Learning Enhancement Team
Steps into Calculus
What is differentiation?

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STUDENT SUPPORT SERVICE

This guide introduces the concept of differentiation and the language associated with it. It makes the connection between the derivative of a function, the rate of change of a physical quantity and the gradient of the graph of a function.

Introduction

The study of how one quantity, such as distance travelled by a car, is affected by another quantity, such as time, involves the use of **differentiation**. If the car is travelling at a steady speed on a motorway, then the problem is straightforward. However, if the car is stopping and starting in a city, then the exact distance it has travelled at any time can only be calculated by measuring its position at many small time intervals. The branch of mathematics concerned with infinitesimally small changes is called **calculus**. Calculus has two branches, differentiation and **integration** and the two subjects were independently unified by Sir Isaac Newton and Gottfried Leibniz in the 17th century.

To find out how a variable, for example *y,* changes with respect to another variable, for example *x,* you need to **differentiate** it. The result is the **derivative** which is written:

$$\boxed{\frac{dy}{dx}}$$

and is pronounced "dee *y* by dee *x*". Differentiation can be thought of as the study of the **rate of change** of one variable with respect to another. How to describe these changes is crucial to the mathematical modelling of processes which occur in many disciplines:

Subject	**Symbol**	**Description**
Physics	$\frac{ds}{dt}$	Velocity which is the rate of change of distance *s* with respect to time *t.*
Chemistry	$\frac{d[A]}{dt}$	The rate of change of the concentration of chemical *A* with respect to time *t.*
Economics	$\frac{dQ}{dP}$	The rate of change in demand *Q* for a good as the price of the good *P* changes.
Biology	$\frac{dP}{dF}$	The rate of growth of a population *P* with respect to food *F* availability.

In this guide you will learn how the rate of change or derivative of a function is the same as the **gradient** of the graph of the function. You will learn how to calculate gradients for straight lines and see how this can be generalised to approximate the gradients of curves. Most importantly, you will learn that the terms:

Rate of Change	$\frac{dy}{dx}$	**Gradient**	$f'(x)$	**Derivative**

are all equivalent.

You might want to be familiar with the concept of linear equations and straight lines in our *What is a straight line?* study guide

This guide is not about learning how to differentiate, you can find this out by reading the study guides: *Differentiating using the Power Rule* and *Differentiating Basic Functions* and doing the worksheets that go with them. Then you will be able to tackle differentiating harder functions by using *The Chain Rule*, *The Product Rule* and *The Quotient Rule*.

Gradients of straight lines

You may be familiar with the road sign which gives the gradient of a slope as a ratio $1:5$ for example. This means that for every 5m you go along you go up 1m. Mathematically, the gradient of a straight line is the amount you go up divided by the amount you go along to the right. Or, in other words, the change in *y* divided by the change in *x.*

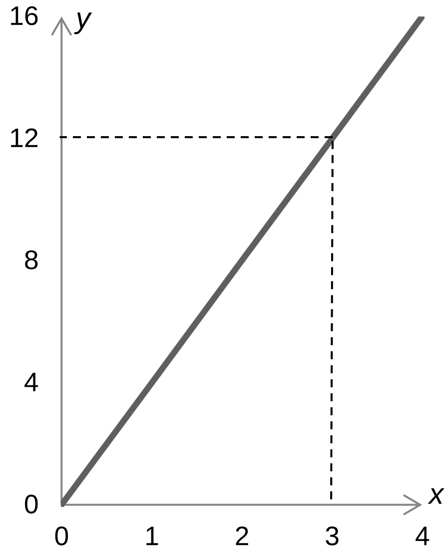
$$\boxed{\text{Gradient} = \frac{\Delta y}{\Delta x}}$$

Here (and throughout mathematics and science) the capital Greek letter **delta** (Δ) is used to indicate a “change in” a variable so Δy means the change in *y* and Δx means the change in *x*. Then, in the road example, if Δx is the distance travelled along the road and Δy is the height you have climbed while going along this distance, then a $1:5$ slope has a gradient of:

$$\frac{\Delta y}{\Delta x} = \frac{1}{5}$$

Another way of thinking about this is that the rate of change of the height of the slope with respect to horizontal distance is $1/5$. You can apply this to find the gradients of the graphs of functions.

Example: What is the gradient of the graph $y = 4x$?



From the graph of $y = 4x$ you can see that the line goes through the coordinates $(0,0)$ and $(3,12)$ so, between these two points, the change in *x* is:

$$\Delta x = 3 - 0 = 3$$

and the change in *y* is:

$$\Delta y = 12 - 0 = 12$$

and so the gradient, or the rate of change of *y* as *x* changes is:

$$\frac{\Delta y}{\Delta x} = \frac{12}{3} = 4$$

You may have been able to see this straight away if you knew that graphs of the form $y = mx$ represent straight lines which pass through the origin and have a gradient *m* (see the study guide: *Finding Equations of Straight Lines*). The graph above shows $y = 4x$ which is of the form $y = mx$ with $m = 4$ and so the gradient of the line is equal to 4.

Whatever two points you choose, the gradient will be the same. For example, using the the coordinates $(1,4)$ and $(2,8)$ the change in *x* is 1 and the change in *y* is 4 and so the gradient is:

$$\frac{\Delta y}{\Delta x} = \frac{4}{1} = 4$$

The gradient is the same everywhere along the line. This is what “straight line” means.

Differentiation and gradients

Differentiating a function gives the gradient of that function. Another way of saying this is that the derivative of a function is the same as the gradient of the graph of that function. For functions of the form $y = mx$ described above, the gradient is always equal to *m* and so the rate of change of *y* as *x* changes, or, in other words, the derivative of *y* with respect to *x* is always equal to *m*.

The notation for the derivative of a function *y* which is differentiated with respect to *x* is:

$$\frac{dy}{dx}$$

This comes from the notation used above to calculate the gradient of a line where the Δ sign has been replaced with a *d*. In both cases the Δ and the *d* represent a change. For a straight line, this is exactly the same as the formula for the gradient.

For a straight line $y = mx$ the gradient is $\frac{dy}{dx} = m$

So in the example above, for $y = 4x$ the gradient is $\frac{dy}{dx} = 4$.

Differentiating functions

If the number of rabbits in a field at time *t* is given by the function $R(t)$ then you can call the rate of change of the population of rabbits with respect to time:

$$\frac{dR}{dt}$$

To start with, if there are very few rabbits, then the population increases slowly and the rate of change of the population is small so that:

$\frac{dR}{dt}$ is small to begin with

As the population of rabbits increases (as there are more and more rabbits breeding) the rate of increase in the rabbit population will also increase so that:

$\frac{dR}{dt}$ increases as time goes on

Furthermore as "rate of change" means the same as "gradient", the gradient of the graph of *R* against *t* is changing and so the function is not a straight line but a curve.

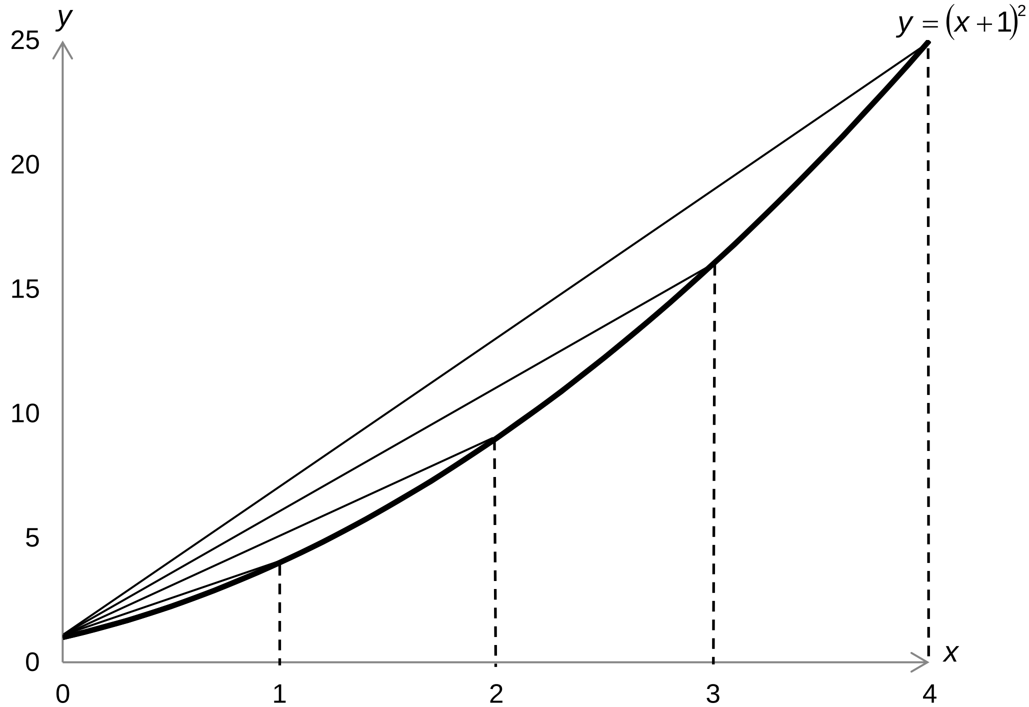
In general, functions $f(x)$ are more complicated than $y = mx$ and have graphs that are not straight lines but curves. This means that the gradient of these functions is not the same everywhere but is also changing with *x*. So the gradient of a curve is another function of *x* which is written as $f'(x)$ and said "*f* prime of *x*" or sometimes "*f* dash of *x*".

For a curve $y = f(x)$ the gradient is another function $\frac{dy}{dx} = f'(x)$

Alternatively the notation, $y'(x)$ is often used. In the rabbit example, the gradient or rate of change of the rabbit population is $R'(t)$. What is this function $f'(x)$? There is no easy answer to this as it depends on what the original function $f(x)$ is. You will need to read the study guides recommended in the introduction to help you calculate $f'(x)$. For now, it is important to realise that the gradient of a function is another function and so changes at different values of *x*.

Example: What is the gradient of the function $f(x) = (x+1)^2$ at $x = 0$?

For the straight line, the gradient was the same everywhere but now, as *x* increases the function's curve becomes steeper. In other words, the gradient $f'(x)$ is also a function of *x*. In order to learn how to find this function $f'(x)$ you can read the study guide: *Differentiating using the Power Rule*. However, there is a way of approximating the gradient of the curve at a single point $x = 0$.



The table below shows the gradients $\dfrac{\Delta y}{\Delta x}$ of the straight lines on the curve on the graph.

x	$y = (x+1)^2$	$\Delta y = y - 1$	Δx	$\dfrac{\Delta y}{\Delta x}$
4	25	24	4	6
3	16	15	3	5
2	9	8	2	4
1	4	3	1	3
0.5	2.25	1.25	0.5	2.5
0.25	1.5625	0.5625	0.25	2.25

It is not possible to calculate the gradient of the straight line when $x = 0$ because this would give $\Delta x = 0$ and dividing by 0 is never allowed. However, you can see from the graph that, as *x* becomes smaller and smaller, each of the lines on the graph gives a better and better approximation to the gradient of the curve at $x = 0$. Also, the table shows that:

As Δx gets closer to 0, the gradient $\dfrac{\Delta y}{\Delta x}$ gets closer to 2

So it seems that a good approximation to the actual gradient at $x = 0$ is 2.

Differentiation from first principles

The example above illustrates the basic idea behind differentiation as described by Newton and Leibniz. This is, for any function, as Δx gets closer to 0, the gradient of the straight lines gets closer to the actual gradient of the function so that:

$$\boxed{\text{As } \Delta x \to 0 \text{ the gradient } \frac{\Delta y}{\Delta x} \to \frac{dy}{dx}}$$

Want to know more?

If you have any questions about this topic or any other maths, you can make an appointment to see a Learning Enhancement Tutor for Maths and Stats in the Student Support Service.

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