A perturbed twofold saddle point-based mixed finite element method for the Navier-Stokes equations with variable viscosity[*]

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Abstract

This paper proposes and analyzes a new mixed variational formulation for the Navier-Stokes equations with constant density and variable viscosity that depends nonlinearly on the velocity gradient. Differently from previous works in which augmented terms are added to the formulation, the present approach employs a technique previously applied to the stationary Boussinesq problem and the Navier-Stokes equations with constant viscosity. Firstly, a modified pseudostress tensor is introduced involving the diffusive and convective terms and the pressure. Secondly, by using an equivalent statement suggested by the incompressibility condition, the pressure is eliminated, and the gradient of velocity is incorporated as an auxiliary unknown to handle the nonlinear viscosity. As a consequence, a Banach spaces-based formulation is obtained, which can be written as a perturbed twofold saddle point operator equation. We analyze the continuous and discrete solvability of this problem using a relevant abstract theory developed specifically for this purpose, by linearizing the perturbation and applying the classical Banach fixed point theorem. In particular, given an integer $\ell \geq 0$, feasible choices of finite element subspaces include piecewise polynomials of degree $\leq \ell$ for the gradient of velocity, Raviart-Thomas spaces of order ℓ for the pseudostress, and piecewise polynomials of degree $\leq \ell$ for the velocity. Finally, optimal a priori error estimates are derived, and several numerical results illustrating the good performance of the scheme and confirming the theoretical rates of convergence, are reported.

Key words: Navier-Stokes equations, nonlinear viscosity, Banach spaces, mixed finite element methods, a priori error analysis

Mathematics subject classifications (2000): 35J66, 65J15, 65N12, 65N15, 65N30, 47J26, 76D07.

1 Introduction

The development of mixed finite element techniques for Newtonian and non-Newtonian incompressible fluids has received special attention in recent years. The Navier-Stokes problem with variable viscosity

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refers to the mathematical description of the motion of a fluid whose viscosity coefficient is not constant but rather varies with respect to position and/or time. This problem is more complex than the conventional Navier-Stokes problem for Newtonian fluids with constant viscosity since it requires a more sophisticated numerical solution. In the context of augmented methods, a mixed finite element approach for solving the Navier-Stokes equations with a viscosity that depends non-linearly on the magnitude of the velocity gradient has been recently introduced and analyzed in [6, 5]. In the first approach, the modified pseudostress tensor used in [8] is employed, which, like the one from [21], involves diffusive and convective terms as well as pressure. The second approach takes into account the dependence of the viscosity on the strain rate tensor, resulting in a more physically relevant model that incorporates both deformation and vorticity as auxiliary unknowns. Additionally, in both works, the pressure unknown is eliminated through an equivalent statement implied by the incompressibility condition. In turn, due to the convective term, and in order to stay within a Hilbertian framework, the velocity is sought in the Sobolev space of order 1, which requires the incorporation into the variational formulation of additional Galerkin-type terms arising from the constitutive and equilibrium equations. Although, the augmentation procedure avoids the need to prove continuous and discrete inf-sup conditions, which yields, in particular, more flexibility for choosing finite element subspaces, the complexity of both the resulting system and its associated computational implementation increases considerably, leading to much more expensive schemes. This latter remark constitutes our main motivation to look now for non-augmented schemes.

In the context of nonlinear twofold saddle point operator equations, also known as dual-dual variational formulations, there has been a diverse range of theories developed over the past two decades. These theories arose from the need of applying dual-mixed methods to a class of nonlinear boundary value problems in continuum mechanics. In [14], the Babuška-Brezzi theory in Hilbert spaces is generalized to a class of nonlinear variational problems, and in [16], a natural extension of the abstract framework for continuous and discrete nonlinear twofold saddle point formulations is derived. More recently, a fully-mixed finite element method has been developed and analyzed for the coupling of the Stokes and Darcy-Forchheimer problems in [1]. This method was later extended to the coupling of the Navier-Stokes and Darcy-Forchheimer problems with constant density and viscosity in [10]. The main novelty of these works is the use of a new approach that leads to Banach spaces and a twofold saddle point structure for the equation of the corresponding operator. The continuous and discrete solubilities of this structure are analyzed in both papers using a suitable abstract theory developed for this purpose in the context of separable reflexive Banach spaces.

According to the previous discussion, the goal of the present paper is to extend the applicability of the Banach spaces framework discussed above by introducing a new fully-mixed formulation for the Navier-Stokes equations with constant density and variable viscosity, without any augmentation procedure. The analysis and results from [10] are used to achieve this goal. The paper proves the well-posedness and uniqueness of both the continuous and discrete formulations using a fixed point argument and an abstract theory for twofold saddle point problems. An a priori analysis is also performed, and optimal rates of convergence are derived. Given an integer $\ell \geq 0$, piecewise polynomials of degree $\leq \ell$ for the gradient of velocity, Raviart-Thomas spaces of order ℓ for the pseudostress, and piecewise polynomials of degree $\leq \ell$ for the velocity are feasible choices. The paper is structured as follows. In the rest of this section, we provide an overview of the standard notation and functional spaces that will be utilized throughout the paper. In Section 2 we introduce the model problem of interest and define the unknown to be considered in the variational formulation. Subsequently, in Section 3 we identify the twofold saddle point structure of the corresponding variational system. We then proceed to analyze the continuous solvability and the equivalent fixed point setting in Section 4, and present the corresponding well-posedness result, assuming sufficiently small data. In Section 5, we investigate the associated Galerkin scheme by utilizing a discrete version of the fixed point strategy developed in Section 4 for the continuous case. Additionally, we derive the associated a priori error estimate in the same section. Furthermore, in Section 6 we specify particular choices of discrete subspaces that satisfy the hypotheses from Section 4 and provide the rates of convergence of the Galerkin schemes. Finally, we present several numerical examples in Section 7, which illustrate the good performance of the fully mixed finite element method and confirm the theoretical rates of convergence.

Preliminary notations

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}_0^{l,t}(\Omega)$, with $l \geq 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, respectively, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

On the other hand, given $t \in (1,+\infty)$, we also introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}, \tag{1.1}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}, \tag{1.2}$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_t;\Omega), \tag{1.3}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega). \tag{1.4}$$

Then, proceeding as in [15, eq. (1.43), Section 1.3.4] (see also [7, Section 4.1] and [11, Section 3.1]), it is easy to show that for each $t \in \begin{cases} (1,+\infty) & \text{if} \quad n = 2 \\ [6/5,+\infty) & \text{if} \quad n = 3 \end{cases}$, there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v + v\,\mathrm{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v) \in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega), \tag{1.5}$$

and analogously

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle = \int_\Omega \Big\{\boldsymbol{\tau}:\nabla\mathbf{v} + \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega), \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$.

2 The model problem

2.1 The Navier-Stokes equations with variable viscosity

The Navier-Stokes problem with variable viscosity and constant density consists of finding the velocity $\mathbf{u}$ and the pressure p of a fluid occupying the region Ω, such that

$$\begin{gathered} -\mathrm{div}(\mu(|\nabla \mathbf{u}|)\nabla \mathbf{u}) + (\nabla \mathbf{u})\mathbf{u} + \nabla p \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \,=\, \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega p \,=\, 0\,, \end{gathered} \tag{2.1}$$

where the given data are a function $\mu : \mathrm{R}^+ \to \mathrm{R}$ describing the nonlinear viscosity, a volume force $\mathbf{f}$, and the boundary velocity $\mathbf{g}$. The right spaces to which $\mathbf{f}$ and $\mathbf{g}$ need to belong are specified later on. Note that $\mathbf{g}$ must formally satisfy the compatibility condition

$$\int_\Gamma \mathbf{g} \cdot \boldsymbol{\nu} \,=\, 0, \tag{2.2}$$

which arises from the incompressibility condition of the fluid, and that uniqueness of a pressure solution of (2.1) is ensured in the space

$$\mathrm{L}^2_0(\Omega) \,=\, \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q \,=\, 0 \right\}. \tag{2.3}$$

Furthermore, we assume that μ is of class C^1, and that there exist constants μ_1, $\mu_2 > 0$, such that

$$\mu_1 \,\le\, \mu(s) \,\le\, \mu_2 \quad \text{and} \quad \mu_1 \,\le\, \mu(s) + s\mu'(s) \,\le\, \mu_2 \qquad s \,\ge\, 0\,, \tag{2.4}$$

which, according to the result provided by [20, Theorem 3.8], implies Lipschitz continuity and strong monotonicity of the nonlinear operator induced by μ, which is defined later on (cf. (3.9)). We will go back to this fact in Section 4. Some examples of nonlinear μ are the following:

$$\mu(s) := 2 + \frac{1}{1+s} \quad \text{and} \quad \mu(s) \,:=\, \alpha_0 + \alpha_1(1+s^2)^{(\beta-2)/2}, \tag{2.5}$$

where $\alpha_0\,, \alpha_1 > 0$ and $\beta \in [1,2]$. The first example is basically academic but the second one corresponds to a particular case of the well-known Carreau law in fluid mechanics. It is easy to see that they both satisfy (2.4) with $(\mu_1,\mu_2) \,=\, (2,3)$ and $(\mu_1,\mu_2) \,=\, (\alpha_0,\alpha_0 + \alpha_1)$, respectively. The forthcoming analysis also applies to the slightly more general case of a viscosity function acting on $\Omega \times \mathrm{R}^+$. Next, proceeding similarly as in [6], we introduce the pseudostress tensor unknown, which is defined by

$$\boldsymbol{\sigma} := \, \mu(|\nabla \mathbf{u}|)\nabla \mathbf{u} - (\mathbf{u} \otimes \mathbf{u}) - p\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

In this way, noting that $\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) \,=\, (\nabla \mathbf{u})\mathbf{u}$, which makes uses of the fact that $\mathrm{div}(\mathbf{u}) \,=\, 0$, we find that the first equation of (2.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of (2.6), which can be understood as the constitutive equation expressing $\boldsymbol{\sigma}$ in terms of $\mathbf{u}$, that the latter and the incompressibility condition are equivalent to the pair

$$\begin{gathered} \boldsymbol{\sigma}^{\mathrm{d}} \,=\, \mu(|\nabla \mathbf{u}|)\nabla \mathbf{u} - (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \quad \text{and} \\ p \,=\, -\frac{1}{n}\mathrm{tr}\big(\boldsymbol{\sigma} \,+\, (\mathbf{u} \otimes \mathbf{u})\big) \quad \text{in} \quad \Omega\,. \end{gathered} \tag{2.7}$$

Thus, eliminating the pressure unknown which, anyway, can be approximated later on by the post-processed formula suggested in (2.7), we arrive, at first instance, at the following system of equations with unknowns $\mathbf{u}$ and $\boldsymbol{\sigma}$:

$$\begin{gathered}\boldsymbol{\sigma}^{\mathrm{d}} \,=\, \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) \,=\, 0\,.\end{gathered} \tag{2.8}$$

Finally, since we are interested in a mixed variational formulation of our nonlinear problem, and in order to employ the integration by parts formula typically required by this approach, we introduce the auxiliary unknown $\mathbf{t} := \nabla\mathbf{u}$ in Ω. Consequently, instead of (2.8), we consider from now the set of equations with unknowns $\mathbf{t}$, $\mathbf{u}$, and $\boldsymbol{\sigma}$, given by

$$\begin{gathered}\mathbf{t} \,=\, \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\sigma}^{\mathrm{d}} \,=\, \mu(|\mathbf{t}|)\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) \,=\, 0\,.\end{gathered} \tag{2.9}$$

3 The fully mixed formulation

In this section we derive a Banach spaces-based fully-mixed formulation of (2.9). The integration by parts formula provided by (1.6), along with the Cauchy-Schwarz and Hölder inequalities, play a key role in this derivation. We begin by looking originally for $\mathbf{u} \in \mathbf{H}^1(\Omega)$. Then, multiplying the first equation of (2.9) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, with $t \in \begin{cases} (1,+\infty) & \text{if} \quad n \,=\, 2 \\ [6/5,+\infty) & \text{if} \quad n \,=\, 3 \end{cases}$, applying the integration by parts formula (1.6), and using the Dirichlet boundary conditions for $\mathbf{u}$, which implicity assumes that $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we find

$$\int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{3.1}$$

It is clear from (3.1) that its first term is well defined for $\mathbf{t} \in \mathbb{L}^2(\Omega)$, which, along with the free trace property of $\mathbf{t}$, suggests to look for $\mathbf{t} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \,:=\, \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) \,=\, 0 \Big\}\,.$$

In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, we realize from the second term and Hölder's inequality that it suffices to look for $\mathbf{u} \in \mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. Next, it follows from the second equation of (2.9), that formally

$$\int_\Omega \mu(|\mathbf{t}|)\mathbf{t} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} - \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s} \,=\, 0 \qquad \forall\,\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.2}$$

from which we notice that the first term is well-defined, whereas the second one makes sense if $\boldsymbol{\sigma}$ is sought in $\mathbb{L}^2(\Omega)$. In turn, for the third one there holds

$$\left|\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s}\right| \,=\, \left|\int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \mathbf{s}\right| \,\leq\, \|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{u}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,, \tag{3.3}$$

which, necessarily yields $t' \,=\, 4$, and thus $t \,=\, 4/3$. Finally, looking for $\boldsymbol{\sigma}$ in the same space of its corresponding test function $\boldsymbol{\tau}$, that is $\boldsymbol{\sigma} \in \mathbb{H}(\mathrm{div}_{4/3};\Omega)$, the equilibrium equation in (2.9) is tested as

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,=\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall\,\mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{3.4}$$

which forces $\mathbf{f}$ to belong to $\mathbf{L}^{4/3}(\Omega)$. Now we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{3.5}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0 \right\}.$$

It follows that $\boldsymbol{\sigma}$ can be uniquely decomposed as $\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 + c_0\mathbb{I}$, where, according to the third equation of the second row of (2.9),

$$\boldsymbol{\sigma}_0 \,\in\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 \,:=\, \frac{1}{n|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, -\frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,. \tag{3.6}$$

In this way, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that (3.2) and (3.4) remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. In addition, thanks to the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using the compatibility condition (2.2), we realize that testing (3.1) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Thus, redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and suitably gathering (3.1), (3.2) and (3.4), we arrive at the following mixed formulation: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \times \mathbf{L}^4(\Omega)$ such that

$$\begin{array}{llll}
\displaystyle\int_\Omega \mu(|\mathbf{t}|)\mathbf{t} : \mathbf{s} \;-\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} & \displaystyle -\int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s} & =\, 0 & \forall\, \mathbf{s} \in \mathbb{L}^2\mathrm{tr}(\Omega), \\[2ex]
\displaystyle -\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \mathbf{t} & \displaystyle -\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) & =\, -\langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle_\Gamma & \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \\[2ex]
\displaystyle \qquad\qquad\; -\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) & & \displaystyle =\, \int_\Omega \mathbf{f}\cdot\mathbf{v} & \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega).
\end{array} \tag{3.7}$$

Next, we observe that (3.7) has a perturbed twofold saddle point structure. Indeed, we first define the Banach spaces

$$\mathbb{H}_1 \,:=\, \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \mathbb{H}_2 \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \quad \text{and} \quad \mathbf{Q} \,:=\, \mathbf{L}^4(\Omega)\,, \tag{3.8}$$

which are endowed with the norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{\mathbf{div}_{4/3};\Omega}$, and $\|\cdot\|_{0,4;\Omega}$, respectively. Next, we introduce the nonlinear operator $\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$, and the bounded linear operators $\mathcal{B}_1 : \mathbb{H}_1 \to \mathbb{H}_2'$ and $\mathcal{B} : \mathbb{H}_2 \to \mathbf{Q}'$, given by

$$\begin{aligned}
[\mathcal{A}(\mathbf{r}), \mathbf{s}] &:= \int_\Omega \mu(|\mathbf{r}|)\mathbf{r} : \mathbf{s} && \forall\, \mathbf{r}, \mathbf{s} \in \mathbb{H}_1, \\
[\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}] &:= -\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \mathbf{s} && \forall\, (\mathbf{s}, \boldsymbol{\tau}) \in \mathbb{H}_1 \times \mathbb{H}_2, \\
[\mathcal{B}(\zeta), \mathbf{v}] &:= -\int_\Omega \mathbf{v}\cdot\mathbf{div}(\zeta) && \forall\, (\zeta, \mathbf{v}) \in \mathbb{H}_2 \times \mathbf{Q}.
\end{aligned} \tag{3.9}$$

Hereafter, $[\cdot\,,\cdot]$ stands for the duality pairing between the corresponding Banach space involved and its dual. In turn, $\mathrm{G} \in \mathbb{H}_2'$, and $\mathrm{F} \in \mathbf{Q}'$ are the bounded linear functionals defined by

$$[\mathrm{G}, \boldsymbol{\tau}] \,:=\, -\langle \boldsymbol{\tau\nu}, \mathbf{g}\rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2\,, \tag{3.10}$$

and

$$[\mathrm{F}, \mathbf{v}] \,:=\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,. \tag{3.11}$$

Regarding the boundedness of G and F, we first observe, using the identity (1.6) and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, that

$$\big|[\mathrm{G}, \boldsymbol{\tau}]\big| \,\leq\, \max\big\{1, \|\mathbf{i}_4\|\big\} \|\mathbf{g}\|_{1/2,\Gamma}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2\,. \tag{3.12}$$

In addition, it follows by Hölder's inequality that

$$\big|[\mathrm{F}, \mathbf{v}]\big| \,\leq\, \|\mathbf{f}\|_{0,4/3;\Omega}\, \|\mathbf{v}\|_{0,4;\Omega} \quad \forall\, \mathbf{v} \in \mathbf{Q}\,. \tag{3.13}$$

According to the above, the fully mixed formulation (3.7) can be rewritten as: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ such that

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}), \mathbf{s}] \quad + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma}] \quad - \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} : \mathbf{s} &&= 0 &&\forall\, \mathbf{s} \in \mathbb{H}_1,\\
&[\mathcal{B}_1(\mathbf{t}), \boldsymbol{\tau}] \qquad\qquad\qquad + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u}] &&= [\mathrm{G}, \boldsymbol{\tau}] &&\forall\, \boldsymbol{\tau} \in \mathbb{H}_2,\\
&\qquad\qquad [\mathcal{B}(\boldsymbol{\sigma}), \mathbf{v}] &&= [\mathrm{F}, \mathbf{v}] &&\forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned} \tag{3.14}$$

4 The continuous solvability analysis

In this section, we analyze the solvability of (3.14). For this purpose, we employ the recently developed theory, in the context of reflexive and separable Banach spaces, which is described in [10, Theorem 3.4].

4.1 The fixed-point strategy

We begin by rewriting (3.14) as an equivalent fixed point equation. To this end, we proceed to linearize the perturbation (third term of the first equation of (3.14)) defining for each $\mathbf{w} \in \mathbf{Q}$ the functional $\mathrm{H}_{\mathbf{w}} : \mathbb{H}_1 \to \mathrm{R}$ by

$$[\mathrm{H}_{\mathbf{w}}, \mathbf{s}] \,:=\, \int_\Omega (\mathbf{w}\otimes\mathbf{w})^{\mathrm{d}} : \mathbf{s} \qquad \forall\, \mathbf{s} \in \mathbb{H}_1, \tag{4.1}$$

and let $\mathbf{T} : \mathbf{Q} \to \mathbf{Q}$ be the operator given by

$$\mathbf{T}(\mathbf{w}) \,=\, \mathbf{u} \qquad \forall\, \mathbf{w} \in \mathbf{Q}, \tag{4.2}$$

where $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ is the unique solution (to be proved later on) of the following system of equations:

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}), \mathbf{s}] \quad + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma}] &&= [\mathrm{H}_{\mathbf{w}}, \mathbf{s}] &&\forall\, \mathbf{s} \in \mathbb{H}_1,\\
&[\mathcal{B}_1(\mathbf{t}), \boldsymbol{\tau}] \qquad\qquad\qquad + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u}] &&= [\mathrm{G}, \boldsymbol{\tau}] &&\forall\, \boldsymbol{\tau} \in \mathbb{H}_2,\\
&\qquad\qquad [\mathcal{B}(\boldsymbol{\sigma}), \mathbf{v}] &&= [\mathrm{F}, \mathbf{v}] &&\forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned} \tag{4.3}$$

Thus, we realize that solving (3.14) is equivalent to finding a fixed point of $\mathbf{T}$, that is $\mathbf{u} \in \mathbf{Q}$ such that

$$\mathbf{T}(\mathbf{u}) \,=\, \mathbf{u}.$$

4.2 Well-definedness of the operator $\mathbf{T}$

We continue by establishing the well-definedness of the operator $\mathbf{T}$, equivalently, that problem (4.3) is well-posed. To this end, we employ the abstract theory mentioned above for this type of twofold saddle-point operator equation. More precisely, let $\mathbb{H}_1$, $\mathbb{H}_2$, and $\mathbf{Q}$ be separable and reflexive Banach spaces so that theirs duals $\mathbb{H}_1'$, $\mathbb{H}_2'$, and $\mathbf{Q}'$ are separable and reflexive as well. In addition, let $\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$ be a nonlinear operator, and let $\mathcal{B}_1 : \mathbb{H}_1 \to \mathbb{H}_2'$ and $\mathcal{B} : \mathbb{H}_2 \to \mathbf{Q}'$ be bounded linear operators. Then, given $(\mathrm{H}, \mathrm{G}, \mathrm{F}) \in \mathbb{H}_1' \times \mathbb{H}_2' \times \mathbf{Q}'$, we are interested in the following nonlinear variational problem: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ such that

$$
\begin{aligned}
&[\mathcal{A}(\mathbf{t}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma}] &&= [\mathrm{H}, \mathbf{s}] \quad &&\forall\, \mathbf{s} \in \mathbb{H}_1, \\
&[\mathcal{B}_1(\mathbf{t}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u}] &&= [\mathrm{G}, \boldsymbol{\tau}] \quad &&\forall\, \boldsymbol{\tau} \in \mathbb{H}_2, \\
&[\mathcal{B}(\boldsymbol{\sigma}), \mathbf{v}] &&= [\mathrm{F}, \mathbf{v}] \quad &&\forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned} \tag{4.4}
$$

The following abstract theorem establishes sufficient conditions for the well-posedness of (4.4).

Theorem 4.1. *Let $\mathcal{K} := N(\mathcal{B})$ and assume that*

i) *$\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$ is Lipschitz continuous, that is there exists a constant $\gamma > 0$ such that*

$$\|\mathcal{A}(\mathbf{r}) - \mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \le \gamma\, \|\mathbf{r} - \mathbf{s}\|_{\mathbb{H}_1} \quad \forall\, \mathbf{r}, \mathbf{s} \in \mathbb{H}_1, \tag{4.5}$$

ii) *for each $\mathbf{s} \in \mathbb{H}_1$, the family of operators $\mathcal{A}(\cdot + \mathbf{s}) : \mathbb{H}_1 \to \mathbb{H}_1'$ is strictly monotone with a monotonicity constant $\alpha > 0$, independent of $\mathbf{s}$, that is*

$$[\mathcal{A}(\mathbf{t} + \mathbf{s}) - \mathcal{A}(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \ge \alpha\, \|\mathbf{t} - \mathbf{r}\|^2_{\mathbb{H}_1} \quad \forall\, \mathbf{t}, \mathbf{r} \in \mathbb{H}_1, \tag{4.6}$$

iii) *there exists a positive constant β such that*

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_2 \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}), \mathbf{v}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \ge \beta\, \|\mathbf{v}\|_{\mathbf{Q}} \quad \forall\, \mathbf{v} \in \mathbf{Q}, \quad \textit{and} \tag{4.7}$$

iv) *there exists a positive constant β_1 such that*

$$\sup_{\substack{\mathbf{s} \in \mathbb{H}_1 \\ \mathbf{s} \neq \mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} \ge \beta_1\, \|\boldsymbol{\tau}\|_{\mathbb{H}_2} \quad \forall\, \boldsymbol{\tau} \in \mathcal{K}. \tag{4.8}$$

Then, for each $(\mathrm{H}, \mathrm{G}, \mathrm{F}) \in \mathbb{H}_1' \times \mathbb{H}_2' \times \mathbf{Q}'$ there exists a unique $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ solution of (4.4). Moreover, there exists a constant $C > 0$, depending only on $\gamma, \alpha, \beta, \beta_1, \|\mathcal{B}_1\|$ and $\|\mathcal{B}_1'\|$, such that

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \le C \Big\{ \|\mathrm{H}\|_{\mathbb{H}_1'} + \|\mathrm{G}\|_{\mathbb{H}_2'} + \|\mathrm{F}\|_{\mathbf{Q}'} + \|\mathcal{A}(\mathbf{0})\|_{\mathbb{H}_1'} \Big\}. \tag{4.9}$$

Proof. It follows from a straightforward application of [10, Theorem 3.4] to the particular case of the exponents $p_1 = p_2 = 2$ considered there. $\square$

Now, if $\mathcal{A}$ becomes linear, the above theorem reduces to the following.

Theorem 4.2. *Let $\mathcal{K} := N(\mathcal{B})$ and assume that* *i)* $\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$ *is linear, bounded and* $\mathbb{H}_1$*-elliptic, that is, there exist constants* $\gamma, \alpha > 0$ *such that*

$$\|\mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \,\leq\, \gamma\,\|\mathbf{s}\|_{\mathbb{H}_1} \qquad \text{and} \qquad [\mathcal{A}(\mathbf{s}), \mathbf{s}] \,\geq\, \alpha\,\|\mathbf{s}\|^2_{\mathbb{H}_1} \quad \forall\, \mathbf{s} \in \mathbb{H}_1, \text{ and} \tag{4.10}$$

ii) the hypotheses iii)-iv) *of Theorem* 4.1 *are satisfied.*

Then, for each $(\mathrm{H}, \mathrm{G}, \mathrm{F}) \in \mathbb{H}_1' \times \mathbb{H}_2' \times \mathbf{Q}'$ *there exists a unique* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ *solution of* (4.4)*. Moreover, there exists a constant* $C > 0$*, depending only on* γ*,* α*,* β*,* β_1*,* $\|\mathcal{B}_1\|$ *and* $\|\mathcal{B}_1'\|$*, such that*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \,\leq\, C \left\{ \|\mathrm{H}\|_{\mathbb{H}_1'} + \|\mathrm{G}\|_{\mathbb{H}_2'} + \|\mathrm{F}\|_{\mathbf{Q}'} \right\}. \tag{4.11}$$

Proof. It suffices to observe that the linearity, boundedness, and ellipticity of $\mathcal{A}$ imply that this operator is Lipschitz continuous and strongly monotone, and that $\mathcal{A}(0) = \mathbf{0}$. Therefore, the proof follows from a straightforward application of Theorem 4.1. $\square$

We remark here that (4.11) is equivalent to an inf-sup condition for the bilinear form arising after adding the left-hand sides of (4.4) in the linear case of $\mathcal{A}$. More precisely, letting

$$[\mathcal{S}(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}), (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})] \,:=\, [\mathcal{A}(\mathbf{r}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}), \mathbf{v}]\,, \tag{4.12}$$

for all $(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}), (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_1' \times \mathbb{H}_2' \times \mathbf{Q}$, there exists a constant $C > 0$, depending only on γ, α, β, β_1, $\|\mathcal{B}_1\|$ and $\|\mathcal{B}_1'\|$, such that

$$\sup_{\substack{(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q} \\ (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \neq \mathbf{0}}} \frac{[\mathcal{S}(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}), (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})]}{\|(\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}}} \,\geq\, C \|(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \tag{4.13}$$

for all $(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$.

We now verify that problem (4.3) satisfies the hypotheses of Theorem 4.1. To this end, in what follows we establish the Lipschitz continuity and strong monotonicity of $\mathcal{A}$, as well as the continuous inf-sup conditions for $\mathcal{B}$ and $\mathcal{B}_1$.

Lemma 4.3. *Let* $\gamma_\mu := \max\{\mu_2, 2\mu_2 - \mu_1\}$*, where* μ_1 *and* μ_2 *are the bounds of* μ *given in* (2.4)*. Then, for each* $\mathbf{r}, \mathbf{s} \in \mathbb{L}^2(\Omega)$*, there hold the following inequalities:*

$$\|\mathcal{A}(\mathbf{r}) - \mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \,\leq\, \gamma_\mu\, \|\mathbf{r} - \mathbf{s}\|_{\mathbb{H}_1}, \tag{4.14}$$

and

$$[\mathcal{A}(\mathbf{r}) - \mathcal{A}(\mathbf{s}), \mathbf{r} - \mathbf{s}]_{\mathbb{H}_1'} \,\geq\, \mu_1\, \|\mathbf{r} - \mathbf{s}\|^2_{\mathbb{H}_1}. \tag{4.15}$$

Proof. See [20, Theorem 3.8] for details. $\square$

Lemma 4.4. *There exists a constant* $\beta > 0$*, such that*

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_2 \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}), \mathbf{v}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \,\geq\, \beta\, \|\mathbf{v}\|_{\mathbf{Q}} \quad \forall\, \mathbf{v} \in \mathbf{Q}. \tag{4.16}$$

Proof. See [11, Lemma 3.3] or [4, Lemma 3.5] for details. $\square$

In turn, in order to prove that $\mathcal{B}_1$ satisfies hypothesis iv), we need to recall a useful estimate for tensors in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Indeed, suitably modifying the proof of [15, Lemma 2.3], one can show that there exists a positive constant c_1, depending only on Ω, such that

$$c_1 \, \|\boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega). \tag{4.17}$$

Lemma 4.5. *There exists a constant $\beta_1 > 0$, such that*

$$\sup_{\substack{\mathbf{s}\in\mathbb{H}_1\\ \mathbf{s}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} \,\geq\, \beta_1 \, \|\boldsymbol{\tau}\|_{\mathbb{H}_2} \quad \forall\, \boldsymbol{\tau} \in \mathcal{K}\,. \tag{4.18}$$

Proof. In order to satisfy the continuous inf-sup condition for $\mathcal{B}_1$, it is necessary to first realize t hat $\mathcal{K} := N(\mathcal{B})$ (cf. (3.9)), is given by

$$\mathcal{K} \,=\, \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \,=\, \mathbf{0} \quad \text{in } \Omega \Big\}\,. \tag{4.19}$$

Then, given $\boldsymbol{\tau} \in \mathcal{K}$ such that $\boldsymbol{\tau}^{\mathtt{d}} \neq \mathbf{0}$, we have that $\boldsymbol{\tau}^{\mathtt{d}} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, so that bounding the supremum in (4.18) by below with $\mathbf{s} \,=\, -\boldsymbol{\tau}^{\mathtt{d}}$, it follows that

$$\sup_{\substack{\mathbf{s}\in\mathbb{H}_1\\ \mathbf{s}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} \,=\, \sup_{\substack{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ \mathbf{s}\neq\mathbf{0}}} \frac{-\displaystyle\int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \mathbf{s}}{\|\mathbf{s}\|_{0,\Omega}} \,\geq\, \frac{\displaystyle\int_\Omega \boldsymbol{\tau}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}}}{\|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,\Omega}} \,=\, \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0;\Omega}\,,$$

which, using (4.17) and the fact that $\mathbf{div}(\boldsymbol{\tau}) \,=\, \mathbf{0}$, implies that $\mathcal{B}_1$ satisfies the inf-sup condition with a constant $\beta_1 \,=\, c_1$. On the other hand, if $\boldsymbol{\tau}^{\mathtt{d}} \,=\, \mathbf{0}$, it is clear from (4.17) that $\boldsymbol{\tau} \,=\, \mathbf{0}$, and so (4.18) is trivially satisfied. □

Consequently, the well-definedness of the operator $\mathbf{T}$ can be stated as follows.

Theorem 4.6. *For each $\mathbf{w} \in \mathbf{Q}$ there exists a unique $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ solution to (4.4), and hence we can define $\mathbf{T}(\mathbf{w}) := \mathbf{u} \in \mathbf{Q}$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, depending only on $\gamma_\mu, \mu_1, \beta, \beta_1, \|\mathcal{B}_1\|$, $\|\mathcal{B}_1'\|$, and $\|\mathbf{i}_4\|$, and hence independent of $\mathbf{w}$, such that*

$$\|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} \,=\, \|\mathbf{u}\|_{\mathbf{Q}} \,\leq \|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \,\leq\, C_{\mathbf{T}} \Big\{ \|\mathbf{w}\|^2_{0,4;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \Big\}\,. \tag{4.20}$$

Proof. It follows from Lemmas 4.3-4.5 and a straightforward application of Theorem 4.1. In turn, estimate (4.20) is a direct consequence of (4.9) (cf. Theorem 4.1) and the boundedness of G (cf. (3.12)) and F (cf. (3.13)). □

4.3 Solvability analysis of the fixed-point scheme

Knowing that the operator $\mathbf{T}$ is well-defined, in this section we address the solvability of the fixed-point equation (4.2). To this end, in what follows we first derive sufficient conditions on $\mathbf{T}$ to map a closed ball of $\mathbf{Q}$ into itself, and then we apply the Banach Theorem to conclude the unique solvability of (4.2). Indeed, given $\delta > 0$, from now on we let

$$\mathrm{W}(\delta) \,:=\, \Big\{ \mathbf{w} \in \mathbf{Q} : \quad \|\mathbf{w}\|_{0,4;\Omega} \,\leq\, \delta \Big\}\,. \tag{4.21}$$

Lemma 4.7. *Assume that* $\delta \leq \dfrac{1}{2C_{\mathbf{T}}}$ *and that there holds*

$$C_{\mathbf{T}}\Big\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\} \leq \frac{\delta}{2}. \tag{4.22}$$

Then $\mathbf{T}\big(\mathrm{W}(\delta)\big) \subseteq \mathrm{W}(\delta)$.

Proof. Given $\mathbf{w} \in \mathrm{W}(\delta)$, we know from Theorem 4.6 that $\mathbf{T}(\mathbf{w})$ is well defined and that there holds

$$\|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} \leq C_{\mathbf{T}}\left\{\|\mathbf{w}\|^2_{0,4;\Omega} + \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \leq C_{\mathbf{T}}\delta^2 + \frac{\delta}{2} \leq \delta,$$

which confirms that $\mathbf{T}(\mathbf{w}) \in W(\delta)$. $\square$

We continue with the continuity property of the operator $\mathbf{T}$.

Lemma 4.8. *There exists a positive constant* $\mathrm{L}_{\mathbf{T}}$*, depending only on* β*,* $\|\mathcal{B}_1\|$*, and* μ_1*, such that*

$$\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widehat{\mathbf{w}})\|_{0,4;\Omega} \leq \mathrm{L}_{\mathbf{T}}\left\{\|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega}\right\}\|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega} \tag{4.23}$$

for all $\mathbf{w}, \widehat{\mathbf{w}} \in \mathbf{Q}$.

Proof. Given $\mathbf{w}, \widehat{\mathbf{w}} \in \mathbf{Q}$, we let $\mathbf{T}(\mathbf{w}) := \mathbf{u}$ and $\mathbf{T}(\widehat{\mathbf{w}}) := \widehat{\mathbf{u}}$, where $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ and $(\widehat{\mathbf{t}}, \widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ are the corresponding unique solutions of (4.3). Then, subtracting both systems, we obtain

$$\begin{array}{llll}
[\mathcal{A}(\mathbf{t}) - \mathcal{A}(\widehat{\mathbf{t}}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}] & & = [\mathrm{H}_{\mathbf{w}} - \mathrm{H}_{\widehat{\mathbf{w}}}, \mathbf{s}] & \forall\, \mathbf{s} \in \mathbb{H}_1, \\
[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\tau}] & + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u} - \widehat{\mathbf{u}}] & = 0 & \forall\, \boldsymbol{\tau} \in \mathbb{H}_2, \\
\qquad [\mathcal{B}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}), \mathbf{v}] & & = 0 & \forall\, \mathbf{v} \in \mathbf{Q}.
\end{array} \tag{4.24}$$

In particular, taking $\mathbf{s} = \mathbf{t} - \widehat{\mathbf{t}}$ and $\boldsymbol{\tau} = \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}$, we realize from the second and third equations of (4.24) that

$$[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}] = -[\mathcal{B}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}), \mathbf{u} - \widehat{\mathbf{u}}] = 0\,,$$

which, along with the first equation of (4.24), yields

$$[\mathcal{A}(\mathbf{t}) - \mathcal{A}(\widehat{\mathbf{t}}), \mathbf{t} - \widehat{\mathbf{t}}] = [\mathrm{H}_{\mathbf{w}} - \mathrm{H}_{\widehat{\mathbf{w}}}, \mathbf{t} - \widehat{\mathbf{t}}]\,,$$

whence, using the stric monotonicity of $\mathcal{A}$ (cf. (4.15)) and the definition of $\mathrm{H}_{\mathbf{w}}$ (cf. (4.1)), we find that

$$\|\mathbf{t} - \widehat{\mathbf{t}}\|_{0,\Omega} \leq \frac{1}{\mu_1}\left\{\|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega}\right\}\|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega}\,. \tag{4.25}$$

In turn, from Lemma 4.4 and the second equation of (4.24), we bound $\|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega}$ as follows:

$$\|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega} \leq \frac{1}{\beta}\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_2 \\ \boldsymbol{\tau} \neq \mathbf{0}}}\frac{[\mathcal{B}(\boldsymbol{\tau}), \mathbf{u} - \widehat{\mathbf{u}}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} = \frac{1}{\beta}\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_2 \\ \boldsymbol{\tau} \neq \mathbf{0}}}\frac{[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \leq \frac{\|\mathcal{B}_1\|}{\beta}\|\mathbf{t} - \widehat{\mathbf{t}}\|_{0,\Omega}\,. \tag{4.26}$$

Finally, by combining (4.25) and (4.26), we have that

$$\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widehat{\mathbf{w}})\|_{0,4;\Omega} = \|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega} \leq \frac{\|\mathcal{B}_1\|}{\beta\mu_1}\left\{\|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega}\right\}\|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega}\,,$$

which confirms the announced property on $\mathbf{T}$ (cf. (4.23)) with $\mathrm{L}_{\mathbf{T}} := \dfrac{\beta\mu_1}{\|\mathcal{B}_1\|}$. $\square$

Owing to the above analysis, we now establish the main result of this section.

Theorem 4.9. *Assume that* $\delta < \dfrac{1}{2}\min\left\{\dfrac{1}{C_{\mathbf{T}}}, \dfrac{\beta\mu_1}{\|\mathcal{B}_1\|}\right\}$ *and the data are sufficiently small so that the hypothesis of Lemma 4.7 holds, that is*

$$C_{\mathbf{T}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \leq \frac{\delta}{2}. \tag{4.27}$$

Then, the operator $\mathbf{T}$ *has a unique fixed point* $\mathbf{u} \in \mathrm{W}(\delta)$. *Equivalently, the problem (3.14) has a unique solution* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$. *Moreover, there holds*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \leq 2C_{\mathbf{T}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\}. \tag{4.28}$$

Proof. We first recall that the choice of δ and assumption (4.27) guarantee, thanks to Lemma 4.7, that $\mathbf{T}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W}(\delta) \to \mathrm{W}(\delta)$ (cf. (4.23)), a straightforward application of the classical Banach theorem yields the existence of a unique fixed point $\mathbf{u} \in \mathrm{W}(\delta)$ of this operator, and hence a unique solution of (3.14). Finally, regarding the a priori estimate, we first observe from (4.20) that

$$\|\mathbf{T}(\mathbf{u})\|_{0,4;\Omega} = \|\mathbf{u}\|_{0,4;\Omega} \leq \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \leq C_{\mathbf{T}}\left\{\|\mathbf{u}\|_{0,4;\Omega}^2 + \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\}, \tag{4.29}$$

from which, using that

$$\|\mathbf{u}\|_{0,4;\Omega}^2 \leq \delta\,\|\mathbf{u}\|_{0,4;\Omega} \leq \frac{1}{2C_{\mathbf{T}}}\,\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}},$$

we arrive at

$$\begin{aligned}\|\mathbf{T}(\mathbf{u})\|_{0,4;\Omega} &= \|\mathbf{u}\|_{0,4;\Omega} \leq \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \\ &\leq \frac{1}{2}\,\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} + C_{\mathbf{T}}\left\{\|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\right\},\end{aligned}$$

which yields (4.28) and concludes the proof. □

5 The Galerkin scheme

In order to approximate the solution of our fully-mixed variational formulation (3.14), we now introduce the associated Galerkin scheme, analyze its solvability by applying a discrete version of the fixed-point approach adopted in the previous section, and derive the corresponding a priori error estimate.

5.1 Preliminaries

We begin by introducing finite element subspaces $\mathbb{H}_{1,h}$, $\widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $\mathbf{L}^4(\Omega)$, respectively. Hereafter, $h := \max\{h_K : K \in \mathcal{T}_h\}$ denotes the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K. Then, letting

$$\mathbb{H}_{2,h} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \cap \widetilde{\mathbb{H}}_{2,h}, \tag{5.1}$$

the Galerkin scheme associated with (3.14) reads: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ such that

$$
\begin{aligned}
&[\mathcal{A}(\mathbf{t}_h), \mathbf{s}_h] \quad + [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] \quad - \int_\Omega (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathtt{d}} : \mathbf{s}_h \quad &&= \, 0 \qquad &&\forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{1,h}, \\
&[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] \qquad\qquad\qquad\qquad + [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] &&= \, [\mathrm{G}, \boldsymbol{\tau}_h] &&\forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{2,h}, \\
&\qquad\qquad [\mathcal{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= \, [\mathrm{F}, \mathbf{v}_h] &&\forall\, \mathbf{v}_h \in \mathbf{Q}_h.
\end{aligned}
\tag{5.2}
$$

Then, we adopt the discrete version of the strategy employed in Section 4.2 to analyse the solvability of (5.2). To this end, we let $\mathbf{T}_h : \mathbf{Q}_h \to \mathbf{Q}_h$ be the discrete operator defined by

$$
\mathbf{T}_h(\mathbf{w}_h) \, = \, \mathbf{u}_h \qquad \forall\, \mathbf{w}_h \in \mathbf{Q}_h, \tag{5.3}
$$

where $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the following system of equations:

$$
\begin{aligned}
&[\mathcal{A}(\mathbf{t}_h), \mathbf{s}_h] \quad +[\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &&= \, [\mathrm{H}_{\mathbf{w}_h}, \mathbf{s}_h] \qquad &&\forall\, \mathbf{s}_h \in \mathbb{H}_{1,h}, \\
&[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] \qquad\qquad\qquad + [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] &&= \, [\mathrm{G}, \boldsymbol{\tau}_h] &&\forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{2,h}, \\
&\qquad\qquad [\mathcal{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= \, [\mathrm{F}, \mathbf{v}_h] &&\forall\, \mathbf{v}_h \in \mathbf{Q}_h.
\end{aligned}
\tag{5.4}
$$

Then, similarly as in the continuous case, we realize that solving (5.2) is equivalent to finding a fixed point of $\mathbf{T}_h$, that is $\mathbf{u}_h \in \mathbf{Q}_h$ such that

$$
\mathbf{T}_h(\mathbf{u}_h) \, = \, \mathbf{u}_h. \tag{5.5}
$$

5.2 Discrete solvability analysis

In this section we proceed analogously to Sections 4.2 and 4.3 and establish the well-posedness of the discrete system (5.2), equivalently of (5.5). To this end, we need to introduce certain hypotheses concerning the arbitrary spaces $\mathbb{H}_{1,h}, \widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$, and the discrete kernel associated with the linear operator $\mathcal{B}$, that is

$$
\mathcal{K}_h \, := \, \Big\{ \boldsymbol{\tau} \in \mathbb{H}_{2,h} : \quad [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h] \, = \, 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \Big\}\,. \tag{5.6}
$$

More precisely, from now on we assume that:

(**H.0**) $\widetilde{\mathbb{H}}_{2,h}$ contains the multiplies of the identity tensor $\mathbb{I}$,

(**H.1**) $\mathbf{div}\big(\widetilde{\mathbb{H}}_{2,h}\big) \, \subseteq \, \mathbf{Q}_h$,

(**H.2**) $\mathcal{K}_h^{\mathtt{d}} \, := \, \Big\{ \boldsymbol{\tau}_h^{\mathtt{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{K}_h \Big\} \subseteq \mathbb{H}_{1,h}$, and

(**H.3**) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$
\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_{2,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{H}_2}} \, \geq \, \boldsymbol{\beta}_{\mathtt{d}} \, \|\mathbf{v}_h\|_{\mathbf{Q}} \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,. \tag{5.7}
$$

We highlight here that as a consequence of (**H.0**) we can employ the discrete version of the decomposition $\mathbb{H}(\mathbf{div}_{4/3}; \Omega) \, = \, \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \, \oplus \, \mathrm{R}\,\mathbb{I}$, namely $\widetilde{\mathbb{H}}_{2,h} \, = \, \mathbb{H}_{2,h} \oplus \mathrm{R}\,\mathbb{I}$, thanks to which $\mathbb{H}_{2,h}$ (cf.

(5.1)) can be used as the subspace where the unknown $\boldsymbol{\sigma}_h$ is sought. However, for the computational implementation of the Galerkin scheme (5.2), which will be addressed later on in Section 7, we will utilize a real Lagrange multiplier to impose the mean value condition on the trace of the unknown tensor lying in $\mathbb{H}_{2,h}$.

In turn, according to the definition of $\mathcal{B}$ (cf. (3.9)), it follows from (5.6) and (**H.1**) that

$$\mathcal{K}_h := \Big\{ \boldsymbol{\tau} \in \mathbb{H}_{2,h} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \Big\}, \tag{5.8}$$

which yields the discrete analogue of (4.18), that is, given $\boldsymbol{\tau}_h \in \mathcal{K}_h$ such that $\boldsymbol{\tau}_h^{\mathtt{d}} \neq \mathbf{0}$, we realize that $\mathbf{s}_h = -\boldsymbol{\tau}_h^{\mathtt{d}} \in \mathbb{H}_{1,h}$ (which follows from (**H.2**)), and thus

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_{1,h} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{\mathbb{H}_1}} \geq \beta_{1,\mathtt{d}} \, \|\boldsymbol{\tau}_h\|_{\mathbb{H}_2} \quad \forall\, \boldsymbol{\tau} \in \mathcal{K}_h \,, \tag{5.9}$$

with constant $\beta_{1,\mathtt{d}} = c_1$ (cf. (4.17)). On the other hand, if $\boldsymbol{\tau}_h^{\mathtt{d}} = \mathbf{0}$, it is clear from (4.17) that $\boldsymbol{\tau}_h = \mathbf{0}$, and so the discrete inf-sup condition for $\mathcal{B}_1$ (cf. (5.9)) is trivially satisfied.

In addition, we recall that the Lipschitz-continuity and strict monotonicity of $\mathcal{A}$ (cf. Lemma 4.3), is also valid on $\mathbb{H}_{1,h} \times \mathbb{H}_{1,h}'$, which means that, with the same constants γ_μ and μ_1, there hold

$$\|\mathcal{A}(\mathbf{r}_h) - \mathcal{A}(\mathbf{s}_h)\|_{\mathbb{H}_1'} \leq \gamma_\mu \, \|\mathbf{r}_h - \mathbf{s}_h\|_{\mathbb{H}_1} \quad \forall\, \mathbf{r}_h, \mathbf{s}_h \in \mathbb{H}_{1,h} \,, \tag{5.10}$$

and

$$[\mathcal{A}(\mathbf{r}_h) - \mathcal{A}(\mathbf{s}_h), \mathbf{r}_h - \mathbf{s}_h]_{\mathbb{H}_1'} \geq \mu_1 \, \|\mathbf{r}_h - \mathbf{s}_h\|_{\mathbb{H}_1}^2 \quad \forall\, \mathbf{r}_h, \mathbf{s}_h \in \mathbb{H}_{1,h} \,. \tag{5.11}$$

In this way, bearing the above discussion in mind, we are now in a position to establish the discrete analogue of Theorem 4.6.

Theorem 5.1. *For each $\mathbf{w}_h \in \mathbf{Q}_h$ there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ solution to (4.4), and hence we can define $\mathbf{T}_h(\mathbf{w}_h) := \mathbf{u}_h \in \mathbf{Q}_h$. Moreover, there exists a positive constant $C_{\mathbf{T},\mathtt{d}}$, depending only on $\gamma_\mu, \mu_1, \boldsymbol{\beta}_{\mathtt{d}}, \beta_{1,\mathtt{d}}, \|\mathcal{B}_1\|$, $\|\mathcal{B}_1'\|$, and $\|\mathbf{i}_4\|$, and hence independent of $\mathbf{w}_h$, such that*

$$\begin{aligned} \|\mathbf{T}_h(\mathbf{w}_h)\|_{0,4;\Omega} &= \|\mathbf{u}_h\|_{\mathbf{Q}} \leq \|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \\ &\leq C_{\mathbf{T},\mathtt{d}} \Big\{ \|\mathbf{w}_h\|_{0,4;\Omega}^2 + \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \,. \end{aligned} \tag{5.12}$$

Proof. Thanks to the discrete inf-sup conditions for $\mathcal{B}$ (cf. (**H.3**)) and $\mathcal{B}_1$ (cf. (5.9)), and the inequalities (5.10) and (5.11), the proof follows from a direct application of Theorem 4.1. We omit further details. $\square$

Having established that the discrete operator $\mathbf{T}_h$ is well defined, we now address the solvability of the corresponding fixed point equation (5.5). Then, letting $\delta_{\mathtt{d}}$ be an arbitrary radius, we now set

$$\mathrm{W}(\delta_{\mathtt{d}}) := \Big\{ \mathbf{w}_h \in \mathbf{Q}_h : \quad \|\mathbf{w}_h\|_{0,4;\Omega} \leq \delta_{\mathtt{d}} \Big\} \,. \tag{5.13}$$

Then, reasoning analogously to the derivation of Lemma 4.7, we deduce that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself under the analogue discrete assumptions, namely

$$C_{\mathbf{T},\mathtt{d}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \leq \frac{\delta_{\mathtt{d}}}{2} \quad \text{and} \quad \delta_{\mathtt{d}} \leq \frac{1}{2C_{\mathbf{T},\mathtt{d}}} \,. \tag{5.14}$$

We emphasize that the above is exactly the same as for the continuous case (cf. Lemma 4.7), except that the constant $C_{\mathbf{T}}$ and the radius δ are replaced by $C_{\mathbf{T},\mathtt{d}}$ and $\delta_{\mathtt{d}}$, respectively. Moreover, employing similar arguments to those from the proof of Lemma 4.8, we are able to prove the discrete version of (4.23) with constant $\mathrm{L}_{\mathbf{T},\mathtt{d}} := \dfrac{\beta_{\mathtt{d}}\mu_1}{\|\mathcal{B}_1\|}$, that is

$$\|\mathbf{T}_h(\mathbf{w}_h) - \mathbf{T}_h(\widehat{\mathbf{w}}_h)\|_{0,4;\Omega} \,\le\, \mathrm{L}_{\mathbf{T},\mathtt{d}} \Big\{ \|\mathbf{w}_h\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}_h\|_{0,4;\Omega} \Big\} \|\mathbf{w}_h - \widehat{\mathbf{w}}_h\|_{0,4;\Omega} \tag{5.15}$$

for all $\mathbf{w}_h \,, \widehat{\mathbf{w}}_h \in \mathbf{Q}_h$, which proves the continuity of $\mathbf{T}_h$.

According to the above, the main result of this section is establish as follows.

Theorem 5.2. *Assume that $\delta_{\mathtt{d}}$ and the data are sufficiently small so that they satisfy (5.14). Then, the operator $\mathbf{T}_h$ has at least one fixed point $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathtt{d}})$. Equivalently, the problem (5.2) has at least one solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$. Moreover, under the further assumption*

$$\delta_{\mathtt{d}} \,<\, \frac{1}{2} \min \Big\{ \frac{1}{C_{\mathbf{T},\mathtt{d}}} \,, \frac{\beta_{\mathtt{d}}\mu_1}{\|\mathcal{B}_1\|} \Big\} , \tag{5.16}$$

this solution is unique. In addition, there holds

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \,\le\, 2C_{\mathbf{T},\mathtt{d}} \Big\{ \|\mathbf{g}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} . \tag{5.17}$$

Proof. The fact that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathtt{d}})$ into itself, together with the continuity of $\mathbf{T}_h$ (cf. (5.15)), allow to apply the Brouwer Theorem to conclude the existence of a solution to (5.5), and hence to (5.2). Next, the assumption (5.16) and the Banach fixed-point Theorem imply the uniqueness. Finally, the a priori estimate is consequence of Theorem 4.1 and analogue algebraic manipulations to those utilized in the proof of Theorem 4.9. □

5.3 A priori error analysis

In this section we consider finite element subspaces satisfying the assumptions specified in Section 5.2, and derive the Céa estimate for the Galerkin error

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \,=\, \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,4/3,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \,, \tag{5.18}$$

where $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X} := \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ and $\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_h := \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ are the unique solutions of (3.14) and (5.2) respectively, with $\mathbf{u} \in \mathrm{W}(\delta)$ and $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathtt{d}})$. In what follows, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$\operatorname{dist}\big(z, Z_h\big) \,:=\, \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z \,. \tag{5.19}$$

In turn, in order to simplify our analysis, we recall a previous result concerning the operator $\mathcal{A}$. More precisely, we employ the following lemma.

Lemma 5.3. *The operator $\mathcal{A}$ defined in (3.9) has a first-order Gâteaux derivative $D\mathcal{A}$. Moreover, for any $\mathbf{s}_1 \in \mathbb{H}_1$, $D\mathcal{A}(\mathbf{s}_1)$ is a bounded and $\mathbb{H}_1$-elliptic bilinear form, with boundedness and ellipticity constants given by γ_μ and μ_1, respectively.*

Proof. See [16, Lemma 3.1]. □

We begin by introducing the global operator $\mathbf{P} : \mathbf{X} \to \mathbf{X}'$, and for each $\mathbf{w} \in \mathbf{Q}$ the linear functional $\mathrm{F}_{\mathbf{w}} : \mathbf{X} \to \mathrm{R}$ associated with the variational formulation (3.14), that is

$$[\mathbf{P}(\mathbf{r},\boldsymbol{\rho},\mathbf{w}),(\mathbf{s},\boldsymbol{\tau},\mathbf{v})] := [\mathcal{A}(\mathbf{r}),\mathbf{s}] + [\mathcal{B}_1(\mathbf{s}),\boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}),\boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}),\mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}),\mathbf{v}]\,, \tag{5.20}$$

$$[\mathrm{F}_{\mathbf{w}},(\mathbf{s},\boldsymbol{\tau},\mathbf{v})] := \int_\Omega (\mathbf{w}\otimes\mathbf{w})^{\mathrm{d}} : \mathbf{s} + [\mathrm{G},\boldsymbol{\tau}] + [\mathrm{F},\mathbf{v}], \tag{5.21}$$

for all $(\mathbf{r},\boldsymbol{\rho},\mathbf{w}),(\mathbf{s},\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}$. In this way, we realize from (3.14) and (5.2) that there holds

$$[\mathbf{P}(\vec{\mathbf{t}}),\vec{\mathbf{s}}_h] = [\mathbf{P}(\vec{\mathbf{t}}_h),\vec{\mathbf{s}}_h] + [\mathrm{F}_{\mathbf{u}} - \mathrm{F}_{\mathbf{u}_h},\vec{\mathbf{s}}_h] \quad \forall\,\vec{\mathbf{s}}_h \in \mathbf{X}_h\,, \tag{5.22}$$

whereas the triangle inequality gives for each $\vec{\mathbf{r}}_h := (\mathbf{r}_h,\boldsymbol{\rho}_h,\mathbf{w}_h) \in \mathbf{X}_h$

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \le \|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{r}}_h - \vec{\mathbf{t}}_h\|_{\mathbf{X}}\,. \tag{5.23}$$

In order to establish a connection between the second term on the right-hand side of the above inequality and the operator $\mathbf{P}$, we proceed almost verbatim as in [16, Theorem 3.3]. In fact, given $\vec{\mathbf{s}}_h$, $\vec{\mathbf{r}}_h \in \mathbf{X}_h$, we can write

$$\begin{aligned}[\mathbf{P}(\vec{\mathbf{t}}_h),\vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h] &= \int_0^1 \frac{d}{d\mu}\{[\mathbf{P}(\mu\vec{\mathbf{t}}_h + (1-\mu)\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h]\}\,d\mu \\ &= \int_0^1 D\mathbf{P}(\mu\vec{\mathbf{t}}_h + (1-\mu)\vec{\mathbf{r}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h,\vec{\mathbf{s}}_h)\,d\mu\,,\end{aligned} \tag{5.24}$$

where $D\mathbf{P} : \mathbf{X} \to \mathcal{L}(\mathbf{X},\mathbf{X}')$ is the first-order Gâteaux derivative of the operator $\mathbf{P} : \mathbf{X} \to \mathbf{X}'$. More precisely, for any $\vec{\mathbf{s}}_1 := (\mathbf{s}_1,\boldsymbol{\tau}_1,\mathbf{v}_1)$, $\vec{\mathbf{r}} := (\mathbf{r},\boldsymbol{\rho},\mathbf{w})$, $\vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau},\mathbf{v}) \in \mathbf{X}$, $D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{r}},\vec{\mathbf{s}})$ is obtained from (5.20) by replacing $[\mathcal{A}(\mathbf{r}),\mathbf{s}]$ by $D\mathcal{A}(\mathbf{s}_1)(\mathbf{r},\mathbf{s})$, that is

$$D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{r}},\vec{\mathbf{s}}) := D\mathcal{A}(\mathbf{s}_1)(\mathbf{r},\mathbf{s}) + [\mathcal{B}_1(\mathbf{s}),\boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}),\boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}),\mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}),\mathbf{v}]\,. \tag{5.25}$$

Thus, for any $\vec{\mathbf{s}}_1 \in \mathbf{X}$, (5.25) induces the definition of an operator in $\mathcal{L}(\mathbf{X},\mathbf{X}')$, which, according to Lemma 5.3, satisfies the hypotheses of the discrete version of the linear Theorem 4.2 with constants independent of h and of $\vec{\mathbf{s}}_1$. Consequently, bearing in mind that in this case the discrete version of the estimate (4.11) is equivalent to a global discrete inf-sup condition (cf. (4.13)), it follows that, there exists $\widehat{C} > 0$, depending only on $\gamma_\mu\,, \mu_1\,, \beta_1\,, \|\mathcal{B}_1\|$ and β, such that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \le \widehat{C} \sup_{\substack{\vec{\mathbf{s}}_h\in\mathbf{X}_h\\ \vec{\mathbf{s}}_h\neq\mathbf{0}}} \frac{D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h,\vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}} \qquad \forall\,\vec{\mathbf{s}}_1 \in \mathbf{X}. \tag{5.26}$$

On the other hand, the continuity of $D\mathcal{A}$ implies the same property for $D\mathbf{P}$, and hence there exists $\mu_0 \in (0,1)$ such that (5.24) becomes

$$[\mathbf{P}(\vec{\mathbf{t}}_h),\vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h] = D\mathbf{P}(\mu_0\vec{\mathbf{t}}_h + (1-\mu_0)\vec{\mathbf{r}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h,\vec{\mathbf{s}}_h)\,. \tag{5.27}$$

It follows from (5.26) (with $\vec{\mathbf{s}}_1 := \mu_0\vec{\mathbf{t}}_h + (1-\mu_0)\vec{\mathbf{r}}_h$) and (5.27) that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \le \widehat{C} \sup_{\substack{\vec{\mathbf{s}}_h\in\mathbf{X}_h\\ \vec{\mathbf{s}}_h\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\mathbf{t}}_h),\vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h]}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}}\,. \tag{5.28}$$

Next, since $\mathbf{P}$ is Lipschitz continuous, with a constant $\widehat{\gamma}$, depending only on $\gamma_\mu\,$, $\|\mathcal{B}_1\|$ and $\|\mathcal{B}_1'\|$, we subtract and add $\mathbf{P}(\vec{\mathbf{t}})$, and use (5.22), to find that

$$\begin{aligned}[\mathbf{P}(\vec{\mathbf{t}}_h),\vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h] &= [\mathbf{P}(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}),\vec{\mathbf{s}}_h] + [\mathbf{P}(\vec{\mathbf{t}}) - \mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h] \\ &= [\mathrm{F}_{\mathbf{u}_h} - \mathrm{F}_{\mathbf{u}},\vec{\mathbf{s}}_h] + [\mathbf{P}(\vec{\mathbf{t}}) - \mathbf{P}(\vec{\mathbf{r}}_h),\vec{\mathbf{s}}_h] \\ &\le \Big\{\Big(\|\mathbf{u}\|_{\mathbf{Q}} + \|\mathbf{u}_h\|_{\mathbf{Q}}\Big)\,\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Q}} + \widehat{\gamma}\,\|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}}\Big\}\,\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}\,,\end{aligned} \tag{5.29}$$

which, replaced back into (5.28), gives

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \,\le\, \widehat{C}\Big\{\Big(\|\mathbf{u}\|_{\mathbf{Q}} + \|\mathbf{u}_h\|_{\mathbf{Q}}\Big)\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Q}} + \widehat{\gamma}\,\|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}}\Big\}\,. \tag{5.30}$$

Finally, the triangle inequality (cf. (5.23)) along with (5.30) and the fact that $\|\mathbf{u}\|_{\mathbf{Q}}$ and $\|\mathbf{u}_h\|_{\mathbf{Q}}$ are bounded by δ and $\delta_{\mathtt{d}}$, respectively, yield

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \,\le\, \big(1+\widehat{\gamma}\,\widehat{C}\big)\inf_{\vec{\mathbf{s}}_h\in\mathbf{X}}\|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbf{X}} + \widehat{C}\big(\delta+\delta_{\mathtt{d}}\big)\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Q}}\,. \tag{5.31}$$

In this way, our main result for the error $\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}}$ is stated as follows.

Theorem 5.4. *Assume that the hypotheses of Theorems 4.9 and 5.2 hold, and let* $\vec{\mathbf{t}} = (\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\in\mathbf{X}$ *and* $\vec{\mathbf{t}}_h = (\mathbf{t}_h\,,\boldsymbol{\sigma}_h\,,\mathbf{u}_h)\in\mathbf{X}_h$ *be the unique solutions of* (3.14) *and* (5.2)*, respectively. Assume further that*

$$(\delta+\delta_{\mathtt{d}}) \,\le\, \frac{1}{2\widehat{C}}\,, \tag{5.32}$$

where $\widehat{C}$ *is the global inf-sup constant of* $D\mathbf{P}$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \,\le\, C\,\mathrm{dist}(\vec{\mathbf{t}},\mathbf{X}_h)\,. \tag{5.33}$$

Proof. It suffices to use (5.32) in (5.31), which yields (5.33) with $C := 2\big(1+\widehat{\gamma}\,\widehat{C}\big)$. □

We end this section by remarking that (2.7) and (3.6) suggest the following postprocessed approximation for the pressure p

$$p_h \,:=\, -\,\frac{1}{n}\mathrm{tr}\Big(\boldsymbol{\sigma}_h \,+\, (\mathbf{u}_h\otimes\mathbf{u}_h)\Big) \,-\, c_{0,h} \quad \text{in} \quad \Omega\,, \tag{5.34}$$

where

$$c_{0,h} \,:=\, -\,\frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\,.$$

Then, applying the Cauchy-Schwarz inequality, performing some algebraic manipulations, and employing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p \,-\, p_h\|_{0,\Omega} \,\le\, C\,\Big\{\|\boldsymbol{\sigma} \,-\, \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\mathbf{u} \,-\, \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,. \tag{5.35}$$

Thus, combining (5.33) and (5.35), we conclude the existence of a positive constant $\widetilde{C}$, independent of h, such that

$$\begin{aligned}&\|\mathbf{t}-\mathbf{t}_h\|_{\mathbb{H}_1} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbb{H}_2} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{Q}} + \|p \,-\, p_h\|_{0,\Omega}\\ &\quad\le\, \widetilde{C}\,\Big\{\mathrm{dist}(\mathbf{t},\mathbb{H}_{1,h}) + \mathrm{dist}(\boldsymbol{\sigma},\mathbb{H}_{2,h}) + \mathrm{dist}(\mathbf{u},\mathbf{Q}_h)\Big\}\,.\end{aligned} \tag{5.36}$$

6 Specific finite element subspaces

In this section, we introduce specific finite element subspaces $\mathbb{H}_{1,h}$, $\widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $\mathbf{L}^4(\Omega)$, respectively. These subspaces satisfy the hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**), which were introduced in Section 5.2 to ensure the well-posedness of our Galerkin scheme.

6.1 Preliminaries

In what follows, given an integer $\ell \geq 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_\ell(K)$ be the space of polinomials of degree $\leq \ell$ defined on K, whose vector and tensor versions are denoted by $\mathbf{P}_\ell(K) := [\mathrm{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) := [\mathrm{P}_\ell(K)]^{n\times n}$, respectively. Next, we define the corresponding local Raviart-Thomas spaces of order ℓ as

$$\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\mathbf{x} \quad \forall\, K \in \mathcal{T}_h\,,$$

and its associated tensor counterpart $\mathbb{RT}_\ell(K)$, where $\mathbf{x}$ is a generic vector in $\mathbf{R} := \mathrm{R}^n$. In turn, we let $\mathbf{P}_\ell(\mathcal{T}_h)$, $\mathbb{P}_\ell(\mathcal{T}_h)$ and $\mathbb{RT}_\ell(\mathcal{T}_h)$ be the global versions of $\mathbf{P}_\ell(K)$, $\mathbb{P}_\ell(K)$ and $\mathbb{RT}_\ell(K)$, respectively, that is

$$\begin{aligned}
\mathbf{P}_\ell(\mathcal{T}_h) &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbb{P}_\ell(\mathcal{T}_h) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,\\
\mathbb{RT}_\ell(\mathcal{T}_h) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,.
\end{aligned}$$

We stress here that there hold $\mathbf{P}_\ell(\mathcal{T}_h) \subseteq \mathbf{L}^4(\Omega)$ and $\mathbb{RT}_\ell(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, inclusions that are implicitly utilized below to introduce the announced specific finite element subspaces. Indeed, we now define

$$\begin{aligned}
\mathbb{H}_{1,h} &:= \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_\ell(\mathcal{T}_h)\,, \quad \widetilde{\mathbb{H}}_{2,h} := \mathbb{RT}_\ell(\mathcal{T}_h),\\
\mathbb{H}_{2,h} &:= \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \cap \widetilde{\mathbb{H}}_{2,h}\,, \quad \text{and} \quad \mathbf{Q}_h := \mathbf{L}^4(\Omega) \cap \mathbf{P}_\ell(\mathcal{T}_h)\,.
\end{aligned} \tag{6.1}$$

6.2 Verification of the hypotheses (H.0) -(H.3)

We now confirm that the subspaces defined by (6.1) satisfy the hypotheses (**H.0**)-(**H.3**). Indeed, it is easily seen that $\widetilde{\mathbb{H}}_{2,h}$ satisfy (**H.0**) and (**H.1**). Next, in order to check (**H.2**), we recall from (5.8) that

$$\mathcal{K}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_{2,h} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \Big\}\,, \tag{6.2}$$

from which, using that the divergence free tensors of $\mathbb{RT}_\ell(\mathcal{T}_h)$ are contained in $\mathbb{P}_\ell(\mathcal{T}_h)$ (cf. [15, Lemma 3.6]), it follows that $\mathcal{K}_h \subseteq \mathbb{P}_\ell(\mathcal{T}_h)$. Hence, noting that certainly $\mathrm{tr}(\boldsymbol{\tau}_h^{\mathrm{d}}) = 0$, for all $\boldsymbol{\tau}_h \in \mathcal{K}_h$, we deduce that $(\mathcal{K}_h)^{\mathrm{d}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_\ell(\mathcal{T}_h) = \mathbb{H}_{1,h}$, which proves (**H.2**). Finally, (**H.3**) is proved precisely in [9, Lemma 5.1] (see also [12, Lemma 6.1]).

6.3 The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (5.2) with the specific finite element subspaces introduced in Section 6.1, for which we previously collect the respective approximation properties. In fact, thanks to [13, Proposition 1.135] and its corresponding vector version, along with interpolation estimates of Sobolev spaces, those of $\mathbb{H}_{1,h}$, $\mathbb{H}_{2,h}$, and $\mathbf{Q}_h$, are given as follows:

$\big(\mathbf{AP}_h^{\mathbf{t}}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{s}, \mathbb{H}_{1,h}\big) := \inf_{\mathbf{s}_h \in \mathbb{H}_{1,h}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leq C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,,$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbb{H}_{2,h}\big) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{2,h}} \|\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leq\, C\, h^l \,\Big\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \Big\},$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{Q}_h\big) \,:=\, \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{v} \,-\, \mathbf{v}_h\|_{0,4;\Omega} \,\leq\, C\, h^l \, \|\mathbf{v}\|_{l,4;\Omega} \,.$$

The rates of convergence of (5.2) are now established by the following theorem.

Theorem 6.1. *Let $\big(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}\big) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ and $\big(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h\big) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ be the unique solutions of (3.14) and (5.2) with $\mathbf{u} \in \mathrm{W}(\delta)$ and $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathbf{d}})$, whose existences are guaranteed by Theorems 4.9 and 5.2, respectively. In turn, let p and p_h given by (2.7) and (5.34), respectively. Assume the hypotheses of Theorem 5.4, and that there exists $l \in [1, \ell+1]$ such that $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, and $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned} &\|((\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) - (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} + \|p - p_h\|_{0,\Omega} \\ &\qquad\qquad \leq\, C\, h^l \,\Big\{ \|\mathbf{u}\|_{l,4;\Omega} \,+\, \|\mathbf{t}\|_{l,\Omega} \,+\, \|\boldsymbol{\sigma}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} \Big\}. \end{aligned}$$

Proof. It follows straightforwardly from the Céa estimate (5.36), and the approximation properties $\big(\mathbf{AP}_h^{\mathbf{t}}\big)$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, and $(\mathbf{AP}_h^{\mathbf{u}})$. $\square$

7 Computational results

We now turn to the computational results, which mainly refer to the numerical verification of the rates of convergence anticipated by Theorem 6.1. The examples in 2D and 3D to be reported below have been developed with the finite element library FEniCS [2]. In all them, the linear systems emanating from the Newton-Raphson linearisation, with the zero vector as initial guess and iterations stopped once the absolute or relative residual drops below 10^{-8}, have been solved with the multifrontal massively parallel sparse direct method MUMPS [3]. In turn, the condition of zero-average pressure, which, owing to (2.7), entails to fix the trace of the tensor quantity $\boldsymbol{\sigma} + (\mathbf{u} \otimes \mathbf{u})$, is imposed by means of a real Lagrange multiplier. Subsequently, errors are defined as follows:

$$e(\mathbf{t}) = \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \qquad e(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,,$$

$$e(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \qquad e(p) = \|p - p_h\|_{0,\Omega}\,,$$

whereas convergence rates are set as

$$r(\star) = \frac{\log(e(\star)/\widehat{e}(\star))}{\log(h/\widehat{h})} \qquad \forall \star \in \big\{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p\big\}\,,$$

where e and $\widehat{e}$ denote errors computed on two consecutive meshes of sizes h and $\widehat{h}$. In addition, we refer to the number of degrees of freedom and the number of Newton iterations as `dof` and `iter`, respectively.

7.1 Example 1: 2D smooth solution

In our first numerical test, we consider the computational domain $\Omega = (0,1)^2$, and set the nonlinear viscosity to

$$\mu(s) := 2 + \frac{1}{1+s} \quad \forall\, s \geq 0\,. \tag{7.1}$$

In addition, we define the manufactured exact solution:

$$p = x^2 - y^2, \quad \mathbf{u} = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u},$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,,$$

so that the load function $\mathbf{f}$ and the Dirichlet datum $\mathbf{g}$ are computed accordingly. Table 7.1 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations for the approximations. The experiments confirm the theoretical rate of convergence $\mathcal{O}(h^{\ell+1})$ for $\ell \in \{0,1\}$, provided by Theorem 6.1. In addition, the number of Newton-Raphson iterations required to reach the convergence criterion based on the residuals with a tolerance of $1e-8$, was less than or equal to 4 in all runs. Sample of approximate solutions with $\ell = 1$ and `dof` $= 279041$ are shown in Figure 7.1.

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$										
`dof`	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	`iter`
121	0.7071	$1.26e+00$	$*$	$1.71e+01$	$*$	$4.11e-01$	$*$	$7.55e-01$	$*$	3
465	0.3536	$6.20e-01$	1.02	$8.99e+00$	0.93	$2.26e-01$	0.86	$3.69e-01$	1.03	3
1825	0.1768	$3.10e-01$	1.00	$4.59e+00$	0.97	$1.16e-01$	0.96	$1.82e-01$	1.02	4
7233	0.0884	$1.55e-01$	1.00	$2.31e+00$	0.99	$5.84e-02$	0.99	$8.86e-02$	1.04	4
28801	0.0442	$7.77e-02$	1.00	$1.16e+00$	1.00	$2.92e-02$	1.00	$4.33e-02$	1.03	4
114945	0.0221	$3.89e-02$	1.00	$5.79e-01$	1.00	$1.46e-02$	1.00	$2.15e-02$	1.01	4

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
`dof`	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	`iter`
289	0.7071	$2.75e-01$	$*$	$4.46e+00$	$*$	$1.55e-01$	$*$	$2.60e-01$	$*$	4
1121	0.3536	$7.35e-02$	1.90	$1.22e+00$	1.87	$4.11e-02$	1.91	$5.62e-02$	2.21	4
4417	0.1768	$1.93e-02$	1.93	$3.58e-01$	1.77	$1.05e-02$	1.97	$1.30e-02$	2.11	4
17537	0.0884	$4.93e-03$	1.97	$1.02e-01$	1.82	$2.64e-03$	1.99	$3.17e-03$	2.04	4
69889	0.0442	$1.24e-03$	1.99	$2.76e-02$	1.88	$6.62e-04$	2.00	$7.84e-04$	2.01	4
279041	0.0221	$3.12e-04$	1.99	$7.31e-03$	1.92	$1.66e-04$	2.00	$1.95e-04$	2.01	4

Table 7.1: Example 1, convergence history and Newton iteration count for the $\mathbb{P}_\ell - \mathbb{RT}_\ell - \mathbf{P}_\ell$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the $\mathrm{P}_\ell-$approximation of the postprocessed pressure field, with $\ell \in \{0,1\}$.

7.2 Example 2: 2D smooth solution in a non-convex domain

Now we illustrate the accuracy of our method in the non-convex domain $\Omega := (-1,1)^2 \setminus [0,1]^2$. The data $\mathbf{f}$ and $\mathbf{g}$ are computed so that the manufactured exact solution is defined as:

$$p = \sin(\pi x)\exp(y), \quad \mathbf{u} = \begin{pmatrix} -\cos(2\pi y)\sin(2\pi x) \\ \sin(2\pi y)\cos(2\pi x) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,.$$

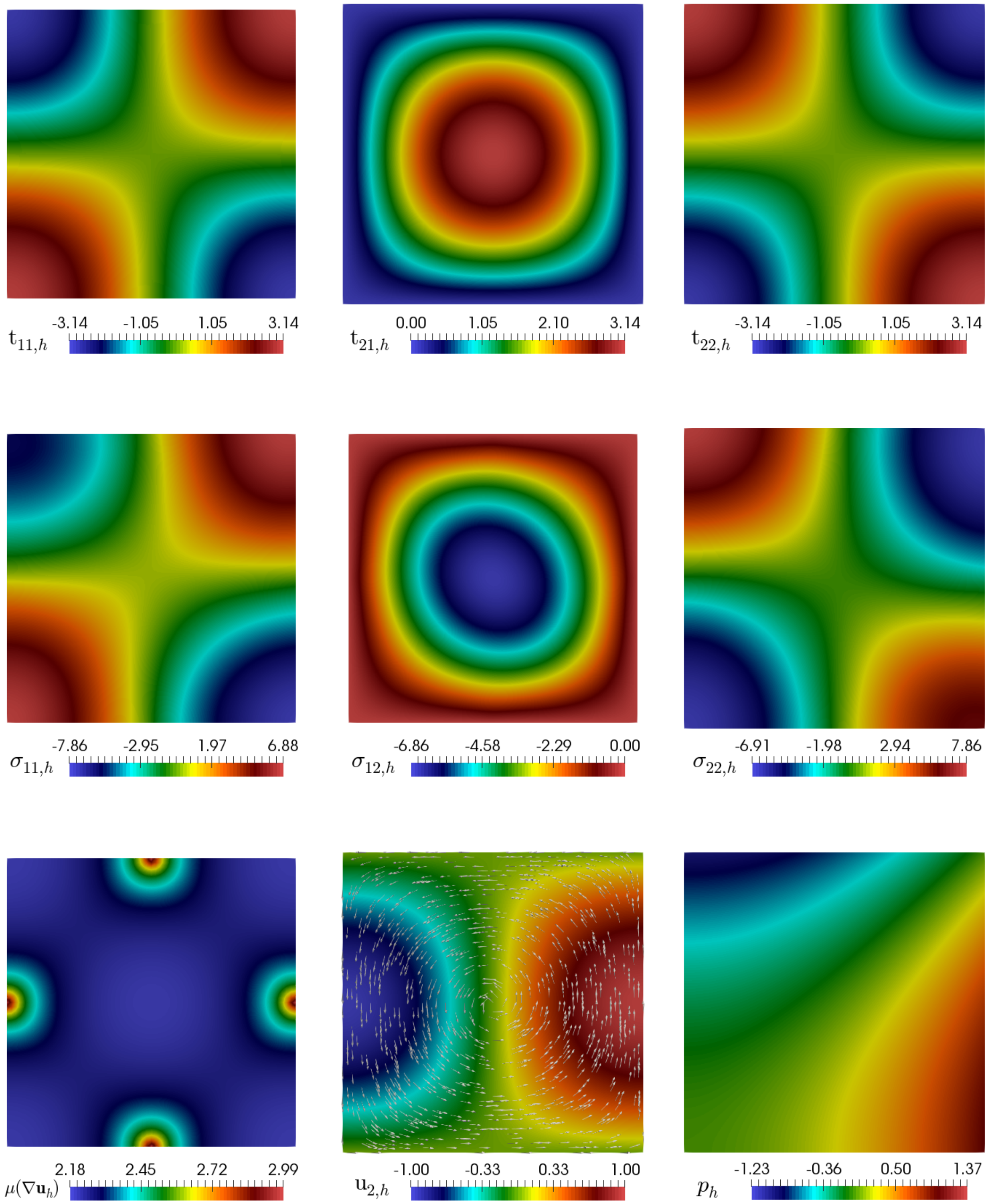

Figure 7.1: Example 1, $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation with `dof` $= 279041$ of velocity gradient components (top panels), pseudostress components (center panels), and viscosity, velocity component with vector directions, and postprocessed pressure field (bottom row).

The variable viscosity is defined in the same way as in Example 1. The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 1$ is shown in Table 7.2. As proven by Theorem 6.1, the mixed finite element method converges optimally with $\mathcal{O}(h^2)$. Additionally, Figure 7.2 displays selected components of the numerical solution, which were obtained using the $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximation with `dof` $= 238603$.

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
`dof`	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	`iter`
383	1.1180	$8.59e+00$	$*$	$2.07e+02$	$*$	$8.07e-01$	$*$	$4.63e+00$	$*$	4
941	0.6212	$3.93e+00$	1.33	$9.23e+01$	1.38	$4.15e-01$	1.13	$2.46e+00$	1.08	4
3646	0.3171	$1.13e+00$	1.85	$2.40e+01$	2.00	$1.29e-01$	1.74	$5.48e-01$	2.23	4
15233	0.1582	$2.85e-01$	1.98	$6.25e+00$	1.94	$3.29e-02$	1.96	$1.29e-01$	2.08	4
59869	0.0795	$7.40e-02$	1.96	$1.94e+00$	1.70	$8.50e-03$	1.97	$3.38e-02$	1.95	4
238603	0.0398	$1.85e-02$	2.00	$6.03e-01$	1.69	$2.16e-03$	1.98	$8.50e-03$	1.99	4

Table 7.2: Example 2, convergence history and Newton iteration count for the fully-mixed $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the P_1-approximation of the postprocessed pressure field.

7.3 Example 3: 2D non-smooth solution in a non-convex domain

The third example is devoted to show that the rates of convergence are affected when the exact solution does not have enough regularity, in particular if it has a singularity near the vertex with major angle of a non-convex domain. In fact, here we consider again the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$, define the manufactured exact solution:

$$p = \frac{1-x}{2(x-0.02)^2 + 2(y-0.02)^2}\,, \quad \mathbf{u} = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,,$$

and compute the data $\mathbf{f}$ and $\mathbf{g}$ accordingly. The variable viscosity is defined in the same way as in Example 1. The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 1$ is shown in Table 7.3. As announced, suboptimal rates arise in this case, which is explained by the fact that the pressure exhibits high gradients near the corner region of the L-shaped domain. This is observed in Figure 7.3 below where selected components of the numerical solution, obtained with the $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximation and `dof` $= 238603$, are displayed.

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
383	1.1180	$1.10e+01$	$*$	$5.58e+02$	$*$	$1.20e+00$	$*$	$4.26e+01$	$*$	3
941	0.6212	$6.98e+00$	0.77	$3.72e+02$	0.69	$5.67e-01$	1.27	$2.35e+01$	1.01	3
3646	0.3171	$7.23e+00$	-0.05	$4.57e+02$	-0.31	$3.66e-01$	0.65	$1.88e+01$	0.33	3
15233	0.1582	$4.50e+00$	0.68	$3.74e+02$	0.29	$1.43e-01$	1.35	$1.17e+01$	0.68	3
59869	0.0795	$2.57e+00$	0.82	$1.83e+02$	1.04	$6.60e-02$	1.12	$6.21e+00$	0.92	3
238603	0.0398	$1.22e+00$	1.07	$7.49e+01$	1.29	$2.05e-02$	1.68	$2.64e+00$	1.23	5

Table 7.3: Example 3, convergence history and Newton iteration count for the $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the P_1-approximation of the postprocessed pressure field.

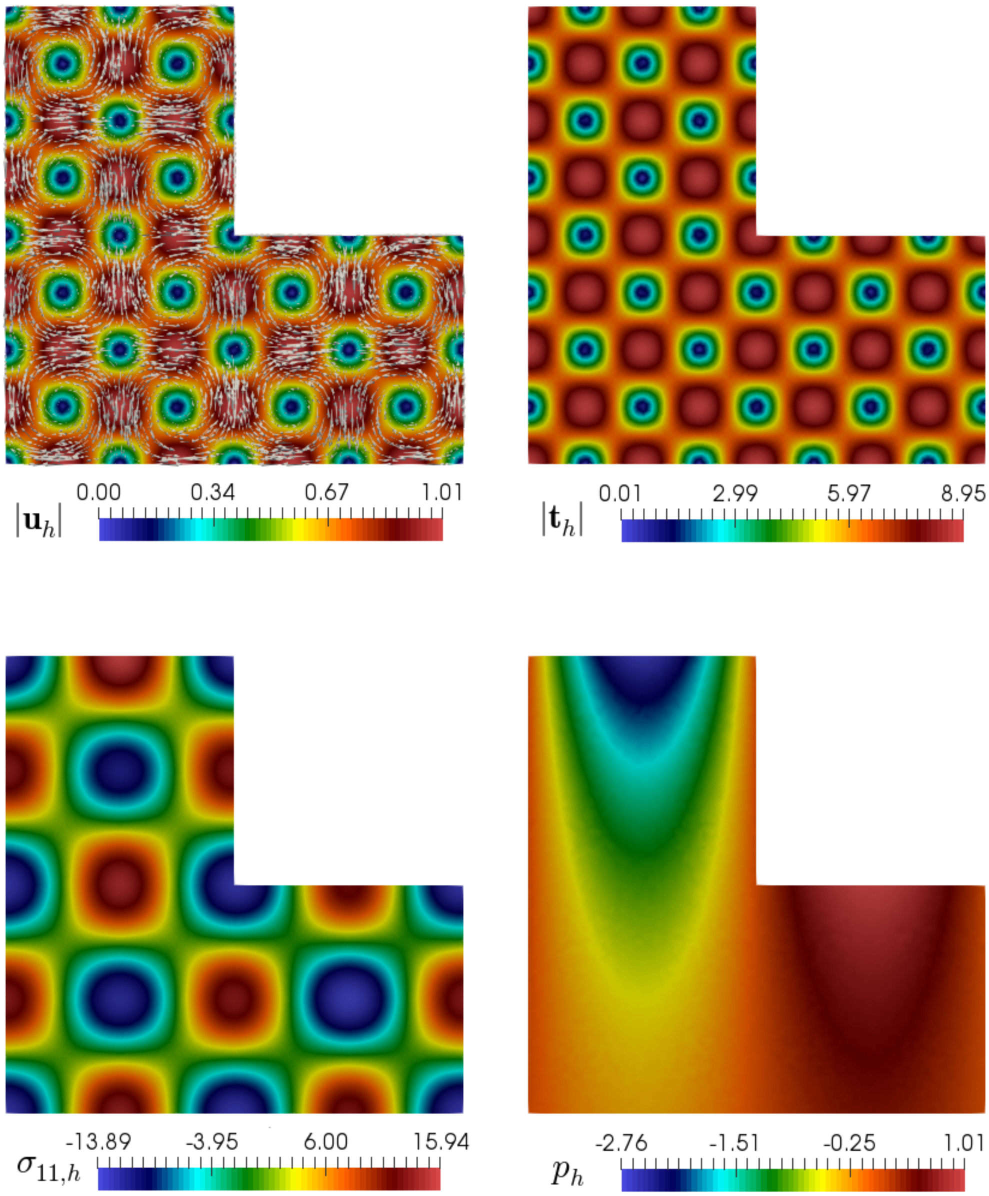

Figure 7.2: Example 2, $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximation with `dof` $= 238603$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

7.4 Example 4: 3D smooth solution

Next we illustrate a three-dimensional problem. In this case, we consider the cube domain $\Omega = (0,1)^3$, and define the nonlinear viscosity as

$$\mu(s) := \alpha_0 + \alpha_1(1+s^2)^{(\beta-2)/2},$$

with $\alpha_0 = 2/5$, $\alpha_1 = 1/2$, and $\beta = 1$. The data are suitably adjusted according to the exact solution defined by the functions:

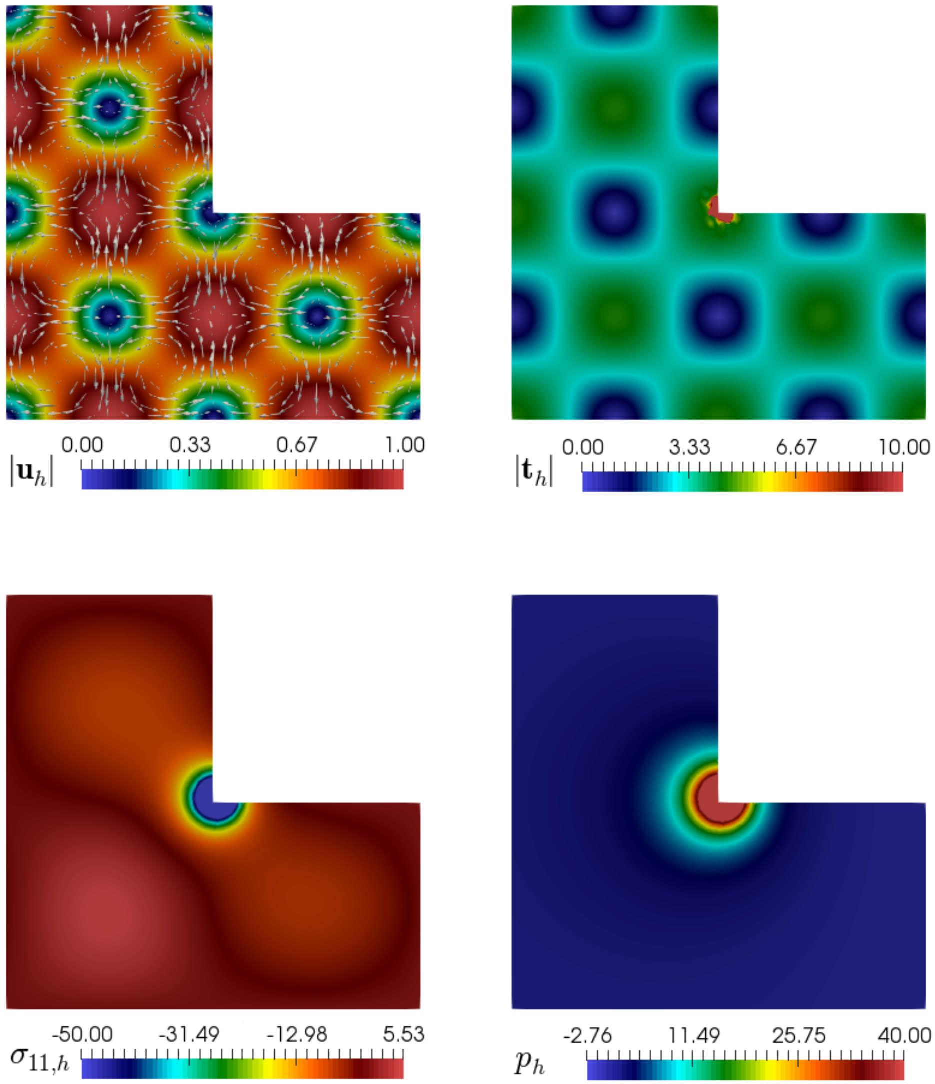

Figure 7.3: Example 3, $\mathbb{P}_1-\mathbb{RT}_1-\mathbf{P}_1$ approximation with `dof` $= 238603$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

$$p = \sin(xyz)\,, \quad \mathbf{u} = \begin{pmatrix} \sin(\pi\,x)\cos(\pi\,y)\cos(\pi\,z) \\ -2\,\cos(\pi\,x)\,\sin(\pi\,y)\,\cos(\pi\,z) \\ \cos(\pi\,x)\,\cos(\pi y)\,\sin(\pi\,z) \end{pmatrix}\,, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,.$$

The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 0$ is shown in Table 7.4, while some components of the approximate solutions with `dof` $= 3360769$ are displayed in Figure 7.4. We observe that the Newton method exhibits a behavior independent of the meshsize, achieving the tolerance of $1e-8$ in four iterations in all cases. Again, the mixed finite element method converges optimally with $\mathcal{O}(h)$, as it was proved by Theorem 6.1.

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
889	0.8660	$2.61e+00$	$*$	$8.08e+00$	$*$	$5.65e-01$	$*$	$2.61e-01$	$*$	4
6817	0.4330	$1.41e+00$	0.89	$4.21e+00$	0.94	$3.01e-01$	0.91	$2.02e-01$	0.37	4
53377	0.2165	$7.31e-01$	0.95	$2.14e+00$	0.97	$1.55e-01$	0.96	$1.15e-01$	0.82	4
422401	0.1083	$3.71e-01$	0.98	$1.07e+00$	1.00	$7.79e-02$	0.99	$5.34e-02$	1.10	4
3360769	0.0541	$1.87e-01$	0.99	$5.36e-01$	1.00	$3.90e-02$	1.00	$2.40e-02$	1.15	4

Table 7.4: Example 4, convergence history and Newton iteration count for the $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$ approximation of the Navier-Stokes model with variable viscosity, and convergence of the P_0-approximation of the postprocessed pressure field.

7.5 Example 5: 3D cavity problem

To conclude the set of numerical examples, we apply our mixed method with $\ell = 0$ to the driven cavity flow problem in the cube domain $\Omega = (0,1)^3$ by using the same sequence of quasi-uniform mesh refinements from Example 4. Again, the viscosity is taken as the Carreau law (2.5) with $\alpha_0 = 1$, $\alpha_1 = 0.1$, and $\beta = 1$. The external body force is zero, and the three-dimensional flow patterns are determined by the boundary conditions only: a unidirectional Dirichlet velocity is set on the top lid $\mathbf{g} := (1,0,0)^{\mathbf{t}}$, and no-slip velocity $\mathbf{u} = \mathbf{0}$ are imposed elsewhere on Γ. Some approximate solutions obtained with dof $= 3360769$ are depicted in Figure 7.5. As expected, abrupt changes are observed near the top corners of the domain, where the Dirichlet datum is discontinuous, and where the pseudostress is concentrated. The maximum number of iterations required over the course of the Newton-Raphson loop was 3.

References

[1] J.A. Almonacid, H.S. Díaz, G.N. Gatica and A. Márquez, *A fully mixed finite element method for the coupling of the Stokes and Darcy-Forchheimer problems.* IMA J. Numer. Anal. 40 (2020), no. 2, 1454–1502.

[2] M.S Alnæs, J. Blechta and J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M.E. Rognes and G.N. Wells, *The FEniCS project version 1.5.* Arch. Numer. Softw. 3 (2015), no. 100, 9–23.

[3] P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent, *Multifrontal parallel distributed symmetric and unsymmetric solvers.* Comput. Methods Appl. Mech. Engrg. 184 (2000), 501–520.

[4] J. Camaño, C. García and R. Oyarzúa, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem.* Numer. Methods Partial Differential Equations 37 (2021), no. 5, 2895–2923.

[5] J. Camaño, G.N. Gatica, R. Oyarzúa and R. Ruiz-Baier, *An augmented stress-based mixed finite element method for the steady state Navier-Stokes equations with nonlinear viscosity.* Numer. Methods Partial Differential Equations 33 (2017), no. 5, 1692–1725.

[6] J. Camaño, G.N. Gatica, R. Oyarzúa and G. Tierra, *An augmented mixed finite element method for the Navier-Stokes equations with variable viscosity.* SIAM J. Numer. Anal. 54 (2016), no. 2, 1069–1092.

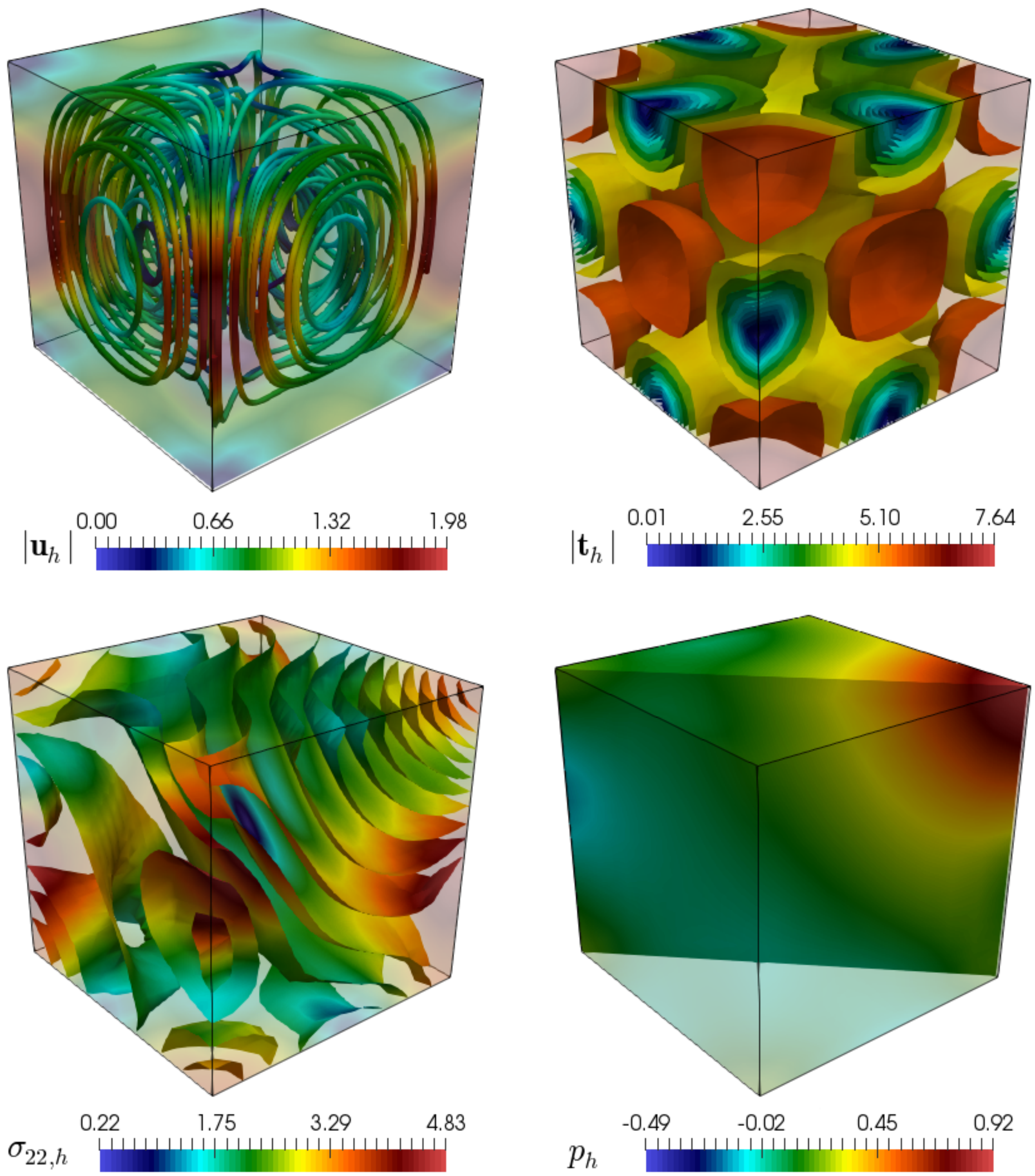

Figure 7.4: Example 4, numerical solutions using $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$ approximations with $\texttt{dof} = 3360769$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

[7] J. Camaño, C. Muñoz and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[8] J. Camaño, R. Oyarzúa and G. Tierra, *Analysis of an augmented mixed-FEM for the Navier-Stokes problem.* Math. Comp. 86 (2017), no. 304, 589–615.

[9] S. Caucao, E. Colmenares, G.N. Gatica and C. Inzunza, *A Banach spaces-based fully mixed finite element method for the stationary chemotaxis-Navier-Stokes problem.* Comp. Math. Appl. 145 (2023), 65–89.

[10] S. Caucao, G.N. Gatica and F. Sandoval, *A fully-mixed finite element method for the coupling of the Navier-Stokes and Darcy-Forchheimer equations.* Numer. Methods Partial Differential

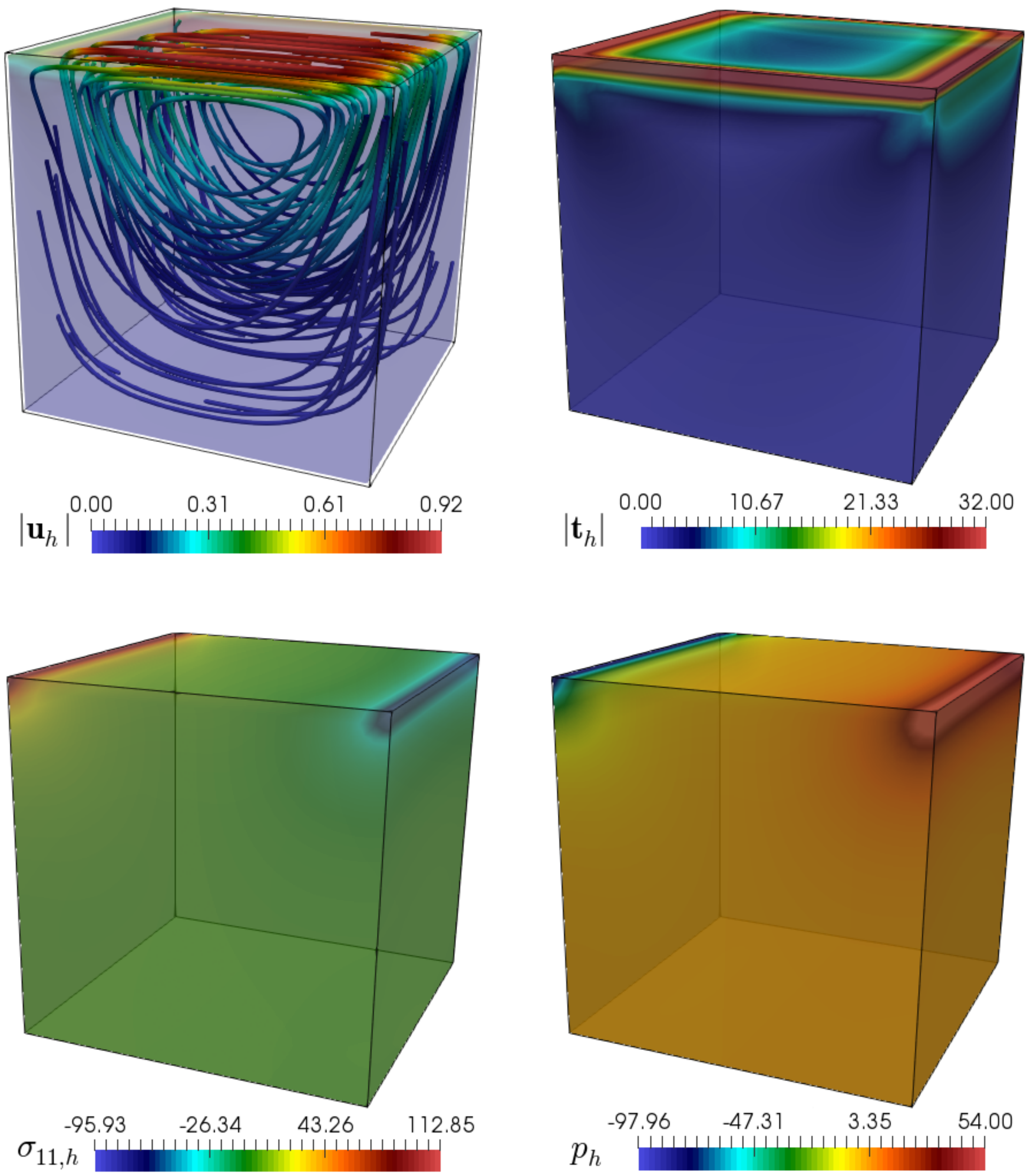

Figure 7.5: Example 5, numerical solutions using $\mathbb{P}_0-\mathbb{RT}_0-\mathbf{P}_0$ approximations with $\texttt{dof} = 3360769$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

Equations 37 (2021), no. 3, 2550–2587.

[11] E. Colmenares, G.N. Gatica and S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem*. ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.

[12] C.I. Correa, G.N. Gatica and R. Ruiz-Baier, *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces*. ESAIM Math. Model. Numer. Anal. 57 (2023), no. 3, 1511–1551.

[13] A. Ern and J.-L Guermond, Theory and Practice of Finite Elements. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[14] G.N. Gatica, *Solvability and Galerkin approximations of a class of nonlinear operator equations.* Z. Anal. Anwendungen 21 (2002), no. 3, 761–781.

[15] G.N. Gatica, A Simple Introduction to the Mixed Finite Element Method. Theory and Applications. SpringerBriefs in Mathematics. Springer, Cham, 2014.

[16] G.N. Gatica, N. Heuer and S. Meddahi, *On the numerical analysis of nonlinear twofold saddle point problems.* IMA J. Numer. Anal. 23 (2003), no. 2, 301–330.

[17] G.N. Gatica, S. Meddahi and R. Ruiz-Baier, *An* L^p *spaces-based formulation yielding a new fully mixed finite element method for the coupled Darcy and heat equations.* IMA J. Numer. Anal. 42 (2022), no. 4, 3154–3206.

[18] G.N. Gatica, N. Núñez and R. Ruiz-Baier, *New non-augmented mixed finite element methods for the Navier-Stokes-Brinkman equations using Banach spaces.* J. Numer. Math., https://doi.org/10.1515/jnma-2022-0073.

[19] G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier and Y.D. Sobral, *Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds.* Comput. Math. Appl. 84 (2021), 244–276.

[20] G.N. Gatica and W.L. Wendland, *Coupling of mixed finite elements and boundary elements for linear and nonlinear elliptic problems.* Appl. Anal. 63 (1996), no. 1-2, 39–75.

[21] J.S. Howell and N.J. Walkington, *Dual-mixed finite element methods for the Navier-Stokes equations.* ESAIM Math. Model. Numer. Anal. 47 (2013), no. 3, 789–805.