

Abstract Binding Trees

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1 Preliminaries

Fix a set \mathcal{S} of *sorts*. We will say s *sort* when $s \in \mathcal{S}$. A valence $\vec{p} \parallel \vec{q}.s$ specifies an expression of sort s which binds symbols in \vec{p} and variables in \vec{q} .

$$\frac{s \text{ sort} \quad p_i \text{ sort } (i \leq m) \quad q_i \text{ sort } (i \leq n)}{p_0, \dots, p_m \parallel q_0, \dots, q_n.s \text{ valence}}$$

An arity $(\vec{v})_s$ specifies an operator of sort s with arguments of valences \vec{v} . We will call the set of valences \mathcal{V} , and the set of arities \mathcal{A} .

$$\frac{s \text{ sort} \quad v_i \text{ valence } (i \leq n)}{(v_0, \dots, v_n)_s \text{ arity}}$$

Let \mathbb{I} be an infinite set of symbols; let \mathbb{F} be the free cocartesian category over \mathbb{I} . Then, fix a covariant presheaf of operators $\mathcal{O} : \mathbb{F} \times \mathcal{A} \rightarrow \mathbf{Set}$ such that the arrows in \mathbb{F} lift to renamings of operators' parameters; via the Grothendieck construction, we can also consider the set $\int \mathcal{O}$ of operators (U, ϱ, ϑ) for $\vartheta \in \mathcal{O}(U, \varrho)$.

$$\frac{\vartheta \in \mathcal{O}(U, \varrho)}{U \Vdash \vartheta : \varrho}$$

The judgment $U \Vdash \vartheta : \varrho$ enjoys the structural properties of weakening and exchange via the functoriality of \mathcal{O} .

Examples Operators are defined by specifying the fibres of $\int \mathcal{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, **exp**; we give its signature by asserting the following about its operators:

$$\begin{aligned} U \Vdash \lambda : (\cdot \parallel \text{exp}.\text{exp})\text{exp} \\ U \Vdash \text{ap} : (\cdot \parallel \cdot.\text{exp}, \cdot \parallel \cdot.\text{exp})\text{exp} \end{aligned}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\begin{aligned}
U \Vdash \text{decl} : (\cdot \parallel \cdot \text{exp}, \text{exp} \parallel \cdot \text{exp}) \text{exp} \\
U, u \Vdash \text{get}\{u\} : (\cdot) \text{exp} \\
U, u \Vdash \text{set}\{u\} : (\cdot \parallel \cdot \text{exp}) \text{exp}
\end{aligned}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a “degeneracy map” on operators, whereas a renaming $u \mapsto v$ will take $\text{get}\{u\}$ to $\text{get}\{v\}$.

2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\begin{aligned}
& \frac{}{\cdot \text{mctx}} \quad \frac{\Omega \text{ mctx} \quad v \text{ valence} \quad M \notin |\Omega|}{\Omega, M : v \text{ mctx}} \\
& \frac{}{\cdot \text{vctx}} \quad \frac{\Gamma \text{ vctx} \quad s \text{ sort} \quad x \notin |\Gamma|}{\Gamma, x : s \text{ vctx}} \\
& \frac{}{\cdot \text{sctx}} \quad \frac{\Upsilon \text{ vctx} \quad s \text{ sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s \text{ sctx}}
\end{aligned}$$

3 Abstract Binding Trees

Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega \text{ mctx}$, $\Upsilon \text{ sctx}$, $\Gamma \text{ vctx}$ and $s \text{ sort}$, meaning that M is an abstract binding tree of sort s , with metavariables in Ω , parameters in Υ , and variables in Γ . Then, the syntax of abstract binding trees is inductively defined in three rules:

$$\begin{array}{c}
\frac{\Gamma \ni x : s}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash x : s} \textit{var} \\[10pt]
\frac{\begin{array}{l} \Omega \ni M : p_0, \dots, p_m \parallel q_0, \dots, q_n . s \\ \Upsilon \ni u_i : p_i \quad (i \leq m) \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash_M \{u_0, \dots, u_m\} (u_0, \dots, u_m; M_0, \dots, M_n) : s} \textit{mvar} \\[10pt]
\frac{\begin{array}{l} |\Upsilon| \Vdash \vartheta : p_0, \dots, p_m \parallel q_0, \dots, q_n . s \\ \Upsilon \ni u_i : p_i \quad (i \leq m) \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash_{\vartheta} \{u_0, \dots, u_m\} (u_0, \dots, u_m; M_0, \dots, M_n) : s} \textit{app}
\end{array}$$