# HOMEWORK PROBLEMS 04, ANLY 561, FALL 2017
# DUE 10/06/17

<u>**Exercises:**</u>

1. Using the chain rule
$$(f \circ \gamma)'(t) = \nabla f(\gamma(t))^T \gamma'(t)$$
for $f \in C^1(\mathbb{R}^2)$ and
$$\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$$
with $\gamma_1, \gamma_2 \in C^1(\mathbb{R})$), justify the chain rule
$$\nabla(f \circ g)(x, y) = Dg(x, y)^T \nabla f(g(x, y))$$
where $f \in C^1(\mathbb{R}^2)$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$,
$$g(x, y) = \begin{pmatrix} g_1(x, y) \\ g_2(x, y) \end{pmatrix},$$
$g_1, g_2 \in C^1(\mathbb{R}^2)$ and
$$Dg(x, y) = \begin{pmatrix} \frac{\partial g_1}{\partial x}(x, y) & \frac{\partial g_1}{\partial y}(x, y) \\ \frac{\partial g_2}{\partial x}(x, y) & \frac{\partial g_2}{\partial y}(x, y) \end{pmatrix}$$
is the **Jacobian** of $g$.

2. For the following two parts, consider the data

| i | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | -1 | -1 | -1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $y_i$ | -1 | -1 | -1 | 1 | -1 | 1 | 1 | 1 | 1 | -1 |

   (a) Show that the Sum of Square Errors (SSE) function,
   $$\ell_{\text{lin}}(\beta_0, \beta_1) = \frac{1}{10} \sum_{i=1}^{10} (y_i - \beta_1 x_i - \beta_0)^2,$$
   is strictly convex, and find the unique minimizer.

   (b) Show that the negative log-likelihood function for logistic regression,
   $$\ell_{\text{log}}(\beta_0, \beta_1) = \frac{1}{10} \sum_{i=1}^{10} \log\left(1 + e^{-y_i(\beta_1 x_i + \beta_0)}\right)$$
   is strictly convex, and write down the necessary and sufficient conditions for optimality of $(\beta_0^*, \beta_1^*)$ in terms of the $x_i$'s and $y_i$'s.

3. Use Lagrange multipliers to show that, if $A$ is a symmetric 2 by 2 matrix, then any solution to
$$\max_{\mathbf{x} \in \mathbb{R}^2} \frac{1}{2} \mathbf{x}^T A \mathbf{x} \text{ subject to } \|x\|^2 = 1$$
is an eigenvector of $A$ corresponding to the largest eigenvalue of $A$.