Homework: Week 6

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1:

a. Using

$$\langle x'|p'\rangle = (2\pi\hbar)^{-1/2} e^{ip'x'/\hbar}$$

prove

$$\langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle$$

where $|\alpha\rangle$ is an arbitrary state ket.

b. Consider a one-dimensional simple harmonic oscillator. Starting with the Schrödinger equation for the state vector, derive the Schrödinger equation for the momentum-space wave function. (Make sure to distinguish the operator p from the eigenvalue p' .) Can you guess the energy eigenfunctions in momentum space?

c. Solving the linear potential problem, i.e. $V(x) = k|x|$ where k is an arbitrary positive constant, in **Momentum space**, and show that the Fourier transform of your solution is indeed the appropriate Airy function. Note that an alternative form of the Airy function is

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(\frac{1}{3}t^3 + xt)dt$$

You need not be concerned with normalising the wave function.

2:

Consider a particle of mass m subject to a one-dimensional potential of the following form:

$$V = \begin{cases} \frac{1}{2}kx^2 & \text{for} \quad x > 0 \\ \infty & \text{for} \quad x < 0 \end{cases}$$

- a: What is the ground state energy?
- b: What is the expectation value $\langle x^2 \rangle$ for the ground state?

3:

Use the WKB method to find the (approximate) energy eigenvalues for the one-dimensional simple harmonic osicllator potential $V(x) = m\omega^2 x^2/2$. (Hint: We have discussed it in the context of Bohr-Sommerfeld quantization rule.)

4:

An electron moves in the presence of a uniform magnetic field in the z-direction $\vec{B} = B\vec{z}$.

- a: Evaluate

  $$[\Pi_x, \Pi_y],$$

  where $\Pi_i := p_i - \frac{eA_i}{c}$, $i = x, y$

- b: By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

  $$E_{k,n} = \frac{\hbar^2 k^2}{2m} + (\frac{|eB|\hbar}{mc})(n + \frac{1}{2})$$

  where $\hbar k$ is the continuous eigenvalues of the p_z operator and n is a non-negative integer including zero.