On estimating the quantum ℓ_α distance

Yupan Liu [1] Qisheng Wang [2]

[1] Graduate School of Mathematics, Nagoya University

[2] School of Informatics, University of Edinburgh

To appear in ESA 2025 & AQIS 2025 (long talk). Available on arXiv:2505.00457.

CQT CS Seminar (online), July 2025

What is quantum state testing

Task: Closeness testing of quantum states

Given two state-preparation circuits Q_0 and Q_1 ("quantum devices") that prepare (the purification of) n-qubit quantum states $\rho_0 \in \mathbb{C}^{N\times N}$ and $\rho_1 \in \mathbb{C}^{N\times N}$, respectively. Decide whether $\mathrm{dist}(\rho_0, \rho_1) \geq a(n)$ or $\mathrm{dist}(\rho_0, \rho_1) \leq b(n)$.

- Classical counterpart (distribution testing) involves probability distributions D_0 and D_1 that are samplable via respective Boolean circuits C_0 and C_1. The task is to decide whether $\mathrm{dist}(D_0, D_1) \geq a(n)$ or $\mathrm{dist}(D_0, D_1) \leq b(n)$.
- Quantum devices Q_b for $b \in \{0,1\}$ can be given either as a query oracle (*black-box model*) or a sequence of $\mathrm{poly}(n)$ elementary quantum gates (*white-box model*).

Typical goal. Minimize the "complexity" of ρ_b (or its corresponding Q_b) for $b \in \{0,1\}$:

Type of query access	Complexity measure
Black-box model	Query complexity (the number of queries)
White-box model	Complexity class

Generalizing the closeness measures via the Schatten α-norm

The most canonical choices of closeness measures are:

- ◇ **Trace distance** $\mathrm{T}(\rho_0,\rho_1) = \frac{1}{2}\mathrm{Tr}(|\rho_0-\rho_1|)$.
- ◇ **Total variation distance** $\mathrm{TV}(D_0,D_1) = \frac{1}{2}\sum_x |D_0(x)-D_1(x)|$.

Generalization. Define the quantum ℓ_α distance via the Schatten norm:

$$\mathrm{T}_\alpha(\rho_0,\rho_1) := \frac{1}{2}\|\rho_0-\rho_1\|_\alpha = \frac{1}{2}\mathrm{Tr}(|\rho_0-\rho_1|^\alpha)^{1/\alpha}.$$

Trace distance ($\alpha=1$). The closeness testing problem in this case is *hard*, with complexity (polynomially) depending on the rank r of the quantum states:

- ▶ The query complexity for estimating $\mathrm{T}(\rho_0,\rho_1)$ to within additive error ε is $\widetilde{O}(r/\varepsilon^2)$ [Wang-Zhang'23] and $\widetilde{\Omega}(r^{1/2})$ [Bun-Kothari-Thaler'17].
- ▶ The promise problem Quantum State Distinguishability ($\mathrm{QSD}[a,b]$) is QSZK-complete* [Watrous'02, Watrous'05], and it is widely believed that BQP $\subsetneq$ QSZK.
  - ◇ The QSZK containment holds only in the *polarizing regime* $a(n)^2-b(n) > 1/O(\log n)$, rather than the *natural regime* $a(n)-b(n) \geq 1/\mathrm{poly}(n)$.
  - ◇ The QSZK containment has recently been slightly improved beyond that in [**L.**'23], while the result is weaker than the classical case [Berman-Degwekar-Rothblum-Vasudevan'19].

Generalizing the closeness measures via the Schatten α-norm (Cont.)

Even $\alpha \in \{2,4,\cdots\}$. The closeness testing problem in this case is *easy*, with complexity *independent* of the rank r of the quantum states:

- The query complexity for estimating $\mathrm{Tr}(\rho_0\rho_1)$ to within additive error ε is $O(1/\varepsilon)$ via the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01].
  - This directly applies to the case $\alpha=2$, since $\mathrm{Tr}\big((\rho_0-\rho_1)^2\big) = \mathrm{Tr}(\rho_0^2)+\mathrm{Tr}(\rho_1^2)-2\mathrm{Tr}(\rho_0\rho_1)$.
  - In the white-box model, the corresponding closeness testing problem is in BQP.
- Similar techniques [Ekert-Alves-Oi-Horodecki-Horodecki-Lwek'02] can estimate $\mathrm{Tr}(\rho_1\rho_2\cdots\rho_k)$ for integer $k>1$, and solve the case of even integers α.

Odd $\alpha=3$. The query complexity of deciding whether $\mathrm{T}_3(\rho_0,\rho_1)=\frac{1}{2}\mathrm{Tr}(|\rho_0-\rho_1|^3)^{1/3}\geq\varepsilon$ or $\rho_0=\rho_1$ is $O(1/\varepsilon^{3/2})$ [Gilyén-Li'19], but this *does not* extend to the estimation task:

- $\mathrm{T}_3(\rho_0,\rho_1)\geq\varepsilon \;\Rightarrow\; \mathrm{Tr}\Big(\big(\frac{\rho_0-\rho_1}{2}\big)^2\big(\frac{\rho_0+\rho_1}{2}\big)\Big)\geq\frac{1}{8}\mathrm{Tr}\big[|\rho_0-\rho_1|^3\big]\geq\varepsilon^3.$

$\alpha>1$ **in general**. For real-valued $\alpha>1$, the query complexity for estimating $\mathrm{T}_\alpha(\rho_0,\rho_1)$ to within additive error ε is $\mathrm{poly}(r,1/\varepsilon)$ [Wang-Guan-Liu-Zhang-Ying'22], which polynomially depends on *the rank* of the states of interests.

Generalizing the closeness measures via the Schatten α-norm (Cont.[2])

What about the complexity of estimating the classical ℓ_α distance?

Similarly, define the classical ℓ_α distance $\mathrm{TV}_\alpha(D_0, D_1) := \frac{1}{2}\left(\sum_x |D_0(x) - D_1(x)|^\alpha\right)^{1/\alpha}$.

- For real-valued $\alpha > 1$, the sample complexity of estimating $\mathrm{TV}_\alpha(D_0, D_1)$ to within additive error ε is $\mathrm{poly}(1/\varepsilon)$, which is *independent* of the support size of distributions D_0 and D_1, and fewer samples are needed as α increases [Waggoner'14].
- **Intuition:** When $\varepsilon = \Theta(1)$, draw $\mathrm{poly}(n)$ samples from D_0 and D_1, and compute the classical ℓ_α distance between the resulting empirical distributions.
- **Issue:** This intuition does not work in the quantum world... ☹

Hope: Estimating the trace of quantum state powers is easy for real-valued $q > 1$.

The query complexity of estimating $\mathrm{Tr}(\rho^q)$ for real-valued $q > 1$ is $\mathrm{poly}(1/\varepsilon)$ [**L.**-Wang'24].

- **Issue:** Since $|\rho_0 - \rho_1|$ is *not* a quantum state, this does not apply to $\mathrm{T}_\alpha(\rho_0, \rho_1)$. ☹

Question: What is the complexity of estimating $\mathrm{T}_\alpha(\rho_0, \rho_1)$ for real-valued $\alpha > 1$?

1. Quantum state testing with respect to the quantum ℓ_α distance
2. Main results: Upper bounds, lower bounds, and complexity classes
3. Proof techniques
4. Open problems

Main results: Upper bounds

Theorem 1 (Quantum estimator for quantum ℓ_α distance).
Given quantum query access to the state-preparation circuit Q_0 and Q_1 for the n-qubit state ρ_0 and ρ_1, for any real-valued $\alpha > 1$, there is a quantum algorithm that estimates $\mathrm{T}_\alpha(\rho_0, \rho_1)$ to within additive error ε, with query complexity $O(1/\varepsilon^{\alpha+1+\frac{1}{\alpha-1}}) = \mathrm{poly}(1/\varepsilon)$.

- The corresponding closeness testing problem $\mathrm{QSD}_\alpha[a(n), b(n)]$ decides whether $\mathrm{T}_\alpha(\rho_0, \rho_1)$ is at least $a(n)$ or at most $b(n)$, e.g., $a(n) = 2/5$ and $b(n) = 1/5$.
- As a corollary, for any real-valued $\alpha > 1$ and all $a(n), b(n) \in [0,1]$ satisfying $a(n) - b(n) \geq 1/\mathrm{poly}(n)$, $\mathrm{QSD}_\alpha[a(n), b(n)]$ is in BQP.
- Using the (multi-)samplizer [Wang-Zhang'23,Wang-Zhang'24], which enables a quantum query-to-sample simulation, the sample complexity used to estimate $\mathrm{T}_\alpha(\rho_0, \rho_1)$ is $\widetilde{O}(1/\varepsilon^{3\alpha+2+\frac{2}{\alpha-1}}) = \mathrm{poly}(1/\varepsilon)$.

While the prior best result [Wang-Guan-Liu-Zhang-Ying'22] has complexity polynomially depending on the rank r of ρ_0 and ρ_1, our work is *rank-independent* and thus provides an *exponential* improvement!

Main results: Complexity classes & lower bounds

Let $\textsc{PureQSD}_\alpha$ be a restricted variant of QSD_α, where the states of interest are *pure*:

Theorem 2 (Computational hardness of QSD_α).

The promise problem QSD_α captures the computational power of the respective complexity classes, depending on the regime of α:

1. **Easy regimes**. For any $1 \leq \alpha \leq \infty$, $\textsc{PureQSD}_\alpha$ (with constant precision) is BQP-hard. Consequently, for real-valued $\alpha > 1$, QSD_α is BQP-complete.
2. **Hard regimes**. For any $\alpha \in \left(1, 1+\frac{1}{n}\right]$, QSD_α is QSZK-complete*.
   - The QSZK containment of $\mathrm{QSD}_\alpha[a,b]$ holds only in the polarizing regime $a(n)^2 - b(n) \geq 1/O(\log n)$.

A sharp phase transition occurs between the case of $\alpha = 1$ and real-valued $\alpha > 1$!

Our reductions used to establish the hardness also leads to quantitative (query & sample complexity) lower bounds for estimating $\mathrm{T}_\alpha(\rho_0,\rho_1)$ to within additive error ε:

The regime of α	$1 < \alpha \leq 1+\frac{1}{n^{1+\delta}}$	$1+\frac{1}{n^{1+\delta}} < \alpha \leq 1+\frac{1}{n}$	Real-valued $\alpha > 1$
Query complexity	$\widetilde{\Omega}(r^{1/2})$	$\Omega(r^{1/3})$	$\Omega(1/\varepsilon)$ & $\mathrm{poly}(1/\varepsilon)$
Sample complexity	$\Omega(r/\varepsilon^2)$		$\Omega(1/\varepsilon^2)$ & $\mathrm{poly}(1/\varepsilon)$

1. Quantum state testing with respect to the quantum ℓ_α distance
2. Main results: Upper bounds, lower bounds, and complexity classes
3. Proof techniques
4. Open problems

Proof techniques: BQP containment of QSD$_\alpha$ for real-valued $\alpha > 1$

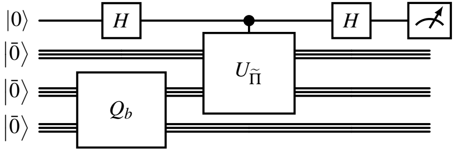
We begin by reviewing the approach in [Wang-Zhang'23] for estimating the trace distance ($\alpha = 1$), which uses the following key identity to decompose $\mathrm{T}(\rho_0,\rho_1)$:

$$\mathrm{T}(\rho_0,\rho_1) = \frac{1}{2}\mathrm{Tr}\left(\rho_0 \mathrm{sgn}\left(\frac{\rho_0-\rho_1}{2}\right)\right) - \frac{1}{2}\mathrm{Tr}\left(\rho_1 \mathrm{sgn}\left(\frac{\rho_0-\rho_1}{2}\right)\right) = \mathrm{Tr}(\Pi\rho_0) - \mathrm{Tr}(\Pi\rho_1).$$

The Holevo-Helstrom measurement $\{\Pi, I-\Pi\}$ and its approx. implementation satisfy:

$$\Pi = \frac{I}{2} + \frac{1}{2}\mathrm{sgn}\left(\frac{\rho_0-\rho_1}{2}\right) \quad \text{and} \quad \widetilde{\Pi} = \frac{I}{2} + \frac{1}{2}P_d^{\mathrm{sgn}}\left(\frac{\rho_0-\rho_1}{2}\right).$$

Implementing Π approximately. Using Quantum Singular Value Transformation [Gilyén-Su-Low-Wiebe'19] with a "good" polynomial approximation $P_d^{\mathrm{sgn}}(x)$ of the sign function $\mathrm{sgn}(x)$ on the interval $[-1,1]\setminus(-\delta,\delta)$, with degree $d = O\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$, one can approx. implement the HH measurement via the Hadamard test [Kitaev'95, Aharonov-Jones-Landau'06]:



- **Algorithmic HH measurement** [Le Gall-**L.**-Wang'23]: Using the *space-efficient* QSVT, for any n-qubit quantum states ρ_0 and ρ_1, $\widetilde{\Pi}$ can be implemented in $\mathrm{poly}(N)$ time and $O(n)$ space, where $N = 2^n$ is the dimension of the states.

BQP containment of QSD_α for real-valued $\alpha > 1$ (Cont.)

Inspired by the identity ($\alpha = 1$) used in [Wang-Zhang'23], we use the following identity to decompose the *powered* quantum ℓ_α distance $\Lambda_\alpha(\rho_0,\rho_1) = 2^{\alpha-1}\mathrm{T}_\alpha(\rho_0,\rho_1)^\alpha$:

$$\Lambda_\alpha(\rho_0,\rho_1) \coloneqq \frac{1}{2}\mathrm{Tr}(|\rho_0-\rho_1|^\alpha) = \frac{1}{2}\mathrm{Tr}\big(\rho_0\cdot\mathrm{sgn}(\nu)|\nu|^{\alpha-1}\big) - \frac{1}{2}\mathrm{Tr}\big(\rho_1\cdot\mathrm{sgn}(\nu)|\nu|^{\alpha-1}\big)$$
$$= \mathrm{Tr}(\Pi_\alpha\rho_0) - \mathrm{Tr}(\Pi_\alpha\rho_1),$$
$$\text{where}\quad \nu = \rho_0-\rho_1 \ \text{ and } \ \Pi_\alpha \coloneqq \frac{I}{2} + \frac{1}{2}\mathrm{sgn}(\nu)|\nu|^{\alpha-1}.$$

Similar to the case $\alpha = 1$, we can approximately implement Π_α via QSVT and the Hadamard test, denoted as $\widetilde{\Pi}_\alpha$, using an *approximate* polynomial approximation $P_d(x)$ of the function $\mathrm{sgn}(x)|x|^\beta$, where $\beta = \alpha - 1 > 0$ is a real number.

Removing the rank dependence. Unlike the case $\alpha = 1$, we need a polynomial $P_d(x)$ that *uniformly* approximate $\mathrm{sgn}(x)|x|^\beta$. The best uniform (polynomial) approximation of x^β was original investigated in [Bernstein'38], and the signed version $\mathrm{sgn}(x)|x|^\beta$ was listed in [Totik'06] and a non-constructive proof is provided in [Ganzburg'08]:

$$\max_{x\in[-1,1]}\left|P^*_{d^*}(x) - \mathrm{sgn}(x)|x|^\beta\right| \to (1/d^*)^\beta, \quad \text{as } d^*\to\infty.$$

By the Chebyshev truncation and the de La Vallée Poussin partial sum, we can make the coefficients of $P^*_{d^*}(x)$ *efficiently computable* with a slightly larger degree $d = 2d^*-1$:

$$\max_{x\in[-1,1]}\left|P_d(x) - \frac{1}{2}\mathrm{sgn}(x)|x|^\beta\right| \le \varepsilon \quad \text{and} \quad \max_{x\in[-1,1]}|P_d(x)| \le 1, \quad \text{where } d = O(1/\varepsilon^{1/\beta}).$$

Proof techniques: Lower bounds via a new inequality between T and T_α

By carefully analyzing the properties of *orthogonal* PSD matrices ς_0 and ς_1 such that $\rho_0 - \rho_1 = \varsigma_0 - \varsigma_1$, we establish a new *rank-dependent* inequality between T and T_α:

Theorem 3 (T vs. T_α). For any quantum states ρ_0 and ρ_1,

$$\forall \alpha \in [1,\infty], \quad 2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1) \le \mathrm{T}(\rho_0,\rho_1) \le 2\big(\mathrm{rank}(\rho_0)^{1-\alpha} + \mathrm{rank}(\rho_1)^{1-\alpha}\big)^{-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1).$$

- The case of $\alpha = 2$ was previously proven in [Coles'11, Coles-Cerezo-Cincio'19].
- The inequalities in Theorem 3 are *sharper* than those between the trace norm and the Schatten norm, such as in [Aubrun-Szarek'17]:
$$\forall \alpha \in [1,\infty], \quad \|A\|_\alpha \le \|A\|_1 \le \mathrm{rank}(A)^{1-\frac{1}{\alpha}} \|A\|_\alpha.$$
- For pure states $|\psi_0\rangle\langle\psi_0|$ and $|\psi_1\rangle\langle\psi_1|$, the following identity holds:
$$\forall \alpha \in [1,\infty], \quad 2^{1-\frac{1}{\alpha}} \cdot \mathrm{T}_\alpha(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|) = \mathrm{T}(|\psi_0\rangle\langle\psi_0|, |\psi_1\rangle\langle\psi_1|).$$

Reductions via inequalities in Theorem 3. Consequently, we obtain:

- Reductions from the case $\alpha = 1$ (e.g., QSD) to the case $\alpha > 1$ (e.g., QSD$_\alpha$), with the relevant $\alpha > 1$ ranges differing between QSD$_\alpha$ and PUREQSD$_\alpha$.
- This implies that the computational hardness and lower bounds for QSD$_\alpha$ and PUREQSD$_\alpha$ follow from the prior works on the trace distance ($\alpha = 1$).

Proof techniques: QSZK containment of QSD_α for $1 < \alpha < 1+\frac{1}{n}$

Directly combining the inequalities in Theorem 3 and the QSZK containment of QSD in [Watrous'02, Watrous'05] implies a QSZK containment of $\mathrm{QSD}_\alpha[a,b]$ under a parameter regime $a(n)^2/2 - b(n) \geq 1/O(\log n)$, which is *even worse* than the polarizing regime. ☹

Lemma 4 (A *partial* polarization lemma for T_α). Given a QSD_α instance (Q_0, Q_1, a, b, k), there exists a deterministic procedure that outputs new quantum circuits $\widetilde{Q}_0$ and $\widetilde{Q}_1$ that prepare the states $\widetilde{\rho}_0$ and $\widetilde{\rho}_1$. The resulting states satisfy the following:

$$\forall \alpha \in \left[1, 1+\frac{1}{n}\right], \quad \begin{aligned} \mathrm{T}_\alpha(\rho_0,\rho_1) \geq a \quad &\Longrightarrow \quad \mathrm{T}_\alpha(\widetilde{\rho}_0,\widetilde{\rho}_1) \geq (1-e^{-k})/2,\\ \mathrm{T}_\alpha(\rho_0,\rho_1) \leq b \quad &\Longrightarrow \quad \mathrm{T}_\alpha(\widetilde{\rho}_0,\widetilde{\rho}_1) \leq 1/16. \end{aligned}$$

When $k \leq O(1)$ or $a^2 - b \geq \Omega(1)$, the time complexity of the procedure is polynomial in the size of Q_0 and Q_1, k, and $\exp\left(\frac{b \log(1/a^2)}{a^2-b}\right)$.

- The polarization lemma for T makes $\mathrm{T}(\widetilde{\rho}_0,\widetilde{\rho}_1)$ either at least $1-2^{-k}$ or at most 2^{-k}:
  - The difference is because the direct product lemma for $1 < \alpha < 1+\frac{1}{n}$ is weaker than $\alpha = 1$;
  - I.e., $\mathrm{T}_\alpha(\rho_0^{\otimes l}, \rho_1^{\otimes l}) \geq \frac{1}{2} - \frac{1}{2}\exp\left(-\frac{l}{2} \cdot \mathrm{T}_\alpha(\rho_0,\rho_1)^2\right)$ vs. $\mathrm{T}(\rho_0^{\otimes l}, \rho_1^{\otimes l}) \geq 1 - \exp(-\frac{l}{2} \cdot \mathrm{T}(\rho_0,\rho_1)^2)$.
- The QSZK containment of QSD_α for $1 < \alpha \leq 1+\frac{1}{n}$ is the following:
  1. Apply the partial polarization lemma for T_α (Lemma 4) to the QSD_α instance;
  2. Combine with the inequalities in Theorem 3 to produce a QSD instance;
  3. Use the QSZK containment of QSD in [Watrous'02, Watrous'05].

1. Quantum state testing with respect to the quantum ℓ_α distance
2. Main results: Upper bounds, lower bounds, and complexity classes
3. Proof techniques
4. Open problems

Conclusions and open problems

Take-home messages on our work

1. For the regime $\alpha \geq 1+\Omega(1)$, estimating the quantum ℓ_α distance $\mathrm{T}_\alpha(\rho_0,\rho_1)$ is computationally *easy* and has *rank-independent* query & sample complexities.
2. For the regime $1 < \alpha \leq 1+\frac{1}{n}$, estimating the quantum ℓ_α distance $\mathrm{T}_\alpha(\rho_0,\rho_1)$ is computationally *hard* and the query & sample complexities are *rank-dependent*.

Discussion and open problems

While $\mathrm{T}_\alpha(\rho_0,\rho_1)$ and its powered version $\Lambda_\alpha(\rho_0,\rho_1)$ are *almost interchangeable* for real-valued $\alpha > 1$, their behavior differs significantly when $\alpha = \infty$:

- $\mathrm{T}_\infty(\rho_0,\rho_1)$ corresponds to the largest eigenvalue $\lambda_{\max}\left(\frac{\rho_0-\rho_1}{2}\right)$.
  - Estimating $\mathrm{TV}_\infty(D_0,D_1)$ to within additive error ε uses $O(1/\varepsilon^2)$ samples [Waggoner'14].
  - The pure state version PureQSD$_\infty$ is BQP-complete, while we only know that the general version QSD$_\infty$ is contained in QMA.
- $\Lambda_\infty(\rho_0,\rho_1) \in \left\{0, \frac{1}{2}, 1\right\}$ for any quantum states ρ_0,ρ_1, and it is nonzero if and only if the states are orthogonal with at least one of them being pure.
  - The pure state version PurePoweredQSD$_\infty[1,0]$ is coNQP-hard, which is a precise variant of BQP that ensures acceptance for all *yes* instances.

Question: What is the computational complexity of estimating $\mathrm{T}_\infty(\rho_0,\rho_1)$?

Thanks!