



# A new non-augmented and momentum-conserving fully-mixed finite element method for a coupled flow-transport problem

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## Abstract

We propose and analyze a new mixed finite element method for the coupling of the Stokes equations with a transport problem modelled by a scalar nonlinear convection–diffusion problem. Our approach is based on the introduction of the Cauchy fluid stress and two vector unknowns involving the gradient and the total flux of the concentration. The introduction of these further unknowns lead to a mixed formulation in a Banach space framework in both Stokes and transport equations, where the aforementioned stress tensor and vector unknowns, together with the velocity and the concentration, are the main unknowns of the system. In this way, and differently from the techniques previously developed for this and related coupled problems, no augmentation procedure needs to be incorporated now into the formulation nor into the solvability analysis. The resulting non-augmented scheme is then written equivalently as a fixed-point equation, so that the well-known Banach theorem, combined with Babuška–Brezzi’s theory in Banach spaces, classical results on nonlinear monotone operators and certain regularity assumptions, are applied to prove the unique solvability of the continuous system. As for the associated Galerkin scheme, whose solvability is established similarly to the continuous case by using the Brouwer fixed-point theorem, we employ Raviart–Thomas approximations of order  $k \geq 0$  for the stress and total flux, and discontinuous piecewise polynomials of degree  $k$  for the velocity, concentration, and concentration gradient. With this choice of spaces, momentum is conserved in both Stokes and transport equations if the external forces belong to the piecewise constants and concentration discrete space, respectively, which constitutes one of the main features of our approach. Finally, we derive optimal a priori error estimates and provide several numerical results

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illustrating the good performance of the scheme and confirming the theoretical rates of convergence.

**Keywords** Stokes equations · Nonlinear transport problem · Fixed point theory · Sedimentation–consolidation process · Mixed finite element methods · A priori error analysis

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## 1 Introduction

The phenomenon of transport of a species density in an immiscible fluid, which involves three main fields, namely the velocity of the flow, pressure and local solids concentration, has a wide range of applications, including processes arising in chemical distillation processes, solid–liquid separation, sedimentation–consolidation processes, aluminum production, natural and thermal convection, and so on. In the present work we are interested in the coupled flow and transport problem determined by a scalar nonlinear convection–diffusion equation interacting with the Stokes equations, which serves as a prototype for certain sedimentation–consolidation processes, see, e.g. [11, 12, 31], and also models the transport of species concentration within a viscous fluid. Indeed, diverse combinations of primal and mixed finite element methods have been proposed lately in the literature for the numerical solution of this and related models, whose most distinctive feature is the fact that, not only the viscosity of the fluid, but also the diffusion coefficient and the function describing hindered settling, depend on the solution to the transport problem, see, e.g. [1, 2, 4, 6, 7, 26]. In particular, the solvability of our model of interest was analyzed in [1] by means of an augmented mixed formulation in the fluid and the usual primal scheme in the transport equation, thus yielding an augmented mixed–primal variational formulation, whose unknowns, given by the Cauchy stress, the velocity of the fluid, and the concentration, are sought in suitable Hilbert spaces. The well-posedness of the continuous and discrete formulations, rewritten as fixed point operator equations, are established by using classical Schauder’s and Brouwer theorems, respectively. In addition, suitable regularity assumptions, Sobolev’s embedding and Rellich–Kondrachov compactness theorems, are also employed in the continuous analysis.

Later on, the approach from [1] was extended in [2] to the case of a strongly coupled flow and transport system modeled by the Brinkman problem with variable viscosity, written in terms of Cauchy pseudo-stresses and bulk velocity of the mixture, coupled with a nonlinear advection–nonlinear diffusion equation describing the transport of the solids volume fraction. The solvability of this model had been previously discussed in [10] for the case of large fluid viscosity, using the technique of parabolic regularization. In addition, the existence of solutions to a related model for chemically reacting non-Newtonian fluid had been established in [9] as well. Regarding the analysis developed in [2], an augmented mixed approach for the

Brinkman problem and the usual primal weak form for the transport equation are employed to derive the variational formulation of the coupled problem. In this way, similarly as in [1], the corresponding continuous and discrete solvability analyses are performed by combining fixed point arguments, elliptic regularity estimates, sufficiently small data assumptions and classical results on Hilbert space frameworks [8, 19, 25]. More recently, in [6] a model describing the flow-transport interaction in a porous-fluidic domain was analyzed employing the techniques from [1] and [2]. In this case, the medium consists of a highly permeable material, where the flow of an incompressible viscous fluid is governed by Brinkman equations (written in terms of vorticity, velocity and pressure, as in [4]), and a porous medium where Darcy's law describes fluid motion using filtration velocity and pressure. Meanwhile, an augmented fully-mixed variational formulation for the model from [1] was introduced and analyzed in [26]. Here, the authors apply a dual-mixed method and augmentation procedure in both Stokes and transport equations. Furthermore, reliable and efficient residual-based *a posteriori* error estimators for the models and corresponding methods studied in [1] and [2] are derived in [3] and [5], respectively.

We point out that while augmentation procedures have played a crucial role in all the aforementioned references, at the continuous level making possible the solvability analyses in suitable Hilbert space frameworks, and at the discrete level allowing arbitrary finite element subspaces, it is no less true that the introduction of additional terms into the formulation certainly leads to much more expensive schemes because of the extra computations that need to be performed in order to set up the stiffness matrix and load vector of the resulting discrete system. As a consequence of this fact, in the last years several efforts have been made aiming to avoid the introduction of augmented terms and appealing to a Banach space framework for analyzing the continuous and discrete formulations of diverse problems in continuum mechanics. The list of works in this direction includes, for instance, [13, 14, 16–18, 20–22], and [29], all of which, irrespective of dealing with different models, namely Poisson, Brinkman–Forchheimer, Navier–Stokes, and Boussinesq equations, share a Banach saddle-point structure for the resulting variational formulations. In the same direction, a Banach spaces-based analysis for the mixed-primal formulation studied in [1] was recently developed in [7]. In particular, the authors considered a dual-mixed formulation of the Stokes equations, where, unlike [1], the velocity of the fluid is sought in  $L^4$ , which consequently forces the Cauchy stress to live in a suitable  $H(\text{div})$ -type Banach space, whereas the usual primal scheme in the transport equation with concentration in  $H^1$  is considered.

According to the above bibliographic discussion, the goal of the present paper is to continue extending the applicability of the aforementioned Banach spaces framework by introducing now a new fully-mixed formulation, without any augmentation procedure, for the coupled problem studied in [1, 26], and [7]. The above is achieved by employing the stress–velocity mixed formulation for the Stokes equations introduced in [7] and a three-field mixed formulation for the transport equations based on the introduction of two additional vector unknowns relating the gradient and total flux of concentration. In this way, similarly to [26], the aforementioned Cauchy stress, total flux, and concentration gradient, together with the velocity and concentration, become the resulting unknowns of the coupled problem but, unlike

[26], only Banach space-based analysis is used. Then, following [14, 20], and [32], we combine a fixed-point argument, suitable regularity assumptions on one of the decoupled problems, Babuška–Brezzi’s theory in Banach spaces, classical results on nonlinear monotone operators, sufficiently small data assumptions, and the well known Banach fixed-point theorem to establish existence and uniqueness of solution of the continuous problem. In particular, since the formulation for the Stokes equation is the same one employed in [7], our present analysis certainly makes use of the corresponding results available there. As for the numerical scheme, whose solvability is established similarly to the continuous case by using the Brouwer fixed-point theorem, we employ Raviart–Thomas spaces of order  $k \geq 0$  for approximating the Cauchy stress and the total flux, and discontinuous piecewise polynomials of degree  $k$  for the velocity, concentration, and concentration gradient fields. We stress that with this choice of spaces momentum is conserved in both Stokes and transport equations if the external forces belong to piecewise constants and concentration discrete space, respectively. Furthermore, applying an ad-hoc Strang-type lemma in Banach spaces, we are able to derive the corresponding *a priori* error estimates and prove that the method is convergent with optimal rate.

The rest of this work is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. The model problem is introduced in Sect. 2, and all the auxiliary variables to be employed in the setting of the formulation are defined there. Next, in Sect. 3 we derive a non-augmented fully-mixed variational formulation and establish the well-posedness of this continuous scheme by means of a fixed-point strategy and Banach’s fixed-point theorem. The corresponding Galerkin system is introduced and analyzed in Sect. 4, where the discrete analogue of the theory used in the continuous case is employed to prove existence of solution. In Sect. 5, an ad-hoc Strang-type lemma in Banach spaces is utilized to derive the corresponding *a priori* error estimate and the consequent rates of convergence. Finally, the performance of the method is illustrated in Sect. 6 with several numerical examples in 2D and 3D, which confirm the aforementioned rates.

## 1.1 Preliminary notations

In what follows  $\Omega \subseteq \mathbb{R}^n$ ,  $n \in \{2, 3\}$ , is a given bounded domain with polyhedral boundary  $\Gamma$ , whose outward unit normal vector is denoted by  $\mathbf{v}$ . Standard notation will be adopted for Lebesgue spaces  $L^p(\Omega)$  and Sobolev spaces  $W^{s,p}(\Omega)$  and  $W_0^{s,p}(\Omega)$ , with  $s \in \mathbb{R}$  and  $p > 1$ , whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by  $\|\cdot\|_{0,p;\Omega}$  and  $\|\cdot\|_{s,p;\Omega}$ , respectively. In particular, given a non-negative integer  $m$ ,  $W^{m,2}(\Omega)$  and  $W_0^{m,2}(\Omega)$  are also denoted by  $H^m(\Omega)$  and  $H_0^m(\Omega)$ , and the notations of its norm and seminorm are simplified to  $\|\cdot\|_{m,\Omega}$  and  $|\cdot|_{m,\Omega}$ , respectively. In addition,  $H^{1/2}(\Gamma)$  is the space of traces of functions of  $H^1(\Omega)$ ,  $H^{-1/2}(\Gamma)$  denotes its dual, and  $\langle \cdot, \cdot \rangle$  stands for the corresponding duality pairing between  $H^{-1/2}(\Gamma)$  and  $H^{1/2}(\Gamma)$ . On the other hand, given any generic scalar functional space  $\mathbf{M}$ , we let  $\mathbf{M}$  and  $\mathbb{M}$  be the corresponding vectorial and tensorial counterparts, whereas  $\|\cdot\|$ , with no subscripts, will be employed for the norm

of any element or operator whenever there is no confusion about the space to which they belong. Furthermore, as usual  $\mathbb{I}$  stands for the identity tensor in  $\mathbb{R}^{n \times n}$ , and  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^n$ . Also, for any vector field  $\mathbf{v} = (v_i)_{i=1,n}$  we let  $\nabla \mathbf{v}$  and  $\text{div}(\mathbf{v})$  be its gradient and divergence, respectively. In addition, for any tensor  $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$  and  $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$ , we let  $\mathbf{div}(\boldsymbol{\tau})$  be the divergence operator  $\text{div}$  acting along the rows of  $\boldsymbol{\tau}$ , and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^t := (\tau_{ji})_{i,j=1,n}, \quad \text{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij} \zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{n} \text{tr}(\boldsymbol{\tau}) \mathbb{I}.$$

Finally, for any pair of normed spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$ , we provide the product space  $X \times Y$  with the natural norm  $\|(x, y)\|_{X \times Y} := \|x\|_X + \|y\|_Y$  for all  $(x, y) \in X \times Y$ .

## 2 The model problem

The following system of partial differential equations describes the stationary state of the transport of species in an immiscible fluid occupying the domain  $\Omega \subseteq \mathbb{R}^n$ :

$$\begin{aligned} \boldsymbol{\sigma} &= \mu(\phi) \nabla \mathbf{u} - p \mathbb{I}, & -\mathbf{div}(\boldsymbol{\sigma}) &= \mathbf{f} \boldsymbol{\phi}, & \text{div}(\mathbf{u}) &= 0, \\ \mathbf{p} &= \vartheta(|\nabla \phi|) \nabla \phi - \phi \mathbf{u} - \gamma(\phi) \mathbf{k}, & -\text{div}(\mathbf{p}) &= g, \\ \int_{\Omega} p &= 0, \end{aligned} \quad (2.1)$$

where the sought quantities are the Cauchy fluid stress  $\boldsymbol{\sigma}$ , the local volume-average velocity of the fluid  $\mathbf{u}$ , the pressure  $p$ , the total (diffusive plus advective) flux for concentration  $\mathbf{p}$ , and the local concentration of species  $\phi$ . In turn,  $\mathbf{f} \in \mathbf{L}^2(\Omega)$  and  $g \in L^{4/3}(\Omega)$  are given functions, and, as observed from the second equation in (2.1), the driving force of the mixture depends linearly on  $\phi$ . In addition, the kinematic effective viscosity,  $\mu$ ; the diffusion coefficient,  $\vartheta$ ; and the one-dimensional flux function describing hindered settling,  $\gamma$ ; depend nonlinearly on  $\phi$ , whereas  $\mathbf{k}$  is a vector pointing in the direction of gravity. Furthermore,  $\vartheta$  is assumed of class  $C^1$  and we suppose that there exist positive constants  $\mu_1, \mu_2, \gamma_1, \gamma_2, \vartheta_1$ , and  $\vartheta_2$ , such that

$$\mu_1 \leq \mu(s) \leq \mu_2 \quad \text{and} \quad \gamma_1 \leq \gamma(s) \leq \gamma_2 \quad \forall s \in \mathbb{R}, \quad (2.2)$$

$$\vartheta_1 \leq \vartheta(s) \leq \vartheta_2 \quad \text{and} \quad \vartheta_1 \leq \vartheta(s) + s \vartheta'(s) \leq \vartheta_2 \quad \forall s \geq 0. \quad (2.3)$$

Examples of nonlinear functions  $\vartheta$  that satisfy the hypothesis (2.3) are  $\vartheta(s) := 2 + \frac{1}{1+s} \forall s \geq 0$ , which is basically of academic character, and the well-known Carreau law in fluid mechanics given by  $\vartheta(s) := m_1 + m_2(1 + s^2)^{(m_3-2)/2} \forall s \geq 0$ , with  $m_1, m_2 > 0$  and  $m_3 \in (0, 2)$ . The latter is indeed considered in the numerical results reported below in Sect. 6.

Additionally, we assume that  $\mu$  and  $\gamma$  are Lipschitz continuous, that is that there exist positive constants  $L_\mu$  and  $L_\gamma$  such that

$$|\mu(s) - \mu(t)| \leq L_\mu |s - t| \quad \forall s, t \in \mathbb{R}, \quad (2.4)$$

and

$$|\gamma(s) - \gamma(t)| \leq L_\gamma |s - t| \quad \forall s, t \in \mathbb{R}. \quad (2.5)$$

Finally, given  $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$  and  $\phi_D \in H^{1/2}(\Gamma)$ , the following Dirichlet boundary conditions complement (2.1):

$$\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma, \quad \phi = \phi_D \quad \text{on } \Gamma, \quad (2.6)$$

where, due to the incompressibility of the fluid, the datum  $\mathbf{u}_D$  must satisfy the compatibility constraint  $\int_\Gamma \mathbf{u}_D \cdot \mathbf{v} = 0$ . On the other hand, it is easy to see that the first and third equations in (2.1) are equivalent to

$$\frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d = \nabla \mathbf{u} \quad \text{and} \quad p = -\frac{1}{n} \text{tr}(\boldsymbol{\sigma}) \quad \text{in } \Omega. \quad (2.7)$$

In this way, and inspired by [20], introducing  $\mathbf{t} := \nabla \phi$  in  $\Omega$  as an additional unknown of the system, (2.1)–(2.6) can be rewritten as follow

$$\begin{aligned} \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d &= \nabla \mathbf{u} \quad \text{in } \Omega, \quad -\text{div}(\boldsymbol{\sigma}) = f\phi \quad \text{in } \Omega, \\ \mathbf{t} &= \nabla \phi \quad \text{in } \Omega, \quad \mathbf{p} = \vartheta(|\mathbf{t}|) \mathbf{t} - \phi \mathbf{u} - \gamma(\phi) \mathbf{k} \quad \text{in } \Omega, \quad -\text{div}(\mathbf{p}) = g \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_D \quad \text{on } \Gamma, \quad \phi = \phi_D \quad \text{on } \Gamma, \quad \int_\Omega \text{tr}(\boldsymbol{\sigma}) = 0. \end{aligned} \quad (2.8)$$

We stress here that the incompressibility condition is implicitly present in the first equation of (2.8), that is in the constitutive equation relating  $\boldsymbol{\sigma}$  and  $\mathbf{u}$ . In addition, the uniqueness condition for  $p$ , originally given by  $\int_\Omega p = 0$ , is now stated as  $\int_\Omega \text{tr}(\boldsymbol{\sigma}) = 0$ , which certainly follows from the postprocessed formula for  $p$  provided by the second expression in (2.7).

### 3 The continuous formulation

In this section we introduce and analyze a new fully-mixed formulation of the coupled problem (2.8).

#### 3.1 A non-augmented fully-mixed approach

We start by recalling the well-known Poincaré inequality, which says that there exists a positive constant  $c_p$ , depending only on  $\Omega$ , such that

$$\|\psi\|_{1,\Omega} \leq c_p |\psi|_{1,\Omega} \quad \forall \psi \in H_0^1(\Omega). \quad (3.1)$$

In turn, we recall that  $H^1(\Omega)$  is continuously embedded into  $L^4(\Omega)$ , which is valid in  $\mathbb{R}^n$ ,  $n \in \{2, 3\}$ . More precisely, we have the following inequality

$$\|\psi\|_{0,4;\Omega} \leq c(\Omega) \|\psi\|_{1,\Omega} \quad \forall \psi \in H^1(\Omega), \quad (3.2)$$

with  $c(\Omega) > 0$ , depending only on  $\Omega$  (see, e.g., [30, Theorem 1.3.4]). Next, we proceed as in [7, Section 3.1], and introduce the Banach space:

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{4/3}(\Omega) \right\},$$

endowed with the norm  $\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}$ . Then, we test the first and second equations of (2.8) against  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$  and  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ , respectively, integrate by parts the first one, and use the identity  $\boldsymbol{\sigma}^d : \boldsymbol{\tau} = \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d$  and the Dirichlet boundary condition  $\mathbf{u} = \mathbf{u}_D$  on  $\Gamma$ , to get

$$\begin{aligned} \int_{\Omega} \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d + \int_{\Omega} \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) &= \langle \boldsymbol{\tau} \mathbf{v}, \mathbf{u}_D \rangle \quad \forall \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega), \\ \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) &= - \int_{\Omega} f \phi \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega), \end{aligned} \quad (3.3)$$

where, as remarked in [14, eq. (2.5)] (see also [20, eq. (3.2)]), the duality  $\langle \boldsymbol{\tau} \mathbf{v}, \mathbf{u}_D \rangle$  is well defined in the sense that  $\boldsymbol{\tau} \mathbf{v} \in \mathbf{H}^{-1/2}(\Gamma)$  for all  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ . At this point, and for convenience of the subsequent analysis, we consider the decomposition (see, for instance, [13, 20]):

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \oplus \mathbb{R} \mathbb{I},$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega) : \int_{\Omega} \text{tr}(\boldsymbol{\tau}) = 0 \right\},$$

and notice from the last equation of (2.8) that  $\boldsymbol{\sigma}$  must lie in  $\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ . In turn, in virtue of the compatibility condition  $\int_{\Gamma} \mathbf{u}_D \cdot \mathbf{v} = 0$ , we realize that the first equation of (3.3) is implicitly satisfied for multiples of the identity  $\mathbb{I}$  since both sides of the equation are nullified in that case, and hence it only suffices to impose the testing against  $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ . Therefore, given  $\phi$  in a suitable space to be defined next, we now consider the following mixed formulation for the flow: Find  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  such that

$$\begin{aligned} \mathbf{a}_{\phi}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \\ \mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) &= \mathbf{G}_{\phi}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega), \end{aligned} \quad (3.4)$$

where  $\mathbf{a}_{\phi} : \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \times \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \rightarrow \mathbb{R}$  and  $\mathbf{b} : \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega) \rightarrow \mathbb{R}$  are the bilinear forms defined, respectively, as

$$\mathbf{a}_{\phi}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_{\Omega} \frac{1}{\mu(\phi)} \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d \quad \text{and} \quad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) := \int_{\Omega} \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}), \quad (3.5)$$

whereas  $\mathbf{F} : \mathbb{H}(\mathbf{div}_{4/3}; \Omega) \rightarrow \mathbb{R}$  and  $\mathbf{G}_{\phi} : \mathbf{L}^4(\Omega) \rightarrow \mathbb{R}$  are the functionals given by

$$\mathbf{F}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau} \mathbf{v}, \mathbf{u}_D \rangle \quad \text{and} \quad \mathbf{G}_\phi(\mathbf{v}) := - \int_{\Omega} f \phi \cdot \mathbf{v}, \quad (3.6)$$

for all  $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ , for all  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ . Notice that using (2.2) in conjunction with the Cauchy–Schwarz and Hölder inequalities, we find that

$$|\mathbf{a}_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq \frac{1}{\mu_1} \|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3}; \Omega} \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3}; \Omega} \quad \text{and} \quad |\mathbf{b}(\boldsymbol{\tau}, \mathbf{v})| \leq \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3}; \Omega} \|\mathbf{v}\|_{0,4; \Omega}, \quad (3.7)$$

for all  $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ , for all  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ . In turn, considering  $\phi \in \mathbf{L}^4(\Omega)$ , employing the duality between  $\mathbf{H}^{-1/2}(\Gamma)$  and  $\mathbf{H}^{1/2}(\Gamma)$ , Cauchy–Schwarz’s inequality, and similar arguments to those in [20, Section 3.1] (see also [13, Lemma 3.5]), we readily show that there exists a positive constant  $\hat{c}(\Omega)$ , depending only on  $c(\Omega)$  (cf. (3.2)), such that

$$|\mathbf{F}(\boldsymbol{\tau})| \leq \hat{c}(\Omega) \|\mathbf{u}_D\|_{1/2, \Gamma} \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3}; \Omega} \quad \text{and} \quad |\mathbf{G}_\phi(\mathbf{v})| \leq \|f\|_{0, \Omega} \|\phi\|_{0,4; \Omega} \|\mathbf{v}\|_{0,4; \Omega}, \quad (3.8)$$

for all  $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega)$ , for all  $\mathbf{v} \in \mathbf{L}^4(\Omega)$ . On the other hand, for the transport equations in (2.8) we proceed as in [20] (see also [21]). In fact, testing the third, fourth and fifth equations of (2.8) against suitable test functions  $\mathbf{q}$ ,  $\mathbf{s}$ , and  $\psi$ , respectively, integrating by parts the first of them, and using the Dirichlet boundary condition  $\phi = \phi_D$  on  $\Gamma$ , we get

$$\begin{aligned} \int_{\Omega} \mathbf{t} \cdot \mathbf{q} + \int_{\Omega} \phi \operatorname{div}(\mathbf{q}) &= \langle \mathbf{q} \cdot \mathbf{v}, \phi_D \rangle, \\ \int_{\Omega} \vartheta(|\mathbf{t}|) \mathbf{t} \cdot \mathbf{s} - \int_{\Omega} \phi \mathbf{u} \cdot \mathbf{s} - \int_{\Omega} \mathbf{p} \cdot \mathbf{s} &= \int_{\Omega} \gamma(\phi) \mathbf{k} \cdot \mathbf{s}, \\ - \int_{\Omega} \psi \operatorname{div}(\mathbf{p}) &= \int_{\Omega} g \psi, \end{aligned} \quad (3.9)$$

for all  $(\mathbf{q}, \mathbf{s}, \psi)$  in spaces to be derived below. In this regard, we begin by noting from the boundedness of  $\vartheta$  (cf. (2.3)) and the fact that  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , that the first term of the first equation of (3.9), and all the terms of the second one, are well defined if  $\mathbf{p}, \mathbf{q}, \mathbf{t}, \mathbf{s} \in \mathbf{L}^2(\Omega)$ , and if  $\phi$ , and consequently the test function  $\psi$ , are chosen to live in  $\mathbf{L}^4(\Omega)$ , which is consistent with (3.8). In this way, since the latter and the remaining terms on the left hand side of (3.9) force both  $\operatorname{div}(\mathbf{p})$  and  $\operatorname{div}(\mathbf{q})$  to live in  $\mathbf{L}^{4/3}(\Omega)$ , we now introduce the Banach space

$$\mathbf{H}(\mathbf{div}_{4/3}; \Omega) := \left\{ \mathbf{q} \in \mathbf{L}^2(\Omega) : \operatorname{div}(\mathbf{q}) \in \mathbf{L}^{4/3}(\Omega) \right\},$$

endowed with the norm  $\|\mathbf{q}\|_{\mathbf{div}_{4/3}; \Omega} := \|\mathbf{q}\|_{0, \Omega} + \|\operatorname{div}(\mathbf{q})\|_{0,4/3; \Omega}$ . Notice that, similarly to (3.3), the right hand side of (3.9) is well defined since  $\mathbf{q} \cdot \mathbf{v}$  belongs to  $\mathbf{H}^{-1/2}(\Gamma)$  for all  $\mathbf{q} \in \mathbf{H}(\mathbf{div}_{4/3}; \Omega)$  and  $\gamma$  is bounded (cf. (2.2)). Therefore, given  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , we obtain the following mixed formulation for the concentration: Find  $((\phi, \mathbf{t}), \mathbf{p}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \times \mathbf{H}(\mathbf{div}_{4/3}; \Omega)$  such that

$$\begin{aligned}
[\mathcal{A}_{\mathbf{u}}(\phi, \mathbf{t}), (\psi, \mathbf{s})] + [\mathcal{B}(\psi, \mathbf{s}), \mathbf{p}] &= [\mathcal{F}_{\phi}, (\psi, \mathbf{s})] \quad \forall (\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega), \\
[\mathcal{B}(\phi, \mathbf{t}), \mathbf{q}] &= [\mathcal{G}, \mathbf{q}] \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega),
\end{aligned}
\tag{3.10}$$

where, given  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , the nonlinear operator  $\mathcal{A}_{\mathbf{u}} : (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \rightarrow (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega))'$  and the linear operator  $\mathcal{B} : (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \rightarrow \mathbf{H}(\operatorname{div}_{4/3}; \Omega)'$ , are defined by

$$[\mathcal{A}_{\mathbf{u}}(\phi, \mathbf{t}), (\psi, \mathbf{s})] := \int_{\Omega} \vartheta(|\mathbf{t}|) \mathbf{t} \cdot \mathbf{s} - \int_{\Omega} \phi \mathbf{u} \cdot \mathbf{s}, \quad [\mathcal{B}(\psi, \mathbf{s}), \mathbf{q}] := - \int_{\Omega} \mathbf{q} \cdot \mathbf{s} - \int_{\Omega} \psi \operatorname{div}(\mathbf{q}), \tag{3.11}$$

for all  $(\phi, \mathbf{t}), (\psi, \mathbf{s}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega))$ , for all  $\mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega)$ , whereas, given  $\phi \in \mathbf{L}^4(\Omega)$ , we set

$$[\mathcal{F}_{\phi}, (\psi, \mathbf{s})] = \int_{\Omega} \gamma(\phi) \mathbf{k} \cdot \mathbf{s} + \int_{\Omega} g \psi \quad \text{and} \quad [\mathcal{G}, \mathbf{q}] := - \langle \mathbf{q} \cdot \mathbf{v}, \phi_D \rangle, \tag{3.12}$$

for all  $(\psi, \mathbf{s}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega))$ , for all  $\mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega)$ . Hereafter,  $[\cdot, \cdot]$  denotes the duality pairing induced by the corresponding operators. Notice that using (2.3), and the Cauchy–Schwarz and Hölder inequalities, there hold

$$\begin{aligned}
|[\mathcal{A}_{\mathbf{u}}(\phi, \mathbf{t}), (\psi, \mathbf{s})]| &\leq (\vartheta_2 + \|\mathbf{u}\|_{0,4;\Omega}) \|(\phi, \mathbf{t})\| \|(\psi, \mathbf{s})\| \\
|[\mathcal{B}(\psi, \mathbf{s}), \mathbf{q}]| &\leq \|(\psi, \mathbf{s})\| \|\mathbf{q}\|_{\operatorname{div}_{4/3}; \Omega}.
\end{aligned}
\tag{3.13}$$

In turn, similarly to (3.8), employing (2.2), the duality between  $\mathbf{H}^{-1/2}(\Gamma)$  and  $\mathbf{H}^{1/2}(\Gamma)$ , and the Cauchy–Schwarz and Hölder inequalities, we readily show that there exists a positive constant  $\tilde{c}(\Omega)$ , depending only on  $c(\Omega)$  (cf. (3.2)), such that

$$\begin{aligned}
|[\mathcal{F}_{\phi}, (\psi, \mathbf{s})]| &\leq (\gamma_2 |\Omega|^{1/2} \|\mathbf{k}\| + \|g\|_{0,4/3;\Omega}) \|(\psi, \mathbf{s})\| \quad \text{and} \\
|[\mathcal{G}, \mathbf{q}]| &\leq \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \|\mathbf{q}\|_{\operatorname{div}_{4/3}; \Omega}.
\end{aligned}
\tag{3.14}$$

In addition, we recall from [27, Theorem 3.8] that, thanks to the assumption on  $\vartheta$  (cf. (2.3)), the nonlinear component of  $\mathcal{A}_{\mathbf{u}}$  given by its first term (cf. (3.11)) is strongly monotone and Lipschitz-continuous in  $\mathbf{L}^2(\Omega)$  with constants  $\vartheta_1$  and  $\tilde{\vartheta}_2 := \max\{\vartheta_2, 2\vartheta_2 - \vartheta_1\}$ , respectively, which means that

$$\int_{\Omega} \left\{ \vartheta(|\mathbf{t}|) \mathbf{t} - \vartheta(|\mathbf{r}|) \mathbf{r} \right\} \cdot (\mathbf{t} - \mathbf{r}) \geq \vartheta_1 \|\mathbf{t} - \mathbf{r}\|_{0,\Omega}^2 \quad \forall \mathbf{t}, \mathbf{r} \in \mathbf{L}^2(\Omega), \quad \text{and} \tag{3.15}$$

$$\left| \int_{\Omega} \left\{ \vartheta(|\mathbf{t}|) \mathbf{t} - \vartheta(|\mathbf{r}|) \mathbf{r} \right\} \cdot \mathbf{s} \right| \leq \tilde{\vartheta}_2 \|\mathbf{t} - \mathbf{r}\|_{0,\Omega} \|\mathbf{s}\|_{0,\Omega} \quad \forall \mathbf{t}, \mathbf{r}, \mathbf{s} \in \mathbf{L}^2(\Omega). \tag{3.16}$$

Then, the non-augmented fully-mixed formulation for (2.8) reduces to (3.4) and (3.10), that is: Find  $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  and  $((\phi, \mathbf{t}), \mathbf{p}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \times \mathbf{H}(\operatorname{div}_{4/3}; \Omega)$  such that

$$\begin{aligned}
\mathbf{a}_\phi(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \mathbb{H}_0(\operatorname{div}_{4/3}; \Omega), \\
\mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{L}^4(\Omega), \\
[\mathcal{A}_\mathbf{u}(\phi, \mathbf{t}), (\boldsymbol{\psi}, \mathbf{s})] + [\mathcal{B}(\boldsymbol{\psi}, \mathbf{s}), \mathbf{p}] &= [\mathcal{F}_\phi, (\boldsymbol{\psi}, \mathbf{s})] & \forall (\boldsymbol{\psi}, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega), \\
[\mathcal{B}(\phi, \mathbf{t}), \mathbf{q}] &= [\mathcal{G}, \mathbf{q}] & \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega).
\end{aligned} \tag{3.17}$$

### 3.2 A fixed point strategy

In what follows we proceed similarly to [1] (see also [2, 7]) and utilize a fixed point strategy to analyze the solvability of (3.17). For this purpose, we first let  $\mathbf{S} : \mathbf{L}^4(\Omega) \rightarrow \mathbb{H}_0(\operatorname{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  be the operator defined by:

$$\mathbf{S}(\phi) = (\mathbf{S}_1(\phi), \mathbf{S}_2(\phi)) := (\boldsymbol{\sigma}, \mathbf{u}) \quad \forall \phi \in \mathbf{L}^4(\Omega),$$

where  $(\boldsymbol{\sigma}, \mathbf{u})$  is the unique solution (to be confirmed below) of (3.4) with the given  $\phi$ . In turn, we let  $\tilde{\mathbf{S}} : \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega) \rightarrow \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}(\operatorname{div}_{4/3}; \Omega)$  be the operator defined by

$$\tilde{\mathbf{S}}(\phi, \mathbf{u}) = (\tilde{\mathbf{S}}_1(\phi, \mathbf{u}), \tilde{\mathbf{S}}_2(\phi, \mathbf{u}), \tilde{\mathbf{S}}_3(\phi, \mathbf{u})) := (\tilde{\phi}, \mathbf{t}, \mathbf{p}) \quad \forall (\phi, \mathbf{u}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega),$$

where  $(\tilde{\phi}, \mathbf{t}, \mathbf{p})$  is the unique solution (to be confirmed below) of the problem: Find  $((\tilde{\phi}, \mathbf{t}), \mathbf{p}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \times \mathbf{H}(\operatorname{div}_{4/3}; \Omega)$  such that

$$\begin{aligned}
[\mathcal{A}_\mathbf{u}(\tilde{\phi}, \mathbf{t}), (\boldsymbol{\psi}, \mathbf{s})] + [\mathcal{B}(\boldsymbol{\psi}, \mathbf{s}), \mathbf{p}] &= [\mathcal{F}_\phi, (\boldsymbol{\psi}, \mathbf{s})] & \forall (\boldsymbol{\psi}, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega), \\
[\mathcal{B}(\tilde{\phi}, \mathbf{t}), \mathbf{q}] &= [\mathcal{G}, \mathbf{q}] & \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega),
\end{aligned} \tag{3.18}$$

with the given  $(\phi, \mathbf{u})$ . Then, we set the operator  $\mathbf{T} : \mathbf{L}^4(\Omega) \rightarrow \mathbf{L}^4(\Omega)$  by

$$\mathbf{T}(\phi) := \tilde{\mathbf{S}}_1(\phi, \mathbf{S}_2(\phi)) \quad \forall \phi \in \mathbf{L}^4(\Omega), \tag{3.19}$$

and realize that solving (3.17) is equivalent to seeking a fixed point of  $\mathbf{T}$ : Find  $\phi \in \mathbf{L}^4(\Omega)$  such that

$$\mathbf{T}(\phi) = \phi. \tag{3.20}$$

### 3.3 Well-posedness of the uncoupled problems

In this section we show that the uncoupled problems (3.4) and (3.18) are well-posed, which means, equivalently, that the operators  $\mathbf{S}$  and  $\tilde{\mathbf{S}}$  are indeed well-defined. We begin with (3.4), whose proof is almost verbatim to the one in [7, Lemma 3.3]. In fact, we first recall from [13, Lemma 3.2] (see also [7, eq. (3.25)]) that there exists  $c_1 > 0$ , depending only on  $\Omega$ , such that

$$c_1 \|\boldsymbol{\tau}\|_{0,\Omega}^2 \leq \|\boldsymbol{\tau}^d\|_{0,\Omega}^2 + \|\operatorname{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega}^2 \quad \forall \boldsymbol{\tau} \in \mathbb{H}_0(\operatorname{div}_{4/3}; \Omega). \tag{3.21}$$

Next, we let  $\mathbb{V}$  be the kernel of the operator induced by the bilinear form  $\mathbf{b}$  (cf. (3.5)), that is

$$\mathbb{V} := \left\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) : \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \text{ in } \Omega \right\}, \quad (3.22)$$

and observe, similarly to [7, Lemma 3.1], thanks to the definition of  $\mathbf{a}_\phi$  (cf. (3.5)) and (3.21), that  $\mathbf{a}_\phi$  is elliptic on  $\mathbb{V}$ , that is

$$\mathbf{a}_\phi(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \alpha \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3}; \Omega}^2 \quad \forall \boldsymbol{\tau} \in \mathbb{V}, \quad (3.23)$$

where  $\alpha = c_1/\mu_2$ . As a straightforward consequence of (3.23) it follows that

$$\sup_{\boldsymbol{\zeta} \in \mathbb{V}} \mathbf{a}_\phi(\boldsymbol{\zeta}, \boldsymbol{\tau}) > 0 \quad \forall \boldsymbol{\tau} \in \mathbb{V} \setminus \{\mathbf{0}\}, \quad \forall \phi \in L^4(\Omega). \quad (3.24)$$

In turn, according to [13, Lemma 3.3], we know that there exists a positive constant  $\beta$ , depending only on  $n$ ,  $c_p$  and  $c(\Omega)$  (cf. (3.1), (3.2)), such that  $\mathbf{b}$  verifies the following inf-sup condition

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}, \mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3}; \Omega}} \geq \beta \|\mathbf{v}\|_{0,4; \Omega} \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega). \quad (3.25)$$

According to the previous results, we are now able to prove the well-definedness of the operator  $\mathbf{S}$ .

**Lemma 3.1** *For each  $\phi \in L^4(\Omega)$  there exists a unique  $\mathbf{S}(\phi) := (\boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  solution to the problem (3.4). Moreover, there exists a positive constant  $C_S$ , depending only on  $\mu_1$ ,  $\alpha$ ,  $\beta$ , and  $\widehat{c}(\Omega)$  (cf. (3.8)), and hence independent of  $\phi$ , such that*

$$\|\mathbf{S}(\phi)\| = \|(\boldsymbol{\sigma}, \mathbf{u})\| \leq C_S \left\{ \|\mathbf{u}_D\|_{1/2, \Gamma} + \|\mathbf{f}\|_{0, \Omega} \|\phi\|_{0,4; \Omega} \right\} \quad \forall \phi \in L^4(\Omega). \quad (3.26)$$

**Proof** Given  $\phi \in L^4(\Omega)$ , we proceed as in [7, Lemma 3.3]. In fact, we first recall from (3.7) and (3.8) that  $\mathbf{a}_\phi$ ,  $\mathbf{b}$ ,  $\mathbf{F}$ ,  $\mathbf{G}_\phi$  are all bounded. Then, thanks to (3.23), (3.24), and (3.25), the proof follows from a straightforward application of the Babuška–Brezzi theory in Banach spaces (see, e.g., [24, Theorem 2.34]) to problem (3.4). In particular, there exists a positive constant  $C$  depending only on  $\|\mathbf{a}_\phi\| \leq \frac{1}{\mu_1}$ ,  $\alpha$ , and  $\beta$ , such that

$$\|(\boldsymbol{\sigma}, \mathbf{u})\| \leq C \left\{ \|\mathbf{F}\| + \|\mathbf{G}_\phi\| \right\}, \quad (3.27)$$

which, combined with (3.8), yields (3.26) with  $C_S = C \max \{1, \widehat{c}(\Omega)\}$ .  $\square$

Now we present an abstract result that will be utilized to establish the unique solvability of the nonlinear problem (3.18), equivalently, the well-definedness of the operator  $\tilde{\mathbf{S}}$ .

**Theorem 3.2** *Let  $H$  and  $Q$  be separable and reflexive Banach spaces, with  $H$  being uniformly convex, and let  $a : H \rightarrow H'$  be a nonlinear operator and  $b \in \mathcal{L}(H, Q')$ . In turn, let  $V$  be the null space of  $b$ , and assume that*

(i)  *$a$  is Lipschitz-continuous, that is there exists  $L > 0$  such that*

$$\|a(u) - a(v)\|_{H'} \leq L \|u - v\|_H \quad \forall u, v \in H,$$

(ii) *The family of operators  $a(\cdot + t) : V \rightarrow V'$ , with  $t \in H$ , is uniformly strongly monotone, that is there exists  $\alpha > 0$  such that*

$$[a(u + t) - a(v + t), u - v] \geq \alpha \|u - v\|_H^2 \quad \forall t \in H, \quad \forall u, v \in V,$$

(iii) *There exists  $\beta > 0$  such that*

$$\sup_{\substack{v \in H \\ v \neq 0}} \frac{[b(v), \tau]}{\|v\|_H} \geq \beta \|\tau\|_Q \quad \forall \tau \in Q.$$

*Then, for each  $(F, G) \in H' \times Q'$  there exists a unique  $(u, \sigma) \in H \times Q$  such that*

$$\begin{aligned} [a(u), v] + [b(v), \sigma] &= [F, v] \quad \forall v \in H, \\ [b(u), \tau] &= [G, \tau] \quad \forall \tau \in Q, \end{aligned}$$

*and there hold*

$$\begin{aligned} \|u\|_H &\leq \frac{1}{\alpha} \|F\|_{H'} + \frac{1}{\beta} \left(1 + \frac{L}{\alpha}\right) \|G\|_{Q'} + \frac{1}{\alpha} \|a(0)\|_{H'} \quad \text{and} \\ \|\sigma\|_Q &\leq \frac{1}{\beta} \left(1 + \frac{L}{\alpha}\right) \|F\|_{H'} + \frac{L}{\beta^2} \left(1 + \frac{L}{\alpha}\right) \|G\|_{Q'} + \frac{1}{\beta} \left(1 + \frac{L}{\alpha}\right) \|a(0)\|_{H'}. \end{aligned} \quad (3.28)$$

**Proof** It follows from a slight adaptation of [32, Proposition 2.3] with  $p = 2$  (see also [15, Theorem 3.1] with  $p_1 = p_2 = 2$ ). Further details are omitted.  $\square$

Next, in order to apply Theorem 3.2 to problem (3.18), we first observe that, thanks to the uniform convexity and separability of  $L^p(\Omega)$  for  $p \in (1, +\infty)$ , all the spaces in (3.18), that is,  $L^4(\Omega)$ ,  $\mathbf{L}^2(\Omega)$ , and  $\mathbf{H}(\text{div}_{4/3}; \Omega)$ , share the same properties.

We continue our analysis by proving that for each  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ ,  $\mathcal{A}_{\mathbf{u}}$  satisfies hypothesis (i) of Theorem 3.2.

**Lemma 3.3** *There holds*

$$\|\mathcal{A}_{\mathbf{u}}(\phi, \mathbf{t}) - \mathcal{A}_{\mathbf{u}}(\tilde{\phi}, \tilde{\mathbf{t}})\| \leq (\tilde{\vartheta}_2 + \|\mathbf{u}\|_{0,4;\Omega}) \|(\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})\|, \quad (3.29)$$

*for all  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , and for all  $(\phi, \mathbf{t}), (\tilde{\phi}, \tilde{\mathbf{t}}) \in L^4(\Omega) \times \mathbf{L}^2(\Omega)$ .*

**Proof** Given  $\mathbf{u} \in \mathbf{L}^4(\Omega)$  and  $(\phi, \mathbf{t}), (\tilde{\phi}, \tilde{\mathbf{t}}), (\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$ , we start from the definition of  $\mathcal{A}_{\mathbf{u}}$  (cf. (3.11)), and then employ estimate (3.16) and Cauchy–Schwarz’s inequality, to deduce that

$$\begin{aligned} |[\mathcal{A}_{\mathbf{u}}(\phi, \mathbf{t}) - \mathcal{A}_{\mathbf{u}}(\tilde{\phi}, \tilde{\mathbf{t}}), (\psi, \mathbf{s})]| &\leq \left| \int_{\Omega} \left\{ \vartheta(|\mathbf{t}|) \mathbf{t} - \vartheta(|\tilde{\mathbf{t}}|) \tilde{\mathbf{t}} \right\} \cdot \mathbf{s} \right| + \left| \int_{\Omega} (\phi - \tilde{\phi}) \mathbf{u} \cdot \mathbf{s} \right| \\ &\leq \tilde{\vartheta}_2 \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega} \|\mathbf{s}\|_{0,\Omega} + \|\phi - \tilde{\phi}\|_{0,4;\Omega} \|\mathbf{u}\|_{0,4;\Omega} \|\mathbf{s}\|_{0,\Omega} \\ &\leq (\tilde{\vartheta}_2 + \|\mathbf{u}\|_{0,4;\Omega}) \|(\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})\| \|\mathbf{s}\|_{0,\Omega}, \end{aligned}$$

which implies (3.29) and completes the proof.  $\square$

Now, let us look at the null space of the operator  $\mathcal{B}$  (cf. (3.11)), that is

$$\tilde{\mathbb{V}} := \left\{ (\psi, \mathbf{t}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : [\mathcal{B}(\psi, \mathbf{t}), \mathbf{q}] = 0 \quad \forall \mathbf{q} \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega) \right\},$$

which, proceeding similarly to [20, eq. (3.35)], reduces to

$$\tilde{\mathbb{V}} = \left\{ (\psi, \mathbf{t}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) : \nabla \psi = \mathbf{t} \quad \text{in } \Omega \quad \text{and} \quad \psi \in H_0^1(\Omega) \right\}. \quad (3.30)$$

The following result states that  $\mathcal{A}_{\mathbf{u}}$  satisfies hypothesis (ii) of Theorem 3.2.

**Lemma 3.4** *There exists a constant  $\tilde{\alpha} > 0$ , depending only on  $\vartheta_1$  (cf. (2.3)),  $c_p$  ((3.1)), and  $c(\Omega)$  (cf. (3.2)), such that for each  $\mathbf{u} \in \mathbf{L}^4(\Omega)$  satisfying  $\|\mathbf{u}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ , the family of operators  $\mathcal{A}_{\mathbf{u}}(\cdot + (\psi, \mathbf{s}))$ , with  $(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$ , is uniformly strongly monotone on  $\tilde{\mathbb{V}}$  with constant  $\tilde{\alpha}$ , that is*

$$[\mathcal{A}_{\mathbf{u}}((\phi, \mathbf{t}) + (\psi, \mathbf{s})) - \mathcal{A}_{\mathbf{u}}((\tilde{\phi}, \tilde{\mathbf{t}}) + (\psi, \mathbf{s})), (\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})] \geq \tilde{\alpha} \|(\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})\|^2 \quad (3.31)$$

for all  $(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$  and for all  $(\phi, \mathbf{t}), (\tilde{\phi}, \tilde{\mathbf{t}}) \in \tilde{\mathbb{V}}$ .

**Proof** Given  $\mathbf{u} \in \mathbf{L}^4(\Omega)$ , we let  $(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)$  and  $(\phi, \mathbf{t}), (\tilde{\phi}, \tilde{\mathbf{t}}) \in \tilde{\mathbb{V}}$ . It follows from (3.30) that  $\mathbf{t} - \tilde{\mathbf{t}} = \nabla(\phi - \tilde{\phi})$  in  $\Omega$  and  $\phi - \tilde{\phi} \in H_0^1(\Omega)$ . Then, according to the definition of  $\mathcal{A}_{\mathbf{u}}$  (cf. (3.11)), and using (3.15), Poincaré’s inequality (cf. (3.1)), the continuous injection of  $H^1(\Omega)$  into  $\mathbf{L}^4(\Omega)$  (cf. (3.2)), and the Cauchy–Schwarz and Young inequalities, we obtain

$$\begin{aligned} &[\mathcal{A}_{\mathbf{u}}((\phi, \mathbf{t}) + (\psi, \mathbf{s})) - \mathcal{A}_{\mathbf{u}}((\tilde{\phi}, \tilde{\mathbf{t}}) + (\psi, \mathbf{s})), (\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})] \\ &= \int_{\Omega} \left\{ \vartheta(\mathbf{t} + \mathbf{s}) (\mathbf{t} + \mathbf{s}) - \vartheta(\tilde{\mathbf{t}} + \mathbf{s}) (\tilde{\mathbf{t}} + \mathbf{s}) \right\} \cdot (\mathbf{t} - \tilde{\mathbf{t}}) - \int_{\Omega} (\phi - \tilde{\phi}) \mathbf{u} \cdot (\mathbf{t} - \tilde{\mathbf{t}}) \\ &\geq \vartheta_1 \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega}^2 - \|\mathbf{u}\|_{0,4;\Omega} \|\phi - \tilde{\phi}\|_{0,4;\Omega} \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega} \\ &= \frac{\vartheta_1}{2} \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega}^2 + \frac{\vartheta_1}{2} |\phi - \tilde{\phi}|_{1,\Omega}^2 - \|\mathbf{u}\|_{0,4;\Omega} \|\phi - \tilde{\phi}\|_{0,4;\Omega} \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega} \\ &\geq \frac{1}{2} \left\{ \vartheta_1 - \|\mathbf{u}\|_{0,4;\Omega} \right\} \|\mathbf{t} - \tilde{\mathbf{t}}\|_{0,\Omega}^2 + \frac{1}{2} \left\{ \vartheta_1 c_p^{-2} c(\Omega)^{-2} - \|\mathbf{u}\|_{0,4;\Omega} \right\} \|\phi - \tilde{\phi}\|_{0,4;\Omega}^2. \end{aligned} \quad (3.32)$$

In this way, defining

$$\tilde{\alpha} := \frac{\vartheta_1}{4} \min \{1, c_p^{-2} c(\Omega)^{-2}\}, \quad (3.33)$$

we arrive at

$$[\mathcal{A}_{\mathbf{u}}((\phi, \mathbf{t}) + (\psi, \mathbf{s})) - \mathcal{A}_{\mathbf{u}}((\tilde{\phi}, \tilde{\mathbf{t}}) + (\psi, \mathbf{s})), (\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})] \geq \frac{1}{2} \{4\tilde{\alpha} - \|\mathbf{u}\|_{0,4;\Omega}\} \|(\phi, \mathbf{t}) - (\tilde{\phi}, \tilde{\mathbf{t}})\|^2,$$

from which we readily conclude the proof.  $\square$

We observe here that, instead of imposing  $\|\mathbf{u}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ , we could have considered the assumption  $\|\mathbf{u}\|_{0,4;\Omega} \leq 4\delta\tilde{\alpha}$ , with  $\delta \in (0, 1)$ . In this regard, we notice that choosing  $\delta$  the closer to 1, the larger the resulting range for  $\|\mathbf{u}\|_{0,4;\Omega}$ , but then the strong monotonicity constant approaches 0. Conversely, the closer to 0, the smaller the range for  $\|\mathbf{u}\|_{0,4;\Omega}$ , but then the strong monotonicity constant approaches  $4\tilde{\alpha}$ . Hence, our above implicit choice  $\delta = \frac{1}{2}$  aims to balance both aspects.

We complete the verification of the hypotheses of Theorem 3.2, with the corresponding inf-sup condition for the operator  $\mathcal{B}$ , whose proof can be found in [20, Lemma 3.3].

**Lemma 3.5** *There holds*

$$\sup_{\substack{(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \\ (\psi, \mathbf{s}) \neq \mathbf{0}}} \frac{|\mathcal{B}(\psi, \mathbf{s}), \mathbf{q}|}{\|(\psi, \mathbf{s})\|} \geq \tilde{\beta} \|\mathbf{q}\|_{\text{div}_{4/3}; \Omega} \quad \forall \mathbf{q} \in \mathbf{H}(\text{div}_{4/3}; \Omega), \quad (3.34)$$

with  $\tilde{\beta} = 1/2$ .

We now establish the unique solvability of the nonlinear problem (3.18).

**Lemma 3.6** *Let  $\tilde{\alpha}$  be the constant given by (3.33). Then, for each  $(\phi, \mathbf{u}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$  such that  $\|\mathbf{u}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ , problem (3.18) has a unique solution  $((\tilde{\phi}, \mathbf{t}), \mathbf{p}) \in (\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega)) \times \mathbf{H}(\text{div}_{4/3}; \Omega)$ . Moreover, there exists a positive constant  $C_{\tilde{\mathfrak{s}}}$ , depending on  $\tilde{\alpha}$  and  $L$ , such that*

$$\|\tilde{\mathbf{S}}(\phi, \mathbf{u})\| = \|(\tilde{\phi}, \mathbf{t}, \mathbf{p})\| \leq C_{\tilde{\mathfrak{s}}} \left\{ \gamma_2 |\Omega|^{1/2} \|\mathbf{k}\| + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \right\}. \quad (3.35)$$

**Proof** Given  $(\phi, \mathbf{u}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^4(\Omega)$  as stated, we first recall from (3.13) and (3.14) that  $\mathcal{B}$ ,  $\mathcal{F}_{\phi}$  and  $\mathcal{G}$  are all linear and bounded. In addition, Lemma 3.3 together with the hypothesis on  $\mathbf{u}$  imply the Lipschitz continuity of  $\mathcal{A}_{\mathbf{u}}$  with constant  $\tilde{L} := \tilde{\vartheta}_2 + 2\tilde{\alpha}$ . Thus, bearing in mind Lemmas 3.4 and 3.5, a straightforward application of Theorem 3.2 to problem (3.18) completes the proof. In particular, noting that  $\mathcal{A}_{\mathbf{u}}((\mathbf{0}, \mathbf{0}))$  is the null functional, and recalling from Lemma 3.5 that  $\tilde{\beta} = 1/2$ , the a priori estimate (3.28) establishes the existence of a constant  $C_{\tilde{\mathfrak{s}}} > 0$ , depending only on  $\tilde{\alpha}$  and  $\tilde{L}$ , such that

$$\|(\tilde{\phi}, \mathbf{t}, \mathbf{p})\| \leq C_{\tilde{\mathbf{S}}} \left\{ \|\mathcal{F}_{\phi}\| + \|\mathcal{G}\| \right\},$$

which, together with the bounds of  $\|\mathcal{F}_{\phi}\|$  and  $\|\mathcal{G}\|$  (cf. (3.14)), yield (3.35) and finish the proof.  $\square$

### 3.4 Solvability analysis of the fixed point equation

Having established in the previous section the well-posedness of the uncoupled problem (3.4) and (3.18), which confirms that the operators  $\mathbf{S}$ ,  $\tilde{\mathbf{S}}$ , and  $\mathbf{T}$  are well defined, we now aim to establish the existence of a unique fixed-point of the operator  $\mathbf{T}$ . We begin by providing suitable conditions under which  $\mathbf{T}$  maps a ball into itself.

**Lemma 3.7** *Given  $r > 0$ , we let  $W := \left\{ \phi \in L^4(\Omega) : \|\phi\|_{0,4;\Omega} \leq r \right\}$ , and assume that the data satisfy*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + r \|f\|_{0,\Omega} \leq \frac{2\tilde{\alpha}}{C_{\tilde{\mathbf{S}}}} \quad (3.36)$$

and

$$\gamma_2 |\Omega|^{1/2} + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \leq \frac{r}{C_{\tilde{\mathbf{S}}}}. \quad (3.37)$$

Then  $\mathbf{T}(W) \subseteq W$ .

**Proof** Given  $\phi \in W$ , we get from (3.26) that

$$\|\mathbf{S}(\phi)\| \leq C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + r \|f\|_{0,\Omega} \right\},$$

and hence, thanks to the constraint (3.36), we observe that  $\|\mathbf{S}_2(\phi)\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ , which says that the pair  $(\phi, \mathbf{S}_2(\phi))$  satisfies the hypothesis of Lemma 3.6. Then, the corresponding estimate (3.35) yields

$$\|\mathbf{T}(\phi)\| = \|\tilde{\mathbf{S}}_1(\phi, \mathbf{S}_2(\phi))\| \leq C_{\tilde{\mathbf{S}}} \left\{ \gamma_2 |\Omega|^{1/2} + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \right\},$$

which, in virtue of the assumption (3.37), proves that  $\mathbf{T}(\phi) \in W$ , thus finishing the proof.  $\square$

We now aim to prove that the operator  $\mathbf{T}$  is Lipschitz continuous, for which, according to its definition (cf. (3.19)), it suffices to show that both  $\mathbf{S}$  and  $\tilde{\mathbf{S}}$  satisfy this property. We begin next with the corresponding result for  $\mathbf{S}$ , for which, inspired by [20, eq. (3.62)], we need to incorporate further regularity on the solution of the problem defining this operator. More precisely, we assume that

$\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}(\Gamma)$  for some  $\varepsilon \in [1/2, 1)$  (when  $n = 2$ ) or  $\varepsilon \in [3/4, 1)$  (when  $n = 3$ ), and that for each  $\psi \in L^4(\Omega)$  with  $\|\psi\|_{0,4;\Omega} \leq r$ ,  $r > 0$  given, there hold  $\mathbf{S}(\psi) := (\zeta, \mathbf{w}) \in (\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \cap \mathbb{H}^\varepsilon(\Omega)) \times \mathbf{W}^{\varepsilon,4}(\Omega)$  and

$$\|\zeta\|_{\varepsilon,\Omega} + \|\mathbf{w}\|_{\varepsilon,4;\Omega} \leq \tilde{C}_S(r) \left\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|f\|_{0,\Omega} \|\psi\|_{0,4;\Omega} \right\}, \quad (3.38)$$

with a positive constant  $\tilde{C}_S(r)$  independent of the given  $\psi$  but depending on the upper bound  $r$  of its  $L^4(\Omega)$ -norm. We notice that the reason of the indicated range for  $\varepsilon$  will be clarified in the proof of the following lemma.

**Lemma 3.8** *There exists a positive constant  $L_S$ , depending on  $\mu_1$ ,  $L_\mu$ ,  $\alpha$ ,  $\beta$  and  $\varepsilon$ , such that*

$$\|\mathbf{S}(\phi) - \mathbf{S}(\psi)\| \leq L_S \left\{ \|f\|_{0,\Omega} + \|\mathbf{S}_1(\psi)\|_{\varepsilon,\Omega} \right\} \|\phi - \psi\|_{0,4;\Omega} \quad \forall \phi, \psi \in L^4(\Omega). \quad (3.39)$$

**Proof** It follows from a slight modification of the proof of [7, Lemma 3.7], considering now (3.38) with the corresponding intervals for  $\varepsilon$ . In fact, given  $\phi, \psi \in L^4(\Omega)$ , we first denote  $(\sigma, \mathbf{u}) = \mathbf{S}(\phi)$  and  $(\zeta, \mathbf{w}) = \mathbf{S}(\psi)$  (cf. (3.4)). Then, using the ellipticity of  $\mathbf{a}_\phi$  (cf. (3.23)), the inf-sup condition for  $\mathbf{b}$  (cf. (3.25)), the a priori estimate (3.27) of problem (3.4), the Lipschitz-continuity of  $\mu$  (cf. (2.4)), the Cauchy–Schwarz inequality, and some algebraic manipulations, we are able to show that there exists a positive constant  $C$ , depending only on  $\mu_1$ ,  $\alpha$ , and  $\beta$ , such that

$$\|\mathbf{S}(\phi) - \mathbf{S}(\psi)\| \leq C \left\{ L_\mu \|\zeta\|_{0,2p;\Omega} \|\phi - \psi\|_{0,2q;\Omega} + \|f\|_{0,\Omega} \|\phi - \psi\|_{0,4;\Omega} \right\}, \quad (3.40)$$

where  $p, q \in [1, +\infty)$  are such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Next, bearing in mind the further regularity assumption given by (3.38), we notice that the Sobolev embedding Theorem (see, e.g., [30, Theorem 1.3.4]) establishes the continuous injection  $i_\varepsilon : \mathbf{H}^\varepsilon(\Omega) \rightarrow L^{\varepsilon^*}(\Omega)$ , where  $\varepsilon^* := \begin{cases} \frac{2}{1-\varepsilon} & \text{if } n = 2, \\ \frac{6}{3-2\varepsilon} & \text{if } n = 3 \end{cases}$ . Thus, choosing  $p$  such that  $2p = \varepsilon^*$ , it follows that  $\zeta := \mathbf{S}_1(\psi)$  belongs to  $\mathbb{L}^{2p}(\Omega)$ , and hence, thanks to the aforementioned continuity, there holds

$$\|\zeta\|_{0,2p;\Omega} \leq \|i_\varepsilon\| \|\zeta\|_{\varepsilon,\Omega}. \quad (3.41)$$

In turn, with this choice of  $2p$  we obtain that  $2q = n/\varepsilon$ , and hence, using now that for the specified range of  $\varepsilon$  the injection  $\tilde{i}_\varepsilon : L^4(\Omega) \rightarrow L^{n/\varepsilon}(\Omega)$  is also continuous, we find that

$$\|\phi - \psi\|_{0,2q;\Omega} = \|\phi - \psi\|_{0,n/\varepsilon;\Omega} \leq \|\tilde{i}_\varepsilon\| \|\phi - \psi\|_{0,4;\Omega}. \quad (3.42)$$

In this way, replacing (3.41) and (3.42) back into (3.40), we get (3.39) and complete the proof.  $\square$

On the other hand, the continuity of  $\tilde{\mathbf{S}}$  is proved next.

**Lemma 3.9** *There exists a positive constant  $L_{\tilde{S}}$ , depending on  $L_\gamma$ ,  $\tilde{L}$  (cf. proof of Lemma 3.6),  $\tilde{\beta}$  and  $\tilde{\alpha}$ , such that for all  $(\phi, \mathbf{u}), (\varphi, \mathbf{w}) \in L^4(\Omega) \times \mathbf{L}^4(\Omega)$ , with  $\|\mathbf{u}\|_{0,4;\Omega}, \|\mathbf{w}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ , there holds*

$$\|\tilde{S}(\phi, \mathbf{u}) - \tilde{S}(\varphi, \mathbf{w})\| \leq L_{\tilde{S}} \left\{ |\Omega|^{1/4} \|\mathbf{k}\| \|\phi - \varphi\|_{0,4;\Omega} + \|\tilde{S}_1(\varphi, \mathbf{w})\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} \right\}. \quad (3.43)$$

**Proof** Let  $(\phi, \mathbf{u}), (\varphi, \mathbf{w}) \in L^4(\Omega) \times \mathbf{L}^4(\Omega)$  such that  $\|\mathbf{u}\|_{0,4;\Omega}, \|\mathbf{w}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$ . Then, subtracting the respective problems from (3.18) defining  $(\tilde{\phi}, \mathbf{t}, \mathbf{p}) := \tilde{S}(\phi, \mathbf{u})$  and  $(\tilde{\varphi}, \mathbf{r}, \mathbf{m}) := \tilde{S}(\varphi, \mathbf{w})$ , we obtain

$$\begin{aligned} [\mathcal{A}_{\mathbf{u}}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r}), (\psi, \mathbf{s})] + [\mathcal{B}(\psi, \mathbf{s}), \mathbf{p} - \mathbf{m}] &= [\mathcal{F}_\phi - \mathcal{F}_\varphi, (\psi, \mathbf{s})], \\ [\mathcal{B}((\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})), \mathbf{q}] &= 0, \end{aligned} \quad (3.44)$$

for all  $(\psi, \mathbf{s}) \in L^4(\Omega) \times \mathbf{L}^2(\Omega)$  and for all  $\mathbf{q} \in \mathbf{H}(\text{div}_{4/3}; \Omega)$ . It follows from the second equation of (3.44) that  $(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r}) \in \tilde{\mathbb{V}}$  (cf. (3.30)), and hence, applying the strong monotonicity of  $\mathcal{A}_{\mathbf{u}}$  on  $\tilde{\mathbb{V}}$  (cf. (3.31) in Lemma 3.4) with  $(\tilde{\varphi}, \mathbf{r}) \in L^4(\Omega) \times \mathbf{L}^2(\Omega)$ ,  $(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r}) \in \tilde{\mathbb{V}}$ , and  $(0, \mathbf{0}) \in \tilde{\mathbb{V}}$ , we get

$$\tilde{\alpha} \|(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})\|^2 \leq [\mathcal{A}_{\mathbf{u}}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_{\mathbf{u}}(\tilde{\varphi}, \mathbf{r}), (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})].$$

Then, adding and subtracting  $\mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r})$  in the first component on the right hand side of the foregoing inequality, and using now the first equation of (3.44), we find that

$$\begin{aligned} \tilde{\alpha} \|(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})\|^2 &\leq [\mathcal{A}_{\mathbf{u}}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r}) - (\mathcal{A}_{\mathbf{u}}(\tilde{\varphi}, \mathbf{r}) - \mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r})), (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})] \\ &= [\mathcal{F}_\phi - \mathcal{F}_\varphi, (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})] - [\mathcal{A}_{\mathbf{u}}(\tilde{\varphi}, \mathbf{r}) - \mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r}), (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})]. \end{aligned} \quad (3.45)$$

Now, according to the definitions of  $\mathcal{F}_\phi$  (cf. (3.12)) and  $\mathcal{A}_{\mathbf{u}}$  (cf. (3.11)), and making use of the Lipschitz-continuity of  $\gamma$  (cf. (2.5)) and the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned} |[\mathcal{F}_\phi - \mathcal{F}_\varphi, (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})]| &= \left| \int_{\Omega} \left\{ \gamma(\phi) - \gamma(\varphi) \right\} \mathbf{k} \cdot (\mathbf{t} - \mathbf{r}) \right| \\ &\leq L_\gamma \|\mathbf{k}\| \|\phi - \varphi\|_{0,\Omega} \|\mathbf{t} - \mathbf{r}\|_{0,\Omega} \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} |[\mathcal{A}_{\mathbf{u}}(\tilde{\varphi}, \mathbf{r}) - \mathcal{A}_{\mathbf{w}}(\tilde{\varphi}, \mathbf{r}), (\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})]| &= \left| \int_{\Omega} \tilde{\varphi} (\mathbf{u} - \mathbf{w}) \cdot (\mathbf{t} - \mathbf{r}) \right| \\ &\leq \|\tilde{\varphi}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} \|\mathbf{t} - \mathbf{r}\|_{0,\Omega}. \end{aligned} \quad (3.47)$$

In this way, replacing (3.46) and (3.47) back into (3.45), and using that  $\tilde{\varphi} = \tilde{S}_1(\varphi, \mathbf{w})$ , we conclude that

$$\|(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})\| \leq \frac{1}{\tilde{\alpha}} \left\{ L_\gamma \|\mathbf{k}\| \|\phi - \varphi\|_{0,\Omega} + \|\tilde{S}_1(\varphi, \mathbf{w})\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} \right\}. \quad (3.48)$$

On the other hand, employing the inf-sup condition of  $\mathcal{B}$  (cf. (3.34)) and the first equation of (3.44), we have at first instance

$$\begin{aligned} \tilde{\beta} \|\mathbf{p} - \mathbf{m}\|_{\text{div}_{4/3}; \Omega} &\leq \sup_{\substack{(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \\ (\psi, \mathbf{s}) \neq \mathbf{0}}} \frac{[\mathcal{B}(\psi, \mathbf{s}), \mathbf{p} - \mathbf{m}]}{\|(\psi, \mathbf{s})\|} \\ &= \sup_{\substack{(\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega) \\ (\psi, \mathbf{s}) \neq \mathbf{0}}} \frac{[\mathcal{F}_\phi - \mathcal{F}_\varphi, (\psi, \mathbf{s})] - [\mathcal{A}_\mathbf{u}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_\mathbf{w}(\tilde{\varphi}, \mathbf{r}), (\psi, \mathbf{s})]}{\|(\psi, \mathbf{s})\|}. \end{aligned} \quad (3.49)$$

Next, adding and subtracting  $\mathcal{A}_\mathbf{u}(\tilde{\varphi}, \mathbf{r})$ , applying the Lipschitz-continuity of  $\mathcal{A}_\mathbf{u}$  with corresponding constant  $\tilde{L} = \vartheta_2 + 2\tilde{\alpha}$  (cf. proof of Lemma 3.6), and using the estimate (3.47) with  $(\psi, \mathbf{s})$  instead of  $(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})$ , we obtain

$$\begin{aligned} |[\mathcal{A}_\mathbf{u}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_\mathbf{w}(\tilde{\varphi}, \mathbf{r}), (\psi, \mathbf{s})]| &\leq |[\mathcal{A}_\mathbf{u}(\tilde{\phi}, \mathbf{t}) - \mathcal{A}_\mathbf{u}(\tilde{\varphi}, \mathbf{r}), (\psi, \mathbf{s})]| + |[\mathcal{A}_\mathbf{u}(\tilde{\varphi}, \mathbf{r}) - \mathcal{A}_\mathbf{w}(\tilde{\varphi}, \mathbf{r}), (\psi, \mathbf{s})]| \\ &\leq \left\{ \tilde{L} \|(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})\| + \|\tilde{\varphi}\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} \right\} \|(\psi, \mathbf{s})\|. \end{aligned} \quad (3.50)$$

In turn, using now (3.46) with  $(\psi, \mathbf{s})$  instead of  $(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})$  as well, we get

$$|[\mathcal{F}_\phi - \mathcal{F}_\varphi, (\psi, \mathbf{s})]| \leq L_\gamma \|\mathbf{k}\| \|\phi - \varphi\|_{0,\Omega} \|\mathbf{s}\|_{0,\Omega}. \quad (3.51)$$

Finally, employing the estimates (3.50) and (3.51) in (3.49), we deduce that

$$\begin{aligned} \tilde{\beta} \|\mathbf{p} - \mathbf{m}\|_{\text{div}_{4/3}; \Omega} &\leq L_\gamma \|\mathbf{k}\| \|\phi - \varphi\|_{0,\Omega} + \|\tilde{\mathbf{S}}_1(\varphi, \mathbf{w})\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{w}\|_{0,4;\Omega} + \tilde{L} \|(\tilde{\phi}, \mathbf{t}) - (\tilde{\varphi}, \mathbf{r})\|, \end{aligned} \quad (3.52)$$

which, together with (3.48) and the fact that  $|\Omega|^{1/4}$  is the boundedness constant of the injection of  $\mathbf{L}^4(\Omega)$  into  $\mathbf{L}^2(\Omega)$ , yield (3.43) and end the proof.  $\square$

As a consequence of Lemmas 3.8 and 3.9, we provide next the Lipschitz continuity of  $\mathbf{T}$ .

**Lemma 3.10** *Assume that the data satisfy (3.36), and let  $C_{\mathbf{T}} := L_{\tilde{\mathbf{S}}} \max\{1, L_{\mathbf{S}}\}$ , where  $L_{\mathbf{S}}$  and  $L_{\tilde{\mathbf{S}}}$  are the constants provided by Lemmas 3.8 and 3.9, respectively. Then, for all  $\phi, \varphi \in \mathbf{L}^4(\Omega)$  there holds*

$$\begin{aligned} \|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{0,4;\Omega} &\leq C_{\mathbf{T}} \left\{ |\Omega|^{1/4} \|\mathbf{k}\| + \|\mathbf{T}(\varphi)\|_{0,4;\Omega} \left( \|f\|_{0,\Omega} + \|\mathbf{S}_1(\varphi)\|_{\epsilon,\Omega} \right) \right\} \|\phi - \varphi\|_{0,4;\Omega}. \end{aligned} \quad (3.53)$$

**Proof** We begin by observing (as we did in the proof of Lemma 3.7) that, given  $\phi, \varphi \in \mathbf{L}^4(\Omega)$ , the hypothesis (3.36) on the data guarantees that the pairs  $(\phi, \mathbf{S}_2(\phi))$  and  $(\varphi, \mathbf{S}_2(\varphi))$  satisfy the hypothesis of Lemma 3.6, whence  $\mathbf{T}(\phi) := \tilde{\mathbf{S}}_1(\phi, \mathbf{S}_2(\phi))$  and  $\mathbf{T}(\varphi) := \tilde{\mathbf{S}}_1(\varphi, \mathbf{S}_2(\varphi))$  are indeed well defined. Having said this, (3.53) follows

from straightforward applications of the Lipschitz-continuity estimates provided by Lemmas 3.8 and 3.9.  $\square$

We are now in position to establish the main result concerning the solvability of (3.17).

**Theorem 3.11** *Given  $r > 0$ , we let  $W := \left\{ \phi \in L^4(\Omega) : \|\phi\|_{0,4;\Omega} \leq r \right\}$ , and assume that the data satisfy (3.36), (3.37), and the further condition*

$$L_T := C_T \left\{ |\Omega|^{1/4} \|\mathbf{k}\| + r \left( (1 + r \tilde{C}_S(r)) \|f\|_{0,\Omega} + \tilde{C}_S(r) \|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma} \right) \right\} < 1, \quad (3.54)$$

where  $\tilde{C}_S(r)$  is given by (3.38). Then, the fully-mixed formulation (3.17) has a unique solution  $(\sigma, \mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  and  $((\phi, \mathbf{t}), \mathbf{p}) \in (L^4(\Omega) \times L^2(\Omega)) \times \mathbf{H}(\mathbf{div}_{4/3}; \Omega)$  with  $\phi \in W$ , and there holds

$$\|(\phi, \mathbf{t}, \mathbf{p})\| \leq C_S \left\{ \gamma_2 |\Omega|^{1/2} \|\mathbf{k}\| + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \right\} \quad (3.55)$$

and

$$\|(\sigma, \mathbf{u})\| \leq C_S \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|f\|_{0,\Omega} \|\phi\|_{0,4;\Omega} \right\}. \quad (3.56)$$

**Proof** The proof follows similar ideas to those in [1, Theorem 3.13]. Indeed, we first recall from Lemma 3.7 that the assumptions (3.36) and (3.37) guarantee that  $T(W) \subseteq W$ , and hence, for each  $\varphi \in W$  we have that both  $\|\varphi\|_{0,4;\Omega}$  and  $\|\mathbf{T}(\varphi)\|_{0,4;\Omega}$  are bounded by  $r$ . Thus, it follows from (3.38) that  $\|\mathbf{S}_1(\varphi)\|_{\epsilon,\Omega} \leq \tilde{C}_S(r)$ .  $\|\mathbf{S}_1(\varphi)\|_{\epsilon,\Omega} \leq \tilde{C}_S(r) \left\{ \|\mathbf{u}_D\|_{1/2+\epsilon,\Gamma} + r \|f\|_{0,\Omega} \right\}$ , so that replacing these estimates into (3.53), we arrive at

$$\|\mathbf{T}(\phi) - \mathbf{T}(\varphi)\|_{0,4;\Omega} \leq L_T \|\phi - \varphi\|_{0,4;\Omega} \quad \forall \phi, \varphi \in W.$$

Therefore, according to the equivalence between (3.17) and (3.20), and noting from (3.54) that  $\mathbf{T}$  becomes a contraction, the existence of a unique solution of (3.17) is obtained from a straightforward application of the Banach fixed-point theorem (see, e.g., [19, Theorem 3.7-1]). Moreover, the estimates (3.55) and (3.56) follow from (3.35) and (3.26), respectively, which complete the proof.  $\square$

## 4 The discrete formulation

In this section we introduce the Galerkin scheme associated with (3.17) and study its solvability and convergence.

#### 4.1 A fully-mixed finite element method

We first let  $\{\mathcal{T}_h\}_{h>0}$  be a regular family of triangulations of  $\Omega$  by triangles  $K$  (respectively tetrahedra  $K$  in  $\mathbb{R}^3$ ), and set  $h := \max \{h_K : K \in \mathcal{T}_h\}$ . In turn, given an integer  $l \geq 0$  and a subset  $S$  of  $\mathbb{R}^n$ , we denote by  $\mathbb{P}_l(S)$  the space of polynomials of total degree at most  $l$  defined on  $S$ . Hence, for each integer  $k \geq 0$  and for each  $K \in \mathcal{T}_h$ , we define the local Raviart–Thomas space of order  $k$  as  $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \tilde{\mathbf{P}}_k(K) \mathbf{x}$ , where, according to the convention in Sect. 1,  $\mathbf{P}_k(K) := [\mathbf{P}_k(K)]^n$ ,  $\tilde{\mathbf{P}}_k(K)$  is the space of polynomials of total degree equal to  $k$  defined on  $T$ , and  $\mathbf{x} := (x_1, \dots, x_n)^\top$  is a generic vector of  $\mathbb{R}^n$ . In this way, introducing the finite element subspaces:

$$\begin{aligned} \mathbb{H}_h^\sigma &:= \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\operatorname{div}_{4/3}; \Omega) : \quad \mathbf{c}^\top \boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall \mathbf{c} \in \mathbb{R}^n, \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{H}_h^u &:= \left\{ \mathbf{v}_h \in \mathbf{L}^4(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{H}_h^\phi &:= \left\{ \psi_h \in \mathbf{L}^4(\Omega) : \quad \psi_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{H}_h^t &:= \left\{ \mathbf{s}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{s}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \\ \mathbf{H}_h^p &:= \left\{ \mathbf{q}_h \in \mathbf{H}(\operatorname{div}_{4/3}; \Omega) : \quad \mathbf{q}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h \right\}, \end{aligned} \quad (4.1)$$

the Galerkin scheme for (3.17) reads: Find  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^\sigma \times \mathbf{H}_h^u$  and  $((\phi_h, \mathbf{t}_h), \mathbf{p}_h) \in (\mathbf{H}_h^\phi \times \mathbf{H}_h^t) \times \mathbf{H}_h^p$ , such that

$$\begin{aligned} \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \mathbb{H}_h^\sigma, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h^u, \\ [\mathcal{A}_{\mathbf{u}_h}(\phi_h, \mathbf{t}_h), (\psi_h, \mathbf{s}_h)] + [\mathcal{B}(\psi_h, \mathbf{s}_h), \mathbf{p}_h] &= [\mathcal{F}_{\phi_h}, (\psi_h, \mathbf{s}_h)] \quad \forall (\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^t, \\ [\mathcal{B}(\phi_h, \mathbf{t}_h), \mathbf{q}_h] &= [\mathcal{G}, \mathbf{q}_h] \quad \forall \mathbf{q}_h \in \mathbf{H}_h^p, \end{aligned} \quad (4.2)$$

where  $\mathbf{a}_{\phi_h}$ ,  $\mathbf{b}$ ,  $\mathbf{F}$ ,  $\mathbf{G}_{\phi_h}$ ,  $\mathcal{A}_{\mathbf{u}_h}$ ,  $\mathcal{B}$ ,  $\mathcal{F}_{\phi_h}$ , and  $\mathcal{G}$  are defined in (3.5), (3.6), (3.11), and (3.12), respectively, with  $\phi = \phi_h$  and  $\mathbf{u} = \mathbf{u}_h$ .

#### 4.2 A discrete fixed-point strategy

In what follows we reformulate (4.2) by adopting the discrete analogue of the fixed point strategy developed in Sect. 3.2. Hence, we now let  $\mathbf{S}_h : \mathbf{H}_h^\phi \rightarrow \mathbb{H}_h^\sigma \times \mathbf{H}_h^u$  be the operator defined by

$$\mathbf{S}_h(\phi_h) = (\mathbf{S}_{1,h}(\phi_h), \mathbf{S}_{2,h}(\phi_h)) := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \quad \forall \phi_h \in \mathbf{H}_h^\phi,$$

where  $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^\sigma \times \mathbf{H}_h^u$  is the unique solution of the first two equations of (4.2) with the given  $\phi_h \in \mathbf{H}_h^\phi$ , that is

$$\begin{aligned} \mathbf{a}_{\phi_h}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \mathbb{H}_h^\sigma, \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h^u, \end{aligned} \quad (4.3)$$

In turn, we let  $\tilde{\mathbf{S}}_h : \mathbf{H}_h^\phi \times \mathbf{H}_h^u \rightarrow \mathbf{H}_h^\phi \times \mathbf{H}_h^t \times \mathbf{H}_h^p$  be the operator defined by

$$\tilde{\mathbf{S}}_h(\phi_h, \mathbf{u}_h) = (\tilde{\mathbf{S}}_{1,h}(\phi_h, \mathbf{u}_h), \tilde{\mathbf{S}}_{2,h}(\phi_h, \mathbf{u}_h), \tilde{\mathbf{S}}_{3,h}(\phi_h, \mathbf{u}_h)) := (\tilde{\phi}_h, \mathbf{t}_h, \mathbf{p}_h) \quad \forall (\phi_h, \mathbf{u}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^u,$$

where  $(\tilde{\phi}_h, \mathbf{t}_h, \mathbf{p}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^t \times \mathbf{H}_h^p$  is the unique solution of the last two equations of (4.2) with the given  $(\phi_h, \mathbf{u}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^u$ , that is:

$$\begin{aligned} [\mathcal{A}_{\mathbf{u}_h}(\tilde{\phi}_h, \mathbf{t}_h), (\psi_h, \mathbf{s}_h)] + [\mathcal{B}(\psi_h, \mathbf{s}_h), \mathbf{p}_h] &= [\mathcal{F}_{\phi_h}, (\psi_h, \mathbf{s}_h)] \quad \forall (\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^t, \\ [\mathcal{B}(\tilde{\phi}_h, \mathbf{t}_h), \mathbf{q}_h] &= [\mathcal{G}, \mathbf{q}_h] \quad \forall \mathbf{q}_h \in \mathbf{H}_h^p. \end{aligned} \quad (4.4)$$

Finally, we define the operator  $\mathbf{T}_h : \mathbf{H}_h^\phi \rightarrow \mathbf{H}_h^\phi$  by

$$\mathbf{T}_h(\phi_h) := \tilde{\mathbf{S}}_{1,h}(\phi_h, \mathbf{S}_{2,h}(\phi_h)) \quad \forall \phi_h \in \mathbf{H}_h^\phi, \quad (4.5)$$

and realize that (4.2) is equivalent to seeking a fixed point of  $\mathbf{T}_h$ : Find  $\phi_h \in \mathbf{H}_h^\phi$  such that

$$\mathbf{T}_h(\phi_h) = \phi_h. \quad (4.6)$$

Certainly, all the above makes sense if we guarantee that the discrete problems (4.3) and (4.4) are well-posed, which is addressed in the first part of the following section.

### 4.3 Solvability analysis of the discrete fixed-point equation

We begin by showing that the discrete operator  $\mathbf{S}_h$  is well-defined. To this end, we now let  $\mathbb{V}_h$  be the discrete kernel of  $\mathbf{b}$ , that is

$$\mathbb{V}_h := \left\{ \boldsymbol{\tau}_h \in \mathbf{H}_h^\sigma : \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{H}_h^u \right\},$$

which, using from (4.1) that  $\mathbf{div}(\mathbb{H}_h^\sigma) \subseteq \mathbf{H}_h^u$ , becomes

$$\mathbb{V}_h := \left\{ \boldsymbol{\tau}_h \in \mathbf{H}_h^\sigma : \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in } \Omega \right\}.$$

It follows that  $\mathbb{V}_h \subseteq \mathbb{V}$  (cf. (3.22)), and hence, similarly to (3.23) we deduce the ellipticity of  $\mathbf{a}_{\phi_h}$  on  $\mathbb{V}_h$  with the same constant  $\alpha = c_1/\mu_2$ , that is,

$$\mathbf{a}_{\phi_h}(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3}, \Omega}^2 \quad \forall \boldsymbol{\tau}_h \in \mathbb{V}_h, \quad (4.7)$$

which certainly implies that the bilinear form  $\mathbf{a}_{\phi_h}$  satisfies the corresponding hypothesis required by the discrete Babuška–Brezzi theory in Banach spaces (cf. [24, Proposition 2.42]). Besides the already proved boundedness of the linear functionals involved (cf. (3.8)), the requirements of this abstract result are completed with the discrete inf-sup condition for the bilinear form  $\mathbf{b}$ , whose proof can be found in [20, Lemma 5.5, Section 5.4].

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^\sigma \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\text{div}_{4/3}; \Omega}} \geq \beta_d \|\mathbf{v}_h\|_{0,4;\Omega} \quad \forall \mathbf{v}_h \in \mathbf{H}_h^u. \quad (4.8)$$

We are now in position to establish next the discrete analogue of Lemma 3.1.

**Lemma 4.1** *For each  $\phi_h \in H_h^\phi$  there exists a unique  $\mathbf{S}_h(\phi_h) := (\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^\sigma \times \mathbf{H}_h^u$  solution to the problem (4.3). Moreover, there exists a positive constant  $C_{\mathbf{S},d}$ , depending only on  $\mu_1$ ,  $\alpha$ ,  $\beta_d$  and  $\widehat{c}(\Omega)$  (cf. (3.8)), and hence independent of  $\phi_h$ , such that*

$$\|\mathbf{S}_h(\phi_h)\| = \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq C_{\mathbf{S},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|f\|_{0,\Omega} \|\phi_h\|_{0,4;\Omega} \right\} \quad \forall \phi_h \in H_h^\phi. \quad (4.9)$$

**Proof** It follows from (4.7), (4.8), and a direct application of the discrete Babuška–Brezzi theory in Banach spaces (cf. [24, Proposition 2.42]). In particular, the corresponding a priori estimate reduces to

$$\|\mathbf{S}_h(\phi_h)\| = \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq \widehat{C} \left\{ \|\mathbf{F}|_{\mathbb{H}_h^\sigma}\| + \|\mathbf{G}_{\phi_h}|_{\mathbf{H}_h^u}\| \right\}, \quad (4.10)$$

with a positive constant  $\widehat{C}$  depending only on  $\mu_1$ ,  $\alpha$ , and  $\beta_d$ . Then, using (3.8) to bound the discrete norms  $\|\mathbf{F}|_{\mathbb{H}_h^\sigma}\| := \sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^\sigma \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{|\mathbf{F}(\boldsymbol{\tau}_h)|}{\|\boldsymbol{\tau}_h\|_{\text{div}_{4/3}; \Omega}}$  and

$$\|\mathbf{G}_{\phi_h}|_{\mathbf{H}_h^u}\| := \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h^u \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{|\mathbf{G}_{\phi_h}(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{0,4;\Omega}},$$

and replacing the resulting estimates back into (4.10), we get (4.9) with  $C_{\mathbf{S},d} := \widehat{C} \max\{1, \widehat{c}(\Omega)\}$ .  $\square$

Next, we aim to prove that (4.4) is well posed, or equivalently that  $\widetilde{\mathbf{S}}_h$  is well defined. Indeed, we remark in advance that the respective proof, being the discrete analogue of the one of Lemma 3.6, makes use again of the abstract result given by Theorem 3.2. Hence, we first set the null space of the operator  $\mathcal{B}$  (cf. (3.11)) restricted to  $H_h^\phi \times \mathbf{H}_h^t$ , which is given by

$$\widetilde{\mathbf{V}}_h = \left\{ (\boldsymbol{\psi}_h, \mathbf{r}_h) \in H_h^\phi \times \mathbf{H}_h^t : [\mathcal{B}(\boldsymbol{\psi}_h, \mathbf{r}_h), \mathbf{q}] := \int_{\Omega} \mathbf{r}_h \cdot \mathbf{q}_h + \int_{\Omega} \boldsymbol{\psi}_h \text{div}(\mathbf{q}_h) = 0 \quad \forall \mathbf{q}_h \in \mathbf{H}_h^p \right\}. \quad (4.11)$$

Then, following the approach from [20, Section 5], we now prove the discrete inf-sup condition for  $\mathcal{B}$  and an intermediate result that will be used to show later on the strong monotonicity of  $\mathcal{A}_{\mathbf{u}_h}$  on  $\widetilde{\mathbf{V}}_h$  (cf. (4.11)).

**Lemma 4.2** *There exist positive constants  $\widetilde{\beta}_d$  and  $\widetilde{C}_\phi$  independent of  $h$ , such that*

$$\sup_{\substack{(\psi_h, \mathbf{r}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^t \\ (\psi_h, \mathbf{r}_h) \neq \mathbf{0}}} \frac{[\mathcal{B}(\psi_h, \mathbf{r}_h), \mathbf{q}_h]}{\|(\psi_h, \mathbf{r}_h)\|} \geq \tilde{\beta}_d \|\mathbf{q}_h\|_{\text{div}_{4/3}; \Omega} \quad \forall \mathbf{q}_h \in \mathbf{H}_h^p, \quad (4.12)$$

and

$$\|\mathbf{r}_h\|_{0, \Omega} \geq \tilde{C}_d \|\psi_h\|_{0,4; \Omega} \quad \forall (\psi_h, \mathbf{r}_h) \in \tilde{\mathbf{V}}_h. \quad (4.13)$$

**Proof** We begin with the introduction of the discrete space  $Z_{0,h}$  defined by

$$Z_{0,h} := \left\{ \mathbf{q}_h \in \mathbf{H}_h^p : [\mathcal{B}(\psi_h, \mathbf{0}), \mathbf{q}_h] = \int_{\Omega} \psi_h \operatorname{div}(\mathbf{q}_h) = 0 \quad \forall \psi_h \in \mathbf{H}_h^\phi \right\},$$

which, using from (4.1) that  $\operatorname{div}(\mathbf{H}_h^p) \subseteq \mathbf{H}_h^\phi$ , becomes

$$Z_{0,h} = \left\{ \mathbf{q}_h \in \mathbf{H}_h^p : \operatorname{div}(\mathbf{q}_h) = 0 \quad \text{in } \Omega \right\}.$$

Next, by using the abstract equivalence result provided by [20, Lemma 5.1] with the setting  $X = \mathbf{H}_h^\phi$ ,  $Y = Y_1 = \mathbf{H}_h^t$ ,  $Y_2 = \{0\}$ ,  $V = \tilde{\mathbf{V}}_h$ ,  $Z = \mathbf{H}_h^p$ , and  $Z_0 = Z_{0,h}$ , where  $X$ ,  $Y$ ,  $Y_1$ ,  $Y_2$ ,  $V$ ,  $Z$ , and  $Z_0$  corresponds to the notation employed there, we deduce that (4.12) and (4.13) are jointly equivalent to the existence of positive constants  $\tilde{\beta}_1$  and  $\tilde{\beta}_2$ , independent of  $h$ , such that there hold

$$\sup_{\substack{\mathbf{q}_h \in \mathbf{H}_h^p \\ \mathbf{q}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\psi_h, \mathbf{0}), \mathbf{q}_h]}{\|\mathbf{q}_h\|_{\text{div}_{4/3}; \Omega}} = \sup_{\substack{\mathbf{q}_h \in \mathbf{H}_h^p \\ \mathbf{q}_h \neq \mathbf{0}}} \frac{\int_{\Omega} \psi_h \operatorname{div}(\mathbf{q}_h)}{\|\mathbf{q}_h\|_{\text{div}_{4/3}; \Omega}} \geq \tilde{\beta}_1 \|\psi_h\|_{0,4; \Omega} \quad \forall \psi_h \in \mathbf{H}_h^\phi \quad (4.14)$$

and

$$\sup_{\substack{\mathbf{r}_h \in \mathbf{H}_h^t \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\mathbf{0}, \mathbf{r}_h), \mathbf{q}_h]}{\|\mathbf{r}_h\|_{0, \Omega}} = \sup_{\substack{\mathbf{r}_h \in \mathbf{H}_h^t \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{\int_{\Omega} \mathbf{r}_h \cdot \mathbf{q}_h}{\|\mathbf{r}_h\|_{0, \Omega}} \geq \tilde{\beta}_2 \|\mathbf{q}_h\|_{\text{div}_{4/3}; \Omega} \quad \forall \mathbf{q}_h \in Z_{0,h}. \quad (4.15)$$

Therefore, since (4.14) and (4.15) have already been proved in [20, Section 5.5], this proof is concluded.  $\square$

We now establish the discrete strong monotonicity and Lipschitz-continuity properties of  $\mathcal{A}_{\mathbf{u}_h}$ .

**Lemma 4.3** *There exists a constant  $\tilde{\alpha}_d > 0$ , depending only on  $\vartheta_1$  (cf. (2.3)) and  $\tilde{C}_d$  (cf. (4.13)), such that for each  $\mathbf{u}_h \in \mathbf{H}_h^u$  satisfying  $\|\mathbf{u}_h\|_{0,4; \Omega} \leq 2\tilde{\alpha}_d$ , the family of*

operators  $\mathcal{A}_{\mathbf{u}_h}(\cdot + (\psi_h, \mathbf{s}_h)) : \widetilde{\mathbb{V}}_h \rightarrow \widetilde{\mathbb{V}}'_h$  with  $(\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}$  is uniformly strongly monotone on  $\widetilde{\mathbb{V}}_h$  with constant  $\widetilde{\alpha}_d$ , that is

$$[\mathcal{A}_{\mathbf{u}_h}((\phi_h, \mathbf{t}_h) + (\psi_h, \mathbf{s}_h)) - \mathcal{A}_{\mathbf{u}_h}((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) + (\psi_h, \mathbf{s}_h)), (\phi_h, \mathbf{t}_h) - (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)] \geq \widetilde{\alpha}_d \|(\phi_h, \mathbf{t}_h) - (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)\|^2 \quad (4.16)$$

for all  $(\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}$  and for all  $(\phi_h, \mathbf{t}_h), (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) \in \widetilde{\mathbb{V}}_h$ . In addition, the operator  $\mathcal{A}_{\mathbf{u}_h} : \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}} \rightarrow (\mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}})'$  is Lipschitz-continuous with constant  $\widetilde{L}_d := \vartheta_2 + 2\widetilde{\alpha}_d$ .

**Proof** Given  $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ , we let  $(\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}$ , and  $(\phi_h, \mathbf{t}_h), (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) \in \widetilde{\mathbb{V}}_h$ . Then, proceeding similarly to the derivation of (3.32) (cf. proof of Lemma 3.4), we arrive at

$$\begin{aligned} & [\mathcal{A}_{\mathbf{u}_h}((\phi_h, \mathbf{t}_h) + (\psi_h, \mathbf{s}_h)) - \mathcal{A}_{\mathbf{u}_h}((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) + (\psi_h, \mathbf{s}_h)), (\phi_h, \mathbf{t}_h) - (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)] \\ & \geq \frac{\vartheta_1}{2} \|\mathbf{t}_h - \widetilde{\mathbf{t}}_h\|_{0,\Omega}^2 + \frac{\vartheta_1}{2} \|\mathbf{t}_h - \widetilde{\mathbf{t}}_h\|_{0,\Omega}^2 - \|\mathbf{u}_h\|_{0,4;\Omega} \|\phi_h - \widetilde{\phi}_h\|_{0,4;\Omega} \|\mathbf{t}_h - \widetilde{\mathbf{t}}_h\|_{0,\Omega}. \end{aligned} \quad (4.17)$$

Thus, applying now (4.13) to  $(\phi_h - \widetilde{\phi}_h, \mathbf{t}_h - \widetilde{\mathbf{t}}_h) \in \widetilde{\mathbb{V}}_h$ , which yields  $\|\mathbf{t}_h - \widetilde{\mathbf{t}}_h\|_{0,\Omega}^2 \geq \widetilde{C}_d^2 \|\phi_h - \widetilde{\phi}_h\|_{0,4;\Omega}^2$ , and defining

$$\widetilde{\alpha}_d := \frac{\vartheta_1}{4} \min \{1, \widetilde{C}_d^2\}, \quad (4.18)$$

we deduce from (4.17) that

$$\begin{aligned} & [\mathcal{A}_{\mathbf{u}_h}((\phi_h, \mathbf{t}_h) + (\psi_h, \mathbf{s}_h)) - \mathcal{A}_{\mathbf{u}_h}((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) + (\psi_h, \mathbf{s}_h)), (\phi_h, \mathbf{t}_h) - (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)] \\ & \geq \frac{1}{2} \left\{ \vartheta_1 - \|\mathbf{u}_h\|_{0,4;\Omega} \right\} \|\mathbf{t}_h - \widetilde{\mathbf{t}}_h\|_{0,\Omega}^2 + \frac{1}{2} \left\{ \vartheta_1 \widetilde{C}_d^2 - \|\mathbf{u}_h\|_{0,4;\Omega} \right\} \|\phi_h - \widetilde{\phi}_h\|_{0,4;\Omega}^2 \\ & \geq \frac{1}{2} \left\{ 4\widetilde{\alpha}_d - \|\mathbf{u}_h\|_{0,4;\Omega} \right\} \|(\phi_h, \mathbf{t}_h) - (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)\|^2, \end{aligned} \quad (4.19)$$

from which, assuming that  $\|\mathbf{u}_h\|_{0,4;\Omega} \leq 2\widetilde{\alpha}_d$ , it follows (4.16). Furthermore, we now observe that for all  $(\phi_h, \mathbf{t}_h), (\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}$  there holds

$$\|\mathcal{A}_{\mathbf{u}_h}(\phi_h, \mathbf{t}_h) - \mathcal{A}_{\mathbf{u}_h}(\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)\|_{(\mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}})'} \leq \|\mathcal{A}_{\mathbf{u}_h}(\phi_h, \mathbf{t}_h) - \mathcal{A}_{\mathbf{u}_h}(\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h)\|,$$

where the latter is the norm of  $(\mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega))'$ , and hence, (3.29) (cf. Lemma 3.3) and the specified bound on  $\|\mathbf{u}_h\|_{0,4;\Omega}$ , imply the required Lipschitz-continuity property of  $\mathcal{A}_{\mathbf{u}_h}$ .  $\square$

We are now in position of establishing the discrete analogue of Lemma 3.6.

**Lemma 4.4** *Let  $\widetilde{\alpha}_d$  be the constant given by (4.18). Then, for each  $(\phi_h, \mathbf{u}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{u}}$  such that  $\|\mathbf{u}_h\|_{0,4;\Omega} \leq 2\widetilde{\alpha}_d$ , problem (4.4) has a unique solution  $((\widetilde{\phi}_h, \widetilde{\mathbf{t}}_h), \mathbf{p}_h) \in (\mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}) \times \mathbf{H}_h^{\mathbf{p}}$ . Moreover, there exists a positive constant  $C_{\widetilde{\mathbf{S}},d}$ , depending only on  $\widetilde{\beta}_d, \widetilde{\alpha}_d$ , and  $\widetilde{L}_d$ , such that*

$$\|\tilde{\mathbf{S}}_h(\phi_h, \mathbf{u}_h)\| = \|(\tilde{\phi}_h, \mathbf{t}_h, \mathbf{p}_h)\| \leq C_{\tilde{\mathbf{S}},d} \left\{ \gamma_2 |\Omega|^{1/2} \|\mathbf{k}\| + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \right\}. \quad (4.20)$$

**Proof** According to Lemma 4.3 and the discrete inf-sup condition for  $\mathcal{B}$  provided by (4.12) (cf. Lemma 4.2), the proof follows from a direct application of Theorem 3.2 to the discrete setting represented by (4.4). In particular, the a priori bound (4.20) is consequence of the abstract estimate (3.28) applied to (4.4), and the bounds for  $\mathcal{F}_{\phi_h}$  and  $\mathcal{G}$  given by (3.14).  $\square$

In order to address the solvability of the fixed point equation (4.6), which is equivalent to our discrete system (4.2), we begin by recalling next the Brouwer Theorem (cf. [19, Theorem 9.9-2]).

**Theorem 4.5** *Let  $W$  be a compact and convex subset of a finite dimensional Banach space  $X$ , and let  $T : W \rightarrow W$  be a continuous mapping. Then  $T$  has at least one fixed point.*

Thus, in what follows we proceed to show that  $\mathbf{T}_h$  satisfies the hypotheses of Theorem 4.5. Firstly, we establish the discrete version of Lemma 3.7.

**Lemma 4.6** *Given  $r > 0$ , we let  $W_h := \left\{ \phi_h \in H_h^\phi : \|\phi_h\|_{0,4;\Omega} \leq r \right\}$ , and assume that the data satisfy*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + r \|f\|_{0,\Omega} \leq \frac{2\tilde{\alpha}_d}{C_{\tilde{\mathbf{S}},d}} \quad (4.21)$$

and

$$\gamma_2 |\Omega|^{1/2} + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \leq \frac{r}{C_{\tilde{\mathbf{S}},d}}. \quad (4.22)$$

Then  $\mathbf{T}_h(W_h) \subseteq W_h$ .

**Proof** Similarly to the proof of Lemma 3.7, it is a direct consequence of Lemmas 4.1 and 4.4.  $\square$

The discrete analogue of Lemma 3.8 is provided now. We stress here that, instead of the regularity assumption employed in the proof of that result, which actually is not needed nor could be applied in the present discrete case, we simply utilize a  $L^4 - L^4 - L^2$  argument.

**Lemma 4.7** *There exists a positive constant  $L_{\mathbf{S},\mathfrak{d}}$  depending on  $\mu_1$ ,  $L_\mu$ ,  $\alpha$ , and  $\beta_{\mathfrak{d}}$  such that*

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\psi_h)\| \leq L_{\mathbf{S},d} \left\{ \|f\|_{0,\Omega} + \|\mathbf{S}_{1,h}(\psi_h)\|_{0,4;\Omega} \right\} \|\phi_h - \psi_h\|_{0,4;\Omega} \quad \forall \phi_h, \psi_h \in H_h^\phi. \quad (4.23)$$

**Proof** It follows from [7, Lemma 4.6] with the same constant  $L_{S,d}$ . We omit further details.  $\square$

The discrete analogue of Lemma 3.9, which establishes the continuity of  $\tilde{S}_h$ , reads as follows.

**Lemma 4.8** *There exists a positive constant  $L_{\tilde{S},d}$  depending on  $L_\gamma, \tilde{L}_d$  (cf. Lemma 4.3),  $\tilde{\beta}_d$  and  $\tilde{\alpha}_d$ , such that for all  $(\phi_h, \mathbf{u}_h), (\varphi_h, \mathbf{w}_h) \in H_h^\phi \times \mathbf{H}_h^u$ , with  $\|\mathbf{u}_h\|_{0,4;\Omega}, \|\mathbf{w}_h\|_{0,4;\Omega} \leq 2\tilde{\alpha}_d$ , there holds*

$$\begin{aligned} & \|\tilde{S}_h(\phi_h, \mathbf{u}_h) - \tilde{S}_h(\varphi_h, \mathbf{w}_h)\| \\ & \leq L_{\tilde{S},d} \left\{ |\Omega|^{1/4} \|\mathbf{k}\| \|\phi_h - \varphi_h\|_{0,4;\Omega} + \|\tilde{S}_{1,h}(\varphi_h, \mathbf{w}_h)\|_{0,4;\Omega} \|\mathbf{u}_h - \mathbf{w}_h\|_{0,4;\Omega} \right\}. \end{aligned} \quad (4.24)$$

**Proof** It proceeds analogously to the proof of Lemma 3.9 by using now the strong monotonicity and Lipschitz-continuity of  $\mathcal{A}_{\mathbf{u}_h}$  (cf. Lemma 4.3), the Lipschitz-continuity of  $\gamma$  (cf. (2.5)), and the discrete inf-sup condition for  $\mathcal{B}$  (cf. (4.12) in Lemma 4.2). We omit further details.  $\square$

The continuity of the discrete fixed-point operator  $\mathbf{T}_h$  is proved next.

**Lemma 4.9** *Assume that the data satisfy (4.21), and let  $C_{T,d} := L_{\tilde{S},d} \max\{1, L_{S,d}\}$ , where  $L_{S,d}$  and  $L_{\tilde{S},d}$  are the constants provided by Lemmas 4.7 and 4.8, respectively. Then, for all  $\phi_h, \varphi_h \in H_h^\phi$  there holds*

$$\begin{aligned} & \|\mathbf{T}_h(\phi_h) - \mathbf{T}_h(\varphi_h)\|_{0,4;\Omega} \\ & \leq C_{T,d} \left\{ |\Omega|^{1/4} \|\mathbf{k}\| + \|\mathbf{T}_h(\varphi_h)\|_{0,4;\Omega} \left( \|\mathbf{f}\|_{0,\Omega} + \|\mathbf{S}_{1,h}(\varphi_h)\|_{0,4;\Omega} \right) \right\} \|\phi_h - \varphi_h\|_{0,4;\Omega}. \end{aligned} \quad (4.25)$$

**Proof** Analogously to the proof of Lemma 3.10, we first observe that, given  $\phi_h, \varphi_h \in H_h^\phi$ , the assumption (4.21) guarantees that  $(\phi_h, \mathbf{S}_{2,h}(\phi_h))$  and  $(\varphi_h, \mathbf{S}_{2,h}(\varphi_h))$  verify the hypothesis of Lemma 4.4. In this way,  $\mathbf{T}_h(\phi_h) := \mathbf{S}_{1,h}(\phi_h, \mathbf{S}_{2,h}(\phi_h))$  and  $\mathbf{T}_h(\varphi_h) := \mathbf{S}_{1,h}(\varphi_h, \mathbf{S}_{2,h}(\varphi_h))$  are well defined, and hence, direct applications of the estimates provided by Lemmas 4.7 and 4.8, lead to (4.25) and conclude the proof.  $\square$

We stress here that, while  $\|\mathbf{T}_h(\varphi_h)\|_{0,4;\Omega}$  can be certainly bounded by  $r$  in (4.25), the lack of a bound independent of  $h$  for the expression  $\|\mathbf{S}_{1,h}(\varphi_h)\|_{0,4;\Omega}$  that also appears there, stops us of deriving a controllable Lipschitz-continuity constant for  $\mathbf{T}_h$ . This is the reason why we are not able to apply the Banach fixed-point theorem to  $\mathbf{T}_h$ , but only the Brouwer one (cf. Theorem 4.5) as we state next.

**Theorem 4.10** *Given  $r > 0$ , we let  $W_h := \{\phi_h \in H_h^\phi : \|\phi_h\|_{1,\Omega} \leq r\}$ , and assume that the data satisfy (4.21) and (4.22). Then the Galerkin scheme (4.2) has at least one solution  $(\sigma_h, \mathbf{u}_h) \in \mathbb{H}_h^\sigma \times \mathbf{H}_h^u$  and  $((\phi_h, \mathbf{t}_h), \mathbf{p}_h) \in (H_h^\phi \times \mathbf{H}_h^t) \times \mathbf{H}_h^p$  with  $\phi_h \in W_{\mathbf{p}}$  and there holds*

$$\|(\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| \leq C_{\mathbb{S},d} \left\{ \gamma_2 |\Omega|^{1/2} \|\mathbf{k}\| + \|g\|_{0,4/3;\Omega} + \tilde{c}(\Omega) \|\phi_D\|_{1/2,\Gamma} \right\} \quad (4.26)$$

and

$$\|(\sigma_h, \mathbf{u}_h)\| \leq C_{\mathbb{S},d} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|f\|_{0,\Omega} \|\phi_h\|_{0,4;\Omega} \right\}. \quad (4.27)$$

**Proof** Thanks to the equivalence between (4.2) and (4.6), the existence of solution follows from Lemmas 4.6 and 4.9, and a direct application of Theorem 4.5. In addition, the a priori estimates (4.26) and (4.27) are consequences of (4.20) and (4.9), respectively.  $\square$

We end this section by stressing that our Galerkin scheme provides exact conservation of momentum when  $f$  is piecewise constant and  $g$  belongs to the concentration discrete space  $H_h^\phi$ . In fact, using again that  $\text{div}(\mathbb{H}_h^\sigma) \subseteq \mathbf{H}_h^\mathbf{u}$  and  $\text{div}(\mathbf{H}_h^\mathbf{p}) \subseteq H_h^\phi$ , and observing in this case that  $f\phi_h \in \mathbf{H}_h^\mathbf{u}$ , we deduce from the second and fourth equations of (4.2), respectively, that

$$\text{div}(\sigma_h) + f\phi_h = \mathbf{0} \quad \text{in } \Omega \quad \text{and} \quad \text{div}(\mathbf{p}_h) + g = 0 \quad \text{in } \Omega. \quad (4.28)$$

Nevertheless, when  $f$  is not piecewise constant or  $g$  does not belong to  $H_h^\phi$ , the corresponding identity of (4.28) can be obtained in an approximate sense only by replacing  $f\phi_h$  or  $g$ , respectively, by  $\tilde{\mathcal{P}}_k(f\phi_h)$  or  $\mathcal{P}_k(g)$ , where  $\mathcal{P}_k$  is the  $L^2(\Omega)$ -orthogonal projection onto discontinuous piecewise polynomials of degree  $k$  and  $\tilde{\mathcal{P}}_k$  is its vectorial version. The verification of the present conservation of momentum is illustrated below in Sect. 6, where we use that for  $k = 0$  there holds  $\mathcal{P}_k(f\phi_h) = \mathcal{P}_k(f)\phi_h$ .

## 5 A priori error analysis

In this section we derive the Céa estimate for our Galerkin scheme (4.2) with the finite element subspaces given by (4.1) (cf. Sect. 4.1), and then use the approximation properties of the latter to establish the corresponding rates of convergence. In what follows,  $(\sigma, \mathbf{u}) \in \mathbb{H}_0(\text{div}_{4/3}; \Omega) \times \mathbf{L}^4(\Omega)$  and  $((\phi, \mathbf{t}), \mathbf{p}) \in (L^4(\Omega) \times L^2(\Omega)) \times \mathbf{H}(\text{div}_{4/3}; \Omega)$ , with  $\phi \in W$ , is the solution of (3.17), whereas  $(\sigma_h, \mathbf{u}_h) \in \mathbb{H}_h^\sigma \times \mathbf{H}_h^\mathbf{u}$  and  $((\phi_h, \mathbf{t}_h), \mathbf{p}_h) \in (H_h^\phi \times \mathbf{H}_h^\mathbf{t}) \times \mathbf{H}_h^\mathbf{p}$ , with  $\phi_h \in W_h$ , is a solution of (4.2). In this way, our goal is to obtain an a priori estimate for the error

$$\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\| + \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\|.$$

We begin our analysis by establishing next an ad-hoc Strang-type estimate. Hereafter, given a subspace  $X_h$  of a generic Banach space  $(X, \|\cdot\|_X)$ , we set as usual  $\text{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X$  for all  $x \in X$ .

**Lemma 5.1** *Let  $H$  and  $Q$  be separable and reflexive Banach spaces, with  $H$  being uniformly convex, and let  $a : H \rightarrow H'$  be a nonlinear operator and  $b \in \mathcal{L}(H, Q')$ , such that  $a$  and  $b$  satisfy the hypotheses of Theorem 3.2 with respective constants*

$L$ ,  $\alpha$  and  $\beta$ . Furthermore, let  $\{H_h\}_{h>0}$  and  $\{Q_h\}_{h>0}$  be sequences of finite dimensional subspaces of  $H$  and  $Q$ , respectively, and for each  $h > 0$  consider a nonlinear operator  $a_h : H \rightarrow H'$ , such that  $a_h|_{H_h} : H_h \rightarrow H'_h$  and  $b|_{H_h} : H_h \rightarrow Q'_h$  satisfy the hypotheses of Theorem 3.2 as well with constants  $L_d$ ,  $\alpha_d$  and  $\beta_d$  all of them independent of  $h$ . In turn, given  $F \in H'$ ,  $G \in Q'$ , and sequences of functionals  $\{F_h\}_{h>0}$  and  $\{G_h\}_{h>0}$  with  $F_h \in H'_h$  and  $G_h \in Q'_h$  for each  $h > 0$ , we let  $(\sigma, u) \in H \times Q$  and  $(\sigma_h, u_h) \in H_h \times Q_h$  be the unique solutions, respectively, to problems

$$\begin{aligned} [a(\sigma), \tau] + [b(\tau), u] &= [F, \tau] \quad \forall \tau \in H, \\ [b(\sigma), v] &= [G, v] \quad \forall v \in Q, \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} [a_h(\sigma_h), \tau_h] + [b(\tau_h), u_h] &= [F_h, \tau_h] \quad \forall \tau_h \in H_h, \\ [b(\sigma_h), v_h] &= [G_h, v_h] \quad \forall v_h \in Q_h. \end{aligned} \quad (5.2)$$

Then, there exist positive constants  $C_{S,i}$ ,  $i \in \{1, 2, 3\}$ , depending only on  $L$ ,  $\alpha_d$ ,  $\beta_d$ ,  $L_d$ , and  $\|b\|$  such that

$$\begin{aligned} \|\sigma - \sigma_h\|_H + \|u - u_h\|_Q &\leq C_{S,1} \text{dist}(\sigma, H_h) + C_{S,2} \text{dist}(u, Q_h) \\ &\quad + C_{S,3} \left\{ \|F - F_h\|_{H'_h} + \|G - G_h\|_{Q'_h} + \|a(\sigma) - a_h(\sigma)\|_{H'_h} \right\}. \end{aligned}$$

**Proof** It is basically a suitable modification of the proof of [20, Lemma 6.1] (see also [7, Lemma 5.2]), which in turn, is a modification of [25, Theorem 2.6]. We omit further details and just stress that the continuity bound and inf-sup condition of the respective linear operator  $a_h$  from [20, Lemma 6.1] are now replaced by the corresponding Lipschitz continuity bound and strong monotonicity property of the present nonlinear operator  $a_h$ , respectively.  $\square$

In order to apply Lemma 5.1, we now observe that the problems (3.17) and (4.2) can be rewritten as two pairs of corresponding continuous and discrete formulations of the type defined by (5.1) and (5.2), namely

$$\begin{aligned} \mathbf{a}_\phi(\sigma, \tau) + \mathbf{b}(\tau, \mathbf{u}) &= \mathbf{F}(\tau) \quad \forall \tau \in \mathbb{H}_0(\text{div}_{4/3}; \Omega), \\ \mathbf{b}(\sigma, \mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{L}^4(\Omega), \\ \mathbf{a}_{\phi_h}(\sigma_h, \tau_h) + \mathbf{b}(\tau_h, \mathbf{u}_h) &= \mathbf{F}(\tau_h) \quad \forall \tau_h \in \mathbb{H}_h^\sigma, \\ \mathbf{b}(\sigma_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}, \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} [\mathcal{A}_u(\phi, \mathbf{t}), (\psi, \mathbf{s})] + [\mathcal{B}(\psi, \mathbf{s}), \mathbf{p}] &= [\mathcal{F}_\phi, (\psi, \mathbf{s})] \quad \forall (\psi, \mathbf{s}) \in \mathbf{L}^4(\Omega) \times \mathbf{L}^2(\Omega), \\ [\mathcal{B}(\phi, \mathbf{t}), \mathbf{q}] &= [\mathcal{G}, \mathbf{q}] \quad \forall \mathbf{q} \in \mathbf{H}(\text{div}_{4/3}; \Omega), \\ [\mathcal{A}_{u_h}(\phi_h, \mathbf{t}_h), (\psi_h, \mathbf{s}_h)] + [\mathcal{B}(\psi_h, \mathbf{s}_h), \mathbf{p}_h] &= [\mathcal{F}_{\phi_h}, (\psi_h, \mathbf{s}_h)] \quad \forall (\psi_h, \mathbf{s}_h) \in \mathbf{H}_h^\phi \times \mathbf{H}_h^{\mathbf{t}}, \\ [\mathcal{B}(\phi_h, \mathbf{t}_h), \mathbf{q}_h] &= [\mathcal{G}, \mathbf{q}_h] \quad \forall \mathbf{q}_h \in \mathbf{H}_h^{\mathbf{p}}. \end{aligned} \quad (5.4)$$

The following lemma provides a preliminary estimate for the error  $\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\|$ .

**Lemma 5.2** *There exists a constant  $\widehat{C}_{\text{ST}} > 0$ , depending only on  $\mu_1$ ,  $\alpha$ ,  $\beta_\phi$  and  $\varepsilon$ , such that*

$$\begin{aligned} \|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\| &\leq \widehat{C}_{\text{ST}} \left\{ \text{dist}(\sigma, \mathbb{H}_h^\sigma) + \text{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \right. \\ &\quad \left. + \left( L_\mu \|\sigma\|_{\varepsilon, \Omega} + \|f\|_{0, \Omega} \right) \|\phi - \phi_h\|_{0,4; \Omega} \right\}. \end{aligned} \quad (5.5)$$

**Proof** It proceeds similarly to the proof of [7, Lemma 5.4]. Indeed, we first observe that, with  $\phi \in L^4(\Omega)$  and  $\phi_h \in H_h^\phi$  given, the continuous and discrete systems of (5.3) satisfy the hypotheses of Theorem 3.2, with constants  $L = L_d = \frac{1}{\mu_1}$ ,  $\|\mathbf{b}\| \leq 1$ ,  $\alpha = \alpha_d$ ,  $\beta$ , and  $\beta_d$  (cf. (3.7), (3.23), (3.25), (4.7), and (4.8)). Hence, applying Lemma 5.1 to the context given by (5.3), we deduce the existence of a constant  $C_{\text{ST}} > 0$ , depending only on  $\mu_1$ ,  $\alpha$ , and  $\beta_d$ , such that

$$\begin{aligned} \|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\| &\leq C_{\text{ST}} \left\{ \text{dist}(\sigma, \mathbb{H}_h^\sigma) + \text{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \right. \\ &\quad \left. + \|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} + \|\mathbf{A}_\phi(\sigma) - \mathbf{A}_{\phi_h}(\sigma)\|_{(\mathbb{H}_h^\sigma)'} \right\}, \end{aligned} \quad (5.6)$$

where  $\mathbf{A}_\phi$  and  $\mathbf{A}_{\phi_h}$ , both belonging to  $\mathcal{L}(\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega), \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)')$ , are the operators induced by  $\mathbf{a}_\phi$  and  $\mathbf{a}_{\phi_h}$ , respectively. Thus, we readily get

$$|\mathbf{G}_\phi(\mathbf{v}_h) - \mathbf{G}_{\phi_h}(\mathbf{v}_h)| = \left| \int_\Omega f(\phi - \phi_h) \cdot \mathbf{v}_h \right| \leq \|f\|_{0, \Omega} \|\phi - \phi_h\|_{0,4; \Omega} \|\mathbf{v}_h\|_{0,4; \Omega},$$

which yields

$$\|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} \leq \|f\|_{0, \Omega} \|\phi - \phi_h\|_{0,4; \Omega}. \quad (5.7)$$

In turn, invoking again the continuous injections  $i_\varepsilon : H^\varepsilon(\Omega) \rightarrow L^{\varepsilon^*}(\Omega)$  and  $\widetilde{i}_\varepsilon : L^4(\Omega) \rightarrow L^{n/\varepsilon}(\Omega)$  specified in the proof of Lemma 3.8, using the Lipschitz-continuity of  $\mu$  (cf. (2.4)), choosing  $p$  and  $q$  as indicated in the proof of that lemma, and employing the Cauchy–Schwarz and Hölder inequalities, we find that for each  $\tau_h \in \mathbb{H}_h^\sigma$  there holds

$$\begin{aligned} |\mathbf{a}_\phi(\sigma, \tau_h) - \mathbf{a}_{\phi_h}(\sigma, \tau_h)| &= \left| \int_\Omega \left\{ \frac{\mu(\phi) - \mu(\phi_h)}{\mu(\phi) \mu(\phi_h)} \right\} \sigma^d : \tau_h^d \right| \\ &\leq \frac{L_\mu}{\mu_1^2} \|\sigma\|_{0,2p; \Omega} \|\phi - \phi_h\|_{0,2q; \Omega} \|\tau_h\|_{0, \Omega} \\ &\leq \frac{L_\mu}{\mu_1^2} \|i_\varepsilon\| \|\widetilde{i}_\varepsilon\| \|\sigma\|_{\varepsilon, \Omega} \|\phi - \phi_h\|_{0,4; \Omega} \|\tau_h\|_{0, \Omega}, \end{aligned}$$

which implies

$$\|\mathbf{A}_\phi(\sigma) - \mathbf{A}_{\phi_h}(\sigma)\|_{(\mathbb{H}_h^\sigma)'} \leq \frac{L_\mu}{\mu_1^2} \|i_\varepsilon\| \|\widetilde{i}_\varepsilon\| \|\sigma\|_{\varepsilon, \Omega} \|\phi - \phi_h\|_{0,4; \Omega}. \quad (5.8)$$

Finally, replacing (5.7) and (5.8) back into (5.6), we obtain (5.5) and finish the proof.  $\square$

Next, we have the following result estimating  $\|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\|$ .

**Lemma 5.3** *There exists a constant  $\tilde{C}_{ST} > 0$ , depending only on  $\tilde{L}$ ,  $\tilde{L}_d$ ,  $\tilde{\alpha}_\phi$  and  $\tilde{\beta}_\phi$  such that*

$$\begin{aligned} \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| &\leq \tilde{C}_{ST} \left\{ \text{dist}((\phi, \mathbf{t}), H_h^\phi \times \mathbf{H}_h^t) + \text{dist}(\mathbf{p}, \mathbf{H}_h^p) \right. \\ &\quad \left. + L_\gamma |\Omega|^{1/4} \|\mathbf{k}\| \|\phi - \phi_h\|_{0,4;\Omega} + \|\phi\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \right\}. \end{aligned} \quad (5.9)$$

**Proof** We begin by observing that, with  $\mathbf{u} \in \mathbf{L}^4(\Omega)$  and  $\mathbf{u}_h \in \mathbf{H}_h^u$  given, which verify  $\|\mathbf{u}\|_{0,4;\Omega} \leq 2\tilde{\alpha}$  and  $\|\mathbf{u}_h\|_{0,4;\Omega} \leq 2\tilde{\alpha}_d$ , the continuous and discrete systems of (5.4) satisfy the hypotheses of Theorem 3.2, with constants  $\tilde{L}$ ,  $\tilde{\alpha}$ ,  $\tilde{\beta} = \frac{1}{2}$ ,  $\tilde{L}_d$ ,  $\tilde{\alpha}_d$ , and  $\tilde{\beta}_d$  (cf. proof of Lemma 3.6, (3.33), (3.34), and Lemmas 4.2 and 4.3). Therefore, applying Lemma 5.1 to the context given by (5.4), we deduce the existence of a constant  $\tilde{C}_{ST} > 0$ , depending only on  $\tilde{L}$ ,  $\tilde{L}_d$ ,  $\tilde{\alpha}_d$ , and  $\tilde{\beta}_d$ , such that

$$\begin{aligned} \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| &\leq \tilde{C}_{ST} \left\{ \text{dist}((\phi, \mathbf{t}), H_h^\phi \times \mathbf{H}_h^t) + \text{dist}(\mathbf{p}, \mathbf{H}_h^p) \right. \\ &\quad \left. + \|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{(H_h^\phi \times \mathbf{H}_h^t)'} + \|\mathcal{A}_u(\phi, \mathbf{t}) - \mathcal{A}_{u_h}(\phi, \mathbf{t})\|_{(H_h^\phi \times \mathbf{H}_h^t)'} \right\}. \end{aligned} \quad (5.10)$$

Then, proceeding exactly as we did in (3.46) and (3.47), that is employing the Lipschitz-continuity of  $\gamma$  (cf. (2.5)) and the Cauchy–Schwarz inequality, and additionally recalling that  $|\Omega|^{1/4}$  is the boundedness constant of the injection of  $L^4(\Omega)$  into  $L^2(\Omega)$ , we find that for each  $(\psi_h, \mathbf{s}_h) \in H_h^\phi \times \mathbf{H}_h^t$  there holds

$$|[\mathcal{F}_\phi - \mathcal{F}_{\phi_h}, (\psi_h, \mathbf{s}_h)]| \leq L_\gamma |\Omega|^{1/4} \|\mathbf{k}\| \|\phi - \phi_h\|_{0,4;\Omega} \|\mathbf{s}_h\|_{0,\Omega},$$

and

$$|[\mathcal{A}_u(\phi, \mathbf{t}) - \mathcal{A}_{u_h}(\phi, \mathbf{t}), (\psi_h, \mathbf{s}_h)]| \leq \|\phi\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \|\mathbf{s}_h\|_{0,\Omega},$$

which yield, respectively,

$$\|\mathcal{F}_\phi - \mathcal{F}_{\phi_h}\|_{(H_h^\phi \times \mathbf{H}_h^t)'} \leq L_\gamma |\Omega|^{1/4} \|\mathbf{k}\| \|\phi - \phi_h\|_{0,4;\Omega}, \quad (5.11)$$

and

$$\|\mathcal{A}_u(\phi, \mathbf{t}) - \mathcal{A}_{u_h}(\phi, \mathbf{t})\|_{(H_h^\phi \times \mathbf{H}_h^t)'} \leq \|\phi\|_{0,4;\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}. \quad (5.12)$$

In this way, replacing (5.11) and (5.12) back into (5.10), we obtain (5.9) and conclude the proof.  $\square$

The required Céa estimate will now follow from (3.38), (5.5) and (5.9). In fact, using from (3.38) that

$$\|\sigma\|_{\varepsilon,\Omega} \leq \tilde{C}_S(r) \left\{ \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} + \|f\|_{0,\Omega} \|\phi\|_{0,4;\Omega} \right\}, \quad (5.13)$$

incorporating the resulting bound for  $\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}$  provided by (5.5) into (5.9), recalling that  $\|\phi\|_{0,4;\Omega}$  is bounded by  $r$ , and performing some algebraic manipulations, we deduce that

$$\begin{aligned} & \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| \\ & \leq \left\{ C_1 \|\mathbf{k}\| + (C_2 + r C_3) \|f\|_{0,\Omega} + C_3 \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \right\} \|\phi - \phi_h\|_{0,4;\Omega} \\ & \quad + C_0 \left\{ \text{dist}((\sigma, \mathbf{u}), \mathbb{H}_h^\sigma \times \mathbf{H}_h^u) + \text{dist}((\phi, \mathbf{t}), \mathbf{H}_h^\phi \times \mathbf{H}_h^t) + \text{dist}(\mathbf{p}, \mathbf{H}_h^p) \right\}, \end{aligned} \quad (5.14)$$

where  $C_i, i \in \{1, 2, 3\}$ , are the positive constants defined by

$$C_1 := \tilde{C}_{ST} L_\gamma |\Omega|^{1/4}, \quad C_2 := \hat{C}_{ST} \tilde{C}_{ST} r, \quad C_3 := \hat{C}_{ST} \tilde{C}_{ST} L_\mu \tilde{C}_S(r) r,$$

and  $C_0$  is another positive constant, which depends on  $\hat{C}_{ST}$ ,  $\tilde{C}_{ST}$ , and  $r$ .

Thus, imposing the term that multiplies  $\|\phi - \phi_h\|_{0,4;\Omega}$  in (5.14) to be sufficiently small, say  $\leq 1/2$ , we derive the *a priori* error estimate for  $\|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\|$ , which, employed then to bound the last term on the right-hand side of (5.5) in conjunction with (5.13), provides the corresponding upper bound for  $\|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\|$ . More precisely, we have proved the following result.

**Theorem 5.4** Assume that the data  $\mathbf{k}, f$  and  $\mathbf{u}_D$  are sufficiently small so that

$$C_1 \|\mathbf{k}\| + (C_2 + r C_3) \|f\|_{0,\Omega} + C_3 \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma} \leq \frac{1}{2}. \quad (5.15)$$

Then, there exist a positive constant  $\tilde{C}_0$ , which depends only on parameters and other constants, all them independent of  $h$ , such that

$$\begin{aligned} & \|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\| + \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| \\ & \leq \tilde{C}_0 \left\{ \text{dist}((\sigma, \mathbf{u}), \mathbb{H}_h^\sigma \times \mathbf{H}_h^u) + \text{dist}((\phi, \mathbf{t}), \mathbf{H}_h^\phi \times \mathbf{H}_h^t) + \text{dist}(\mathbf{p}, \mathbf{H}_h^p) \right\}. \end{aligned} \quad (5.16)$$

We recall next the approximation properties of the subspaces defined by (4.1), which follow from interpolation estimates of Sobolev spaces and the approximation properties of the orthogonal projectors and the interpolation operators involved in their definitions (see, e.g. [8, 14, 20, 24, 25]).

( $\mathbf{AP}_h^\sigma$ ): there exists  $C > 0$ , independent of  $h$ , such that for each  $l \in (0, k+1]$ , and for each  $\tau \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$  with  $\mathbf{div}(\tau) \in \mathbf{W}^{l,4/3}(\Omega)$ , there holds

$$\text{dist}(\tau, \mathbb{H}_h^\sigma) := \inf_{\tau_h \in \mathbb{H}_h^\sigma} \|\tau - \tau_h\|_{\mathbf{div}_{4/3}; \Omega} \leq C h^l \left\{ \|\tau\|_{l,\Omega} + \|\mathbf{div}(\tau)\|_{l,4/3;\Omega} \right\}.$$

( $\mathbf{AP}_h^u$ ): there exists  $C > 0$ , independent of  $h$ , such that for each  $l \in [0, k+1]$ , and for each  $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$  there holds

$$\text{dist}(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) := \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leq C h^l \|\mathbf{v}\|_{l,4;\Omega}.$$

( $\mathbf{AP}_h^{\phi}$ ): there exists  $C > 0$ , independent of  $h$ , such that for each  $l \in [0, k + 1]$ , and for each  $\psi \in W^{l,4}(\Omega)$  there holds

$$\text{dist}(\psi, \mathbf{H}_h^{\phi}) := \inf_{\psi_h \in \mathbf{H}_h^{\phi}} \|\psi - \psi_h\|_{0,4;\Omega} \leq C h^l \|\psi\|_{l,4;\Omega}.$$

( $\mathbf{AP}_h^{\mathbf{t}}$ ): there exists  $C > 0$ , independent of  $h$ , such that for each  $l \in [0, k + 1]$ , and for each  $\mathbf{s} \in \mathbf{H}^l(\Omega)$  there holds

$$\text{dist}(\mathbf{s}, \mathbf{H}_h^{\mathbf{t}}) := \inf_{\mathbf{s}_h \in \mathbf{H}_h^{\mathbf{t}}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leq C h^l \|\mathbf{s}\|_{l,\Omega}.$$

( $\mathbf{AP}_h^{\mathbf{p}}$ ): there exists  $C > 0$ , independent of  $h$ , such that for each  $l \in (0, k + 1]$ , and for each  $\mathbf{q} \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\text{div}_{4/3}; \Omega)$  with  $\text{div}(\mathbf{q}) \in W^{l,4/3}(\Omega)$ , there holds

$$\text{dist}(\mathbf{q}, \mathbf{H}_h^{\mathbf{p}}) := \inf_{\mathbf{q}_h \in \mathbf{H}_h^{\mathbf{p}}} \|\mathbf{q} - \mathbf{q}_h\|_{\text{div}_{4/3};\Omega} \leq C h^l \left\{ \|\mathbf{q}\|_{l,\Omega} + \|\text{div}(\mathbf{q})\|_{l,4/3;\Omega} \right\}.$$

Finally, we conclude this section with the rates of convergence of our Galerkin scheme (4.2).

**Theorem 5.5** *In addition to the hypotheses of Theorems 3.11, 4.10, and 5.4, assume that there exists  $l \in (0, k + 1]$  such that  $\sigma \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\text{div}_{4/3}; \Omega)$ ,  $\text{div}(\sigma) \in W^{l,4/3}(\Omega)$ ,  $\mathbf{u} \in W^{l,4}(\Omega)$ ,  $\phi \in W^{l,4}(\Omega)$ ,  $\mathbf{t} \in \mathbf{H}^l(\Omega)$ ,  $\mathbf{p} \in \mathbf{H}^l(\Omega) \cap \mathbf{H}(\text{div}_{4/3}; \Omega)$  and  $\text{div}(\mathbf{p}) \in W^{l,4/3}(\Omega)$ . Then, there exists a positive constant  $C$ , independent of  $h$ , such that*

$$\begin{aligned} & \|(\sigma, \mathbf{u}) - (\sigma_h, \mathbf{u}_h)\| + \|(\phi, \mathbf{t}, \mathbf{p}) - (\phi_h, \mathbf{t}_h, \mathbf{p}_h)\| \leq C h^l \left\{ \|\sigma\|_{l,\Omega} + \|\text{div}(\sigma)\|_{l,4/3;\Omega} \right. \\ & \quad \left. + \|\mathbf{u}\|_{l,4;\Omega} + \|\phi\|_{l,4;\Omega} + \|\mathbf{t}\|_{l,\Omega} + \|\mathbf{p}\|_{l,\Omega} + \|\text{div}(\mathbf{p})\|_{l,4/3;\Omega} \right\}. \end{aligned} \quad (5.17)$$

**Proof** The result follows from a direct application of Theorem 5.4 and the approximation properties of the finite element subspaces. Further details are omitted.  $\square$

## 6 Numerical results

In this section we present three examples illustrating the performance of our non-augmented and momentum-conserving fully-mixed finite element method (4.2) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (4.1) (cf. Sect. 4.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by  $k = 0$  and  $k = 1$ , as simply  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  and  $\mathbb{RT}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$ , respectively. Our implementation is based on a FreeFem++ code [28], in conjunction with the direct linear solver UMFPAK [23]. The way we solve the fixed-point problem (4.6) is explained as follows. Given an initial guess  $\phi_h^{(0)}$  for the concentration (usually the

null function), we first solve the linear system (4.3) with the given  $\phi_h := \phi_h^{(0)}$ , whose solution is denoted  $(\sigma_h^{(1)}, \mathbf{u}_h^{(1)})$ . Next, we look at the nonlinear system (4.4) with the given  $(\mathbf{u}_h, \phi_h) = (\mathbf{u}_h^{(1)}, \phi_h^{(0)})$ , so that, starting from a null initial guess, we perform just one Newton iteration to obtain  $(\phi_h^{(1)}, \mathbf{t}_h^{(1)}, \mathbf{p}_h^{(1)})$  as an approximate solution of it. Then, the process continues with  $\phi_h^{(m)}$  for each  $m \geq 1$ . In this way, for a fixed tolerance  $\text{tol} = 1\text{E} - 6$ , the above iterations are terminated, which yields the number of Newton iterations reported in the tables below, once the relative error of the entire coefficient vectors between two consecutive iterates, say  $\mathbf{coeff}^m$  and  $\mathbf{coeff}^{m+1}$ , is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leq \text{tol},$$

where  $\|\cdot\|$  stands for the usual Euclidean norm in  $\mathbf{R}^{\text{DOF}}$ , with DOF denoting the total number of degrees of freedom defining the finite element subspaces  $\mathbb{H}_h^\sigma$ ,  $\mathbf{H}_h^{\mathbf{u}}$ ,  $\mathbf{H}_h^\phi$ ,  $\mathbf{H}_h^{\mathbf{t}}$ , and  $\mathbf{H}_h^{\mathbf{p}}$  (cf. (4.1)).

We now introduce some additional notation. The individual errors are denoted by:

$$\begin{aligned} \mathbf{e}(\sigma) &:= \|\sigma - \sigma_h\|_{\text{div}_{4/3}; \Omega}, & \mathbf{e}(\mathbf{u}) &:= \|\mathbf{u} - \mathbf{u}_h\|_{0,4; \Omega}, & \mathbf{e}(p) &:= \|p - p_h\|_{0, \Omega}, \\ \mathbf{e}(\phi) &:= \|\phi - \phi_h\|_{0,4; \Omega}, & \mathbf{e}(\mathbf{t}) &:= \|\mathbf{t} - \mathbf{t}_h\|_{0, \Omega}, & \mathbf{e}(\mathbf{p}) &:= \|\mathbf{p} - \mathbf{p}_h\|_{\text{div}_{4/3}; \Omega}, \end{aligned}$$

where  $p_h$  stands for the post-processed pressure suggested by the second formula of (2.7), that is

$$p_h = -\frac{1}{n} \text{tr}(\sigma_h). \quad (6.1)$$

It follows that

$$\|p - p_h\|_{0, \Omega} = \frac{1}{n} \|\text{tr}(\sigma - \sigma_h)\|_{0, \Omega} \leq \frac{1}{\sqrt{n}} \|\sigma - \sigma_h\|_{\text{div}_{4/3}; \Omega},$$

which shows that the rate of convergence for  $p$  is at least the one for  $\sigma$ , which is indeed confirmed below by the numerical results reported.

Next, as usual, for each  $\diamond \in \{\sigma, \mathbf{u}, p, \phi, \mathbf{t}, \mathbf{p}\}$  we let  $r(\diamond)$  be the experimental rate of convergence given by

$$r(\diamond) := \frac{\log(\mathbf{e}(\diamond)/\widehat{\mathbf{e}}(\diamond))}{\log(h/\widehat{h})},$$

where  $h$  and  $\widehat{h}$  denote two consecutive meshsizes with errors  $\mathbf{e}$  and  $\widehat{\mathbf{e}}$ , respectively.

The examples to be considered in this section are described next. In all of them, for the sake of simplicity, we take  $\mathbf{k} = (0, -1)$ , and similarly to [1, 7, 26], we choose the coefficients  $\mu$ ,  $\gamma$ , and  $\vartheta$ , respectively, as:

$$\mu(\phi) = (1 - c\phi)^{-2}, \quad \gamma(\phi) = c\phi(1 - c\phi)^2, \quad \text{and} \quad \vartheta(|\mathbf{t}|) = m_1 + m_2(1 + |\mathbf{t}|^2)^{m_3/2-1},$$

where  $c = m_1 = m_2 = 1/2$  and  $m_3 = 3/2$ . In addition, the mean value of  $\text{tr}(\sigma_h)$  over  $\Omega$  is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (4.3) for  $\sigma_h$  and  $\mathbf{u}_h$ ).

## 6.1 Example 1: Two-dimensional smooth exact solution

In this first example, we illustrate the accuracy of our method in 2D by considering a manufactured exact solution defined on  $\Omega := (0, 1)^2$ . More precisely, the source terms  $\mathbf{f}$  and  $g$  in (2.8) are adjusted in such a way that  $\phi$ ,  $\mathbf{t}$ ,  $\mathbf{u}$ , and  $\sigma$ , are given by the smooth functions

$$\begin{aligned} \phi(x_1, x_2) &= 15 - 15 \exp(-x_1(x_1 - 1)x_2(x_2 - 1)), \quad \mathbf{t} = \nabla \phi, \\ \mathbf{u}(x_1, x_2) &= \begin{pmatrix} \sin(2\pi x_1) \cos(2\pi x_2) \\ -\cos(2\pi x_1) \sin(2\pi x_2) \end{pmatrix}, \quad \text{and} \quad \sigma = \mu(\phi) \nabla \mathbf{u} - (x_1^2 - x_2^2) \mathbb{I}. \end{aligned}$$

Notice that  $\phi$  vanishes at  $\Gamma$  and  $\mathbf{u}_D$  is imposed accordingly to the exact solution. Tables 1 and 2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations. Notice that we are able not only to approximate the original unknowns but also the pressure field through the formula (6.1). The results confirm that the optimal rates of convergence  $\mathcal{O}(h^{k+1})$ , provided by Theorem 5.5 are attained for  $k = 0, 1$ . The Newton method exhibits a behavior independent of the meshsize, converging in six iterations in all cases. We notice here that for 2D domains, the number of degrees of freedom of the Raviart–Thomas elements of order  $k \geq 0$  and the discontinuous piecewise polynomials of degree  $\leq k$ , which are employed in (4.2), are given, respectively, by

$$k(k+1) \times \{\#T\} + (k+1) \times \{\#E\}$$

and

$$\frac{1}{2}(k+1)(k+2) \times \{\#T\},$$

where  $\#T$  and  $\#E$  denote, respectively, the number of triangles and edges of  $\mathcal{T}_h$ . Then, multiplying the former by 3 (factors 2 for  $\mathbb{H}_h^\sigma$  and 1 for  $\mathbf{H}_h^p$ ) and the latter by 5 (factors 2 for  $\mathbf{H}_h^u$ , 1 for  $H_h^\phi$ , and 2 for  $\mathbf{H}_h^t$ ), we find that the total number of degrees of freedom of the fully-mixed method (4.2) becomes

$$\text{DOF} = \left(\frac{11}{2}k + 5\right)(k+1) \times \{\#T\} + 3(k+1) \times \{\#E\}.$$

In this way, subtracting the evaluation of the foregoing formula for  $k = 0$  from that for  $k = 1$ , we obtain that the respective increase in DOF reduces to

$$16 \times \{\#T\} + 3 \times \{\#E\}.$$

**Table 1** Example 1, Number of degrees of freedom, number of triangles, number of edges, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed  $\mathbb{R}\mathbb{I}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation

DOF	# $T$	# $E$	$h$	$e(\sigma)$	$r(\sigma)$	$e(u)$	$r(u)$	$e(p)$	$r(p)$
1492	152	244	0.1964	3.51E+01	–	2.13E-01	–	1.42E+00	–
3340	344	540	0.1267	2.52E+01	0.7620	1.42E-01	0.9297	8.79E-01	1.0964
9164	952	1468	0.0776	1.65E+01	0.8644	8.49E-02	1.0469	4.99E-01	1.1525
29913	3126	4761	0.0448	9.67E+00	0.9702	4.69E-02	1.0834	2.42E-01	1.3182
104490	10956	16570	0.0244	5.59E+00	0.9021	2.49E-02	1.0431	1.24E-01	1.1011
391679	41146	61983	0.0135	3.18E+00	0.9551	1.29E-02	1.1050	6.38E-02	1.1243
$e(\phi)$	$r(\phi)$	$e(t)$	$r(t)$	$e(p)$	$r(p)$	iter			
7.74E-02	–	3.67E-01	–	7.57E-01	–	6			
5.08E-02	0.9601	2.40E-01	0.9729	5.17E-01	0.8708	6			
3.24E-02	0.9187	1.42E-01	1.0645	3.07E-01	1.0624	6			
1.69E-02	1.1815	7.81E-02	1.0914	1.66E-01	1.1147	6			
9.22E-03	1.0017	4.22E-02	1.0141	9.00E-02	1.1026	6			
4.79E-03	1.1059	2.17E-02	1.1221	4.64E-02	1.1166	6			

**Table 2** Example 1, Number of degrees of freedom, number of triangles, number of edges, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed  $\mathbb{R}\mathbb{T}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$  approximation

DOF	# $T$	# $E$	$h$	$e(\sigma)$	$r(\sigma)$	$e(u)$	$r(u)$	$e(p)$	$r(p)$
4656	152	244	0.1964	5.08E+00	–	2.74E-02	–	1.41E-01	–
10464	344	540	0.1267	2.24E+00	1.8707	1.19E-02	1.8999	6.05E-02	1.9294
28800	952	1468	0.0776	8.23E-01	2.0370	4.32E-03	2.0661	2.11E-02	2.1450
94212	3126	4761	0.0448	2.63E-01	2.0838	1.31E-03	2.1806	6.75E-03	2.0775
329496	10956	16570	0.0244	7.70E-02	2.0202	3.76E-04	2.0507	1.90E-03	2.0886
1235964	41146	61983	0.0135	2.17E-02	2.1386	9.99E-05	2.2400	5.12E-04	2.2139
$e(\phi)$	$r(\phi)$	$e(t)$	$r(t)$	$e(p)$	$r(p)$	iter			
6.21E-03	–	2.46E-02	–	1.17E-01	–	6			
2.63E-03	1.9627	1.13E-02	1.7721	5.29E-02	1.8052	6			
8.75E-04	2.2375	4.10E-03	2.0641	1.72E-02	2.2909	6			
2.57E-04	2.2359	1.27E-03	2.1376	4.93E-03	2.2776	6			
7.25E-05	2.0827	3.68E-04	2.0404	1.39E-03	2.0876	6			
1.97E-05	2.1976	9.85E-05	2.2248	3.71E-04	2.2261	6			

**Table 3** Example 1, Conservation of momentum for the fully-mixed  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation

$h$	0.1964	0.1267	0.0776	0.0448	0.0244	0.0135
$\ \mathbf{div}(\boldsymbol{\sigma}_h) + \mathcal{P}_0(\mathbf{f}) \phi_h\ _{\ell^\infty}$	3.97E-06	4.36E-06	4.18E-06	4.27E-06	4.26E-06	4.25E-06
$\ \mathbf{div}(\mathbf{p}_h) + \mathcal{P}_0(\mathbf{g})\ _{\ell^\infty}$	1.07E-14	1.95E-14	4.80E-14	9.50E-14	1.79E-13	4.00E-13

**Table 4** Example 1, Comparison between the augmented fully-mixed approximation [26, eq. (4.1)] and the present non-augmented mixed formulation (4.2) for  $k = 1$ 

$h$	Augmented mixed formulation				Non-augmented mixed formulation				
	DOF	Error-A	Rate	CPU[s]	DOF	Error-B	Rate	CPU[s]	iter
0.3536	821	2.23E+01	–	0.1064	1008	2.20E+01	–	0.0590	6
0.2828	1245	1.51E+01	1.7515	0.2681	1560	1.39E+01	1.8893	0.1503	6
0.2020	2357	7.99E+00	1.8885	0.5888	3024	7.64E+00	1.7787	0.3386	6
0.1286	5637	3.30E+00	1.9558	1.4146	7392	3.13E+00	1.9762	0.8700	6
0.0744	16421	1.11E+00	1.9865	5.5152	21888	1.08E+00	1.9444	2.6995	6
0.0404	54885	3.29E-01	1.9962	211.29	73920	3.24E-01	1.9717	66.996	6

In particular, we observe from Tables 1 and 2 that for  $h = 0.0135$ , the number of triangles is 41, 146 and the number of edges is 61, 983, so that the number of degrees of freedom of our method  $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{P}_k - \mathbf{P}_k - \mathbf{RT}_k$  increases from 391, 679 to 1, 235, 964 as the polynomial degree changes from  $k = 0$  to  $k = 1$ . Indeed, the corresponding increment is precisely  $16 \times \{41, 146\} + 3 \times \{61, 983\} = 844, 285$ . In Figure 1 we display the solution obtained with the fully-mixed  $\mathbb{RT}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$  approximation with meshsize  $h = 0.0135$  and 41, 146 triangle elements (actually representing 1, 235, 964 DOF). In addition, in the case  $k = 0$ , and since both  $\mathbf{f}$  and  $\mathbf{g}$  are not piecewise constant, we observe, as explained at the end of Sect. 4, that our Galerkin scheme provides conservation of momentum in an approximate sense. We illustrate this fact in Table 3, where the computed  $\ell^\infty$ -norm for both  $\mathbf{div}(\boldsymbol{\sigma}_h) + \mathcal{P}_0(\mathbf{f}) \phi_h$  and  $\mathbf{div}(\mathbf{p}_h) + \mathcal{P}_0(\mathbf{g})$ , are displayed. As expected, these values are certainly close to zero. We end this example with a comparison between the present approach (4.2) and the augmented one from [26] in terms of degrees of freedom, accuracy, and CPU time with respect to the same meshsizes for the polynomial degree  $k = 1$ . To this end, we consider the total errors for the augmented fully-mixed formulation [26, eq. (4.1)] and the numerical scheme (4.2), respectively,

$$\text{error-A} := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}; \Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + \|\phi - \phi_h\|_{1, \Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0, \Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{div}; \Omega}$$

and

$$\text{error-B} := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3}; \Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0, 4; \Omega} + \|\phi - \phi_h\|_{0, 4; \Omega} + \|\mathbf{t} - \mathbf{t}_h\|_{0, \Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{div}_{4/3}; \Omega}.$$

We first observe from Table 4 that in terms of accuracy both methods have a similar behavior and achieve the Newton convergence in six iterations in all cases. In turn, the degrees of freedom of (4.2) are a bit higher than the ones from [26], which is justified by the fact that [26, eq. (4.1)] considers continuous piecewise polynomials of degree  $\leq k + 1$  to approximate the velocity and concentration fields instead of discontinuous piecewise polynomials of degree  $\leq k$  as in (4.2). Note that both methods employ Raviart–Thomas approximations of order  $k$  for the stress and total flux, and discontinuous piecewise polynomials of degree  $\leq k$  for the concentration gradient. Nevertheless, the CPU times of the augmented method [26, eq. (4.1)] are much higher than the ones required in (4.2), especially as the meshsize tends to 0. The latter is explained, on one hand, by the introduction of additional terms into the augmented formulation [26, eq. (4.1)], and on the other hand, by the extra computations that need to be performed in order to set up the stiffness matrix and load vector of the resulting discrete system. In other words, the fact that, differently from [26, eq. (4.1)], our present method does not require to perform any assembling of the degrees of freedom of  $\mathbf{u}_h$  nor of  $\phi_h$ , implies a significant saving of CPU time.

## 6.2 Example 2: Three-dimensional smooth exact solution

In our second example, we consider the cube domain  $\Omega = (0, 1)^3$ . We consider the external force  $\mathbf{f} = (0, 0, -9.81)^t$ , and the terms on the right-hand side are adjusted so that the exact solution is given by the functions

$$\begin{aligned} \phi(x_1, x_2, x_3) &= 15 - 15 \exp(x_1(x-1)x_2(y-1)x_3(z-1)), \quad \mathbf{t} = \nabla \phi, \\ \mathbf{u}(x_1, x_2, x_3) &= \begin{pmatrix} \sin(\pi x_1) \cos(\pi x_2) \cos(\pi x_3) \\ -2 \cos(\pi x_1) \sin(\pi x_2) \cos(\pi x_3) \\ \cos(\pi x_1) \cos(\pi x_2) \sin(\pi x_3) \end{pmatrix}, \quad \text{and} \quad \sigma = \mu(\phi) \nabla \mathbf{u} - (x_1 - 0.5)^3 \sin(x_3 + x_2) \mathbb{I}. \end{aligned}$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. The numerical solutions are shown in Figure 2, which were built using the fully-mixed  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation with meshsize  $h = 0.0544$  and 105, 456 tetrahedral elements (actually representing 1, 598, 064 DOF). The convergence history for a set of quasi-uniform mesh refinements using  $k = 0$  is shown in Table 5. Again, the mixed finite element method converges optimally with order  $\mathcal{O}(h)$ , as it was proved by Theorem 5.5.

## 6.3 Example 3: Fluid flow in a two-dimensional inverted L-shaped domain

In our last example we study the behavior of the model for fluid flow in an inverted L-shaped domain given by  $\Omega = (0, 2) \times (-1, 1) \setminus (0, 1) \times (-1, 0)$  with boundary  $\Gamma$ , and whose input, upper and lower parts are given by  $\Gamma_{\text{in}} = \{0\} \times (0, 1)$ ,  $\Gamma_{\text{top}} = (0, 2) \times \{1\}$  and  $\Gamma_{\text{bottom}} = (1, 2) \times \{-1\}$ , respectively. The right-hand side data are chosen as  $\mathbf{f} = (0, -9.81)$  and  $g = 0$ , and the boundary conditions are

**Table 5** Example 2, Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation

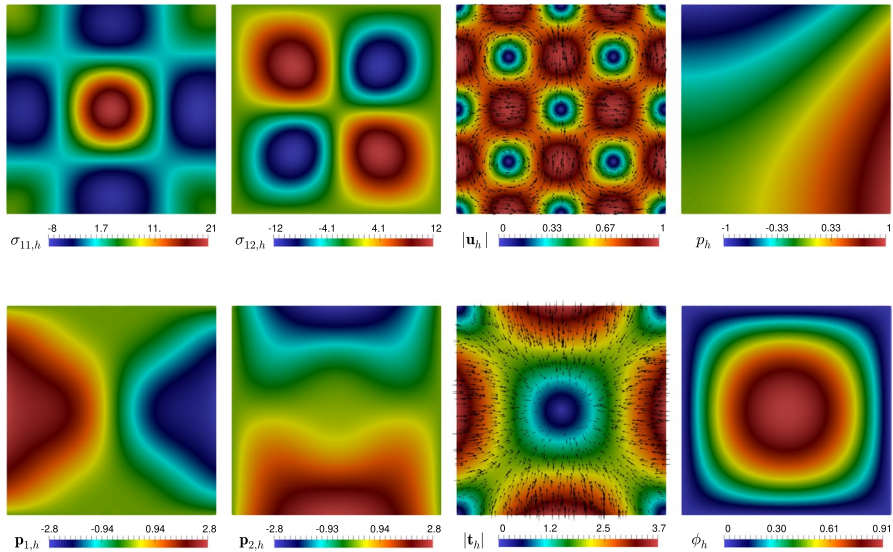
DOF	$h$	$e(\sigma)$	$r(\sigma)$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$		
2646	0.4714	1.12E+01	–	3.93E–01	–	8.52E–01	–		
20304	0.2357	5.77E+00	0.9627	2.05E–01	0.9344	4.16E–01	1.0327		
92400	0.1414	3.46E+00	0.9999	1.24E–01	0.9818	2.29E–01	1.1745		
374784	0.0884	2.16E+00	1.0061	7.80E–02	0.9937	1.31E–01	1.1791		
1598064	0.0544	1.32E+00	1.0049	4.80E–02	0.9977	7.60E–02	1.1271		
$e(\phi)$	$r(\phi)$	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\mathbf{p})$	$r(\mathbf{p})$				
4.32E–02	–	2.01E–01	–	7.42E–01	–			5	
2.33E–02	0.8905	1.06E–01	0.9162	3.87E–01	0.9385			5	
1.42E–02	0.9651	6.47E–02	0.9728	2.34E–01	0.9830			5	
8.96E–03	0.9859	4.07E–02	0.9875	1.47E–01	0.9932			5	
5.54E–03	0.9914	2.52E–02	0.9897	9.06E–02	0.9958			6	

**Table 6** Example 3, Number of degrees of freedom, meshsizes, errors, rates of convergence, and Newton iteration count for the fully-mixed  $\mathbb{RT}_0$  approximation

Dof	$h$	$e(\sigma)$	$r(\sigma)$	$e(u)$	$r(u)$	$e(p)$	$r(p)$
252	0.7500	5.67E+00	–	6.94E-01	–	2.94E+00	–
1055	0.3750	3.02E+00	0.9114	3.37E-01	1.0420	1.42E+00	1.0507
4219	0.1901	1.61E+00	0.9259	1.81E-01	0.9133	6.90E-01	1.0611
16475	0.1012	9.56E-01	0.8255	8.55E-02	1.1926	4.05E-01	0.8479
66295	0.0542	5.41E-01	0.9108	4.45E-02	1.0444	2.28E-01	0.9203
264469	0.0277	2.92E-01	0.9210	2.24E-02	1.0273	1.19E-01	0.9708
$e(\phi)$	$r(\phi)$	$e(t)$	$r(t)$	$e(p)$	$r(p)$	$r(p)$	iter
8.44E-02	–	1.51E-01	–	9.70E-02	–	–	14
4.66E-02	0.8574	8.80E-02	0.7825	6.23E-02	0.6401	0.6401	13
2.20E-02	1.1064	4.74E-02	0.9092	3.42E-02	0.8836	0.8836	13
1.15E-02	1.0283	2.60E-02	0.9541	1.89E-02	0.9371	0.9371	13
5.58E-03	1.1555	1.36E-02	1.0327	1.00E-02	1.0189	1.0189	13
2.92E-03	0.9691	7.17E-03	0.9600	5.24E-03	0.9672	0.9672	13

**Table 7** Example 3, Conservation of momentum for the fully-mixed  $\mathbb{R}\mathbb{T}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation

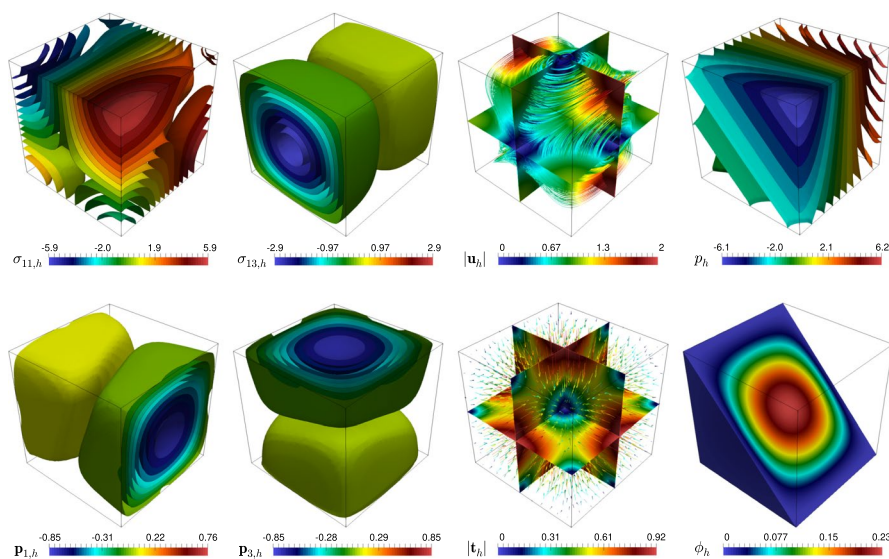
$h$	0.7500	0.3750	0.1901	0.1012	0.0542	0.0277
$\ \mathbf{div}(\boldsymbol{\sigma}_h) + \mathbf{f}\phi_h\ _{\ell^\infty}$	1.28E-06	1.95E-06	1.42E-06	1.23E-06	1.17E-06	1.16E-06
$\ \mathbf{div}(\mathbf{p}_h) + \mathbf{g}\ _{\ell^\infty}$	4.16E-17	2.22E-16	8.88E-16	3.55E-15	7.11E-15	2.84E-14

**Fig. 1** Example 1,  $\mathbb{R}\mathbb{T}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{P}_1 - \mathbf{RT}_1$  approximation of Cauchy stress components, magnitude of the velocity, and pressure field (top plots); total flux components, magnitude of the concentration gradient and concentration field (bottom plots)

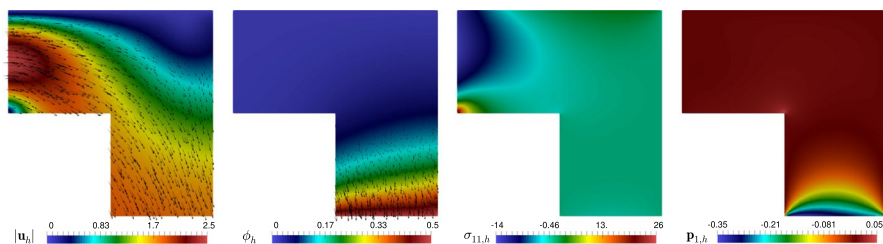
$$\mathbf{u} = (-10x_2(x_2 - 1), 0) \quad \text{on } \Gamma_{\text{in}}, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_{\text{top}}, \quad \boldsymbol{\sigma}\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma \setminus \Gamma_{\text{in}} \cup \Gamma_{\text{top}},$$

$$\phi = 0.5 \quad \text{on } \Gamma_{\text{bottom}}, \quad \phi = 0 \quad \text{on } \Gamma_{\text{top}}, \quad \mathbf{p} \cdot \mathbf{v} = 0 \quad \text{on } \Gamma \setminus \Gamma_{\text{bottom}} \cup \Gamma_{\text{top}}.$$

In particular, the first one corresponds to inflow driven through a parabolic fluid velocity on part of the left boundary of the domain. We stress that slight modifications allow to extend our analysis to these new boundary conditions. Here, since the analytical solution is unknown, we construct the convergence history by considering a solution calculated with 111, 506 triangle elements (representing 1,060,843 DOF) as the exact solution, and employing the  $\mathbb{R}\mathbb{T}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation on a sequence of quasi-uniform triangulations. In Table 6 we show that the mixed finite element method converges optimally with order  $\mathcal{O}(h)$ . In addition, similarly to Example 1, but now using the fact that  $\mathbf{f}$  and  $\mathbf{g}$  are constant data, we compute the  $\ell^\infty$ -norm for both  $\mathbf{div}(\boldsymbol{\sigma}_h) + \mathbf{f}\phi_h$  and  $\mathbf{div}(\mathbf{p}_h) + \mathbf{g}$ , whose values displayed in Table 7 are close to zero, which illustrates again that this method conserves momentum. On the other hand, in Figure 3 we display the computed magnitude of the velocity, concentration field, first Cauchy stress component, and first total flux component,



**Fig. 2** Example 2,  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation of Cauchy stress components, magnitude of the velocity, and pressure field (top plots); total flux components, magnitude of the concentration gradient and concentration field (bottom plots)



**Fig. 3** Example 3,  $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{P}_0 - \mathbf{RT}_0$  approximation of the magnitude of the velocity, concentration field, first Cauchy stress component, and first total flux component

respectively. As expected, the flow start moving from left to right and then is driven into the bottom of the domain due to the effect of gravity force imposed by the source term  $f$ . In turn, the concentration is zero on the top of the domain and goes increasing towards the bottom of the inverted L-shaped domain. Finally, we observe that the first components of both the Cauchy stress and total flux satisfy homogeneous boundary conditions on the corresponding sides of the boundary.

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