Abstract Binding Trees

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1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say s *sort* when $s \in S$. A valence $\{\vec{p}\}[\vec{q}].s$ specifies an expression of sort s which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s \ sort \quad p_i \ sort \ (i \le m) \quad q_i \ sort \ (i \le n)}{\{p_0, \dots, p_m\}[q_0, \dots, q_n].s \ valence}$$

An arity $(\vec{v})s$ specifies an operator of sort s with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s \ sort \quad v_i \ valence \ (i \le n)}{(v_0, \dots, v_n)s \ arity}$$

Let $\mathbb{I}$ be an infinite set of symbols. Let $\mathbb{F}$ be the category of finite subsets of $\mathbb{I}$ and their injective maps; then the comma construction $\mathbf{SCtx} \triangleq \mathbb{F} \downarrow \mathscr{S}_\equiv$, with $\mathscr{S}_\equiv$ the discrete category on the set $\mathscr{S}$, is the category of contexts of symbols, whose objects are finite sets of symbols U and sort-assignments $\mathfrak{s} : U \to \mathscr{S}$, and whose morphisms are sort-preserving renamings; we will write Υ for a symbol context $(U, \mathfrak{s})$.

Then, fix a covariant presheaf (copresheaf) of operators $\mathscr{O} : \mathbf{SCtx} \times \mathscr{A}_\equiv \to \mathbf{Set}$ such that the arrows in $\mathbf{SCtx}$ lift to renamings of operators' parameters. Via the Grothendieck construction[1] $\oint(-) : \mathbf{Set}^C \to \mathbf{Cat}$ on operators we have a category of objects $\langle\langle \Upsilon, a \rangle, \vartheta\rangle \in \oint \mathscr{O}$ for $\vartheta \in \mathscr{O}\langle \Upsilon, a \rangle$ and morphisms $\oint \mathscr{O}[\langle\langle \Upsilon, a \rangle, \vartheta\rangle, \langle\langle \Upsilon', a' \rangle, \vartheta'\rangle]$ for $f : \langle \Upsilon, a \rangle \to \langle \Upsilon', a' \rangle$ such that $f^*\vartheta = \vartheta' \in \mathscr{O}\langle \Upsilon', a' \rangle$. Equivalently, $\oint \mathscr{O}$ is the pullback of $\mathscr{O}$ along the universal $\mathbf{Set}$-bundle where $\pi_\mathscr{O}$ is a discrete Grothendieck opfibration and $|-|_\bullet$ is the forgetful functor from pointed sets:

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[1] In this case, $C \oint \Psi$ represents the category of elements of a copresheaf $\Psi : C \to \mathbf{Set}$ but we keep the C implicit and simply refer to it as the Grothendieck construction. Alternatively, this construction can be understood as a coend $C \oint \Psi \cong \int^{c \in C} c/C \otimes \Psi(c)_\equiv$.

$$
\begin{array}{ccc}
\oint \mathscr{O} & \xrightarrow{\ \pi_\in\ } & \bullet/\mathbf{Set} \\
{\scriptstyle \pi_{\mathscr{O}}}\big\downarrow & & \big\downarrow{\scriptstyle |-|_\bullet} \\
\mathbf{SCtx} \times \mathscr{A}_\equiv & \xrightarrow[\ \mathscr{O}\ ]{} & \mathbf{Set}
\end{array}
$$

$$
\frac{\vartheta \in \mathscr{O}\langle \Upsilon, a\rangle}{\Upsilon \Vdash \vartheta : a}
$$

The judgment $\Upsilon \Vdash \vartheta : a$ enjoys the structural properties of weakening and exchange via the functoriality of $\mathscr{O}$.

Examples Operators are defined by specifying the fibers of $\pi_{\mathscr{O}}$ in which they reside. For instance, consider the lambda calculus with a single sort, exp; we give its signature by asserting the following about its operators:

$$\Upsilon \Vdash \lambda : (\{\cdot\}\,[\mathsf{exp}]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon \Vdash \mathsf{ap} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\Upsilon \Vdash \mathsf{decl} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\mathsf{exp}\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{get}[u] : (\cdot)\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{set}[u] : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Note that the functoriality of $\mathscr{O}$ guarantees for any renaming $\varrho : \Upsilon \to \Upsilon'$ and $f : a \to a'$, a restriction map $\langle \varrho, f\rangle^*$ such that $\Upsilon' \Vdash \langle \varrho, f\rangle^*(\vartheta) : a$ when $\Upsilon \Vdash \vartheta : a'$. Because the category $\mathscr{A}_\equiv$ is discrete and has only identity arrows, we will often write ϱ^* to mean $\langle \varrho, \mathbf{id}\rangle^*$ through an abuse of notation. In particular, the renaming $\Upsilon, u \mapsto \Upsilon, v$ shall take $\mathsf{get}[u]$ to $\mathsf{get}[v]$.

2   Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences

to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot \; mctx} \qquad \frac{\Omega \; mctx \quad v \; valence \quad \text{M} \notin |\Omega|}{\Omega, \text{M} : v \; mctx}$$

$$\frac{}{\cdot \; vctx} \qquad \frac{\Gamma \; vctx \quad s \; sort \quad x \notin |\Gamma|}{\Gamma, x : s \; vctx}$$

$$\frac{}{\cdot \; sctx} \qquad \frac{\Upsilon \; vctx \quad s \; sort \quad u \notin |\Upsilon|}{\Upsilon, u : s \; sctx}$$

3   Abstract Binding Trees

Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega \; mctx$, $\Upsilon \; sctx$, $\Gamma \; vctx$ and $s \; sort$, meaning that M is an abstract binding tree of sort s, with metavariables in Ω, parameters in Υ, and variables in Γ. Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash E : v$ presuppose $v \; valence$. Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\frac{\Gamma \ni x : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash x : s} \; var$$

$$\frac{\Omega \ni \text{M} : \{p_0, \dots, p_m\}\left[q_0, \dots, q_n\right].s \quad \Upsilon \ni u_i : p_i \;\; (i \le m) \quad \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \;\; (i \le n)}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \text{M} \{u_0, \dots, u_m\}(M_0, \dots, M_n) : s} \; mvar$$

$$\frac{\Upsilon \Vdash \vartheta : v_1, \dots, v_n \quad \Omega \rhd \Upsilon \parallel \Gamma \vdash E_i : q_i \;\; (i \le n)}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \vartheta (E_0, \dots, E_n) : s} \; app$$

$$\frac{\Omega \rhd \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash M : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \left(\{\vec{u}\}\left[\vec{x}\right].M\right) : \left(\{\vec{p}\}\left[\vec{q}\right].s\right)} \; abs$$

Abstract binding trees are identified up to α-equivalence.

3.1 Renaming of Symbols

The only place that symbols appear in our calculus is as parameters to operators (unlike variables, symbols are not terms). Therefore, the functorial action of the operator presheaf can be lifted into terms by recursion on their structure, via a pair of judgments $M \upharpoonright \varrho \rightsquigarrow N$ and $E \upharpoonright \varrho \rightsquigarrow F$, presupposing $\varrho : \Upsilon \to \Upsilon'$, and $\Omega \rhd \Upsilon \parallel \Gamma \vdash M : s$ and $\Omega \rhd \Upsilon \parallel \Gamma \vdash E : v$ respectively:

$$\overline{x \upharpoonright \varrho \rightsquigarrow x}$$

$$\frac{\varrho(\vec{u}) \equiv \vec{v} \quad M_i \upharpoonright \varrho \rightsquigarrow N_i \quad (i \leq n)}{\mathrm{M}\left\{\vec{u}\right\}(M_0, \dots, M_n) \upharpoonright \varrho \rightsquigarrow \mathrm{M}\left\{\vec{v}\right\}(N_0, \dots, N_n)}$$

$$\frac{\varrho^*(\vartheta) \equiv \vartheta' \quad E_i \upharpoonright \varrho \rightsquigarrow F_i \quad (i \leq n)}{\vartheta(E_0, \dots, E_n) \upharpoonright \varrho \rightsquigarrow \vartheta'(F_0, \dots, F_n)}$$

$$\frac{M \upharpoonright \varrho \setminus \vec{u} \rightsquigarrow N}{\left(\left\{\vec{u}\right\}\left[\vec{x}\right].M\right) \upharpoonright \varrho \rightsquigarrow \left(\left\{\vec{u}\right\}\left[\vec{x}\right].N\right)}$$

Above, the notation $\varrho \setminus \vec{u}$ means the omission of the variables $\vec{u}$ from the renaming ϱ. Because terms are identified up to α-equivalence, the renaming judgment is functional in its input, and so we are justified in writing $M \upharpoonright \varrho$ for N when $M \upharpoonright \varrho \rightsquigarrow N$.

3.2 Substitution of Variables

Variable substitution in abstract binding trees is defined inductively by a pair of judgments, $[N/x]M \rightsquigarrow M'$ and $[N/x]E \rightsquigarrow E'$:

$$\frac{x = y}{[N/x]y \rightsquigarrow N} \qquad \frac{x \mathbin{\#} y}{[N/x]y \rightsquigarrow y}$$

$$\frac{[N/x]M_i \rightsquigarrow M_i' \quad (i \leq n)}{[N/x]\,\mathrm{M}\left\{\vec{u}\right\}(M_0, \dots, M_n) \rightsquigarrow \mathrm{M}\left\{\vec{u}\right\}\left(M_0', \dots, M_n'\right)}$$

$$\frac{[N/x]E_i \rightsquigarrow E_i' \quad (i \leq n)}{[N/x]\,\vartheta(E_0, \dots, E_n) \rightsquigarrow \vartheta\left(E_0', \dots, E_n'\right)}$$

$$\frac{x \notin \vec{y} \quad \vec{y} \mathbin{\#} \mathbf{FV}(N) \quad [N/x]M \rightsquigarrow M'}{[N/x]\left\{\vec{u}\right\}\left[\vec{y}\right].M \rightsquigarrow \left\{\vec{u}\right\}\left[\vec{y}\right].M'} \qquad \frac{x \in \vec{y} \quad \vec{y} \mathbin{\#} \mathbf{FV}(N)}{[N/x]\left\{\vec{u}\right\}\left[\vec{y}\right].M \rightsquigarrow \left\{\vec{u}\right\}\left[\vec{y}\right].M}$$

Going forward, we will write $[N/x]M$ for M' when $[N/x]M \leadsto M'$, and $[\vec{N}/\vec{x}]M$ for the simultaneous substitution of $\vec{N}$ for $\vec{x}$ in M.

3.3 Substitution of Metavariables

Metavariable substitution is defined inductively by the judgment $[E/\text{M}]M \leadsto M'$:

$$\frac{}{[E/\text{M}]x \leadsto x}$$

$$\frac{\text{M} \mathbin{\#} \text{N} \quad [E/\text{N}]M_i \leadsto M_i' \quad (i \le n)}{[E/\text{M}]\,\text{N}\,\{\vec{u}\}\,(M_0, \dots, M_n) \leadsto \text{N}\,\{\vec{u}\}\,(M_0', \dots, M_n')}$$