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A computational comparison of formulations for the economic lot-sizing with remanufacturing

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ABSTRACT

An important way to try reducing environmental damage in the manufacture of industrialized goods is through the use of production systems which deal with the reuse of returned materials such as reverse logistics. In this paper, we consider a production planning problem arising in the context of reverse logistics, namely the economic lot-sizing with remanufacturing (ELSR). In the ELSR, deterministic demand for a single item over a finite time horizon has to be satisfied, which can be performed from either newly produced or remanufactured items, and the goal consists in minimizing the total production costs. Our objective is to devise approaches to solve larger (more difficult) instances of the problem available in the literature to optimality using a standard mixed-integer programming (MIP) solver. We present a multicommodity extended formulation and a strengthened Wagner–Whitin based formulation, which makes use of *a priori* addition of newly described valid inequalities in the space of original variables. We also propose a novel dynamic heuristic measure based on the cost structure to automatically determine the size of a partial version of the Wagner–Whitin based formulation. Computational results show that the novel partial Wagner–Whitin based formulation with the size automatically determined in a heuristic way outperforms all the other tested approaches, including a best performing shortest path formulation available in the literature, when we consider the number of instances solved to proven optimality using a standard MIP solver. This new approach allowed to solve to optimality more than 96% of the tested instances for the ELSR with separate setups, including several instances that could not be solved otherwise.

1. Introduction

The economic lot-sizing with remanufacturing (ELSR) has received great attention in recent years, and one of the reasons is the increasing interest in the search for better ways to provide sustainable production systems that can be implemented effectively. The problem consists in planning the production of new items from raw materials together with the remanufacture of returned items in order to satisfy the deterministic demands over a finite discrete time horizon while minimizing the total production costs. The problem was independently shown to be NP-Hard in Baki, Chaouch, and Abdul-Kader (2014) and Retel Helmrich, Jans, van den Heuvel, and Wagelmans (2014) (also Retel Helmrich (2013)).

A basic production planning problem in the literature is the uncapacitated lot-sizing (ULS). Wagner and Whitin (1958) considered the ULS in a seminal work on the algorithmic treatment given to production planning problems. Barany, Van Roy, and Wolsey (1984a,b) proposed the (l, S)-inequalities and showed that together with basic constraints they give the convex hull of the set of feasible solutions. Extended formulations were proposed in Krarup, Bilde, and location (1977) (multicommodity or facility location) and Eppen and Martin (1987) (shortest path). Since then, valid inequalities have been widely used to treat several production planning models (see Pochet & Wolsey (2006) for an extensive review).

The economic lot-sizing with remanufacturing is an extension of the ULS in which remanufacturing options are available and has been recently studied in several works. Richter and Sombrutzki (2000) treated a simple version of the problem in which both production and remanufacture are unlimited, i.e. the amount of returned items available at the beginning of the planning horizon is enough to satisfy the entire demand, and the costs are time invariant. The authors analyzed the properties of optimal

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solutions and proposed a dynamic programming algorithm. Richter and Weber (2001) extended this uncapacitated model to treat time variant costs in a reverse Wagner–Whitin model with variable manufacturing and remanufacturing costs.

Teunter, Bayindir, and van den Heuvel (2006) treated the economic lot-sizing with remanufacturing with both separate and joint setup costs. The authors obtained a polynomial time algorithm for the case with joint setup and stationary costs and proposed heuristics for both joint and separate setup variants. Later, Schulz (2011) generalized the Silver-Meal heuristic presented in (Teunter et al., 2006). Baki et al. (2014) proposed an alternative mixed-integer programming (MIP) formulation for the problem and showed that it provided better linear relaxation bounds than a standard formulation. They also developed a dynamic programming based heuristic and performed extensive numerical experiments, using several instances with a small planning horizon of 12 periods and some with larger planning horizons, to validate the good performance of the heuristic.

Retel Helmrich et al. (2014) compared MIP approaches to the ELSR and proposed a shortest path formulation, an approximate shortest path formulation and valid inequalities based on the (l, S, WW)-inequalities for the ULS which were added *a priori* to a standard formulation in order to obtain a Wagner–Whitin based formulation. They performed computational experiments comparing the approaches and also showed that the problem with joint setups is NP-Hard when the costs are time variant. Recently, Sifaleras, Konstantaras, and Mladenović (2015) developed a variable neighborhood search heuristic (VNS) to the problem. The proposed VNS heuristic outperformed the state-of-the-art heuristic methods from the literature in the reported computational experiments using a set of benchmark instances (6480 instances with 12 periods each) proposed by Schulz (2011). They also presented a new benchmark set of larger (more difficult) instances, with 52 periods, and demonstrated the robustness of the approach using these new instances. Some authors also considered multi-item extensions of the ELSR. Sahling (2013) proposed a column generation approach for a multi-item extension of the ELSR which also included big bucket capacity constraints on production and remanufacture. More recently, Sifaleras and Konstantaras (in press) studied another multi-item variant of the problem and proposed a variable neighborhood descent (VND) heuristic which was shown to outperform the use of a standard MIP solver through computational experiments.

Our work concentrates on mixed-integer programming approaches in an attempt to, using a standard MIP solver, solve to optimality the largest instances of the economic lot-sizing with remanufacturing available in the literature. Therefore, we limited ourselves to two benchmark sets of instances: the first one proposed by Sifaleras et al. (2015) (108 instances with 52 periods each), and the second proposed by Retel Helmrich et al. (2014) (120 instances with 25 periods, 120 instances with 50 periods and 120 instances with 75 periods).

The economic lot-sizing with remanufacturing can be formally defined as follows. There is a single item with deterministic demand over a finite discrete time horizon of NT periods. The demand for each period $t \in \{1 \dots NT\}$ is d_t and the amount of returned items arriving at each period is r_t. Production of new items is unlimited while the remanufacture is restricted to the availability of returned items. There are fixed and variable production costs (respectively f^p_t and $\overline{p}^p_t$) as well as fixed and variable remanufacture costs (respectively f^r_t and $\overline{p}^r_t$). The storage of finished items implies a cost of h^p_t per unit and that of returned items a cost of h^r_t per unit. The ELSR has two main variants, namely the economic lot-sizing with remanufacturing and separate setups (ELSRs) in which there are separate setups for production and for remanufacture, and the economic lot-sizing with remanufacturing and joint setups (ELSRj) in which both production and remanufacture share the same setup. It is assumed that there is no initial stock of either finished or returned items and no final stocks of finished items. In addition, all data is non negative, the cumulated demand in the interval $[k, t]$ is defined as $d_{kt} = \sum_{l=k}^{t} d_l$ for $1 \leqslant k \leqslant t \leqslant NT$ and the cumulated amount of returned items in the interval $[k, t]$ as $r_{kt} = \sum_{l=k}^{t} r_l$ for $1 \leqslant k \leqslant t \leqslant NT$.

The remainder of the paper is organized as follows. In Section 2 we formally define the economic lot-sizing with remanufacturing and separate setups using a standard MIP formulation. The shortest path formulation of Retel Helmrich et al. (2014) is presented in Section 3. A new multicommodity formulation is introduced in Section 4, and a Wagner–Whitin based formulation is given in Section 5 together with new valid inequalities to the problem and a heuristic technique to determine the size of a partial formulation based on the problem's cost structure. In Section 6 we show how the approaches devised for ELSRs can be adapted to deal with ELSRj. Computational experiments are summarized in Section 7. The results show that the multicommodity formulation outperforms the considered shortest path formulation (which was the best approach in (Retel Helmrich et al., 2014) when we consider the number of instances solved to optimality) for most of the cases, and that the partial Wagner–Whitin formulation with automatically determined size outperforms all the other approaches allowing to solve several additional instances to optimality. Some final remarks are discussed in Section 8.

2. The economic lot-sizing with remanufacturing and separate setups

In this section, we present a formal description of the economic lot-sizing with remanufacturing and separate setups (ELSRs) using a standard mixed-integer programming formulation.

With the purpose of formulating the problem as a mixed-integer program, consider x^p_t to be the amount of items produced in period t, x^r_t to be the amount of items remanufactured in period t, s^p_t to be the amount of finished items in stock at the end of period t, s^r_t to be the amount of returned items in stock at the end of period t, y^p_t to be equal to 1 if production happens in period t and to be 0 otherwise, and y^r_t to be equal to 1 if remanufacture happens in period t and to be 0 otherwise. Using the variables just described, the problem can be formulated as

$$\min \sum_{t=1}^{NT} (h^p_t s^p_t + \overline{p}^p_t x^p_t + f^p_t y^p_t) + \sum_{t=1}^{NT} (h^r_t s^r_t + \overline{p}^r_t x^r_t + f^r_t y^r_t) \tag{1}$$

$$s^p_{t-1} + x^p_t + x^r_t = d_t + s^p_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{2}$$

$$s^r_{t-1} + r_t = x^r_t + s^r_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{3}$$

$$x^p_t \leqslant d_{t,NT} y^p_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{4}$$

$$x^r_t \leqslant \min\{r_{1t},\ d_{t,NT}\} y^r_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{5}$$

$$x^p,\ x^r,\ s^p,\ s^r \in \mathbb{R}^{NT}_+, \tag{6}$$

$$y^p,\ y^r \in \{0,1\}^{NT}. \tag{7}$$

The objective function minimizes the total cost. Constraints (2) are balance constraints regarding the finished items while constraints (3) are balance constraints for the returned items. Constraints (4) force the production setup variables to be equal to one if production occurs. Constraints (5) enforce the remanufacture setup variables to be equal to one if remanufacture occurs. Note that, differently from Retel Helmrich et al. (2014), in (5) the cumulated returns r_{1t} are also considered as upper bounds on remanufacture in an attempt to obtain better linear relaxation bounds, especially when the amount of returned items is small. Constraints (6) and (7) are, respectively, non negativity and integrality constraints on the variables.

Observe that the following equivalent objective function for the problem can be obtained by removing the stock variables (s^p_t and s^r_t) from (1) together with their costs (see Appendix A):

$$\min \sum_{t=1}^{NT}(p^p_t x^p_t + f^p_t y^p_t) + \sum_{t=1}^{NT}(p^r_t x^r_t + f^r_t y^r_t). \tag{8}$$

3. The shortest path formulation

We briefly describe the shortest path formulation presented in Retel Helmrich et al. (2014) to ELSRs, in whose work a more detailed explanation can be obtained.

Consider z^{sp}_{ij} to be the fraction of demand in periods from i to j that is fulfilled by new items produced in period i, z^{sr}_{ij} to be the fraction of demand in periods from i to j that is fulfilled by items remanufactured in period i, z^r_{ij} to be the fraction of returned items in periods from i to j that is remanufactured in period j, and l_t to be the fraction of returned items in periods from t to NT that is added to the final inventory of returned items at the end of period NT. Defining the following cost parameters:

$$C^{sp}_{ij} = \overline{p}^p_i d_{ij} + \sum_{t=i}^{j-1} h^p_t d_{t+1,j}, \quad \text{for } 1 \leqslant i \leqslant j \leqslant NT,$$

$$C^{sr}_{ij} = \overline{p}^r_i d_{ij} + \sum_{t=i}^{j-1} h^p_t d_{t+1,j}, \quad \text{for } 1 \leqslant i \leqslant j \leqslant NT,$$

$$C^{r}_{ij} = \sum_{t=i}^{j-1} h^r_t r_{it}, \quad \text{for } 1 \leqslant i \leqslant j \leqslant NT,$$

$$C^{l}_{t} = \sum_{j=t}^{NT} h^r_j d_{tj}, \quad \text{for } 1 \leqslant t \leqslant NT,$$

a shortest path formulation for ELSRs was presented in (Retel Helmrich et al., 2014) as

$$\min \sum_{t=1}^{NT}\left(f^p_t y^p_t + f^r_t y^r_t + C^l_t l_t\right) + \sum_{i=1}^{NT}\sum_{j=i}^{NT}\left(C^{sp}_{ij} z^{sp}_{ij} + C^{sr}_{ij} z^{sr}_{ij} + C^r_{ij} z^r_{ij}\right) \tag{9}$$

$$\sum_{j=1}^{NT}\left(z^{sp}_{1j} + z^{sr}_{1j}\right) = 1, \tag{10}$$

$$\sum_{i=1}^{t-1}\left(z^{sp}_{i,t-1} + z^{sr}_{i,t-1}\right) = \sum_{j=t}^{NT}\left(z^{sp}_{tj} + z^{sr}_{tj}\right), \quad \text{for } 2 \leqslant t \leqslant NT, \tag{11}$$

$$\sum_{j=t}^{NT} z^{sp}_{tj} \leqslant y^p_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{12}$$

$$\sum_{j=t}^{NT} z^{sr}_{tj} \leqslant y^r_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{13}$$

$$\sum_{j=1}^{NT} z^r_{1j} + l_1 = 1, \tag{14}$$

$$\sum_{i=1}^{t-1} z^r_{i,t-1} = \sum_{j=t}^{NT} z^r_{tj} + l_t, \quad \text{for } 2 \leqslant t \leqslant NT, \tag{15}$$

$$\sum_{i=1}^{t} z^r_{it} \leqslant y^r_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{16}$$

$$\sum_{i=1}^{t} r_{it} z^r_{it} = \sum_{j=t}^{NT} d_{tj} z^{sr}_{tj}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{17}$$

$$z^{sm}_{ij},\ z^{sr}_{ij},\ z^r_{ij} \geqslant 0, \quad \text{for } 1 \leqslant i \leqslant j \leqslant NT, \tag{18}$$

$$y^p,\ y^r \in \{0,1\}^{NT}. \tag{19}$$

The objective function minimizes the total cost. The shortest path constraints for finished items are given by (10)–(13), with (10) and (11) being flow conservation constraints and (12) and (13) setup enforcing constraints. The shortest path constraints for the returned items are given by (14)–(16), with (14) and (15) being flow conservation constraints and (16) setup enforcing constraints. Constraints (17) link the z^r with the z^{sr} variables. Constraints (18) and (19) are non negativity and integrality constraints on the variables.

4. A multicommodity formulation

Multicommodity reformulations have been applied successfully to several production planning problems (Pochet & Wolsey, 2006) but, to the best of our knowledge, this approach has not yet been proposed to the economic lot-sizing with remanufacturing. In such a formulation, the demand for each period can be seen as a separate commodity. Define the variables w^p_{kt} to be the amount of new items produced in period k to satisfy demand in period t, w^r_{kt} to be the amount remanufactured in period k to satisfy demand in period t, and o^r_{kt} to be the amount of returned items arriving in period k to be remanufactured in period t. A multicommodity formulation for the problem can be obtained as

$$\min \sum_{t=1}^{NT}(p^p_t x^p_t + f^p_t y^p_t) + \sum_{t=1}^{NT}(p^r_t x^r_t + f^r_t y^r_t)$$

$$\sum_{k=1}^{t}(w^p_{kt} + w^r_{kt}) \geqslant d_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{20}$$

$$\sum_{k=1}^{t} o^r_{kt} = \sum_{k=t}^{NT} w^r_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{21}$$

$$\sum_{t=k}^{NT} o^r_{kt} \leqslant r_k, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \tag{22}$$

$$w^p_{kt} \leqslant d_t y^p_k, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \tag{23}$$

$$w^r_{kt} \leqslant \min\{r_{1k},\ d_t\} y^r_k, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \tag{24}$$

$$o^r_{kt} \leqslant r_k y^r_t, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \tag{25}$$

$$x^p_t = \sum_{k=t}^{NT} w^p_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{26}$$

$$x^r_t = \sum_{k=t}^{NT} w^r_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{27}$$

$$w^p,\ w^r,\ o^r \in \mathbb{R}^{NT \times NT}_{+}, \tag{28}$$

$$y^p,\ y^r \in \{0,1\}^{NT}. \tag{29}$$

The objective function minimizes the total cost. Constraints (20) guarantee that the demand for a given period is fulfilled by production and/or remanufacturing happening latest in that period. Constraints (21) associate the returned items to be remanufactured in a given period to the real amount remanufactured in that period. Constraints (22) imply that the amount to be remanufactured does not exceed the amount of available returned items. Constraints (23)–(25) are setup enforcing constraints on production and remanufacture. Constraints (26) and (27) link the multicommodity variables with the original ones. Constraints (28) and (29) are non negativity and integrality constraints on the variables.

5. A Wagner–Whitin based formulation

In this section, we present a Wagner–Whitin based formulation which is obtained using the original formulation strengthened with the *a priori* addition of a polynomial number of valid inequalities, certain of them already available in the literature and two new families of valid inequalities.

5.1. Valid inequalities available in the literature

The two families of valid inequalities to ELSRs available in the literature are variants of the Wagner–Whitin (l, S)-inequalities for the uncapacitated lot-sizing, and were used in the Wagner–Whitin based formulation presented in (Retel Helmrich et al. (2014)). The first family of valid inequalities is related to the demands

$$s^p_{t-1} + \sum_{k=t}^{l} d_{kl}(y^p_k + y^r_k) \geqslant d_{tl}, \quad \text{for } 2 \leqslant t \leqslant l \leqslant NT, \tag{30}$$

while the second one is associated to the returned items

$$s^r_l + \sum_{k=t}^{l} r_{tk} y^r_k \geqslant r_{tl}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT. \tag{31}$$

5.2. New valid inequalities

Here, we introduce two new families of valid inequalities. We remark that the goal is to obtain a polynomial number of inequalities in order to be able to add them *a priori* to the original formulation to be used in a standard MIP solver.

The first family is a simple generalization of (30) whose aim is to improve the obtained linear relaxation bounds, and is described in Corollary 1.

Corollary 1. Inequalities (30) can be improved as

$$s^p_{t-1} + \sum_{k=t}^{l} d_{kl} y^p_k + \sum_{k=t}^{l} \min\{r_{1k}, d_{kl}\} y^r_k \geqslant d_{tl}, \quad \text{for } 2 \leqslant t \leqslant l \leqslant NT. \tag{32}$$

Corollary 1 follows from (30) and the simple fact that the amount remanufactured in a given period t is limited by the cumulated quantity of returned items arriving until that period.

We now describe a new family of valid inequalities. Let $\underline{d}^p \in \mathbb{R}^{NT}_+$ be the vector of minimum demands that must be attended by production of new items, i.e. cannot be satisfied from remanufactured items given the limited availability of returned items, such that the amount $\underline{d}^p_{1k} = \sum_{l=1}^{k} \underline{d}^p_l$ is minimum for every $k \in \{1, \dots, NT\}$. In order to determine $\bar{d}^p$, consider rr_k to be the residual amount of returned items in period k given that a maximum quantity of demand can be satisfied from returned items, i.e. the amount of r_{1k} that cannot be used to satisfy part of the demand until period $k-1$, which can be calculated as

$$rr_k = \begin{cases} r_k + \max\{0, rr_{k-1} - d_{k-1}\}, & \text{if } 2 \leqslant k \leqslant NT, \\ r_k, & \text{if } k = 1. \end{cases}$$

Using the values rr_k, each element $\underline{d}^p_k$, for $1 \leqslant k \leqslant NT$, can therefore be determined as

$$\underline{d}^p_k = \max\{0, d_k - rr_k\}.$$

Define X^{ELSRs} as the set of integer feasible solutions to (2)–(7). The new family of valid inequalities associates the production variables with the vector $\underline{d}^p$, and is presented in Proposition 2.

Proposition 2. The inequalities

$$\sum_{k=1}^{t-1} x^p_k + \sum_{k=t}^{l} \underline{d}^p_{kl} y^p_k \geqslant \underline{d}^p_{1l}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT, \tag{33}$$

are valid for X^{ELSRs}, in which $\underline{d}^p_{kl} = \sum_{j=k}^{l} \underline{d}^p_j$.

Proof. The proof is straightforward and it follows from the fact that the demands in $\underline{d}^p$ must be satisfied from production, with $\underline{d}^p_k$ representing the demand that must be produced from raw material latest at period k, and (33) are simply (l, S)-inequalities for the vector $\underline{d}^p$. □

5.3. The strengthened Wagner–Whitin based formulation

The Wagner–Whitin based formulation presented in this section is obtained using the original formulation strengthened with a subset of the inequalities (31)–(33). Define K^p_t and K^r_t, for $1 \leqslant t \leqslant NT$, to be integer values in the interval $[0, NT-1]$. The formulation is

$$\min \sum_{t=1}^{NT} (h^p_t s^p_t + \bar{p}^p_t x^p_t + f^p_t y^p_t) + \sum_{t=1}^{NT} (h^r_t s^r_t + \bar{p}^r_t x^r_t + f^r_t y^r_t) \tag{34}$$

$$(2)–(7)$$

$$s_{t-1} + \sum_{k=t}^{l} d_{kl} y^p_k + \sum_{k=t}^{l} \min\{r_{1k}, d_{kl}\} y^r_k \geqslant d_{tl}, \quad \text{for } 2 \leqslant t \leqslant l \leqslant NT,\ l \leqslant t + K^p_t, \tag{35}$$

$$s^r_l + \sum_{k=t}^{l} r_{tk} y^r_k \geqslant r_{tl}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT,\ l \leqslant t + K^r_t, \tag{36}$$

$$\sum_{k=1}^{t-1} x^p_k + \sum_{k=t}^{l} \underline{d}^p_{kl} y^p_k \geqslant \underline{d}^p_{1l}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT,\ l \leqslant t + K^p_t. \tag{37}$$

Observe that the values K^p_t and K^r_t can be used to limit the size of the reformulation and when they are large enough, all the Wagner–Whitin based inequalities are considered. In the next subsection we show how these values can be heuristically estimated in order to determine a partial reformulation.

5.4. Using a partial reformulation: determining K^p_t and K^r_t

The technique of partial (or approximate) formulations (Van Vyve & Wolsey, 2006) is usually applied to large formulations in an attempt to make them more tractable computationally. It consists in determining a parameter K, which is normally a constant, that limits the size of the formulation. In this section we propose a dynamic way to determine this parameter K based on the problem's cost structure and apply it to the Wagner–Whitin based formulation presented in Section 5.3.

Our goal is to devise a way to heuristically set the size of the Wagner–Whitin based formulation, i.e. number of inequalities to be added a priori, considering the problem's cost structure. For the ease of explanation, we use the storage costs h^p and we limit our analysis to time invariant costs since the setup and storage costs are stationary while the production costs are zero in the available instances that were considered in the numerical experiments. An estimation of the interval in which a production setup is likely to occur can be inferred from the amount of periods necessary for the cumulated storage cost of the items to become larger than the cost of performing a new setup, and a possible measure can be

$$K^{p'} = \arg\min_{k \in \{1,\dots,NT\}} \left(\frac{d_{1,NT}}{NT} \times k \times h^p \geqslant f^p \right)$$

for production and

$$K^{r'} = \arg\min_{k \in \{1,\dots,NT\}} \left(\frac{r_{1,NT}}{NT} \times k \times h^p \geqslant f^r \right)$$

for remanufacture.

Based on the fact that the goal is to devise a formulation that is strong enough but at the same time keeping a size that is not too large, the values K^p and K^r can be obtained from $K^{p'}$ and $K^{r'}$ as

$$K^p = \max\left\{5, \left\lceil 0.5 \times K^{p'} \right\rceil\right\}$$

and

$$K^r = \max\left\{5, \left\lceil 0.5 \times K^{r'} \right\rceil\right\}.$$

Considering that the costs are time invariant, it is reasonable to set $K^p_t = K^p$ and $K^r_t = K^r$ for every t.

6. Adapting the approaches to the economic lot-sizing with remanufacturing and joint setups

In this section, we show how the approaches presented in the previous sections can be easily adapted to the economic lot-sizing with remanufacturing and joint setups (ELSRj), which is a variant of the ELSR in which production and remanufacturing of items share the same setup. After we formulate the problem as a mixed-integer program, in Section 6.1 we present the shortest path formulation proposed by Retel Helmrich et al. (2014), in Section 6.2 we adapt the multicommodity formulation presented in Section 4 to deal with this case and in Section 6.3 we modify the Wagner–Whitin based formulation presented in Section 5 to this variant.

Note that the similarity of the two variants (ELSRs and ELSRj) indicate that their formulations should not have many differences. To formulate ELSRj as a mixed-integer program, define the variables y_t to be equal to 1 if production or remanufacture happens in period t and to be 0 otherwise, and $x^p_t, x^r_t, s^p_t, s^r_t$ to represent the same as for ELSRs in Section 2. A standard MIP formulation for ELSRj can be described as

$$\min \sum_{t=1}^{NT}(h^p_t s^p_t + \overline{p}^p_t x^p_t) + \sum_{t=1}^{NT}(h^r_t s^r_t + \overline{p}^r_t x^r_t) + \sum_{t=1}^{NT} f_t y_t$$

$$s^p_{t-1} + x^p_t + x^r_t = d_t + s^p_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{38}$$

$$s^r_{t-1} + r_t = x^r_t + s^r_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{39}$$

$$x^p_t + x^r_t \leqslant d_{t,NT} y_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{40}$$

$$x^r_t \leqslant r_{1t} y_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{41}$$

$$x^p,\ x^r \in \mathbb{R}^{NT}_+, \tag{42}$$

$$y \in \{0,1\}^{NT}. \tag{43}$$

This formulation only differs from the standard formulation for ELSRs, described as (2)–(7), in the absence of the y^r_t variables and the substitution of the setup enforcing constraints (4) and (5) by constraints (40) and (41). Note that the standard formulation presented here differs from the one in (Retel Helmrich et al., 2014) in the fact that it contains the returned items related setup enforcing constraints (41) on remanufacture.

6.1. The shortest path formulation

The shortest path formulation for ELSRj proposed by Retel Helmrich et al. (2014) is a simplification of the one available for the ELSRs, which was presented in Section 3. Since both the production and remanufacturing processes share a joint setup, only one type of flow variable is used. Consider z^s_{ij} to be the fraction of demand in periods from i to j that is fulfilled by items produced or remanufactured in period i. Also, define the cost parameter:

$$\overline{C}^r_{ij} = \left(\overline{p}^r_j - \overline{p}^p_j\right) r_{i,j} + \sum_{t=i}^{j-1} h^r_t r_{i,t}. \tag{44}$$

The shortest path formulation can be described as

$$\min \sum_{t=1}^{NT}\left(f_t y_t + C^l_t l_t\right) + \sum_{i=1}^{NT}\sum_{j=i}^{NT}\left(C^{sp}_{ij} z^s_{ij} + \overline{C}^r_{ij} z^r_{ij}\right), \tag{45}$$

$$\sum_{j=1}^{NT} z^s_{1j} = 1, \tag{46}$$

$$\sum_{i=1}^{t-1} z^s_{i,t-1} = \sum_{j=t}^{NT} z^s_{tj}, \quad \text{for } 2 \leqslant t \leqslant NT, \tag{47}$$

$$\sum_{j=t}^{NT} z^s_{tj} \leqslant y_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{48}$$

$$\sum_{j=1}^{NT} z^r_{1j} + l_1 = 1, \tag{49}$$

$$\sum_{i=1}^{t-1} z^r_{i,t-1} = \sum_{j=t}^{NT} z^r_{tj} + l_t, \quad \text{for } 2 \leqslant t \leqslant NT, \tag{50}$$

$$\sum_{i=1}^{t} z^r_{it} \leqslant y_t, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{51}$$

$$\sum_{i=1}^{t} r_{it} z^r_{it} = \sum_{j=t}^{NT} d_{tj} z^s_{tj}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{52}$$

$$z^s_{ij},\ z^r_{ij} \geqslant 0, \quad \text{for } 1 \leqslant i \leqslant j \leqslant NT, \tag{53}$$

$$y \in \{0,1\}^{NT}. \tag{54}$$

We refer the reader to Retel Helmrich et al. (2014), in whose work a detailed explanation of the formulation can be found.

6.2. A multicommodity formulation

A multicommodity formulation can be obtained for ELSRj in a similar way it was done for ELSRs in Section 4. The differences rely in the absence of the y^r_t variables and in the fact that the setup enforcing constraints (23) and (24) are replaced by (55).

$$\begin{aligned}
&\min \sum_{t=1}^{NT} p^p_t x^p_t + \sum_{t=1}^{NT} p^r_t x^r_t + \sum_{t=1}^{NT} f_t y_t \\
&\sum_{k=1}^{t} (w^p_{kt} + w^r_{kt}) \geqslant d_t, \quad \text{for } 1 \leqslant t \leqslant NT, \\
&\sum_{k=1}^{t} o^r_{kt} = \sum_{k=t}^{NT} w^r_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \\
&\sum_{t=k}^{NT} o^r_{kt} \leqslant r_k, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \\
&w^p_{kt} + w^r_{kt} \leqslant d_t y_k, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \\
&o^r_{kt} \leqslant r_k y_t, \quad \text{for } 1 \leqslant k \leqslant t \leqslant NT, \\
&x^p_t = \sum_{k=t}^{NT} w^p_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \\
&x^r_t = \sum_{k=t}^{NT} w^r_{tk}, \quad \text{for } 1 \leqslant t \leqslant NT, \\
&w^p,\ w^r,\ o^r \in \mathbb{R}^{NT\times NT}_+, \\
&y \in \{0,1\}^{NT}.
\end{aligned} \tag{55}$$

6.3. The Wagner–Whitin based formulation

The joint setup counterparts of inequalities (30) and (31) presented in Retel Helmrich et al. (2014) are, respectively,

$$s^p_{t-1} + \sum_{k=t}^{l} d_{kl} y_k \geqslant d_{tl}, \quad \text{for } 2 \leqslant t \leqslant l \leqslant NT, \tag{56}$$

and

$$s_l^r + \sum_{k=t}^{l} r_{tk} y_k \geqslant r_{tl}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT. \tag{57}$$

In addition, the joint setup counterpart of the family represented by (33) is

$$\sum_{k=1}^{t-1} x_k^p + \sum_{k=t}^{l} \underline{d}_{kl}^p y_k \geqslant \underline{d}_{1l}^p, \text{ for} 1 \leqslant t \leqslant l \leqslant NT, \tag{58}$$

whose validity follows from Proposition 2.

Using inequalities (56)–(58), a Wagner–Whitin based formulation can be obtained as

$$\min \sum_{t=1}^{NT} (h_t^p s_t^p + \overline{p}_t^p x_t^p) + \sum_{t=1}^{NT} (h_t^r s_t^r + \overline{p}_t^r x_t^r) + \sum_{t=1}^{NT} f_t y_t \tag{59}$$

(38)–(43)

$$s_{t-1}^p + \sum_{k=t}^{l} d_{kl} y_k \geqslant d_{tl}, \quad \text{for } 2 \leqslant t \leqslant l \leqslant NT, \ l \leqslant t + K_t^p \tag{60}$$

$$s_l^r + \sum_{k=t}^{l} r_{tk} y_k \geqslant r_{tl}, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT, \ l \leqslant t + K_t^r. \tag{61}$$

$$\sum_{k=1}^{t-1} x_k^p + \sum_{k=t}^{l} \underline{d}_{kl}^p y_k \geqslant \underline{d}_{1l}^p, \quad \text{for } 1 \leqslant t \leqslant l \leqslant NT, \ l \leqslant t + K_t^p. \tag{62}$$

Observe that the values K_t^p and K_t^r can be used to limit the size of the reformulation and the approach presented in Section 5.4 can be used to generate a partial reformulation.

7. Computational experiments

In this section we report on the computational experiments carried out to assess the effectiveness of the proposed methods. All experiments were performed on a machine running under Xubuntu x86_64 GNU/Linux, with an Intel Core i5-3470 3.20 GHz processor, 4 Gb of RAM memory using FICO Xpress 7.7. The codes were written in C++ and compiled with g++ 4.8.2. The solver's default settings were used, with exception of the optimality tolerance that was set to 10^{-6}, and a time limit of one hour (3600s) was imposed to every execution. We performed computational experiments with two different instance sets with large instances available in the literature, which are briefly described in Section 7.1. The results are summarized in Section 7.2 for ELSRs and in Section 7.3 for ELSRj. Detailed results can be obtained from the authors.

7.1. Instances description

In our experiments, we considered two sets of larger (and more difficult) instances available in the literature. A brief description follows in the next paragraphs.

The first set, which we denote *Type S* instances, is composed of 108 instances for ELSRs and is described in Sifaleras et al. (2015). There is a planning horizon of $NT = 52$ periods and all the costs are time invariant. The demand d_t for each period follows a normal distribution with mean of 100 units, while the quantity of returned items in each period, r_t, follows a normal distribution with mean in $\{30, 50, 70\}$. Fixed setup costs, f^p and f^r, assume values in $\{200, 500, 2000\}$. Variable production and remanufacturing costs are considered to be zero and per unit holding cost of final items, h^p, is unitary while the per unit holding cost for returned items, h^r, is drawn from $\{0.2, 0.5, 0.8\}$. The instances can be organized in instance groups of four elements with the same characteristics, and each group is identified in the remainder of the paper as $S\_(f^r)\_(f^p)\_(h^r)$.

The second set, which we denote *Type H* instances, is composed of 360 instances for both ELSRs and ELSRj, and a detailed description can be encountered in Retel Helmrich et al. (2014). There is a time horizon of $NT \in \{25, 50, 75\}$ periods and the demands d_t are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$ for each period. The returns r_t are also normally distributed with three different configurations $(\mu = 10, \sigma = 5)$, $(\mu = 50, \sigma = 25)$, and $(\mu = 90, \sigma = 45)$. All cost parameters are time invariant. Fixed setup costs, f^p and f^r, assume values in $\{125, 250, 500, 1000\}$, considering that $f^p = f^r$ for ELSRs. Variable production and remanufacturing costs are considered to be zero and there are unitary per unit holding costs for both produced and remanufactured items. The instances can be organized in instance groups of ten elements with the same characteristics, and each group is identified in the remainder of the paper as $H\_(NT)\_(\mu)\_(f^p)$.

7.2. Results for ELSRs

We summarize here the computational results for the economic lot-sizing with remanufacturing and separate setups. We present results regarding Type S instances in Section 7.2.1 and Type H instances in Section 7.2.2. The following approaches were considered in the experiments:

- *SP*: shortest path formulation of Retel Helmrich et al. (2014), presented in Section 3;
- *MC*: multicommodity formulation;
- *WW*: Wagner–Whitin based formulation with only inequalities (31) and (32);
- *WWd*: complete Wagner–Whitin based formulation;
- *pWWd*: partial Wagner–Whitin based formulation.

We remark that the standard formulation without inequalities was not considered in the experiments, since it is known that big M formulations usually have bad performances (Pochet & Wolsey, 2006), what was observed for ELSR in (Retel Helmrich et al., 2014) with a few exceptions for ELSRj. We also note that other formulations were tested in Retel Helmrich et al. (2014), but we compare our approaches only with *SP* since it performed best in their experiments when we consider the number of instances solved to optimality.

7.2.1. Type S instances

Table 1 summarizes the results for Type S instances. The first column, *Instances*, gives the instance group. The next five columns give the geometric means of the linear relaxation gaps (in %), gap_{LP}, for each of the tested approaches (*SP*, *MC*, *WW*, *WWd* and *pWWd*), considering that the value gap_{LP} for each instance is determined as $100 \times (best - lb)/best$, in which *best* is the best known integer solution and *lb* the linear relaxation bound obtained using the approach. And finally, information regarding the integer solutions is presented for each of the formulations *SP*, *MC*, *WWd* and *pWWd*. For each approach, column *time* indicates the geometric mean of the run times in seconds; *gap* gives the geometric mean of the gaps for the instance group, considering that the value *gap* for each instance is determined as $100 \times (best - bestbound)/best$, in which *bestbound* is the best known bound achieved using the approach at the end of the execution; and *opt* shows the number of instances solved to optimality for each instance group. The value '–' in the column corresponding to the *gap* indicates that all instances in the group were solved to optimality. The last row in

Table 1
Type S ELSRs instances.

Instances	Linear relaxations					Integer solutions											
	SP	*MC*	*WW*	*WWd*	*pWWd*	*SP*			*MC*			*WWd*			*pWWd*		
	gap_{LP}	gap_{LP}	gap_{LP}	gap_{LP}	gap_{LP}	*time*	*gap*	*opt*	*time*	*gap*	*opt*	*time*	*gap*	*opt*	*time*	*gap*	*opt*
S_200_200_0.2	1.9	1.9	7.8	4.3	4.3	28.0	–	4	30.5	–	4	89.0	–	4	14.7	–	4
S_200_200_0.5	3.5	3.5	13.5	6.0	6.1	285.5	0.8	3	328.8	–	4	372.7	–	4	70.9	–	4
S_200_200_0.8	4.2	4.2	16.7	5.8	5.9	34.9	1.5	2	35.1	1.1	2	224.1	1.5	3	48.2	1.0	3
S_200_500_0.2	2.6	2.6	30.6	9.2	13.1	41.0	–	4	43.2	–	4	146.6	–	4	19.9	–	4
S_200_500_0.5	4.3	4.3	33.6	7.8	8.4	837.9	0.4	2	664.9	0.5	3	1010.0	–	4	175.5	–	4
S_200_500_0.8	5.2	5.2	34.7	6.7	7.1	2580.0	0.6	1	1559.2	0.5	2	1412.8	–	4	133.4	–	4
S_200_2000_0.2	3.0	3.0	50.7	17.5	24.3	23.1	–	4	31.0	–	4	54.9	–	4	7.0	–	4
S_200_2000_0.5	5.4	5.4	51.5	7.4	8.6	119.6	–	4	164.3	–	4	256.2	–	4	11.9	–	4
S_200_2000_0.8	8.2	8.2	52.2	4.4	4.5	411.4	–	4	386.1	–	4	362.2	–	4	12.0	–	4
S_500_200_0.2	1.2	1.2	3.1	3.1	3.9	6.2	–	4	6.3	–	4	16.4	–	4	6.5	–	4
S_500_200_0.5	1.5	1.5	2.1	2.1	2.1	11.9	–	4	13.4	–	4	39.6	–	4	11.0	–	4
S_500_200_0.8	1.9	1.9	2.7	2.6	2.6	14.7	–	4	19.3	–	4	54.4	–	4	14.0	–	4
S_500_500_0.2	1.6	1.6	11.4	3.9	4.2	11.4	–	4	13.6	–	4	31.1	–	4	9.4	–	4
S_500_500_0.5	2.7	2.7	11.0	5.9	6.4	99.9	–	4	122.6	–	4	237.6	–	4	44.4	–	4
S_500_500_0.8	3.0	3.0	13.7	6.3	7.0	88.0	0.8	3	89.2	0.8	3	172.6	0.8	3	76.6	–	4
S_500_2000_0.2	1.9	1.9	34.0	10.9	15.9	10.3	–	4	15.8	–	4	43.6	–	4	10.6	–	4
S_500_2000_0.5	3.5	3.5	35.9	7.5	8.1	62.5	–	4	74.0	–	4	123.3	–	4	22.9	–	4
S_500_2000_0.8	4.7	4.7	36.8	6.7	6.9	63.5	0.8	3	200.2	-	4	128.8	–	4	45.4	–	4
S_2000_200_0.2	0.4	0.4	9.3	9.3	9.3	1.4	–	4	1.2	–	4	5.4	–	4	4.1	–	4
S_2000_200_0.5	0.8	0.8	5.7	5.7	5.7	2.2	–	4	3.3	–	4	11.7	–	4	6.2	–	4
S_2000_200_0.8	0.7	0.7	3.5	3.5	3.5	2.1	–	4	2.8	–	4	8.8	–	4	5.2	–	4
S_2000_500_0.2	0.5	0.5	3.7	3.7	3.7	1.4	–	4	2.2	–	4	6.4	–	4	4.7	–	4
S_2000_500_0.5	0.8	0.8	1.8	1.8	1.8	2.9	–	4	3.7	–	4	8.8	–	4	5.6	–	4
S_2000_500_0.8	0.9	0.9	1.5	1.5	1.5	3.8	–	4	3.7	–	4	9.2	–	4	6.2	–	4
S_2000_2000_0.2	0.9	0.9	4.3	2.8	3.1	2.6	–	4	3.8	–	4	12.4	–	4	5.2	–	4
S_2000_2000_0.5	1.3	1.3	7.9	4.6	5.3	4.2	–	4	5.7	–	4	19.9	–	4	8.6	–	4
S_2000_2000_0.8	1.6	1.6	9.8	5.1	5.7	5.3	–	4	8.2	–	4	27.2	–	4	9.4	–	4
#opt								98			102			106			107

the table gives the total number of instances solved to optimality using the formulations.

The results in Table 1 show that *SP* and *MC* obtained the same linear relaxation bounds for all instance groups, and they were much better than those obtained with *WW*. It is noteworthy that the use of the new proposed inequalities (33) allowed large improvements in *WWd* when compared to *WW*.

The results regarding the integer solutions show that the four approaches performed well, with *pWWd* performing best among them when we compare the number of instances solved to optimality. All instances but one could be solved to optimality when using *pWWd*, and the remaining open gap for the unsolved instance was only 1.0%. It is possible to see that, in general, the linear relaxation bounds obtained by *pWWd* are weaker than those obtained by the complete reformulation *WWd*, but the impressive reduction in time shows that the heuristic to limit the size of the partial formulation allowed the solver to perform in a much more effective way. For most of the instance groups, the mean time to prove optimality did not even reach 30 s and for only four instance groups it was over 60 s, but never reaching 180 s. One can note that the instances with higher storage costs for returned items were more difficult to tackle, since five out of the six unsolved instances with *MC* and the two that remained unsolved with *WWd* have storage cost 0.8. Note that even though *SP* and *MC* provided the same linear relaxation bounds, four additional instances could be solved to optimality by the solver with *MC* when compared to *SP*.

7.2.2. Type H instances

Table 2 summarizes the results comparing the approaches for Type H instances. The relative behavior of the approaches when compared to each other was similar to that presented for Type S instances. Again, *SP* and *MC* obtained the same linear relaxation bounds for all the instance groups.

The results in Table 2 show that all the 120 smaller instances with 25 periods could be solved to optimality in around one second on average for most of the instance groups using the four different approaches. When we take into consideration all the 360 tested instances, *MC* could solve more of them to optimality (310) than *SP* (298). Using *WWd* the number of instances solved to optimality increased to 318 and using *pWWd* this number reached a much higher number of 345 instances. In most of the cases the difference in time taken to solve the instances with *SP* and *MC* only differs by a few seconds, and on average it usually takes longer for *WWd* to find the optimal solutions although it can reach optimality to a larger number of them.

We can conclude from the results presented in Table 2 that our approach of using a partial formulation with heuristic determined size allowed the solver to prove optimality for several additional instances when compared to the other tested techniques when considering Type H instances.

7.3. Results for ELSRj

We now comment on the results for the economic lot-sizing with remanufacturing and joint setups. The results using the joint setup versions of *SP*, *MC*, *WWd* and *pWWd* are summarized in Table 3. Given the fact that the available instances for this variant are much easier than those for ELSRs, we restrain our comparisons to the ones with 50 and 75 periods.

The results in Table 3 show that the linear relaxations of the approaches are equal to the optimal solution values in almost all instances with lower returns (mean 10), with *WWd* being the only one to achieve zero linear relaxation gap for all of them. The bounds obtained by *MC* and *SP* are the same for all instance groups, and those obtained by *WWd* are always better or equal than those of *SP* and *MC*. It is noteworthy that for most of the instance groups, the use of *pWWd* did not increase the linear relaxation gaps when

Table 2
Type H ELSRs instances.

Instances	Linear relaxations					Integer solutions											
	SP	*MC*	*WW*	*WWd*	*pWWd*	*SP*			*MC*			*WWd*			*pWWd*		
	gap_{LP}	gap_{LP}	gap_{LP}	gap_{LP}	gap_{LP}	*time*	*gap*	*opt*	*time*	*gap*	*opt*	*time*	*gap*	*opt*	*time*	*gap*	*opt*
H_25_10_125	0.9	0.9	18.5	4.7	4.7	1.0	–	10	1.0	–	10	1.0	–	10	1.0	–	10
H_25_10_250	0.6	0.6	15.3	4.2	4.2	1.0	–	10	1.0	–	10	1.0	–	10	1.1	–	10
H_25_10_500	0.7	0.7	13.2	4.1	4.1	1.0	–	10	1.0	–	10	1.1	–	10	1.0	–	10
H_25_10_1000	0.0	0.0	10.1	2.9	3.1	1.0	–	10	1.0	–	10	1.0	–	10	1.0	–	10
H_25_50_125	5.7	5.7	18.2	4.6	4.7	1.0	–	10	1.0	–	10	1.2	–	10	1.0	–	10
H_25_50_250	5.4	5.4	15.3	5.7	6.0	1.0	–	10	1.0	–	10	1.3	–	10	1.0	–	10
H_25_50_500	4.1	4.1	12.8	5.7	6.1	1.0	–	10	1.0	–	10	1.2	–	10	1.0	–	10
H_25_50_1000	3.5	3.5	10.6	5.7	6.9	1.0	–	10	1.0	–	10	1.1	–	10	1.0	–	10
H_25_90_125	8.8	8.8	11.1	6.1	6.1	1.0	–	10	1.0	–	10	1.2	–	10	1.0	–	10
H_25_90_250	8.7	8.7	11.1	6.8	7.2	1.1	–	10	1.1	–	10	1.3	–	10	1.0	–	10
H_25_90_500	7.5	7.5	10.3	6.9	7.4	1.0	–	10	1.0	–	10	1.2	–	10	1.0	–	10
H_25_90_1000	6.0	6.0	8.9	6.5	6.9	1.0	–	10	1.0	–	10	1.1	–	10	1.0	–	10
H_50_10_125	1.6	1.6	20.2	5.6	5.6	1.4	–	10	1.1	–	10	6.7	–	10	1.4	–	10
H_50_10_250	1.0	1.0	17.1	5.0	5.0	1.4	–	10	1.4	–	10	9.6	–	10	2.3	–	10
H_50_10_500	1.0	1.0	15.8	5.1	5.2	1.6	–	10	1.8	–	10	10.9	–	10	3.1	–	10
H_50_10_1000	0.5	0.5	13.5	4.7	4.9	1.3	–	10	1.4	–	10	9.3	–	10	3.4	–	10
H_50_50_125	6.8	6.8	19.6	6.0	6.0	22.8	–	10	8.2	–	10	24.4	–	10	3.4	–	10
H_50_50_250	6.2	6.2	16.8	6.9	7.2	74.5	0.4	9	71.3	–	10	120.4	–	10	20.4	–	10
H_50_50_500	4.7	4.7	13.9	6.3	7.2	17.2	–	10	20.9	–	10	76.5	–	10	13.2	–	10
H_50_50_1000	3.7	3.7	12.3	6.1	8.1	10.8	–	10	14.7	–	10	40.9	–	10	9.4	–	10
H_50_90_125	6.9	6.9	9.3	5.4	5.5	13.7	–	10	5.6	–	10	14.7	–	10	1.5	–	10
H_50_90_250	7.8	7.8	10.1	6.9	7.3	72.1	–	10	32.9	–	10	45.2	–	10	5.6	–	10
H_50_90_500	7.6	7.6	10.0	7.2	8.1	49.6	1.1	9	41.0	0.6	9	71.0	0.8	9	18.0	–	10
H_50_90_1000	6.1	6.1	8.7	6.6	7.5	19.8	0.4	9	25.8	0.0	9	54.4	–	10	10.0	–	10
H_75_10_125	1.3	1.3	20.2	5.5	5.5	2.9	–	10	2.7	–	10	20.8	–	10	2.4	–	10
H_75_10_250	1.1	1.1	17.8	5.3	5.3	4.4	–	10	5.3	–	10	55.8	–	10	4.6	–	10
H_75_10_500	0.9	0.9	16.0	5.1	5.1	4.5	–	10	5.5	–	10	49.2	–	10	6.2	–	10
H_75_10_1000	0.7	0.7	14.5	5.1	5.4	3.5	–	10	5.1	–	10	60.8	–	10	8.7	–	10
H_75_50_125	7.4	7.4	19.9	6.2	6.3	796.0	1.0	1	386.7	0.5	5	268.2	–	10	31.5	–	10
H_75_50_250	6.2	6.2	16.6	6.7	7.0	–	1.2	0	2415.0	0.9	1	1585.9	0.9	2	429.7	0.4	8
H_75_50_500	5.0	5.0	14.6	6.6	7.6	523.3	0.9	2	237.0	0.9	3	1124.8	1.5	3	575.7	1.1	8
H_75_50_1000	4.3	4.3	13.0	6.4	8.5	398.0	0.4	6	646.5	0.6	7	2735.6	0.8	4	475.8	–	10
H_75_90_125	6.8	6.8	9.3	5.4	5.6	225.0	1.7	5	71.4	0.6	6	122.3	–	10	11.4	–	10
H_75_90_250	7.6	7.6	10.0	6.9	7.3	310.6	2.4	3	125.9	2.1	4	187.3	1.2	5	107.5	1.1	9
H_75_90_500	8.1	8.1	10.7	7.9	8.8	124.0	2.1	1	733.3	2.2	3	618.0	2.0	2	112.7	1.7	4
H_75_90_1000	6.5	6.5	9.0	6.8	7.7	116.8	1.8	3	118.2	2.0	3	382.1	2.5	3	137.0	2.0	6
#opt								298			310			318			345

Table 3
Type H ELSRj instances.

	Linear relaxations				Integer solutions			
	SP	*MC*	*WWd*	*pWWd*	*SP*	*MC*	*WWd*	*pWWd*
Instances	gap_{LP}	gap_{LP}	gap_{LP}	gap_{LP}	*time*	*time*	*time*	*time*
H_50_10_125	0.1	0.1	–	–	1.0	1.0	1.0	<1.0
H_50_10_250	–	–	–	–	1.0	1.0	1.0	1.0
H_50_10_500	–	–	–	–	1.0	1.0	1.0	1.0
H_50_10_1000	–	–	–	0.0	1.0	1.0	1.0	<1.0
H_50_50_125	1.0	1.0	0.8	0.8	1.0	1.0	1.0	1.0
H_50_50_250	0.4	0.4	0.4	0.4	1.0	1.0	1.0	1.0
H_50_50_500	0.2	0.2	0.2	0.2	1.0	1.0	1.4	1.2
H_50_50_1000	0.0	0.0	0.0	0.0	1.0	1.0	<1.0	1.0
H_50_90_125	3.2	3.2	3.1	3.1	2.0	1.5	3.7	1.1
H_50_90_250	3.2	3.2	3.2	3.2	1.9	3.1	4.4	1.2
H_50_90_500	2.5	2.5	2.5	2.5	2.2	1.0	6.0	1.7
H_50_90_1000	1.5	1.5	1.5	1.5	1.6	1.6	6.9	1.4
H_75_10_125	–	–	–	–	1.0	1.0	1.0	1.0
H_75_10_250	–	–	–	–	1.0	1.0	1.0	1.0
H_75_10_500	–	–	–	–	1.0	1.0	1.0	<1.0
H_75_10_1000	–	–	–	0.1	1.0	1.0	1.0	1.0
H_75_50_125	0.9	0.9	0.9	0.9	1.0	1.0	1.1	1.2
H_75_50_250	0.3	0.3	0.3	0.3	1.0	1.0	1.0	1.1
H_75_50_500	0.1	0.1	0.1	0.1	1.0	1.0	1.7	1.8
H_75_50_1000	0.2	0.2	0.2	0.2	1.0	1.0	2.2	1.1
H_75_90_125	3.2	3.2	3.1	3.1	5.1	10.8	10.7	1.8
H_75_90_250	3.2	3.2	3.2	3.2	5.7	1.0	14.8	2.6
H_75_90_500	3.3	3.3	3.3	3.3	11.0	1.2	33.5	5.5
H_75_90_1000	2.1	2.1	2.1	2.1	6.2	6.5	28.8	4.5

compared with the full *WWd*, meaning that our heuristic measure allowed us to have a smaller partial formulation that is almost as strong as the full formulation.

Since all instances could be solved to optimality when using the four approaches, we only report the mean time to solve them. Observe that, in general *MC* requires less time than *SP* and the time to solve the instances using *WWd* is usually larger than those using *SP* and *MC*. For almost all instance groups, *pWWd* solves the instances in less time when compared to *WWd*. Note that *SP*, *MC* and *pWWd* could solve all instance groups in very small mean times, with a special attention to *pWWd* which did not reach more than 5.5 s for any instance group.

8. Final remarks

We considered the economic lot-sizing with remanufacturing and proposed a multicommodity reformulation and new valid inequalities that were used in a Wagner–Whitin based formulation for the problem. We also proposed a novel heuristic technique to automatically determine the size of a partial reformulation based on the problem's cost structure, which was applied to the Wagner–Whitin based formulation.

Computational results showed that our approaches outperformed a shortest path formulation presented in Retel Helmrich et al. (2014), which was the best approach in their experiments when we consider the number of instances solved to optimality. For all the tested instances, the linear relaxation bounds obtained by the proposed multicommodity formulation and by the shortest path formulation were the same and therefore the question remains whether one can show if they are always the same or, if not, under which circumstances this observation holds. It is noteworthy that although the same linear relaxation bound was obtained using both approaches, the solver could solve more instances to optimality when using the multicommodity formulation than when using the shortest path one.

The newly proposed valid inequalities allowed us to achieve, in certain cases, better bounds than those obtained with the other approaches, and made it possible to obtain an effective Wagner–Whitin based formulation. As far as we know, a polyhedral study of ELSR has not yet been done and this appear to be an interesting direction for future research.

Our partial Wagner–Whitin based formulation, with size determined in an automated heuristic way based on the problem's cost structure, allowed us to solve to optimality more than 96% (452 out of 468) of the available instances for the problem with separate setups, outperforming by far the other tested approaches and solving several instances that could not be solved otherwise. This approach also performed slightly better than the other approaches for ELSRj, whose available instances were considerably easier. We believe that such an approach could be used effectively for other production planning problems such as the ones considered in (Retel Helmrich, Jans, van den Heuvel, & Wagelmans, 2015; Sahling, 2013; Sifaleras & Konstantaras, in press), and that it could also be used to improve the performance of rolling horizon approaches by determining the size of intervals to be solved at each iteration. In addition, it is possible that the use of a heuristic to generate starting feasible solutions such as the ones proposed in (Baki et al., 2014; Sifaleras et al., 2015) could help our approach to solve some additional instances that remained unsolved.

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Appendix A. Objective function without storage costs

Observe that the stock of finished items can be calculated as

$$s_t^p = \sum_{k=1}^{t} x_k^p + \sum_{k=1}^{t} x_k^r - d_{1t}, \quad \text{for } 1 \leqslant t \leqslant NT, \tag{63}$$

and the stock of returned items as

$$s_t^r = r_{1t} - \sum_{k=1}^{t} x_k^r, \quad \text{for } 1 \leqslant t \leqslant NT. \tag{64}$$

Using (63) and (64) to remove the stock variables from (1) one can make the necessary algebraic manipulations in order to obtain the new objective function

$$\sum_{t=1}^{NT} (p_t^p x_t^p + f_t^p y_t^p) + \sum_{t=1}^{NT} (p_t^r x_t^r + f_t^r y_t^r) + M,$$

in which

$$p_t^p = \overline{p}_t^p + \sum_{j=t}^{NT} h_j^p, \quad 1 \leqslant t \leqslant NT$$

$$p_t^r = \overline{p}_t^r + \sum_{j=t}^{NT} h_j^p - \sum_{j=t}^{NT} h_j^r, \quad 1 \leqslant t \leqslant NT$$

and M is the constant

$$M = \sum_{t=1}^{NT} (-h_t^p d_{1t} + h_t^r r_{1t}).$$

Note that the only costs involved in the manipulations were the variable costs related to production and remanufacturing and the storage costs. Therefore a similar procedure can be done to remove the stock costs from the objective function of the economic lot-sizing with remanufacturing and joint setups.

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