'(1) \,=\, b\,. \tag{1.21}$$

The primal formulation of (1.21) is given by: Find $u \in H := H^1(\Omega)$ such that

$$A(u,v) \,=\, F(v) \qquad \forall\, v \,\in\, H\,, \tag{1.22}$$

where $A : H \times H \to \mathbb{R}$ is the bilinear form defined by

$$A(u,v) \,:=\, \int_0^1 \Big\{ u'\,v' \,+\, \kappa\, u\, v \Big\} \qquad \forall\, u,\, v \,\in\, H\,,$$

and $F : H \to \mathbb{R}$ is the linear functional given by

$$F(v) \,:=\, \int_0^1 f\,v \,+\, \Big\{ b\,v(1) \,-\, a\,v(0) \Big\} \qquad \forall\, v \,\in\, H\,.$$

It is important to observe here that the boundary conditions of (1.21) are incorporated automatically, through the integration by parts procedure, into the functional F of the variational formulation (1.22). This is actually a characteristic feature of the Neumann boundary conditions in primal formulations, which explains the name *natural boundary conditions* given to them.

Now, in order to demonstrate the well-posedness of (1.22), that is unique solvability and continuous dependence on the data, it suffices to verify the hypotheses of the classical Lax-Milgram Lemma (cf. Theorem 1.1). In fact, since $\kappa > 0$, there holds

$$A(v,v) \,=\, \int_0^1 \Big\{ (v')^2 \,+\, \kappa\, v^2 \Big\} \,\geq\, \min\{1,\kappa\}\,\|v\|^2_{1,\Omega} \qquad \forall\, v \,\in\, H\,,$$

which proves that A is H-elliptic. In addition, utilizing the Cauchy-Schwarz inequality, we deduce that

$$|\,A(u,v)\,| \;\leq\; \max\{1,\kappa\}\, \|u\|_{1,\Omega}\, \|v\|_{1,\Omega} \qquad \forall\, u\,,\, v \,\in\, H\,,$$

which shows that A is bounded. For the boundedness of F we observe, also as a consequence of the Cauchy-Schwarz inequality, that

$$\left| \int_0^1 f\, v \right| \;\leq\; \|f\|_{0,\Omega}\, \|v\|_{0,\Omega} \;\leq\; \|f\|_{0,\Omega}\, \|v\|_{1,\Omega} \qquad \forall\, v \,\in\, H\,. \tag{1.23}$$

In turn, given $v \in C^1(\bar{\Omega})$ (restrictions to Ω of functions that are of class C^1 in an open set containing $\bar{\Omega}$), we have that

$$v(0) \;=\; v(x) \;-\; \int_0^x v'(t)\, dt \qquad \forall\, x \,\in\, \Omega\,,$$

from where

$$\begin{aligned} |v(0)|^2 \;\leq&\;\; 2\, \left\{ |v(x)|^2 \,+\, \left| \int_0^x v'(t)\, dt \right|^2 \right\} \\ \leq&\;\; 2\, \left\{ |v(x)|^2 \,+\, \left(\int_0^x 1\, dt \right) \left(\int_0^x (v'(t))^2\, dt \right) \right\} \\ \leq&\;\; 2\, \left\{ |v(x)|^2 \,+\, x\, |v|^2_{1,\Omega} \right\} \quad \forall\, x \,\in\, \Omega\,. \end{aligned}$$

Then, integrating with respect to $x \in \Omega$, we find that

$$|v(0)|^2 \;\leq\; 2 \left\{ \|v\|^2_{0,\Omega} \,+\, \frac{1}{2}\, |v|^2_{1,\Omega} \right\} \;\leq\; 2\, \|v\|^2_{1,\Omega}\,,$$

and hence

$$|v(0)| \;\leq\; \sqrt{2}\, \|v\|_{1,\Omega} \qquad \forall\, v \,\in\, C^1(\bar{\Omega})\,.$$

Analogously, it is proved that

$$|v(1)| \;\leq\; \sqrt{2}\, \|v\|_{1,\Omega} \qquad \forall\, v \,\in\, C^1(\bar{\Omega})\,,$$

and finally, the fact that $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ allows to show that both inequalities are extended to H. In this way, it follows that

$$\Big|\, b\, v(1) \,-\, a\, v(0) \,\Big| \;\leq\; \sqrt{2}\, (a \,+\, b)\, \|v\|_{1,\Omega} \qquad \forall\, v \,\in\, H\,,$$

which, together with (1.23), shows that F is bounded.

On the other hand, one of the main motivations for using mixed variational formulations, which also constitutes one of the most important features of this methodology, is the possibility of introducing additional variables (unknowns) having either a physical or mathematical interest, which usually depend on the original unknowns.

In order to illustrate the above, let us additionally define $\sigma := u'$ in Ω, so that the boundary value problem (1.21) is reformulated as the first order system:

$$\sigma = u' \quad \text{in} \quad \Omega\,, \quad -\sigma' + \kappa\, u = f \quad \text{in} \quad \Omega\,, \quad \sigma(0) = a\,, \quad \sigma(1) = b\,. \tag{1.24}$$

Note that we now have two unknowns, σ and u, and the Neumann boundary conditions for u become Dirichlet boundary conditions for σ. Then, multiplying the equation $\sigma = u'$ in Ω by $\tau \in H_0^1(\Omega)$ and integrating by parts, we arrive at:

$$\int_0^1 \sigma\,\tau \,+\, \int_0^1 u\,\tau' = 0 \qquad \forall\,\tau \in H_0^1(\Omega)\,.$$

In addition, multiplying $-\sigma' + \kappa\, u = f$ in Ω by $v \in L^2(\Omega)$, we obtain:

$$\int_0^1 \sigma'\,v \,-\, \kappa \int_0^1 u\,v = -\int_0^1 f\,v \qquad \forall\,v \in L^2(\Omega)\,.$$

In this way, a mixed variational formulation of (1.21) would be given, at first glance, by: Find $(\sigma, u) \in H^1(\Omega) \times L^2(\Omega)$ such that $\sigma(0) = a$, $\sigma(1) = b$,

$$\begin{aligned} \int_0^1 \sigma\,\tau \,+\, \int_0^1 u\,\tau' \quad &= \quad 0 && \forall\,\tau \in H_0^1(\Omega)\,, \\ \int_0^1 \sigma'\,v \,-\, \kappa \int_0^1 u\,v \quad &= \quad -\int_0^1 f\,v && \forall\,v \in L^2(\Omega)\,. \end{aligned} \tag{1.25}$$

However, the system (1.25) is not *symmetric* with respect to unknowns and test functions since σ lies in an affine space (translated from $H^1(\Omega)$) and the respective test function τ belongs to $H_0^1(\Omega)$. The above is caused by the fact that, differently from what happens for a primal formulation, the Neumann boundary conditions are not natural for the mixed formulations, reason for which they are given in this case the name of *essential boundary conditions*. In order to circumvent this difficulty, in what follows we proceed in two diferent ways.

TRANSLATION OF THE UNKNOWN σ.

Let us define the auxiliary function $\sigma_0(x) := a + (b-a)\,x \quad \forall\, x \in \Omega$ and the translated unknown $\tilde{\sigma} = \sigma - \sigma_0$. Note that σ_0 satisfies the boundary conditions from (1.24). It follows that the system (1.24) is rewritten as:

$$\begin{gathered} \tilde{\sigma} = u' - \sigma_0 \quad \text{in} \quad \Omega\,, \qquad -\tilde{\sigma}' + \kappa\, u = f + (b-a) \quad \text{in} \quad \Omega\,, \\ \tilde{\sigma}(0) = 0\,, \quad \tilde{\sigma}(1) = 0\,. \end{gathered} \tag{1.26}$$

Thus, proceeding as before, the mixed variational formulation of (1.26) reduces to: Find $(\tilde{\sigma}, u) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \int_0^1 \tilde{\sigma}\,\tau + \int_0^1 u\,\tau' &= -\int_0^1 \sigma_0\,\tau && \forall\, \tau \in H_0^1(\Omega)\,, \\ \int_0^1 \tilde{\sigma}'\,v - \kappa \int_0^1 u\,v &= -\int_0^1 f\,v - (b-a)\int_0^1 v && \forall\, v \in L^2(\Omega)\,, \end{aligned} \tag{1.27}$$

from which it is clear that the unknowns and tests functions live now in the same product space. Unfortunately, this procedure is not applicable, from a practical point of view, to higher dimensional problems. Indeed, in those cases it is known the existence of such function σ_0, but, in general, it is not possible to obtain it explicitly.

<u>Use of a Lagrange multiplier</u>.

Starting from the system (1.24), and instead of employing a test function $\tau \in H_0^1(\Omega)$, one simply considers a function $\tau \in H^1(\Omega)$ and introduces the auxiliary unknown (the Lagrange multiplier) $\varphi := (\varphi_1, \varphi_2) \in \mathbb{R}^2$, with $\varphi_1 := u(1)$ and $\varphi_2 := -u(0)$. This induces the weak imposition of the boundary conditions from (1.24) through the simple equation:

$$\psi \cdot (\sigma(0), \sigma(1)) = \psi \cdot (a, b) \qquad \forall\, \psi \in \mathbb{R}^2\,.$$

Consequently, and defining the spaces $H := H^1(\Omega)$ and $Q := L^2(\Omega) \times \mathbb{R}^2$, we arrive at the following mixed variational formulation of (1.24): Find $(\sigma, (u, \varphi)) \in H \times Q$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, (u, \varphi)) &= F(\tau) && \forall\, \tau \in H\,, \\ b(\sigma, (v, \psi)) - c((u, \varphi), (v, \psi)) &= G(v, \psi) && \forall\, (v, \psi) \in Q\,, \end{aligned} \tag{1.28}$$

where $a : H \times H \to \mathbb{R}$, $b : H \times Q \to \mathbb{R}$, and $c : Q \times Q \to \mathbb{R}$ are the bilinear forms defined by

$$a(\sigma, \tau) := \int_0^1 \sigma\,\tau \qquad \forall\, (\sigma, \tau) \in H \times H\,,$$

$$b(\tau,(v,\psi)) := \int_0^1 \tau' \, v \, - \, \psi \cdot (\tau(0),\tau(1)) \qquad \forall\, (\tau,(v,\psi)) \in H \times Q\,,$$

$$c((u,\varphi),(v,\psi)) := \kappa \int_0^1 u\, v \qquad \forall\, ((u,\varphi),(v,\psi)) \in Q \times Q\,,$$

$F : H \to \mathbb{R}$ is the null functional, and $G : Q \to \mathbb{R}$ is given by

$$G(v,\psi) := - \int_0^1 f\, v \, - \, \psi \cdot (a,b) \qquad \forall\, (v,\psi) \in Q\,.$$

The structure of the problems (1.27) and (1.28), in particular the one arising with $\kappa = 0$, corresponds to the typical form of a mixed variational formulation. The main aspects of the respective abstract theory are reviewed below in Chapter 2. Furthermore, we remark that the idea of introducing Lagrange multipliers to deal with essential boundary conditions will also be employed in subsequent chapters when analyzing more complex boundary value problems in 2D and 3D.

1.2.2 A model in $\mathbb{R}^n$

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz-continuous boundary Γ. Then, given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$ (see definition of $H^{1/2}(\Gamma)$ and further details in Section 1.3.2), we consider the Poisson problem:

$$- \Delta u = f \quad \text{in} \quad \Omega\,, \quad u = g \quad \text{on} \quad \Gamma\,. \tag{1.29}$$

The primal formulation of (1.29), which is derived by using one of the Green identities (cf. Corollary 1.2 or Theorem 1.8), reduces to: Find $u \in H^1(\Omega)$ such that $u = g$ on Γ and

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \forall\, v \in H_0^1(\Omega) \tag{1.30}$$

(see the beginning of Section 1.3 for the definitions of $H^1(\Omega)$ and $H_0^1(\Omega)$ in this n-dimensional case). Similarly to the analysis for (1.25), we remark here that the Dirichlet boundary condition is not natural but only essential for a primal formulation of (1.30). However, in what follows we show that it does become natural when the corresponding mixed formulation is utilized instead. In fact, defining the additional unknown $\boldsymbol{\sigma} = \nabla u$ in Ω, problem (1.29) is rewritten as the first order system:

$$\boldsymbol{\sigma} = \nabla u \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\boldsymbol{\sigma}) = -f \quad \text{in} \quad \Omega\,, \quad u = g \quad \text{in} \quad \Gamma\,.$$

Then, multiplying the equation $\boldsymbol{\sigma} = \nabla u$ in Ω by $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, integrating by parts, and using the Dirichlet boundary condition for u, we arrive at:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_\Omega u \,\mathrm{div}\,(\boldsymbol{\tau}) = \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega)\,, \tag{1.31}$$

where $\boldsymbol{\nu}$ is the normal vector exterior to Γ and $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (see definition of this duality in Section 1.3.4). We do recall here, which will be utilized below in Section 1.3.4, that

$$H(\mathrm{div};\Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^n : \quad \mathrm{div}\,(\boldsymbol{\tau}) \in L^2(\Omega) \right\}, \tag{1.32}$$

where $\mathrm{div}\,(\boldsymbol{\tau}) \in L^2(\Omega)$ is meant in the distributional sense, that is, that there exists $z \in L^2(\Omega)$ such that

$$- \int_\Omega \nabla \varphi \cdot \boldsymbol{\tau} = \int_\Omega z \, \varphi \qquad \forall\, \varphi \in C_0^\infty(\Omega)\,.$$

Furthermore, it is easy to show, using that $L^2(\Omega)$ is Hilbert, that $H(\mathrm{div};\Omega)$ endowed with the inner product

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathrm{div},\Omega} := \int_\Omega \left\{ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \mathrm{div}\,(\boldsymbol{\sigma})\, \mathrm{div}\,(\boldsymbol{\tau}) \right\} \qquad \forall\, \boldsymbol{\sigma},\ \boldsymbol{\tau} \in H(\mathrm{div};\Omega)\,,$$

and induced norm $\| \cdot \|_{\mathrm{div},\Omega}$, is also Hilbert. As announced above, further details on this space, including the proof of the integration by parts formula yielding (1.31), are presented in Section 1.3.4.

On the other hand, multiplying the equation $\mathrm{div}\,(\boldsymbol{\sigma}) = -f$ in Ω by $v \in L^2(\Omega)$, we get

$$\int_\Omega v \,\mathrm{div}\,(\boldsymbol{\sigma}) = - \int_\Omega f\, v \qquad \forall\, v \in L^2(\Omega)\,. \tag{1.33}$$

Therefore, the mixed variational formulation of (1.29) is obtained by gathering (1.31) and (1.33), which leads to: Find $(\boldsymbol{\sigma}, u) \in H \times Q$ such that

$$\begin{aligned} a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u) &= \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle && \forall\, \boldsymbol{\tau} \in H\,, \\ b(\boldsymbol{\sigma}, v) &= - \int_\Omega f\, v && \forall\, v \in Q\,, \end{aligned} \tag{1.34}$$

where $H := H(\mathrm{div};\Omega)$, $Q := L^2(\Omega)$, and $a : H \times H \to \mathbb{R}$, $b : H \times Q \to \mathbb{R}$, are the bilinear forms defined by:

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau} \qquad \forall\, (\boldsymbol{\sigma}, \boldsymbol{\tau}) \in H \times H\,,$$

$$b(\boldsymbol{\tau}, v) \,:=\, \int_\Omega v \,\mathrm{div}\,(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau}, v) \in H \times Q\,.$$

Similarly as mentioned for problems (1.27) and (1.28), the structure of (1.34) also corresponds to the typical form of a mixed variational formulation (see the corresponding details in Chapter 2).

As a final comment, and according to what has been observed with the examples of this section, we remark that Dirichlet and Neumann boundary conditions exchange their roles (natural versus essential) when primal and mixed variational formulations are employed. The following table summarizes this fact:

FORMULATION $\longrightarrow$	PRIMAL	MIXED
Dirichlet condition	essential	natural
Neumann condition	natural	essential

1.3 Traces and Green's identities

In this section we present some results on traces, integration by parts formulae, and Green's identities for some Sobolev spaces, and particularly for $H(\mathrm{div};\Omega)$. In what follows, given a bounded domain Ω of $\mathbb{R}^n$ with Lipschitz-continuous boundary Γ, the Sobolev space of order 1 is defined as:

$$H^1(\Omega) \,:=\, \left\{ v \,\in\, L^2(\Omega) : \quad \frac{\partial v}{\partial x_i} \,\in\, L^2(\Omega) \quad \forall\, i \,\in\, \{1, 2, ..., n\} \right\},$$

where $\dfrac{\partial v}{\partial x_i} \,\in\, L^2(\Omega)$ is meant in the distributional sense, that is, that there exists $z_i \in L^2(\Omega)$ such that

$$-\int_\Omega v \,\frac{\partial \varphi}{\partial x_i} \;=\; \int_\Omega z_i\,\varphi \qquad \forall\,\varphi \,\in\, C_0^\infty(\Omega)\,.$$

It is easy to show, using that $L^2(\Omega)$ is Hilbert, that $H^1(\Omega)$ endowed with the inner product

$$\langle v, w\rangle_{1,\Omega} \,:=\, \int_\Omega \Big\{ \nabla v \cdot \nabla w \,+\, vw \Big\} \quad \forall\, v,\, w \in H^1(\Omega)$$

is also Hilbert. The induced semi-norm and norm are given, respectively, by

$$|v|_{1,\Omega} \,:=\, \|\nabla v\|_{0,\Omega} \quad \mathrm{y} \quad \|v\|_{1,\Omega} \,:=\, \Big\{ |v|_{1,\Omega}^2 \,+\, \|v\|_{0,\Omega}^2 \Big\}^{1/2} \quad \forall\, v \in H^1(\Omega)\,.$$