HW3

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1 ANLY 561 HW3

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In [12]:
```
'''
This code imports numpy packages and allows us to pass data from python to global java
objects. It was developed by znah@github
'''

import json
import numpy as np
import numpy.random as rd
from ipywidgets import widgets
from IPython.display import HTML, Javascript, display

def json_numpy_serializer(o):
    if isinstance(o, np.ndarray):
        return o.tolist()
    raise TypeError("{} of type {} is not JSON serializable".format(repr(o), type(o)))

def jsglobal(**params):
    code = [];
    for name, value in params.items():
        jsdata = json.dumps(value, default=json_numpy_serializer)
        code.append("window.{}={};".format(name, jsdata))
    display(Javascript("\n".join(code)))
```

In [13]:
```
%%javascript

// Loading the compiled MathBox bundle.
require.config({
    baseUrl:'', paths: {mathBox: 'http://localhost:8888/tree/Desktop/static/mathbox/bu
    // online compilation
    //baseUrl: '', paths: {mathBox: '../static/mathbox/build/mathbox-bundle'}
    // online compilation without local library-- remove baseUrl
    //paths: {mathBox: '//cdn.rawgit.com/unconed/mathbox/eaeb8e15/build/mathbox-bundle
});
```

```
// Minified graphing functions

window.with_mathbox=function(element,func){require(['mathBox'],function(){var mathbox=
var intervalId=setInterval(function(){if(three.element.offsetParent===null){clearInte
var visible=isInViewport(three.canvas);if(three.Loop.running!=visible){visible?three.l
view.area({id:'yaxis',width:1,height:1,axes:[1,3],expr:function(emit,x,y,i,j){emit(4,0
window.addSequence=function(view,seq,col){var idx=0;var d=new Date();var start=d.getTi
start=now}
emit(seq[idx][1],seq[idx][2],seq[idx][0])},items:1,channels:3}).point({color:col,point
```

```
<IPython.core.display.Javascript object>
```

1.1 Problem 1

1.1.1 $f(x,y) = x^2 + y^2$

Comment: It is strictly convex.

1.1.2 $f(x,y) = x^2$

Comment: It is convex but not strictly convex.

1.1.3 $f(x,y) = x^2 - y^2$

Comment: It is NOT convex.

1.1.4 $f(x,y) = -x^2$

Comment: It is NOT convex.

1.1.5 $f(x,y) = -x^2 - y^2$

Comment: It is NOT convex.

1.2 Problem 2

Try to prove that $f(x_1, x_2) \geq f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$ or $f(x_1, x_2) > f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$
OR
disprove it.

1.2.1 $f(x,y) = x^2 + y^2$

Proof:
$f(x_1, x_2) = x_1^2 + x_2^2$
$f(y_1, y_2) = y_1^2 + y_2^2$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1)$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 2y_2(x_2 - y_2)$

So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
$= y_1^2 + y_2^2 + 2y_1(x_1 - y_1) + 2y_2(x_2 - y_2)$
$= y_1^2 + y_2^2 + 2y_1x_1 - 2y_1^2 + 2y_2x_2 - 2y_2^2$
$= 2y_1x_1 + 2y_2x_2 - y_2^2 - y_1^2$
$f(x_1, x_2) - [f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)]$
$= x_1^2 + x_2^2 - 2y_1x_1 - 2y_2x_2 + y_2^2 + y_1^2$
$= (y_1 - x_1)^2 + (y_2 - x_2)^2 > 0$ since $\mathbf{x} \neq \mathbf{y}, y_1 - x_1 \neq 0$, and $y_2 - x_2 \neq 0$ at the same time.
Therefore,
$f(x_1, x_2) > f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$, so this function is strictly convex.

1.2.2 $f(x, y) = x^2$

Proof:
$f(x_1, x_2) = x_1^2$
$f(y_1, y_2) = y_1^2$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1)$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
$= y_1^2 + 2y_1(x_1 - y_1)$
$= y_1^2 + 2y_1x_1 - 2y_2^2$
$= 2y_1x_1 - y_1^2$
$f(x_1, x_2) - [f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)]$
$= x_1^2 - 2y_1x_1 + y_1^2$
$= (y_1 - x_1)^2 \geq 0$ since y_1 could be equal to x_1 as long as $y_2 \neq x_2$.
Therefore,
$f(x_1, x_2) \geq f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in \mathbb{R}$, so this function is convex.

1.2.3 $f(x, y) = x^2 - y^2$

Proof:
As $(x_1, x_2) = (1, -1)$ and $(y_1, y_2) = (1, 1)$
$f(x_1, x_2) = x_1^2 - x_2^2 = 0$
$f(y_1, y_2) = y_1^2 - y_2^2 = 0$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1) = 0$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = -2y_2(x_2 - y_2) = 4$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 4$
$f(x_1, x_2) = 0 < 4 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

1.2.4 $f(x,y) = -x^2$

Proof:
As $(x_1, x_2) = (1, 0)$ and $(y_1, y_2) = (-1, 0)$
$f(x_1, x_2) = -x_1^2 = -1$
$f(y_1, y_2) = y_1^2 = -1$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = -2y_1(x_1 - y_1) = 4$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 3$
$f(x_1, x_2) = -1 < 3 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

1.2.5 $f(x,y) = -x^2 - y^2$

Proof:
As $(x_1, x_2) = (1, 0)$ and $(y_1, y_2) = (-1, 0)$
$f(x_1, x_2) = -x_1^2 - x_2^2 = -1$
$f(y_1, y_2) = y_1^2 - y_2^2 = -1$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = -2y_1(x_1 - y_1) = 4$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = -2y_2(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 3$
$f(x_1, x_2) = -1 < 3 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

1.3 Problem 3

$f(x,y) = \frac{y^2}{\sqrt{x^2+y^2}}$

1.3.1 Part a

Let $x = rcos(\theta), y = rsin(\theta)$, then

$$f(x,y) = f(r,\theta) = \frac{r^2 sin^2(\theta)}{\sqrt{r^2 cos^2(\theta) + r^2 sin^2(\theta)}} = \frac{r^2 sin^2(\theta)}{\sqrt{r^2(cos^2(\theta) + sin^2(\theta))}}$$
$$= \frac{r^2 sin^2(\theta)}{\sqrt{r^2}} = \frac{r^2 sin^2(\theta)}{r} = rsin^2(\theta)$$

Since $sin^2(\theta)$ is continuous, so $rsin^2(\theta)$ is continuous. Then $f(x,y) = \frac{y^2}{\sqrt{x^2+y^2}}$ is continuous. Therefore, $f(x,y)$ is continuous at $(0,0)$.

1.3.2 Part b

$$g(t) = \frac{b^2t^2}{\sqrt{a^2t^2 + b^2t^2}} = \frac{b^2t^2}{t\sqrt{a^2 + b^2}} = \frac{b^2}{\sqrt{a^2 + b^2}}t$$

So $g(t)$ is a just a line with slope $\frac{b^2}{\sqrt{a^2+b^2}}$.
Let $x, y \in \mathbb{R}$.
Let $k \in (0, 1)$
$g((1-k)x + ky)$
$= \frac{b^2}{\sqrt{a^2+b^2}}((1-k)x + ky)$
$= (1-k)\frac{b^2}{\sqrt{a^2+b^2}}x + k\frac{b^2}{\sqrt{a^2+b^2}}y$
$= (1-k)g(x) + kg(y)$
So, by definition, $g(t)$ is convex.

1.3.3 Part c

```
In [19]: %%javascript

         with_mathbox(element, function(mathbox) {

             var fcn = function(x, y) {
               return (y*y) /Math.sqrt(x*x+y*y);
             };

             var view = plotGraph(mathbox, fcn);

         })
```

```
<IPython.core.display.Javascript object>
```

Proof:
As $(x_1, x_2) = (2, 1)$ and $(y_1, y_2) = (-2, 1)$
$f(x_1, x_2) = \frac{1}{\sqrt{5}}$
$f(y_1, y_2) = \frac{1}{\sqrt{5}}$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = (-y_1y_2^2(y_1^2 + y_2^2)^{-\frac{3}{2}})(x_1 - y_1) = 8 * 5^{-\frac{3}{2}} > 0$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 0$ since $x_2 = y_2$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = \frac{1}{\sqrt{5}} + 8 * 5^{-\frac{3}{2}}$
$f(x_1, x_2) = \frac{1}{\sqrt{5}} < \frac{1}{\sqrt{5}} + 8 * 5^{-\frac{3}{2}} = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

1.4 Problem 4

$$p(x, y) = f(x^{(0)}, y^{(0)}) + \partial_1 f(x^{(0)}, y^{(0)})(x - x^{(0)}) + \partial_2 f(x^{(0)}, y^{(0)})(y - y^{(0)}) + \frac{1}{2}\Big(\partial_{1,1} f(x^{(0)}, y^{(0)})(x - x^{(0)})^2 + 2\partial_{1,2}(x$$

In this problem,

$$f(x_1,x_2) = -\log\left(\det\begin{pmatrix}1+x_1^2 & x_1x_2\\ x_1x_2 & 1+x_2^2\end{pmatrix}\right) = -\log\left((1+x_1^2)(1+x_2^2)-x_1^2x_2^2\right)$$

$$\partial_1 f(x_1,x_2) = \frac{-2x_1}{1+x_1^2+x_2^2}\,,\partial_2 f(x_1,x_2) = \frac{-2x_2}{1+x_1^2+x_2^2}$$

$$\partial_{1,1} f(x_1,x_2) = \partial_1\partial_1 f(x_1,x_2) = \partial_1\frac{-2x_1}{(1+x_1^2)(1+x_2^2)-x_1^2x_2^2} = \frac{2(x_1^2-x_2^2-1)}{(x_1^2+x_2^2+1)^2}$$

$$\partial_{2,2} f(x_1,x_2) = \partial_2\partial_2 f(x_1,x_2) = \partial_2\frac{-2x_2}{(1+x_1^2)(1+x_2^2)-x_1^2x_2^2} = \frac{2(x_2^2-x_1^2-1)}{(x_2^2+x_1^2+1)^2}$$

$$\partial_{1,2} f(x_1,x_2) = \partial_1\partial_2 f(x_1,x_2) = \partial_1\frac{-2x_2}{(1+x_1^2)(1+x_2^2)-x_1^2x_2^2} = \frac{4x_1x_2}{(x_2^2+x_1^2+1)^2}$$

$f(1,1) = -\log 3$, $\partial_1 f(1,1) = -\frac{2}{3}$, $\partial_2 f(1,1) = -\frac{2}{3}$, $\partial_{1,1} f(x_1,x_2) = -\frac{2}{9}$, $\partial_{2,2} f(x_1,x_2) = -\frac{2}{9}$, $\partial_{1,2} f(x_1,x_2) = \frac{4}{9}$
Thus, the second order Taylor approximation to f at $(1,1)$ is $p(x_1,x_2) = -\log 3 - \frac{2}{3}(x_1-1) - \frac{2}{3}(x_2-1) + \frac{1}{2}\left(-\frac{2}{9}(x_1-1)^2 + \frac{8}{9}(x_1-1)(x_2-1) - \frac{2}{9}(x_2-1)^2\right) = -\log 3 - \frac{2}{3}(x_1-1) - \frac{2}{3}(x_2-1) - \frac{1}{9}(x_1-1)^2 + \frac{4}{9}(x_1-1)(x_2-1) - \frac{1}{9}(x_2-1)^2$
$= -\log 3 - \frac{1}{9}(x_1^2+x_2^2-4x_1x_2+8x_1+8x_2-14)$

1.5 Problem 5

1.5.1 Part a

if $A,B \in SPD(2)$, then $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}$
$\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T(A\mathbf{x}+B\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} \geq 0$ (By the distributive property of matrix-vector multiplication)
Therefore, $A+B \in SPD(2)$.

1.5.2 Part b

proof: Take $A,B \in SPD(2)$ and $t \in [0,1]$,
then $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}$
$\mathbf{x}^T(tA+(1-t)B)\mathbf{x} = \mathbf{x}^T(tA\mathbf{x}+(1-t)B\mathbf{x}) = \mathbf{x}^T tA\mathbf{x} + \mathbf{x}^T(1-t)B\mathbf{x} = t\mathbf{x}^T A\mathbf{x} + (1-t)\mathbf{x}^T B\mathbf{x}$ (By the distributive property of matrix-vector multiplication)
$t,(1-t) \geq 0$ since $t \in [0,1]$.
so $t\mathbf{x}^T A\mathbf{x} \geq 0$ and $(1-t)\mathbf{x}^T B\mathbf{x} \geq 0$
Therefore, $\mathbf{x}^T(tA+(1-t)B)\mathbf{x} \geq 0$, also $tA+(1-t)B \in SPD(2)$
$SPD(2)$ is a convex subset.

1.5.3 Part c

proof: if $X \in M_{2,2}$, then X is a 2x2 matrix, and X^T is also a 2x2 matrix. Then by the rule of matrix multiplication, $X^T X$ is also a 2x2 matrix.
Take a $\mathbf{v} \in \mathbb{R}$, then $\mathbf{v}^T X^T X\mathbf{v} = (X\mathbf{v})^T X\mathbf{v} = X\mathbf{v} \bullet X\mathbf{v} \geq 0$.
Therefore, $X^T X \in SPD(2)$.

1.5.4 Part d

proof:
A is positive semidefinite, and B is positive definite.
$\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T(A\mathbf{x}+B\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x}$ (By the distributive property of matrix-vector multiplication)
We have $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} > 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}$
So $\mathbf{x}^T(A+B)\mathbf{x} > 0$
Therefore, $A+B$ is positive definite.

1.5.5 Part e

proof:
If A is positive definite, then all eigenvalues of A are positive. So 0 is not an eigenvalue of A, then the determinant of A is not zero, therefore A^{-1} exists.