Abstract Binding Trees

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1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say s *sort* when $s \in S$. A valence $\{\vec{p}\}[\vec{q}].s$ specifies an expression of sort s which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s \; sort \quad p_i \; sort \; (i \leq m) \quad q_i \; sort \; (i \leq n)}{\{p_0, \dots, p_m\}[q_0, \dots, q_n].s \; valence}$$

An arity $(\vec{v})s$ specifies an operator of sort s with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s \; sort \quad v_i \; valence \; (i \leq n)}{(v_0, \dots, v_n)s \; arity}$$

Let $\mathbb{I}$ be an infinite set of symbols. Let $\mathbb{F}$ be the category of finite subsets of $\mathbb{I}$ and their injective maps; then the comma construction $\mathbf{SCtx} \triangleq \mathbb{F} \downarrow \mathscr{S}_{\equiv}$, with $\mathscr{S}_{\equiv}$ the discrete category on the set $\mathscr{S}$, is the category of contexts of symbols, whose objects are finite sets of symbols U and sort-assignments $\mathfrak{s} : U \to \mathscr{S}$, and whose morphisms are sort-preserving renamings; we will write Υ for a symbol context $(U, \mathfrak{s})$.

Then, fix a covariant presheaf (copresheaf) of operators $\mathscr{O} : \mathbf{SCtx} \times \mathscr{A}_{\equiv} \to \mathbf{Set}$ such that the arrows in $\mathbf{SCtx}$ lift to renamings of operators' parameters. Via the Grothendieck construction[1] $\oint(-) : \mathbf{Set}^C \to \mathbf{Cat}$ on operators we have a category of objects $\langle\langle \Upsilon, a \rangle, \vartheta \rangle \in \oint \mathscr{O}$ for $\vartheta \in \mathscr{O}\langle \Upsilon, a \rangle$ and morphisms $\oint \mathscr{O}[\langle\langle \Upsilon, a \rangle, \vartheta \rangle, \langle\langle \Upsilon', a' \rangle, \vartheta' \rangle]$ for $f : \langle \Upsilon, a \rangle \to \langle \Upsilon', a' \rangle$ such that $f^* \vartheta = \vartheta' \in \mathscr{O}\langle \Upsilon', a' \rangle$. Equivalently, $\oint \mathscr{O}$ is the pullback of $\mathscr{O}$ along the universal $\mathbf{Set}$-bundle where $\pi_{\mathscr{O}}$ is a discrete Grothendieck opfibration and $|-|_{\bullet}$ is the forgetful functor from pointed sets:

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[1]In this case, $C \oint \Psi$ represents the category of elements of a copresheaf $\Psi : C \to \mathbf{Set}$ but we keep the C implicit and simply refer to it as the Grothendieck construction. Alternatively, this construction can be understood as a coend $C \oint \Psi \cong \int^{c \in C} c/C \otimes \Psi(c)_{\equiv}$.

$$\oint \mathscr{O} \xrightarrow{\ \pi_\in\ } \bullet/\mathbf{Set}$$

$$\pi_\mathscr{O} \downarrow \qquad\qquad \downarrow |-|_\bullet$$

$$\mathbf{SCtx} \times \mathscr{A}_\equiv \xrightarrow[\ \mathscr{O}\ ]{} \mathbf{Set}$$

$$\frac{\vartheta \in \mathscr{O}\langle \Upsilon, a\rangle}{\Upsilon \Vdash \vartheta : a}$$

The judgment $\Upsilon \Vdash \vartheta : a$ enjoys the structural properties of weakening and exchange via the functoriality of $\mathscr{O}$.

Examples Operators are defined by specifying the fibers of $\pi_\mathscr{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, exp; we give its signature by asserting the following about its operators:

$$\Upsilon \Vdash \lambda : (\{\cdot\}\,[\mathsf{exp}]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon \Vdash \mathsf{ap} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\Upsilon \Vdash \mathsf{decl} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\mathsf{exp}\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{get}[u] : (\cdot)\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{set}[u] : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Note that the functoriality of $\mathscr{O}$ guarantees for any renaming $\varrho : \Upsilon \to \Upsilon'$ a restriction map ϱ^* such that $\Upsilon' \Vdash \varrho^*(\vartheta) : a$ when $\Upsilon \Vdash \vartheta : a$. In particular, the renaming $\Upsilon, u \mapsto \Upsilon, v$ shall take $\mathsf{get}[u]$ to $\mathsf{get}[v]$.

2   Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot\ mctx} \qquad \frac{\Omega\ mctx \quad v\ valence \quad \text{M} \notin |\Omega|}{\Omega, \text{M} : v\ mctx}$$

$$\frac{}{\cdot\ vctx} \qquad \frac{\Gamma\ vctx \quad s\ sort \quad x \notin |\Gamma|}{\Gamma, x : s\ vctx}$$

$$\frac{}{\cdot\ sctx} \qquad \frac{\Upsilon\ vctx \quad s\ sort \quad u \notin |\Upsilon|}{\Upsilon, u : s\ sctx}$$

3 Abstract Binding Trees

Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega\ mctx$, $\Upsilon\ sctx$, $\Gamma\ vctx$ and $s\ sort$, meaning that M is an abstract binding tree of sort s, with metavariables in Ω, parameters in Υ, and variables in Γ. Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash E : v$ presuppose $v\ valence$. Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\frac{\Gamma \ni x : s}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash x : s}\ var$$

$$\frac{\Omega \ni \text{M} : \{p_0, \dots, p_m\}\big[q_0, \dots, q_n\big].s \qquad \Upsilon \ni u_i : p_i \ \ (i \leq m) \qquad \Omega \triangleright \Upsilon \parallel \Gamma \vdash M_i : q_i \ \ (i \leq n)}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \text{M}\{u_0, \dots, u_m\}(M_0, \dots, M_n) : s}\ mvar$$

$$\frac{\Upsilon \Vdash \vartheta : v_1, \dots, v_n \qquad \Omega \triangleright \Upsilon \parallel \Gamma \vdash E_i : q_i \ \ (i \leq n)}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \vartheta(E_0, \dots, E_n) : s}\ app$$

$$\frac{\Omega \triangleright \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash M : s}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \big(\{\vec{u}\}\big[\vec{x}\big].M\big) : \big(\{\vec{p}\}\big[\vec{q}\big].s\big)}\ abs$$

Abstract binding trees are identified up to α-equivalence.

3.1 Substitution of Variables

Variable substitution in abstract binding trees is defined inductively by a pair of judgments, $[N/x]M \rightsquigarrow M'$ and $[N/x]E \rightsquigarrow E'$:

$$\frac{x = y}{[N/x]\,y \rightsquigarrow N} \qquad \frac{x \mathbin{\#} y}{[N/x]\,y \rightsquigarrow y}$$

$$\frac{[N/x]\,M_i \rightsquigarrow M_i' \quad (i \le n)}{[N/x]\,\mathrm{M}\,\{\vec{u}\}\,(M_0, \dots, M_n) \rightsquigarrow \mathrm{M}\,\{\vec{u}\}\,(M_0', \dots, M_n')}$$

$$\frac{[N/x]\,E_i \rightsquigarrow E_i' \quad (i \le n)}{[N/x]\,\vartheta\,(E_0, \dots, E_n) \rightsquigarrow \vartheta\,(E_0', \dots, E_n')}$$

$$\frac{x \notin \vec{y} \quad \vec{y} \mathbin{\#} \mathbf{FV}(N) \quad [N/x]\,M \rightsquigarrow M'}{[N/x]\,\{\vec{u}\}\,[\vec{y}].M \rightsquigarrow \{\vec{u}\}\,[\vec{y}].M'} \qquad \frac{x \in \vec{y} \quad \vec{y} \mathbin{\#} \mathbf{FV}(N)}{[N/x]\,\{\vec{u}\}\,[\vec{y}].M \rightsquigarrow \{\vec{u}\}\,[\vec{y}].M}$$

Because terms are identified up to α-equivalence, the variable substitution judgment is functional in its inputs, and so we are justified in writing $[N/x]\,M$ for M' when $[N/x]\,M \rightsquigarrow M'$. We write $\left[\vec{N}/\vec{x}\right]M$ for the simultaneous substitution of $\vec{N}$ for $\vec{x}$ in M.

3.2 Substitution of Metavariables

Metavariable substitution is defined inductively by the judgment $[E/\mathrm{M}]\,M \rightsquigarrow M'$:

$$\frac{}{[E/\mathrm{M}]\,x \rightsquigarrow x}$$

$$\frac{\mathrm{M} \mathbin{\#} \mathrm{N} \quad [E/\mathrm{N}]\,M_i \rightsquigarrow M_i' \quad (i \le n)}{[E/\mathrm{M}]\,\mathrm{N}\,\{\vec{u}\}\,(M_0, \dots, M_n) \rightsquigarrow \mathrm{N}\,\{\vec{u}\}\,(M_0', \dots, M_n')}$$

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