ANLY 561 HW

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Problem 1

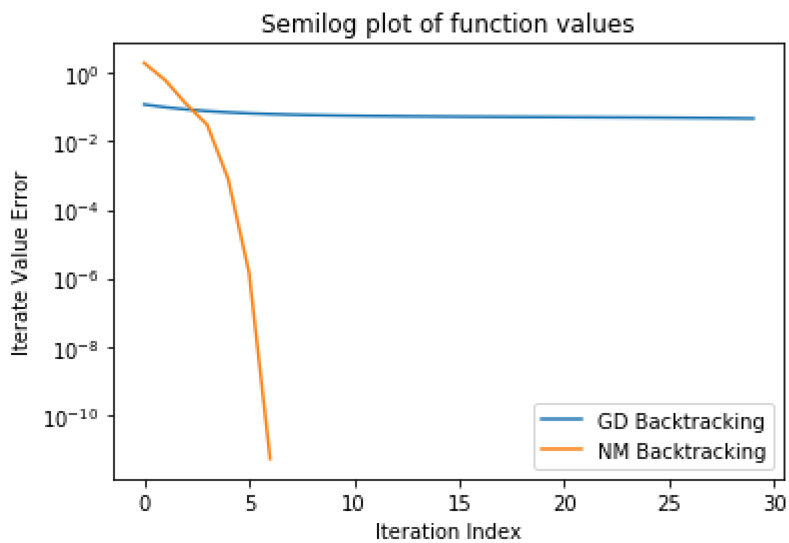
In [8]:
```
import numpy as np
import matplotlib.pyplot as plt

def backtracking(x0, dx, f, df0, alpha=0.2, beta=0.8, verbose=False):
    '''
    Backtracking for general functions with illustrations
    :param x0: Previous point from backtracking, or initial guess
    :param dx: Incremental factor for updating x0
    :param f: Objective function
    :param df0: Gradient of f at x0
    :param alpha: Sloping factor of stopping criterion
    :param beta: "Agressiveness" parameter for backtracking steps
    :param verbose: Boolean for providing plots and data
    :return: x1, the next iterate in backtracking
    '''

    # Note that the definition below requires that dx and df0 have the same shape
    delta = alpha * np.sum(dx * df0) # A general, but memory intensive inner product

    t = 1 # Initialize t=beta^0
    f0 = f(x0) # Evaluate for future use
    x = x0 + dx # Initialize x_{0, inner}
    fx = f(x)

    if verbose:
        n=0
        xs = [x]
        fs = [fx]
        ts = [1] * 3

    while (not np.isfinite(fx)) or f0 + delta * t < fx:
        t = beta * t
        x = x0 + t * dx
        fx = f(x)
    ######################################

        if verbose:
            n += 1
            xs.append(x)
            fs.append(fx)
            ts.append(t)
            ts.pop(0)

    if verbose:
        # Display the function along the line search direction as a function of t
        s = np.linspace(-0.1*ts[-1], 1.1*ts[0], 100)
        xi = [0, 1.1*ts[0]]
        fxi = [f0, f0 + 1.1*ts[0]*delta]
        y = np.zeros(len(s))

        for i in range(len(s)):
            y[i] = f(x0 + s[i]*dx) # Slow for vectorized functions

        plt.figure('Backtracking illustration')
        arm, =plt.plot(xi, fxi, '--', label='Armijo Criterion')
        fcn, =plt.plot(s, y, label='Objective Function')
```

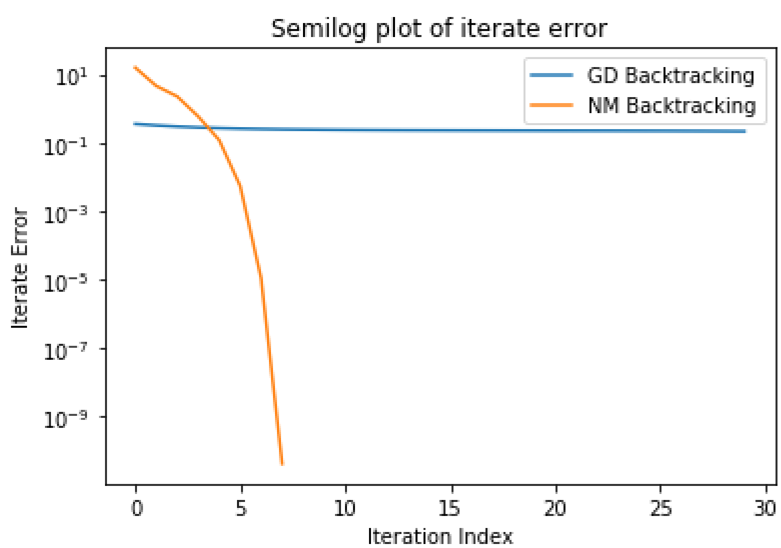
```
        plt.plot([s[0], s[-1]], [0, 0], 'k--')
        pts =plt.scatter(ts, [0 for p in ts], label='Backtracking points for n=%d, %d, %d'
        plt.scatter(ts, [f(x0 + q*dx) for q in ts] , label='Backtracking values for n=%d,
        init =plt.scatter([0], [f0], color='black', label='Initial point')
        plt.xlabel('$t$')
        plt.ylabel('$f(x^{(k)}+t\Delta x^{(k+1)})$')
        plt.legend(handles=[arm, fcn, pts, init])
        plt.show()

        return x, xs, fs

    else:
        return x


def logistic_objective(x,y):
    N = len(x)
    def fun(beta):
        result = 0
        for i in range(len(x)):
            result += np.log(1 + np.exp(-y[i]*(beta[0]+beta[1]*x[i])))
        return result/N
    return fun


def dlogistic_objective(x,y):
    N = len(x)
    def fun(beta):
        result0 = 0
        result1 = 0
        for i in range(len(x)):
            result0 +=  -y[i]/(np.exp(y[i]*(beta[0]+beta[1]*x[i]))+1)
            result1 +=  -x[i]*y[i]/(np.exp(y[i]*(beta[0]+beta[1]*x[i]))+1)
        return np.array([result0/N,result1/N])
    return fun


def d2logistic_objective(x,y):
    N = len(x)
    def fun(beta):
        result00 = 0
        result11 = 0
        result01 = 0
        for i in range(len(x)):
            result00 += y[i]*y[i]*np.exp(y[i]*(beta[0]+beta[1]*x[i]))/(np.exp(y[i]*(beta[0
            result11 += y[i]*y[i]*x[i]*x[i]*np.exp(y[i]*(beta[0]+beta[1]*x[i]))/(np.exp(y[
            result01 += y[i]*y[i]*x[i]*np.exp(y[i]*(beta[0]+beta[1]*x[i]))/(np.exp(y[i]*(b
        return np.array([[result00/N,result01/N],[result01/N,result11/N]])
    return fun

x = np.array([-1,-1,-1,0,0,0,1,1,1,1])
y = np.array([-1,-1,-1,1,-1,1,1,1,1,-1])

fun = logistic_objective(x,y)
dfun = dlogistic_objective(x,y)
d2fun = d2logistic_objective(x,y)
```

```python
alpha = 0.2
beta = 0.8

iter = 30 # 30 iterations of each
beta0 = np.array([10,10])
'''
backtracking with gradient desent
'''
x_gd_bt = [beta0]
f_gd_bt = [fun(beta0)]
b = beta0
for i in range(iter):
    b = backtracking(b, - dfun(b), fun, dfun(b))
    x_gd_bt.append(b)
    f_gd_bt.append(fun(b))

'''
backtracking with Newton's method
'''
x_nm_bt = [beta0]
f_nm_bt = [fun(beta0)]
b = beta0
for i in range(iter):
    b = backtracking(b, - np.linalg.solve(d2fun(b), dfun(b)), fun, dfun(b))
    x_nm_bt.append(b)
    f_nm_bt.append(fun(b))


# Compare convergence of function values with semilog plot
f_gd_bt_error = []
for i in range(len(x_gd_bt)-1):
    f_gd_bt_error.append(np.abs(f_gd_bt[i+1] - f_gd_bt[i]))

f_nm_bt_error = []
for i in range(len(x_nm_bt)-1):
    f_nm_bt_error.append(np.abs(f_nm_bt[i+1] - f_nm_bt[i]))

gd_bt, = plt.semilogy(f_gd_bt_error, label='GD Backtracking')
nm_bt, = plt.semilogy(f_nm_bt_error, label='NM Backtracking')

plt.xlabel('Iteration Index')
plt.ylabel('Iterate Value Error')
plt.legend(handles=[gd_bt, nm_bt])
plt.title('Semilog plot of function values')
plt.show()

# Compare convergece of iterates to the minimizer
x_gd_bt_norm = []
for i in range(len(x_gd_bt)-1):
    x_gd_bt_norm.append(np.linalg.norm(x_gd_bt[i+1] - x_gd_bt[i]))

x_nm_bt_norm = []
for i in range(len(x_nm_bt)-1):
    x_nm_bt_norm.append(np.linalg.norm(x_nm_bt[i+1] - x_nm_bt[i]))

gd_bt, = plt.semilogy(x_gd_bt_norm, label='GD Backtracking')
nm_bt, = plt.semilogy(x_nm_bt_norm, label='NM Backtracking')
```

```
plt.xlabel('Iteration Index')
plt.ylabel('Iterate Error')
plt.legend(handles=[gd_bt, nm_bt])
plt.title('Semilog plot of iterate error')
plt.show()
```

```
C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:77: RuntimeWarning: ov
erflow encountered in exp
```





Problem 2

Part (a)

$$\min_{(x,y)\in\mathbb{R}^2} 2x + 3y \text{ subject to } -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1$$

$f(x, y) = 2x + 3y$, $h_1(x, y) = x - 1$, $h_2(x, y) = -x - 1$, $h_3(x, y) = y - 1$, $h_4(x, y) = -y - 1$

KKT conditions:

(Stationarity)

$\nabla f(\mathbf{x}^*) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\nabla h_1(\mathbf{x}^*) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\nabla h_2(\mathbf{x}^*) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\nabla h_3(\mathbf{x}^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\nabla h_4(\mathbf{x}^*) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so

$$-\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

(Primal Feasibility)

$$h_1(\mathbf{x}^*) = x^* - 1 \le 0, h_2(\mathbf{x}^*) = -x^* - 1 \le 0,$$
$$h_3(\mathbf{x}^*) = y^* - 1 \le 0, h_4(\mathbf{x}^*) = -y^* - 1 \le 0$$

(Dual Feasibility)

$$\lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0, \lambda_4 \ge 0$$

(Complementary Slackness)

$$\lambda_1 h_1(\mathbf{x}^*) = \lambda_1(x^* - 1) = 0, \lambda_2 h_2(\mathbf{x}^*) = \lambda_2(-x^* - 1) = 0,$$
$$\lambda_3 h_3(\mathbf{x}^*) = \lambda_3(y^* - 1) = 0, \lambda_4 h_4(\mathbf{x}^*) = \lambda_4(-y^* - 1) = 0,$$

Part (b)

We have

$$\begin{aligned} \lambda_1 - \lambda_2 &= -2 & \lambda_3 - \lambda_4 &= -3 \\ \lambda_1(x^* - 1) &= 0 & \lambda_3(y^* - 1) &= 0 \\ \lambda_2(-x^* - 1) &= 0 & \lambda_4(-y^* - 1) &= 0 \end{aligned}$$

From the first column of equations, we get $\lambda_2 = \lambda_1 + 2$, then

$$(\lambda_1 + 2)(-x^* - 1) = 0 \text{ and } \lambda_1(x^* - 1) = 0$$

since $\lambda_1 \ge 0$, and $\lambda_1 + 2 \ge 2$, so

$$\begin{aligned} (-x^* - 1) = 0 &\implies x^* = -1 \\ \lambda_1(x^* - 1) = 0 &\implies \lambda_1 = 0 \\ \lambda_1 - \lambda_2 = -2 &\implies \lambda_2 = 2 \end{aligned}$$

Similarly, from the second column of equations, we get $\lambda_4 = \lambda_3 + 3$, then

$$(\lambda_3 + 3)(-y^* - 1) = 0 \text{ and } \lambda_3(y^* - 1) = 0$$

since $\lambda_3 \ge 0$, and $\lambda_3 + 3 \ge 3$, so

$$\begin{aligned} (-y^* - 1) = 0 &\implies y^* = -1 \\ \lambda_3(y^* - 1) = 0 &\implies \lambda_3 = 0 \\ \lambda_3 - \lambda_4 = -3 &\implies \lambda_4 = 3 \end{aligned}$$

Then we have to check the Primal Feasibility,

$$h_1(\mathbf{x}^*) = -1 - 1 = -2 < 0, h_2(\mathbf{x}^*) = -(-1) - 1 = 0,$$
$$h_3(\mathbf{x}^*) = -1 - 1 = -2 < 0, h_4(\mathbf{x}^*) = -(-1) - 1 = 0,$$

Therefore, $(-1, -1)$ is the only point which satisfies the KKT conditions.

As $\mathbf{x}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, We have

$$\begin{aligned} \lambda_1 - \lambda_2 &= -2 & \lambda_3 - \lambda_4 &= -3 \\ \lambda_1(x^* - 1) &= 0 & \lambda_3(y^* - 1) &= 0 \\ \lambda_2(-x^* - 1) &= 0 & \lambda_4(-y^* - 1) &= 0 \end{aligned}$$

Then we have $\lambda_1 = -2, \lambda_2 = 0, \lambda_3 = -3, \lambda_4 = 0$.
Check the Stationarity:

$$-2\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}-1\\0\end{pmatrix} + (-3)\begin{pmatrix}0\\1\end{pmatrix} + 0\begin{pmatrix}0\\-1\end{pmatrix} = -\begin{pmatrix}2\\3\end{pmatrix}$$

Check the Primal Feasibility:

$$h_1(\mathbf{x}^*) = 1 - 1 = 0, h_2(\mathbf{x}^*) = -1 - 1 = -2 < 0,$$
$$h_3(\mathbf{x}^*) = 1 - 1 = 0, h_4(\mathbf{x}^*) = -1 - 1 = -2 < 0,$$

Check the Dual Feasibility, however

$$\lambda_1 = -2 < 0, \lambda_3 = -3 < 0$$

Therefore, $(1, 1)$ satisfies all the KKT conditions except dual feasibility.

Part (c)

As $\mathbf{x}^* = \begin{pmatrix}0\\0\end{pmatrix}$,

$$h_1(\mathbf{x}^*) = 0 - 1 = -1 < 0, h_2(\mathbf{x}^*) = -(0) - 1 = -1 < 0,$$
$$h_3(\mathbf{x}^*) = 0 - 1 = -1 < 0, h_4(\mathbf{x}^*) = -(0) - 1 = -1 < 0,$$

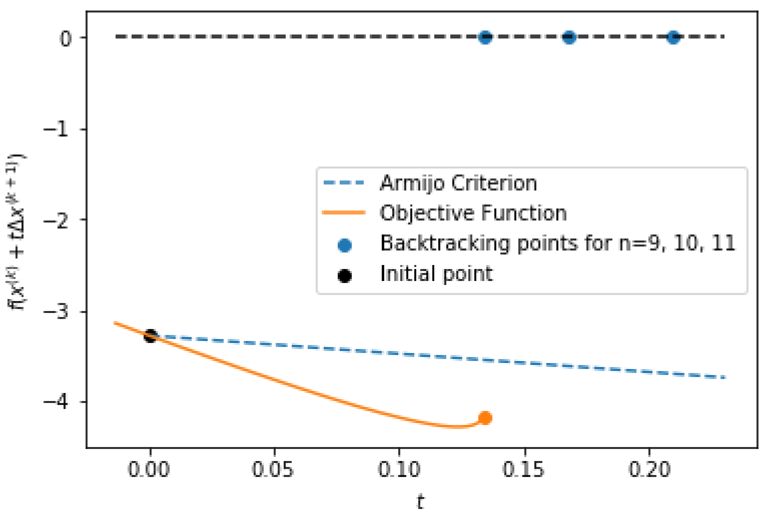
We see that $\mathbf{x}^* = \begin{pmatrix}0\\0\end{pmatrix}$ is strictly feasible. So the point $(0, 0)$ is an interior point of this program.

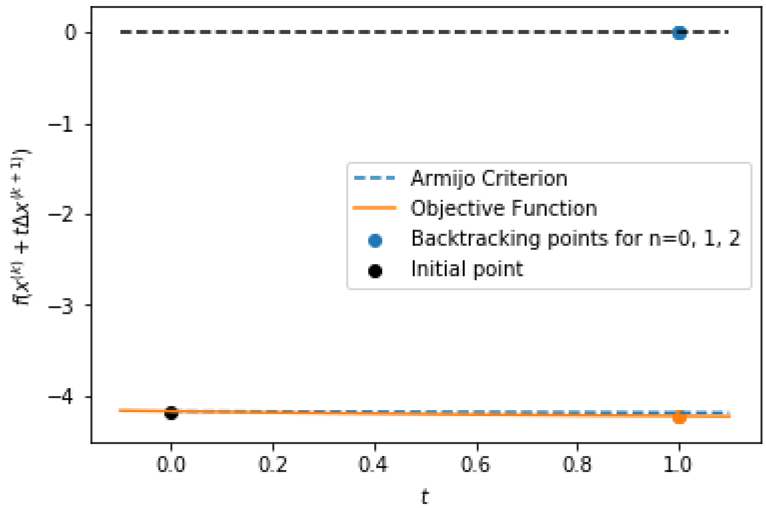
In [9]:
```
from pprint import pprint

# log barrier
fun = lambda x: 2*x[0]+3*x[1]
dfun = lambda x: np.array([2,3])

h1 = lambda x: x[0]-1
dh1 = lambda x: np.array([1,0])

h2 = lambda x: -x[0]-1
dh2 = lambda x: np.array([-1,0])

h3 = lambda x: x[1]-1
dh3 = lambda x: np.array([0,1])

h4 = lambda x: -x[1]-1
dh4 = lambda x: np.array([0,-1])

lb1 = lambda x: fun(x) - 1*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-h4(x))
dlb1 = lambda x: dfun(x) - 1*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb1 = lambda x: np.array([[0, 0],[0, 0]]) - 1*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/(h2(x)

alpha = 0.2
beta = 0.8

# 10 centering steps
x0 = np.array([0, 0])
x_nm_btlb = [x0]

x = x0
for i in range(10):
    x = backtracking(x, - np.linalg.solve(d2lb1(x), dlb1(x)), lb1, dlb1(x))
    x_nm_btlb.append(x)

# The process of 10 centering steps
print('\nThe process of 10 centering steps:\nx:')
pprint(x_nm_btlb)


# outer loop 1
lb2 = lambda x: fun(x) - 0.1*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-h4(x
dlb2 = lambda x: dfun(x) - 0.1*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb2 = lambda x: np.array([[0, 0],[0, 0]]) - 0.1*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/(h2(

x2 = np.zeros((3, 4))
x2[:2,0] = x_nm_btlb[-1] # initialize at the output of previous step

x2[:2, 1], xs, fs = backtracking(x2[:2, 0], - np.linalg.solve(d2lb2(x2[:2,0]), dlb2(x2[:2,
x2[:2, 2], xs, fs = backtracking(x2[:2, 1], - np.linalg.solve(d2lb2(x2[:2,1]), dlb2(x2[:2,
x2[:2, 3], xs, fs = backtracking(x2[:2, 2], - np.linalg.solve(d2lb2(x2[:2,2]), dlb2(x2[:2,

print('The result for the 1st outer loop:')
pprint(x2)


# outer loop 2
```

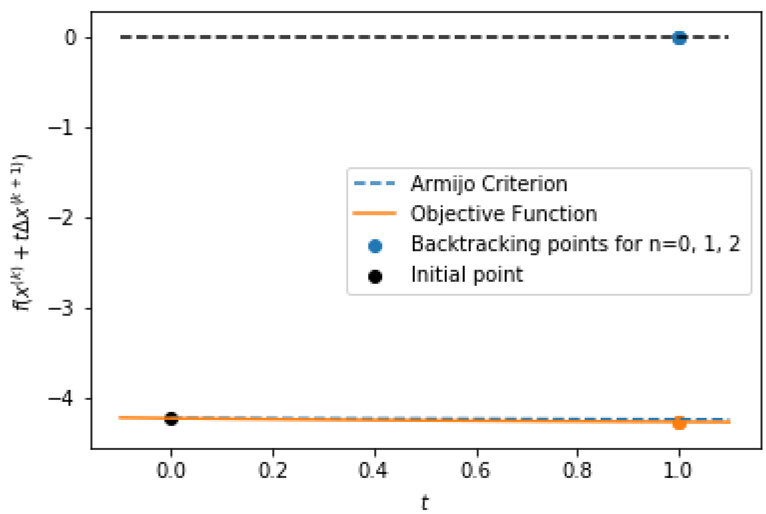
```
lb3 = lambda x: fun(x) - 0.01*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-h4(
dlb3 = lambda x: dfun(x) - 0.01*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb3 = lambda x: np.array([[0, 0],[0, 0]]) - 0.01*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/(h2

x3 = np.zeros((3, 4))
x3[:2,0] = x2[:2, 3] # initialize at the output of previous step

x3[:2, 1], xs, fs = backtracking(x3[:2, 0], - np.linalg.solve(d2lb3(x3[:2,0]), dlb3(x3[:2,
x3[:2, 2], xs, fs = backtracking(x3[:2, 1], - np.linalg.solve(d2lb3(x3[:2,1]), dlb3(x3[:2,
x3[:2, 3], xs, fs = backtracking(x3[:2, 2], - np.linalg.solve(d2lb3(x3[:2,2]), dlb3(x3[:2,

print('The result for the 2nd outer loop:')
pprint(x3)

# outer loop 3
lb4 = lambda x: fun(x) - 0.001*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-h4
dlb4 = lambda x: dfun(x) - 0.001*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb4 = lambda x: np.array([[0, 0],[0, 0]]) - 0.01*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/(h2

x4 = np.zeros((3, 4))
x4[:2,0] = x3[:2, 3] # initialize at the output of previous step

x4[:2, 1], xs, fs = backtracking(x4[:2, 0], - np.linalg.solve(d2lb4(x4[:2,0]), dlb4(x4[:2,
x4[:2, 2], xs, fs = backtracking(x4[:2, 1], - np.linalg.solve(d2lb4(x4[:2,1]), dlb4(x4[:2,
x4[:2, 3], xs, fs = backtracking(x4[:2, 2], - np.linalg.solve(d2lb4(x4[:2,2]), dlb4(x4[:2,

print('The result for the 3rd outer loop:')
pprint(x4)



# outer loop 4
lb5 = lambda x: fun(x) - 0.0001*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-h
dlb5 = lambda x: dfun(x) - 0.0001*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb5 = lambda x: np.array([[0, 0],[0, 0]]) - 0.0001*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/(

x5 = np.zeros((3, 4))
x5[:2,0] = x4[:2, 3] # initialize at the output of previous step

x5[:2, 1], xs, fs = backtracking(x5[:2, 0], - np.linalg.solve(d2lb5(x5[:2,0]), dlb5(x5[:2,
x5[:2, 2], xs, fs = backtracking(x5[:2, 1], - np.linalg.solve(d2lb5(x5[:2,1]), dlb5(x5[:2,
x5[:2, 3], xs, fs = backtracking(x5[:2, 2], - np.linalg.solve(d2lb5(x5[:2,2]), dlb5(x5[:2,

print('The result for the 4th outer loop:')
pprint(x5)



# outer loop 5
lb6 = lambda x: fun(x) - 0.00001*(np.log(-h1(x))+ np.log(-h2(x))+ np.log(-h3(x))+ np.log(-
dlb6 = lambda x: dfun(x) - 0.00001*(dh1(x)/h1(x)+dh2(x)/h2(x)+dh3(x)/h3(x)+dh4(x)/h4(x))
d2lb6 = lambda x: np.array([[0, 0],[0, 0]]) - 0.00001*np.array([-dh1(x)/(h1(x)**2)+dh2(x)/

x6 = np.zeros((3, 4))
x6[:2,0] = x5[:2, 3] # initialize at the output of previous step

x6[:2, 1], xs, fs = backtracking(x6[:2, 0], - np.linalg.solve(d2lb6(x6[:2,0]), dlb6(x6[:2,
x6[:2, 2], xs, fs = backtracking(x6[:2, 1], - np.linalg.solve(d2lb6(x6[:2,1]), dlb6(x6[:2,
x6[:2, 3], xs, fs = backtracking(x6[:2, 2], - np.linalg.solve(d2lb6(x6[:2,2]), dlb6(x6[:2,
```

```
print('The result for the 5th outer loop:')
pprint(x6)
```

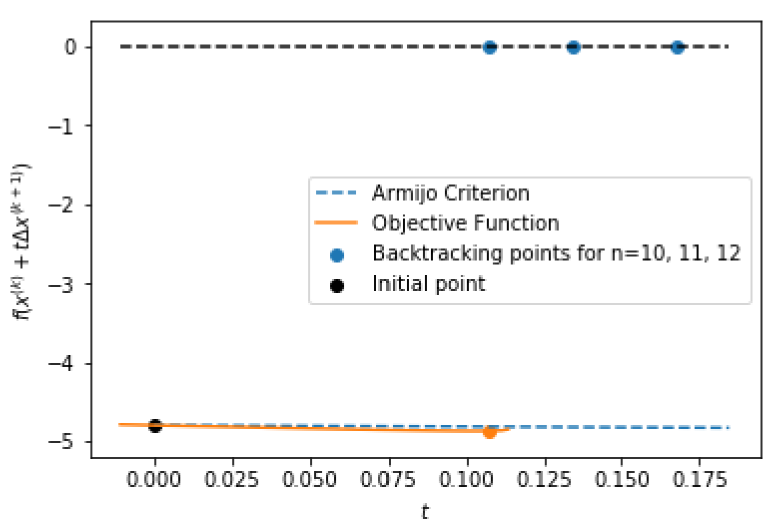
```
C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:19: RuntimeWarning: di
vide by zero encountered in log
C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:19: RuntimeWarning: in
valid value encountered in log
C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:41: RuntimeWarning: in
valid value encountered in log


The process of 10 centering steps:
x:
[array([0, 0]),
 array([-0.64, -0.96]),
 array([-0.61922543, -0.92563064]),
 array([-0.61803746, -0.87080766]),
 array([-0.61803399, -0.80094212]),
 array([-0.61803399, -0.74346044]),
 array([-0.61803399, -0.72255929]),
 array([-0.61803399, -0.72077048]),
 array([-0.61803399, -0.72075922]),
 array([-0.61803399, -0.72075922]),
 array([-0.61803399, -0.72075922])]
```

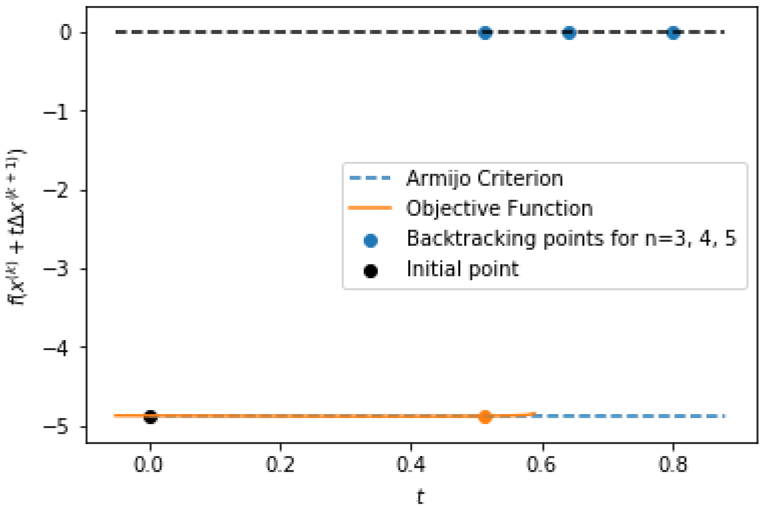


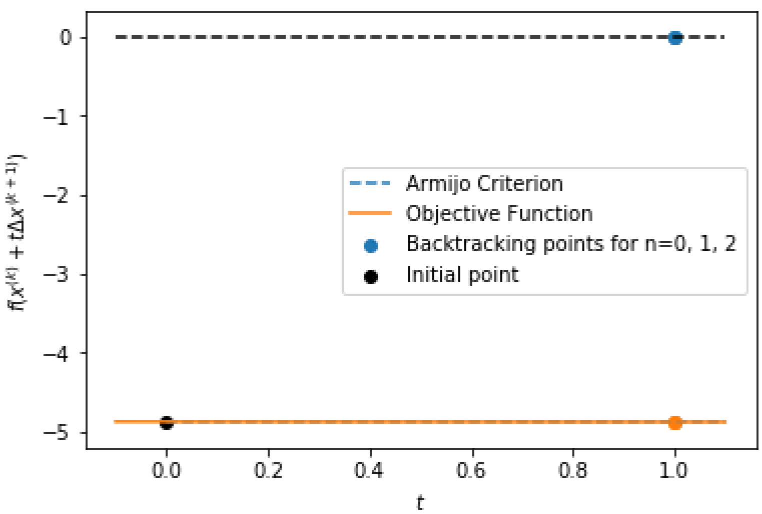




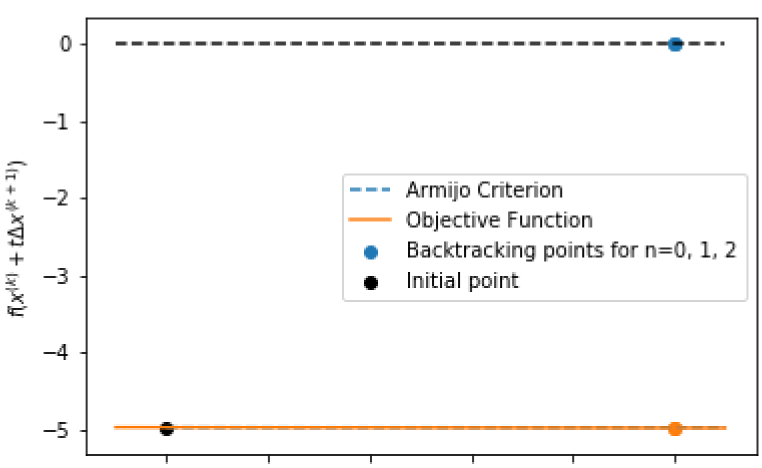

```
The result for the 1st outer loop:
array([[-0.61803399, -0.9519058 , -0.95125806, -0.95124922],
       [-0.72075922, -0.99608229, -0.99263273, -0.98692108],
       [ 0.        ,  0.        ,  0.        ,  0.        ]])

C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:57: RuntimeWarning: in
valid value encountered in log
```

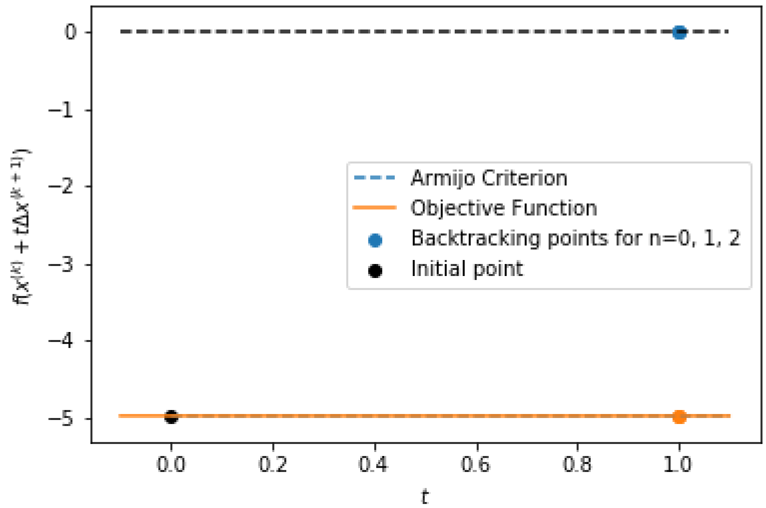


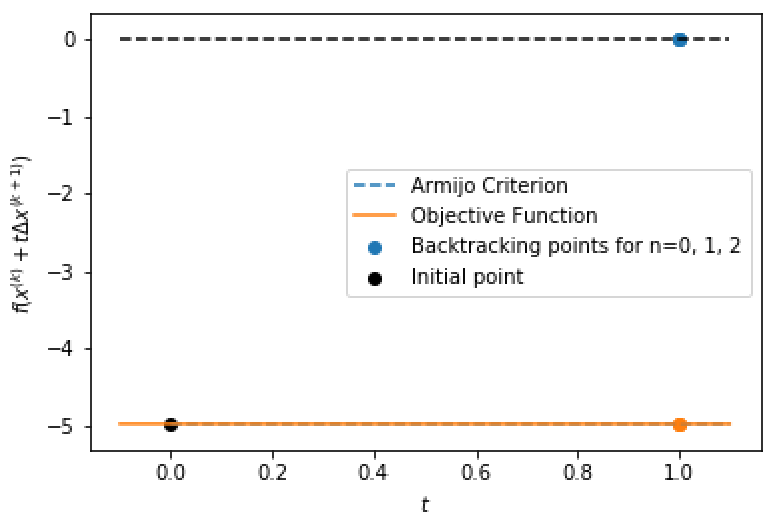





```
The result for the 2nd outer loop:
array([[-0.95124922, -0.9971547 , -0.99652898, -0.99547359],
       [-0.98692108, -0.99103598, -0.99880923, -0.99804456],
       [ 0.        ,  0.        ,  0.        ,  0.        ]])
```

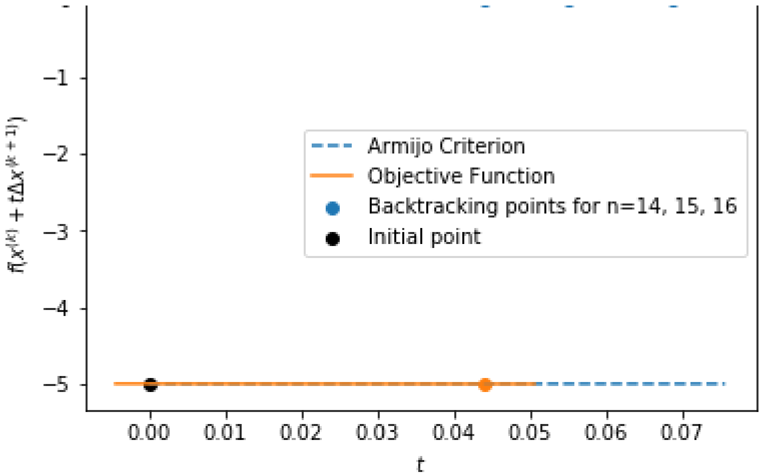




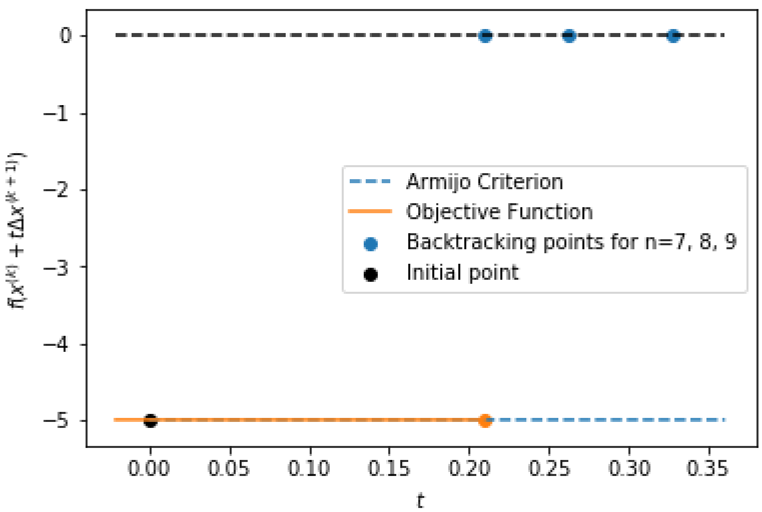



```
The result for the 3rd outer loop:
array([[-0.99547359, -0.99911963, -0.99918664, -0.99923765],
       [-0.99804456, -0.99899633, -0.99919822, -0.99931093],
       [ 0.        ,  0.        ,  0.        ,  0.        ]])
```

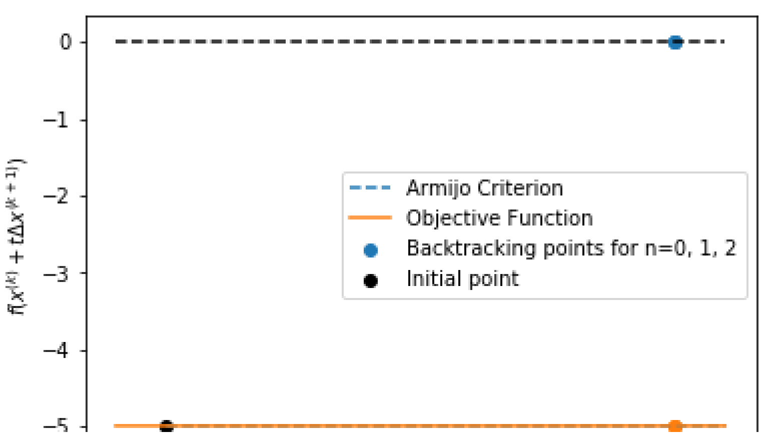
```
C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:88: RuntimeWarning: in
valid value encountered in log
```



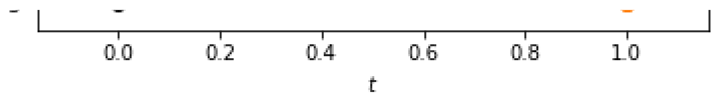
Armijo Criterion
Objective Function
Backtracking points for n=14, 15, 16
Initial point
$f(x^{(k)} + t\Delta x^{(k+1)})$
t



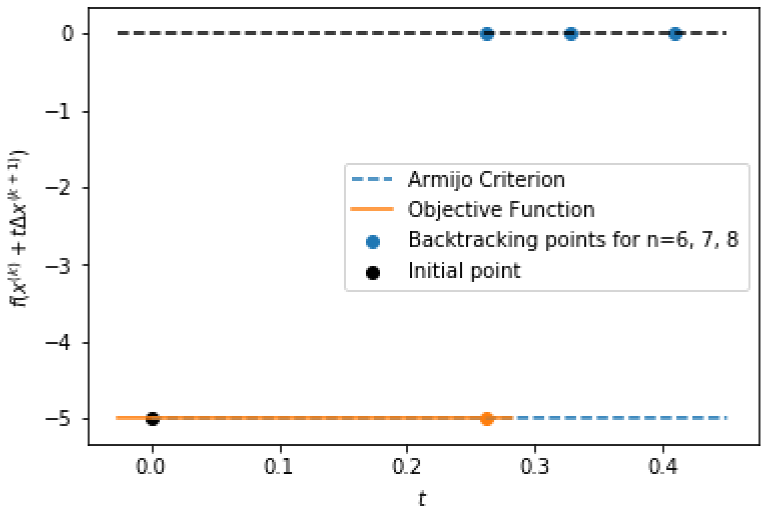
Armijo Criterion
Objective Function
Backtracking points for n=7, 8, 9
Initial point
$f(x^{(k)} + t\Delta x^{(k+1)})$
t

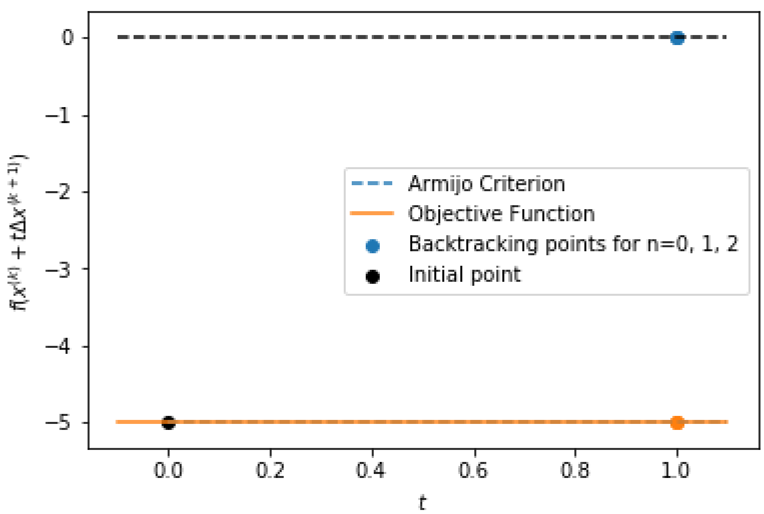


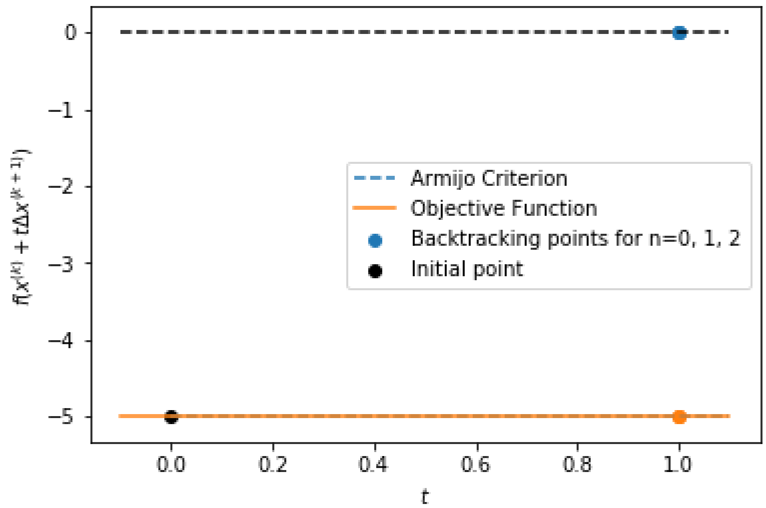
Armijo Criterion
Objective Function
Backtracking points for n=0, 1, 2
Initial point
$f(x^{(k)} + t\Delta x^{(k+1)})$




```
The result for the 4th outer loop:
array([[-0.99923765, -0.99971534, -0.99999552, -0.99999144],
       [-0.99931093, -0.99990712, -0.99994192, -0.99998504],
       [ 0.        ,  0.        ,  0.        ,  0.        ]])

C:\Users\45336\Anaconda3\lib\site-packages\ipykernel_launcher.py:104: RuntimeWarning: i
nvalid value encountered in log
```








```
The result for the 5th outer loop:
array([[-0.99999144, -0.99999304, -0.99999577, -0.99999512],
       [-0.99998504, -0.99999871, -0.99999792, -0.99999714],
       [ 0.        ,  0.        ,  0.        ,  0.        ]])
```

Problem 3

Given that

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x1}(\mathbf{x}) \\ \frac{\partial f}{\partial x2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial xn}(\mathbf{x}) \end{pmatrix}$$

Prove that

$$\nabla (g \circ f)(\mathbf{x}) = g'(f(\mathbf{x}))\nabla f(\mathbf{x})$$

PROOF:

$$\nabla (g \circ f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial (g \circ f)}{\partial x1}(\mathbf{x}) \\ \frac{\partial (g \circ f)}{\partial x2}(\mathbf{x}) \\ \vdots \\ \frac{\partial (g \circ f)}{\partial xn}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial f}(f(\mathbf{x}))\frac{\partial f}{\partial x1}(\mathbf{x}) \\ \frac{\partial g}{\partial f}(f(\mathbf{x}))\frac{\partial f}{\partial x2}(\mathbf{x}) \\ \vdots \\ \frac{\partial g}{\partial f}(f(\mathbf{x}))\frac{\partial f}{\partial xn}(\mathbf{x}) \end{pmatrix} \text{(Using the chain rule)}$$

Since function $g : \mathbb{R} \rightarrow \mathbb{R}$, so $\frac{\partial g}{\partial f}(f(\mathbf{x})) = \frac{dg}{df}(f(\mathbf{x})) = g'(f(\mathbf{x}))$. Then,

$$\nabla (g \circ f)(\mathbf{x}) = g'(f(\mathbf{x})) \begin{pmatrix} \frac{\partial f}{\partial x1}(\mathbf{x}) \\ \frac{\partial f}{\partial x2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial xn}(\mathbf{x}) \end{pmatrix} = g'(f(\mathbf{x}))\nabla f(\mathbf{x})$$

Problem 4

Part (a)

$$\nabla f_i(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_i}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f_i}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f_i}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

And

$$\nabla f_i(\mathbf{x})^T = \begin{pmatrix} \frac{\partial f_i}{\partial x_1}(\mathbf{x}) & \frac{\partial f_i}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_i}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

Since

$$Df(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

Obviously, the ith row of $Df(\mathbf{x})$ is simply $\nabla f_i(\mathbf{x})^T$ for all $i = 1, \cdots, m$ so that

$$Df(\mathbf{x}) = \begin{pmatrix} \nabla f_1(\mathbf{x})^T \\ \nabla f_2(\mathbf{x})^T \\ \vdots \\ \nabla f_m(\mathbf{x})^T \end{pmatrix}$$

Part (b)

At first,

$$\frac{\partial(g \circ f)}{\partial x_1}(\mathbf{x}) = \frac{\partial g}{\partial f_1}(f_1(\mathbf{x}))\frac{\partial f_1}{\partial x_1}(\mathbf{x}) + \frac{\partial g}{\partial f_2}(f_2(\mathbf{x}))\frac{\partial f_2}{\partial x_1}(\mathbf{x}) + \cdots + \frac{\partial g}{\partial f_m}(f_m(\mathbf{x}))\frac{\partial f_m}{\partial x_1}(\mathbf{x})$$

$$= \begin{pmatrix} \frac{\partial g}{\partial f_1}(f_1(\mathbf{x})) & \frac{\partial g}{\partial f_2}(f_2(\mathbf{x})) & \cdots & \frac{\partial g}{\partial f_m}(f_m(\mathbf{x})) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) \end{pmatrix}$$

$$= (\nabla g(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right)^T \nabla g(f(\mathbf{x}))$$

Then

$$\nabla(g \circ f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial(g \circ f)}{\partial x 1}(\mathbf{x}) \\ \frac{\partial(g \circ f)}{\partial x 2}(\mathbf{x}) \\ \vdots \\ \frac{\partial(g \circ f)}{\partial x n}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial f}{\partial x 1}(\mathbf{x})\right)^T \nabla g(f(\mathbf{x})) \\ \left(\frac{\partial f}{\partial x 2}(\mathbf{x})\right)^T \nabla g(f(\mathbf{x})) \\ \vdots \\ \left(\frac{\partial f}{\partial x n}(\mathbf{x})\right)^T \nabla g(f(\mathbf{x})) \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial f}{\partial x 1}(\mathbf{x})\right)^T \\ \left(\frac{\partial f}{\partial x 2}(\mathbf{x})\right)^T \\ \vdots \\ \left(\frac{\partial f}{\partial x n}(\mathbf{x})\right)^T \end{pmatrix} \nabla g(f(\mathbf{x}))$$

$$= \begin{pmatrix} \frac{\partial f 1}{\partial x 1}(\mathbf{x}) & \frac{\partial f 2}{\partial x 1}(\mathbf{x}) & \cdots & \frac{\partial f m}{\partial x 1}(\mathbf{x}) \\ \frac{\partial f 1}{\partial x 2}(\mathbf{x}) & \frac{\partial f 2}{\partial x 2}(\mathbf{x}) & \cdots & \frac{\partial f m}{\partial x 2}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f 1}{\partial x n}(\mathbf{x}) & \frac{\partial f 2}{\partial x n}(\mathbf{x}) & \cdots & \frac{\partial f m}{\partial x n}(\mathbf{x}) \end{pmatrix} \nabla g(f(\mathbf{x})) = Df(\mathbf{x})^T \nabla g(f(\mathbf{x}))$$

Part (c)

We have $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^k \to \mathbb{R}^m$, so $g \circ f : \mathbb{R}^n \to \mathbb{R}^m$, so the number of f_i is k and the number of g_i is m.

$$D(g \circ f)(\mathbf{x}) = \begin{pmatrix} \frac{\partial(g 1 \circ f)}{\partial x 1}(\mathbf{x}) & \frac{\partial(g 1 \circ f)}{\partial x 2}(\mathbf{x}) & \cdots & \frac{\partial(g 1 \circ f)}{\partial x n}(\mathbf{x}) \\ \frac{\partial(g 2 \circ f)}{\partial x 1}(\mathbf{x}) & \frac{\partial(g 2 \circ f)}{\partial x 2}(\mathbf{x}) & \cdots & \frac{\partial(g 2 \circ f)}{\partial x n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(g m \circ f)}{\partial x 1}(\mathbf{x}) & \frac{\partial(g m \circ f)}{\partial x 2}(\mathbf{x}) & \cdots & \frac{\partial(g m \circ f)}{\partial x n}(\mathbf{x}) \end{pmatrix}$$

We will first focus on one single element in this matrix,

$$\frac{\partial(g_1 \circ f)}{\partial x_1}(\mathbf{x}) = \frac{\partial g_1}{\partial f_1}(f_1(\mathbf{x}))\frac{\partial f_1}{\partial x_1}(\mathbf{x}) + \frac{\partial g}{\partial f_2}(f_2(\mathbf{x}))\frac{\partial f_2}{\partial x_1}(\mathbf{x}) + \cdots + \frac{\partial g_1}{\partial f_k}(f_k(\mathbf{x}))\frac{\partial f_k}{\partial x_1}(\mathbf{x})$$

$$= \begin{pmatrix} \frac{\partial g_1}{\partial f_1}(f_1(\mathbf{x})) & \frac{\partial g_1}{\partial f_2}(f_2(\mathbf{x})) & \cdots & \frac{\partial g_1}{\partial f_k}(f_k(\mathbf{x})) \end{pmatrix} \begin{pmatrix} \frac{\partial f 1}{\partial x 1}(\mathbf{x}) \\ \frac{\partial f 2}{\partial x 1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f k}{\partial x 1}(\mathbf{x}) \end{pmatrix} = (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right)$$

So

$$D(g \circ f)(\mathbf{x}) = \begin{pmatrix} (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 1}(\mathbf{x})\right) & (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 2}(\mathbf{x})\right) & \cdots & (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x n}(\mathbf{x})\right) \\ (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 1}(\mathbf{x})\right) & (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 2}(\mathbf{x})\right) & \cdots & (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x n}(\mathbf{x})\right) \\ \vdots & \vdots & \ddots & \vdots \\ (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 1}(\mathbf{x})\right) & (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x 2}(\mathbf{x})\right) & \cdots & (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x n}(\mathbf{x})\right) \end{pmatrix}$$

$$Dg(f(\mathbf{x}))Df(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial f_1}(f_1(\mathbf{x})) & \frac{\partial g_1}{\partial f_2}(f_2(\mathbf{x})) & \cdots & \frac{\partial g_1}{\partial f_k}(f_k(\mathbf{x})) \\ \frac{\partial g_2}{\partial f_1}(f_1(\mathbf{x})) & \frac{\partial g_2}{\partial f_2}(f_2(\mathbf{x})) & \cdots & \frac{\partial g_2}{\partial f_k}(f_k(\mathbf{x})) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial f_1}(f_1(\mathbf{x})) & \frac{\partial g_m}{\partial f_2}(f_2(\mathbf{x})) & \cdots & \frac{\partial g_m}{\partial f_k}(f_k(\mathbf{x})) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(\mathbf{x}) & \frac{\partial f_k}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_k}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

We observe that the ith row of $Dg(f(\mathbf{x}))$ is just $(\nabla g_i(f(\mathbf{x})))^T$ for all $i = 1, 2, \cdots, m$, and the jth column of $Df(\mathbf{x})$ is just $\frac{\partial f}{\partial x_j}(\mathbf{x})$ for all $j = 1, 2, \cdots, n$, thus

$$Dg(f(\mathbf{x}))Df(\mathbf{x}) = \begin{pmatrix} (\nabla g_1(f(\mathbf{x})))^T \\ (\nabla g_2(f(\mathbf{x})))^T \\ \vdots \\ (\nabla g_m(f(\mathbf{x})))^T \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) & \frac{\partial f}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

$$= \begin{pmatrix} (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right) & (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_2}(\mathbf{x})\right) & \cdots & (\nabla g_1(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_n}(\mathbf{x})\right) \\ (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right) & (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_2}(\mathbf{x})\right) & \cdots & (\nabla g_2(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_n}(\mathbf{x})\right) \\ \vdots & \vdots & \ddots & \vdots \\ (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_1}(\mathbf{x})\right) & (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_2}(\mathbf{x})\right) & \cdots & (\nabla g_m(f(\mathbf{x})))^T \left(\frac{\partial f}{\partial x_n}(\mathbf{x})\right) \end{pmatrix}$$

Therefore, $D(g \circ f)(\mathbf{x}) = Dg(f(\mathbf{x}))Df(\mathbf{x})$.

In [ ]: