

Quantum state testing beyond the polarizing regime and quantum triangular discrimination

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- ① Quantum state testing and the class QSZK
- ② Main result: quantum state testing beyond the polarizing regime
- ③ Which parameter regime is easy for the class QSZK?
- ④ Open problems

Statistical Difference Problem meets statistical zero-knowledge

Definition 1.1 (Statistical zero-knowledge, informal) An interactive proof protocol admits the (*statistical*) *zero-knowledge* property if verifier's view (\mathcal{P}_0) is (*statistically*) *indistinguishable* from "verifier's view" (\mathcal{P}_1) generated by a poly-time simulator.

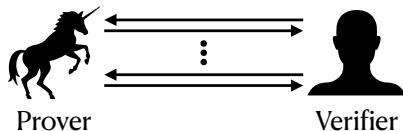


Figure: Verifier's view \mathcal{P}_0

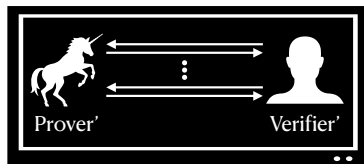


Figure: Simulated "Verifier's view" (\mathcal{P}_1)

- ▶ **Public-coin is sufficient.** All SZK protocols can be transformed into a form that all messages from verifier (V) to prover (P) are *public coins* [Okamoto'00].
- ▶ Intuitively, the views \mathcal{P}_0 and \mathcal{P}_1 can be treated as distributions p_0 and p_1 , respectively. Then *statistical indistinguishability* is on $SD(p_0, p_1) := \frac{1}{2} \|p_0 - p_1\|_1$.

Definition 1.2 (Statistical Difference Problem, SDP) [SV03]. Given efficiently samplable (namely, using polynomial-size Boolean circuits) distributions p_0 and p_1 , decide whether $SD(p_0, p_1) \geq \alpha$ or $SD(p_0, p_1) \leq \beta$.

Theorem 1.3 [Sahai-Vadhan'03, Goldreich-Sahai-Vadhan'98]. Statistical Difference Problem is SZK-complete. Specifically, (α, β) -SDP is in SZK if $\alpha^2 - \beta > 0$ for constant α and β .

From quantum ℓ_1 norm to Quantum State Distinguishability Problem

An n -qubit quantum state ρ is a $2^n \times 2^n$ matrix such that $\text{Tr}(\rho) = 1$ and $\rho \succeq 0$.

Classical and quantum ℓ_1 norms

- ▶ Classical: statistical distance $\text{SD}(p_0, p_1) = \frac{1}{2} \sum_{x \in \mathcal{S}} |p_0(x) - p_1(x)|$.
- ▶ Q (option 1): trace distance $\text{td}(\rho_0, \rho_1) := \frac{1}{2} \text{Tr}|\rho_0 - \rho_1|$.
- ▶ Q (option 2): “measured ℓ_1 distance” $\text{td}^{\text{meas}}(\rho_0, \rho_1) := \sup_{\mathcal{E}} \left\{ \text{SD}(p_0^{(\mathcal{E})}, p_1^{(\mathcal{E})}) \right\}$:
 - ◇ measurement $\mathcal{E} = (E_1, \dots, E_{2^n})$ such that $\sum_i E_i = I$ and $E_i \succeq 0 \ \forall i$;
 - ◇ induced distribution $p_k^{(\mathcal{E})} = (\text{Tr}(\rho_k E_1), \dots, \text{Tr}(\rho_k E_{2^n}))$ for $k \in \{0, 1\}$.

Theorem 1.4 [Helstrom'76]. For any ρ_0 and ρ_1 , $\text{td}(\rho_0, \rho_1) = \text{td}^{\text{meas}}(\rho_0, \rho_1)$.

Definition 1.5 (Quantum State Distinguishability Problem, QSDP) [Wat02]. Given efficiently preparable (namely, using polynomial-size quantum circuits) quantum states ρ_0 and ρ_1 , decide whether $\text{td}(\rho_0, \rho_1) \geq \alpha$ or $\text{td}(\rho_0, \rho_1) \leq \beta$.

Theorem 1.6 [Watrous'02, Watrous'09]. QUANTUM STATE DISTINGUISHABILITY PROBLEM is QSZK-complete. Specifically, (α, β) -QSDP is in QSZK if $\alpha^2 - \beta > 0$ for const α and β .

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Polarization lemma: SZK and QSZK containments

Polarization lemma [Sahai-Vadhan'03]

There is an efficient algorithm to construct p'_k for $k \in \{0,1\}$ such that

- ◇ Yes: $\text{SD}(p_0, p_1) \geq \alpha$;
- ◇ Yes: $\text{SD}(p'_0, p'_1) \geq 1 - \epsilon$;
- ◇ No: $\text{SD}(p_0, p_1) \leq \beta$;
- ◇ No: $\text{SD}(p'_0, p'_1) \leq \epsilon$;

where the dimension of p_k is 2^n .

where the dimension of p'_k is $2^n \cdot \ln(1/\epsilon)$.

The construction of (p'_0, p'_1) is based on an appropriate composition of:

- ▶ Direct product lemma (yes instances): $(p_0^{\otimes l}, p_1^{\otimes l})$;
- ▶ XOR lemma (no instances): $\left(2^{-l} \sum_{i_1 \oplus \dots \oplus i_l = 0} p_{i_1} \otimes \dots \otimes p_{i_l}, 2^{-l} \sum_{i_1 \oplus \dots \oplus i_l = 1} p_{i_1} \otimes \dots \otimes p_{i_l}\right)$.

By inspection, the proof techniques in [SV03] can achieve:

Theorem 2.1 [SV03,GSV98]. (α, β) -SDP is in SZK if $\alpha^2(n) - \beta(n) \geq 1/O(\log n)$.

Theorem 2.2 [Wat02,Wat06]. (α, β) -QSDP is in QSZK if $\alpha^2(n) - \beta(n) \geq 1/O(\log n)$.

- ▶ **Q1:** What about the parameter regime $\alpha^2 < \beta < \alpha$?
- ▶ **Q2:** Could we make the promise gap $\alpha^2(n) - \beta(n) \geq 1/\text{poly}(n)$?

Main result on quantum state testing and QSZK

Theorem 2.3 [[Berman-Degwekar-Rothblum-Vasudevan'19](#)]. (α, β) -SDP is in SZK if $\alpha^2(n) - \beta(n) \geq 1/\text{poly}(n)$. Additionally, there are two new SZK-complete problems:

- ◇ JENSEN-SHANNON DIVERGENCE PROBLEM: (α, β) -JSP is in SZK if $\alpha - \beta \geq 1/\text{poly}$.
- ◇ TRIANGULAR DISCRIMINATION PROBLEM: (α, β) -TDP is in SZK if $\alpha - \beta \geq 1/\text{poly}$;

- ▶ Examining *existing approaches* to polarization: TDP \leftrightarrow original polarization lemma [[SV03](#)] and JSP \leftrightarrow entropy extraction based polarization [[GV99](#)];
- ▶ Proof by reductions: Entropy Difference \rightarrow JSP \rightarrow TDP \rightarrow $1/\text{poly}$ -SDP.

Theorem 2.4. (α, β) -QSDP is in QSZK if $\alpha^2 - \sqrt{2\ln 2}\beta \geq 1/\text{poly}$. In addition, there are two new QSZK-complete problems:

- ◇ QUANTUM JENSEN-SHANNON DIVERGENCE PROBLEM:
 (α, β) -QJSP is in QSZK if $\alpha(n) - \beta(n) \geq 1/\text{poly}(n)$.
- ◇ MEASURED QUANTUM TRIANGULAR DISCRIMINATION PROBLEM:
 (α, β) -MEASQTDP is in QSZK if $\alpha(n) - \beta(n) \geq 1/O(\log n)$;

What we need: Quantum counterparts of classical distances...

Quantum analogues of the triangular discrimination

Triangular discrimination: $\text{TD}(p_0, p_1) := \frac{1}{2} \sum_{x \in \mathcal{S}} \frac{(p_0(x) - p_1(x))^2}{p_0(x) + p_1(x)}$.

Quantum analogues of the triangular discrimination

- ◇ Option 1: $\text{QTD}(\rho_0, \rho_1) := \frac{1}{2} \text{Tr} \left((\rho_0 - \rho_1)(\rho_0 + \rho_1)^{-1/2}(\rho_0 - \rho_1)(\rho_0 + \rho_1)^{-1/2} \right)$;
- ◇ Option 2: $\text{QTD}^{\text{meas}}(\rho_0, \rho_1) := \sup_{\text{measurement } \mathcal{E}} \left\{ \text{TD}(p_0^{(\mathcal{E})}, p_1^{(\mathcal{E})}) \right\}$.

Theorem 2.5. Inequalities on quantum analogues of triangular discrimination:

	Classical	Quantum
SD vs. TD	$\text{SD}^2 \leq \text{TD} \leq \text{SD}$ [Topsøe'00]	$\text{td}^2 \leq \text{QTD}^{\text{meas}} \leq \text{QTD} \leq \text{td}$
JS vs. TD	$\frac{1}{2} \text{TD} \leq \text{JS} \leq \ln 2 \cdot \text{TD}$ [Topsøe'00]	$\frac{1}{2} \text{QTD}^2 \leq \text{QJS} \leq \text{QTD}$
H ² vs. TD	$\text{H}^2 \leq \text{TD} \leq 2\text{H}^2$ [Le Cam'86]	$\frac{1}{2} \text{B}^2 \leq \text{QTD}^{\text{meas}} \leq \text{B}^2$ $\frac{1}{2} \text{B}^2 \leq \text{QTD} \leq \text{B}$

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Easy regimes for the class QSZK

Theorem 3.1 (Easy regimes for the class QSZK).

Parameter regimes	$(1 - \epsilon, \epsilon)\text{-}\overline{\text{SDP}}$	$(1 - \epsilon, \epsilon)\text{-}\overline{\text{QSDP}}$
$\epsilon = 0$	in NP Folklore	in BQP This work
$\epsilon \leq 2^{-n/2-1}$ [Boulund-Chen-Holden-Thaler-Vasudevan'19]	in PP	in PP This work
$\epsilon \geq 2^{-n^{1/2-\gamma}}$ for $\gamma \in (0, 1/2)$	SZK-hard Implicitly in [Sahai-Vadhan'03]	QSZK-hard Implicitly in [Watrous'02]

The proof is mainly based on different usages of the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01].

Corollary 3.2. Length-preserving polarization seems impossible unless $\text{QSZK} \subseteq \text{PP}$.

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Conclusions and open problems

Take-home messages

- 1 A classical distance may have *several* quantum counterpart, and a *unique* counterpart will make our life much easier.
- 2 We define two quantum counterparts of the triangular discrimination, and demonstrate inequalities between these distances and other common distances.
- 3 By employing these inequalities, we improve the QSZK containment of QSDP to *non-polarizing regimes* via two new QSZK-complete problems.
- 4 Easy regimes for QSZK indicates that length-preserving polarization seems impossible unless $\text{QSZK} \subseteq \text{PP}$.

Open problems

- 1 Is there any *other applications* of these quantum analogues of the triangular discrimination? For instance, triangular discrimination can be used to improve the communication complexity lower bound of the point chasing problem [Yehudayoff'20].
- 2 Better upper bound for (α, β) -QSDP with $\alpha - \beta \geq 1/\text{poly}$?
The best known bound is PSPACE which is implicitly shown in [Watrous'02].
- 3 Quantum analogue of the set lower bound protocol and better bounds for QSZK?

Thanks!