

Space-bounded quantum state testing
via space-efficient quantum singular value transformation

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What is quantum state testing

Task: Quantum state testing (with two-sided error).

Given two quantum devices Q_0 and Q_1 that prepare $\text{poly}(n)$ -qubit quantum (mixed) states $\rho_0 \in \mathbb{C}^{N \times N}$ and $\rho_1 \in \mathbb{C}^{N \times N}$, respectively, which may be viewed as “sample access” to ρ_0 and ρ_1 . Decide whether $\text{dist}(\rho_0, \rho_1) \leq \epsilon_1$ or $\text{dist}(\rho_0, \rho_1) \geq \epsilon_2$.

The one-sided error variant and the classical counterpart are as follows:

► **Quantum state certification** [Bădescu-O'Donnell-Wright'19]:

Given “sample access” to ρ_0 and ρ_1 , decide whether $\rho_0 = \rho_1$ or $\text{dist}(\rho_0, \rho_1) \geq \epsilon$.

► **Distribution testing** (a.k.a. closeness testing of distributions, see [Canonne'20]):

Given sample accesses to probability distributions D_0 and D_1 , decide whether $\text{dist}(D_0, D_1) \leq \epsilon_1$ or $\text{dist}(D_0, D_1) \geq \epsilon_2$.

Typical goal: Minimize the number of copies (*sample complexity*) of ρ_0 and ρ_1 .

In this work: Viewing quantum state testing as a computational (promise) problem.

Quantum state testing w.r.t. various distance-like measures

Classical and quantum distance-like measures that are considered:

	Quantum	Classical
ℓ_1 norm	trace distance $\text{td}(\rho_0, \rho_1) := \frac{1}{2} \rho_0 - \rho_1 $	total variation distance (a.k.a. statistical distance)
ℓ_2 norm	Hilbert-Schmidt distance $\text{HS}^2(\rho_0, \rho_1) := \frac{1}{2} (\rho_0 - \rho_1)^2$	Euclidean distance
Entropy	von Neumann entropy $S(\rho) := -\text{Tr}(\rho \ln \rho)$	Shannon entropy
Jensen-Shannon divergence	Quantum Jensen-Shannon divergence $\text{QJS}_2(\rho_0, \rho_1) := S_2\left(\frac{\rho_0 + \rho_1}{2}\right) - \frac{S_2(\rho_0) + S_2(\rho_1)}{2}$ where $S_2(\rho) := -\text{Tr}(\rho \log_2 \rho)$ [Majtey-Lamberti-Prato'05, Briët-Harremoës'09]	Jensen-Shannon divergence

Remark. Quantum Jensen-Shannon divergence can be viewed as a *distance version* of the quantum (von Neumann) entropy difference.

Main result: Space-bounded state certification (one-sided error scenario)

Task 1.1 (Space-bounded quantum state certification). Given two *polynomial-size* $O(\log n)$ -qubit quantum circuits Q_0 and Q_1 that prepare $O(\log n)$ -qubit quantum (mixed) states ρ_0 and ρ_1 , respectively, with access to their “source codes”.

Decide whether $\rho_0 = \rho_1$ or $\text{dist}(\rho_0, \rho_1) \geq \alpha$.

Theorem 1.2 (Space-bounded quantum state certification is coRQ_{UL} -complete).

The following space-bounded quantum state certification problems are coRQ_{UL} -complete. For any $\alpha(n) \geq 1/\text{poly}(n)$, decide whether

- ① $\overline{\text{CERTQSD}}_{\log}$: $\rho_0 = \rho_1$ or $\text{td}(\rho_0, \rho_1) \geq \alpha(n)$;
- ② $\overline{\text{CERTQHS}}_{\log}$: $\rho_0 = \rho_1$ or $\text{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$.

Remark. coRQ_{UL} is a complexity class with *perfect completeness*, namely the acceptance probability $p_{\text{acc}} = 1$ for yes instances whereas $p_{\text{acc}} \leq 1/2$ for no instances.

Main result: Space-bounded quantum state testing (two-sided error scenario)

Task 1.3 (Space-bounded quantum state testing). Given two *polynomial-size* $O(\log n)$ -qubit quantum circuits Q_0 and Q_1 that prepare $O(\log n)$ -qubit quantum (mixed) states ρ_0 and ρ_1 , respectively, with access to their “source codes”. Decide whether $\text{dist}(\rho_0, \rho_1) \leq \beta$ or $\text{dist}(\rho_0, \rho_1) \geq \alpha$.

Theorem 1.4 (Space-bounded quantum state testing is BQL-complete). The following (log)space-bounded quantum state testing problems are BQL-complete. For any α, β s.t. $\alpha(n) - \beta(n) \geq 1/\text{poly}(n)$ or any $g(n) \geq 1/\text{poly}(n)$, decide whether

- ① GAPQSD_{log}: $\text{td}(\rho_0, \rho_1) \geq \alpha$ or $\text{td}(\rho_0, \rho_1) \leq \beta$;
- ② GAPQED_{log}: $S(\rho_0) - S(\rho_1) \geq g$ or $S(\rho_1) - S(\rho_0) \geq g$;
- ③ GAPQJS_{log}: $\text{QJS}_2(\rho_0, \rho_1) \geq \alpha$ or $\text{QJS}_2(\rho_0, \rho_1) \leq \beta$;
- ④ GAPQHS_{log}: $\text{HS}^2(\rho_0, \rho_1) \geq \alpha$ or $\text{HS}^2(\rho_0, \rho_1) \leq \beta$.

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BQL and BQ_UL: Two-sided error space-bounded quantum computation

BQL (and BQ_UL if only allow *unitary* gates), defined in [Watrous'99, Watrous'03], captures quantum computation that performs by a *logspace-uniform* $O(\log n)$ -qubit quantum circuit. This class admits the following properties:

- ▶ Quadratic advantage in space (?): $\text{BQL} \subseteq \text{DSPACE}[\log^2(n)]$ [Wat99, Wat03].
- ▶ Gateset-indep.: Space-efficient Solovay-Kitaev theorem [van Melkebeek-Watson'12].
- ▶ Error reduction for BQ_UL [Fefferman-Kobayashi-Lin-Morimae-Nishimura'16].
- ▶ Intermediate measurements are useless: $\text{BQL} = \text{BQ}_U\text{L}$ [Fefferman-Remscrem'21].

History of the only family of (natural) BQL-complete problem:

- ▶ Inverting a well-conditioned matrix is in BQL [Ta-Shma'13], whereas it only has $\text{DSPACE}[\log^2(n)]$ containment without the help of quantum.
- ▶ Inverting a well-conditioned matrix is BQ_UL-complete [Fefferman-Lin'18].
- ▶ A well-conditioned version of DET*-complete problems are BQL-complete [Fefferman-Remscrem'21], such as well-conditioned integer determinant, well-conditioned matrix powering, well-conditioned iterative matrix product.

Takeaway. This work (Theorem 1.4) presents a *new family* of natural BQL-complete problems that emerge from quantum property testing.

RQ_{UL} and coRQ_{UL}: One-sided error space-bounded quantum computation

RQ_{UL} and coRQ_{UL}, defined in [Watrous'01], capture *one-sided error* quantum computation that performs by a *logspace-uniform* $O(\log n)$ -qubit quantum circuits. These classes admit the following properties:

- ▶ Error reduction for RQ_{UL} and coRQ_{UL} [Watrous'01].
- ▶ Gateset-dependence which is because of perfect completeness or soundness.
- ▶ Undirected graph connectivity (USTCON) is in $\text{RQ}_{\text{UL}} \cap \text{coRQ}_{\text{UL}}$ [Watrous'01], although USTCON is actually in L [Reingold'08] .

Open problems on RQ_{UL} and coRQ_{UL}:

- ▶ A (natural) complete problem for the class RQ_{UL} or coRQ_{UL} remains *unknown*. A “verification” version of well-conditioned iterative matrix product problem is coRQL-hard [Fefferman-Remscrem'21] while there is no containment (*hard direction*).
- ▶ $\text{RQ}_{\text{UL}} \stackrel{?}{=} \text{RQL}$ and $\text{coRQ}_{\text{UL}} \stackrel{?}{=} \text{coRQL}$.

Takeaway. This work (Theorem 1.2) demonstrates *the first family* of natural coRQ_{UL}-complete problems that arise from quantum property testing as well.

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Time-bounded quantum state testing: ℓ_1 norm scenario

Task 3.1 (Time-bounded quantum state testing). Given two *polynomial-size* quantum circuits Q_0 and Q_1 that prepare $\text{poly}(n)$ -qubit quantum (mixed) states ρ_0 and ρ_1 , respectively, with access to their “source codes”.
Decide whether $\text{dist}(\rho_0, \rho_1) \leq \beta$ or $\text{dist}(\rho_0, \rho_1) \geq \alpha$.

Time-bounded distribution testing. Given two *efficiently samplable* distributions D_0 and D_1 (prepared by circuits), decide whether $\text{dist}(D_0, D_1) \leq \beta$ or $\text{dist}(D_0, D_1) \geq \alpha$.

Computational hardness of these tasks with respect to ℓ_1 norm:

- ▶ Statistical Difference Problem (SDP) is SZK-complete
when *constant* $\alpha^2 - \beta > 0$ [Sahai-Vadhan'03, Goldreich-Sahai-Vadhan'98].
- ▶ Quantum State Distinguishability Problem (QSDP) is QSZK-complete
when *constant* $\alpha^2 - \beta > 0$ [Watrous'02, Watrous'09].
- ▶ Open problem: (α, β) -QSDP is in QSZK when $\alpha(n) - \beta(n) \geq 1/\text{poly}(n)$.
Recent progress: [Berman-Degwekar-Rothblum-Vasudevan'19] (classical) and [Liu'23] (quantum).

Structural complexity-theoretic results regarding QSZK:

- ▶ $\text{BQP} \subseteq \text{QSZK} \subseteq \text{QIP}(2) \subseteq \text{PSPACE}$ [Watrous'02, Watrous'09].
- ▶ $\exists \mathcal{O}$ s.t. $\text{QSZK}^{\mathcal{O}} \not\subseteq \text{PP}^{\mathcal{O}}$ [Bouland-Chen-Holden-Thaler-Vasudevan'19].

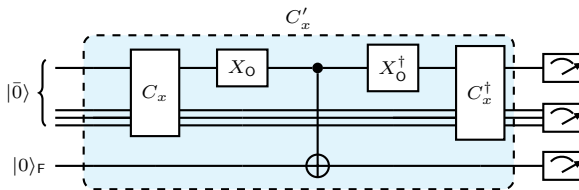
Time-bounded quantum state testing: ℓ_2 norm scenario

Proposition 3.2 [BCWdW01, RASW23]. Quantum Hilbert-Schmidt distance problem, namely time-bounded quantum state testing w.r.t. ℓ_2 norm, is BQP-complete.

BQP containment. As all three terms in $\text{HS}^2(\rho_0, \rho_1) = \frac{1}{2} \text{Tr}(\rho_0^2) + \frac{1}{2} \text{Tr}(\rho_1^2) - \text{Tr}(\rho_0 \rho_1)$ can be estimated by the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01], we have a hybrid algorithm succeeds w.p. $\frac{1}{2} + \frac{1}{2} \text{HS}^2(\rho_0, \rho_1)$:

- ❶ Toss two random coins $r_0, r_1 \in \{0, 1\}$;
- ❷ Perform the SWAP test on quantum states according to r_0 and r_1 .

BQP hardness (adapted from [Rethinasamy-Agarwal-Sharma-Wilde'23]). Consider a BQP circuit C_x , we can construct $C'_x := C_x^\dagger X_O^\dagger \text{CNOT}_{O \rightarrow F} X_O C_x$ with an ancillary qubit on F such that $\Pr[C'_x \text{ accepts}] = \|(|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F) C'_x (|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F)\|_2^2 = \Pr^2[C_x \text{ accepts}]$.



By defining two pure states $\rho_0 := |\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F$ and $\rho_1 := C'_x (|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F) (C'_x)^\dagger$, we have $\Pr[C'_x \text{ accepts}] = \text{Tr}(\rho_0 \rho_1) = 1 - \text{HS}^2(\rho_0, \rho_1)$. \square

Computational hardness of time-bounded testing depends on distance

Computational hardness of Task 3.1: a “dichotomy” theorem

- ▶ Time-bounded state testing w.r.t. ℓ_1 norm or entropy difference (QSZK-complete) is seemingly *much harder* than only preparing these states (in BQP).
- ▶ Time-bounded state testing w.r.t. ℓ_2 norm (BQP-complete) is computationally as *easy as* just preparing these states (in BQP).

Interestingly, the computational hardness “dichotomy” is linked to the dependence of the sample complexity for distribution testing and state testing on *the dimension N* :

	ℓ_1 norm	ℓ_2 norm	Entropy
Classical sample complexity	$\text{poly}(N, 1/\epsilon)$ [CDVV14]	$\text{poly}(1/\epsilon)$ [CDVV14]	$\text{poly}(N, 1/\epsilon)$ [JVHW15, WY16]
Quantum sample complexity	$\text{poly}(N, 1/\epsilon)$ [BOW19]	$\text{poly}(1/\epsilon)$ [BOW19]	$\text{poly}(N, 1/\epsilon)$ [AISW20, OW21]

Summary: Time-bounded and space-bounded quantum state testing

Computational hardness of time-bounded and space-bounded quantum state testing:

	ℓ_1 norm	ℓ_2 norm	Entropy
Classical Time-bounded	SZK-complete* [SV03,GSV98]	BPP-complete Folklore	SZK-complete [GV99,GSV98]
Quantum Time-bounded	QSZK-complete* [Wat02,Wat09]	BQP-complete [BCWdW01, RASW23]	QSZK-complete [BASTS10]
Classical Space-bounded	BPL-hard [†]	BPL-complete [†] Folklore	BPL-complete [†] Implied by [ABIS19]
Quantum Space-bounded	BQL-complete This work	BQL-complete [BCWdW01] and this work	BQL-complete This work

Remark[†]. Space-bounded distribution testing can be viewed as a “white-box” version of *streaming distribution testing* with i.i.d. samples.

Takeaways. For space-bounded state testing and certification problems, the computational hardness of these problems is as easy as just preparing quantum states, which is *independent of the choice* of aforementioned distance-like measures.

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Quantum singular value transformation in a nutshell

QSVT [Gilyén-Su-Low-Wiebe'19] is a systematic approach to (time-efficiently) *manipulating singular values* $\{\sigma_i\}_i$ of an Hermitian matrix A using a corresponding projected unitary encoding $A = \tilde{\Pi}U\Pi$ for orthogonal projectors $\tilde{\Pi}$ and Π .

Quantum singular value transformation, revisited

Given a singular value decomposition $A = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$ associated with an $s(n)$ -qubit *projected unitary encoding*, we can *approximately* implement a QSVT $f^{(\text{SV})}(A) = \sum_i f(\sigma_i) |\tilde{\psi}_i\rangle\langle\psi_i|$ by employing a polynomial P_d of degree $d = O\left(\frac{1}{\delta} \log \frac{1}{\epsilon}\right)$ satisfying that

- ▶ P_d well-approximates f on the interval of interest \mathcal{I} :
 $\max_{x \in \mathcal{I} \setminus \mathcal{I}_\delta} |P_d(x) - f(x)| \leq \epsilon$ where $\mathcal{I}_\delta \subseteq \mathcal{I} \subseteq [-1, 1]$ and typically $\mathcal{I}_\delta := (-\delta, \delta)$.
- ▶ P_d is bounded: $\max_{x \in [-1, 1]} |P_d(x)| \leq 1$.

Moreover, all coefficients of P_d (namely, *classical pre-processing*) can be computed in deterministic $\text{poly}(d)$ time (and thus space). Hence, the transformation $P_d^{(\text{SV})}(A)$ can be implemented by a $\text{poly}(d)$ -size quantum circuit acts on $O(\max\{\log d, s(n)\})$ qubits.

Remark. Quantum circuit implementation in QSVT is already space-efficient!

Space-efficient quantum singular value transformation

Question 4.1 (Space-efficient QSVT). Can we implement a degree- d QSVT for any $s(n)$ -qubit projected unitary encoding with $d \leq 2^{O(s(n))}$, using only $O(s(n))$ space in both **classical pre-processing** and quantum circuit implementation?

Theorem 4.2 (Space-bounded QSVT, [Metger-Yuen'23]). Implement a degree- d QSVT associated with *sign function* or *square-root function* for any $O(\log n)$ qubit *block-encoding* with $d \leq \text{poly}(n)$ requires **$O(\text{poly log } n)$ space for classical pre-processing** and $O(\log n)$ qubits in quantum circuit implementation.

Remark. Theorem 4.2 can be easily extended to continuous functions bounded on $[-1, 1]$.

Theorem 4.3 (Space-efficient QSVT, [This work](#)). Implement a degree- d QSVT associated with *piecewise-smooth functions* for any $O(\log n)$ qubit *bitstring indexed encoding* with $d \leq \text{poly}(n)$ requires **(randomized) $O(\log n)$ space for classical pre-processing** and $O(\log n)$ qubits in quantum circuit implementation. Moreover, the implementation requires $O(d^2 \|\mathbf{c}\|_1)$ uses of U , U^\dagger , $C_\Pi \text{NOT}$, $C_{\bar{\Pi}} \text{NOT}$, among with other gates, where \mathbf{c} is the coeffs of *Chebyshev interpolation polynomial*.

E.g. Normalized log function $\ln_\beta(x) := \frac{2 \ln(1/x)}{2 \ln(2/\beta)}$ on the interval $\mathcal{I} = [\beta, 1]$ for any $\beta \geq 1/\text{poly}(n)$.

Space-efficient quantum singular value transformation: Proof sketch

Bounded functions. We mainly follow the construction in [MY23]:

Near-minimax approximation by Chebyshev interpolation [Powell'67]

For any continuous function $f: [-1, 1] \rightarrow \mathbb{R}$, if there is a degree- d polynomial P_d satisfying $\max_{x \in [-1, 1]} |f(x) - P_d(x)| \leq \epsilon$, then we have a Chebyshev interpolation polynomial $\hat{P}_d := \frac{c_0}{2} + \sum_{k=1}^d c_k T_k$, where $c_k := \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_k(x)}{\sqrt{1-x^2}} dx$ and T_k is the k -th Chebyshev polynomial (of the first kind), such that $\max_{x \in [-1, 1]} |\hat{P}_d(x) - f(x)| \leq O(\epsilon \log d)$.

- 1 Space-efficient QSVT implementation for $T_k^{(\text{SV})}(\tilde{\Pi}U\Pi)$ [GSLW19]
- 2 For any bounded functions, any coefficient c_k is *space-efficiently computable* by the standard numerical integral technique. \rightarrow A careful analysis is required!
- 3 Implement $\hat{P}_d^{(\text{SV})}(\tilde{\Pi}U\Pi)$ from $T_k^{(\text{SV})}(\tilde{\Pi}U\Pi)$ by LCU [Berry-Childs-Cleve-Kothari-Somma'15]
 \rightarrow Query complexity $O(d^2)$ and the operator norm of $\hat{P}_d(\tilde{\Pi}U\Pi)$ is at most $\|c\|_1$
- 4 Renormalizing the resulting (bitstring indexed) encoding $\hat{P}_d^{(\text{SV})}(\tilde{\Pi}U\Pi)$
 \rightarrow Query complexity $O(d^2 \|c\|_1)$ where $\|c\|_1 \leq O(d)$ in general.

Piecewise-smooth functions. We adapt the construction (i.e., a reduction to a linear combination of bounded functions) in [van Apeldoorn-Gilyén-Gribling-de Wolf'20].

📌 We thus reduce the main challenge to *stochastic matrix powering problem*, essential for the BPL vs. L problem [Saks-Zhou'99, Cohen-Doron-Sberlo-Ta-Shma'23, Putterman-Pyre'23]. \square

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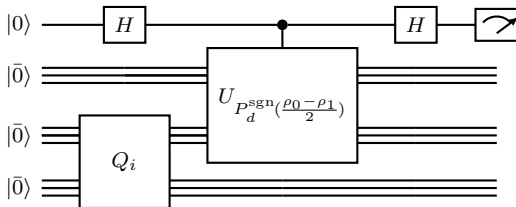
Proof overview: two-sided error scenario

Proof of Theorem 1.4 ①: $\text{GAPQSD}_{\log} \in \text{BQL}$. Inspired by the approach in

[Gilyén-Poremba'22, Wang-Zhang'23], note that $\text{sgn}(x) \approx_{\epsilon, \delta} P_d^{\text{sgn}}(x)$ and

$$\text{td}(\rho_0, \rho_1) = \frac{1}{2} \text{Tr}|\rho_0 - \rho_1| = \frac{1}{2} \left(\text{Tr} \left(\text{sgn}^{(\text{SV})} \left(\frac{\rho_0 - \rho_1}{2} \right) \rho_0 \right) - \text{Tr} \left(\text{sgn}^{(\text{SV})} \left(\frac{\rho_0 - \rho_1}{2} \right) \rho_1 \right) \right).$$

We consider the following quantum tester $\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\text{sgn}})$ where $\rho_- := \frac{\rho_0 - \rho_1}{2}$:



Here, Q_i prepares a purification of the state ρ_i for $i \in \{0, 1\}$, and ρ_- is block-encoded in U_{ρ_-} . We say that the tester \mathcal{T} accepts if the measurement outcome is “0”.

By using the space-efficient QSVT (Theorem 4.3) associated with a bounded approx polynomial P_d^{sgn} of sgn , we implement $U_{P_d^{\text{sgn}}(\frac{\rho_0 - \rho_1}{2})}$. Consequently, the “acceptance probability” of ρ_i is $\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\text{sgn}}) \text{ accepts}] = \frac{1}{2} \left(1 + \text{Tr} \left(P_d^{\text{sgn}} \left(\frac{\rho_0 - \rho_1}{2} \right) \rho_i \right) \right).$

Therefore, for $i \in \{0, 1\}$, it suffices to estimate $\text{Tr} \left(P_d^{\text{sgn}} \left(\frac{\rho_0 - \rho_1}{2} \right) \rho_i \right) \pm \varepsilon$ with high probability using $O(1/\varepsilon^2)$ sequential repetitions. \square

Proof overview: one-sided error scenario

Proof of Theorem 1.2 ①: $\overline{\text{CERTQSD}}_{\log} \in \text{coRQUL}$.

Our construction is mainly based on the previous quantum tester $\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\text{sgn}})$, then achieving perfect completeness by standard techniques.

📌 We first notice that our space-efficient QSVT in Theorem 4.3 preserves the parity. In particular, the QSVT implementation associated with \hat{P}_d^{sgn} satisfies $\hat{P}_d^{\text{sgn}}(\mathbf{0}) = \mathbf{0}$. This enables us to construct the algorithm \mathcal{A} specified below:

- ◇ For yes instances ($\rho_0 = \rho_1$), we thus have $\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\text{sgn}}) \text{ accepts}] = \frac{1}{2}$. Then we obtain an algorithm \mathcal{A} accept with certainty via *exact amplitude amplification* [Boyer-Brassard-Høyer-Tapp'98, Brassard-Høyer-Mosca-Tapp'02].
- ◇ For no instances ($\text{td}(\rho_0, \rho_1) \geq \alpha$), we have

$$|\Pr[\mathcal{T}(Q_i, U_{\rho_-}, P_d^{\text{sgn}}) \text{ accepts}] - \frac{1}{2}| \geq \Omega(\alpha).$$

By a direct (still, a bit complicated) calculation, we can make sure the algorithm \mathcal{A} accepts w.p. at most $1 - \Omega(\alpha^2)$.

Finally, we conclude a coRQUL containment from \mathcal{A} by applying *error reduction* for coRQUL , which can be deduced from our space-efficient QSVT (Theorem 4.3). \square

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Conclusions and open problems

Take-home messages on our work

- ① Space-bounded quantum state certification problems w.r.t. trace distance and Hilbert-Schmidt distance are coRQ_{UL} -complete (Theorem 1.2).
This is the *first* family of natural coRQ_{UL} -complete problem!
- ② Space-bounded quantum state testing problems w.r.t. common distance-like measures (i.e., trace distance, squared Hilbert-Schmidt distance, quantum entropy difference, quantum Jensen-Shannon divergence) are BQL-complete (Theorem 1.4).
- ③ Quantum singular value transformation on bitstring indexed encoding can be done in *quantum logspace*, with a *randomized* classical pre-processing (Theorem 4.3).

Open problems

- ① Space-efficient QSVT with $O(d)$ queries *instead of* $O(d^2\|\mathbf{c}\|_1)$ in Theorem 4.3, as well as make the pre-processing *deterministic rather than randomized*.
- ② Are space-bounded state testings with respect to other (proper) *quantum analogs of symmetric f -divergence* also in BQL?
- ③ Is space-bounded distribution testing problem w.r.t. the *total variation distance* BPL-complete? What about the streaming distribution testing counterpart?

Thanks!