Assignment 1. solutions

$\boxed{1}$ i) Let $f(t,x) = x^k$. By Itô,

$$dB_t^k = k B_t^{k-1} dB_t + \frac{1}{2} k(k-1) B_t^{k-2} dt.$$

$$B_t^k = \int_0^t k B_s^{k-1} dB_s + \frac{1}{2} k(k-1) \int_0^t B_s^{k-2} ds.$$

$$\mathbb{E}[B_t^k] = \underbrace{\mathbb{E}\left[k \int_0^t B_s^{k-1} dB_s\right]}_{\text{m.g., } 0} + \frac{1}{2} k(k-1) \int_0^t \mathbb{E}[B_s^{k-2}] ds.$$

Let $\beta_k(t) := \mathbb{E}[B_t^k]$.

$$\beta_k(t) = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s)\, ds$$

ii) Induction.

$\boxed{2}$. i). Let $Y_t := \int_0^t \frac{1}{1-s} dW_s$, Consider $f(t,y) = (1-t)y$.

$$d f(t, Y_t) = d\left((1-t) Y_t\right) = -Y_t\, dt + (1-t)\, dY_t$$

$$= -Y_t\, dt + (1-t) \cdot \frac{1}{1-t} dW_t$$

$$= \frac{(1-t) Y_t}{t-1} dt + dW_t. \qquad \square$$

ii). X_t is Gaussian since W_t has independent increments, and

thus is characterized by $\mathbb{E}[X_t]$ and $\mathrm{Var}[X_t]$

$$\mathbb{E}[X_t] = \mathbb{E}\left[\int_0^t \frac{1-t}{1-s}\, dW_s\right] = 0.$$

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[\left(\int_0^t \frac{1-t}{1-s}\, dW_s\right)^2\right]$$

$$= \mathbb{E}\left[\int_0^t \left(\frac{1-t}{1-s}\right)^2 ds\right]$$

Itô isometry.

$$= (1-t)^2 \left(\frac{1}{1-t} - 1\right)$$

$$= (1-t)\, t.$$

Therefore, $X_t \sim N(0, t(1-t))$.

It follows that $\lim_{t \uparrow 1} \mathbb{E}[X_t^2] = 0$.

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3. i). $\mu(t, x) = \mu x$. $\sigma(x) = \sigma$ satisfy the Lipschitz and linear growth conditions.

ii). Let $Y_t := e^{-\mu t} X_t$, $Y_0 = x$.

$$dY_t = -\mu Y_t\, dt + e^{-\mu t} dX_t$$

$$= -\mu Y_t\, dt + e^{-\mu t}(\mu X_t\, dt + \sigma\, dW_t)$$

$$= \sigma e^{-\mu t}\, dW_t$$

$$Y_t = x + \sigma \int_0^t e^{-\mu s}\, dW_s$$

$$X_t = e^{\mu t} \cdot Y_t$$

$$= x e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dW_s.$$

ii'). $\mathbb{E}[X_T] = x e^{\mu T} + \sigma \underbrace{\mathbb{E}[\int_0^T \sim dW_s]}_{0} = x e^{\mu T}.$

$$\mathbb{E}[X_T^2] = \mathbb{E}[(x e^{\mu t} + \sigma \int_0^t e^{\mu(t-s)} dW_s)^2]$$

$$= x^2 e^{2\mu t} + \mathbb{E}[\int_0^t \sigma^2 e^{2\mu(t-s)} ds]$$

$$= x^2 e^{2\mu t} + \sigma^2 e^{2\mu t} \cdot \frac{1}{2\mu}(1 - e^{-2\mu t})$$

$$= x^2 e^{2\mu t} + \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)$$

iii). Directly from Feynman-Kac.

iv).

$$\begin{cases} u_t + \mu x u_x + \frac{1}{2}\sigma^2 u_{xx} = 0. \\ u(T, x) = e^{i\xi x} \end{cases}$$

$$u(t, x) = \exp(\beta(t) + i\xi \alpha(t) x).$$

$$\begin{cases} u_t = (\beta_t + i\xi x \alpha_t) u. \\ u_x = i\xi \alpha(t) \cdot u. \\ u_{xx} = -\xi^2 \alpha^2(t) \cdot u. \end{cases}$$

$$\beta_t + i\xi x \alpha_t + \mu x i\xi \alpha(t) - \frac{1}{2}\sigma^2 \xi^2 \alpha^2(t) = 0$$

$$\Rightarrow i\xi(\alpha_t + \mu \alpha(t)) x + \beta_t - \frac{1}{2}\sigma^2 \xi^2 \alpha^2(t) = 0$$

for all $t \le T$, all x.

$$\Rightarrow \quad \alpha_t + \mu\alpha(t) \overset{(*)}{=} 0, \quad \beta_t \overset{(**)}{=} \frac{1}{2}\sigma^2\xi^2\alpha^2(t).$$

(*) yields
$$\begin{cases} \alpha(t) = Ce^{-\mu t}. \\ \alpha(T) = 1 \end{cases} \Rightarrow \alpha(t) = e^{\mu(T-t)}$$

(**) yields.
$$\begin{cases} \beta'(t) = \frac{1}{2}\sigma^2\xi^2 e^{2\mu(T-t)} \\ \beta(T) = 0 \end{cases}$$

$$\beta(t) - \beta(0) = \frac{1}{2}\sigma^2\xi^2 \int_0^t e^{2\mu(T-s)}\,ds$$

$$\Rightarrow \beta(t) = -\frac{1}{4\mu}\left(\sigma^2\xi^2(e^{2\mu(T-t)} - 1)\right).$$

$$\Rightarrow u(t,x) = \exp\left(i\xi e^{\mu(T-t)}x - \frac{1}{4\mu}\sigma^2\xi^2(e^{2\mu(T-t)} - 1)\right).$$

$$\phi_{X_T}(\xi) = u(0,x)$$
$$= \exp\left(i\xi e^{\mu T}x - \frac{1}{4\mu}\sigma^2\xi^2(e^{2\mu T} - 1)\right).$$

ii)

Let $S := B_{\sqrt{n}}(0)$ be the ball centered at 0 with radius $\sqrt{n}$.

Then $H \subset S$ and $\tau_H(\omega) \le \tau_S(\omega)$, for any $\omega \in \Omega$.

Thus $\mathbb{E}[\tau_H] \le \mathbb{E}[\tau_S]$.

Assume W_t starts with $W_0 = x$.

Sol 1 (Dynkin's).

Let $f = |x|^2$ and $\tau := \tau_S \wedge t$.

$$\mathbb{E}_x[f(W_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2}\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}(W_s)\,ds\right]$$

$$= |x|^2 + \mathbb{E}_x\left[\int_0^\tau \frac{1}{2}\cdot 2n\,ds\right]$$

$$= |x|^2 + n\,\mathbb{E}_x[\tau]$$

$$f(W_\tau) \le f(W_{\tau_S}) = n$$

$$\Rightarrow |x|^2 + n\,\mathbb{E}_x[\tau] \le n$$

$|x|^2 > 0$.

$$\Rightarrow \mathbb{E}_x[\tau] \le \frac{1}{n}(n - |x|^2) < 1.$$

Sol 2 (Martingale / optional sampling).

Let $M_t := W_t^2 - nt$.

Claim: M_t is a martingale.

Prf. $\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}[W_t^2 - nt \mid \mathcal{F}_s]$

$$= \mathbb{E}[(W_t - W_s + W_s)^2 - nt \mid \mathcal{F}_s]$$

$$= n(t-s) + W_s^2 - nt$$

$$= W_s^2 - ns = M_s.$$

$$M_0 = |x|^2 - 0 \underset{\substack{\uparrow \\ \text{optional sampling}}}{=} \mathbb{E}[M_{\tau_s}]$$

$$= \mathbb{E}[W_{\tau_s}^2 - n\tau_s]$$

$$= \mathbb{E}[W_{\tau_s}^2] - n\,\mathbb{E}[\tau_s]$$

$$= n - n\,\mathbb{E}[\tau_s].$$

$$\Rightarrow \mathbb{E}[\tau_s] = \frac{n - |x|^2}{n} < 1.$$

Sol 3. Let $u(x) = \mathbb{E}_x[\tau_s]$, then $u(x)$ solves the boundary problem

$$\begin{cases} \frac{1}{2}\Delta u + 1 = 0 & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

Ansatzs : $u(x) = C(|x|^2 - n)$

$$\frac{1}{2} \cdot 2Cn + 1 = 0 \Rightarrow C = -\frac{1}{n}.$$

$$\Rightarrow u(x) = \frac{1}{n}(n - |x|^2) \underset{\substack{\uparrow \\ |x| < 1.}}{<} 1.$$