

# Abstract Binding Trees

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## 1 Preliminaries

Fix a set  $\mathcal{S}$  of *sorts*. We will say  $s$  *sort* when  $s \in \mathcal{S}$ . A valence  $\{\vec{p}\}[\vec{q}].s$  specifies an expression of sort  $s$  which binds symbols in  $\vec{p}$  and variables in  $\vec{q}$ .

$$\frac{s \text{ sort} \quad p_i \text{ sort } (i \leq m) \quad q_i \text{ sort } (i \leq n)}{\{p_0, \dots, p_m\}[q_0, \dots, q_n].s \text{ valence}}$$

An arity  $(\vec{v})s$  specifies an operator of sort  $s$  with arguments of valences  $\vec{v}$ . We will call the set of valences  $\mathcal{V}$ , and the set of arities  $\mathcal{A}$ .

$$\frac{s \text{ sort} \quad v_i \text{ valence } (i \leq n)}{(v_0, \dots, v_n)s \text{ arity}}$$

Let  $\mathbb{I}$  be an infinite set of symbols. Let  $\mathbf{F}$  be the category of finite subsets of  $\mathbb{I}$  and their injective maps; then the comma construction  $\mathbf{SCtx} \triangleq \mathbf{F} \downarrow \mathcal{S}_{\equiv}$ , with  $\mathcal{S}_{\equiv}$  the discrete category on the set  $\mathcal{S}$ , is the category of contexts of symbols, whose objects are finite sets of symbols  $U$  and sort-assignments  $\mathfrak{s} : U \rightarrow \mathcal{S}$ , and whose morphisms are sort-preserving renamings; we will write  $\Upsilon$  for a symbol context  $(U, \mathfrak{s})$ .

Then, fix a covariant presheaf (copresheaf) of operators  $\mathcal{O} : \mathbf{SCtx} \times \mathcal{A}_{\equiv} \rightarrow \mathbf{Set}$  such that the arrows in  $\mathbf{SCtx}$  lift to renamings of operators' parameters. Via the Grothendieck construction<sup>1</sup>  $\oint(-) : \mathbf{Set}^{\mathbf{C}} \rightarrow \mathbf{Cat}$  on operators we have a category of objects  $\langle \langle \Upsilon, \varrho \rangle, \vartheta \rangle \in \oint \mathcal{O}$  for  $\vartheta \in \mathcal{O}(\Upsilon, \varrho)$  and morphisms  $\oint \mathcal{O}[\langle \langle \Upsilon, \varrho \rangle, \vartheta \rangle, \langle \langle \Upsilon', \varrho' \rangle, \vartheta' \rangle]$  for  $f : \langle \Upsilon, \varrho \rangle \rightarrow \langle \Upsilon', \varrho' \rangle$  such that  $f^*\vartheta = \vartheta' \in \mathcal{O}(\Upsilon', \varrho')$ . Equivalently,  $\oint \mathcal{O}$  is the pullback of  $\mathcal{O}$  along the universal  $\mathbf{Set}$ -bundle where  $\pi_{\mathcal{O}}$  is a discrete Grothendieck opfibration and  $|-|_{\bullet}$  is the forgetful functor from pointed sets:

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<sup>1</sup>In this case,  $\mathbf{C} \oint \Psi$  represents the category of elements of a copresheaf  $\Psi : \mathbf{C} \rightarrow \mathbf{Set}$  but we keep the  $\mathbf{C}$  implicit and simply refer to it as the Grothendieck construction. Alternatively, this construction can be understood as a coend  $\mathbf{C} \oint \Psi \cong \int^{c \in \mathbf{C}} c / \mathbf{C} \otimes \Psi(c)_{\equiv}$ .

$$\begin{array}{ccc}
\mathfrak{S} \mathcal{O} & \xrightarrow{\pi_{\epsilon}} & \bullet / \mathbf{Set} \\
\pi_{\mathcal{O}} \downarrow & & \downarrow |- \cdot \\
\mathbf{SCtx} \times \mathcal{A} & \xrightarrow{\mathcal{O}} & \mathbf{Set}
\end{array}$$

$$\frac{\mathfrak{D} \in \mathcal{O} \langle \Upsilon, \varrho \rangle}{\Upsilon \Vdash \mathfrak{D} : \varrho}$$

The judgment  $\Upsilon \Vdash \mathfrak{D} : \varrho$  enjoys the structural properties of weakening and exchange via the functoriality of  $\mathcal{O}$ .

**Examples** Operators are defined by specifying the fibers of  $\pi_{\mathcal{O}}$  in which they reside. For instance, consider the lambda calculus with a single sort,  $\mathbf{exp}$ ; we give its signature by asserting the following about its operators:

$$\begin{aligned}
&\Upsilon \Vdash \lambda : (\{\cdot\} [\mathbf{exp}] . \mathbf{exp}) \mathbf{exp} \\
&\Upsilon \Vdash \mathbf{ap} : (\{\cdot\} [\cdot] . \mathbf{exp}, \{\cdot\} [\cdot] . \mathbf{exp}) \mathbf{exp}
\end{aligned}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\begin{aligned}
&\Upsilon \Vdash \mathbf{decl} : (\{\cdot\} [\cdot] . \mathbf{exp}, \{\mathbf{exp}\} [\cdot] . \mathbf{exp}) \mathbf{exp} \\
&\Upsilon, u : \mathbf{exp} \Vdash \mathbf{get}[u] : (\cdot) \mathbf{exp} \\
&\Upsilon, u : \mathbf{exp} \Vdash \mathbf{set}[u] : (\{\cdot\} [\cdot] . \mathbf{exp}) \mathbf{exp}
\end{aligned}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a “degeneracy map” on operators, whereas a renaming  $u \mapsto v$  will take  $\mathbf{get}[u]$  to  $\mathbf{get}[v]$ .

## 2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context  $\Omega$  consists of bindings of valences to metavariables; a variable context  $\Gamma$  is a collection of bindings of sorts to variables, and a parameter context  $\Upsilon$  is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot \text{ mctx}} \quad \frac{\Omega \text{ mctx} \quad v \text{ valence} \quad M \notin |\Omega|}{\Omega, M : v \text{ mctx}}$$

$$\frac{}{\cdot \text{ vctx}} \quad \frac{\Gamma \text{ vctx} \quad s \text{ sort} \quad x \notin |\Gamma|}{\Gamma, x : s \text{ vctx}}$$

$$\frac{}{\cdot \text{ sctx}} \quad \frac{\Upsilon \text{ vctx} \quad s \text{ sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s \text{ sctx}}$$

### 3 Abstract Binding Trees

Let the judgment  $\Omega \triangleright \Upsilon \parallel \Gamma \vdash M : s$  presuppose  $\Omega \text{ mctx}$ ,  $\Upsilon \text{ sctx}$ ,  $\Gamma \text{ vctx}$  and  $s \text{ sort}$ , meaning that  $M$  is an abstract binding tree of sort  $s$ , with metavariables in  $\Omega$ , parameters in  $\Upsilon$ , and variables in  $\Gamma$ . Let the judgment  $\Omega \triangleright \Upsilon \parallel \Gamma \vdash E : v$  presuppose  $v \text{ valence}$ . Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\begin{array}{c} \frac{\Gamma \ni x : s}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash x : s} \text{ var} \\[10pt] \frac{\begin{array}{l} \Omega \ni M : \{p_0, \dots, p_m\} [q_0, \dots, q_n] . s \\ \Upsilon \ni u_i : p_i \quad (i \leq m) \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash M \{u_0, \dots, u_m\} (M_0, \dots, M_n) : s} \text{ mvar} \\[10pt] \frac{\begin{array}{l} \Upsilon \Vdash \vartheta : v_1, \dots, v_n \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash E_i : q_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \vartheta (E_0, \dots, E_n) : s} \text{ app} \\[10pt] \frac{\Omega \triangleright \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash M : s}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash (\{\vec{u}\} [\vec{x}]. M) : (\{\vec{p}\} [\vec{q}]. s)} \text{ abs} \end{array}$$

Abstract binding trees are identified up to  $\alpha$ -equivalence.

#### 3.1 Substitution of Variables

Variable substitution in abstract binding trees is defined inductively by a pair of judgments,  $[N/x]M \rightsquigarrow M'$  and  $[N/x]E \rightsquigarrow E'$ :

$$\begin{array}{c}
\frac{x = y}{[N/x]y \rightsquigarrow N} \quad \frac{x \# y}{[N/x]y \rightsquigarrow y} \\
\\
\frac{[N/x]M_i \rightsquigarrow M'_i \ (i \leq n)}{[N/x]_{\mathbf{M}} \{\vec{u}\} (M_0, \dots, M_n) \rightsquigarrow_{\mathbf{M}} \{\vec{u}\} (M'_0, \dots, M'_n)} \\
\\
\frac{[N/x]E_i \rightsquigarrow E'_i \ (i \leq n)}{[N/x]_{\mathfrak{D}} (E_0, \dots, E_n) \rightsquigarrow_{\mathfrak{D}} (E'_0, \dots, E'_n)}
\end{array}$$

$$\frac{x \notin \vec{y} \quad \vec{y} \# \mathbf{FV}(N) \quad [N/x]M \rightsquigarrow M'}{[N/x]_{\{\vec{u}\}[\vec{y}]} . M \rightsquigarrow_{\{\vec{u}\}[\vec{y}]} . M'} \quad \frac{x \in \vec{y} \quad \vec{y} \# \mathbf{FV}(N)}{[N/x]_{\{\vec{u}\}[\vec{y}]} . M \rightsquigarrow_{\{\vec{u}\}[\vec{y}]} . M}$$

Because terms are identified up to  $\alpha$ -equivalence, the variable substitution judgment is functional in its inputs, and so we are justified in writing  $[N/x]M$  for  $M'$  when  $[N/x]M \rightsquigarrow M'$ . We write  $[\vec{N}/\vec{x}]M$  for the simultaneous substitution of  $\vec{N}$  for  $\vec{x}$  in  $M$ .

### 3.2 Substitution of Metavariables

Metavariable substitution is defined inductively by the judgment  $[E/\mathbf{M}]M \rightsquigarrow M'$ :

$$\begin{array}{c}
\overline{[E/\mathbf{M}]x \rightsquigarrow x} \\
\\
\frac{\mathbf{M} \# \mathbf{N} \quad [E/\mathbf{N}]M_i \rightsquigarrow M'_i \ (i \leq n)}{[E/\mathbf{M}]_{\mathbf{N}} \{\vec{u}\} (M_0, \dots, M_n) \rightsquigarrow_{\mathbf{N}} \{\vec{u}\} (M'_0, \dots, M'_n)}
\end{array}$$