HOMEWORK PROBLEMS 08, ANLY 561, FALL 2017
DUE 11/10/17 BY MIDNIGHT

Readings: §5.9, 6.1, 6.2, 6.3 of **Goodfellow, Bengio, and Courville**. For more on the primal-dual method, read §11.7 of **Boyd and Vandenberghe**.

Exercises:

1. For any fixed nonzero $\mathbf{v} \in \mathbb{R}^n$ and any fixed $b \in \mathbb{R}$, the set of points $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\mathbf{v}^T\mathbf{x} - b = 0$$

is called an **affine hyperplane** of dimension $n-1$ in $\mathbb{R}^n$. Let $\mathbf{y} \in \mathbb{R}^n$ and find the solution $\mathbf{x}^*$ (in terms of $\mathbf{v}$, b, and $\mathbf{y}$) to the program

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{y} - \mathbf{x}\|_2^2 \text{ subject to } \mathbf{v}^T\mathbf{x} - b = 0.$$

This $\mathbf{x}^*$ is called the **projection** of $\mathbf{y}$ onto the affine hyperplane determined by $\mathbf{v}$ and b.

2. Use the code from Lecture 03 Part IX (Support Vector Machines and Duality) to generate a dataset $\{\mathbf{x}^{(i)}\}_{i=0}^{19} \subset \mathbb{R}^2$ with labels $\{y^{(i)}\}_{i=0}^{19} \in \{-1, 1\}$ such that $\mathbf{x}^{(i)}$ is drawn uniformly from the unit disc centered at $(-2,-2)$ and $y^{(i)} = -1$for $i = 0, \ldots, 9$, and $x^{(i)}$ is drawn uniformly from the unit disc centered at $(2,2)$ and $y^{(i)} = +1$ for $i = 10, \ldots, 19$.

   (a) Code up a Phase I method to find an interior point for the (3D) program

   $$\min_{\mathbf{v}\in\mathbb{R}^2, b\in\mathbb{R}} \frac{1}{2}\|\mathbf{v}\|^2 \text{ subject to } h_i(\mathbf{v}, b) \leq 0$$

   where $h_i(\mathbf{v}, b) = 1 - y^{(i)}(\mathbf{v}^T x^{(i)} - b)$. That is, use the log barrier method to solve the 4D program

   $$\min_{\mathbf{v}\in\mathbb{R}^2, b\in\mathbb{R}, z\in\mathbb{R}} \widetilde{f}(\mathbf{v}, b, z) \text{ subject to } \widetilde{h}_i(\mathbf{v}, b, z) \leq 0$$

   where $\widetilde{f}(\mathbf{v}, b, z) = z$ and $\widetilde{h}_i(\mathbf{v}, b, z) = 1 - y^{(i)}(\mathbf{v}^T x^{(i)} - b) - z$, but stop as soon as you find an iterate $(\mathbf{v}^{(k)}, b^{(k)}, z^{(k)})$ such that $z^{(k)} < 0$ (which then implies that $(\mathbf{v}^{(k)}, b^{(k)})$ is an interior point). Initialize the Phase I method with the data $\mathbf{v}^{(0)} = (-20, 20)$, $b^{(0)} = 10$, and determine $z^{(0)}$ (in your code) such that $(\mathbf{v}^{(0)}, b^{(0)}, z^{(0)})$ is an interior point of the 4D convex program.

   (b) Use the $(\mathbf{v}^{(k)}, b^{(k)})$ from part (a) to initialize Phase II for your generated data. That is, use this interior point to initialize the log barrier method for the 3D program. Use 3 centering steps, $M = 10$, and 5 iterations in the outer loop with 2 inner loop iterations each. For each backtracking step, use Newton search directions, $\alpha = 0.1$, and $\beta = 0.5$.

   (c) Use the $(\mathbf{v}^{(k)}, b^{(k)})$ from part (a) to initialize the primal-dual algorithm. Take 10 steps with $\nu = 10$.

   (d) Compare the answers you get from parts (b) and (c), and provide a simultaneous plot of your data, the separating line corresponding to the interior point $(\mathbf{v}^{(k)}, b^{(k)})$ from part (a), and the two separating lines from parts (b) and (c).