Abstract Binding Trees

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1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say $s\ sort$ when $s \in S$. A valence $\{\vec{p}\}[\vec{q}].s$ specifies an expression of sort s which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s\ sort \quad p_i\ sort\ \ (i \le m) \quad q_i\ sort\ \ (i \le n)}{\{p_0, \dots, p_m\}[q_0, \dots, q_n].s\ valence}$$

An arity $(\vec{v})s$ specifies an operator of sort s with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s\ sort \quad v_i\ valence\ \ (i \le n)}{(v_0, \dots, v_n)s\ arity}$$

Let $\mathbb{I}$ be an infinite set of symbols. Let $\mathbb{F}$ be the category of finite subsets of $\mathbb{I}$ and their injective maps; then the comma construction $\mathbf{SCtx} \triangleq \mathbb{F} \downarrow \mathscr{S}_{\equiv}$, with $\mathscr{S}_{\equiv}$ the discrete category on the set $\mathscr{S}$, is the category of contexts of symbols, whose objects are finite sets of symbols U and sort-assignments $\mathfrak{s} : U \to \mathscr{S}$, and whose morphisms are sort-preserving renamings; we will write Υ for a symbol context $(U, \mathfrak{s})$.

Then, fix a covariant presheaf (copresheaf) of operators $\mathscr{O} : \mathbf{SCtx} \times \mathscr{A} \to \mathbf{Set}$ such that the arrows in $\mathbf{SCtx}$ lift to renamings of operators' parameters. Via the Grothendieck construction[1] $\oint(-) : \mathbf{Set}^C \to \mathbf{Cat}$ on operators we have a category of objects $\langle\langle\Upsilon, \varrho\rangle, \vartheta\rangle \in \oint \mathscr{O}$ for $\vartheta \in \mathscr{O}\langle\Upsilon, \varrho\rangle$ and morphisms $\oint \mathscr{O}\left[\langle\langle\Upsilon, \varrho\rangle, \vartheta\rangle, \langle\langle\Upsilon', \varrho'\rangle, \vartheta'\rangle\right]$ for $f : \langle\Upsilon, \varrho\rangle \to \langle\Upsilon', \varrho'\rangle$ such that $f^*\vartheta = \vartheta' \in \mathscr{O}\langle\Upsilon', \varrho'\rangle$. Equivalently, $\oint \mathscr{O}$ is the pullback of $\mathscr{O}$ along the universal $\mathbf{Set}$-bundle where $\pi_{\mathscr{O}}$ is a discrete Grothendieck opfibration and $ap_{\{\bullet\}}$ is the forgetful functor from pointed sets:

[1] In this case, $C \oint \Psi$ represents the category of elements of a copresheaf $\Psi : C \to \mathbf{Set}$ but we keep the C implicit and simply refer to it as the Grothendieck construction. Alternatively, this construction can be understood as a coend $C \oint \Psi \cong \int^{c \in C} c/C \otimes \Psi(c)_{\equiv}$.

$$
\begin{tikzcd}
\oint \mathscr{O} \arrow[r, "\pi_{\{\bullet\}}"] \arrow[d, "\pi_{\mathscr{O}}"'] & \{\bullet\}/\mathbf{Set} \arrow[d, "ap_{\{\bullet\}}"] \\
\mathbf{SCtx} \times \mathscr{A} \arrow[r, "\mathscr{O}"'] & \mathbf{Set}
\end{tikzcd}
$$

$$\frac{\vartheta \in \mathscr{O}\langle \Upsilon, \varrho \rangle}{\Upsilon \Vdash \vartheta : \varrho}$$

The judgment $\Upsilon \Vdash \vartheta : \varrho$ enjoys the structural properties of weakening and exchange via the functoriality of $\mathscr{O}$.

Examples Operators are defined by specifying the fibers of $\pi_{\mathscr{O}}$ in which they reside. For instance, consider the lambda calculus with a single sort, exp; we give its signature by asserting the following about its operators:

$$\Upsilon \Vdash \lambda : (\{\cdot\}\,[\mathsf{exp}]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon \Vdash \mathsf{ap} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\Upsilon \Vdash \mathsf{decl} : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp}, \{\mathsf{exp}\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{get}[u] : (\cdot)\,\mathsf{exp}$$
$$\Upsilon, u : \mathsf{exp} \Vdash \mathsf{set}[u] : (\{\cdot\}\,[\cdot]\,.\,\mathsf{exp})\,\mathsf{exp}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a "degeneracy map" on operators, whereas a renaming $u \mapsto v$ will take $\mathsf{get}[u]$ to $\mathsf{get}[v]$.

2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot\ \mathit{mctx}} \qquad \frac{\Omega\ \mathit{mctx} \quad v\ \mathit{valence} \quad \textsc{m} \notin |\Omega|}{\Omega, \textsc{m} : v\ \mathit{mctx}}$$

$$\frac{}{\cdot\ \mathit{vctx}} \qquad \frac{\Gamma\ \mathit{vctx} \quad s\ \mathit{sort} \quad x \notin |\Gamma|}{\Gamma, x : s\ \mathit{vctx}}$$

$$\frac{}{\cdot\ \mathit{sctx}} \qquad \frac{\Upsilon\ \mathit{vctx} \quad s\ \mathit{sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s\ \mathit{sctx}}$$

3 Abstract Binding Trees

Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash M : s$ presuppose Ω *mctx*, Υ *sctx*, Γ *vctx* and s *sort*, meaning that M is an abstract binding tree of sort s, with metavariables in Ω, parameters in Υ, and variables in Γ. Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash E : v$ presuppose v *valence*. Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\frac{\Gamma \ni x : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash x : s}\ \mathit{var}$$

$$\frac{\begin{array}{l}\Omega \ni \textsc{m} : \{p_0, \ldots, p_m\}[q_0, \ldots, q_n].s \\ \Upsilon \ni u_i : p_i \quad (i \leq m) \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \leq n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \textsc{m}\{u_0, \ldots, u_m\}(M_0, \ldots, M_n) : s}\ \mathit{mvar}$$

$$\frac{\begin{array}{l}\Upsilon \Vdash \vartheta : v_1, \ldots, v_n \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash E_i : q_i \quad (i \leq n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \vartheta(E_0, \ldots, E_n) : s}\ \mathit{app}$$

$$\frac{\Omega \rhd \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash M : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \left(\{\vec{u}\}[\vec{x}].M\right) : \left(\{\vec{p}\}[\vec{q}].s\right)}\ \mathit{abs}$$

Abstract binding trees are identified up to α-equivalence.

3.1 Substitution of Variables

Variable substitution in abstract binding trees is defined inductively by a pair of judgments, $[N/x]M \rightsquigarrow M'$ and $[N/x]E \rightsquigarrow E'$:

$$\frac{x = y}{[N/x]\, y \rightsquigarrow N} \qquad \frac{x \# y}{[N/x]\, y \rightsquigarrow y}$$

$$\frac{[N/x]\, M_i \rightsquigarrow M_i' \quad (i \leq n)}{[N/x]\, \textsc{m}\left\{\vec{u}\right\}(M_0, \ldots, M_n) \rightsquigarrow \textsc{m}\left\{\vec{u}\right\}\left(M_0', \ldots, M_n'\right)}$$

$$\frac{[N/x]\, E_i \rightsquigarrow E_i' \quad (i \leq n)}{[N/x]\, \vartheta\,(E_0, \ldots, E_n) \rightsquigarrow \vartheta\left(E_0', \ldots, E_n'\right)}$$

$$\frac{x \notin \vec{y} \quad \vec{y} \# \mathbf{FV}(N) \quad [N/x]\, M \rightsquigarrow M'}{[N/x]\left\{\vec{u}\right\}\left[\vec{y}\right].M \rightsquigarrow \left\{\vec{u}\right\}\left[\vec{y}\right].M'} \qquad \frac{x \in \vec{y} \quad \vec{y} \# \mathbf{FV}(N)}{[N/x]\left\{\vec{u}\right\}\left[\vec{y}\right].M \rightsquigarrow \left\{\vec{u}\right\}\left[\vec{y}\right].M}$$

Because terms are identified up to α-equivalence, the variable substitution judgment is functional in its inputs, and so we are justified in writing $[N/x]\,M$ for M' when $[N/x]\,M \rightsquigarrow M'$. We write $\left[\vec{N}/\vec{x}\right]M$ for the simultaneous substitution of $\vec{N}$ for $\vec{x}$ in M.

3.2 Substitution of Metavariables

Metavariable substitution is defined inductively by the judgment $[E/\textsc{m}]\,M \rightsquigarrow M'$:

$$\frac{}{[E/\textsc{m}]\, x \rightsquigarrow x}$$

$$\frac{\textsc{m} \# \textsc{n} \quad [E/\textsc{n}]\, M_i \rightsquigarrow M_i' \quad (i \leq n)}{[E/\textsc{m}]\, \textsc{n}\left\{\vec{u}\right\}(M_0, \ldots, M_n) \rightsquigarrow \textsc{n}\left\{\vec{u}\right\}\left(M_0', \ldots, M_n'\right)}$$