# Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

∗ Name: Haoyi You Student ID: 519030910193 Email: yuri-you@sjtu.edu.cn

1. Prove that for any integer $n > 2$, there is a prime $p$ satisfying $n < p < n!$. (Hint: consider a prime factor $p$ of $n! - 1$ and prove by contradiction)

   **Proof.** Assume there exists n satisfying that there is no integer t with $n < t < n!$. Obviously, integer $k = n! - 1$ is not a prime.Let $k = p_1^{\alpha_1} p_2^{\alpha_2} ... p_l^{\alpha_l}$,with $p_i (1 \leq i \leq l)$ are all prime number. From the assumption we know that for any $i$,$p_i \leq n$,but it is certain that $(n! - 1, p_i) = (-1, p_i) = 1$. So the assume fails. QED.

   □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   **Proof.** First,we know $7 = 2 \times 2 + 3 \times 1$,$8 = 2 \times 4 + 3 \times 0$.

   Assumption: There exist integers that do not satisfy the given condition, assume $n_1 \geq 9$ be the smallest one of them.

   However, let $n_2 = n_1 - 2 \geq 7$.If $n_2$ satisfies the condition and there exists integers $i_{n_2}$ and $j_{n_2}$, such that $n_2 = i_{n_2} \times 2 + j_{n_2} \times 3$.We can get $n_1 = (i_{n_2} + 1) \times 2 + j_{n_2} \times 3$.

   As a result,$n_2$ does not satisfy the condition, either.But we have assumed $n_1$ be the smallest one and we find $n_2 \leq n_1$.From the minimal counterexample principle,the assumption fails.QED.

   □

3. Suppose the function $f$ be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   **Proof.** Assumption: $f(n) = 3^n - 2^n$.

   First,when $n = 1$ and $n = 2$,the assumption is valid.

   Then,we suppose when $n \leq k$ the assumption is valid.

   So for $n \leq k + 1$

   $$\begin{aligned} f(k+1) = 5f(k) - 6f(k-1) = 5 * (3^k - 2^k) - 6 * (3^{k-1} - 2^{k-1}) \\ = 3^{k-1} * (5 * 3 - 6) + 2^{k-1} * (6 - 5 * 2) = 9 * 3^{k-1} - 4 * 2^{k-1} = 3^{k+1} - 2^{k+1}. \end{aligned} \tag{1}$$

   According to the principle of mathematical induction,the assumption is valid.QED. □

4. An $n$-team basketball tournament consists of some set of $n \geq 2$ teams. Team $p$ beats team $q$ iff $q$ does not beat $p$, for all teams $p \neq q$. A sequence of distinct teams $p_1$, $p_2$,..., $p_k$, such that team $p_i$ beats team $p_{i+1}$ for $1 \leq i < k$ is called a ranking of these teams. If also team $p_k$ beats team $p_1$, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

**Proof.** Assumption: for any $n \geq 2$ team,either there is a "champion" team that beats every other team, or there is a 3-cycle.

First,when n=2,team p beats team q, p is the "champion" team.

Then, we suppose when $n \leq k$ the assumption is valid.

So for $n \leq k+1$,there are teams $p_1$, $p_2$,..., $p_{k+1}$.For first k of these teams,the assumption is valid.

If there is a 3-cycle,there is also a 3-cycle when we add the team $p_{k+1}$.

If there is a "champion" team, we let $p_1$ beats the other k-1 teams. If $p_1$ also beats $p_{k+1}$,$p_1$ is also the "champion" team.

Otherwise, consider the relations between $p_{k+1}$ and $p_i (2 \leq i \leq k)$. If $p_{k+1}$ beats all of them, $p_{k+1}$ is the "champion" team. Otherwise there exists m that $p_m$ beats $p_{k+1}$,so we know that $p_1$ beats $p_m$,$p_m$ beats $p_{k+1}$,$p_{k+1}$ beats $p_1$,which is a 3-cycle.

According to the principle of mathematical induction,the assumption is valid.QED. □

**Remark:** You need to include your .pdf and .tex files in your uploaded .rar or .zip file.