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ABSTRACT ALGEBRA

Contents

Operations

[Pinter, 2016, chapter 2]

A. Examples of Operations

1 $a * b = \sqrt{|ab|}$ is not an operation on $\mathbb{Q}$, because e.g. $2 * 1 = \sqrt{|2|} \notin \mathbb{Q}$.

2 $a * b = a \ln b$ is not an operation on $\mathbb{R}_{>0}$, because e.g. $\forall a, b \in \mathbb{R}_{>0}(b \leq 1 \rightarrow a \ln b \notin \mathbb{R}_{>0})$

3 If $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, $*$ is not an operation on $\mathbb{R}$, because $\forall a, b \in \mathbb{R}(a \neq 0 \wedge b \neq 0 \rightarrow x = \pm ab)$

4 Subtraction is an operation on $\mathbb{Z}$, because $\forall a, b \in \mathbb{Z}(a - b \in \mathbb{Z})$.

5 Subtraction is not an operation on $\mathbb{Z}_{\geq 0}$, because e.g. $0 - 1 \notin \mathbb{Z}_{\geq 0}$.

6 $a * b = |a - b|$ is an operation on $\mathbb{Z}_{\geq 0}$, because $\forall a, b \in \mathbb{Z}_{\geq 0}(|a - b| \in \mathbb{Z}_{\geq 0})$.

B. Properties of Operations

1 $x * y = x + 2y + 4$

(i) $*$ is not commutative.

$$
\begin{aligned}
x * y &= x + 2y + 4 \\
y * x &= y + 2x + 4 \\
x * y &\neq y * x
\end{aligned}
$$

(ii) $*$ is not associative.

$$
\begin{aligned}
x * (y * z) &= x * (y + 2z + 4) \\
&= x + 2(y + 2z + 4) + 4 \\
&= x + 2y + 4z + 12 \\
(x * y) * z &= (x + 2y + 4) * z \\
&= x + 2y + 4 + 2z + 4 \\
&= x + 2y + 2z + 8 \\
x + 2y + 4z + 12 &\neq x + 2y + 2z + 8
\end{aligned}
$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x * e &= x \\ x + 2e + 4 &= x \\ 2e + 4 &= 0 \\ e &= -2 \\ e * x &= x \\ e + 2x + 4 &= x \\ e &= -x - 4 \neq -2 \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

2 $x * y = x + 2y - xy$

(i) $*$ is not commutative.

$$\begin{aligned} x * y &= x + 2y - xy \\ y * x &= y + 2x - yx \\ x * y &\neq y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x * (y * z) &= x * (y + 2z - yz) \\ &= x + 2(y + 2z - yz) - x(y + 2z - yz) \\ &= x + 2y + 4z - 2yz - xy - 2xz + xyz \\ (x * y) * z &= (x + 2y - xy) * z \\ &= (x + 2y - xy) + 2z - (x + 2y - xy)z \\ &= x + 2y + 2z - 2yz - xy - xz + xyz \\ x * (y * z) &\neq (x * y) * z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x * e &= x \\ x + 2e - xe &= x \\ 2e - xe &= 0 \\ e(2 - x) &= 0 \\ e &= 0 \\ e * x &= x \\ e + 2x - ex &= x \\ e + x - ex &= 0 \\ e(1 - x) &= -x \\ e &= -x(1 - x) \neq 0 \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

3 $x * y = |x + y|$

(i) $*$ is commutative.

$$
\begin{aligned}
x * y &= |x + y| \\
y * x &= |y + x| = |x + y| \\
x * y &= y * x
\end{aligned}
$$

(ii) $*$ is not associative.

$$
\begin{aligned}
x * (y * z) &= x * |y + z| = |x + |y + z|| \\
(x * y) * z &= |x + y| * z = ||x + y| + z| \\
x = 0, y < 0 \to x * (y * z) &= |y + z| \\
(x * y) * z &= ||y| + z| \\
y < 0 \to y \neq |y| \to |y + z| &\neq ||y| + z| \\
x * (y * z) &\neq (x * y) * z
\end{aligned}
$$

(iii) $\mathbb{R}$ has an identity element with respect to $*$.

$$
\begin{aligned}
x * e &= x \\
|x + e| &= x \\
e &= 0 \\
e * x &= x \\
|e + x| &= x \\
e &= 0
\end{aligned}
$$

(iv) Every $x \in \mathbb{R}$ has an inverse with respect to $*$.

$$
\begin{aligned}
x * x' &= 0 \\
|x + x'| &= 0 \\
x' &= -x \\
x * (-x) &= |x - x| = 0 \\
(-x) * x &= |-x + x| = 0 \\
x * x' &= x' * x
\end{aligned}
$$

4 $x * y = |x - y|$

(i) $*$ is commutative.

$$
\begin{aligned}
x * y &= |x - y| \\
y * x &= |y - x| \\
x = y \to x * y &= 0 \\
y * x &= 0
\end{aligned}
$$

If $x < y$ then $x = y + k$, and:

$$\begin{aligned} x * y &= |(y+k) - y| = |k| \\ y * x &= |y - (y+k)| = |-k| = |k| \\ x * y &= y * x \end{aligned}$$

If $x = y$:

$$\begin{aligned} x * y &= |y - y| = 0 \\ y * x &= |y - y| = 0 \\ x * y &= y * x \end{aligned}$$

If $x > y$ then $y = x + k$, and:

$$\begin{aligned} x * y &= |x - (x+k)| = |-k| = |k| \\ y * x &= |(x+k) - x| = |k| \\ x * y &= y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x * (y * z) &= x * |y - z| \\ &= |x - |y - z|| \\ (x * y) * z &= |x - y| * z \\ &= ||x - y| - z| \end{aligned}$$

If $x = 0$ and $y < 0$:

$$\begin{aligned} x * (y * z) = |-|y - z|| = |y - z| &= \sqrt{(y - z)^2} \\ (x * y) * z = ||-y| - z| = ||y| - z| &= \sqrt{(|y| - z)^2} \\ |y| &\neq y \\ x * (y * z) &\neq (x * y) * z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x * e &= x \\ |x - e| &= x \\ e &= 2x \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

5 $x * y = xy + 1$

(i) $*$ is commutative.

$$\begin{aligned} x * y &= xy + 1 \\ y * x &= yx + 1 = xy + 1 \\ x * y &= y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x*(y*z) &= x*(yz+1) \\ &= x(yz+1)+1 = xyz+x+1 \\ (x*y)*z &= (xy+1)*z \\ &= (xy+1)z+1 = xyz+z+1 \\ x*(y*z) &\neq (x*y)*z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x*e &= x \\ xe+1 &= x \\ xe &= x-1 \\ x &= 1-\frac{1}{x} \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

6 $x*y = \max\{\, x, y \,\}$ = the larger of the two numbers x and y

(i) $*$ is commutative.

$$\begin{aligned} x*y &= \max\{\, x, y \,\} \\ y*x &= \max\{\, y, x \,\} = \max\{\, x, y \,\} \\ x*y &= y*x \end{aligned}$$

(ii) $*$ is associative.

$$\begin{aligned} x*(y*z) &= x*\max\{\, y, z \,\} \\ &= \max\{\, x, \max\{\, y, z \,\} \,\} = \max\{\, x, y, z \,\} \\ (x*y)*z &= (\max\{\, x, y \,\})*z \\ &= \max\{\, \max\{\, x, y \,\}, z \,\} = \max\{\, x, y, z \,\} \\ x*(y*z) &= (x*y)*z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x*e &= x \\ \max\{\, x, e \,\} &= x \\ e &= \{n \in \mathbb{R} : n \leq x\} \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

7 $x*y = \frac{xy}{x+y+1}$

(i) $*$ is commutative.

$$\begin{aligned} x*y &= \frac{xy}{x+y+1} \\ y*x &= \frac{yx}{y+x+1} = \frac{xy}{x+y+1} \\ x*y &= y*x \end{aligned}$$

(ii) $*$ is associative.

$$
\begin{aligned}
x*(y*z) &= x*(\frac{yz}{y+z+1}) \\
&= \frac{\frac{xyz}{y+z+1}}{x+\frac{yz}{y+z+1}+1} \\
&= \frac{xyz}{x(y+z+1)+yz+(y+z+1)} \\
&= \frac{xyz}{xy+xz+yz+x+y+z+1} \\
(x*y)*z &= \left(\frac{xy}{x+y+1}\right)*z \\
&= \frac{\frac{xyz}{x+y+1}}{\frac{xy}{x+y+1}+z+1} \\
&= \frac{xyz}{xy+z(x+y+1)+z+(x+y+1)} \\
&= \frac{xyz}{xy+xz+yz+x+y+z+1} \\
x*(y*z) &= (x*y)*z
\end{aligned}
$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$
\begin{aligned}
x*e &= x \\
\frac{xe}{x+e+1} &= x \\
xe &= x(x+e+1) \\
e &= e+x+1
\end{aligned}
$$

(iv) Since there is no identity element, there can be no inverses.

C. Operations no a Two-Element Set Let A be the two-element set $A = \{\, a, b \,\}$.

1

Table 1.1: 0_1

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	a

Table 1.2: 0_2

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	a
(b,b)	b

Table 1.3: 0_3

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	a

Table 1.4: 0_4

(x,y)	$x*y$
(a,a)	a
(a,b)	a
(b,a)	b
(b,b)	b

Table 1.5: 0_5

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	a

Table 1.6: 0_6

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	a
(b,b)	b

Table 1.7: 0_7

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	a

Table 1.8: 0_8

(x,y)	$x*y$
(a,a)	a
(a,b)	b
(b,a)	b
(b,b)	b

Table 1.9: 0_9

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	a

Table 1.10: 0_{10}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	b

Table 1.11: 0_{11}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	a

Table 1.12: 0_{12}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	b

Table 1.13: 0_{13}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	a

Table 1.14: 0_{14}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	b

Table 1.15: 0_{15}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	a

Table 1.16: 0_{16}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	b

2 Commutativity

- 0_1 is commutative: $a*b=a=b*a$
- 0_2 is commutative: $a*b=a=b*a$
- 0_3 is not commutative: $a*b=a\neq b=b*a$
- 0_4 is not commutative: $a*b=a\neq b=b*a$
- 0_5 is not commutative: $a*b=b\neq a=b*a$
- 0_6 is not commutative: $a*b=b\neq a=b*a$
- 0_7 is commutative: $a*b=b=b*a$
- 0_8 is commutative: $a*b=b=b*a$
- 0_9 is commutative: $a*b=a=b*a$
- 0_{10} is commutative: $a*b=a=b*a$
- 0_{11} is not commutative: $a*b=a\neq b=b*a$
- 0_{12} is not commutative: $a*b=a\neq b=b*a$
- 0_{13} is not commutative: $a*b=b\neq a=b*a$
- 0_{14} is not commutative: $a*b=b\neq a=b*a$
- 0_{15} is commutative: $a*b=b=b*a$
- 0_{16} is commutative: $a*b=b=b*a$

3 Associativity

- 0_1 is associative:

$$\forall x,y\in A(x*y=a\rightarrow x*(y*z)=x*a=a=a*z=(x*y)*z)$$

- 0_2 is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*a = a*b = (a*a)*b \\
a*(b*a) &= a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = a*a = (b*a)*a \\
b*(a*b) &= b*a = a*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- 0_3 is not associative: $b*(a*b) = b*a = b \neq a = b*b = (b*a)*b$
- 0_4 is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*a = a*b = (a*a)*b \\
a*(b*a) &= a*b = a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*a = b*b = (b*a)*b \\
b*(b*a) &= b*b = b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- 0_5 is not associative: $b*(a*b) = b*b = a \neq b = a*b = (b*a)*b$
- 0_6 is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*a = b*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- 0_7 is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*b = b*a = (a*b)*a \\
a*(b*b) &= a*a = b*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*b = a*a = (b*b)*a \\
b*(b*b) &= b*a = a*b = (b*b)*b
\end{aligned}
$$

- 0_8 is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*b = b*a = (a*b)*a \\
a*(b*b) &= a*b = b*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*b = b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- 0_9 is not associative: $a*(a*b) = a*a = b \neq a = b*b = (a*a)*b$
- 0_{10} is associative.

$$
\begin{aligned}
a*(a*a) &= a*b = b*a = (a*a)*a \\
a*(a*b) &= a*a = b*b = (a*a)*b \\
a*(b*a) &= a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*b = a*a = (b*a)*a \\
b*(a*b) &= b*a = a*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- 0_{11} is not associative: $a*(a*a) = a*b = a \neq b = b*a = (a*a)*a$
- 0_{12} is not associative: $a*(b*a) = a*b = a \neq b = a*a = (a*b)*a$
- 0_{13} is not associative: $a*(a*a) = a*b = b \neq a = b*a = (a*a)*a$
- 0_{14} is not associative: $a*(b*a) = a*a = b \neq a = b*a = (a*b)*a$
- 0_{15} is not associative: $a*(a*a) = a*b = b \neq a = b*b = (a*a)*b$
- 0_{16} is associative:

  $\forall x, y \in A(x*y = b \to x*(y*z) = x*b = b = b*z = (x*y)*z)$

4 Identity

- A does not have an identity element with respect to 0_1.
- A has an identity element with respect to 0_2.

$$\begin{aligned} x * e &= x \\ a * b &= a \\ b * b &= b \\ e &= b \\ e * x &= x \\ b * a &= a \\ b * b &= b \\ e &= b \end{aligned}$$

- A does not have an identity element with respect to 0_3.
- A does not have an identity element with respect to 0_4.
- A does not have an identity element with respect to 0_5.
- A does not have an identity element with respect to 0_6.
- A does not have an identity element with respect to 0_7.
- A has an identity element with respect to 0_8.

$$\begin{aligned} x * e &= x \\ a * a &= a \\ b * a &= b \\ e &= a \\ e * x &= x \\ a * a &= a \\ a * b &= b \\ e &= a \end{aligned}$$

- A does not have an identity element with respect to 0_9.
- A has an identity element with respect to 0_{10}.

$$\begin{aligned} x * e &= x \\ a * b &= a \\ b * b &= b \\ e &= b \\ e * x &= x \\ b * a &= a \\ b * b &= b \\ e &= b \end{aligned}$$

- A does not have an identity element with respect to 0_{11}.
- A does not have an identity element with respect to 0_{12}.
- A does not have an identity element with respect to 0_{13}.
- A does not have an identity element with respect to 0_{14}.
- A does not have an identity element with respect to 0_{15}.
- A does not have an identity element with respect to 0_{16}.

5 Since A only has identity elements with respect to 0_2, 0_8, and 0_{10}, the rest cannot have inverses. As it turns out, with respect to those three operations, it is not the case that every $x \in A$ has an inverse.

D. Automata: The Algebra of Input/Output Sequences

Let A be an alphabet and A^* be the set of all sequences of symbols in the alphabet A. There is an operation on A^* called *concatenation*: If $\mathbf{a}$ and $\mathbf{b}$ are in A^*, say $\mathbf{a} = a_1a_2...a_n$ and $\mathbf{b} = b_1b_2...b_m$, then

$$\mathbf{ab} = a_1a_2...a_nb_1b_2...b_m$$

The symbol λ denotes the empty sequence.

1 Concatenation is associative.

$$a(bc) = a(b_1b_2...b_mc_1c_2...c_k) = a_1a_2...a_nb_1b_2...b_mc_1c_2...c_k$$
$$(ab)c = (a_1a_2...a_nb_1b_2...b_m)c = a_1a_2...a_nb_1b_2...b_mc_1c_2...c_k$$
$$a(bc) = (ab)c$$

2 Concatenation is not commutative.

$$ab = a_1a_2...a_nb_1b_2...b_m$$
$$ba = b_1b_2...b_ma_1a_2...a_n$$
$$ab \neq ba$$

3 λ is the identity element for concatenation: $x\lambda = \lambda x = x$

The Definition of Groups

[Pinter, 2016, chapter 3]

A. Examples of Abelian Groups

1 $\langle \mathbb{R}, x * y = x + y + k \rangle$

(i) $*$ is commutative: $x * y = x + y + k = y + x + k = y * x$

(ii) $*$ is associative.

$$\begin{aligned}
x(yz) &= x(y + z + k) = x + y + z + 2k \\
(xy)z &= (x + y + k)z = (xy)z \\
x(yz) &= (xy)z
\end{aligned}$$

(iii) $\mathbb{R}$ has an identity element with respect to $*$.

$$\begin{aligned}
xe &= x \\
x + e + k &= x \\
e &= -k \\
(-k)x &= x \\
-k + x + k &= x
\end{aligned}$$

(iv) $\forall x \in \mathbb{R}(\exists x' \in \mathbb{R}(x * x' = -k))$

$$\begin{aligned}
xx' &= -k \\
x + x' + k &= -k \\
x' &= -x - 2k \\
x'x &= xx' \qquad \text{due to commutativity}
\end{aligned}$$

2 $\langle \mathbb{R}^*, x * y = \frac{xy}{2} \rangle$

(i) $*$ is commutative: $x * y = \frac{xy}{2} = \frac{yx}{2} = y * x$

(ii) $*$ is associative.

$$\begin{aligned}
x * (y * z) = x * (\frac{yz}{2}) &= \frac{xyz}{4} \\
(x * y) * z = (\frac{xy}{2}) * z &= \frac{xyz}{4}
\end{aligned}$$

(iii) $\mathbb{R}^*$ has an identity element with respect to $*$.

$$\begin{aligned}
x * e &= \frac{xe}{2} = \frac{ex}{2} = e * x = x \\
e &= 2
\end{aligned}$$

(iv) $\forall x \in \mathbb{R}(\exists x' \in \mathbb{R}(x * x' = 2))$

$$\begin{aligned} x * x' &= \frac{xx'}{2} = \frac{x'x}{2} = x' * x = e = 2 \\ x' &= \frac{4}{x} \end{aligned}$$

3 $\langle \{\, x \in \mathbb{R} : x \neq -1 \,\}, x * y = x + y + xy \rangle$

(i) $*$ is commutative: $x * y = x + y + xy = y + x + yx = y * x$

(ii) $*$ is associative.

$$x * (y * z) = x * (y + z + yz) = x + (y + z + yz) + x(y + z + yz) = x + y + z + xy + xz + yz + xyz$$
$$(x * y) * z = (x + y + xy) * z = (x + y + xy) + z + (x + y + xy)z = x + y + z + xy + xz + yz + xyz$$

(iii) $\{\, x \in \mathbb{R} : x \neq -1 \,\}$ has an identity element with respect to $*$.

$$\begin{aligned} x * e &= x + e + xe = e + x + ex = e * x = x \\ e(x+1) &= 0 \\ e &= 0 \end{aligned}$$

(iv) Every element of $\{\, x \in \mathbb{R} : x \neq -1 \,\}$ has an inverse with respect to $*$.

$$\begin{aligned} x * x' &= x + x' + xx' = x' + x + x'x = e = 0 \\ x'(x+1) &= -x \\ x' &= -\frac{x}{x+1} \end{aligned}$$

Bibliography

Charles C. Pinter. *A Book of Abstract Algebra*. Dover, Mineola, NY, second edition, 2016.