Abstract Binding Trees

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1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say s *sort* when $s \in S$. A valence $\vec{p} \parallel \vec{q}.s$ specifies an expression of sort s which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s \ sort \quad p_i \ sort \ (i \leq m) \quad q_i \ sort \ (i \leq n)}{p_0, \dots, p_m \parallel q_0, \dots, q_n.s \ valence}$$

An arity $(\vec{v})s$ specifies an operator of sort s with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s \ sort \quad v_i \ valence \ (i \leq n)}{(v_0, \dots, v_n)s \ arity}$$

Let $\mathbb{I}$ be an infinite set of symbols; let $\mathbb{F}$ be the free cocartesian category over $\mathbb{I}$. Then, fix a covariant presheaf of operators $\mathscr{O} : \mathbb{F} \times \mathscr{A} \to \mathbf{Set}$ such that the arrows in $\mathbb{F}$ lift to renamings of operators' parameters; via the Grothendieck construction, we can also consider the set $\int \mathscr{O}$ of operators (U, ϱ, ϑ) for $\vartheta \in \mathscr{O}(U, \varrho)$.

$$\frac{\vartheta \in \mathscr{O}(U, \varrho)}{U \Vdash \vartheta : \varrho}$$

The judgment $U \Vdash \vartheta : \varrho$ enjoys the structural properties of weakening and exchange via the functoriality of $\mathscr{O}$.

Examples   Operators are defined by specifying the fibres of $\int \mathscr{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, exp; we give its signature by asserting the following about its operators:

$$U \Vdash \lambda : (\cdot \parallel \mathsf{exp}.\mathsf{exp})\mathsf{exp})$$
$$U \Vdash \mathsf{ap} : (\cdot \parallel \cdot.\mathsf{exp}, \cdot \parallel \cdot.\mathsf{exp})\mathsf{exp}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$U \Vdash \mathsf{decl} : (\cdot \parallel \cdot . \, \mathsf{exp}, \mathsf{exp} \parallel \cdot . \, \mathsf{exp}) \, \mathsf{exp}$$
$$U, u \Vdash \mathsf{get}\{u\} : (\cdot) \, \mathsf{exp}$$
$$U, u \Vdash \mathsf{set}\{u\} : (\cdot \parallel \cdot . \, \mathsf{exp}) \, \mathsf{exp}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a "degeneracy map" on operators, whereas a renaming $u \mapsto v$ will take $\mathsf{get}\{u\}$ to $\mathsf{get}\{v\}$.

2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot \; mctx} \qquad \frac{\Omega \; mctx \quad v \; valence \quad \textsc{m} \notin |\Omega|}{\Omega, \textsc{m} : v \; mctx}$$

$$\frac{}{\cdot \; vctx} \qquad \frac{\Gamma \; vctx \quad s \; sort \quad x \notin |\Gamma|}{\Gamma, x : s \; vctx}$$

$$\frac{}{\cdot \; sctx} \qquad \frac{\Upsilon \; vctx \quad s \; sort \quad u \notin |\Upsilon|}{\Upsilon, u : s \; sctx}$$

3 Abstract Binding Trees

Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega \; mctx$, $\Upsilon \; sctx$, $\Gamma \; vctx$ and $s \; sort$, meaning that M is an abstract binding tree of sort s, with metavariables in Ω, parameters in Υ, and variables in Γ. Then, the syntax of abstract binding trees is inductively defined in three rules:

$$\frac{\Gamma \ni x : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash x : s} \;\; var$$

$$\frac{\begin{array}{c} \Omega \ni \textsc{m} : p_0, \ldots, p_m \parallel q_0, \ldots, q_n . s \\ \Upsilon \ni u_i : p_i \;\; (i \le m) \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \;\; (i \le n) \end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \textsc{m} \{u_0, \ldots, u_m\} (M_0, \ldots, M_n) : s} \;\; mvar$$

$$\frac{\begin{array}{c} |\Upsilon| \Vdash \vartheta : (v_0, \ldots, v_n) s \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \;\; (i \le n) \end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \vartheta (M_0, \ldots, M_n) : s} \;\; app$$