

We end this section by mentioning that a complete analysis of the linear elasticity problem with pure Neumann boundary conditions can be found in [35].

2.4.4 Primal-mixed formulation of the Poisson problem

The present set of application examples is finished with the Poisson problem analyzed in Section 2.4.1, but utilizing now what we call the primal-mixed formulation. Recall that the geometry is given by a bounded domain Ω of \mathbb{R}^n , $n \geq 2$, with Lipschitz-continuous boundary Γ . Then, given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we are interested in the boundary value problem:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma. \quad (2.67)$$

Multiplying the partial differential equation by $v \in H^1(\Omega)$ and applying the improved Green identity (1.52) (cf. Theorem 1.8), we obtain:

$$\int_{\Omega} f v = - \int_{\Omega} \Delta u = \int_{\Omega} \nabla u \cdot \nabla v - \langle \gamma_1(u), \gamma_0(v) \rangle,$$

where $\gamma_1 : H_A^1(\Omega) \rightarrow H^{-1/2}(\Gamma)$ is the linear and bounded operator given by $\gamma_{\mathbf{n}} \circ \nabla$, and $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Then, introducing the auxiliary unknown $\xi := -\gamma_1(u) \in H^{-1/2}(\Gamma)$, we can write

$$\int_{\Omega} \nabla u \cdot \nabla v + \langle \xi, \gamma_0(v) \rangle = \int_{\Omega} f v \quad \forall v \in H^1(\Omega). \quad (2.68)$$

In turn, the non-homogeneous Dirichlet boundary condition $\gamma_0(u) = g$ is weakly imposed as:

$$\langle \lambda, \gamma_0(u) \rangle = \langle \lambda, g \rangle \quad \forall \lambda \in H^{-1/2}(\Gamma). \quad (2.69)$$

In this way, placing together (2.68) and (2.69), we arrive at the primal-mixed variational formulation of (2.67): Find $(u, \xi) \in H \times Q$ such that

$$\begin{aligned} a(u, v) + b(v, \xi) &= F(v) \quad \forall v \in H, \\ b(u, \lambda) &= G(\lambda) \quad \forall \lambda \in Q, \end{aligned} \quad (2.70)$$

where $H := H^1(\Omega)$, $Q := H^{-1/2}(\Gamma)$, a and b are the bilinear forms defined by

$$\begin{aligned} a(u, v) &:= \int_{\Omega} \nabla u \cdot \nabla v \quad \forall (u, v) \in H \times H, \\ b(v, \lambda) &:= \langle \lambda, \gamma_0(v) \rangle \quad \forall (v, \lambda) \in H \times Q, \end{aligned}$$

and the functionals $F \in H'$ and $G \in Q'$ are given by

$$F(v) := \int_{\Omega} f v \quad \forall v \in H, \quad G(\lambda) := \langle \lambda, g \rangle \quad \forall \lambda \in Q.$$

It is clear that a and b are bounded since, applying Cauchy-Schwarz inequality, we get

$$|a(u, v)| \leq |u|_{1,\Omega} |v|_{1,\Omega} \leq \|u\|_{1,\Omega} \|v\|_{1,\Omega},$$

and employing also the trace inequality, we have

$$|b(v, \lambda)| \leq \|\lambda\|_{-1/2,\Gamma} \|\gamma_0(v)\|_{1/2,\Gamma} \leq \|v\|_{1,\Omega} \|\lambda\|_{-1/2,\Gamma},$$

which shows that $\|\mathbf{A}\| \leq 1$ and $\|\mathbf{B}\| \leq 1$, where $\mathbf{A} : H \rightarrow H$ and $\mathbf{B} : H \rightarrow Q$ are the operators induced by a and b , respectively. Moreover, if $\mathcal{R} : H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$ denotes the respective Riesz mapping, we have

$$b(v, \lambda) = \langle \lambda, \gamma_0(v) \rangle = \langle \mathcal{R}(\lambda), \gamma_0(v) \rangle_{1/2,\Gamma} = \langle \mathcal{R}^* \gamma_0(v), \lambda \rangle_{-1/2,\Gamma},$$

where $\langle \cdot, \cdot \rangle_{r,\Gamma}$ is the inner product of $H^r(\Gamma)$, $r \in \{-1/2, 1/2\}$, which shows that the operator \mathbf{B} reduces to

$$\mathbf{B}(v) = \mathcal{R}^* \gamma_0(v) \quad \forall v \in H.$$

Thus, since the adjoint $\mathcal{R}^* : H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ is certainly bijective, it follows that

$$V := \mathbf{N}(\mathbf{B}) = \left\{ v \in H : \mathbf{B}(v) = 0 \right\} = \left\{ v \in H^1(\Omega) : \gamma_0(v) = 0 \right\} = H_0^1(\Omega),$$

and hence, thanks to the Friedrichs-Poincaré inequality, there exists $\alpha > 0$ such that

$$a(v, v) = |v|_{1,\Omega}^2 \geq \alpha \|v\|_{1,\Omega}^2 \quad \forall v \in V,$$

which proves the V -ellipticity of a .

On the other hand, for the surjectivity of \mathbf{B} it suffices to see that this operator is given by the compose of the operators \mathcal{R}^* (which is bijective) and γ_0 (which is surjective). For example, given $\lambda \in H^{-1/2}(\Gamma)$, we have that $z := \tilde{\gamma}_0^{-1}(\mathcal{R}^*)^{-1}(\lambda) \in H_0^1(\Omega)^\perp$ satisfies $\mathbf{B}(z) = \lambda$, confirming the above assertion. Finally, utilizing the Cauchy-Schwarz inequality and the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, it follows easily that F and G are bounded with $\|F\| \leq \|f\|_{0,\Omega}$ and $\|G\| \leq \|g\|_{1/2,\Gamma}$.

Consequently, applying once again Theorem 2.3, we deduce that there exists a unique $(u, \xi) \in H \times Q$ solution of (2.70), which satisfies

$$\|(u, \xi)\|_{H \times Q} \leq C \left\{ \|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma} \right\},$$

with $C > 0$ depending on the constant β for the continuous inf-sup condition of b , $\|\mathbf{A}\| \leq 1$, and the ellipticity constant α . Then, in order to get an explicit value for β , we proceed as in all the previous examples. Indeed, given $\lambda \in Q := H^{-1/2}(\Gamma)$, we have, making use of (1.36) (cf. Lemma 1.3), that

$$\begin{aligned}
\sup_{\substack{v \in H \\ v \neq 0}} \frac{b(v, \lambda)}{\|v\|_H} &= \sup_{\substack{v \in H^1(\Omega) \\ v \neq 0}} \frac{\langle \lambda, \gamma_0(v) \rangle}{\|v\|_{1,\Omega}} \geq \frac{\langle \lambda, \gamma_0 \tilde{\gamma}_0^{-1}(\mathcal{R}\lambda) \rangle}{\|\tilde{\gamma}_0^{-1}(\mathcal{R}(\lambda))\|_{1,\Omega}} \\
&= \frac{\langle \lambda, \mathcal{R}(\lambda) \rangle}{\|\tilde{\gamma}_0^{-1}(\mathcal{R}(\lambda))\|_{1,\Omega}} = \frac{\|\mathcal{R}(\lambda)\|_{1/2,\Gamma}^2}{\|\mathcal{R}(\lambda)\|_{1/2,\Gamma}} = \|\mathcal{R}(\lambda)\|_{1/2,\Gamma} = \|\lambda\|_{-1/2,\Gamma},
\end{aligned}$$

which yields $\beta = 1$.

For further applications in continuum mechanics of the Babuška-Brezzi theory and related abstract developments, we refer to the classical books [16] and [41], and to the recent updated version of [16] given by [15], which, among several new features, provides interesting new results on electromagnetism problems.

2.5 The Galerkin scheme

Let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be sequences of finite dimensional subspaces of H and Q , respectively. Then, given $F \in H'$ and $G \in Q'$, the Galerkin scheme of (2.1) reads: Find $(\sigma_h, u_h) \in H_h \times Q_h$ such that

$$\begin{aligned}
a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= F(\tau_h) \quad \forall \tau_h \in H_h, \\
b(\sigma_h, v_h) &= G(v_h) \quad \forall v_h \in Q_h.
\end{aligned} \tag{2.71}$$

For the analysis of (2.71) we basically follow the same approach of Section 2.1. In fact, let $\mathbf{A}_h : H_h \rightarrow H_h$ and $\mathbf{B}_h : H_h \rightarrow Q_h$ be the linear and bounded operators induced by a and b on $H_h \times H_h$ and $H_h \times Q_h$, respectively, that is:

$$\mathbf{A}_h := \mathcal{R}_{H_h} \circ \mathcal{A}_h \quad \text{and} \quad \mathbf{B}_h := \mathcal{R}_{Q_h} \circ \mathcal{B}_h,$$

where $\mathcal{R}_{H_h} : H'_h \rightarrow H_h$ and $\mathcal{R}_{Q_h} : Q'_h \rightarrow Q_h$ are the respective Riesz mappings, and the operators $\mathcal{A}_h : H_h \rightarrow H'_h$ and $\mathcal{B}_h : Q_h \rightarrow Q'_h$ are defined by:

$$\mathcal{A}_h(\sigma_h)(\tau_h) := a(\sigma_h, \tau_h) \quad \forall \sigma_h \in H_h, \quad \forall \tau_h \in H_h,$$

and

$$\mathcal{B}_h(\tau_h)(v_h) := b(\tau_h, v_h) \quad \forall \tau_h \in H_h, \quad \forall v_h \in Q_h.$$

The following theorem establishes sufficient conditions for (2.71) to be well-posed.

Theorem 2.4. *Let $V_h := N(\mathbf{B}_h) = \{ \tau_h \in H_h : b(\tau_h, v_h) = 0 \quad \forall v_h \in Q_h \}$, and let $\Pi_h : H_h \rightarrow V_h$ be the orthogonal projection operator. Assume that:*

- i) $\Pi_h \mathbf{A}_h : V_h \rightarrow V_h$ is injective.
- ii) there exists $\beta_h > 0$ such that