

Space-bounded quantum state testing  
via space-efficient quantum singular value transformation

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Available at [arXiv:2308.05079](https://arxiv.org/abs/2308.05079)

Tsinghua University, Aug 2023

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## ① Quantum state testing: $\ell_1$ norm vs. $\ell_2$ norm

### What is quantum state testing

Time-bounded quantum state testing

## ② Main result: Space-bounded quantum state testing and certification

## ③ Implications on space-bounded quantum computation

## ④ Proof technique: Space-efficient quantum singular value transformation

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# What is quantum state testing

**Task:** Quantum state testing (with two-sided error).

Given two quantum devices  $Q_0$  and  $Q_1$  that prepare  $\text{poly}(n)$ -qubit (mixed) quantum states  $\rho_0 \in \mathbb{C}^{N \times N}$  and  $\rho_1 \in \mathbb{C}^{N \times N}$ , respectively, which may be viewed as “sample access” to  $\rho_0$  and  $\rho_1$ . Decide whether  $\text{dist}(\rho_0, \rho_1) \leq \epsilon_1$  or  $\text{dist}(\rho_0, \rho_1) \geq \epsilon_2$ .

The one-sided error variant and the classical counterpart are as follows.

► **Quantum state certification** [Bădescu-O'Donnell-Wright'19]:

Given “sample access” to  $\rho_0$  and  $\rho_1$ , decide whether  $\rho_0 = \rho_1$  or  $\text{dist}(\rho_0, \rho_1) \geq \epsilon$ .

► **Distribution testing** (a.k.a. closeness testing of distributions, see [Canonne'20]):

Given sample accesses to probability distributions  $D_0$  and  $D_1$ , decide whether  $\text{dist}(D_0, D_1) \leq \epsilon_1$  or  $\text{dist}(D_0, D_1) \geq \epsilon_2$ .

**Goal.** Minimize the number of required copies (*sample complexity*) of  $\rho_0$  and  $\rho_1$ .

## Quantum state testing: sample complexity perspective

Classical and quantum distances that are considered:

	Quantum	Classical
$\ell_1$ norm	trace distance $\text{td}(\rho_0, \rho_1) := \frac{1}{2} \rho_0 - \rho_1 $	total variation distance (a.k.a., statistical distance)
$\ell_2$ norm	Hilbert-Schmidt distance $\text{HS}^2(\rho_0, \rho_1) := \frac{1}{2}(\rho_0 - \rho_1)^2$	Euclidean distance
Entropy	von Neumann entropy $S(\rho) := -\text{Tr}(\rho \ln \rho)$	Shannon entropy

Sample complexity for distribution testing and quantum state testing:

	$\ell_1$ norm	$\ell_2$ norm	Entropy
Classical sample complexity	$\text{poly}(\mathbf{N}, 1/\epsilon)$ [CDVV14]	$\text{poly}(1/\epsilon)$ [CDVV14]	$\text{poly}(\mathbf{N}, 1/\epsilon)$ [JVHW15, WY16]
Quantum sample complexity	$\text{poly}(\mathbf{N}, 1/\epsilon)$ [BOW19]	$\text{poly}(1/\epsilon)$ [BOW19]	$\text{poly}(\mathbf{N}, 1/\epsilon)$ [AISW20, OW21]

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## Time-bounded quantum state testing: $\ell_1$ norm scenario

**Task 1.1 (Time-bounded quantum state testing).** Given two *polynomial-size* quantum circuits  $Q_0$  and  $Q_1$  that prepare  $\text{poly}(n)$ -qubit (mixed) states  $\rho_0$  and  $\rho_1$ , respectively, with access to their “source codes”.

Decide whether  $\text{dist}(\rho_0, \rho_1) \leq \beta$  or  $\text{dist}(\rho_0, \rho_1) \geq \alpha$ .

**Time-bounded distribution testing:** Given two *efficiently samplable* distributions  $D_0$  and  $D_1$ , decide whether  $\text{dist}(D_0, D_1) \leq \beta$  or  $\text{dist}(D_0, D_1) \geq \alpha$ .

Computational hardness of these tasks w.r.t.  $\ell_1$  norm:

- ▶ Statistical Difference Problem (SDP) is SZK-complete when constant  $\alpha^2 - \beta > 0$ .  
[Sahami-Vadhan'03, Goldreich-Sahami-Vadhan'98]
- ▶ Quantum State Distinguishability Problem (QSDP) is QSZK-complete when constant  $\alpha^2 - \beta > 0$ . [Watrous'02, Watrous'09]
- ▶ Open problem:  $(\alpha, \beta)$ -QSDP is in QSZK when  $\alpha(n) - \beta(n) \geq 1/\text{poly}(n)$ .

Structural complexity-theoretic results regarding QSZK:

- ▶  $\text{BQP} \subseteq \text{QSZK} \subseteq \text{QIP}(2) \subseteq \text{PSPACE}$ . [Watrous'02, Watrous'09]
- ▶  $\exists \mathcal{O}$  s.t.  $\text{QSZK}^{\mathcal{O}} \not\subseteq \text{PP}^{\mathcal{O}}$ . [Bouland-Chen-Holden-Thaler-Vasudevan'19]

## Time-bounded quantum state testing: $\ell_2$ norm scenario

**Proposition 1.2** [BCWdW01, ARSW21]. Quantum Hilbert-Schmidt distance problem, namely time-bounded quantum state testing w.r.t.  $\ell_2$  norm, is BQP-complete.

**BQP containment.** As all three terms in  $\text{HS}^2(\rho_0, \rho_1) = \frac{1}{2}\text{Tr}(\rho_0^2) + \frac{1}{2}\text{Tr}(\rho_1^2) - \text{Tr}(\rho_0\rho_1)$  can be estimated by the SWAP test [Buhrman-Cleve-Watrous-de Wolf'01], we have a hybrid algorithm successes w.p.  $\frac{1}{2} + \frac{1}{2}\text{HS}^2(\rho_0, \rho_1)$ :

- ① Toss two random coins  $r_0, r_1 \in \{0, 1\}$ ;
- ② Perform the SWAP test on quantum states according to  $r_0$  and  $r_1$ .

**BQP hardness** [Agarwal-Rethinasamy-Sharma-Wilde'21]. Consider a BQP circuit  $C_x$ , we can construct  $C'_x := C_x^\dagger X_0^\dagger \text{CNOT}_{O \rightarrow F} X_0 C_x$  with an ancillary qubit on F such that  $\Pr[C'_x \text{ accepts}] = \|(|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F) C'_x (|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F)\|_2^2 = \Pr^2[C_x \text{ accepts}]$ . By defining two pure states  $\rho_0 := |\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F$  and  $\rho_1 := C'_x (|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_F)$ , we have  $\Pr[C'_x \text{ accepts}] = \text{Tr}(\rho_0\rho_1) = 1 - \text{HS}^2(\rho_0, \rho_1)$ .  $\square$

### Takeaways: Computational hardness of Task 1.1 depends on distance

- ▶ Time-bounded quantum state testing w.r.t.  $\ell_1$  norm (QSZK-complete) is seemingly *much harder* than only preparing these states (in BQP).
- ▶ Time-bounded quantum state testing w.r.t.  $\ell_2$  norm (BQP-complete) is as easy as only preparing these states (in BQP).



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## Main result: Space-bounded quantum state testing (two-sided error scenario)

**Task 2.1 (Space-bounded quantum state testing).** Given two *polynomial-size*  $O(\log n)$ -qubit quantum circuits  $Q_0$  and  $Q_1$  that prepare  $O(\log n)$ -qubit (mixed) states  $\rho_0$  and  $\rho_1$ , respectively, with access to their “source codes”.  
Decide whether  $\text{dist}(\rho_0, \rho_1) \leq \beta$  or  $\text{dist}(\rho_0, \rho_1) \geq \alpha$ .

**Theorem 2.2 (Space-bounded quantum state testing is BQL-complete).** The following (log)space-bounded quantum state testing problems are BQL-complete. For any  $\alpha, \beta$  s.t.  $\alpha(n) - \beta(n) \geq 1/\text{poly}(n)$  or any  $g(n) \geq 1/\text{poly}(n)$ , decide whether

- ①  $\text{GAPQSD}_{\log}$ :  $\text{td}(\rho_0, \rho_1) \geq \alpha$  or  $\text{td}(\rho_0, \rho_1) \leq \beta$ ;
- ②  $\text{GAPQED}_{\log}$ :  $S(\rho_0) - S(\rho_1) \geq g$  or  $S(\rho_1) - S(\rho_0) \geq g$ ;
- ③  $\text{GAPQJS}_{\log}$ :  $\text{QJS}_2(\rho_0, \rho_1) \geq \alpha$  or  $\text{QJS}_2(\rho_0, \rho_1) \leq \beta$ ;
- ④  $\text{GAPQHS}_{\log}$ :  $\text{HS}^2(\rho_0, \rho_1) \geq \alpha$  or  $\text{HS}^2(\rho_0, \rho_1) \leq \beta$ .

Here, quantum Jensen-Shannon divergence QJS is defined as

$$\text{QJS}_2(\rho_0, \rho_1) := S_2\left(\frac{\rho_0 + \rho_1}{2}\right) - \frac{1}{2}(S_2(\rho_0) + S_2(\rho_1)) \text{ where } S_2(\rho) := -\text{Tr}(\rho \log_2 \rho).$$

## Summary: Time-bounded and space-bounded quantum state testing

Computational hardness of Time-bounded and space-bounded quantum state testing:

	$\ell_1$ norm	$\ell_2$ norm	Entropy
Classical Time-bounded	SZK-complete [SV03,GSV98]	BPP-complete Folklore	SZK-complete [GV99,GSV98]
Quantum Time-bounded	QSZK-complete [Wat02,Wat09]	BQP-complete [BCWdW01, ARSW21]	QSZK-complete [BASTS10]
Classical Space-bounded	BPL-hard <sup>†</sup>	BPL-complete <sup>†</sup> Folklore	BPL-complete <sup>†</sup> [ABIS19]
Quantum Space-efficient	BQL-complete This work	BQL-complete [BCWdW01] and this work	BQL-complete This work

**Remark<sup>†</sup>.** Space-bounded distribution testing can be viewed as a “white-box” version of *streaming distribution testing* with i.i.d. samples.

**Takeaways.** For (log)space-bounded state testing and certification problems, the computational hardness of these problems is as easy as only preparing quantum states, which is **independent of the choice** of aforementioned distance-like measures.

## Main result: Space-bounded state certification (one-sided error scenario)

**Task 2.3 (Space-bounded quantum state certification).** Given two *polynomial-size*  $O(\log n)$ -qubit quantum circuits  $Q_0$  and  $Q_1$  that prepare  $O(\log n)$ -qubit (mixed) states  $\rho_0$  and  $\rho_1$ , respectively, with access to their “source codes”.

Decide whether  $\rho_0 = \rho_1$  or  $\text{dist}(\rho_0, \rho_1) \geq \alpha$ .

**Theorem 2.4 (Space-bounded quantum state certification is  $\text{coRQ}_{\text{UL}}$ -complete).**

The following space-bounded quantum state certification problems are  $\text{coRQ}_{\text{UL}}$ -complete. For any  $\alpha(n) \geq 1/\text{poly}(n)$ , decide whether

- ①  $\overline{\text{CERTQSD}}_{\log}$ :  $\rho_0 = \rho_1$  or  $\text{td}(\rho_0, \rho_1) \geq \alpha(n)$ ;
- ②  $\overline{\text{CERTQHS}}_{\log}$ :  $\rho_0 = \rho_1$  or  $\text{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$ .

**Remark.**  $\text{coRQ}_{\text{UL}}$  is a complexity class with *perfect completeness*, namely the acceptance probability  $p_{\text{acc}} = 1$  for yes instances whereas  $p_{\text{acc}} \leq 1/2$  for no instances.

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# BQL and BQ<sub>U</sub>L: Two-sided error space-bounded quantum computation

BQL (and BQ<sub>U</sub>L if only allow *unitary* gates), defined in [Watrous'99, Watrous'03], captures quantum computation that performs by a *logspace-uniform*  $O(\log n)$ -qubit quantum circuit. This class admits the following properties:

- ▶ Quadratic advantage in space (?):  $\text{BQL} \subseteq \text{DSPACE}[\log^2(n)]$  [Wat99, Wat03].
- ▶ Gateset-indep.: Space-efficient Solovay-Kitaev theorem [van Melkebeek-Watson'12].
- ▶ Error reduction for BQ<sub>U</sub>L [Fefferman-Kobayashi-Lin-Morimae-Nishimura'16].
- ▶ Intermediate measurements are useless:  $\text{BQL} = \text{BQ}_U\text{L}$  [Fefferman-Remscrem'21].

History of the only family of (natural) BQL-complete problem:

- ▶ Inverting a well-conditioned matrix is in BQL [Ta-Shma'13], whereas it only has  $\text{DSPACE}[\log^2(n)]$  containment without the help of quantum.
- ▶ Inverting a well-conditioned matrix is BQ<sub>U</sub>L-complete [Fefferman-Lin'18].
- ▶ A well-conditioned version of DET\*-complete problems are BQL-complete [Fefferman-Remscrem'21], such as well-conditioned integer determinant, well-conditioned matrix powering, well-conditioned iterative matrix product.

**Takeaway.** This work (Theorem 2.2) presents a *new family* of natural BQL-complete problems that emerge from quantum property testing.

# RQ<sub>UL</sub> and coRQ<sub>UL</sub>: One-sided error space-bounded quantum computation

RQ<sub>UL</sub> and coRQ<sub>UL</sub>, defined in [Watrous'01], capture *one-sided error* quantum computation that performs by a *logspace-uniform*  $O(\log n)$ -qubit quantum circuits. These classes admit the following properties:

- ▶ Error reduction for RQ<sub>UL</sub> and coRQ<sub>UL</sub> [Watrous'01].
- ▶ Gateset-dependence which is because of perfect completeness or soundness.
- ▶ Undirected graph connectivity (USTCON) is in  $\text{RQ}_{\text{UL}} \cap \text{coRQ}_{\text{UL}}$  [Watrous'01], although USTCON is actually in L [Reingold'08] .

## Open problems on RQ<sub>UL</sub> and coRQ<sub>UL</sub>:

- ▶ A (natural) complete problem for the class RQ<sub>UL</sub> or coRQ<sub>UL</sub> remains *unknown*. A “verification” version of well-conditioned iterative matrix product problem is coRQL-hard [Fefferman-Remscrim'21] while there is no containment (*hard direction*).
- ▶  $\text{RQ}_{\text{UL}} \stackrel{?}{=} \text{RQL}$  and  $\text{coRQ}_{\text{UL}} \stackrel{?}{=} \text{coRQL}$ .

**Takeaway.** This work (Theorem 2.4) demonstrates *the first family* of natural coRQ<sub>UL</sub>-complete problems that arise from quantum property testing as well.

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## Quantum singular value transformation in a nutshell

QSVT [Gilyén-Su-Low-Wiebe'19] is a systematic approach to (Time-boundedly) *manipulating singular values*  $\{\sigma_i\}_i$  of an Hermitian matrix  $A$  using a corresponding projected unitary encoding  $A = \tilde{\Pi}U\Pi$  for projectors  $\tilde{\Pi}$  and  $\Pi$ .

### Quantum singular value transformation, revisited

Given a singular value decomposition  $A = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$  associated with an  $s(n)$ -qubit *projected unitary encoding*, we can *approximately* implement a QSVT  $f^{(\text{SV})}(A) = \sum_i f(\sigma_i) |\tilde{\psi}_i\rangle\langle\psi_i|$  by employing a polynomial  $\hat{P}_d$  of degree  $d = O\left(\frac{1}{\delta} \log \frac{1}{\epsilon}\right)$  satisfying that

- ▶  $\hat{P}_d$  well-approximates  $f$  on the interval of interest  $\mathcal{I}$ :  
 $\max_{x \in \mathcal{I} \setminus \mathcal{I}_\delta} |\hat{P}_d(x) - f(x)| \leq \epsilon$  where  $\mathcal{I}_\delta \subseteq \mathcal{I} \subseteq [-1, 1]$  and typically  $\mathcal{I}_\delta := (-\delta, \delta)$ .
- ▶  $\hat{P}_d$  is bounded:  $\max_{x \in [-1, 1]} |\hat{P}_d(x)| \leq 1$ .

Moreover, all coefficients of  $\hat{P}_d$  (namely, *classical pre-processing*) can be computed in deterministic  $\text{poly}(d)$  time (and thus space). Hence, the transformation  $\hat{P}_d^{(\text{SV})}(A)$  can be implemented by a  $\text{poly}(d)$ -size quantum circuit acts on  $O(\max\{\log d, s(n)\})$  qubits.

**Question 4.1.** Can we implement a degree- $d$  QSVT for any  $s(n)$ -qubit projected unitary encoding with  $d \leq 2^{O(s(n))}$ , using only  $O(s(n))$  space in both *classical pre-processing* and quantum circuit implementation?

# Space-efficient quantum singular value transformation

**Theorem 4.2** (Space-bounded QSVT) [Metger-Yuen'23]. Implement a degree- $d$  QSVT associated with *sign function* or *square-root function* for any  $O(\log n)$  qubit *block-encoding* with  $d \leq \text{poly}(n)$  requires  **$O(\text{poly log } n)$  space for classical pre-processing** and  $O(\log n)$  qubits in quantum circuit implementation.

**Remark.** Theorem 4.2 can be easily extended to continuous functions bounded on  $[-1, 1]$ .

**Theorem 4.3** (Space-efficient QSVT). Implement a degree- $d$  QSVT associated with *piecewise-smooth functions* for any  $O(\log n)$  qubit *bitstring indexed encoding* with  $d \leq \text{poly}(n)$  requires **(randomized)  $O(\log n)$  space for classical pre-processing** and  $O(\log n)$  qubits in quantum circuit implementation.

Moreover, the implementation requires  $O(d^2 \|\mathbf{c}\|_1)$  uses of  $U$ ,  $U^\dagger$ ,  $C_{\Pi}\text{NOT}$ ,  $C_{\bar{\Pi}}\text{NOT}$ , among with other gates, where  $\mathbf{c}$  is the coeffs of *Chebyshev interpolation polynomial*.

E.g. Normalized log function  $\ln_\beta(x) := \frac{2 \ln(1/x)}{2 \ln(2/\beta)}$  on the interval  $\mathcal{I} = [\beta, 1]$  for any  $\beta \geq 1/\text{poly}(n)$ .

## Proof Sketch.

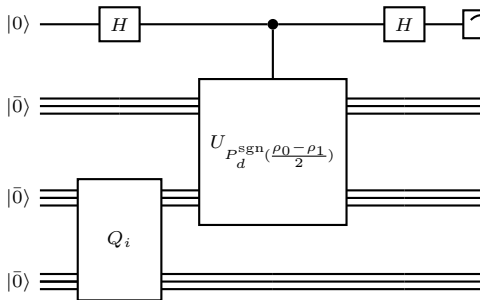
- ▶ For bounded functions, we mainly follow the construction in [MY23] with a careful analysis;
- ▶ For piecewise-smooth functions, we adapt the construction (i.e., a reduction to a linear combination of bounded functions) in [van Apeldoorn-Gilyén-Gribling-de Wolf'20].  
The main challenge can be boiled down to **stochastic matrix powering problem** which is crucial for the BPL vs. L problem [Saks-Zhou'99, Cohen-Doron-Sberlo-Ta-Shma'23, Putterman-Pyre'23].  $\square$

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## Proof overview

### Proof of Theorem 2.2 ①: $\text{GAPQSD}_{\log} \in \text{BQL}$

Inspired by the approach in [Gilyén-Poremba'22, Wang-Zhang'23], note that  $\text{sgn}(x) \approx_{\epsilon, \delta} P_d^{\text{sgn}}(x)$  and  $\text{td}(\rho_0, \rho_1) = \frac{1}{2} \text{Tr}|\rho_0 - \rho_1| = \frac{1}{2} \left( \text{Tr} \left( \text{sgn}^{(\text{SV})} \left( \frac{\rho_0 - \rho_1}{2} \right) \rho_0 \right) - \text{Tr} \left( \text{sgn}^{(\text{SV})} \left( \frac{\rho_0 - \rho_1}{2} \right) \rho_1 \right) \right)$ .



- ◇  $Q_i (i = 0, 1)$  prepares the state  $\rho_i$ .
- ◇ We implement  $U_{P_d^{\text{sgn}}(\frac{\rho_0 - \rho_1}{2})}$  by the space-efficient QSVT associated with  $P_d^{\text{sgn}}$  (Thm 4.3).
- ◇ “Acceptance probability” of  $\rho_i$ :  
 $\Pr[\text{outcome} = 0] = \frac{1}{2} \left( 1 + \text{Tr} \left( P_d^{\text{sgn}} \left( \frac{\rho_0 - \rho_1}{2} \right) \rho_i \right) \right)$ .
- ◇ For  $i \in \{0, 1\}$ , estimating  $\text{Tr} \left( P_d^{\text{sgn}} \left( \frac{\rho_0 - \rho_1}{2} \right) \rho_i \right) \pm \epsilon$  using  $O(1/\epsilon^2)$  repetitions. □

### Proof of Theorem 2.4 ①: $\overline{\text{CERTQSD}}_{\log} \in \text{coRQ}_{\text{UL}}$

- ◇ If  $\rho_0 = \rho_1$ ,  $\Pr[\text{outcome} = 0] = \frac{1}{2}$ . Then we obtain an algorithm  $\mathcal{A}$  accept with certainty via *exact amplitude amplification* [Boyer-Brassard-Høyer-Tapp'98, Brassard-Høyer-Mosca-Tapp'02].
- ◇ If  $\text{td}(\rho_0, \rho_1) \geq \alpha$ ,  $|\Pr[\text{outcome} = 0] - \frac{1}{2}| \geq \Omega(\alpha)$ .  
 Make sure the algorithm  $\mathcal{A}$  accepts w.p. at most  $1 - \Omega(\alpha^2)$  by a direct calculation. □

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# Conclusions and open problems

## Take-home messages on our work

- ① Space-bounded quantum state testing problems w.r.t. common distances (i.e., trace distance, Hilbert-Schmidt distance, entropy difference, quantum Jensen-Shannon divergence) are BQL-complete (Theorem 2.2).
- ② Space-bounded quantum state certification problems w.r.t. trace distance and Hilbert-Schmidt distance are  $\text{coRQ}_{\text{UL}}$ -complete (Theorem 2.4).  
This is the *first* family of natural  $\text{coRQ}_{\text{UL}}$ -complete problem!
- ③ Quantum singular value transformation on bitstring indexed encoding can be done in *quantum logspace*, with a *randomized* classical pre-processing (Theorem 4.3).

## Open problems

- ① Space-efficient QSVT with  $O(d)$  queries *instead of*  $O(d^2\|\mathbf{c}\|_1)$  in Theorem 4.3, as well as make the pre-processing *deterministic rather than randomized*.
- ② Are space-bounded state testings with respect to other (proper) *quantum analogs of symmetric  $f$ -divergence* also in BQL?
- ③ Is space-bounded distribution testing problem w.r.t. the *total variation distance* BPL-complete? What about the streaming distribution testing counterpart?

Thanks!