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# ABSTRACT ALGEBRA

# *Contents*

# *Operations*

[Pinter, 2016, chapter 2]

## A. Examples of Operations

**1** $a * b = \sqrt{|ab|}$ is not an operation on $\mathbb{Q}$, because e.g. $2 * 1 = \sqrt{|2|} \notin \mathbb{Q}$.

**2** $a * b = a \ln b$ is not an operation on $\mathbb{R}_{>0}$, because e.g. $\forall a, b \in \mathbb{R}_{>0}(b \leq 1 \to a \ln b \notin \mathbb{R}_{>0})$

**3** If $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, $*$ is not an operation on $\mathbb{R}$, because $\forall a, b \in \mathbb{R}(a \neq 0 \wedge b \neq 0 \to x = \pm ab)$

**4** Subtraction is an operation on $\mathbb{Z}$, because $\forall a, b \in \mathbb{Z}(a - b \in \mathbb{Z})$.

**5** Subtraction is not an operation on $\mathbb{Z}_{\geq 0}$, because e.g. $0 - 1 \notin \mathbb{Z}_{\geq 0}$.

**6** $a * b = |a - b|$ is an operation on $\mathbb{Z}_{\geq 0}$, because $\forall a, b \in \mathbb{Z}_{\geq 0}(|a - b| \in \mathbb{Z}_{\geq 0})$.

## B. Properties of Operations

**1** $x * y = x + 2y + 4$

(i) $*$ is not commutative.

$$\begin{aligned}
x * y &= x + 2y + 4 \\
y * x &= y + 2x + 4 \\
x * y &\neq y * x
\end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned}
x * (y * z) &= x * (y + 2z + 4) \\
&= x + 2(y + 2z + 4) + 4 \\
&= x + 2y + 4z + 12 \\
(x * y) * z &= (x + 2y + 4) * z \\
&= x + 2y + 4 + 2z + 4 \\
&= x + 2y + 2z + 8 \\
x + 2y + 4z + 12 &\neq x + 2y + 2z + 8
\end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned}
x * e &= x \\
x + 2e + 4 &= x \\
2e + 4 &= 0 \\
e &= -2 \\
e * x &= x \\
e + 2x + 4 &= x \\
e &= -x - 4 \neq -2
\end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

**2** $x * y = x + 2y - xy$

(i) $*$ is not commutative.

$$\begin{aligned}
x * y &= x + 2y - xy \\
y * x &= y + 2x - yx \\
x * y &\neq y * x
\end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned}
x * (y * z) &= x * (y + 2z - yz) \\
&= x + 2(y + 2z - yz) - x(y + 2z - yz) \\
&= x + 2y + 4z - 2yz - xy - 2xz + xyz \\
(x * y) * z &= (x + 2y - xy) * z \\
&= (x + 2y - xy) + 2z - (x + 2y - xy)z \\
&= x + 2y + 2z - 2yz - xy - xz + xyz \\
x * (y * z) &\neq (x * y) * z
\end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned}
x * e &= x \\
x + 2e - xe &= x \\
2e - xe &= 0 \\
e(2 - x) &= 0 \\
e &= 0 \\
e * x &= x \\
e + 2x - ex &= x \\
e + x - ex &= 0 \\
e(1 - x) &= -x \\
e &= -x(1 - x) \neq 0
\end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

**3** $x * y = |x + y|$

(i) $*$ is commutative.

$$\begin{aligned} x * y &= |x + y| \\ y * x &= |y + x| = |x + y| \\ x * y &= y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x * (y * z) &= x * |y + z| = |x + |y + z|| \\ (x * y) * z &= |x + y| * z = ||x + y| + z| \\ x = 0, y < 0 \rightarrow x * (y * z) &= |y + z| \\ (x * y) * z &= ||y| + z| \\ y < 0 \rightarrow y \neq |y| \rightarrow |y + z| &\neq ||y| + z| \\ x * (y * z) &\neq (x * y) * z \end{aligned}$$

(iii) $\mathbb{R}$ has an identity element with respect to $*$.

$$\begin{aligned} x * e &= x \\ |x + e| &= x \\ e &= 0 \\ e * x &= x \\ |e + x| &= x \\ e &= 0 \end{aligned}$$

(iv) Every $x \in \mathbb{R}$ has an inverse with respect to $*$.

$$\begin{aligned} x * x' &= 0 \\ |x + x'| &= 0 \\ x' &= -x \\ x * (-x) &= |x - x| = 0 \\ (-x) * x &= |-x + x| = 0 \\ x * x' &= x' * x \end{aligned}$$

**4** $x * y = |x - y|$

(i) $*$ is commutative.

$$\begin{aligned} x * y &= |x - y| \\ y * x &= |y - x| \\ x = y \rightarrow x * y &= 0 \\ y * x &= 0 \end{aligned}$$

If $x < y$ then $x = y + k$, and:

$$\begin{aligned} x * y &= |(y + k) - y| = |k| \\ y * x &= |y - (y + k)| = |-k| = |k| \\ x * y &= y * x \end{aligned}$$

If $x = y$:

$$\begin{aligned} x * y &= |y - y| = 0 \\ y * x &= |y - y| = 0 \\ x * y &= y * x \end{aligned}$$

If $x > y$ then $y = x + k$, and:

$$\begin{aligned} x * y &= |x - (x + k)| = |-k| = |k| \\ y * x &= |(x + k) - x| = |k| \\ x * y &= y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x * (y * z) &= x * |y - z| \\ &= |x - |y - z|| \\ (x * y) * z &= |x - y| * z \\ &= ||x - y| - z| \end{aligned}$$

If $x = 0$ and $y < 0$:

$$\begin{aligned} x * (y * z) = |-|y - z|| = |y - z| &= \sqrt{(y - z)^2} \\ (x * y) * z = ||-y| - z| = ||y| - z| &= \sqrt{(|y| - z)^2} \\ |y| &\neq y \\ x * (y * z) &\neq (x * y) * z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x * e &= x \\ |x - e| &= x \\ e &= 2x \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

**5** $x * y = xy + 1$

(i) $*$ is commutative.

$$\begin{aligned} x * y &= xy + 1 \\ y * x &= yx + 1 = xy + 1 \\ x * y &= y * x \end{aligned}$$

(ii) $*$ is not associative.

$$\begin{aligned} x*(y*z) &= x*(yz+1) \\ &= x(yz+1)+1 = xyz+x+1 \\ (x*y)*z &= (xy+1)*z \\ &= (xy+1)z+1 = xyz+z+1 \\ x*(y*z) &\neq (x*y)*z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x*e &= x \\ xe+1 &= x \\ xe &= x-1 \\ x &= 1-\frac{1}{x} \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

**6** $x*y = \max\{\,x,y\,\}$ = the larger of the two numbers $x$ and $y$

(i) $*$ is commutative.

$$\begin{aligned} &x*y = \max\{\,x,y\,\} \\ &y*x = \max\{\,y,x\,\} = \max\{\,x,y\,\} \\ &x*y = y*x \end{aligned}$$

(ii) $*$ is associative.

$$\begin{aligned} x*(y*z) &= x*\max\{\,y,z\,\} \\ &= \max\{\,x,\max\{\,y,z\,\}\,\} = \max\{\,x,y,z\,\} \\ (x*y)*z &= (\max\{\,x,y\,\})*z \\ &= \max\{\,\max\{\,x,y\,\},z\,\} = \max\{\,x,y,z\,\} \\ x*(y*z) &= (x*y)*z \end{aligned}$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$\begin{aligned} x*e &= x \\ \max\{\,x,e\,\} &= x \\ e &= \{n\in\mathbb{R} : n\leq x\} \end{aligned}$$

(iv) Since there is no identity element, there can be no inverses.

**7** $x*y = \frac{xy}{x+y+1}$

(i) $*$ is commutative.

$$\begin{aligned} &x*y = \frac{xy}{x+y+1} \\ &y*x = \frac{yx}{y+x+1} = \frac{xy}{x+y+1} \\ &x*y = y*x \end{aligned}$$

(ii) $*$ is associative.

$$
\begin{aligned}
x*(y*z) &= x*(\frac{yz}{y+z+1}) \\
&= \frac{\frac{xyz}{y+z+1}}{x+\frac{yz}{y+z+1}+1} \\
&= \frac{xyz}{x(y+z+1)+yz+(y+z+1)} \\
&= \frac{xyz}{xy+xz+yz+x+y+z+1} \\
(x*y)*z &= \left(\frac{xy}{x+y+1}\right)*z \\
&= \frac{\frac{xyz}{x+y+1}}{\frac{xy}{x+y+1}+z+1} \\
&= \frac{xyz}{xy+z(x+y+1)+z+(x+y+1)} \\
&= \frac{xyz}{xy+xz+yz+x+y+z+1} \\
x*(y*z) &= (x*y)*z
\end{aligned}
$$

(iii) $\mathbb{R}$ does not have an identity element with respect to $*$.

$$
\begin{aligned}
x*e &= x \\
\frac{xe}{x+e+1} &= x \\
xe &= x(x+e+1) \\
e &= e+x+1
\end{aligned}
$$

(iv) Since there is no identity element, there can be no inverses.

**C. Operations no a Two-Element Set** Let $A$ be the two-element set $A = \{a, b\}$.

1

Table 1.1: $0_1$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $a$ |

Table 1.2: $0_2$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $b$ |

Table 1.3: $0_3$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $a$ |

Table 1.4: $0_4$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $b$ |

Table 1.5: $0_5$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $a$ |

Table 1.6: $0_6$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $b$ |

Table 1.7: $0_7$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $a$ |

Table 1.8: $0_8$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $a$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $b$ |

Table 1.9: $0_9$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $a$ |

Table 1.10: $0_{10}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $b$ |

Table 1.11: $0_{11}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $a$ |

Table 1.12: $0_{12}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $a$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $b$ |

Table 1.13: $0_{13}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $a$ |

Table 1.14: $0_{14}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $a$ |
| $(b,b)$ | $b$ |

Table 1.15: $0_{15}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $a$ |

Table 1.16: $0_{16}$

| $(x,y)$ | $x*y$ |
|---|---|
| $(a,a)$ | $b$ |
| $(a,b)$ | $b$ |
| $(b,a)$ | $b$ |
| $(b,b)$ | $b$ |

## 2 Commutativity

- $0_1$ is commutative: $a*b=a=b*a$
- $0_2$ is commutative: $a*b=a=b*a$
- $0_3$ is not commutative: $a*b=a\neq b=b*a$
- $0_4$ is not commutative: $a*b=a\neq b=b*a$
- $0_5$ is not commutative: $a*b=b\neq a=b*a$
- $0_6$ is not commutative: $a*b=b\neq a=b*a$
- $0_7$ is commutative: $a*b=b=b*a$
- $0_8$ is commutative: $a*b=b=b*a$
- $0_9$ is commutative: $a*b=a=b*a$
- $0_{10}$ is commutative: $a*b=a=b*a$
- $0_{11}$ is not commutative: $a*b=a\neq b=b*a$
- $0_{12}$ is not commutative: $a*b=a\neq b=b*a$
- $0_{13}$ is not commutative: $a*b=b\neq a=b*a$
- $0_{14}$ is not commutative: $a*b=b\neq a=b*a$
- $0_{15}$ is commutative: $a*b=b=b*a$
- $0_{16}$ is commutative: $a*b=b=b*a$

## 3 Associativity

- $0_1$ is associative:

$$\forall x,y\in A(x*y=a\to x*(y*z)=x*a=a=a*z=(x*y)*z)$$

- $0_2$ is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*a = a*b = (a*a)*b \\
a*(b*a) &= a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = a*a = (b*a)*a \\
b*(a*b) &= b*a = a*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- $0_3$ is not associative: $b*(a*b) = b*a = b \neq a = b*b = (b*a)*b$
- $0_4$ is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*a = a*b = (a*a)*b \\
a*(b*a) &= a*b = a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*a = b*b = (b*a)*b \\
b*(b*a) &= b*b = b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- $0_5$ is not associative: $b*(a*b) = b*b = a \neq b = a*b = (b*a)*b$
- $0_6$ is associative.

$$
\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*a = b*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}
$$

- $0_7$ is associative.

$$\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*b = b*a = (a*b)*a \\
a*(b*b) &= a*a = b*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*b = a*a = (b*b)*a \\
b*(b*b) &= b*a = a*b = (b*b)*b
\end{aligned}$$

- $0_8$ is associative.

$$\begin{aligned}
a*(a*a) &= a*a = (a*a)*a \\
a*(a*b) &= a*b = (a*a)*b \\
a*(b*a) &= a*b = b*a = (a*b)*a \\
a*(b*b) &= a*b = b*b = (a*b)*b \\
b*(a*a) &= b*a = (b*a)*a \\
b*(a*b) &= b*b = (b*a)*b \\
b*(b*a) &= b*b = b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}$$

- $0_9$ is not associative: $a*(a*b) = a*a = b \neq a = b*b = (a*a)*b$
- $0_{10}$ is associative.

$$\begin{aligned}
a*(a*a) &= a*b = b*a = (a*a)*a \\
a*(a*b) &= a*a = b*b = (a*a)*b \\
a*(b*a) &= a*a = (a*b)*a \\
a*(b*b) &= a*b = (a*b)*b \\
b*(a*a) &= b*b = a*a = (b*a)*a \\
b*(a*b) &= b*a = a*b = (b*a)*b \\
b*(b*a) &= b*a = (b*b)*a \\
b*(b*b) &= b*b = (b*b)*b
\end{aligned}$$

- $0_{11}$ is not associative: $a*(a*a) = a*b = a \neq b = b*a = (a*a)*a$
- $0_{12}$ is not associative: $a*(b*a) = a*b = a \neq b = a*a = (a*b)*a$
- $0_{13}$ is not associative: $a*(a*a) = a*b = b \neq a = b*a = (a*a)*a$
- $0_{14}$ is not associative: $a*(b*a) = a*a = b \neq a = b*a = (a*b)*a$
- $0_{15}$ is not associative: $a*(a*a) = a*b = b \neq a = b*b = (a*a)*b$
- $0_{16}$ is associative:

  $\forall x, y \in A(x*y = b \to x*(y*z) = x*b = b = b*z = (x*y)*z)$

**4** Identity

- $A$ does not have an identity element with respect to $0_1$.
- $A$ has an identity element with respect to $0_2$.

$$
\begin{aligned}
x * e &= x \\
a * b &= a \\
b * b &= b \\
e &= b \\
e * x &= x \\
b * a &= a \\
b * b &= b \\
e &= b
\end{aligned}
$$

- $A$ does not have an identity element with respect to $0_3$.
- $A$ does not have an identity element with respect to $0_4$.
- $A$ does not have an identity element with respect to $0_5$.
- $A$ does not have an identity element with respect to $0_6$.
- $A$ does not have an identity element with respect to $0_7$.
- $A$ has an identity element with respect to $0_8$.

$$
\begin{aligned}
x * e &= x \\
a * a &= a \\
b * a &= b \\
e &= a \\
e * x &= x \\
a * a &= a \\
a * b &= b \\
e &= a
\end{aligned}
$$

- $A$ does not have an identity element with respect to $0_9$.
- $A$ has an identity element with respect to $0_{10}$.

$$
\begin{aligned}
x * e &= x \\
a * b &= a \\
b * b &= b \\
e &= b \\
e * x &= x \\
b * a &= a \\
b * b &= b \\
e &= b
\end{aligned}
$$

- $A$ does not have an identity element with respect to $0_{11}$.
- $A$ does not have an identity element with respect to $0_{12}$.
- $A$ does not have an identity element with respect to $0_{13}$.
- $A$ does not have an identity element with respect to $0_{14}$.
- $A$ does not have an identity element with respect to $0_{15}$.
- $A$ does not have an identity element with respect to $0_{16}$.

**5** Since $A$ only has identity elements with respect to $0_2$, $0_8$, and $0_{10}$, the rest cannot have inverses. As it turns out, with respect to those three operations, it is not the case that every $x \in A$ has an inverse.

**D. Automata: The Algebra of Input/Output Sequences**

Let $A$ be an alphabet and $A^*$ be the set of all sequences of symbols in the alphabet $A$. There is an operation on $A^*$ called *concatenation*: If $\mathbf{a}$ and $\mathbf{b}$ are in $A^*$, say $\mathbf{a} = a_1a_2...a_n$ and $\mathbf{b} = b_1b_2...b_m$, then

$$\mathbf{ab} = a_1a_2...a_nb_1b_2...b_m$$

The symbol $\lambda$ denotes the empty sequence.

**1** Concatenation is associative.

$$a(bc) = a(b_1b_2...b_mc_1c_2...c_k) = a_1a_2...a_nb_1b_2...b_mc_1c_2...c_k$$
$$(ab)c = (a_1a_2...a_nb_1b_2...b_m)c = a_1a_2...a_nb_1b_2...b_mc_1c_2...c_k$$
$$a(bc) = (ab)c$$

**2** Concatenation is not commutative.

$$ab = a_1a_2...a_nb_1b_2...b_m$$
$$ba = b_1b_2...b_ma_1a_2...a_n$$
$$ab \neq ba$$

**3** $\lambda$ is the identity element for concatenation: $x\lambda = \lambda x = x$

# *Bibliography*

Charles C. Pinter. *A Book of Abstract Algebra*. Dover, Mineola, NY, second edition, 2016.