Bayesian dynamic pricing in discrete time

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Outline

1. **Introduction**
   - Incomplete information
   - Applications
2. **Problem formulation and results**
   - Formulation
   - Construct the value function
   - Consequence of convexity
3. **Some remarks**

Stopping with incomplete information

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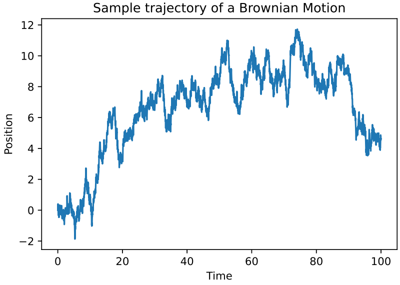

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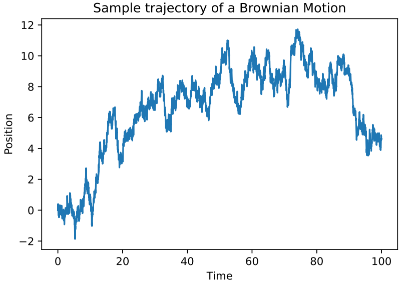

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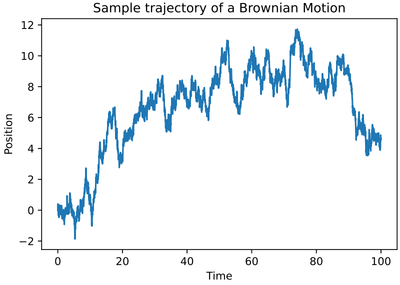

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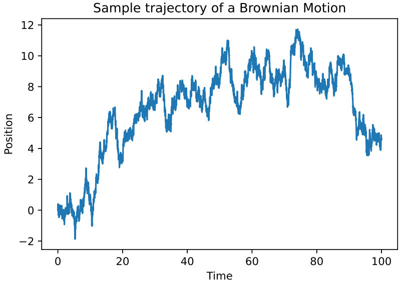

  – Is X a Gaussian with mean 0 or mean 1?

- Combine optimal stopping and filtering: optimising while **learning**.
- Applications: e.g. **statistics**, and **option pricing**.

In statistics: sequential testing

Classical problem: Testing the unknown drift of a BM.

- Observe the trajectory of a BM with unknown drift:

$$X_t = \theta t + W_t.$$

where $\mathbb{P}(\theta = 1) = \pi = 1 - \mathbb{P}(\theta = 0)$, $\pi \in (0, 1)$.

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- The time to stop observing is part of the decision.

In statistics: sequential testing

- The minimised cost V:

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- Standard method applies: explicit solution. c.f. Shiryaev (1969).

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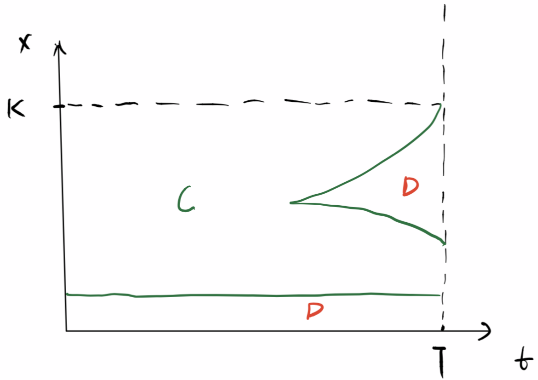
- Solution: rely on the Π process again:

$$V = \sup_{\tau} \mathbb{E}_{t,x,\pi}[e^{-r\tau}(K - X_\tau)^+].$$

c.f. Decamps et al. (2005), Gapeev (2012).

In finance: American option pricing

- Can reduce to a one (spatial)-dimensional problem.



- The behavior of boundries depends on the parameter.
  c.f. Ekström et al. (2019)

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- We consider a **learning-and-earning problem** in discrete time.

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- We know this conditional distribution:

$$F_\theta(x,p) = \mathbb{P}(X(\theta,p) \leq x)$$

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- The seller seeks to maximise the discounted profit:

$$V = \sup_{\{p_n\}_{n\geq 0}} \mathbb{E}\left[\sum_{n=0}^{\infty} a^n p_n X_n^{p_n}\right].$$

where $0 < a < 1$.

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- Observing one customer at a time.
- "incomplete learning", myopic strategies, examples (linear).

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- By conditioning, the value function can be written as

$$V(\pi) = \sup_{\{p_k\}_{k=0}^{\infty}} \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} a^k p_k \left(\Pi_k^{p_k} \int_{\mathbb{R}} x F_1(dx, p_k) + (1 - \Pi_k^{p_k}) \int_{\mathbb{R}} x F_0(dx, p_k) \right) \right]$$

- Only π dependent! Time-homogeneous.

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Definition

Define an operator T:

$$(Tu)(\pi) = \sup_p \left\{ a\mathbb{E}_\pi \left[u(\Pi_1^p) \right] + p \left(\pi \int_{\mathbb{R}} xF_1(dx,p) + (1-\pi) \int_{\mathbb{R}} xF_0(dx,p) \right) \right\}.$$

- Define a sequence of functions $\{u_n\}_{n\geq 0}$ by letting

$$u_0 = 0, \quad u_{n+1} = Tu_n.$$

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Theorem

The value function V is convex and continuous on $[0, 1]$.

Consequence of convexity: how to choose a model

How to find a model where V is larger? Convexity gives us the answer.

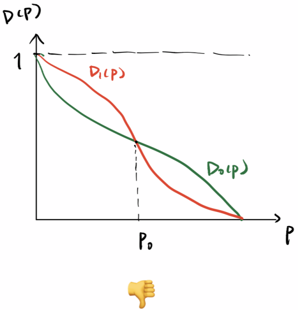
Theorem

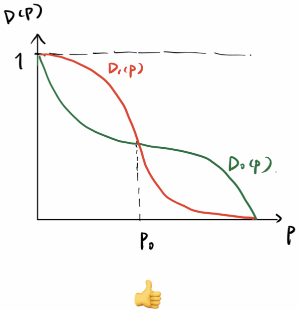
For any $(\pi, p) \in [0,1] \times \mathbb{R}$, take $\Pi_0 = \tilde{\Pi}_0 = \pi$. Assume the following two conditions hold, then $V_n(\pi) \geq \tilde{V}_n(\pi)$ for all $n \geq 0$ and $\pi \in [0,1]$.

1. *(Earning)* $\int_{\mathbb{R}} xF_1(dx,p) \geq \int_{\mathbb{R}} x\tilde{F}_1(dx,p)$, *and* $\int_{\mathbb{R}} xF_0(dx,p) \geq \int_{\mathbb{R}} x\tilde{F}_0(dx,p)$.
2. *(Learning)* Π_1^p *dominates* $\tilde{\Pi}_1^p$ *in convex order.*

Properties of the optimal strategy

Example: Bernoulli observations





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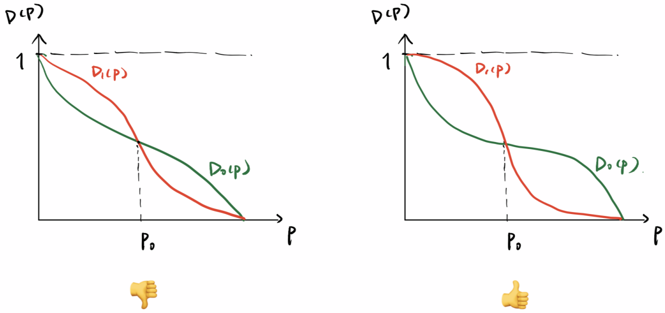

Other results

- Conditions for monotonicity of V.
- Conditions for monotonicity of the optimal strategy.

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Generalisations?

- Can relax the prior to an **arbitrary** distribution, and let X be in the **exponential family**.
- But the problems becomes time-dependent and very difficult.

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- Arbitrary prior?

Thank you for your attention!

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