

Abstract Binding Trees

Jonathan Sterling

1 Preliminaries

Fix a set \mathcal{S} of *sorts*. We will say s *sort* when $s \in \mathcal{S}$. A valence $\{\vec{p}\}[\vec{q}] . s$ specifies an expression of sort s which binds symbols in \vec{p} and variables in \vec{q} .

$$\frac{s \text{ sort} \quad p_i \text{ sort } (i \leq m) \quad q_i \text{ sort } (i \leq n)}{\{p_0, \dots, p_m\}[q_0, \dots, q_n] . s \text{ valence}}$$

An arity $(\vec{v})s$ specifies an operator of sort s with arguments of valences \vec{v} . We will call the set of valences \mathcal{V} , and the set of arities \mathcal{A} .

$$\frac{s \text{ sort} \quad v_i \text{ valence } (i \leq n)}{(v_0, \dots, v_n)s \text{ arity}}$$

Let \mathbb{I} be an infinite set of symbols; let \mathbb{F} be the free cocartesian category over \mathbb{I} . Then, fix a covariant presheaf of operators $\mathcal{O} : \mathbb{F} \times \mathcal{A} \rightarrow \mathbf{Set}$ such that the arrows in \mathbb{F} lift to renamings of operators' parameters; via the Grothendieck construction, we can also consider the set $\int \mathcal{O}$ of operators (U, ϱ, ϑ) for $\vartheta \in \mathcal{O}(U, \varrho)$.

$$\frac{\vartheta \in \mathcal{O}(U, \varrho)}{U \Vdash \vartheta : \varrho}$$

The judgment $U \Vdash \vartheta : \varrho$ enjoys the structural properties of weakening and exchange via the functoriality of \mathcal{O} .

Examples Operators are defined by specifying the fibres of $\int \mathcal{O}$ in which they reside. For instance, consider the lambda calculus with a single sort, **exp**; we give its signature by asserting the following about its operators:

$$\begin{aligned} U \Vdash \lambda : (\{\cdot\}[\mathbf{exp}] . \mathbf{exp}) \mathbf{exp} \\ U \Vdash \mathbf{ap} : (\{\cdot\}[\cdot] . \mathbf{exp}, \{\cdot\}[\cdot] . \mathbf{exp}) \mathbf{exp} \end{aligned}$$

So far, we have made no use of symbols and parameters; however, consider the extension of the calculus with assignables (references):

$$\begin{aligned}
U &\Vdash \text{decl} : (\{\cdot\} [\cdot] \text{ . exp}, \{\text{exp}\} [\cdot] \text{ . exp}) \text{ exp} \\
U, u &\Vdash \text{get}[u] : (\cdot) \text{ exp} \\
U, u &\Vdash \text{set}[u] : (\{\cdot\} [\cdot] \text{ . exp}) \text{ exp}
\end{aligned}$$

Declaring a new assignable consists in providing an initial value, and an expression with a free symbol (which shall represent the assignable in scope). Weakening can be seen as inducing a “degeneracy map” on operators, whereas a renaming $u \mapsto v$ will take $\text{get}[u]$ to $\text{get}[v]$.

2 Contexts

In general, we will have three kinds of context: metavariable contexts, variable contexts, and symbol (parameter) contexts. A metavariable context Ω consists of bindings of valences to metavariables; a variable context Γ is a collection of bindings of sorts to variables, and a parameter context Υ is a collection of bindings of sorts to symbols.

$$\begin{aligned}
&\frac{}{\cdot \text{ mctx}} \quad \frac{\Omega \text{ mctx} \quad v \text{ valence} \quad M \notin |\Omega|}{\Omega, M : v \text{ mctx}} \\
&\frac{}{\cdot \text{ vctx}} \quad \frac{\Gamma \text{ vctx} \quad s \text{ sort} \quad x \notin |\Gamma|}{\Gamma, x : s \text{ vctx}} \\
&\frac{}{\cdot \text{ sctx}} \quad \frac{\Upsilon \text{ vctx} \quad s \text{ sort} \quad u \notin |\Upsilon|}{\Upsilon, u : s \text{ sctx}}
\end{aligned}$$

3 Abstract Binding Trees

Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash M : s$ presuppose $\Omega \text{ mctx}$, $\Upsilon \text{ sctx}$, $\Gamma \text{ vctx}$ and $s \text{ sort}$, meaning that M is an abstract binding tree of sort s , with metavariables in Ω , parameters in Υ , and variables in Γ . Let the judgment $\Omega \triangleright \Upsilon \parallel \Gamma \vdash E : v$ presuppose $v \text{ valence}$. Then, the syntax of abstract binding trees is inductively defined in four rules:

$$\begin{array}{c}
\frac{\Gamma \ni x : \textcolor{red}{s}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash x : \textcolor{blue}{s}} \textit{var} \\[10pt]
\frac{\begin{array}{l} \Omega \ni \mathsf{M} : \{p_0, \dots, p_m\} [q_0, \dots, q_n] . \textcolor{red}{s} \\ \Upsilon \ni u_i : \textcolor{red}{p}_i \quad (i \leq m) \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash M_i : \textcolor{red}{q}_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \mathsf{M} \{u_0, \dots, u_m\} (M_0, \dots, M_n) : \textcolor{blue}{s}} \textit{mvar} \\[10pt]
\frac{\begin{array}{l} |\Upsilon| \Vdash \vartheta : \textcolor{red}{v}_1, \dots, \textcolor{red}{v}_n \\ \Omega \triangleright \Upsilon \parallel \Gamma \vdash E_i : q_i \quad (i \leq n) \end{array}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash \vartheta (E_0, \dots, E_n) : \textcolor{blue}{s}} \textit{app} \\[10pt]
\frac{\Omega \triangleright \Upsilon, \vec{u} : \vec{p} \parallel \Gamma, \vec{x} : \vec{q} \vdash E : \textcolor{red}{s}}{\Omega \triangleright \Upsilon \parallel \Gamma \vdash (\{\vec{u}\} [\vec{x}] . E) : (\{\vec{p}\} [\vec{q}] . \textcolor{blue}{s})} \textit{abs}
\end{array}$$