ANLY 561 HW3

Name:Yuqi Wang
NetID:yw545

In [12]:

```
'''
This code imports numpy packages and allows us to pass data from python to global javascri
objects. It was developed by znah@github
'''

import json
import numpy as np
import numpy.random as rd
from ipywidgets import widgets
from IPython.display import HTML, Javascript, display

def json_numpy_serializer(o):
    if isinstance(o, np.ndarray):
        return o.tolist()
    raise TypeError("{} of type {} is not JSON serializable".format(repr(o), type(o)))

def jsglobal(**params):
    code = [];
    for name, value in params.items():
        jsdata = json.dumps(value, default=json_numpy_serializer)
        code.append("window.{}={};".format(name, jsdata))
    display(Javascript("\n".join(code)))
```

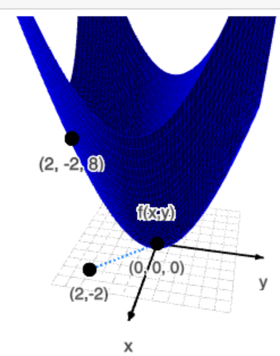
In [13]:
```
%%javascript

// Loading the compiled MathBox bundle.
require.config({
    baseUrl:'', paths: {mathBox: 'http://localhost:8888/tree/Desktop/static/mathbox/build/m
    // online compilation
    //baseUrl: '', paths: {mathBox: '../static/mathbox/build/mathbox-bundle'}
    // online compilation without local library-- remove baseUrl
    //paths: {mathBox: '//cdn.rawgit.com/unconed/mathbox/eaeb8e15/build/mathbox-bundle'}
});

// Minified graphing functions

window.with_mathbox=function(element,func){require(['mathBox'],function(){var mathbox=math
var intervalId=setInterval(function(){if(three.element.offsetParent===null){clearInterval(
var visible=isInViewport(three.canvas);if(three.Loop.running!=visible){visible?three.Loop.
view.area({id:'yaxis',width:1,height:1,axes:[1,3],expr:function(emit,x,y,i,j){emit(4,0,0);
window.addSequence=function(view,seq,col){var idx=0;var d=new Date();var start=d.getTime()
start=now}
emit(seq[idx][1],seq[idx][2],seq[idx][0])},items:1,channels:3}).point({color:col,points:'<
```

<IPython.core.display.Javascript object>

Problem 1

$$f(x, y) = x^2 + y^2$$

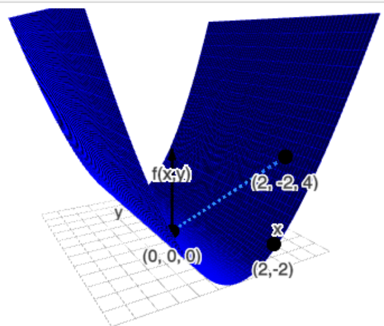
In [14]:
```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (x*x + y*y);
    };

    var view = plotGraph(mathbox, fcn);

    addSegment(view, [2, -2, 0], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 0], 0x000000, '(2,-2)');
    addPoint(view, [0, 0, 0], 0x000000, '(0, 0, 0)');

    addSegment(view, [2, -2, fcn(2, -2)], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 8], 0x000000, '(2, -2, 8)');

})
```



Comment: It is strictly convex.

$$f(x, y) = x^2$$

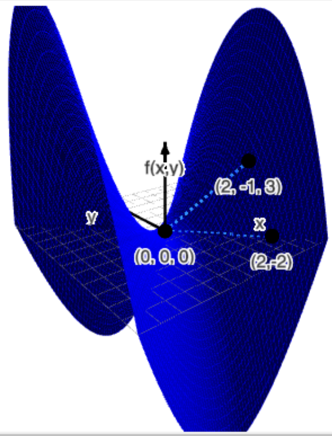
In [52]:
```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (x*x);
    };

    var view = plotGraph(mathbox, fcn);

    addSegment(view, [2, -2, 0], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 0], 0x000000, '(2,-2)');
    addPoint(view, [0, 0, 0], 0x000000, '(0, 0, 0)');

    addSegment(view, [2, -2, fcn(2, -2)], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 4], 0x000000, '(2, -2, 4)');

})
```



Comment: It is convex but not strictly convex.

$$f(x, y) = x^2 - y^2$$

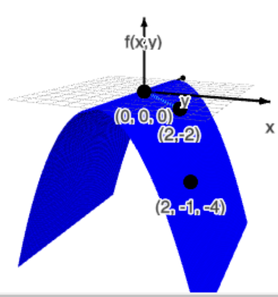
In [53]:
```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (x*x-y*y);
    };

    var view = plotGraph(mathbox, fcn);

    addSegment(view, [2, -2, 0], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 0], 0x000000, '(2,-2)');
    addPoint(view, [0, 0, 0], 0x000000, '(0, 0, 0)');

    addSegment(view, [2, -1, fcn(2, -1)], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -1, 3], 0x000000, '(2, -1, 3)');

})
```



Comment: It is NOT convex.

$$f(x, y) = -x^2$$

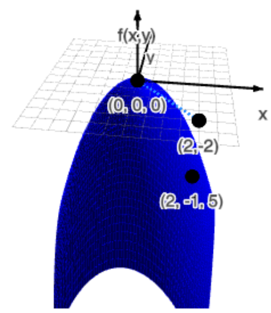
In [54]:
```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (-x*x);
    };

    var view = plotGraph(mathbox, fcn);

    addSegment(view, [2, -2, 0], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 0], 0x000000, '(2,-2)');
    addPoint(view, [0, 0, 0], 0x000000, '(0, 0, 0)');

    addSegment(view, [2, -1, fcn(2, -1)], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -1, -4], 0x000000, '(2, -1, -4)');

})
```



Comment: It is NOT convex.

$$f(x, y) = -x^2 - y^2$$

In [55]:
```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (-x*x-y*y);
    };

    var view = plotGraph(mathbox, fcn);

    addSegment(view, [2, -2, 0], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -2, 0], 0x000000, '(2,-2)');
    addPoint(view, [0, 0, 0], 0x000000, '(0, 0, 0)');

    addSegment(view, [2, -1, fcn(2, -1)], [0, 0, 0], 0x3090FF);
    addPoint(view, [2, -1, -5], 0x000000, '(2, -1, 5)');

})
```



Comment: It is NOT convex.

Problem 2

Try to prove that $f(x_1, x_2) \geq f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$ or $f(x_1, x_2) > f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$
OR
disprove it.

$$f(x, y) = x^2 + y^2$$

Proof:
$f(x_1, x_2) = x_1^2 + x_2^2$
$f(y_1, y_2) = y_1^2 + y_2^2$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1)$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 2y_2(x_2 - y_2)$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$

$= y_1^2 + y_2^2 + 2y_1(x_1 - y_1) + 2y_2(x_2 - y_2)$
$= y_1^2 + y_2^2 + 2y_1x_1 - 2y_1^2 + 2y_2x_2 - 2y_2^2$
$= 2y_1x_1 + 2y_2x_2 - y_2^2 - y_1^2$
$f(x_1, x_2) - [f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)]$
$= x_1^2 + x_2^2 - 2y_1x_1 - 2y_2x_2 + y_2^2 + y_1^2$
$= (y_1 - x_1)^2 + (y_2 - x_2)^2 > 0$ since $\mathbf{x} \neq \mathbf{y}, y_1 - x_1 \neq 0, \text{and } y_2 - x_2 \neq 0 \text{ at the same time.}$
Therefore,
$f(x_1, x_2) > f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in X$, so this function is strictly convex.

$f(x, y) = x^2$

Proof:
$f(x_1, x_2) = x_1^2$
$f(y_1, y_2) = y_1^2$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1)$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
$= y_1^2 + 2y_1(x_1 - y_1)$
$= y_1^2 + 2y_1x_1 - 2y_2^2$
$= 2y_1x_1 - y_1^2$
$f(x_1, x_2) - [f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)]$
$= x_1^2 - 2y_1x_1 + y_1^2$
$= (y_1 - x_1)^2 \geq 0$ since y_1 could be equal to x_1 as long as $y_2 \neq x_2$.
Therefore,
$f(x_1, x_2) \geq f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$ for all $\mathbf{x} \neq \mathbf{y} \in \mathbb{R}^2$, so this function is convex.

$f(x, y) = x^2 - y^2$

Proof:
As $(x_1, x_2) = (1, -1)$ and $(y_1, y_2) = (1, 1)$
$f(x_1, x_2) = x_1^2 - x_2^2 = 0$
$f(y_1, y_2) = y_1^2 - y_2^2 = 0$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = 2y_1(x_1 - y_1) = 0$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = -2y_2(x_2 - y_2) = 4$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 4$
$f(x_1, x_2) = 0 < 4 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}^2 \text{ s.t. } f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

$$f(x, y) = -x^2$$

Proof:
As $(x_1, x_2) = (1, 0)$ and $(y_1, y_2) = (-1, 0)$
$f(x_1, x_2) = -x_1^2 = -1$
$f(y_1, y_2) = y_1^2 = -1$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = -2y_1(x_1 - y_1) = 4$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 3$
$f(x_1, x_2) = -1 < 3 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}^2$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

$$f(x, y) = -x^2 - y^2$$

Proof:
As $(x_1, x_2) = (1, 0)$ and $(y_1, y_2) = (-1, 0)$
$f(x_1, x_2) = -x_1^2 - x_2^2 = -1$
$f(y_1, y_2) = y_1^2 - y_2^2 = -1$
$\partial_1 f(y_1, y_2)(x_1 - y_1) = -2y_1(x_1 - y_1) = 4$
$\partial_2 f(y_1, y_2)(x_2 - y_2) = -2y_2(x_2 - y_2) = 0$
So
$f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2) = 3$
$f(x_1, x_2) = -1 < 3 = f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$
Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}^2$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

Problem 3

$f(x, y) = \frac{y^2}{\sqrt{x^2+y^2}}$

Part a

Let $x = rcos(\theta), y = rsin(\theta)$, then

$$f(x, y) = f(r, \theta) = \frac{r^2 sin^2(\theta)}{\sqrt{r^2 cos^2(\theta) + r^2 sin^2(\theta)}} = \frac{r^2 sin^2(\theta)}{\sqrt{r^2(cos^2(\theta) + sin^2(\theta))}}$$
$$= \frac{r^2 sin^2(\theta)}{\sqrt{r^2}} = \frac{r^2 sin^2(\theta)}{r} = rsin^2(\theta)$$

Now if we let $r \to 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = rsin^2(\theta) = 0$$

Therefore, $f(x,y)$ is continuous at $(0,0)$.

Part b

$$g(t) = \frac{b^2t^2}{\sqrt{a^2t^2+b^2t^2}} = \frac{b^2t^2}{t\sqrt{a^2+b^2}} = \frac{b^2}{\sqrt{a^2+b^2}}t$$

So $g(t)$ is a just a line with slope $\frac{b^2}{\sqrt{a^2+b^2}}$.

Let $x, y \in \mathbb{R}$.
Let $k \in (0,1)$
$g((1-k)x+ky)$
$= \frac{b^2}{\sqrt{a^2+b^2}}((1-k)x+ky)$
$= (1-k)\frac{b^2}{\sqrt{a^2+b^2}}x + k\frac{b^2}{\sqrt{a^2+b^2}}y$
$= (1-k)g(x)+kg(y)$
So, by definition, $g(t)$ is convex.

Part c

In [19]:

```
%%javascript

with_mathbox(element, function(mathbox) {

    var fcn = function(x, y) {
      return (y*y) /Math.sqrt(x*x+y*y);
    };

    var view = plotGraph(mathbox, fcn);

})
```

<IPython.core.display.Javascript object>

Proof:
As $(x_1,x_2) = (2,1)$ and $(y_1,y_2) = (-2,1)$
$f(x_1,x_2) = \frac{1}{\sqrt{5}}$
$f(y_1,y_2) = \frac{1}{\sqrt{5}}$
$\partial_1 f(y_1,y_2)(x_1-y_1) = (-y_1y_2^2(y_1^2+y_2^2)^{-\frac{3}{2}})(x_1-y_1) = 8 * 5^{-\frac{3}{2}} > 0$
$\partial_2 f(y_1,y_2)(x_2-y_2) = 0$ since $x_2 = y_2$
So
$f(y_1,y_2) + \partial_1 f(y_1,y_2)(x_1-y_1) + \partial_2 f(y_1,y_2)(x_2-y_2) = \frac{1}{\sqrt{5}} + 8 * 5^{-\frac{3}{2}}$
$f(x_1,x_2) = \frac{1}{\sqrt{5}} < \frac{1}{\sqrt{5}} + 8 * 5^{-\frac{3}{2}} = f(y_1,y_2) + \partial_1 f(y_1,y_2)(x_1-y_1) + \partial_2 f(y_1,y_2)(x_2-y_2)$

Therefore,
$\exists \mathbf{x} \neq \mathbf{y} \in \mathbb{R}^2$ s.t. $f(x_1, x_2) < f(y_1, y_2) + \partial_1 f(y_1, y_2)(x_1 - y_1) + \partial_2 f(y_1, y_2)(x_2 - y_2)$, so this function is NOT convex.

Problem 4

$$p(x, y) = f(x^{(0)}, y^{(0)}) + \partial_1 f(x^{(0)}, y^{(0)})(x - x^{(0)}) + \partial_2 f(x^{(0)}, y^{(0)})(y - y^{(0)})$$
$$+ \frac{1}{2}\left(\partial_{1,1} f(x^{(0)}, y^{(0)})(x - x^{(0)})^2 + 2\partial_{1,2}(x^{(0)}, y^{(0)})(x - x^{(0)})(y - y^{(0)}) + \partial_{2,2} f(x^{(0)}, y^{(0)})(y - y^{(0)})^2\right)$$

In this problem,

$$f(x_1, x_2) = -\log\left(\det\begin{pmatrix} 1 + x_1^2 & x_1 x_2 \\ x_1 x_2 & 1 + x_2^2 \end{pmatrix}\right) = -\log\left((1 + x_1^2)(1 + x_2^2) - x_1^2 x_2^2\right)$$

$$\partial_1 f(x_1, x_2) = \frac{-2x_1}{1 + x_1^2 + x_2^2},\ \partial_2 f(x_1, x_2) = \frac{-2x_2}{1 + x_1^2 + x_2^2}$$
$$\partial_{1,1} f(x_1, x_2) = \partial_1 \partial_1 f(x_1, x_2) = \partial_1 \frac{-2x_1}{(1 + x_1^2)(1 + x_2^2) - x_1^2 x_2^2} = \frac{2(x_1^2 - x_2^2 - 1)}{(x_1^2 + x_2^2 + 1)^2}$$
$$\partial_{2,2} f(x_1, x_2) = \partial_2 \partial_2 f(x_1, x_2) = \partial_2 \frac{-2x_2}{(1 + x_1^2)(1 + x_2^2) - x_1^2 x_2^2} = \frac{2(x_2^2 - x_1^2 - 1)}{(x_2^2 + x_1^2 + 1)^2}$$
$$\partial_{1,2} f(x_1, x_2) = \partial_1 \partial_2 f(x_1, x_2) = \partial_1 \frac{-2x_2}{(1 + x_1^2)(1 + x_2^2) - x_1^2 x_2^2} = \frac{4x_1 x_2}{(x_2^2 + x_1^2 + 1)^2}$$

$f(1, 1) = -\log 3$, $\partial_1 f(1, 1) = -\frac{2}{3}$, $\partial_2 f(1, 1) = -\frac{2}{3}$, $\partial_{1,1} f(x_1, x_2) = -\frac{2}{9}$, $\partial_{2,2} f(x_1, x_2) = -\frac{2}{9}$, $\partial_{1,2} f(x_1, x_2) = \frac{4}{9}$
Thus, the second order Taylor approximation to f at $(1, 1)$ is
$p(x_1, x_2) = -\log 3 - \frac{2}{3}(x_1 - 1) - \frac{2}{3}(x_2 - 1) + \frac{1}{2}$
$\left(-\frac{2}{9}(x_1 - 1)^2 + \frac{8}{9}(x_1 - 1)(x_2 - 1) - \frac{2}{9}(x_2 - 1)^2\right)$
$= -\log 3 - \frac{2}{3}(x_1 - 1) - \frac{2}{3}(x_2 - 1) - \frac{1}{9}(x_1 - 1)^2 + \frac{4}{9}(x_1 - 1)(x_2 - 1) - \frac{1}{9}(x_2 - 1)^2$
$= -\log 3 - \frac{1}{9}(x_1^2 + x_2^2 - 4x_1 x_2 + 8x_1 + 8x_2 - 14)$

Problem 5

Part a

if $A, B \in SPD(2)$, then $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$
$\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^T(A\mathbf{x} + B\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} \geq 0$ (By the distributive property of matrix-vector multiplication)
Therefore, $A + B \in SPD(2)$.

Part b

proof: Take $A, B \in SPD(2)$ and $t \in [0, 1]$,
then $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$
$\mathbf{x}^T(tA + (1-t)B)\mathbf{x} = \mathbf{x}^T(tA\mathbf{x} + (1-t)B\mathbf{x}) = \mathbf{x}^T tA\mathbf{x} + \mathbf{x}^T(1-t)B\mathbf{x} = t\mathbf{x}^T A\mathbf{x} + (1-t)\mathbf{x}^T B\mathbf{x}$
(By the distributive property of matrix-vector multiplication)
$t, (1-t) \geq 0$ since $t \in [0, 1]$.
so $t\mathbf{x}^T A\mathbf{x} \geq 0$ and $(1-t)\mathbf{x}^T B\mathbf{x} \geq 0$
Therefore, $\mathbf{x}^T(tA + (1-t)B)\mathbf{x} \geq 0$, also $tA + (1-t)B \in SPD(2)$
$SPD(2)$ is a convex subset.

Part c

proof: if $X \in M_{2,2}$, then X is a 2x2 matrix, and X^T is also a 2x2 matrix. Then by the rule of matrix multiplication, $X^T X$ is also a 2x2 matrix.
Take a $\mathbf{v} \in \mathbb{R}^2$, then $\mathbf{v}^T X^T X\mathbf{v} = (X\mathbf{v})^T X\mathbf{v} = X\mathbf{v} \bullet X\mathbf{v} \geq 0$.
Therefore, $X^T X \in SPD(2)$.

Part d

proof:
A is positive semidefinite, and B is positive definite.
$\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^T(A\mathbf{x} + B\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x}$ (By the distributive property of matrix-vector multiplication)
We have $\mathbf{x}^T A\mathbf{x} \geq 0$, and $\mathbf{x}^T B\mathbf{x} > 0$ for all $\mathbf{x} \neq 0 \in \mathbb{R}^2$
So $\mathbf{x}^T(A + B)\mathbf{x} > 0$
Therefore, $A + B$ is positive definite.

Part e

proof:
If A is positive definite, then all eigenvalues of A are positive. So 0 is not an eigenvalue of A, then the determinant of A is not zero, therefore A^{-1} exists.