ANLY 561 HW

Name:Yuqi Wang
NetID:yw545

Problem 1

Part (a)

Observe that this program

$$\max_{x\in\mathbb{R}^2} x_1 \text{ subject to } x_2 + (x_1 - 1)^3 \leq 0, x_1 \geq 0, x_2 \geq 0.$$

is equivalent to the program

$$\min_{x\in\mathbb{R}^2} -x_1 \text{ subject to } x_2 + (x_1 - 1)^3 \leq 0, x_1 \geq 0, x_2 \geq 0.$$

Now we let $f(x_1, x_2) = -x_1$, $h_1(x_1, x_2) = x_2 + (x_1 - 1)^3$, $h_2(x_1, x_2) = -x_1$, $h_3(x_1, x_2) = -x_2$, then we have

$$\nabla f(x_1, x_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_1(x_1, x_2) = \begin{pmatrix} 3(x_1 - 1)^2 \\ 1 \end{pmatrix}, \nabla h_2(x_1, x_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_3(x_1, x_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

At $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\nabla h_1(1, 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \nabla h_2(1, 0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla h_3(1, 0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

So we can see that $\nabla h_1(1, 0)$ and $\nabla h_3(1, 0)$ are linearly dependent because if we form a eqution that $c_1\nabla h_1(1, 0) + c_2\nabla h_3(1, 0) = \mathbf{0}$, c_1 could be any number (so not necessary to be 0), then $c_2 = -c_1$ could satisfy the equation.

Since $\nabla h_1(1, 0)$ and $\nabla h_3(1, 0)$ are linearly dependent, the Linear Independence Constraint Qualification fails at $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Part (b)

The KKT conditions:
(Stationarity)

$$-\begin{pmatrix}-1\\0\end{pmatrix}=\lambda_1\begin{pmatrix}3(x_1-1)^2\\1\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\0\end{pmatrix}+\lambda_3\begin{pmatrix}0\\-1\end{pmatrix}$$

(Primal Feasibility)

$$h_1(x_1,x_2)=x_2+(x_1-1)^3\leq 0, h_2(x_1,x_2)=-x_1\leq 0,, h_3(x_1,x_2)=-x_2\leq 0$$

(Dual Feasibility)

$$\lambda_1\geq 0,\lambda_2\geq 0,\lambda_3\geq 0$$

(Complementary Slackness)

$$\lambda_1 h_1(x_1,x_2)=\lambda_1(x_2+(x_1-1)^3)=0,\lambda_2 h_2(x_1,x_2)=\lambda_2(-x_1)=0,\lambda_3 h_3(x_1,x_2)=\lambda_3(-x_2)=0$$

Part (c)

As $x=\begin{pmatrix}1\\0\end{pmatrix}$,

$$-\begin{pmatrix}-1\\0\end{pmatrix}=\lambda_1\begin{pmatrix}0\\1\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\0\end{pmatrix}+\lambda_3\begin{pmatrix}0\\-1\end{pmatrix}$$

implies that λ_2 has to be $-1<0$, so it does not satisfy the Dual Feasibility. Therefore, $x=\begin{pmatrix}1\\0\end{pmatrix}$ does not satisfy the KKT conditions.

Part (d)

Consider the program

$$\max_{x \in \mathbb{R}^2} x_1 \text{ subject to } x_2 + (x_1 - 1)^3 \leq 0, x_1 \geq 0, x_2 \geq 0.$$

and we suppose that $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is NOT the solution to the above program, then there must exist a $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ such that $x_2^* + (x_1^* - 1)^3 \leq 0$, $x_1^* \geq 0$, $x_2^* \geq 0$ and also $x_1^* > 1$. Since $x_1^* > 1$, then $(x_1^* - 1)^3 > 0$. If we want to satisfy $x_2^* + (x_1^* - 1)^3 \leq 0$, x_2^* has to be less than 0. However, we also have a condition that $x_2^* \geq 0$. Thus, we get a contradiction.

So $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the solution to this program.

Problem 2

Part (a)

Consider the program

$$\min_{x\in\mathbb{R}^2} 3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_2x_3 + 3x_3^2 \text{ subject to } 0 \le x_1 \le 1, 0 \le x_2 \le 1, 0 \le x_3 \le 1$$

Now we let $f(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_2x_3 + 3x_3^2$, $h_1(x_1, x_2, x_3) = -x_1$, $h_2(x_1, x_2, x_3) = x_1 - 1$, $h_3(x_1, x_2, x_3) = -x_2$, $h_4(x_1, x_2, x_3) = x_2 - 1$, $h_5(x_1, x_2, x_3) = -x_3$, $h_6(x_1, x_2, x_3) = x_3 - 1$, then we have

$$\nabla^2 f(x_1, x_2, x_3) = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

So $\nabla^2 f$ has 1 by 1 leading principal minor: 6.
$\nabla^2 f$ has 2 by 2 leading principal minor: $6 * 6 - (-2)(-2) = 36 - 4 = 32$.
$\nabla^2 f$ has 3 by 3 leading principal minor: $6 * (6 * 6 - (-2)(-2)) - (-2)((-2)6 - 0) = 192 - 24 = 168$.
By Sylvester's Criterion, $\nabla^2 f(x_1, x_2, x_3)$ is positive definite. Then by Second Order Conditions for Convexity, $f(x_1, x_2, x_3)$ is strictly convex.
In addition, $h_1(x_1, x_2, x_3) = -x_1$, $h_2(x_1, x_2, x_3) = x_1 - 1$, $h_3(x_1, x_2, x_3) = -x_2$, $h_4(x_1, x_2, x_3) = x_2 - 1$, $h_5(x_1, x_2, x_3) = -x_3$, $h_6(x_1, x_2, x_3) = x_3 - 1$ are all affine so they are all convex. So the program is a convex program. And as $x = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$, it satisfies $h_1 < 0, h_2 < 0, h_3 < 0, h_4 < 0, h_5 < 0, h_6 < 0$, which means that Slater's condition holds. Therefore, by the Sufficient Conditions for Convex Programming, the solution to this program must satisfy the KKT conditions.

Part (b)

The KKT conditions:
(Stationarity)

$$-\begin{pmatrix} 6x_1 - 2x_2 \\ -2x_1 + 6x_2 - 2x_3 \\ -2x_2 + 6x_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_5 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \lambda_6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(Primal Feasibility)

$$h_1(x_1, x_2, x_3) = -x_1 \leq 0, h_2(x_1, x_2, x_3) = x_1 - 1 \leq 0, h_3(x_1, x_2, x_3) = -x_2 \leq 0,$$
$$h_4(x_1, x_2, x_3) = x_2 - 1 \leq 0, h_5(x_1, x_2, x_3) = -x_3 \leq 0, h_6(x_1, x_2, x_3) = x_3 - 1 \leq 0$$

(Dual Feasibility)

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0, \lambda_5 \geq 0, \lambda_6 \geq 0$$

(Complementary Slackness)

$$\lambda_1 h_1(x_1, x_2, x_3) = \lambda_1(-x_1) = 0, \lambda_2 h_2(x_1, x_2, x_3) = \lambda_2(x_1 - 1) = 0, \lambda_3 h_3(x_1, x_2, x_3) = \lambda_3(-x_2) = 0,$$
$$\lambda_4 h_4(x_1, x_2, x_3) = \lambda_4(x_2 - 1) = 0, \lambda_5 h_5(x_1, x_2, x_3) = \lambda_5(-x_3) = 0, \lambda_6 h_6(x_1, x_2, x_3) = \lambda_6(x_3 - 1) = 0$$

Part (c)

We have

$$\begin{pmatrix} 2x_2 - 6x_1 = -\lambda_1 + \lambda_2 & 2x_1 - 6x_2 + 2x_3 = -\lambda_3 + \lambda_4 & 2x_2 - 6x_3 = -\lambda_5 + \lambda_6 \\ \lambda_1(-x_1) = 0 & \lambda_2(x_1 - 1) = 0 & \lambda_3(-x_2) = 0 \\ \lambda_4(x_2 - 1) = 0 & \lambda_5(-x_3) = 0 & \lambda_6(x_3 - 1) = 0 \end{pmatrix}$$

and

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0, \lambda_5 \geq 0, \lambda_6 \geq 0$$
$$0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1$$

- Case 1: $x_1 = 0$, then $x_1 - 1 \neq 0$, so $\lambda_2 = 0$,
  - Case 1-1: $x_2 = 0$, $x_3 = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$, it satisfies the KKT conditions.
  - Case 1-2: $x_2 = 0$, $x_3 = 1$, then $\lambda_4 = \lambda_5 = 0$, so $-\lambda_3 = 2x_3 = 2 \implies \lambda_3 = -2 < 0$, so the KKT conditions fail.
  - Case 1-3: $x_2 = 1$, $x_3 = 0$, then $\lambda_3 = \lambda_6 = 0$, so $\lambda_4 = -6 \implies \lambda_4 < 0$, so the KKT conditions fail.
  - Case 1-4: $x_2 = 1$, $x_3 = 1$, then $\lambda_3 = \lambda_5 = 0$, so $\lambda_6 = 2 - 6 = -4 \implies \lambda_6 < 0$, so the KKT conditions fail.

- Case 2: $x_1 = 1$, then $\lambda_1 = 0$,
  - Case 2-1: $x_2 = 0$, $x_3 = 0$, then $\lambda_4 = \lambda_6 = 0$, so $-\lambda_3 = 2 - 0 + 0 = 2 \implies \lambda_3 < 0$, so the KKT conditions fail.
  - Case 2-2: $x_2 = 0$, $x_3 = 1$, then $\lambda_4 = \lambda_5 = 0$, so $-\lambda_3 = 2 - 0 + 2 = 4 \implies \lambda_3 < 0$, so the KKT conditions fail.
  - Case 2-3: $x_2 = 1$, $x_3 = 0$, then $\lambda_3 = \lambda_6 = 0$, so $-\lambda_5 = 2 - 0 = 2 \implies \lambda_5 < 0$, so the KKT conditions fail.
  - Case 2-4: $x_2 = 1$, $x_3 = 1$, then $\lambda_4 = \lambda_5 = 0$, so $\lambda_6 = 2 - 6 = -4 \implies \lambda_6 < 0$, so the KKT conditions fail.

- Case 3: $\lambda_1 = \lambda_2 = 0$ and $x_1 \neq 0$ or 1, then $6x_1 = 2x_2 \implies 3x_1 = x_2$, then we will have $x_2 \neq 0 \implies \lambda_3 = 0$.
  - Case 3-1: if $x_2 = 1$ and $x_1 = \frac{1}{3}$, then $\frac{2}{3} - 6 + 2x_3 = \lambda_4$. Since $x_3 \leq 1$, then $2x_3 \leq 2$, implies that $\frac{2}{3} - 6 + 2x_3 = \lambda_4 < 0$, so the KKT conditions fail.
  - Case 3-2: if $x_2 < 1$, then $\lambda_3 = \lambda_4 = 0$, so $2x_1 - 6x_2 + 2x_3 = 0 = -16x_1 + 2x_3$. And also λ_5, λ_6 cannot be 0 at the same time in this case, so $x_3 = 0$ or 1. x_3 cannot be 0 since $x_1 \neq 0$ and $0 = -16x_1 + 2x_3$. Then $x_3 = 1$, so $x_1 = \frac{1}{8}$, and $x_2 = \frac{3}{8}$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$, $\lambda_6 = 2x_2 - 6x_3 = \frac{3}{4} - 6 < 0$, so the KKT conditions fail.

- Case 4: $\lambda_1 = \lambda_2 = 0$ and $x_1 = 0$ or 1, then $6x_1 = 2x_2 \implies 3x_1 = x_2$, since $x_2 \leq 1$, $x_1 \neq 1$, so $x_1 = x_2 = 0 \implies \lambda_4 = 0$. In order to avoid duplication of cases, $x_3 \neq 0$.
  Then $2x_3 = -\lambda_3$. Since $0 < x_3 \leq 1$, then $\lambda_3 < 0$, so the KKT conditions fail.

- Case 5: $\lambda_3 = \lambda_4 = 0$ and $x_2 \neq 0$ or 1, then $2x_1 - 6x_2 + 2x_3 = 0 \implies 2x_3 = 6x_2 - 2x_1$. In order to avoid duplication of cases, $\lambda_1, \lambda_2 \neq 0$ at the same time.
  - Case 5-1: $x_1 = 0, \lambda_2 = 0$, then $2x_2 = -\lambda_1 \implies \lambda_1 < 0$, so the KKT conditions fail.
  - Case 5-2: $x_1 = 1, \lambda_1 = 0$, then $2x_2 - 6 = \lambda_2 \implies \lambda_2 < 0$, so the KKT conditions fail.

- Case 6: $\lambda_3 = \lambda_4 = 0$ and $x_2 = 0$ or 1, then $2x_1 - 6x_2 + 2x_3 = 0 \implies 2x_3 = 6x_2 - 2x_1$. In order to avoid duplication of cases, $\lambda_1, \lambda_2 \neq 0$ at the same time.
  - Case 6-1: $x_2 = 0$, then $2x_3 = -2x_1$, and we have $x_1 = 0$ or 1. if $x_1 = 0$, then $x_2 = 0$, $x_3 = 0$, this is the same as Case 1-1. If $x_1 = 1$, $x_3 < 0$, so the KKT conditions fail.
  - Case 6-2: $x_2 = 1$, then $2x_3 = 6 - 2x_1$. Since $x_1 = 0$ or 1, then $2x_3 = 6$ or 4. In both cases, x_3 is not in the feasible range, so the KKT conditions fail.

- Case 7: $\lambda_5 = \lambda_6 = 0$ and $x_3 \neq 0$ or 1, then $6x_3 = 2x_2 \implies 3x_3 = x_2$, then we will have $x_2 \neq 0 \implies \lambda_3 = 0$.
  - Case 7-1: if $x_2 = 1$ and $x_3 = \frac{1}{3}$, then $\frac{2}{3} - 6 + 2x_1 = \lambda_4$. Since $x_1 \leq 1$, then $2x_1 \leq 2$, implies that $\frac{2}{3} - 6 + 2x_1 = \lambda_4 < 0$, so the KKT conditions fail.
  - Case 7-2: if $x_2 < 1$, then $\lambda_3 = \lambda_4 = 0$, so $2x_1 - 6x_2 + 2x_3 = 0 = -16x_3 + 2x_1$. And also λ_1, λ_2 cannot be 0 at the same time in this case, so $x_1 = 0$ or 1. x_1 cannot be 0 since $x_3 \neq 0$ and $0 = -16x_1 + 2x_3$. Then $x_1 = 1$, so $x_3 = \frac{1}{8}$, and $x_2 = \frac{3}{8}$, and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ $-\lambda_1 = 2x_2 - 6x_1 = \frac{3}{4} - 6 < 0$, so the KKT conditions fail.

- Case 8: $\lambda_5 = \lambda_6 = 0$ and $x_3 = 0$ or 1, then $6x_3 = 2x_2 \implies 3x_3 = x_2$, since $x_2 \leq 1$, $x_3 \neq 1$, so $x_3 = x_2 = 0 \implies \lambda_4 = 0$. In order to avoid duplication of cases, $x_1 \neq 0$.
  Then $2x_1 = -\lambda_3$. Since $0 < x_1 \leq 1$, then $\lambda_3 < 0$, so the KKT conditions fail.

Therefore, we can get that as $x_1 = 0, x_2 = 0, x_3 = 0$, $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the only point that satisfies the KKT conditions.

Part (d)

$x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ **is the solution to this program** not only because that $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the only point that satisfies the KKT conditions, but also:

Consider the program

$$\min_{x \in \mathbb{R}^2} 3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_2x_3 + 3x_3^2 \text{ subject to } 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 1$$

then, observe that

$$f(x_1, x_2, x_3) = 3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_2x_3 + 3x_3^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + 2x_1^2 + x_2^2 + 2x_3^2$$

We know that $(x_1 - x_2)^2 \geq 0$, $(x_2 - x_3)^2 \geq 0$, $2x_1^2 \geq 0$, $x_2^2 \geq 0$, $2x_3^2 \geq 0$, so

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + 2x_1^2 + x_2^2 + 2x_3^2 \geq 0$$

So the minimum value of $f(x_1, x_2, x_3)$ is 0. Then the minimum value occurs as $(x_1 - x_2)^2 = 0$, $(x_2 - x_3)^2 = 0$, $2x_1^2 = 0$, $x_2^2 = 0$, $2x_3^2 = 0$, which means

$$x_1 = 0, x_2 = 0, x_3 = 0.$$

Thus, $x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the solution to this program.