Variance of balanced sky S^{bal}

Definition

Let:

- D be a matrix of independent Poisson random variables (the detector image)
- Λ (with values λ_{ij}) the mean matrix of D
- R, B constant matrices (the decoding and bulk arrays respectively).

The balanced reconstructed sky is:

$$S^{bal} = (R \star D)_{kl} - (R \star B)_{kl} \cdot \frac{\sum D}{\sum B}$$

Let:

$$\begin{aligned} S &= (R \star D)_{kl} \text{ (unbalanced sky image)} \\ \xi &= (R \star B)_{kl} \text{ (balancing array)} \\ \beta &= \sum B \text{ (total active elements)} \\ \delta &= \sum D \text{ (total detector counts)} \end{aligned}$$

Then each element of S^{bal} is:

$$S_{kl}^{bal} = S_{kl} - \xi_{kl} \cdot \frac{\delta}{\beta}$$

Since S_{kl}^{bal} is a linear combination of Poisson variables, and δ is their sum, we can write

$$\mathrm{Var}(S_{kl}^{bal}) = \mathrm{Var}(S_{kl}) + \left(\frac{\xi_{kl}}{\beta}\right)^2 \mathrm{Var}(\delta) - 2 \cdot \frac{\xi_{kl}}{\beta} \cdot \mathrm{Cov}(S_{kl}, \delta)$$

Where:

$$\begin{aligned} \mathrm{Var}(S_{kl}) &= \sum_{i,j} R_{ij}^2 \lambda_{i+k,j+l} \\ \mathrm{Var}(\delta) &= \sum_{i,j} \lambda_{ij} \\ \mathrm{Cov}(S_{kl}, \delta) &= \sum_{i,j} R_{ij} \lambda_{i+k,j+l} \end{aligned}$$

So:

$$\mathrm{Var}(S_{kl}^{bal}) = \sum_{i,j} R_{ij}^2 \lambda_{i+k,j+l} + \left(\frac{\xi_{kl}}{\beta}\right)^2 \sum_{i,j} \lambda_{ij} - 2 \cdot \frac{\xi_{kl}}{\beta} \cdot \sum_{i,j} R_{ij} \lambda_{i+k,j+l}$$

Variance in Cross-Correlation Notation

Using cross-correlation notation and considering Λ as the mean matrix of D

$$\mathrm{Var}(S_{kl}^{bal}) = (R^2 \star \Lambda)_{kl} + \left(\frac{(R \star B)_{kl}}{\sum B}\right)^2 \cdot \sum \Lambda - 2 \cdot \frac{(R \star B)_{kl}}{\sum B} \cdot (R \star \Lambda)_{kl}$$

Where $\star$ denotes 2D cross-correlation and R^2 is the element-wise square of R.

Relationship between D and Λ

Now:

Let $D = \{D_{ij}\}$ be a matrix of independent Poisson-distributed random variables, where each $D_{ij} \sim \mathrm{Poisson}(\lambda_{ij})$. Let $\Lambda = \{\lambda_{ij}\}$ be the matrix of corresponding Poisson means.

- $\sum D = \sum_{i,j} D_{ij}$ is a **random variable**, representing the total sum of all Poisson random variables in D.
- $\sum \Lambda = \sum_{i,j} \lambda_{ij}$ is a **deterministic quantity**, representing the sum of the expected values of the Poisson variables.

Expectation

By the linearity of expectation, we have:

$$\mathbb{E}\left[\sum D\right] = \sum_{i,j} \mathbb{E}[D_{ij}] = \sum_{i,j} \lambda_{ij} = \sum \Lambda$$

Conclusion

Although $\sum \Lambda$ and $\sum D$ are closely related, they are not equal in general. Specifically:

$$\sum \Lambda = \mathbb{E}[\sum D]$$

That is, $\sum \Lambda$ is the **expected value** of the random variable $\sum D$. In any specific realization of the random matrix D, the actual value of $\sum D$ may differ from $\sum \Lambda$ due to the inherent randomness of the Poisson distribution.

Conclusions

Considering that, for a large number of elements δ approximates $\sum \Lambda$ we can rewrite the variance formula as:

$$\mathrm{Var}(S^{bal}) = (R^2 \star \Lambda) + \frac{\delta}{\beta^2} \cdot (R \star B)^2 - 2 \cdot \frac{(R \star B)}{\beta} \cdot (R \star \Lambda)$$