On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces*

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Abstract

In this paper we provide sufficient conditions for perturbed saddle-point formulations in Banach spaces and their associated Galerkin schemes to be well-posed. Our approach, which extends a similar procedure employed with Hilbert spaces, proceeds in two slightly different ways depending on whether the kernel of the adjoint operator induced by one of the bilinear forms is trivial or not. If the latter holds, we make use of an equivalence result between a couple of inf-sup conditions involved, which, differently from the Hilbertian case, turns out to hold with different constants. While this fact causes no inconvenient in the continuous analysis, it does become a delicate issue at the discrete level, and hence the corresponding inf-sup conditions need to be assumed separately with respective constants independent of the finite element subspaces employed. In turn, if that kernel is trivial, then we employ a suitable characterization of a closed range injective adjoint operator, so that only one of the aforementioned inf-sup conditions is then required for the analysis. The applicability of the continuous solvability is illustrated with a mixed formulation arising from the coupled Poisson-Nernst-Planck and Stokes equations.

Key words: saddle-point formulation, Banach spaces, inf-sup conditions, Poisson-Nernst-Planck equations

Mathematics subject classifications (2000): 65J05, 65N12, 65N15, 65N30, 76D07

1 Introduction

The purpose of this note is to analyze the solvability of the continuous and discrete schemes arising from perturbed saddle-point problems formulated in terms of Banach spaces. More precisely, given reflexive Banach spaces H and Q, bounded bilinear forms $a : \mathrm{H} \times \mathrm{H} \longrightarrow \mathrm{R}$, $b : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, and $c : \mathrm{Q} \times \mathrm{Q} \longrightarrow \mathrm{R}$, and functionals $f \in \mathrm{H}'$ and $g \in \mathrm{Q}'$, the formulation of interest consists of seeking $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \qquad \forall\, \tau \in \mathrm{H}\,, \\ b(\sigma,v) - c(u,v) &= g(v) \qquad \forall\, v \in \mathrm{Q}\,. \end{aligned} \tag{1.1}$$

In the particular case in which H and Q are Hilbert spaces, the well-posedness of (1.1) and its associated Galerkin scheme is very well established nowadays. We refer to [3, Theorem 1.2, Section II.1.2]

*This work was partially supported by ANID-Chile through the projects ACE210010, Centro de Modelamiento Matemático (FB210005), and Anillo of Computational Mathematics for Desalination Processes (ACT210087); and by Centro de Investigación en Ingeniería Matemática (CI^{2}MA).

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and [3, Proposition 2.11, Section II.2.4] for a through analysis of it, including the derivation of the corresponding Cea estimate. While several possible cases of the bilinear form c, which constitutes the so-called perturbation, are considered, the most frequent ones in applications are those in which, either the null space of the adjoint of the operator induced by b is trivial, or the bilinear form c is coercive on that kernel. Similar results to those in [3], though with slightly different proofs and providing further details, but still within a Hilbertian framework, are discussed in [2, Theorem 4.3.1, Sections 4.3.1] and [2, Theorem 5.5.1, Proposition 5.5.2, Section 5.5.1]. In turn, denoting by V and W the null spaces of the operator induced by b and its adjoint, respectively, we stress that a key result for the solvability analysis of (1.1) in the Hilbertian context is given by the identity (see, e.g. [2, eq. (4.3.18)])

$$\inf_{\substack{\tau\in \mathrm{V}^\perp\\ \tau\neq 0}} \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \,=\, \inf_{\substack{v\in \mathrm{W}^\perp\\ v\neq 0}} \sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}} \,>\, 0\,, \tag{1.2}$$

whose discrete version is also satisfied (see, e.g. [2, eq. (5.5.12)]).

According to the above discussion, and since the respective results do not seem to be available in the literature, the present work aims to extend the aforementioned theory to the Banach case. In this regard, we warn in advance that (1.2) is not going to hold for the continuous formulation nor for the discrete one, and hence the analysis and results to be presented below will take this fact into consideration, mainly when we deal with the Galerkin scheme of (1.1). Indeed, in this case the discrete inf-sup conditions arising from both sides of (1.2) require to be assumed separately with constants independent of the meshsizes. However, in the particular, though very frequent case in which W is the null subspace, we are able to apply a suitable characterization of closed range injective adjoint operators, so that for the solvability analysis it suffices to assume only the inf-sup condition arising from the right-hand side of (1.2). An analogous reasoning is valid if the discrete version of W, say W_h, is the null subspace as well.

The rest of the paper is organized as follows. In Section 2 we present some preliminary results on the spaces H and Q and the operators induced by the bilinear form b. In particular, we address here a key equivalence result between the inf-sup conditions involving b. Next, in Section 3 we establish the theorems providing the unique solvability of (1.1) and its associated Galerkin scheme. The presentation considers first a general situation in which nothing is said about W, and then the particular case in which it is assumed that $\mathrm{W} = \{0\}$. We proceed analogously for the discrete solvability. Finally, an application of the continuous theory to a mixed formulation arising from one of the equations forming part of the coupled Poisson-Nernst-Planck/Stokes model, is discussed in Section 4.

2 Preliminary results

In this section we present some previous results concerning the spaces and operators involved, which will be employed later on. To this end, we first let $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ and $\mathbf{B}^{\mathbf{t}} : \mathrm{Q} \to \mathrm{H}'$ be the bounded linear operators induced by b, that is

$$\mathbf{B}(\tau)(v) \,:=\, b(\tau,v) \quad \forall\,\tau\in\mathrm{H}\,,\ \forall\,v\in\mathrm{Q} \quad \text{and} \quad \mathbf{B}^{\mathbf{t}}(v)(\tau) \,:=\, b(\tau,v) \quad \forall\,v\in\mathrm{Q}\,,\ \forall\,\tau\in\mathrm{H}\,, \tag{2.1}$$

and introduce the respective null spaces

$$\mathrm{V} \,:=\, \mathrm{N}(\mathbf{B}) \,:=\, \Big\{\tau\in\mathrm{H} : \quad b(\tau,v) = 0 \quad \forall\,v\in\mathrm{Q}\Big\} \tag{2.2}$$

and

$$\mathrm{W} \,:=\, \mathrm{N}(\mathbf{B}^{\mathbf{t}}) \,:=\, \Big\{v\in\mathrm{Q} : \quad b(\tau,v) = 0 \quad \forall\,\tau\in\mathrm{H}\Big\}\,. \tag{2.3}$$

Next, we assume that V and W admit topological complements, which means that there exist closed subspaces $\mathrm{V}^{\perp}$ and $\mathrm{W}^{\perp}$ of H and Q, respectively, such that

$$\mathrm{H} \,=\, \mathrm{V} \oplus \mathrm{V}^{\perp} \qquad \text{and} \qquad \mathrm{Q} \,=\, \mathrm{W} \oplus \mathrm{W}^{\perp}\,, \tag{2.4}$$

and let $i : \mathrm{V}^{\perp} \to \mathrm{H}$ and $j : \mathrm{W}^{\perp} \to \mathrm{Q}$ be the respective injections. Notice that these complements are denoted using the symbol $^{\perp}$ just to keep the analogy with the orthogonal decomposition theorem in the Hilbert spaces case, but certainly we are aware of the fact that in the present discussion we have no inner products and hence no orthogonality concepts.

Furthermore, a direct application of the open mapping theorem implies the existence of positive constants C_{H} and C_{Q}, depending only on H and Q, respectively, such that

$$\|\tau_0\|_{\mathrm{H}} + \|\bar{\tau}\|_{\mathrm{H}} \,\le\, C_{\mathrm{H}}\, \|\tau\|_{\mathrm{H}} \quad \text{and} \quad \|v_0\|_{\mathrm{Q}} + \|\bar{v}\|_{\mathrm{Q}} \,\le\, C_{\mathrm{Q}}\, \|v\|_{\mathrm{Q}} \tag{2.5}$$

for all $\tau = \tau_0 + \bar{\tau} \in \mathrm{V} \oplus \mathrm{V}^{\perp}$, and for all $v = v_0 + \bar{v} \in \mathrm{W} \oplus \mathrm{W}^{\perp}$. As a consequence of these boundedness properties, we have the following result.

Lemma 2.1. *There hold*

$$\frac{1}{C_{\mathrm{H}}}\, \|\tau\|_{\mathrm{H}} \,\le\, \mathrm{dist}(\tau, \mathrm{V}) \,\le\, \|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}^{\perp}\,, \tag{2.6}$$

and

$$\frac{1}{C_{\mathrm{Q}}}\, \|v\|_{\mathrm{Q}} \,\le\, \mathrm{dist}(v, \mathrm{W}) \,\le\, \|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{W}^{\perp}\,. \tag{2.7}$$

Proof. We begin by noticing that the upper bounds of (2.6) and (2.7) are straightforward, and that they are actually valid for all $(\tau, v) \in \mathrm{H} \times \mathrm{Q}$. In addition, being the respective lower bounds proved analogously, it suffices to provide the proof for one of them, say (2.6). To this end, we first recall that if X is a reflexive Banach space and T is a closed subspace of X', there holds

$$\mathrm{dist}(x, {}^{\circ}\mathrm{T}) \,=\, \sup_{\substack{\mathrm{F} \in \mathrm{T} \\ \mathrm{F} \neq 0}} \frac{|\mathrm{F}(x)|}{\|\mathrm{F}\|_{X'}} \qquad \forall\, x \in \mathrm{X}\,.$$

Thus, applying this identity to $\mathrm{X} = \mathrm{H}$ and $\mathrm{T} = \mathrm{V}^{\circ}$, and using that ${}^{\circ}\big(\mathrm{V}^{\circ}\big) = \mathrm{V}$, we deduce that

$$\mathrm{dist}(\tau, \mathrm{V}) \,=\, \sup_{\substack{\mathrm{F} \in \mathrm{V}^{\circ} \\ \mathrm{F} \neq 0}} \frac{|\mathrm{F}(\tau)|}{\|\mathrm{F}\|_{\mathrm{H}'}} \qquad \forall\, \tau \in \mathrm{H}\,. \tag{2.8}$$

Next, we restrict to $\tau \in \mathrm{V}^{\perp}$. Then, given $\mathrm{G} \in \mathrm{H}'$, we define the functional $g : \mathrm{H} \to \mathrm{R}$ by $g(\zeta) := \mathrm{G}(\bar{\zeta})$ for all $\zeta = \zeta_0 + \bar{\zeta} \in \mathrm{H} = \mathrm{V} \oplus \mathrm{V}^{\perp}$. It follows that g is linear, $g|_{\mathrm{V}} \equiv 0$, and, using (2.5),

$$|g(\zeta)| \,=\, |\mathrm{G}(\bar{\zeta})| \,\le\, \|\mathrm{G}\|_{\mathrm{H}'}\, \|\bar{\zeta}\|_{\mathrm{H}} \,\le\, C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}\, \|\zeta\|_{\mathrm{H}} \quad \forall\, \zeta \in \mathrm{H}\,,$$

which says that g is bounded, with $\|g\|_{\mathrm{H}'} \,\le\, C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}$, and hence $g \in \mathrm{V}^{\circ}$. In this way, according to (2.8), and noting that $g(\tau) = \mathrm{G}(\tau)$, we find that

$$\mathrm{dist}(\tau, \mathrm{V}) \,\ge\, \frac{|g(\tau)|}{\|g\|_{\mathrm{H}'}} \,\ge\, \frac{|\mathrm{G}(\tau)|}{C_{\mathrm{H}}\, \|\mathrm{G}\|_{\mathrm{H}'}}$$

from which, taking supremum with respect to $\mathrm{G} \in \mathrm{H}'$, we conclude that

$$\mathrm{dist}(\tau, \mathrm{V}) \,\ge\, \frac{1}{C_{\mathrm{H}}}\, \|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}^{\perp}\,,$$

thus finishing the proof of (2.6). □

Some equivalence properties connecting $\mathbf{B}$ and $\mathbf{B}^{\mathsf{t}}$ are established next.

Lemma 2.2. *The following statements are equivalent:*

i) $\mathbf{B}^{\mathsf{t}} \circ j : \mathrm{W}^{\perp} \to \mathrm{H}'$ *is injective and of closed range, that is there exists a constant* $\widetilde{\beta} > 0$ *such that*

$$\|\mathbf{B}^{\mathsf{t}}(v)\|_{\mathrm{H}'} \,:=\, \sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \,\geq\, \widetilde{\beta}\,\|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{W}^{\perp}\,. \tag{2.9}$$

ii) $j' \circ \mathbf{B} : \mathrm{H} \to \big(\mathrm{W}^{\perp}\big)'$ *is surjective.*

iii) $\mathbf{B} \circ i : \mathrm{V}^{\perp} \to \mathrm{Q}'$ *is injective and of closed range, that is there exists a constant* $\widehat{\beta} > 0$ *such that*

$$\|\mathbf{B}(\tau)\|_{\mathrm{Q}'} \,:=\, \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}} \,\geq\, \widehat{\beta}\,\|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}^{\perp}\,. \tag{2.10}$$

iv) $i' \circ \mathbf{B}^{\mathsf{t}} : \mathrm{Q} \to \big(\mathrm{V}^{\perp}\big)'$ *is surjective.*

Proof. Let $\mathcal{J}_{\mathrm{H}} : \mathrm{H} \to \mathrm{H}''$ and $\mathcal{J}_{\mathrm{Q}} : \mathrm{Q} \to \mathrm{Q}''$ be the isometric and bijective linear mappings given by

$$\mathcal{J}_{\mathrm{H}}(\tau)(\mathrm{F}) \,:=\, \mathrm{F}(\tau) \quad \forall\, \tau \in \mathrm{H}\,, \quad \forall\, \mathrm{F} \in \mathrm{H}' \quad \text{and} \quad \mathcal{J}_{\mathrm{Q}}(v)(\mathrm{G}) \,:=\, \mathrm{G}(v) \quad \forall\, v \in \mathrm{Q}\,, \quad \forall\, \mathrm{G} \in \mathrm{Q}'\,,$$

and observe, as suggested by the diagrams

$$\mathrm{H} \xrightarrow{\mathcal{J}_{\mathrm{H}}} \mathrm{H}'' \xrightarrow{(\mathbf{B}^{\mathsf{t}})'} \mathrm{Q}' \quad \text{and} \quad \mathrm{Q} \xrightarrow{\mathcal{J}_{\mathrm{Q}}} \mathrm{Q}'' \xrightarrow{\mathbf{B}'} \mathrm{H}'\,,$$

that there holds

$$\mathbf{B} \,=\, (\mathbf{B}^{\mathsf{t}})' \circ \mathcal{J}_{\mathrm{H}} \quad \text{and} \quad \mathbf{B}^{\mathsf{t}} \,=\, \mathbf{B}' \circ \mathcal{J}_{\mathrm{Q}}\,. \tag{2.11}$$

Indeed, given $\tau \in \mathrm{H}$ and $v \in \mathrm{Q}$, we obtain

$$\big((\mathbf{B}^{\mathsf{t}})' \circ \mathcal{J}_{\mathrm{H}}\big)(\tau)(v) \,=\, (\mathbf{B}^{\mathsf{t}})'\big(\mathcal{J}_{\mathrm{H}}(\tau)\big)(v) \,=\, \mathcal{J}_{\mathrm{H}}(\tau)\big(\mathbf{B}^{\mathsf{t}}(v)\big) \,=\, \mathbf{B}^{\mathsf{t}}(v)(\tau) \,=\, \mathbf{B}(\tau)(v)\,,$$

which proves the first identity of (2.11). The second one proceeds similarly or as a consequence of the first one after exchanging $\mathbf{B}$ with $\mathbf{B}^{\mathsf{t}}$ and the roles of the spaces H and Q. It follows from (2.11) that

$$j' \circ \mathbf{B} \,=\, \big(j' \circ (\mathbf{B}^{\mathsf{t}})'\big) \circ \mathcal{J}_{\mathrm{H}} \,=\, \big(\mathbf{B}^{\mathsf{t}} \circ j\big)' \circ \mathcal{J}_{\mathrm{H}}\,, \tag{2.12}$$

and hence, bearing in mind the bijectivity of $\mathcal{J}_{\mathrm{H}}$, we deduce that $j' \circ \mathbf{B} : \mathrm{H} \to \big(\mathrm{W}^{\perp}\big)'$ is surjective if and only if $\big(\mathbf{B}^{\mathsf{t}} \circ j\big)' : \mathrm{H}'' \to \big(\mathrm{W}^{\perp}\big)'$ is surjective as well, which, in turn, is equivalent to stating that $\mathbf{B}^{\mathsf{t}} \circ j : \mathrm{W}^{\perp} \to \mathrm{H}'$ is injective and of closed range. The above shows the equivalence between i) and ii). Analogously, employing from the second identity in (2.11) that

$$i' \circ \mathbf{B}^{\mathsf{t}} \,=\, \big(i' \circ \mathbf{B}'\big) \circ \mathcal{J}_{\mathrm{Q}} \,=\, \big(\mathbf{B} \circ i\big)' \circ \mathcal{J}_{\mathrm{Q}}\,, \tag{2.13}$$

we are able to prove that iii) and iv) are equivalent. In order to conclude the proof, it suffices to see, for instance, that i) and iii) share the same property, which is addressed in what follows. Indeed, let us assume now that i) holds. Then, knowing that $\mathbf{B}^{\mathsf{t}} \circ j$ has closed range, we have that $R(\mathbf{B}^{\mathsf{t}} \circ j) \,=\, {}^{\circ}\mathrm{N}\big((\mathbf{B}^{\mathsf{t}} \circ j)'\big)$, where, according to (2.12), $\mathrm{N}\big((\mathbf{B}^{\mathsf{t}} \circ j)'\big) \,=\, \mathcal{J}_{\mathrm{H}}\big(\mathrm{N}(j' \circ \mathbf{B})\big)$. A simple computation yields

$$\mathrm{N}(j' \circ \mathbf{B}) \,=\, \Big\{\tau \in \mathrm{H} : \quad j'\big(\mathbf{B}(\tau)\big)(v) = \mathbf{B}(\tau)(v) = 0 \quad \forall\, v \in \mathrm{W}^{\perp}\Big\} \,=\, \mathrm{V}\,,$$

and hence $R(\mathbf{B}^{\mathtt{t}} \circ j) = {}^{\circ}\mathcal{J}_{\mathrm{H}}(\mathrm{V}) = \mathrm{V}^{\circ}$. In this way, we conclude that $\mathbf{B}^{\mathtt{t}} \circ j : \mathrm{W}^{\perp} \to \mathrm{V}^{\circ}$ is bijective, and (2.9) implies that $\|(\mathbf{B}^{\mathtt{t}} \circ j)^{-1}\| \leq \frac{1}{\widetilde{\beta}}$. It follows that $(\mathbf{B}^{\mathtt{t}} \circ j)' : (\mathrm{V}^{\circ})' \to (\mathrm{W}^{\perp})'$ is bijective as well, and

$$\|\big((\mathbf{B}^{\mathtt{t}} \circ j)'\big)^{-1}\| = \|\big((\mathbf{B}^{\mathtt{t}} \circ j)^{-1}\big)'\| = \|(\mathbf{B}^{\mathtt{t}} \circ j)^{-1}\| \leq \frac{1}{\widetilde{\beta}},$$

which says, equivalently, that

$$\|(\mathbf{B}^{\mathtt{t}} \circ j)'(\mathcal{G})\|_{(\mathrm{W}^{\perp})'} \geq \widetilde{\beta}\, \|\mathcal{G}\|_{(\mathrm{V}^{\circ})'} \qquad \forall\, \mathcal{G} \in (\mathrm{V}^{\circ})'. \tag{2.14}$$

In particular, taking $\mathcal{G} = \mathcal{J}_{\mathrm{H}}(\tau)|_{\mathrm{V}^{\circ}}$, with $\tau \in H$, we obtain

$$\|(\mathbf{B}^{\mathtt{t}} \circ j)'(\mathcal{G})\|_{(\mathrm{W}^{\perp})'} = \sup_{\substack{v \in \mathrm{W}^{\perp} \\ v \neq 0}} \frac{(\mathbf{B}^{\mathtt{t}} \circ j)'(\mathcal{J}_{\mathrm{H}}(\tau))(v)}{\|v\|_{\mathrm{Q}}} = \sup_{\substack{v \in \mathrm{W}^{\perp} \\ v \neq 0}} \frac{\mathcal{J}_{\mathrm{H}}(\tau)\big(\mathbf{B}^{\mathtt{t}}(v)\big)}{\|v\|_{\mathrm{Q}}} = \sup_{\substack{v \in \mathrm{W}^{\perp} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}}, \tag{2.15}$$

whereas, making use of (2.8) in the last equality below, we find that

$$\|\mathcal{J}_{\mathrm{H}}(\tau)|_{\mathrm{V}^{\circ}}\|_{(\mathrm{V}^{\circ})'} := \sup_{\substack{\mathrm{F} \in \mathrm{V}^{\circ} \\ \mathrm{F} \neq 0}} \frac{\mathcal{J}_{\mathrm{H}}(\tau)(\mathrm{F})}{\|\mathrm{F}\|_{\mathrm{H}'}} = \sup_{\substack{\mathrm{F} \in \mathrm{V}^{\circ} \\ \mathrm{F} \neq 0}} \frac{\mathrm{F}(\tau)}{\|\mathrm{F}\|_{\mathrm{H}'}} = \mathrm{dist}(\tau, \mathrm{V}). \tag{2.16}$$

In this way, replacing (2.15) and (2.16) back into (2.14), we conclude that

$$\|\mathbf{B}(\tau)\|_{\mathrm{Q}'} \geq \sup_{\substack{v \in \mathrm{W}^{\perp} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}} \geq \widetilde{\beta}\, \mathrm{dist}(\tau, \mathrm{V}) \qquad \forall\, \tau \in \mathrm{H}, \tag{2.17}$$

which, together with the lower bound of (2.6), yields iii) (cf. (2.10)) with $\widehat{\beta} := \frac{\widetilde{\beta}}{C_{\mathrm{H}}}$. Conversely, let us assume that iii) holds. Then, in order to prove i), we proceed analogously to the opposite implication. In particular, using now (2.13) one deduces that $R(\mathbf{B} \circ i) = \mathrm{W}^{\circ}$, so that $\mathbf{B} \circ i : \mathrm{V}^{\perp} \longrightarrow \mathrm{W}^{\circ}$ and $(\mathbf{B} \circ i)' : \big(\mathrm{W}^{\circ}\big)' \longrightarrow \big(\mathrm{V}^{\perp}\big)'$ are bijective with $\|\big((\mathbf{B} \circ i)'\big)^{-1}\| = \|\big(\mathbf{B} \circ i\big)^{-1}\| \leq \frac{1}{\widehat{\beta}}$. In this way, we get the analogue of (2.17), that is

$$\|\mathbf{B}^{\mathtt{t}}(v)\|_{\mathrm{H}'} \geq \sup_{\substack{\tau \in \mathrm{V}^{\perp} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geq \widehat{\beta}\, \mathrm{dist}(v, \mathrm{W}) \qquad \forall\, v \in \mathrm{Q}, \tag{2.18}$$

from which, along with (2.7), we arrive at (2.9) with $\widetilde{\beta} := \frac{\widehat{\beta}}{C_{\mathrm{Q}}}$. Further details are omitted. $\square$

We find it important to emphasize here, as announced in Section 1, that the equivalence between the inf-sup conditions (2.9) (cf. i)) and (2.10) (cf. iii)) holds with different constants $\widetilde{\beta}$ and $\widehat{\beta}$. Indeed, from the proof of Lemma 2.2 we notice that, starting from i), we first derive the inequality (2.17) with the same constant $\widetilde{\beta}$, thus yielding the partial implication summarized as

$$\left\{ \sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geq \widetilde{\beta}\, \|v\|_{\mathrm{Q}} \quad \forall\, v \in \mathrm{W}^{\perp} \right\} \quad \Rightarrow \quad \left\{ \sup_{\substack{v \in \mathrm{Q} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}} \geq \widetilde{\beta}\, \mathrm{dist}(\tau, \mathrm{V}) \quad \forall\, \tau \in \mathrm{H} \right\}. \tag{2.19}$$

Similarly, starting from iii), we obtain (2.18) with the same constant $\widehat{\beta}$, which gives rise to the partial implication

$$\left\{ \sup_{\substack{v \in \mathrm{Q} \\ v \neq 0}} \frac{b(\tau, v)}{\|v\|_{\mathrm{Q}}} \geq \widehat{\beta}\, \|\tau\|_{\mathrm{H}} \quad \forall\, \tau \in \mathrm{V}^{\perp} \right\} \quad \Rightarrow \quad \left\{ \sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geq \widehat{\beta}\, \mathrm{dist}(v, \mathrm{W}) \quad \forall\, v \in \mathrm{Q} \right\}. \tag{2.20}$$

However, as observed in the aforementioned proof, the expressions given by $\widetilde{\beta}\,\mathrm{dist}(\tau,\mathrm{V})$ in (2.19) and $\widehat{\beta}\,\mathrm{dist}(v,\mathrm{W})$ in (2.20) are then bounded below, respectively, by $\widehat{\beta}\,\|\tau\|_{\mathrm{H}}$ for each $\tau\in\mathrm{V}^{\perp}$, with $\widehat{\beta}=\frac{\widetilde{\beta}}{C_{\mathrm{H}}}$, and by $\widetilde{\beta}\,\|v\|_{\mathrm{Q}}$ for each $v\in\mathrm{W}^{\perp}$, with $\widetilde{\beta}=\frac{\widehat{\beta}}{C_{\mathrm{Q}}}$. Differently from the above, when H and Q are Hilbert spaces, there hold $\mathrm{dist}(\tau,\mathrm{V})=\|\tau\|_{\mathrm{H}}$ for each $\tau\in\mathrm{V}^{\perp}$, and $\mathrm{dist}(v,\mathrm{W})=\|v\|_{\mathrm{Q}}$ for each $v\in\mathrm{W}^{\perp}$, so that in this case the equivalence between i) and iii) does hold with the same constant $\widetilde{\beta}=\widehat{\beta}$, as it has already been established in the available bibliography (see, e.g. [2, eq. (4.3.18), Theorem 4.3.1], [7, Lemma 2.1], and [3, Proposition 1.2 and eqs. (1.15) and (1.16), Chapter II]). Moreover, this fact can be written, equivalently, as

$$\inf_{\substack{\tau\in\mathrm{V}^{\perp}\\ \tau\neq 0}}\sup_{\substack{v\in\mathrm{Q}\\ v\neq 0}}\frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}}\;=\;\inf_{\substack{v\in\mathrm{W}^{\perp}\\ v\neq 0}}\sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}}\frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}}}\;=\;\widetilde{\beta}\;>\;0\,,$$

which is exactly what what was highlighted in (1.2) (cf. Section 1).

3 The main results

In this section we address the well-posedness of (1.1) and its associated Galerkin scheme.

3.1 An equivalent setting

We begin by observing that the perturbed saddle-point formulation (1.1) can be re-stated, equivalently, as: Find $(\sigma,u)\in\mathrm{H}\times\mathrm{Q}$ such that

$$\mathbf{A}\big((\sigma,u),(\tau,v)\big)\;=\;\mathbf{F}(\tau,v)\qquad\forall\,(\tau,v)\in\mathrm{H}\times\mathrm{Q}\,,\tag{3.1}$$

where $\mathbf{A}:(\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q})\to\mathrm{R}$ and $\mathbf{F}:\mathrm{H}\times\mathrm{Q}\to\mathrm{R}$ are the bounded bilinear form and linear functional, respectively, defined by

$$\mathbf{A}\big((\zeta,w),(\tau,v)\big)\;:=\;a(\zeta,\tau)+b(\tau,w)+b(\zeta,v)-c(w,v)\quad\forall\,(\zeta,w),\,(\tau,v)\in\mathrm{H}\times\mathrm{Q}\,,\tag{3.2}$$

and

$$\mathbf{F}(\tau,v)\;:=\;f(\tau)+g(v)\qquad\forall\,(\tau,v)\in\mathrm{H}\times\mathrm{Q}\,.\tag{3.3}$$

Throughout the rest of the paper we consider the product norm

$$\|(\tau,v)\|_{\mathrm{H}\times\mathrm{Q}}\;:=\;\|\tau\|_{\mathrm{H}}+\|v\|_{\mathrm{Q}}\qquad\forall\,(\tau,v)\in\mathrm{H}\times\mathrm{Q}\,.$$

Thus, resorting to the Banach-Nečas-Babuška theorem (cf. [6, Theorem 2.6]), also known as the generalized Lax-Milgram lemma, we deduce that (3.1) (equivalently (1.1)) is well-posed if and only if the following hypotheses are satisfied:

1) there exists a constant $\alpha>0$ such that

$$S(\zeta,w):=\sup_{\substack{(\tau,v)\in\mathrm{H}\times\mathrm{Q}\\ (\tau,v)\neq\mathbf{0}}}\frac{\mathbf{A}\big((\zeta,w),(\tau,v)\big)}{\|(\tau,v)\|_{\mathrm{H}\times\mathrm{Q}}}\;\geq\;\alpha\,\|(\zeta,w)\|_{\mathrm{H}\times\mathrm{Q}}\qquad\forall\,(\zeta,w)\in\mathrm{H}\times\mathrm{Q}\,.\tag{3.4}$$

2) for each $(\tau,v)\in\mathrm{H}\times\mathrm{Q}$, $(\tau,v)\neq\mathbf{0}$:

$$\sup_{(\zeta,w)\in\mathrm{H}\times\mathrm{Q}}\mathbf{A}\big((\zeta,w),(\tau,v)\big)\;>\;0\,.\tag{3.5}$$

Certainly, when $\mathbf{A}$ is symmetric, which is equivalent to assume that a and c are, 2) is redundant and hence it suffices to prove 1). In this regard, we stress that the supremum in (3.4) is equivalent to the expression $\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}$, where

$$\mathrm{F}_{(\zeta,w)}(\tau) \,:=\, \mathbf{A}\big((\zeta,w),(\tau,0)\big) \qquad \forall\,\tau \in \mathrm{H}\,, \tag{3.6}$$

and

$$\mathrm{G}_{(\zeta,w)}(v) \,:=\, \mathbf{A}\big((\zeta,w),(0,v)\big) \qquad \forall\,v \in \mathrm{Q}\,. \tag{3.7}$$

More precisely, it is easy to see that

$$\frac{1}{2}\Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\} \,\le\, S(\zeta,w) \,\le\, \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \quad \forall\,(\zeta,w) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.8}$$

Consequently, a necessary and sufficient condition for 1) is given by the existence of a constant $\widetilde{C} > 0$ such that

$$\|(\zeta,w)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \widetilde{C}\,\Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} + \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\} \qquad \forall\,(\zeta,w) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.9}$$

The above is basically the same procedure that was utilized in the proof of [3, Theorem 1.2, Chapter II] for the Hilbert version of (1.1), as well as the one that, except for some necessary modifications, we adopt below in Section 3.2 for the proof of the main theorem.

From now on we denote by $\|a\|$, $\|b\|$, and $\|c\|$, the smallest positive constants such that

$$\begin{aligned} |a(\zeta,\tau)| \,&\le\, \|a\|\,\|\zeta\|_{\mathrm{H}}\,\|\tau\|_{\mathrm{H}} \qquad && \forall\,(\zeta,\tau) \in \mathrm{H}\times\mathrm{H}\,,\\ |b(\tau,v)| \,&\le\, \|b\|\,\|\tau\|_{\mathrm{H}}\,\|v\|_{\mathrm{Q}} \qquad && \forall\,(\tau,v) \in \mathrm{H}\times\mathrm{Q}\,,\\ |c(w,v)| \,&\le\, \|c\|\,\|w\|_{\mathrm{Q}}\,\|v\|_{\mathrm{Q}} \qquad && \forall\,(w,v) \in \mathrm{Q}\times\mathrm{Q}\,. \end{aligned} \tag{3.10}$$

3.2 Continuous solvability

The main result providing sufficient conditions for the solvability of (1.1) is established now. While some of the definitions and hypotheses have already been introduced, for sake of clearness we include them again in its statement.

Theorem 3.1. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$, $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$, *and* $c : \mathrm{Q}\times\mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms (cf. (3.10)). In addition, let* $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ *and* $\mathbf{B}^{\mathrm{t}} : \mathrm{Q} \to \mathrm{H}'$ *be the bounded linear operators induced by* b *(cf. (2.1)), and let* $\mathrm{V} := \mathrm{N}(\mathbf{B})$ *and* $\mathrm{W} := \mathrm{N}(\mathbf{B}^{\mathrm{t}})$ *be the respective null spaces (cf. (2.2), (2.3)). Assume that:*

- **i)** *there exist closed subspaces* $\mathrm{V}^{\perp}$ *and* $\mathrm{W}^{\perp}$ *of* H *and* Q*, respectively, such that* $\mathrm{H} = \mathrm{V} \oplus \mathrm{V}^{\perp}$ *and* $\mathrm{Q} = \mathrm{W} \oplus \mathrm{W}^{\perp}$,
- **ii)** a *and* c *are symmetric and positive semi-definite, the latter meaning that*

$$a(\tau,\tau) \,\ge\, 0 \quad \forall\,\tau \in \mathrm{H} \quad \textit{and} \quad c(v,v) \,\ge\, 0 \quad \forall\,v \in \mathrm{Q}\,, \tag{3.11}$$

- **iii)** *there exists a constant* $\widetilde{\alpha} > 0$ *such that*

$$\sup_{\substack{\tau\in\mathrm{V}\\ \tau\neq 0}} \frac{a(\vartheta,\tau)}{\|\tau\|_{\mathrm{H}}} \,\ge\, \widetilde{\alpha}\,\|\vartheta\|_{\mathrm{H}} \qquad \forall\,\vartheta \in \mathrm{V}\,, \tag{3.12}$$

iv) *there exists a constant* $\widetilde{\beta} > 0$ *such that* (*cf.* (2.9))

$$\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \,\ge\, \widetilde{\beta}\,\|v\|_{\mathrm{Q}} \qquad \forall\, v\in \mathrm{W}^{\perp}\,, \tag{3.13}$$

v) *and there exists a constant* $\widetilde{\gamma} > 0$ *such that*

$$\sup_{\substack{v\in \mathrm{W}\\ v\neq 0}} \frac{c(z,v)}{\|v\|_{\mathrm{Q}}} \,\ge\, \widetilde{\gamma}\,\|z\|_{\mathrm{Q}} \qquad \forall\, z\in \mathrm{W}\,. \tag{3.14}$$

Then, for each pair $(f,g)\in \mathrm{H}'\times\mathrm{Q}'$ *there exists a unique* $(\sigma,u)\in \mathrm{H}\times\mathrm{Q}$ *solution to* (1.1) (*equivalently* (3.1)). *Moreover, there exists a constant* $\widetilde{C} > 0$, *depending only on* $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, $\widetilde{\beta}$, C_{H} (*cf.* (2.5)), *and* $\widetilde{\gamma}$, *such that*

$$\|(\sigma,u)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \widetilde{C}\,\Big\{\|f\|_{\mathrm{H}'} \,+\, \|g\|_{\mathrm{Q}'}\Big\}\,. \tag{3.15}$$

Proof. Because of the assumed symmetry of a and c (cf. **ii)**), and as previously remarked, the proof reduces to show (3.9). Hence, given $(\zeta,w)\in \mathrm{H}\times\mathrm{Q}$, we first define the functionals $\mathrm{F}_{(\zeta,w)}\in\mathrm{H}'$ and $\mathrm{G}_{(\zeta,w)}\in\mathrm{Q}'$ according to (3.6) and (3.7), respectively, that is

$$\mathrm{F}_{(\zeta,w)}(\tau) \,:=\, a(\zeta,\tau) + b(\tau,w) \qquad \forall\,\tau\in\mathrm{H}\,, \tag{3.16}$$

and

$$\mathrm{G}_{(\zeta,w)}(v) \,:=\, b(\zeta,v) - c(w,v) \qquad \forall\, v\in\mathrm{Q}\,. \tag{3.17}$$

Now, according to **i)**, we decompose ζ and w as

$$\zeta \,=\, \zeta_0 + \bar{\zeta} \quad \text{and} \quad w \,=\, w_0 + \bar{w}\,, \tag{3.18}$$

with $\zeta_0\in\mathrm{V}$, $\bar{\zeta}\in\mathrm{V}^{\perp}$, $w_0\in\mathrm{W}$, and $\bar{w}\in\mathrm{W}^{\perp}$. Therefore, the rest of the proof consists of bounding each one of the four components specified in (3.18). We begin by observing from (3.16) that $\mathrm{F}_{(\zeta,w)}(\tau) = a(\zeta,\tau)$ for all $\tau\in\mathrm{V}$, so that applying (3.12) (cf. **iii)**) to $\vartheta=\zeta_0$, we get

$$\widetilde{\alpha}\,\|\zeta_0\|_{\mathrm{H}} \,\le\, \sup_{\substack{\tau\in \mathrm{V}\\ \tau\neq 0}} \frac{a(\zeta_0,\tau)}{\|\tau\|_{\mathrm{H}}} \,=\, \sup_{\substack{\tau\in \mathrm{V}\\ \tau\neq 0}} \frac{\mathrm{F}_{(\zeta,w)}(\tau) - a(\bar{\zeta},\tau)}{\|\tau\|_{\mathrm{H}}}\,,$$

from which it readily follows that

$$\|\zeta_0\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\alpha}}\,\|\bar{\zeta}\|_{\mathrm{H}}\,. \tag{3.19}$$

In turn, in order to bound $\bar{\zeta}$, we employ the equivalence between i) and iii) of Lemma 2.2, thanks to which and (3.13) (cf. **iv)**), there holds (cf. (2.10))

$$\widehat{\beta}\,\|\tau\|_{\mathrm{H}} \,\le\, \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\tau,v)}{\|v\|_{\mathrm{Q}}} \qquad \forall\,\tau\in\mathrm{V}^{\perp}\,,$$

with $\widehat{\beta} := \frac{\widetilde{\beta}}{C_{\mathrm{H}}}$. Thus, noting from (3.17) that $\mathrm{G}_{(\zeta,w)}(v) = b(\bar{\zeta},v) - c(w,v)$ for all $v\in\mathrm{Q}$, and applying the foregoing inequality to $\tau=\bar{\zeta}\in\mathrm{V}^{\perp}$, we find that

$$\widehat{\beta}\,\|\bar{\zeta}\|_{\mathrm{H}} \,\le\, \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\bar{\zeta},v)}{\|v\|_{\mathrm{Q}}} \,=\, \sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{\mathrm{G}_{(\zeta,w)}(v) + c(w,v)}{\|v\|_{\mathrm{Q}}}\,,$$

from which, using that $c(w,v) \le \|c\|^{1/2}\,|w|_c\,\|v\|_{\mathrm{Q}}$, with $|w|_c := \big(c(w,w)\big)^{1/2}$, which takes into account the positive semi-definiteness of c (cf. **ii)**), we deduce that

$$\|\bar{\zeta}\|_{\mathrm{H}} \,\le\, \frac{1}{\widehat{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \frac{\|c\|^{1/2}}{\widehat{\beta}}\,|w|_c\,. \tag{3.20}$$

Thus, as a direct consequence of (3.19) and (3.20), we have the following preliminary bound

$$\|\zeta\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{1}{\widehat{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{\|c\|^{1/2}}{\widehat{\beta}}\,|w|_c\,. \tag{3.21}$$

Certainly, it remains to bound $|w|_c$ in terms of $\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}$ and $\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}$, which will be done later on. Meanwhile, we address the estimate of $\|w\|_{\mathrm{Q}}$. In fact, from the definition of $\mathrm{G}_{(\zeta,w)}$ (cf. (3.17)) we have $\mathrm{G}_{(\zeta,w)}(v) = -c(w,v) = -c(w_0,v) - c(\bar{w},v)$ for all $v \in \mathrm{W}$, and hence, applying (3.14) (cf. **v)**) to $z = w_0 \in \mathrm{W}$, we get

$$\widetilde{\gamma}\,\|w_0\|_{\mathrm{Q}} \,\le\, \sup_{\substack{v\in\mathrm{W}\\ v\neq 0}} \frac{c(w_0,v)}{\|v\|_{\mathrm{Q}}} \,=\, \sup_{\substack{v\in\mathrm{W}\\ v\neq 0}} \frac{-\mathrm{G}_{(\zeta,w)}(v) - c(\bar{w},v)}{\|v\|_{\mathrm{Q}}}\,, \tag{3.22}$$

which yields

$$\|w_0\|_{\mathrm{Q}} \,\le\, \frac{1}{\widetilde{\gamma}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \frac{\|c\|}{\widetilde{\gamma}}\,\|\bar{w}\|_{\mathrm{Q}}\,. \tag{3.23}$$

Furthermore, it is clear from (3.16) that $\mathrm{F}_{(\zeta,w)}(\tau) = a(\zeta,\tau) + b(\tau,\bar{w})$ for all $\tau \in \mathrm{H}$, so that making use of (3.13) (cf. **iv)**) with $v = \bar{w} \in \mathrm{W}^{\perp}$, we arrive at

$$\widetilde{\beta}\,\|\bar{w}\|_{\mathrm{Q}} \le \sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,\bar{w})}{\|\tau\|_{\mathrm{H}}} \,=\, \sup_{\substack{\tau\in\mathrm{H}\\ \tau\neq 0}} \frac{\mathrm{F}_{(\zeta,w)}(\tau) - a(\zeta,\tau)}{\|\tau\|_{\mathrm{H}}}\,, \tag{3.24}$$

which implies that

$$\|\bar{w}\|_{\mathrm{Q}} \,\le\, \frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.25}$$

In this way, (3.23) and (3.25) give

$$\|w\|_{\mathrm{Q}} \,\le\, \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{1}{\widetilde{\gamma}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.26}$$

On the other hand, we now aim to bound $|w|_c^2 := c(w,w)$. Indeed, evaluating $\mathrm{F}_{(\zeta,w)}$ (cf. (3.16)) and $\mathrm{G}_{(\zeta,w)}$ (cf. (3.17)) in ζ and w, respectively, and subtracting the resulting expressions, we obtain

$$a(\zeta,\zeta) + c(w,w) \,=\, \mathrm{F}_{(\zeta,w)}(\zeta) \,-\, \mathrm{G}_{(\zeta,w)}(w)\,,$$

from which, according to the positive semi-definiteness of a (cf. **ii)**), it follows that

$$|w|_c^2 \,\le\, \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}\,\|\zeta\|_{\mathrm{H}} \,+\, \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\,\|w\|_{\mathrm{Q}}\,. \tag{3.27}$$

Moreover, employing the bounds for $\|\zeta\|_{\mathrm{H}}$ and $\|w\|_{\mathrm{Q}}$ provided by (3.21) and (3.26), using Young's inequality conveniently, and performing several algebraic manipulations, we deduce from (3.27) that

$$|w|_c^2 \,\le\, C_1\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}^2 \,+\, C_2\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}^2 \,+\, \frac{1}{2}\,|w|_c^2\,, \tag{3.28}$$

where C_1 and C_2, positive constants depending on $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widehat{\beta}$, and $\widetilde{\gamma}$, are given explicitly as

$$C_1 \,:=\, \left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\left\{\left(1+\frac{\|a\|}{\widetilde{\alpha}}\right)\frac{\|c\|}{\widehat{\beta}^2} + \left(1+\frac{\|c\|}{\widetilde{\gamma}}\right)\frac{1}{2\widetilde{\beta}} + \frac{1}{2\widehat{\beta}}\right\} \,+\, \frac{1}{\widetilde{\alpha}} \tag{3.29}$$

and

$$C_2 := \Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\Big(1+\frac{\|c\|}{\widetilde{\gamma}}\Big)\left\{\Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\Big(1+\frac{\|c\|}{\widetilde{\gamma}}\Big)\frac{\|a\|^2\|c\|}{\widetilde{\beta}^2\widehat{\beta}^2} + \frac{\|a\|}{\widetilde{\beta}\widehat{\beta}} + \frac{1}{2\widehat{\beta}}\right\} + \Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\frac{1}{2\widehat{\beta}} + \frac{1}{\widetilde{\gamma}}\,. \tag{3.30}$$

Finally, it is easy to see from (3.28) that

$$|w|_c \,\le\, \Big(2\,\max\big\{C_1, C_2\big\}\Big)^{1/2}\Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\}\,, \tag{3.31}$$

which, replaced back into (3.21), completes the upper bound of $\|\zeta\|_{\mathrm{H}}$. In turn, employing the latter in (3.26) leads to the respective estimate for $\|w\|_{\mathrm{Q}}$, and the proof is concluded. □

Bearing in mind the equivalence (3.8), we notice here that the proof of the previous theorem establishes, equivalently, that the global inf-sup condition for $\mathbf{A}$ holds, namely

$$\sup_{\substack{(\tau,v)\in \mathrm{H}\times\mathrm{Q}\\ (\tau,v)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\zeta,w),(\tau,v)\big)}{\|(\tau,v)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{1}{2\,\widetilde{C}}\,\|(\zeta,w)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\zeta,w)\in \mathrm{H}\times\mathrm{Q}\,. \tag{3.32}$$

On the other hand, and related to a previous remark (right after the proof of Lemma 2.2) on the constants $\widetilde{\beta}$ and $\widehat{\beta}$ that appear in the inf-sup conditions (2.9) and (2.10), respectively, we stress here that the fact that they do not coincide does not yield any difficulty in the solvability result provided by Theorem 3.1. The reason is certainly because the difference between them is determined only by the reciprocals of the constants C_{H} and C_{Q}, which depend on the continuous spaces H and Q, which are fixed. However, this issue becomes a delicate point for the associated Galerkin scheme, to be addressed next, since the finite element subspaces employed are varying, and hence, the respective constants could vary as well with them, particularly with their dimensions. According to it, in this case we can not employ the equivalence between i) and iii) from Lemma 2.2 as such, but rather assume (which means proving when dealing with specific subspaces) that both discrete inf-sup conditions are satisfied with constants independent of those dimensions.

3.3 Discrete solvability

We now let $\big\{\mathrm{H}_h\big\}_{h>0}$ and $\big\{\mathrm{Q}_h\big\}_{h>0}$ be families of finite dimensional subspaces of H and Q, respectively, and introduce the Galerkin scheme associated with (1.1): Find $(\sigma_h, u_h)\in \mathrm{H}_h\times\mathrm{Q}_h$ such that

$$\begin{aligned} a(\sigma_h,\tau_h) + b(\tau_h,u_h) &= f(\tau_h) \qquad \forall\,\tau_h\in\mathrm{H}_h\,,\\ b(\sigma_h,v_h) - c(u_h,v_h) &= g(v_h) \qquad \forall\,v_h\in\mathrm{Q}_h\,. \end{aligned} \tag{3.33}$$

Then, we let $\mathbf{B}_h : \mathrm{H}_h \to \mathrm{Q}_h'$ and $\mathbf{B}_h^{\mathtt{t}} : \mathrm{Q}_h\to\mathrm{H}_h'$ be the discrete versions of the bounded linear operators induced by b (cf. (2.1)), and define the respective discrete null spaces

$$\mathrm{V}_h \,:=\, \mathrm{N}(\mathbf{B}_h) \,:=\, \Big\{\tau_h\in\mathrm{H}_h: \quad b(\tau_h,v_h)=0 \quad \forall\,v_h\in\mathrm{Q}_h\Big\} \tag{3.34}$$

and

$$\mathrm{W}_h \,:=\, \mathrm{N}(\mathbf{B}_h^{\mathtt{t}}) \,:=\, \Big\{v_h\in\mathrm{Q}_h: \quad b(\tau_h,v_h)=0 \quad \forall\,\tau_h\in\mathrm{H}_h\Big\}\,. \tag{3.35}$$

In this case, the existence of closed subspaces $\mathrm{V}_h^\perp$ and $\mathrm{W}_h^\perp$ of H_h and Q_h, respectively, satisfying the decompositions $\mathrm{H}_h = \mathrm{V}_h\oplus\mathrm{V}_h^\perp$ and $\mathrm{Q}_h = \mathrm{W}_h\oplus\mathrm{W}_h^\perp$, is guaranteed by the fact that both H_h and Q_h

are finite dimensional. As a consequence, the solvability result for (3.33), which is stated next, does not need to incorporate the aforementioned existence as an assumption (see hypothesis **i)** in Theorem 3.1) but rather as a fact. In this way, the discrete version of that theorem reads as follows. Hereafter, the expression "independent of h" means independent of the finite element subspaces H_h and Q_h.

Theorem 3.2. *In addition to the previous notations and definitions, assume that:*

i) *a and c are symmetric and positive semi-definite (cf. (3.11)),*

ii) *there exists a constant $\widetilde{\alpha}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{V}_h \\ \tau_h \neq 0}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_{\mathrm{H}}} \,\geq\, \widetilde{\alpha}_{\mathtt{d}} \, \|\vartheta_h\|_{\mathrm{H}} \qquad \forall\, \vartheta_h \in \mathrm{V}_h \,, \tag{3.36}$$

iii) *there exists a constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{H}_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{H}}} \,\geq\, \widetilde{\beta}_{\mathtt{d}} \, \|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h \in \mathrm{W}_h^{\perp} \,, \tag{3.37}$$

iv) *there exists a constant $\widehat{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{v_h \in \mathrm{Q}_h \\ v_h \neq 0}} \frac{b(\tau_h, v_h)}{\|v_h\|_{\mathrm{Q}}} \,\geq\, \widehat{\beta}_{\mathtt{d}} \, \|\tau_h\|_{\mathrm{H}} \qquad \forall\, \tau_h \in \mathrm{V}_h^{\perp} \,, \tag{3.38}$$

v) *and there exists a constant $\widetilde{\gamma}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{v_h \in \mathrm{W}_h \\ v_h \neq 0}} \frac{c(z_h, v_h)}{\|v_h\|_{\mathrm{Q}}} \,\geq\, \widetilde{\gamma}_{\mathtt{d}} \, \|z_h\|_{\mathrm{Q}} \qquad \forall\, z_h \in \mathrm{W}_h \,. \tag{3.39}$$

Then, for each pair $(f, g) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ solution to (3.33). Moreover, there exists a constant $\widetilde{C}_{\mathtt{d}} > 0$, depending only on $\|a\|$, $\|c\|$, $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\widehat{\beta}_{\mathtt{d}}$, and $\widetilde{\gamma}_{\mathtt{d}}$, such that

$$\|\sigma_h\|_{\mathrm{H}} \,+\, \|u_h\|_{\mathrm{Q}} \,\leq\, \widetilde{C}_{\mathtt{d}} \Big\{ \|f\|_{\mathrm{H}'} \,+\, \|g\|_{\mathrm{Q}'} \Big\} \,. \tag{3.40}$$

Proof. It follows analogously to the proof of Theorem 3.1, except for the fact that, instead of considering **iv)** as a consequence of **iii)**, the former is assumed here independently. Alternatively, this proof follows from a direct application of a slight modification of Theorem 3.1 in which a continuous version of the present hypothesis **iv)** is added. □

Similarly as noticed right after the proof of Theorem 3.1, we stress here that the previous theorem provides, equivalently, the global discrete inf-sup condition for $\mathbf{A}$, that is

$$\sup_{\substack{(\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{Q}_h \\ (\tau_h, v_h) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\zeta_h, w_h), (\tau_h, v_h)\big)}{\|(\tau_h, v_h)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geq\, \frac{1}{2\,\widetilde{C}_{\mathtt{d}}} \, \|(\zeta_h, w_h)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\zeta_h, w_h) \in \mathrm{H}_h \times \mathrm{Q}_h \,. \tag{3.41}$$

Having established the well-posedness of the continuous and discrete formulations of interest, we now prove the respective Cea estimate. In what follows, given a subspace X_h of a generic Banach space $(X, \|\cdot\|_X)$, we set $\operatorname{dist}(x, X_h) := \inf\limits_{x_h \in X_h} \|x - x_h\|_X$ for each $x \in X$.

Theorem 3.3. *Assume the hypotheses of Theorems 3.1 and 3.2, and let $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ and $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of (1.1) and (3.33), respectively. Then, there exists a constant $\widehat{C}_{\mathtt{d}} > 0$, depending only on $\|a\|$, $\|b\|$, $\|c\|$, $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, $\widehat{\beta}_{\mathtt{d}}$, and $\widetilde{\gamma}_{\mathtt{d}}$, such that*

$$\|\sigma - \sigma_h\|_{\mathrm{H}} + \|u - u_h\|_{\mathrm{Q}} \leq \widehat{C}_{\mathtt{d}} \Big\{ \mathrm{dist}(\sigma, \mathrm{H}_h) + \mathrm{dist}(u, \mathrm{Q}_h) \Big\} . \tag{3.42}$$

Proof. Due to the equivalence between (1.1) and (3.1), it is clear that (3.33) can be, equivalently, rewritten as: Find $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ such that

$$\mathbf{A}\big((\sigma_h, u_h), (\tau_h, v_h)\big) = \mathbf{F}(\tau_h, v_h) \qquad \forall\, (\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{Q}_h \,, \tag{3.43}$$

and hence, the derivation of (3.42) proceeds in the usual way for formulations of this kind. More precisely, we first apply the triangle inequality to obtain

$$\|(\sigma, u) - (\sigma_h, u_h)\|_{\mathrm{H}\times\mathrm{Q}} \leq \|(\sigma, u) - (\zeta_h, w_h)\|_{\mathrm{H}\times\mathrm{Q}} + \|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{\mathrm{H}\times\mathrm{Q}} \,,$$

for each $(\zeta_h, w_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, then we employ the global discrete inf-sup condition (3.41), which gives

$$\|(\sigma_h, u_h) - (\zeta_h, w_h)\|_{\mathrm{H}\times\mathrm{Q}} \leq 2\, \widetilde{C}_{\mathtt{d}} \sup_{\substack{(\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{Q}_h \\ (\tau_h, v_h) \neq \mathbf{0}}} \frac{\mathbf{A}\big((\sigma_h, u_h) - (\zeta_h, w_h), (\tau_h, v_h)\big)}{\|(\tau_h, v_h)\|_{\mathrm{H}\times\mathrm{Q}}} \,,$$

and finally we use that $\mathbf{A}\big((\sigma_h, u_h), (\tau_h, v_h)\big) = \mathbf{A}\big((\sigma, u), (\tau_h, v_h)\big)$ for each $(\tau_h, v_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, along with the boundedness of $\mathbf{A}$. In this way, we readily arrive at (3.42) with $\widehat{C}_{\mathtt{d}} := 1 + 2\, \widetilde{C}_{\mathtt{d}}\, \|\mathbf{A}\|$. Alternatively, we can also derive (3.42) by proceeding similarly to [7, Theorem 2.5], that is by employing the corresponding Galerkin projection. □

3.4 Continuous solvability when $\mathrm{W} = \{0\}$

We now assume the particular case $\mathrm{W} = \{0\}$, equivalently $\mathrm{W}^{\perp} = \mathrm{Q}$, which means that the hypothesis **iv)** of Theorem 3.1 reduces to the existence of a constant $\widetilde{\beta} > 0$ such that

$$\|\mathbf{B}^{\mathtt{t}}(v)\|_{\mathrm{H}'} := \sup_{\substack{\tau \in \mathrm{H} \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_{\mathrm{H}}} \geq \widetilde{\beta}\, \|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q} \,. \tag{3.44}$$

Moreover, recalling from (2.11) that $\mathbf{B}^{\mathtt{t}} = \mathbf{B}' \circ \mathcal{J}_{\mathrm{Q}}$, and using the reflexivity of Q and the fact that $\mathcal{J}_{\mathrm{Q}}$ is an isometry, we observe that (3.44) can be rewritten, equivalently, as

$$\|\mathbf{B}'(\mathcal{G})\|_{\mathrm{H}'} \geq \widetilde{\beta}\, \|\mathcal{G}\|_{\mathrm{Q}''} \qquad \forall\, \mathcal{G} \in \mathrm{Q}'' \,. \tag{3.45}$$

Note that the above establishes that $\mathbf{B}' : \mathrm{Q}'' \to \mathrm{H}'$ is injective and of closed range, which is equivalent to saying that $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ is surjective. Thus, applying the converse implication of the characterization provided in [6, Lemma A.42], which is originally proved in [1], we deduce from (3.45) that for each $G \in \mathrm{Q}'$ there exists $\vartheta \in \mathrm{H}$ such that

$$\mathbf{B}(\vartheta) = G \qquad \text{and} \qquad \|\vartheta\|_{\mathrm{H}} \leq \frac{1}{\widetilde{\beta}}\, \|G\|_{\mathrm{Q}'} \,. \tag{3.46}$$

In this way, having the above result to our disposal in the present case, we can improve the statement of Theorem 3.1 as follows, highlighting in advance that no topological complement of V nor a continuous inf-sup condition for c are needed now.

Theorem 3.4. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$, $b : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, *and* $c : \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms (cf. (3.10)). In addition, let* $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ *be one of the bounded linear operators induced by* b *(cf. (2.1)), and let* $\mathrm{V} := \mathrm{N}(\mathbf{B})$ *be the respective null space (cf. (2.2)). Assume that:*

i) *a and c are symmetric and positive semi-definite, the latter meaning that*

$$a(\tau,\tau) \,\ge\, 0 \quad \forall\, \tau \in \mathrm{H} \quad \textit{and} \quad c(v,v) \,\ge\, 0 \quad \forall\, v \in \mathrm{Q}\,, \tag{3.47}$$

ii) *there exists a constant* $\widetilde{\alpha} > 0$ *such that*

$$\sup_{\substack{\tau\in \mathrm{V}\\ \tau\neq 0}} \frac{a(\vartheta,\tau)}{\|\tau\|_{\mathrm{H}}} \,\ge\, \widetilde{\alpha}\,\|\vartheta\|_{\mathrm{H}} \qquad \forall\, \vartheta \in \mathrm{V}\,, \tag{3.48}$$

iii) *and there exists a constant* $\widetilde{\beta} > 0$ *such that*

$$\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \,\ge\, \widetilde{\beta}\,\|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q}\,, \tag{3.49}$$

Then, for each pair $(f,g) \in \mathrm{H}' \times \mathrm{Q}'$ *there exists a unique* $(\sigma,u) \in \mathrm{H} \times \mathrm{Q}$ *solution to (1.1) (equivalently (3.1)). Moreover, there exists a constant* $\widetilde{C} > 0$*, depending only on* $\|a\|$, $\|c\|$, $\widetilde{\alpha}$, *and* $\widetilde{\beta}$*, such that*

$$\|(\sigma,u)\|_{\mathrm{H}\times\mathrm{Q}} \,\le\, \widetilde{C}\,\Big\{ \|f\|_{\mathrm{H}'} \,+\, \|g\|_{\mathrm{Q}'} \Big\}\,. \tag{3.50}$$

Proof. We proceed analogously to the proof of Theorem 3.1, though with a key difference in the decomposition to be introduced below. Indeed, given $(\zeta,w) \in \mathrm{H}\times\mathrm{Q}$, we first define the functionals $\mathrm{F}_{(\zeta,w)} \in \mathrm{H}'$ and $\mathrm{G}_{(\zeta,w)} \in \mathrm{Q}'$ as we did in (3.16) and (3.17), respectively, and aim to establish the inequality (3.9). To this end, and bearing in mind **iii)**, we apply (3.46) to $G := \mathbf{B}(\zeta) \in \mathrm{Q}'$, thus yielding the existence of $\bar{\zeta} \in \mathrm{H}$ such that

$$\mathbf{B}(\bar{\zeta}) \,=\, \mathbf{B}(\zeta) \qquad \text{and} \qquad \|\bar{\zeta}\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\beta}}\,\|\mathbf{B}(\zeta)\|_{\mathrm{Q}'} \,=\, \frac{1}{\widetilde{\beta}}\,\|\mathbf{B}(\bar{\zeta})\|_{\mathrm{Q}'}\,. \tag{3.51}$$

As a consequence, ζ can be decomposed as

$$\zeta \,=\, \zeta_0 + \bar{\zeta}\,, \tag{3.52}$$

with $\zeta_0 := \zeta - \bar{\zeta} \in \mathrm{V}$. As previously announced, we stress here that there is no need to identify a topological complement to which $\bar{\zeta}$ belongs, but rather to be able to bound $\|\bar{\zeta}\|_{\mathrm{H}}$, which is indeed guaranteed by the inequality from (3.51). Then, observing from (3.16) that $\mathrm{F}_{(\zeta,w)}(\tau) = a(\zeta,\tau)$ for all $\tau \in \mathrm{V}$, and applying (3.48) (cf. **i)**) to $\vartheta = \zeta_0$, we deduce, exactly as for the derivation of (3.19), that

$$\|\zeta_0\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\alpha}}\,\|\bar{\zeta}\|_{\mathrm{H}}\,. \tag{3.53}$$

Next, noting from (3.17) that $b(\bar{\zeta},v) = \mathrm{G}_{(\zeta,w)}(v) + c(w,v)$ for all $v \in \mathrm{Q}$, it follows from the inequality in (3.51) that

$$\|\bar{\zeta}\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\beta}}\,\|\mathbf{B}(\bar{\zeta})\|_{\mathrm{Q}'} \,=\, \frac{1}{\widetilde{\beta}}\,\sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{b(\bar{\zeta},v)}{\|v\|_{\mathrm{Q}}} \,=\, \frac{1}{\widetilde{\beta}}\,\sup_{\substack{v\in \mathrm{Q}\\ v\neq 0}} \frac{\mathrm{G}_{(\zeta,w)}(v) + c(w,v)}{\|v\|_{\mathrm{Q}}}\,,$$

from which, similarly to the derivation of (3.20), we arrive at

$$\|\bar{\zeta}\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \frac{\|c\|^{1/2}}{\widetilde{\beta}}\,|w|_c\,, \tag{3.54}$$

and hence, thanks to (3.53) and (3.54), the analogue of (3.21) becomes

$$\|\zeta\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\alpha}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\,\frac{1}{\widetilde{\beta}}\,\|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'} \,+\, \Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\,\frac{\|c\|^{1/2}}{\widetilde{\beta}}\,|w|_c\,. \tag{3.55}$$

Furthermore, we know from (3.16) that $b(\tau,w) = \mathrm{F}_{(\zeta,w)}(\tau) - a(\zeta,\tau)$ for all $\tau\in\mathrm{H}$, so that applying (3.49) (cf. **iii)**) with $v = w \in \mathrm{Q}$, we readily deduce that

$$\|w\|_{\mathrm{Q}} \,\le\, \frac{1}{\widetilde{\beta}}\,\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \frac{\|a\|}{\widetilde{\beta}}\,\|\zeta\|_{\mathrm{H}}\,. \tag{3.56}$$

The rest of the proof proceeds exactly as the one of Theorem 3.1. In particular, we obtain (cf. (3.27))

$$|w|_c^2 \,\le\, \|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'}\,\|\zeta\|_{\mathrm{H}} \,+\, \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\,\|w\|_{\mathrm{Q}}\,, \tag{3.57}$$

and then, employing the bounds for $\|\zeta\|_{\mathrm{H}}$ and $\|w\|_{\mathrm{Q}}$ provided by (3.55) and (3.56), and applying Young's inequality conveniently, we arrive at

$$|w|_c \,\le\, \Big(2\,\max\big\{\widetilde{C}_1,\widetilde{C}_2\big\}\Big)^{1/2}\,\Big\{\|\mathrm{F}_{(\zeta,w)}\|_{\mathrm{H}'} \,+\, \|\mathrm{G}_{(\zeta,w)}\|_{\mathrm{Q}'}\Big\}\,, \tag{3.58}$$

where

$$\widetilde{C}_1 \,:=\, \frac{1}{\widetilde{\alpha}} \,+\, \frac{1}{\widetilde{\beta}}\Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big) \,+\, \frac{\|c\|}{\widetilde{\beta}^2}\Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)^2 \tag{3.59}$$

and

$$\widetilde{C}_2 \,:=\, \frac{1}{\widetilde{\beta}}\Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\,\Big\{1+\frac{\|a\|}{\widetilde{\beta}} \,+\, \frac{\|a\|^2\,\|c\|}{\widetilde{\beta}^3}\Big(1+\frac{\|a\|}{\widetilde{\alpha}}\Big)\Big\}\,. \tag{3.60}$$

Finally, (3.58), (3.55), and (3.56) complete the proof. □

3.5 Discrete solvability when $\mathrm{W}_h = \{0\}$

In what follows we consider the same notations and definitions given at the beginning of Section 3.3. Then, similarly to the analysis in Section 3.4, we now assume that $\mathrm{W}_h = \{0\}$, which means that the hypothesis **iii)** of Theorem 3.2 reduces to the existence of a constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that

$$\|\mathbf{B}_h^{\mathtt{t}}(v_h)\|_{\mathrm{H}_h'} \,:=\, \sup_{\substack{\tau_h\in\mathrm{H}_h\\ \tau_h\neq 0}} \frac{b(\tau_h,v_h)}{\|\tau_h\|_{\mathrm{H}}} \,\ge\, \widetilde{\beta}_{\mathtt{d}}\,\|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h\in\mathrm{Q}_h\,. \tag{3.61}$$

Therefore, noting that the discrete version of the respective identity in (2.11) becomes $\mathbf{B}_h^{\mathtt{t}} = \mathbf{B}_h'\circ\mathcal{J}_{\mathrm{Q}_h}$, we realize that (3.61) is equivalent to stating

$$\|\mathbf{B}_h'(\mathcal{G}_h)\|_{\mathrm{H}_h'} \,\ge\, \widetilde{\beta}_{\mathtt{d}}\,\|\mathcal{G}_h\|_{\mathrm{Q}_h''} \qquad \forall\,\mathcal{G}_h\in\mathrm{Q}_h''\,, \tag{3.62}$$

so that applying again the converse implication of [6, Lemma A.42], we conclude that for each $G_h\in\mathrm{Q}_h'$ there exists $\vartheta_h\in\mathrm{H}_h$ such that

$$\mathbf{B}_h(\vartheta_h) \,=\, G_h \qquad \text{and} \qquad \|\vartheta_h\|_{\mathrm{H}} \,\le\, \frac{1}{\widetilde{\beta}_{\mathtt{d}}}\,\|G_h\|_{\mathrm{Q}_h'}\,. \tag{3.63}$$

Consequently, we are now in position to present the discrete version of Theorem 3.4.

Theorem 3.5. *Assume that:*

i) *a and c are symmetric and positive semi-definite (cf. (3.11)),*

ii) *there exists a constant $\widetilde{\alpha}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{V}_h \\ \tau_h \neq 0}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_{\mathrm{H}}} \geq \widetilde{\alpha}_{\mathtt{d}} \, \|\vartheta_h\|_{\mathrm{H}} \qquad \forall \, \vartheta_h \in \mathrm{V}_h \,, \tag{3.64}$$

iii) *and there exists a constant $\widetilde{\beta}_{\mathtt{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{H}_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{H}}} \geq \widetilde{\beta}_{\mathtt{d}} \, \|v_h\|_{\mathrm{Q}} \qquad \forall \, v_h \in \mathrm{Q}_h \,. \tag{3.65}$$

Then, for each pair $(f, g) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ solution to (3.33). Moreover, there exists a constant $\widetilde{C}_{\mathtt{d}} > 0$, depending only on $\|a\|$, $\|c\|$, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, such that

$$\|\sigma_h\|_{\mathrm{H}} + \|u_h\|_{\mathrm{Q}} \leq \widetilde{C}_{\mathtt{d}} \Big\{ \|f\|_{\mathrm{H}'} + \|g\|_{\mathrm{Q}'} \Big\} \,. \tag{3.66}$$

Proof. It proceeds analogously to the proof of Theorem 3.4, bearing in mind that, instead of (3.46), we now apply (3.63). In this way, given $(\zeta_h, w_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, we deduce the existence of $\bar{\zeta}_h \in \mathrm{H}_h$ such that

$$\mathbf{B}_h(\bar{\zeta}_h) = \mathbf{B}_h(\zeta_h) \qquad \text{and} \qquad \|\bar{\zeta}_h\|_{\mathrm{H}} \leq \frac{1}{\widetilde{\beta}_{\mathtt{d}}} \, \|\mathbf{B}_h(\zeta_h)\|_{\mathrm{Q}'} = \frac{1}{\widetilde{\beta}_{\mathtt{d}}} \, \|\mathbf{B}_h(\bar{\zeta}_h)\|_{\mathrm{Q}'} \,, \tag{3.67}$$

so that ζ_h can be decomposed as

$$\zeta_h = \zeta_{0,h} + \bar{\zeta}_h \,, \tag{3.68}$$

with $\zeta_{0,h} := \zeta_h - \bar{\zeta}_h \in \mathrm{V}_h$. The rest of the proof is as the one of Theorem 3.4. Further details are omitted. □

Needless to say, we remark that the global inf-sup conditions stated in (3.32) and (3.41) are also consequence of the proofs of Theorems 3.4 and 3.5, respectively. We end this section with the corresponding Cea estimate, whose proof is exactly as that of Theorem 3.3.

Theorem 3.6. *Assume the hypotheses of Theorems 3.4 and 3.5, and let $(\sigma, u) \in \mathrm{H} \times \mathrm{Q}$ and $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ be the unique solutions of (1.1) and (3.33), respectively. Then, there exists a constant $\widehat{C}_{\mathtt{d}} > 0$, depending only on $\|a\|$, $\|b\|$, $\|c\|$, $\widetilde{\alpha}_{\mathtt{d}}$, and $\widetilde{\beta}_{\mathtt{d}}$, such that*

$$\|\sigma - \sigma_h\|_{\mathrm{H}} + \|u - u_h\|_{\mathrm{Q}} \leq \widehat{C}_{\mathtt{d}} \Big\{ \operatorname{dist}(\sigma, \mathrm{H}_h) + \operatorname{dist}(u, \mathrm{Q}_h) \Big\} \,. \tag{3.69}$$

4 Application to the Poisson-Nernst-Planck/Stokes equations

The coupling of the Stokes and Poisson-Nernst-Planck equations is an electrohydrodinamic model describing the stationary flow of a Newtonian and incompressible fluid occupying a domain $\Omega \subseteq \mathrm{R}^n$, $n \in \{2, 3\}$, with polygonal (resp. polyhedral) boundary Γ in R^2 (resp R^3) (see, e.g. [9], [10]). The dynamics of it is determined by the concentration of ionized particles ξ_1 and ξ_2, the electric current field $\boldsymbol{\varphi}$, and the velocity $\mathbf{u}$ and pressure p of the fluid. In particular, knowing the vector fields $\boldsymbol{\varphi}$ and

$\mathbf{u}$, a simplified version of the equation satisfied by each concentration, say ξ, in which the diffusion and dielectric coefficients are assumed to be equal to 1, is expressed in mixed form as

$$\begin{aligned} \boldsymbol{\sigma} \,=\, \nabla\xi \,+\, \xi\,(\boldsymbol{\varphi} \,-\, \mathbf{u}) \quad \text{in} \quad \Omega\,, \\ \xi \,-\, \mathrm{div}(\boldsymbol{\sigma}) \,=\, f \quad \text{in} \quad \Omega\,, \quad \xi \,=\, g \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{4.1}$$

where ∇ and div are the usual gradient and divergence operators acting on scalar and vector fields, respectively, and f and g are given data belonging to suitable function spaces. On purpose of this, in what follows we adopt standard notation for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in (1,+\infty)$, and Sobolev spaces $\mathrm{H}^m(\Omega)$ and $\mathrm{H}^m_0(\Omega)$, with integer $m \geq 0$, whose corresponding norms and seminorms (in the case of the latter), either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{m,\Omega}$, and $|\cdot|_{m,\Omega}$, respectively. Furthermore, as usual we let $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ be the space of traces of $\mathrm{H}^1(\Omega)$ and its dual, with norms $\|\cdot\|_{1/2,\Gamma}$ and $\|\cdot\|_{-1/2,\Gamma}$, respectively, and denote by $\langle\cdot,\cdot\rangle_\Gamma$ the corresponding duality pairing. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ be its vector counterpart.

Now, in order to derive the variational formulation of (4.1), we stress that the right spaces where the unknowns are going to be sought is mainly determined by the term depending on $\boldsymbol{\varphi}$ and $\mathbf{u}$. Indeed, using the Cauchy-Schwarz and Hölder inequalities, we observe that

$$\left|\int_\Omega \xi\,(\boldsymbol{\varphi} \,-\, \mathbf{u})\cdot\boldsymbol{\tau}\right| \,\leq\, \|\xi\|_{0,\rho;\Omega}\left(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\right)\|\boldsymbol{\tau}\|_{0,\Omega} \tag{4.2}$$

for all $\xi \in \mathrm{L}^\rho(\Omega)$, for all $\boldsymbol{\varphi}$, $\mathbf{u} \in \mathbf{L}^r(\Omega)$, and for all $\boldsymbol{\tau} \in \mathbf{L}^2(\Omega)$, where $\rho = 2\ell$ and $r = 2j$, with ℓ, $j \in (1,+\infty)$ conjugate to each other, that is such that $\frac{1}{\ell} + \frac{1}{j} = 1$. Next, we let $\varrho \in (1,+\infty)$ be the conjugate of ρ, introduce the Banach space

$$\mathbf{H}(\mathrm{div}_\varrho;\Omega) \,:=\, \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^\varrho(\Omega)\Big\}\,, \tag{4.3}$$

which is endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega} \,:=\, \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,\varrho;\Omega} \qquad \forall\,\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)\,,$$

and recall from [5, Section 3.1] (see also [4, Section 4.1] or [8, eq. (2.11)]) that for $\varrho \geq \frac{2n}{n+2}$ there holds

$$\langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle_\Gamma \,=\, \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v \,+\, v\,\mathrm{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)\times\mathrm{H}^1(\Omega)\,, \tag{4.4}$$

where $\boldsymbol{\nu}$ stands for the unit outward normal on Γ. Note that the integration by parts formula (4.4) states implicitly that $\boldsymbol{\tau}\cdot\boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ for each $\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. In addition, being $\rho = 2\ell > 2$, it follows that $\varrho \in (1,2)$, and hence the feasible range for ϱ becomes $\big(\frac{2n}{n+2}, 2\big)$. Thus, testing the first equation of (4.1) against $\boldsymbol{\tau} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$, and then applying (4.4) with $v = \xi$, which requires to assume that, originally $\xi \in \mathrm{H}^1(\Omega)$, and that $g \in \mathrm{H}^{1/2}(\Gamma)$, we obtain

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau} \,+\, \int_\Omega \xi\,\mathrm{div}(\boldsymbol{\tau}) \,-\, \int_\Omega \xi\,(\boldsymbol{\varphi} \,-\, \mathbf{u})\cdot\boldsymbol{\tau} \,=\, \langle\boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle_\Gamma\,. \tag{4.5}$$

In turn, assuming that $f \in \mathrm{L}^\varrho(\Omega)$, and testing the second equation of (4.1) against $\eta \in \mathrm{L}^\rho(\Omega)$, we get

$$\int_\Omega \eta\,\mathrm{div}(\boldsymbol{\sigma}) \,-\, \int_\Omega \xi\,\eta \,=\, -\int_\Omega f\,\eta\,. \tag{4.6}$$

In this way, placing together (4.5) and (4.6), we arrive at the following mixed variational formulation for (4.1): Find $(\boldsymbol{\sigma}, \xi) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\begin{aligned} a(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\xi) \,-\, \int_\Omega \xi\,(\boldsymbol{\varphi} \,-\, \mathbf{u})\cdot\boldsymbol{\tau} \quad &= \quad \mathrm{F}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau} \in \mathrm{H}\,, \\ b(\boldsymbol{\sigma},\eta) \,-\, c(\xi,\eta) \quad &= \quad \mathrm{G}(\eta) \qquad \forall\,\eta \in \mathrm{Q}\,, \end{aligned} \tag{4.7}$$

where

$$\mathrm{H} := \mathbf{H}(\mathbf{div}_\varrho;\Omega), \qquad \mathrm{Q} := \mathrm{L}^\rho(\Omega), \tag{4.8}$$

and the bilinear forms $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$, $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$, and $c : \mathrm{Q}\times\mathrm{Q} \to \mathrm{R}$, and the functionals $\mathrm{F} : \mathrm{H} \longrightarrow \mathrm{R}$ and $\mathrm{G} : \mathrm{Q} \longrightarrow \mathrm{R}$, are defined, respectively, as

$$a(\boldsymbol{\zeta},\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau}) \in \mathrm{H}\times\mathrm{H}, \tag{4.9}$$

$$b(\boldsymbol{\tau},\eta) := \int_\Omega \eta\,\mathrm{div}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\tau},\eta) \in \mathrm{H}\times\mathrm{Q}, \tag{4.10}$$

$$c(\lambda,\eta) := \int_\Omega \lambda\,\eta \qquad \forall\,(\lambda,\eta) \in \mathrm{Q}\times\mathrm{Q}, \tag{4.11}$$

$$\mathrm{F}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, g\rangle \qquad \forall\,\boldsymbol{\tau} \in \mathrm{H}, \tag{4.12}$$

and

$$\mathrm{G}(\eta) := -\int_\Omega f\,\eta \qquad \forall\,\eta \in \mathrm{Q}. \tag{4.13}$$

Equivalently, introducing the bilinear forms $\mathbf{A}$, $\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}} : (\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q}) \to \mathrm{R}$ given by

$$\mathbf{A}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big) := a(\boldsymbol{\zeta},\boldsymbol{\tau}) + b(\boldsymbol{\tau},\lambda) + b(\boldsymbol{\zeta},\eta) - c(\lambda,\eta) \tag{4.14}$$

and

$$\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big) := \mathbf{A}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big) - \int_\Omega \lambda\,(\boldsymbol{\varphi} - \mathbf{u})\cdot\boldsymbol{\tau} \tag{4.15}$$

for all $(\boldsymbol{\zeta},\lambda)$, $(\boldsymbol{\tau},\eta) \in \mathrm{H}\times\mathrm{Q}$, we deduce that (4.7) can be re-stated as: Find $(\boldsymbol{\sigma},\xi) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\sigma},\xi),(\boldsymbol{\tau},\eta)\big) = \mathrm{F}(\boldsymbol{\tau}) + \mathrm{G}(\eta) \qquad \forall\,(\boldsymbol{\tau},\eta) \in \mathrm{H}\times\mathrm{Q}. \tag{4.16}$$

According to the above, in what follows we show first that the bilinear forms forming part of $\mathbf{A}$ verify the assumptions of Theorem 3.4. Later on, we combine this fact with the effect of the extra term completing the definition of $\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}$ to conclude the solvability of (4.7) (or (4.16)).

We begin by observing that the reflexivity of $\mathbf{L}^2(\Omega)$, $\mathrm{L}^\varrho(\Omega)$, and $\mathrm{L}^\rho(\Omega)$, imply that H and Q are both reflexive Banach spaces. In addition, straightforward applications of the Cauchy-Schwarz and Hölder inequalities show that a, b, and c, are all bounded with $\|a\| \le 1$, $\|b\| \le 1$, and $\|c\| \le |\Omega|^{(\rho-2)/\rho}$. Also, it is clear from (4.9) and (4.11) that a and b are symmetric and positive semi-definite (assumption **i)** of Theorem 3.4). Next, bearing in mind the definitions of b (cf. (4.10)) and the null space V of the operator $\mathbf{B}$ induced by b (cf. (2.2)), we find that

$$\mathrm{V} = \Big\{ \tau \in \mathbf{H}(\mathrm{div}_\varrho;\Omega) : \quad \mathrm{div}(\boldsymbol{\tau}) = 0 \Big\}, \tag{4.17}$$

and thus

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \|\boldsymbol{\tau}\|_{0,\Omega}^2 = \|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega}^2 \qquad \forall\,\boldsymbol{\tau} \in \mathrm{V},$$

from which it readily follows that a satisfies the continuous inf-sup condition (3.48) with constant $\widetilde{\alpha} = 1$ (assumption **ii)** of Theorem 3.4). It remains to show that b satisfies the continuous inf-sup condition (3.49) (assumption **iii)** of Theorem 3.4). While the corresponding proof is actually available in the literature (see, e.g. [8, Lemma 2.9] and the references mentioned there), we provide it again below for sake of completeness of the presentation.

Lemma 4.1. *There exists a constant $\widetilde{\beta} > 0$, depending only on Ω and ρ, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathrm{H}\\ \boldsymbol{\tau}\neq 0}} \frac{b(\boldsymbol{\tau},\eta)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \ge \widetilde{\beta}\,\|\eta\|_{\mathrm{Q}} \qquad \forall\,\eta \in \mathrm{Q}. \tag{4.18}$$

Proof. Given $\eta \in \mathrm{Q} := \mathrm{L}^\rho(\Omega)$, we first define $\eta_\varrho := |\eta|^{\rho-2}\,\eta$ and observe, thanks to simple algebraic computations, that $\eta_\varrho \in \mathrm{L}^\varrho(\Omega)$ and

$$\int_\Omega \eta\,\eta_\varrho \,=\, \|\eta\|_{0,\rho;\Omega}\,\|\eta_\varrho\|_{0,\varrho;\Omega}\,. \tag{4.19}$$

Then, we let $\widetilde{\boldsymbol{\tau}} := \nabla z \in \mathbf{L}^2(\Omega)$, where $z \in \mathrm{H}_0^1(\Omega)$ is the unique solution of the variational problem

$$\int_\Omega \nabla z \cdot \nabla w \,=\, -\int_\Omega \eta_\varrho\, w \qquad \forall\, w \in \mathrm{H}_0^1(\Omega)\,. \tag{4.20}$$

Indeed, Hölder's inequality and the continuous injection i_ρ of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$ guarantee that the right hand side of (4.20) constitutes a functional in $\mathrm{H}_0^1(\Omega)'$, so that the classical Lax-Milgram Lemma confirms the unique solvability of this problem. In turn, it follows from (4.20) that

$$\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \,=\, \eta_\varrho \quad \text{in} \quad \Omega\,, \tag{4.21}$$

which yields $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_\varrho;\Omega)$. Moreover, according to the continuous dependence result for (4.20) and the resulting bound for the norm of the aforementioned functional, we deduce the existence of a constant $c_\rho > 0$, depending on $\|i_\rho\|$, such that $\|z\|_{1,\Omega} \,\le\, c_\rho\,\|\eta_\varrho\|_{0,\varrho;\Omega}$, and hence

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega} \,=\, |z|_{1,\Omega} \,+\, \|\eta_\varrho\|_{0,\varrho;\Omega} \,\le\, \big(1 + c_\rho\big)\,\|\eta_\varrho\|_{0,\varrho;\Omega}\,. \tag{4.22}$$

Finally, according to the definition of b (cf. (4.10)), and employing (4.21), (4.19), and (4.22), we obtain

$$\sup_{\substack{\boldsymbol{\tau}\in \mathrm{H}\\ \boldsymbol{\tau}\neq 0}} \frac{b(\boldsymbol{\tau},\eta)}{\|\boldsymbol{\tau}\|_{\mathrm{H}}} \,\ge\, \frac{b(\widetilde{\boldsymbol{\tau}},\eta)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \,=\, \frac{\|\eta\|_{0,\rho;\Omega}\,\|\eta_\varrho\|_{0,\varrho;\Omega}}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \,\ge\, \widetilde{\beta}\,\|\eta\|_{0,\rho;\Omega}\,,$$

with $\widetilde{\beta} := \big(1 + c_\rho\big)^{-1}$, thus proving the required continuous inf-sup condition (4.18). $\square$

Having proved that a, b, and c verify the hypotheses of Theorem 3.4, we deduce that the global inf-sup condition (3.32) also holds for the present bilinear form $\mathbf{A}$ (cf. (4.14)), which means in this case that there exists a constant $\widehat{c} > 0$, depending only on $\|a\|$ (≤ 1), $\|c\|$ ($\le |\Omega|^{(\rho-2)/\rho}$), $\widetilde{\alpha} = 1$, and $\widetilde{\beta} = \big(1 + c_\rho\big)^{-1}$, such that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \widehat{c}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\lambda) \in \mathrm{H}\times\mathrm{Q}\,. \tag{4.23}$$

Thus, it readily follows from (4.15), (4.2), and (4.23) that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \Big\{\widehat{c} \,-\, \big(\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega}\big)\Big\}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \quad \forall\,(\boldsymbol{\zeta},\lambda) \in \mathrm{H}\times\mathrm{Q}\,, \tag{4.24}$$

from which, under the assumption that, say $\|\boldsymbol{\varphi}\|_{0,r;\Omega} + \|\mathbf{u}\|_{0,r;\Omega} \,\le\, \frac{\widehat{c}}{2}$, we conclude that

$$\sup_{\substack{(\boldsymbol{\tau},\eta)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\tau},\eta)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\widehat{c}}{2}\,\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\zeta},\lambda) \in \mathrm{H}\times\mathrm{Q}\,. \tag{4.25}$$

Similarly, using the symmetry of $\mathbf{A}$ and (4.23), and under the same hypothesis on $\boldsymbol{\varphi}$ and $\mathbf{u}$, we find that

$$\sup_{\substack{(\boldsymbol{\zeta},\lambda)\in \mathrm{H}\times\mathrm{Q}\\ (\boldsymbol{\zeta},\lambda)\neq \mathbf{0}}} \frac{\mathbf{A}_{\boldsymbol{\varphi},\mathbf{u}}\big((\boldsymbol{\zeta},\lambda),(\boldsymbol{\tau},\eta)\big)}{\|(\boldsymbol{\zeta},\lambda)\|_{\mathrm{H}\times\mathrm{Q}}} \,\ge\, \frac{\widehat{c}}{2}\,\|(\boldsymbol{\tau},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\boldsymbol{\tau},\eta) \in \mathrm{H}\times\mathrm{Q}\,. \tag{4.26}$$

On the other hand, recalling from the proof of Lemma 4.1 that i_ρ is the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$, it is easy to see from (4.4) that there exists a constant $C_\rho > 0$, depending on $\|i_\rho\|$, such that $\|\boldsymbol{\tau}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma} \le C_\rho\,\|\boldsymbol{\tau}\|_{\mathrm{div}_\varrho;\Omega}$ for all $\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}_\varrho;\Omega)$, and hence we deduce from (4.12) that $\mathrm{F}\in\mathrm{H}'$ with $\|\mathrm{F}\|_{\mathrm{H}'} \le C_\rho\,\|g\|_{-1/2,\Gamma}$. In turn, (4.13) and Hölder's inequality yield $\mathrm{G}\in\mathrm{Q}'$ with $\|\mathrm{G}\|_{\mathrm{Q}} \le \|f\|_{0,\varrho;\Omega}$.

In this way, we are now in position of establishing the well-posedness of (4.7) (equivalently (4.16)).

Theorem 4.2. *Let $\boldsymbol{\varphi}$, $\mathbf{u}\in\mathbf{L}^r(\Omega)$ such that $\|\boldsymbol{\varphi}\|_{0,r;\Omega}+\|\mathbf{u}\|_{0,r;\Omega} \le \frac{\widehat{c}}{2}$. Then, there exists a unique $(\boldsymbol{\sigma},\xi)\in\mathrm{H}\times\mathrm{Q}$ solution to (4.7), and there holds*

$$\|\boldsymbol{\sigma}\|_{\mathrm{div}_\varrho;\Omega} + \|\xi\|_{0,\rho;\Omega} \le \frac{2}{\widehat{c}}\max\big\{1, C_\rho\big\}\Big\{\|g\|_{-1/2,\Gamma} + \|f\|_{0,\varrho;\Omega}\Big\}.$$

Proof. Thanks to (4.25), (4.26), and the boundedness of F and G, it follows from a straightforward application of the Banach-Nečas-Babuška Theorem (also known as generalized Lax-Milgram Lemma) (cf. [6, Theorem 2.6]). □

We end the paper by remarking that the continuous and discrete analyses of the full Poisson-Nernst-Planck and Stokes coupled model, which certainly contain those of (4.7), will be provided in a forthcoming work.

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