Abstract Binding Trees

Jonathan Sterling

1 Preliminaries

Fix a set $\mathscr{S}$ of *sorts*. We will say s *sort* when $s \in S$. A valence $\vec{p} \parallel \vec{q}.s$ specifies an expression of sort s which binds symbols in $\vec{p}$ and variables in $\vec{q}$.

$$\frac{s\ \mathit{sort} \quad p_i\ \mathit{sort}\ \ (i \leq m) \quad q_i\ \mathit{sort}\ \ (i \leq n)}{p_0, \dots, p_m \parallel q_0, \dots, q_n.s\ \mathit{valence}}$$

An arity $\left(\vec{v}\right)s$ specifies an operator of sort s with arguments of valences $\vec{v}$. We will call the set of valences $\mathscr{V}$, and the set of arities $\mathscr{A}$.

$$\frac{s\ \mathit{sort} \quad v_i\ \mathit{valence}\ \ (i \leq n)}{(v_0, \dots, v_n)s\ \mathit{arity}}$$

Let $\mathbb{I}$ be the infinite set of symbols; let $\mathbb{F}$ be the free cocartesian category over $\mathbb{I}$. Then, fix a covariant presheaf of operators $\mathscr{O} : \mathbb{F} \times \mathscr{A} \to \mathbf{Set}$ such that the arrows in $\mathbb{F}$ lift to renamings of operators' parameters; via the Grothendieck construction, we can also consider the set $\int \mathscr{O}$ of operators (U, ϱ, ϑ) for $\vartheta \in \mathscr{O}(U, \varrho)$.

$$\frac{\vartheta \in \mathscr{O}(U, \varrho)}{U \vdash \vartheta : \varrho}$$

The judgment $U \vdash \vartheta : \varrho$ supports the structural principles of weakening and exchange, because of the functoriality of $\mathscr{O}$.

Examples Operators are defined by specifying the fibres of $\int \mathscr{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, exp, about whose operators we may assert the following:

$$U \vdash \lambda : (\cdot \parallel \mathsf{exp}.\mathsf{exp})\,\mathsf{exp})$$
$$U \vdash \mathsf{ap} : (\cdot \parallel \cdot.\mathsf{exp}, \cdot \parallel \cdot.\mathsf{exp})\,\mathsf{exp}$$

However, consider the extension of the calculus with assignables. Then, we shall have the following:

$$U \vdash \mathsf{decl} : (\cdot \parallel \cdot\,.\,\mathsf{exp}, \mathsf{exp} \parallel \cdot\,.\,\mathsf{exp})\,\mathsf{exp}$$
$$U, u \vdash \mathsf{get}\{u\} : (\cdot)\,\mathsf{exp}$$
$$U, u \vdash \mathsf{set}\{u\} : (\cdot \parallel \cdot\,.\,\mathsf{exp})\,\mathsf{exp}$$

Weakening can be seen as inducing a “degeneracy map” on operators, whereas a renaming $u \mapsto v$ will take $\mathsf{get}\{u\}$ to $\mathsf{get}\{v\}$.

2 Contexts

In general, we will have three kinds of context: metacontexts, variable contexts, and parameter contexts. A metacontext Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\frac{}{\cdot\ \mathit{mctx}} \qquad \frac{\Omega\ \mathit{mctx} \quad v\ \mathit{valence} \quad \textsc{m} \notin |\Omega|}{\Omega, \textsc{m} : v\ \mathit{mctx}}$$

$$\frac{}{\cdot\ \mathit{vctx}} \qquad \frac{\Gamma\ \mathit{vctx} \quad s\ \mathit{sort} \quad x \notin |\Gamma|}{\Gamma, x : s\ \mathit{vctx}}$$

$$\frac{}{\cdot\ \mathit{sctx}} \qquad \frac{\Upsilon\ \mathit{vctx} \quad s\ \mathit{sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s\ \mathit{sctx}}$$

3 Abstract Binding Trees

Let the judgment $\Omega \rhd \Upsilon \parallel \Gamma \vdash M : s$ presuppose Ω *mctx*, Υ *sctx*, Γ *vctx* and s *sort*, meaning that M is an abstract binding tree of sort s, with metavariables in Ω, parameters in Υ, and variables in Γ.

$$\frac{\Gamma \ni x : s}{\Omega \rhd \Upsilon \parallel \Gamma \vdash x : s}\ \mathit{var}$$

$$\frac{\begin{array}{l}\Omega \ni \mathrm{M} : p_0, \dots, p_m \parallel q_0, \dots, q_n . s \\ \Upsilon \ni u_i : p_i \quad (i \le m) \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \le n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \mathrm{M}\,\{u_0, \dots, u_m\}\,(u_0, \dots, u_m;\ M_0, \dots, M_n) : s}\ \mathit{mvar}$$

$$\frac{\begin{array}{l}|\Upsilon| \vdash \vartheta : p_0, \dots, p_m \parallel q_0, \dots, q_n . s \\ \Upsilon \ni u_i : p_i \quad (i \le m) \\ \Omega \rhd \Upsilon \parallel \Gamma \vdash M_i : q_i \quad (i \le n)\end{array}}{\Omega \rhd \Upsilon \parallel \Gamma \vdash \vartheta\,\{u_0, \dots, u_m\}\,(u_0, \dots, u_m;\ M_0, \dots, M_n) : s}\ \mathit{app}$$