# Abstract Binding Trees

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## 1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say $s$ *sort* when $s \in S$. A valence $\left\{\vec{p}\right\}\left[\vec{q}\right].s$ specifies an expression of sort $s$ which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s\ \mathit{sort} \quad p_i\ \mathit{sort}\ \ (i \leq m) \quad q_i\ \mathit{sort}\ \ (i \leq n)}{\left\{p_0, \dots, p_m\right\}\left[q_0, \dots, q_n\right].s\ \mathit{valence}}$$

An arity $\left(\vec{v}\right)s$ specifies an operator of sort $s$ with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s\ \mathit{sort} \quad v_i\ \mathit{valence}\ \ (i \leq n)}{(v_0, \dots, v_n)\, s\ \mathit{arity}}$$

Let $\mathbb{I}$ be an infinite set of symbols; let $\mathbb{F}$ be the free cocartesian category over $\mathbb{I} \times \mathscr{S}$. Then, fix a covariant presheaf of operators $\mathscr{O} : \mathbb{F} \times \mathscr{A} \to \mathbf{Set}$ such that the arrows in $\mathbb{F}$ lift to renamings of operators' parameters; via the Grothendieck construction, we can also consider the set $\int \mathscr{O}$ of operators $(\Upsilon, \varrho, \vartheta)$ for $\vartheta \in \mathscr{O}(\Upsilon, \varrho)$.

$$\frac{\vartheta \in \mathscr{O}(\Upsilon, \varrho)}{\Upsilon \Vdash \vartheta : \varrho}$$

The judgment $\Upsilon \Vdash \vartheta : \varrho$ enjoys the structural properties of weakening and exchange via the functoriality of $\mathscr{O}$.

**Examples** Operators are defined by specifying the fibres of $\int \mathscr{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, $\mathsf{exp}$; we give its signature by asserting the following about its operators:

$$\Upsilon \Vdash \lambda : (\{\cdot\}\,[\mathsf{exp}]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon \Vdash \mathsf{ap} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\Upsilon \Vdash \mathsf{decl} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\mathsf{exp}\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{get}[u] : (\cdot)\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{set}[u] : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a "degeneracy map" on operators, whereas a renaming $u \mapsto v$ will take $\mathsf{get}[u]$ to $\mathsf{get}[v]$.

## 2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context $\Omega$ consists of bindings of valences to metavariables; a variable context $\Gamma$ is a collection of bindings of sorts to variables, and a parameter context $\Upsilon$ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot\ \mathit{mctx}} \qquad \frac{\Omega\ \mathit{mctx} \quad v\ \mathit{valence} \quad \textsc{m} \notin |\Omega|}{\Omega, \textsc{m} : v\ \mathit{mctx}}$$

$$\frac{}{\cdot\ \mathit{vctx}} \qquad \frac{\Gamma\ \mathit{vctx} \quad s\ \mathit{sort} \quad x \notin |\Gamma|}{\Gamma, x : s\ \mathit{vctx}}$$

$$\frac{}{\cdot\ \mathit{sctx}} \qquad \frac{\Upsilon\ \mathit{vctx} \quad s\ \mathit{sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s\ \mathit{sctx}}$$

## 3 Abstract Binding Trees

Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega$ *mctx*, $\Upsilon$ *sctx*, $\Gamma$ *vctx* and $s$ *sort*, meaning that $M$ is an abstract binding tree of sort $s$, with metavariables in $\Omega$, parameters in $\Upsilon$, and variables in $\Gamma$. Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash E : v$ presuppose $v$ *valence*. Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\frac{\Gamma \ni x : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash x : s}\ var$$

$$\frac{\begin{array}{l}\Omega \ni \mathrm{M} : \left\{p_0, \dots, p_m\right\}\left[q_0, \dots, q_n\right].s \\ \Upsilon \ni u_i : p_i \quad (i \leq m) \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \leq n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \mathrm{M}\{u_0, \dots, u_m\}(M_0, \dots, M_n) : s}\ mvar$$

$$\frac{\begin{array}{l}\Upsilon \Vdash \vartheta : v_1, \dots, v_n \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash E_i : q_i \quad (i \leq n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \vartheta(E_0, \dots, E_n) : s}\ app$$

$$\frac{\Omega \rhd \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash E : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash (\left\{\vec{u}\right\}\left[\vec{x}\right].E) : (\left\{\vec{p}\right\}\left[\vec{q}\right].s)}\ abs$$