Repetition and important formulas from modul 3

1 Important terms

Slumpvariabel/random variable A *random variable* is a function defined on the sample space that assigns each possible, random outcome of an experiment a certain value.

Example: In the case of tossing a coin, the sample space (utfallsrum) is $\Omega = \{\text{head}, \text{tail}\}$. On this sample space one can define the random variable $X =$ "Payoff in one toss" by

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = \text{head}, \\ 0 & \text{if } \omega = \text{tail}. \end{cases}$$

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Fördelningsfunktion/cumulative distribution function (cdf) for a random variable X is defined as

$$F(x) = P(X \leq x), \quad -\infty \leq x \leq \infty$$

and measures the probability that the random outcome is smaller than a specific value x.

Sannolikhetsfunction/probability mass function (pmf) For a discrete random variable X, i.e. if the sample space (utfallsrum) is discrete, the pmf

$$p(x) = P(X = k), \quad k = 0, 1, \ldots,$$

gives the probability that the random outcome is exactly a specific value.

Täthetsfunktion/probability density function (pdf) This is the corresponding function for continuous random variables X, i.e. if the sample space is continuous (for example if the outcome can take any value in an interval or the real line). It is given by

$$f(x) = F'(x), \quad \text{i.e. } F(x) = \int_{-\infty}^{x} f(y)dy$$

if F can be derived. It tells us how much "density" is at a certain value but note that in contrast to the discrete case, the probability for a single value is zero: It holds for $a \leq b$

$$P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)dx$$

and therefore

$$P(X = x) = P(x \leq X \leq x) = F(x) - F(x) = 0.$$

Properties The most important properties are:

- $F(x)$ is an increasing function that goes to zero for $x \to -\infty$ and to one for $x \to \infty$.
- In the discrete case it holds
$$F(k) = P(X \leq k) = \sum_{i=0}^{k} p(i) \quad \text{and} \quad \sum_{i=0}^{\infty} p(i) = 1.$$
- In the continuous case it holds
$$f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(y)dy = 1.$$

2 Discrete distributions

Discrete distributions are characterized by the fact that the sample space (utfallsrum) is a collection of discrete (but possibly infinitely many) elements.

Binomial distribution The probability mass function (sannolikhetsfunktion) for a binomially distributed random variable X is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0, 1, 2, \ldots, n\}.$$

It is denoted by $X \sim \text{Bin}(n, p)$ with parameters n and p. A visualization of the pmf and cdf can be found in fig. 1.

It models n independent repeats of "identical" experiments and p is the probability that a certain events occurs.

Poisson distribution The probability mass function (sannolikhetsfunktion) for a Poisson distributed random variable X is given by

$$p(x) = P(X = x) = e^{-m} \frac{m^x}{x!}, \quad x \in \{0, 1, 2, \ldots\} = \mathbb{N}_0.$$

It is denoted by $X \sim \text{Po}(m)$ with parameter m. A visualization of the pmf and cdf can be found in fig. 2.

It is used to model the number of times an event occurs in a certain time interval, e.g. telephone calls arriving in a system, bus arrivals at a bus stop, etc.

Poisson approximation A binomially distributed random variable $X \sim \text{Bin}(n, p)$ with $n > 10$ and $p < 0.1$ is approximately Poisson distributed with parameter $m = np$.

3 Continuous distributions

Continuous distributions are characterized by the fact that the sample space (utfallsrum) is an interval or the real line. In other words: an outcome can be any value in $\mathbb{R}$ or a subset thereof.

Uniform distribution The probability density function (täthetsfunktion) for a uniformly distributed random variable X is given by

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

It is denoted by $X \sim \text{Re}(a, b)$ or $X \sim U(a, b)$ with parameters a and b. A visualization of the pdf and cdf can be found in fig. 3.

Most importantly it is used to generate pseudo random numbers.

Exponential distribution The probability density function (täthetsfunktion) for an exponentially distributed random variable X is given by

$$f(x) = \frac{1}{a} e^{-x/a}, \quad x \geq 0.$$

It is denoted by $X \sim \text{Exp}(a)$ with parameter $a > 0$. A visualization of the pdf and cdf can be found in fig. 4.

This distribution models for example waiting times (e.g. life time of a bulb) or the time between two Poisson events (e.g. the time between to phone calls).

Normal distribution The probability density function (täthetsfunktion) for a normally distributed random variable X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty \leq x \leq \infty.$$

It is denoted by $X \sim \mathcal{N}(\mu, \sigma^2)$ with parameters μ (mean) and σ^2 (variance). A visualization of the pdf and cdf can be found in fig. 5.

This is one of the most important distributions because of its properties. Why this is the case will become clear during the course.

Standardization The normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$ takes a special role and is called the *standard normal distribution*. Its cumulative distribution function is usually denoted by $\Phi(x)$ instead of $F(x)$ to highlight its special and important role. Any normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ can be transformed to a standard normally distributed random variable via

$$Z := \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

This transformation is very important when calculating $P(X \leq x)$:

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P\left(Z \leq \underbrace{\frac{x - \mu}{\sigma}}_{=:z}\right) = \Phi\left(z\right).$$

Since it is hard to calculated the integral over the density function of the normal distribution one rather uses tables to look up the corresponding values. You can find the table for example in the end of the book (Tabell 1).

How to read the tabel? First, round the value of z to two decimals. The first decimal is then given by the row and the second decimal by the column. Example: If we want to know $\Phi(1.57)$ we take the row where it says "1.5" and the column with ".07". Then the corresponding value is 0.9418.

If we need to calculate $\Phi(-0.38)$ we cannot find the value -0.38 in the table. Instead we use the symmetry of the distribution, i.e. $\Phi(-z) = 1 - \Phi(z)$ (try to really understand this by drawing a picture or looking at one!). Hence $\Phi(-0.38) = 1 - \Phi(0.38) = 1 - 0.6480 = 0.352$.

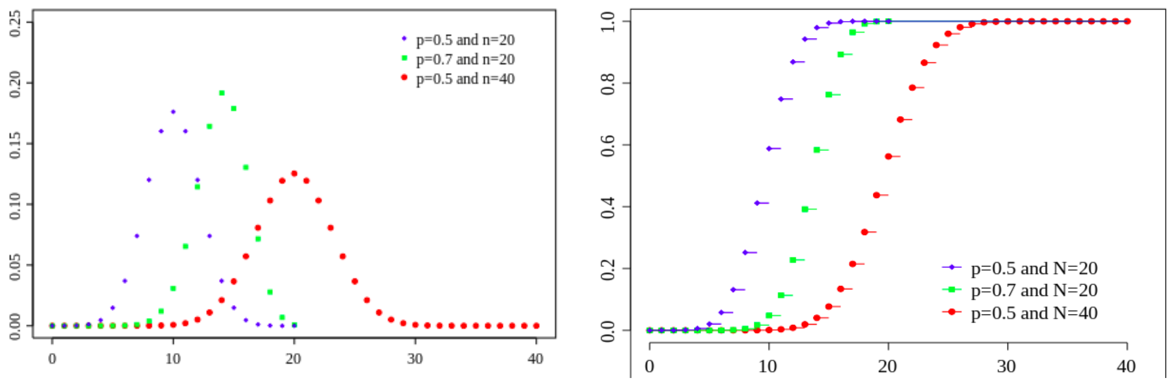

Figure 1: Probability mass function (left) and cumulative distribution function (right) of a binomial random variable. Adapted from `https://en.wikipedia.org/wiki/Binomial_distribution`, 20.03.20.

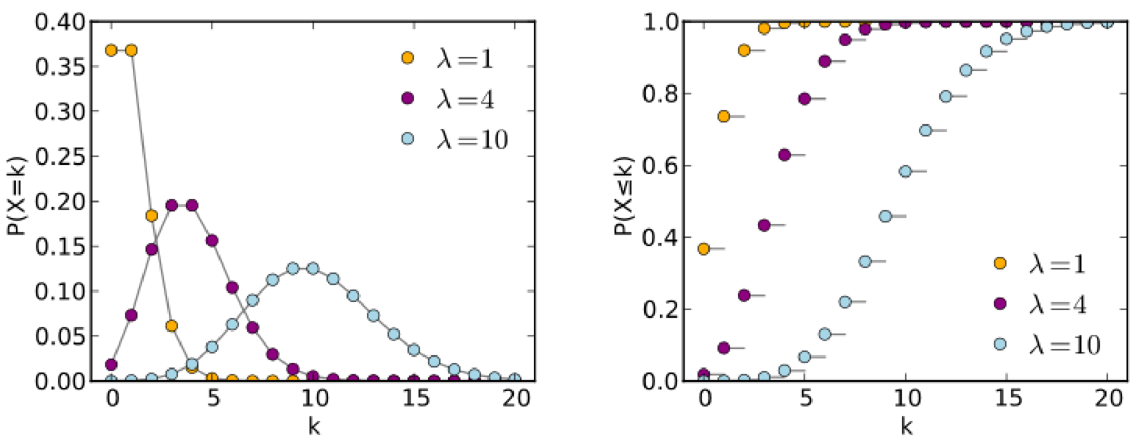

Figure 2: Probability mass function (left) and cumulative distribution function (right) of a Poisson random variable (note that the parameter here is called λ instead of m). Adapted from `https://en.wikipedia.org/wiki/Poisson_distribution`, 20.03.20.

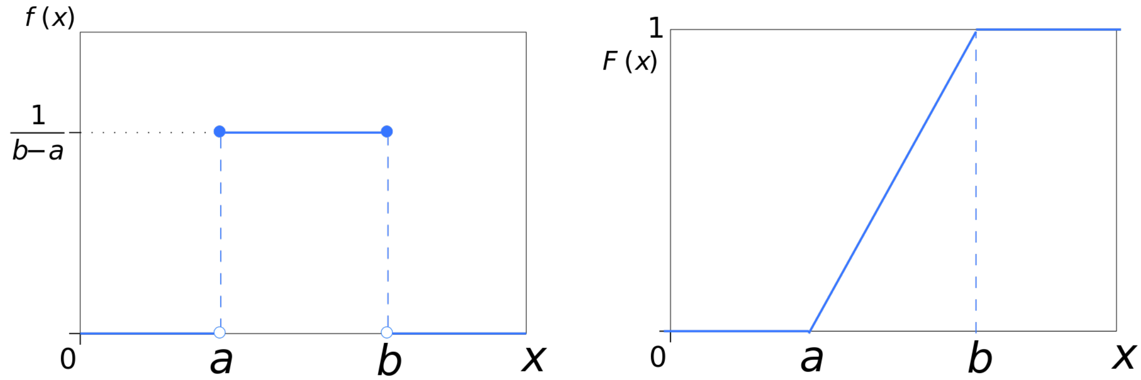

Figure 3: Probability density function (left) and cumulative distribution function (right) of a uniform random variable. Adapted from `https://en.wikipedia.org/wiki/Uniform_distribution_(continuous)`, 20.03.20.

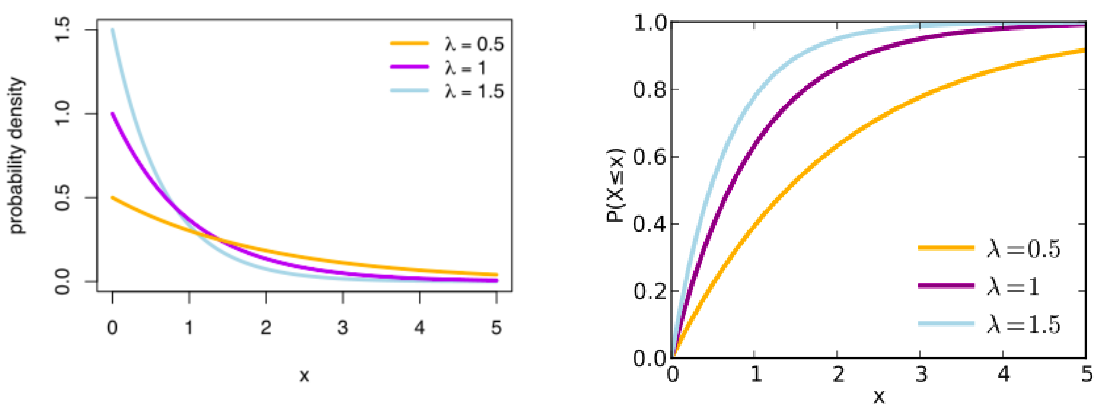

Figure 4: Probability density function (left) and cumulative distribution function (right) of an exponential random variable (note again that the parameter is called λ instead of a). Adapted from `https://en.wikipedia.org/wiki/Uniform_distribution_(continuous)`, 20.03.20.

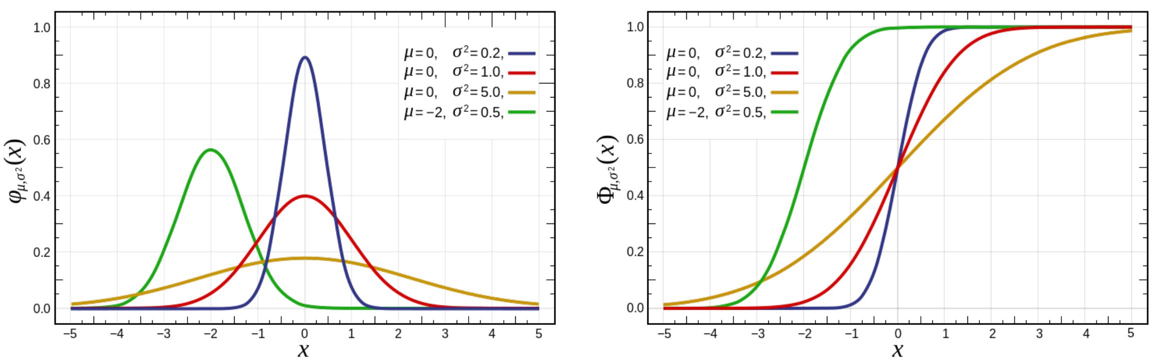

Figure 5: Probability density function (left) and cumulative distribution function (right) of an normal random variable. Adapted from `https://en.wikipedia.org/wiki/Normal_distribution`, 20.03.20.