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Repetition och viktiga formler från kapitel 3

- Understand the basic concepts of probability,
- Understand the concept of conditional probability,
- Distinguish disjoint (oförenliga) and independent (oberoende) events,
- Basic calculations.

1 Sets, Events and Random Variables

Sets (Mängder): A *set* is a well-defined collection of objects, usually denoted by a capital letter, such as A, B. A set can be described by listing all of its elements enclosed in braces. e.g.

$$A = \{2, 4, 6, 8, 10\}.$$

Some of the set operations are:

- Intersection (snitt): $A \cap B$, the set of elements both in A and B;
- Union (union): $A \cup B$, the set of elements either in A or B;
- Complement (komplement): A^* or A^C: the set of elements not in A.

Events (Händelser): An *outcome (utfall)* is a possible result of a random experiment. A collection of outcomes is called an *event*, and the collection of all possible outcomes is called a *sample space*. e.g. All possible outcomes from rolling a fair die is $1, 2, 3, 4, 5, 6$, and the event A of "getting an even number from rolling a fair die" is

$$A = \{2, 4, 6\}.$$

Special types of events:

- *Independent events* (Oberoende händelser): The occurrence of A does not affect the occurrence of B, see later sections for definition.
- *Disjoint events* (Oförenliga händelser): $A \cap B = \varnothing$.

Probability (Sannolikheter): The *probability* of an event refers to the likelihood that the event will occur. It is usually described by the proportion of the size of an event with respect to the size of all possible outcomes, hence a number between 0 and 1.

Random variables (slumpvariabler): The value of a *random variable* is determined by the outcome of the experiment. We will learn the definition more carefully in the following chapters. For example, tossing a fair coin twice, and define a random variable $X :=$"number of heads", where X can take value $0, 1$ or 2.

2 Kolmogorov Axioms of Probability

First Axiom: The probability of an event is a non-negative number

$$\mathbb{P}(A) \in \mathbb{R}, \;\; \mathbb{P}(A) \geq 0, \forall A.$$

Second Axiom: Probability of the sample space is 1:

$$\mathbb{P}(\Omega) = 1.$$

Or in other words, a *certain event (säker händelse)* happens with probability 1, and as a consequence, an *impossible event (omölig händelse)* happens with probability 0.

Third Axiom: If A and B are *disjoint sets* (synonymous with mutually exclusive events), i.e. $A \cap B = \varnothing$, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

The *additivity* stated in the third axiom gives us a way to calculate probability of a certain event: Separate the event into **disjoint** sets, and sum up the corresponding probability values. As a consequence, we have

Comlementation rule: $\mathbb{P}(A^*) = 1 - \mathbb{P}(A)$.

Addition rule for two arbitrary events: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

3 Conditional Probability

Conditional Probability (Betingade Sannolikheter): Let A and B be events, such that $\mathbb{P}(B) > 0$. We define the *conditional probability* of A given B by:

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)},$$

which answers the question

"What is the probability that A occurs given that B occurs?"

As a consequence, if $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, one can write

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A),$$

which can be visualised by a "tree diagram (träddiagram)", see P45 in JR for example.

Note that the axioms still hold for conditional probability:

- For any event A, $0 \leq \mathbb{P}(A|B) \leq 1$.
- $\mathbb{P}(\Omega|B) = 1$.
- For disjoint events A_1, A_2,

$$\mathbb{P}(A_1 \cup A_2|B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B).$$

Bayes' Formula: The Bayes' formula is a direct application of the definition of conditional probability:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)}{\mathbb{P}(A)}\mathbb{P}(A|B).$$

This is useful because many problems involve reversing the order of conditional probabilities, in other words, you know $\mathbb{P}(A|B)$ but want $\mathbb{P}(B|A)$. For a geometric proof and illustration of the Bayes' formula one can see the video by 3B1B: The quick proof of Bayes' theorem

Independent Events (Oberoende händelser) : Any two events A and B are said to be *independent* if and only if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

If $\mathbb{P}(A) > 0$, rearranging the above equation we get:

$$\mathbb{P}(B|A) = \mathbb{P}(B),$$

which means that the occurrence of A doesn't change the probability of B.

Mutual Independence of Multiple Sets: The concept of independence of two events can be generalised to three or more events. The events A, B and C are called *mutually independent* if and only if the two following conditions hold:

- $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$,
- A and B are independent, A and C are independent, and B and C are independent.

One can easily see that pairwise independence does not imply mutual independence. An example see Quiz M2.

4 Other Topics

4.1 Venn Diagrams

A Venn diagram is a visual representation in which the sample space is depicted as a box and events are represented as circles within the same sample space, which helps a lot with visualising the abstract concept of events. See here for more information.

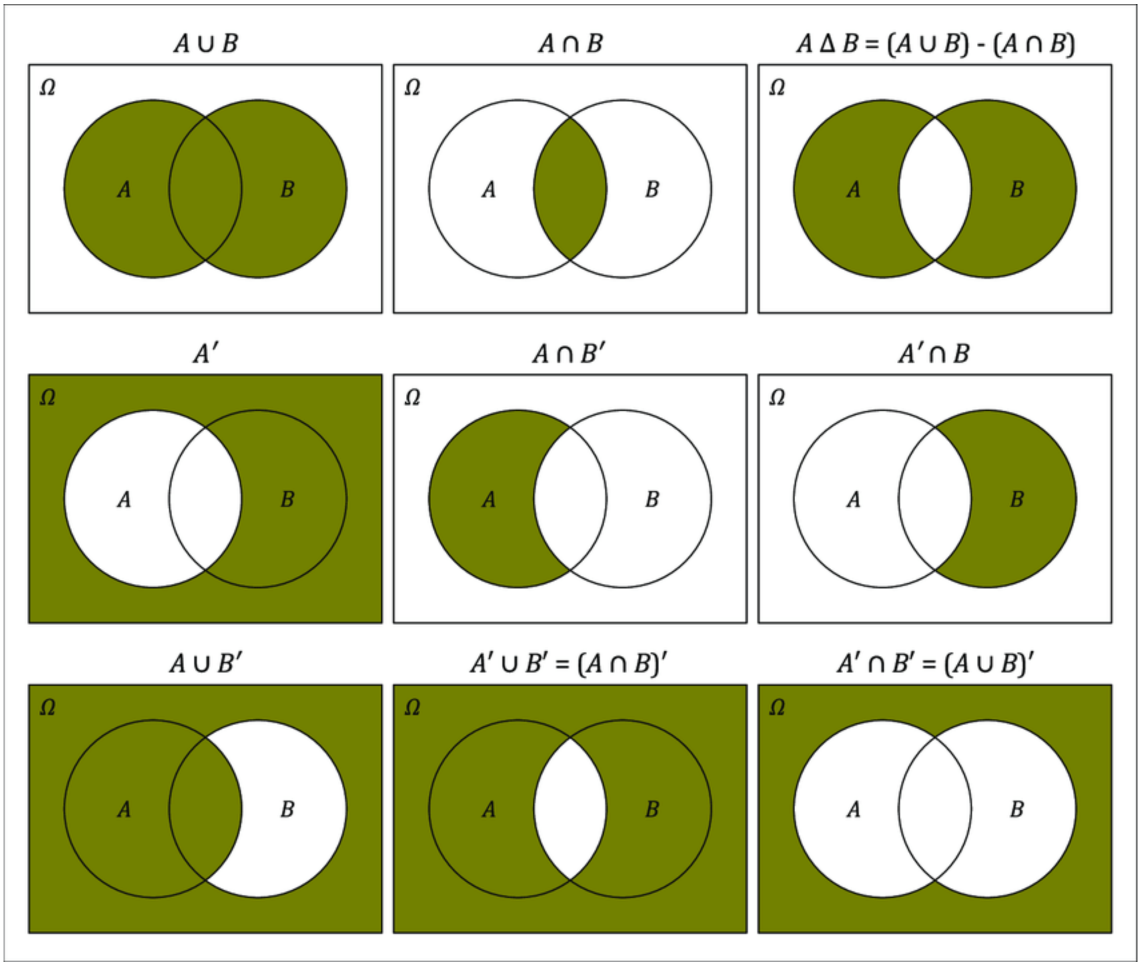

Figure 1: Some Set Operations under Venn Diagrams

One can try to use the Venn diagram to visualise the results in section "Conditional Probability"

4.2 Independent Events and Disjoint Events

Independent events in probability means "unrelated events", which means the outcome of one event does not impact the other.
Disjoint events means two events that do not occur at the same time, or, do not overlap. See the Venn diagram below:

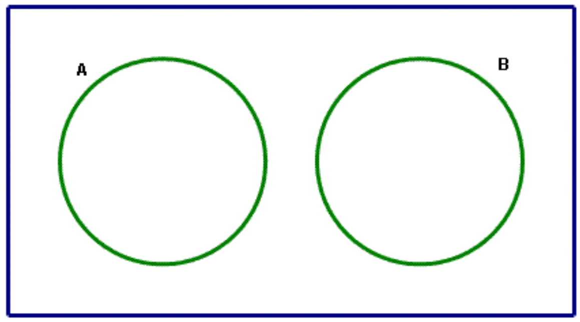

Figure 2: Venn Diagram of Two Disjoint Sets

Students often confuse these two concepts since the diagram above seems to indicate that the two sets are "unrelated". However, keep in mind that the "if and only if" rule to check if two sets are independent is the definition. Let's try to answer the following question:

Are disjoint events independent?

The answer is no. Let us consider the non-trivial case where $\mathbb{P}(A) > 0, \mathbb{P}(B) > 0$. Suppose A and B are disjoint , then

$$\mathbb{P}(A \cap B) = \mathbb{P}(\varnothing) = 0 \neq \mathbb{P}(A)\mathbb{P}(B).$$

Therefore A and B are not independent. In other words, the occurrence of A prevents B from happening, which means A and B are **anything but** independent.