Investigation of 2^{nd} order upwind schemes for 2-D Euler equations

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Two finite volume schemes have been verified in this report using method of manufactured solution. Van leer method and Roe's method are verified for 2-D Euler equations for supersonic and subsonic flow using a manufactured solution source term. Order of accuracy has been reported as well as convergence norm. finally, a case study has been solved for a 2-D nozzle with a supersonic flow inlet. The primitive variables fields have been plotted and the total pressure loss has been calculated.

Nomenclature

e	=	energy, m^2/s^2
f	=	source term
h	=	grid spacing
R	=	Specific gas constant
r	=	grid refinement factor
t	=	time
ρ	=	kg/m^3
u, v	=	Cartesian component in x and y directions, m/s
x, y	=	Cartesian spatial coordinates
γ	=	ratio of specific heats

I. Introduction

Finite volume method has been widely used for solving coupled differential equations. In computational fluid dynamics, this method has many advantages such as: flux continuity over the cells and the ability to simulate shockwaves. Many numerical schemes have been developed within the framework of finite volume method to solve compressible and incompressible flow problems. This report examines two upwind schemes: Van leer and Roe's method. Using method of manufactured solution, the C++ code that implements both schemes has been verified. Finally, 2-D nozzle case with supersonic inlet has been solved to calculate the pressure loss.

II. Mathematical formulation

2-D Euler equations has been solved for a manufactured solution [1]. The source term has been implemented in the weak form of Euler equations after choosing an appropriate solution function. The source term as well as the manufactured solution can be found in Appendix A. more details on the manufactured solution method and code verification can be found in [1]. The implemented 2-D mass, x-momentum, y-momentum and energy equations are

$$\frac{\partial}{\partial t}\oiiint \vec{U}dV + \oiint \vec{F}.d\vec{A} = \oiiint \vec{S}dV$$

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$$\vec{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \vec{F} = \begin{bmatrix} \rho(u+v) \\ \rho u^2 + p + \rho uv \\ \rho vu + \rho v^2 + p \\ \rho u e_t + pu + \rho v e_t + pv \end{bmatrix}, \vec{S} = \begin{bmatrix} f_m \\ f_x \\ f_y \\ f_e \end{bmatrix}$$

where $\vec{U}$ is the vector of the unsteady terms, $\vec{F}$ the convective flux in the *x* and *y* directions, respectively and $\vec{S}$ is the manufacture solution source term. For a calorically per- fect gas, the Euler equations are closed with two auxilia- rye relations for energy

$$e = \frac{1}{\gamma - 1} RT$$

$$e_t = e + \frac{u^2 + v^2}{2}$$

for perfect air, the equation of state is

$$p = \rho RT$$

Where the ratio of specific heats $(\gamma) = 1.4$ for air and specific gas constant for air (R) = 287.0 *Nm/(kg·K)*.

III. Discretization

The previous 2D Euler equations is discretized using finite volume discretization. The flux has been discretized using two upwind schemes, Roe's method and Van leer method [2],[3]. To implement a second order discretization, MUSCLE extrapolation has been used. To preserve monotonicity and total variation diminishing (TVD) condition; flux limiter has been used [4]. The governing equations are also integrated in time using 2-stages Runge Kutta method [5]. For code verification purposes, two cases have been solved, one is subsonic flow and the other is supersonic flow. The code is written using C++ programming language and is solved on systematically refined curvilinear grids. Depending on the scheme and whether it's the subsonic or the supersonic case, different grids has been used for stability purposes. The following table shows the curvilinear grids with grid spacing h, where h is the ratio of the finest grid to the mesh level k.

Table 1 curvilinear grids for the 2-D manufactured solutions

Meh level k	Number of nodes in each direction	Grid spacing, h
1	257	1
2	129	2
3	65	4
4	33	8

IV. Code verification results

The numerical analysis is conducted using the source term to solve for the manufactured solution in both supersonic and subsonic regime. The following figures show the solution for the permeative variables in the flow field. Boundary conditions is implemented using ghost cells approach along all the boundaries. The value of the primitive variables at the ghost cells are calculated using the manufactured solution functions.

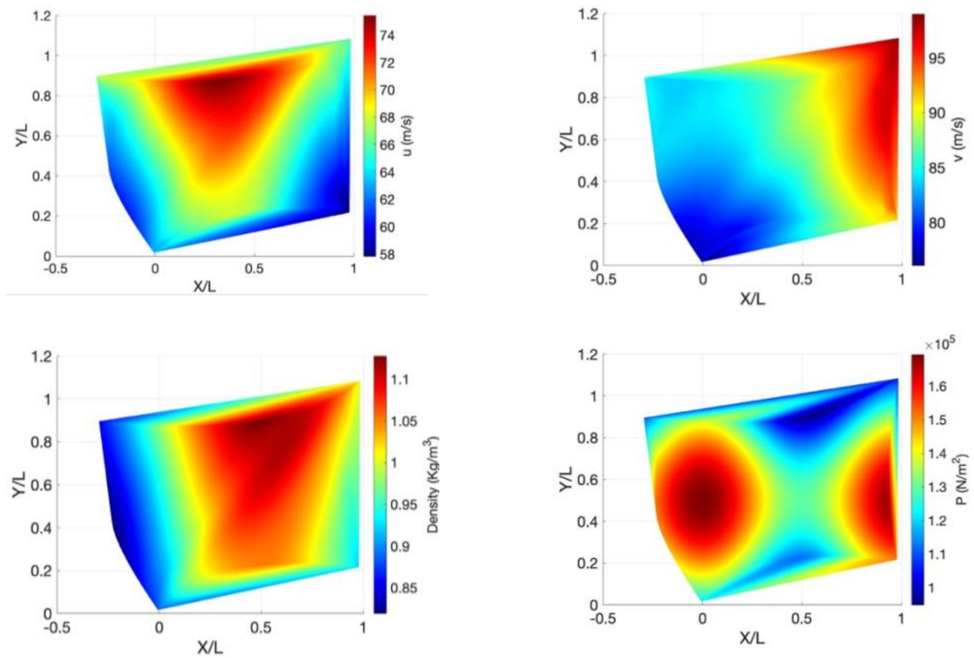


Figure 2 solution of the primitive variables for 2D Euler equations with manufactured solution source term

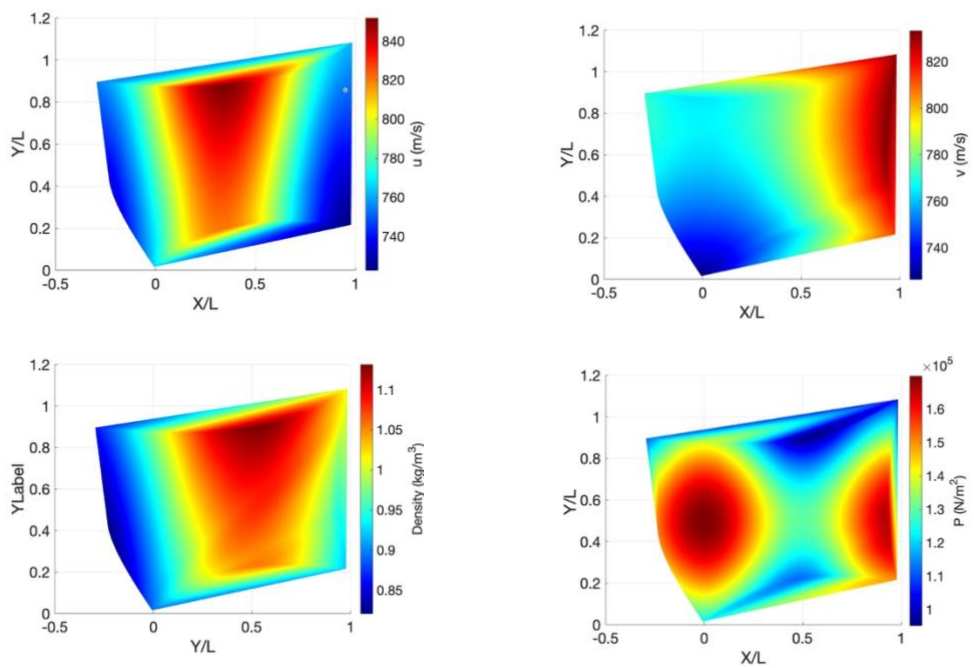


Figure 1 solution of the primitive variables for 2D Euler equations with manufactured solution source term

The order of accuracy has been calculated first using Van leer flux limiter but it turns out that it doesn't produce the desired second order.

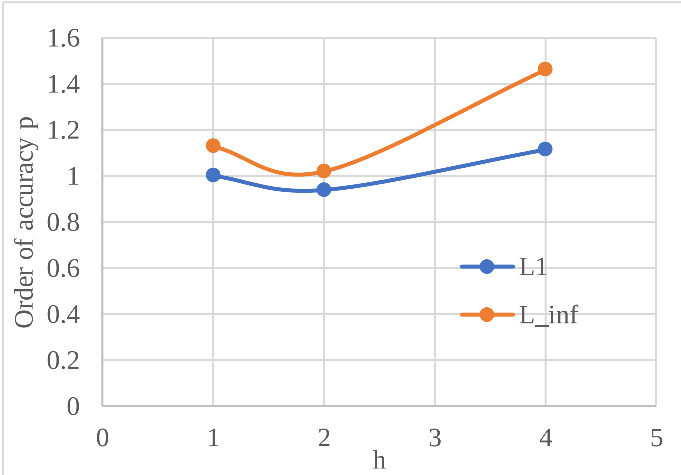


Figure 3 Order of accuracy calculated by L1 and Lmax of the mass equation norms using Van leer flux limiter for subsonic flow

The calculations have been carried out again by utilizing minmod flux limiter which produced the correct order of accuracy (2nd order). The following figure shows the results using Roe's method and Van leer method for a subsonic flow field.

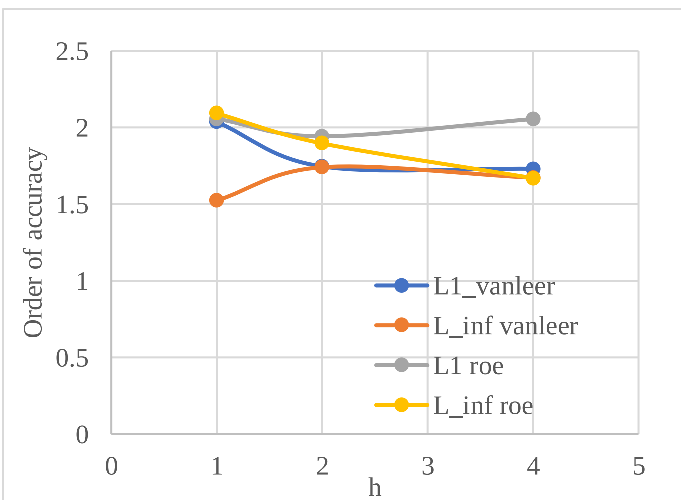


Figure 4 Order of accuracy calculated by L1 and Lmax of the mass equation norms using Van leer flux limiter for subsonic flow

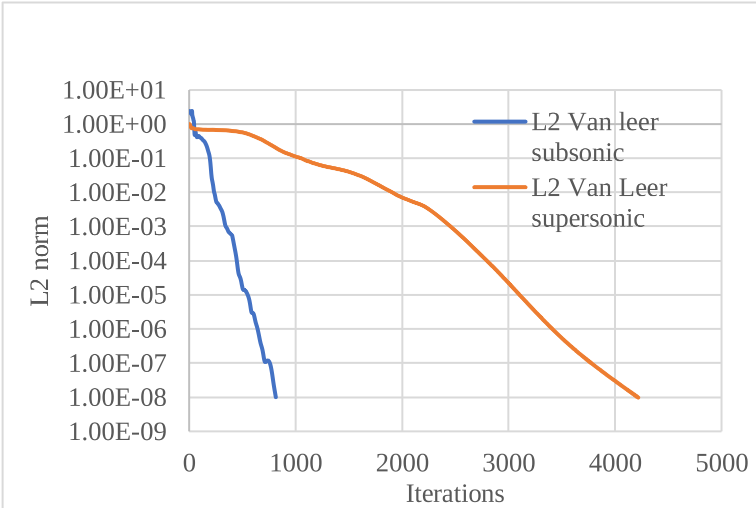


Figure 5 iterative convergence of mass equation using L2 norm for supersonic and subsonic solutions

Iterative convergence shows faster convergence for the subsonic case.

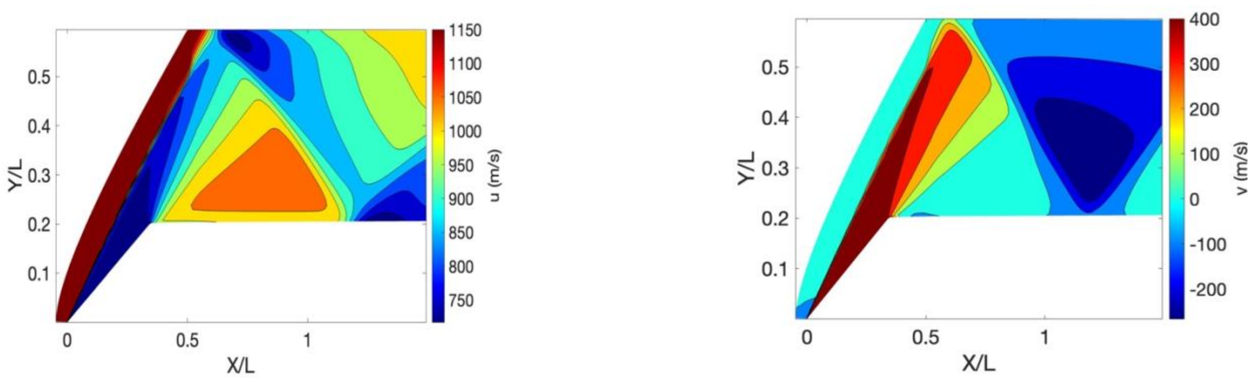
V. Study case

A study case is presented in this section for a 2-D nozzle with 30˚ inlet velocity. The problem is solved on a 209x65 grid nodes using 2nd order Roe's Reimann solver method. The inlet boundary is the left face at 30˚ from y-axis while the outlet is the right face parallel to the y-axis. The free stream conditions are shown below.

Table 2 free stream conditions

Mach number	Pressure	Temperature
4.0	12270 Pa	217 K

Since the flow is supersonic, the inlet flow values have been calculated from the boundary conditions (free stream values). For the outlet boundary condition, the primitive variables at the boundary have been extrapolated from the solution then flux is calculated. Walls boundary condition has been implemented at the rest of the boundary faces by setting the convective velocity in the flux term to zero. MUSCLE extrapolation with ghost cells implementation has been used to achieve 2nd order accuracy at the outlet boundary. The following figures show the solution of the flow field for the nozzle geometry.



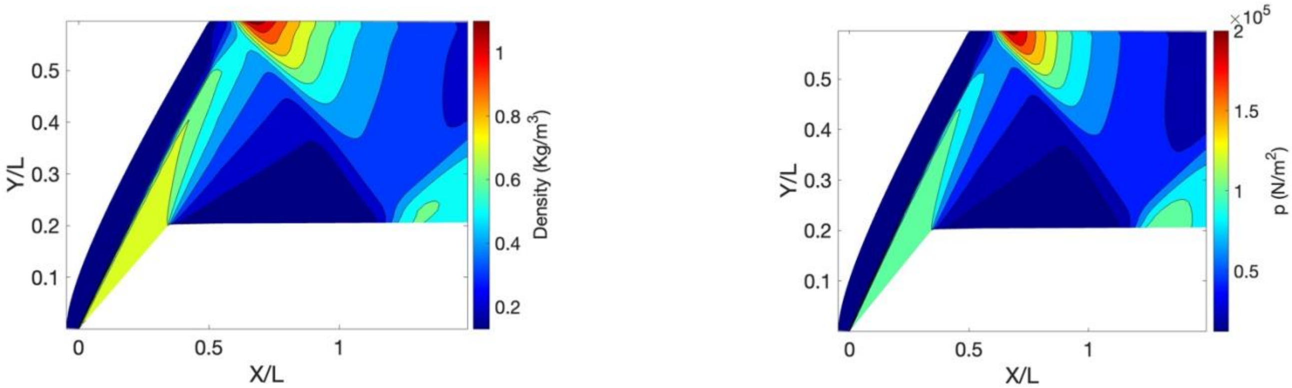


Figure 6 2-D Euler solution for a 2-D nozzle with 30° inlet velocity

The pressure and the density seem to be almost constant aside from the sharp gradient at the north part of the wall. The pressure loss has been calculated by subtracting the average of the outlet pressure from the average of the inlet pressure. The calculated pressure loss = $\frac{1}{H}\int_{y\,min}^{y\,max}(p_{0,\infty} - p_{exit})$ =185691 Pa, where H is the length of the exit boundary.

VI. Conclusion

This project examined the solution for 2-D compressible Euler equations using Van leer and Roe schemes. Second order accuracy has been achieved using MUSCL extrapolation and minmod flux limiter. The code has been verified using method of manufactured solution for supersonic and subsonic flow. A study case for a 2-D nozzle has been solved for the flow field and the pressure loss has been calculated.

Appendix

A. Manufactured solution source terms

For Mass equation:

$$\frac{3\pi uvelx\text{Cos}\left[\frac{3\pi x}{2L}\right]\left(rho\theta + rhoy\text{Cos}\left[\frac{\pi y}{2L}\right] + rhox\text{Sin}\left[\frac{\pi x}{L}\right]\right)}{2L} + \frac{2\pi vvely\text{Cos}\left[\frac{2\pi y}{3L}\right]\left(rho\theta + rhoy\text{Cos}\left[\frac{\pi y}{2L}\right] + rhox\text{sin}\left[\frac{\pi x}{L}\right]\right)}{3L} + \frac{\pi rhox\text{Cos}\left[\frac{\pi x}{L}\right]\left(uvel\theta + uvely\text{Cos}\left[\frac{3\pi y}{5L}\right] + uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)}{L} - \frac{\pi rhoy\text{Sin}\left[\frac{\pi y}{2L}\right]\left(vvel\theta + vvelx\text{Cos}\left[\frac{\pi x}{2L}\right] + vvely\text{sin}\left[\frac{2\pi y}{3L}\right]\right)}{2L}$$

For X-momentum:

$$\frac{1}{L}\left(3\pi\ \text{uvelx Cos}\left[\frac{3\pi x}{2L}\right]\left(\text{rho} + \text{rhoy Cos}\left[\frac{\pi y}{2L}\right] + \text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\left(\text{uvelo} + \text{uvely Cos}\left[\frac{3\pi y}{5L}\right] + uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)\right) + \frac{1}{3L}\left(2\pi vvely\text{Cos}\left[\frac{2\pi y}{3L}\right]\left(\text{rho} + \text{rhoy Cos}\left[\frac{\pi y}{2L}\right] + \text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\left(uvel0 + uvely\text{Cos}\left[\frac{3\pi y}{5L}\right] + uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)\right) + \frac{\pi\text{rhox Cos}\left[\frac{\pi x}{L}\right]\left(uvel0 + uvely\text{Cos}\left[\frac{3\pi y}{5L}\right] + uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)^2}{L} - \frac{2\pi pressx\text{Sin}\left[\frac{2\pi x}{L}\right]}{L} - \frac{1}{2L}\left(\pi\ \text{rhoy}\left(\text{uvel0} + \text{uvely Cos}\left[\frac{3\pi y}{5L}\right] + uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)\text{Sin}\left[\frac{\pi y}{2L}\right]\left(vvel\theta + vvelx\text{Cos}\left[\frac{\pi x}{2L}\right] + vvely\text{Sin}\left[\frac{2\pi y}{3L}\right]\right)\right) - \frac{1}{5L}\left(3\pi\ \text{uvely}\left(\text{rho} + \text{rhoy Cos}\left[\frac{\pi y}{2L}\right] + \text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\text{Sin}\left[\frac{3\pi y}{5L}\right]\left(\text{vel}\ \theta + \text{vel Le Cos}\left[\frac{\pi x}{2L}\right] + vvely\text{Sin}\left[\frac{2\pi y}{3L}\right]\right)\right)$$

Y-momentum:

$$\frac{\pi\ \text{pressy Cos}\left[\frac{\pi y}{L}\right]}{L}-$$

$$\frac{1}{2L}\left(\pi vvelx\text{Sin}\left[\frac{\pi x}{2L}\right]\left(\text{rho0}+\ \text{rhoy Cos}\left[\frac{\pi y}{2L}\right]+\text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\left(uvel0+\ \text{uvely Cos}\left[\frac{3\pi y}{5L}\right]+uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)\right)+$$
$$\frac{1}{2L}\left(3\pi uvelx\text{Cos}\left[\frac{3\pi x}{2L}\right]\left(\text{rho}+\text{rhoy Cos}\left[\frac{\pi y}{2L}\right]+\text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\left(vvel0+vvelx\text{Cos}\left[\frac{\pi x}{2L}\right]+vvely\text{Sin}\left[\frac{2\pi y}{3L}\right]\right)\right)+$$
$$\frac{1}{3L}\left(4\pi\ \text{vvely Cos}\left[\frac{2\pi y}{3L}\right]\left(\text{rho}\ \theta+\text{rhoy Cos}\left[\frac{\pi y}{2L}\right]+\text{rhox Sin}\left[\frac{\pi x}{L}\right]\right)\left(\text{vvelo}+\text{vel}\ x\times\text{Cos}\left[\frac{\pi x}{2L}\right]+vvely\text{Sin}\left[\frac{2\pi y}{3L}\right]\right)\right)+$$
$$\frac{1}{L}\left(\pi\text{rhox Cos}\left[\frac{\pi x}{L}\right]\left(uvel0+uvely\text{Cos}\left[\frac{3\pi y}{5L}\right]+uvelx\text{Sin}\left[\frac{3\pi x}{2L}\right]\right)\left(vvel0+vvelx\text{Cos}\left[\frac{\pi x}{2L}\right]+vvely\text{Sin}\left[\frac{2\pi y}{3L}\right]\right)\right)-$$
$$\frac{\pi\ \text{rhoy Sin}\left[\frac{\pi y}{2L}\right]\left(\text{vel}\ \theta+\text{vel}\ lx\text{Cos}\left[\frac{\pi x}{2L}\right]+\text{vely Sin}\left[\frac{2\pi y}{3L}\right]\right)^2}{2L}$$

Energy:

$$\frac{3\pi\text{uvelxcos}\left(\frac{3\pi x}{2L}\right)\left(\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(\frac{\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)}{(\text{gamma}-1)\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\text{wvel0}^2+\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{2L}$$

$$+\frac{2\pi\text{vvelycos}\left(\frac{2\pi y}{3L}\right)\left(\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(\frac{\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)}{(\text{gamma}-1)\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\text{wvel0}^2+\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{3L}$$

$$+\Bigg(\frac{\pi\text{rhoxcos}\left(\frac{\pi x}{L}\right)\left(\frac{\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)}{(\text{gamma}-1)\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\text{wvel0}^2+\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{L}$$

$$+\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\Bigg(-\frac{2\pi\text{pressxsin}\left(\frac{2\pi x}{L}\right)}{(\text{gamma}-1)L\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}$$

$$+\frac{1}{2}\left(\frac{3\pi\text{uvelxcos}\left(\frac{3\pi x}{2L}\right)\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)}{L}-\frac{\pi\text{vvelxsin}\left(\frac{\pi x}{2L}\right)\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)}{L}\right)-\frac{\pi\text{rhoxcos}\left(\frac{\pi x}{L}\right)\left(\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)\right)}{(\text{gamma}-1)L\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)^2}\Bigg)$$

$$-\frac{2\pi\text{pressxsin}\left(\frac{2\pi x}{L}\right)}{L}\Bigg)\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)$$

$$+\Bigg(\frac{\pi\text{pressycos}\left(\frac{\pi y}{L}\right)}{L}-\frac{\pi\text{rhoysin}\left(\frac{\pi y}{2L}\right)\left(\frac{\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)}{(\text{gamma}-1)\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\text{wvel0}^2+\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{2L}$$

$$+\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\Bigg(\frac{\pi\text{pressycos}\left(\frac{\pi y}{L}\right)}{(\text{gamma}-1)L\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{\pi\text{rhoysin}\left(\frac{\pi y}{2L}\right)\left(\text{press0}+\text{pressxcos}\left(\frac{2\pi x}{L}\right)+\text{pressysin}\left(\frac{\pi y}{L}\right)\right)}{2(\text{gamma}-1)L\left(\text{rho0}+\text{rhoycos}\left(\frac{\pi y}{2L}\right)+\text{rhoxsin}\left(\frac{\pi x}{L}\right)\right)^2}$$

$$+\frac{1}{2}\left(\frac{4\pi\text{vvelycos}\left(\frac{2\pi y}{3L}\right)\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)}{3L}-\frac{6\pi\text{uvely}\left(\text{uvel0}+\text{uvelycos}\left(\frac{3\pi y}{5L}\right)+\text{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)\sin\left(\frac{3\pi y}{5L}\right)}{5L}\right)\Bigg)\Bigg)\left(\text{vvel0}+\text{vvelxcos}\left(\frac{\pi x}{2L}\right)+\text{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)$$

$$\frac{3\pi \mathrm{uvelxcos}\left(\frac{3\pi x}{2L}\right)\left(\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(\frac{\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)}{(\mathrm{gamma}-1)\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\mathrm{wvel0}^2+\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)\right)}{2L}$$

$$+\frac{2\pi \mathrm{vvelycos}\left(\frac{2\pi y}{3L}\right)\left(\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(\frac{\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)}{(\mathrm{gamma}-1)\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\mathrm{wvel0}^2+\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)\right)}{3L}$$

$$+\left(\frac{\pi \mathrm{rhoxcos}\left(\frac{\pi x}{L}\right)\left(\frac{\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)}{(\mathrm{gamma}-1)\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\mathrm{wvel0}^2+\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{L}\right.$$

$$+\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(-\frac{2\pi \mathrm{pressxsin}\left(\frac{2\pi x}{L}\right)}{(\mathrm{gamma}-1)L\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}\right.$$

$$\left.+\frac{1}{2}\left(\frac{3\pi \mathrm{uvelxcos}\left(\frac{3\pi x}{2L}\right)\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)}{L}-\frac{\pi \mathrm{vvelxsin}\left(\frac{\pi x}{2L}\right)\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)}{L}\right)-\frac{\pi \mathrm{rhoxcos}\left(\frac{\pi x}{L}\right)\left(\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)\right)}{(\mathrm{gamma}-1)L\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)^2}\right)$$

$$\left.-\frac{2\pi \mathrm{pressxsin}\left(\frac{2\pi x}{L}\right)}{L}\right)\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)$$

$$+\left(\frac{\pi \mathrm{pressycos}\left(\frac{\pi y}{L}\right)}{L}-\frac{\pi \mathrm{rhoysin}\left(\frac{\pi y}{2L}\right)\left(\frac{\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)}{(\mathrm{gamma}-1)\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{1}{2}\left(\mathrm{wvel0}^2+\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)^2+\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)^2\right)\right)}{2L}\right.$$

$$+\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)\left(\frac{\pi \mathrm{pressycos}\left(\frac{\pi y}{L}\right)}{(\mathrm{gamma}-1)L\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)}+\frac{\pi \mathrm{rhoysin}\left(\frac{\pi y}{2L}\right)\left(\mathrm{press0}+\mathrm{pressxcos}\left(\frac{2\pi x}{L}\right)+\mathrm{pressysin}\left(\frac{\pi y}{L}\right)\right)}{2(\mathrm{gamma}-1)L\left(\mathrm{rho0}+\mathrm{rhoycos}\left(\frac{\pi y}{2L}\right)+\mathrm{rhoxsin}\left(\frac{\pi x}{L}\right)\right)^2}\right.$$

$$\left.\left.+\frac{1}{2}\left(\frac{4\pi \mathrm{vvelycos}\left(\frac{2\pi y}{3L}\right)\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)}{3L}-\frac{6\pi \mathrm{uvely}\left(\mathrm{uvel0}+\mathrm{uvelycos}\left(\frac{3\pi y}{5L}\right)+\mathrm{uvelxsin}\left(\frac{3\pi x}{2L}\right)\right)\sin\left(\frac{3\pi y}{5L}\right)}{5L}\right)\right)\right)\left(\mathrm{vvel0}+\mathrm{vvelxcos}\left(\frac{\pi x}{2L}\right)+\mathrm{vvelysin}\left(\frac{2\pi y}{3L}\right)\right)$$

The general form of the manufactured solution is:

$$\rho(x,y)=\rho_0+\rho_x\sin\left(\frac{a_{\rho x}\pi x}{L}\right)+\rho_y\cos\left(\frac{a_{\rho y}\pi y}{L}\right)$$
$$u(x,y)=u_0+u_x\sin\left(\frac{a_{ux}\pi x}{L}\right)+u_y\cos\left(\frac{a_{uy}\pi y}{L}\right)$$
$$v(x,y)=v_0+v_x\cos\left(\frac{a_{vx}\pi x}{L}\right)+v_y\sin\left(\frac{a_{vy}\pi y}{L}\right)$$

$$p(x,y)=p_0+p_x\cos\left(\frac{a_{px}\pi x}{L}\right)+p_y\sin\left(\frac{a_{py}\pi y}{L}\right)$$

Values for the constant for supersonic and subsonic flows can be found in [1].

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