VE444: Networks

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Matching Markets

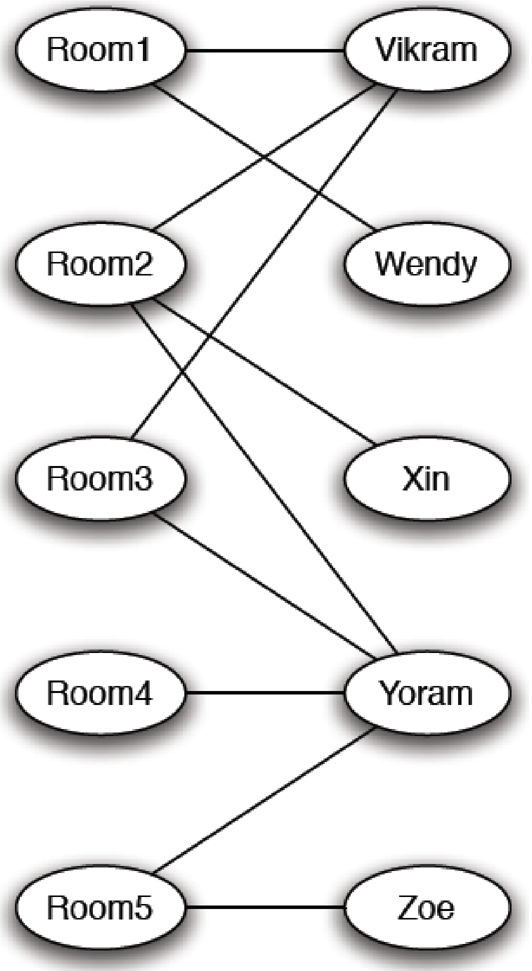
Matching is common

- Matching is a common phenomenon in our society
  - Student-university matching
  - Employee-employer matching
  - Wife-husband matching

- 2012 Nobel Prize in Economy:
  - Lloyd S. Shapley and Alvin E. Roth
  - For the theory of stable allocations and the practice of market design

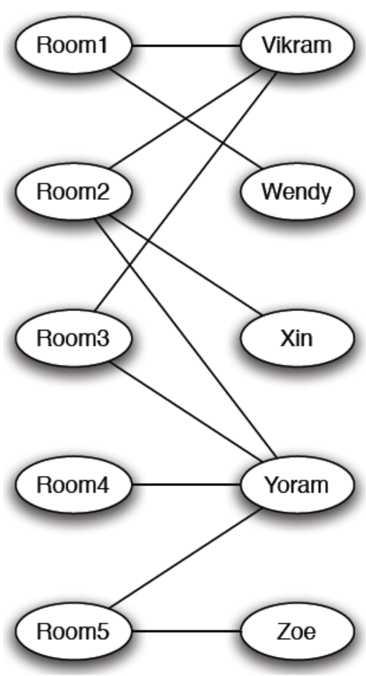
Start from a simple example

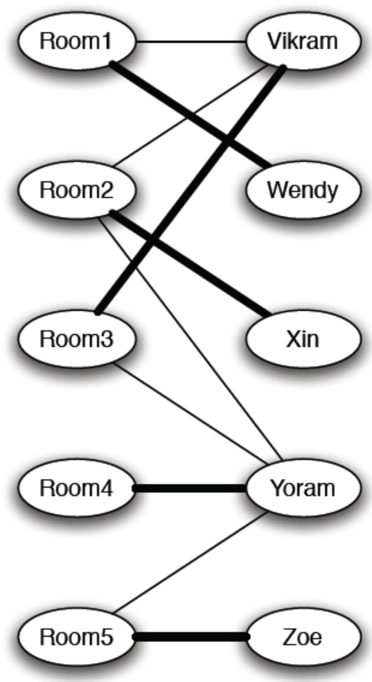
- 5 students and 5 rooms, every one shows his/her preference
- Is there a matching that satisfies all students?
- A bipartite graph



Perfect Matching

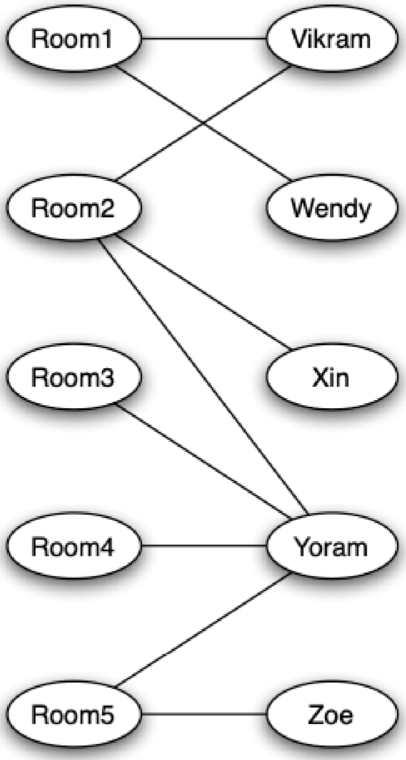
- Perfect Matching: When there are an equal number of nodes on each side of a bipartite graph, a perfect matching is an assignment of nodes on the left to nodes on the right, in such a way that
  - each node is connected by an edge to the node it is assigned to, and
  - no two nodes on the left are assigned to the same node on the right.





Constricted Set

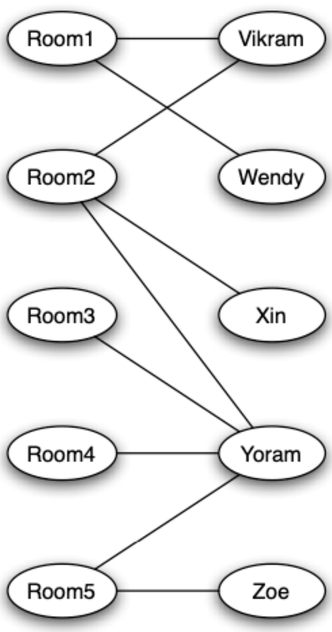
- How to prove to others that a bipartite graph has no perfect matching?

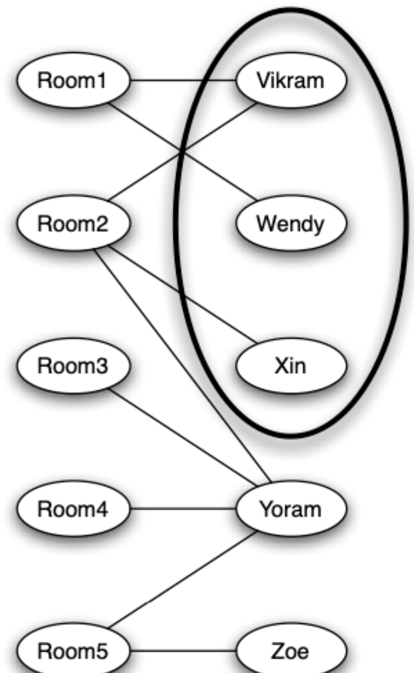


Neighbor set N(S) : collection of all neighbors of a right/left side node set S.
Constricted sets: a set, S, on one side is constricted if S is strictly larger than N(S)

Constricted Set

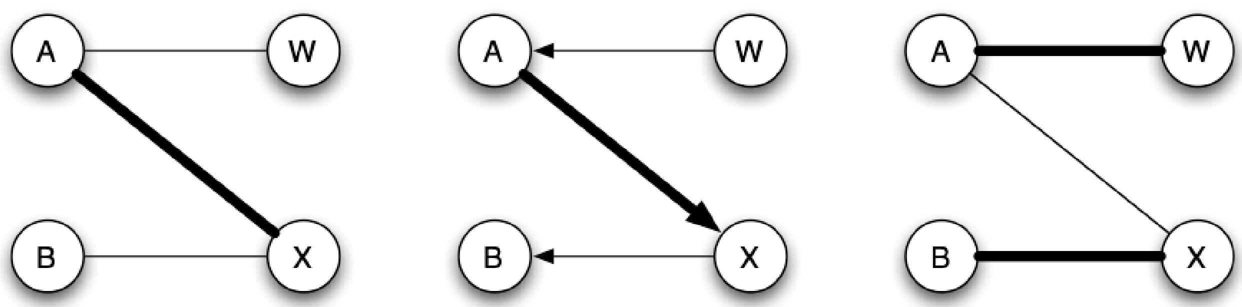
- Matching Theorem: If a bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching, then it must contain a constricted set.



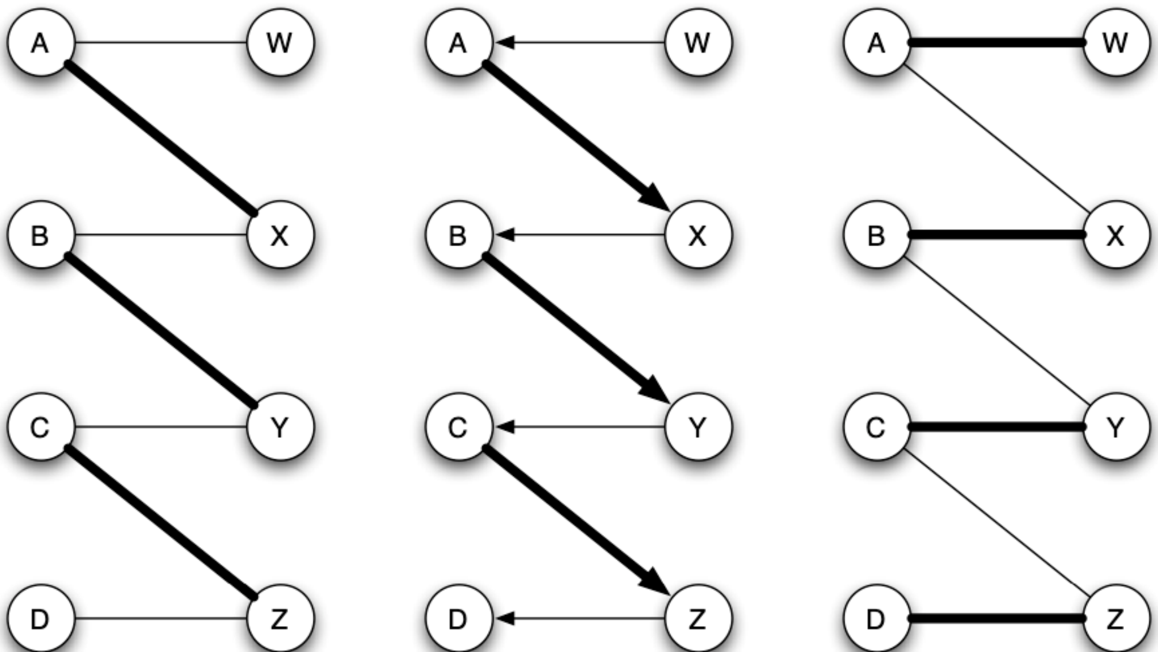


Matching theorem Proof: augmenting path

- Enlarging the existing matching
  - Matching edges
  - Non-matching edges
- If there is an alternating path whose endpoints are unmatched nodes, then the matching can be enlarged
  - Alternating path: a path that alternates between nonmatching and matching edges
  - Augmenting path: an alternating path with unmatched endpoints
  - Flip the roles of edges in the augmenting path to enlarge the matching

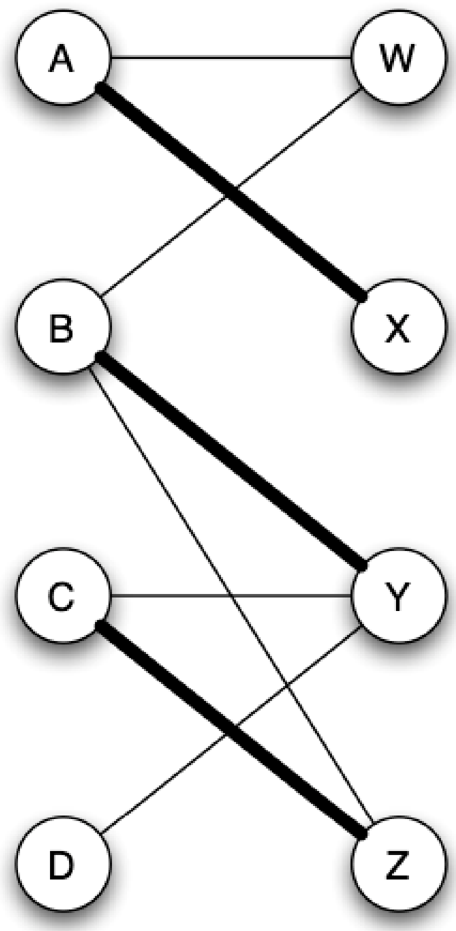


Augmenting path example



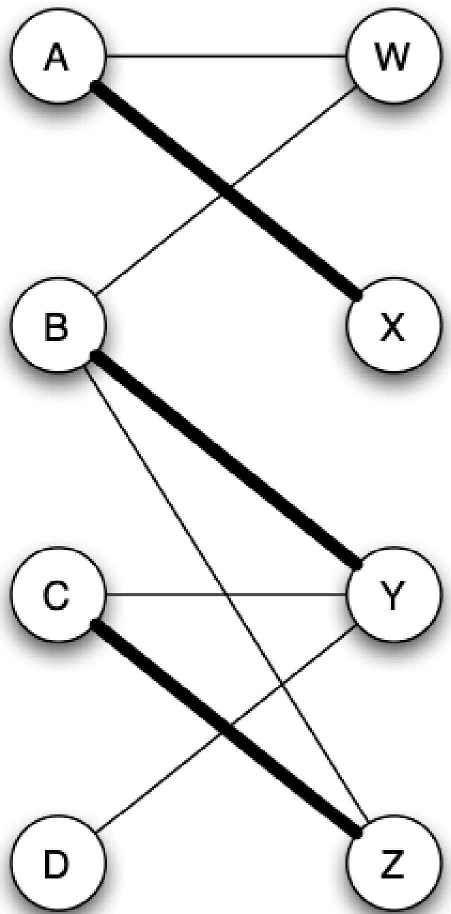
- Find the path is not easy.

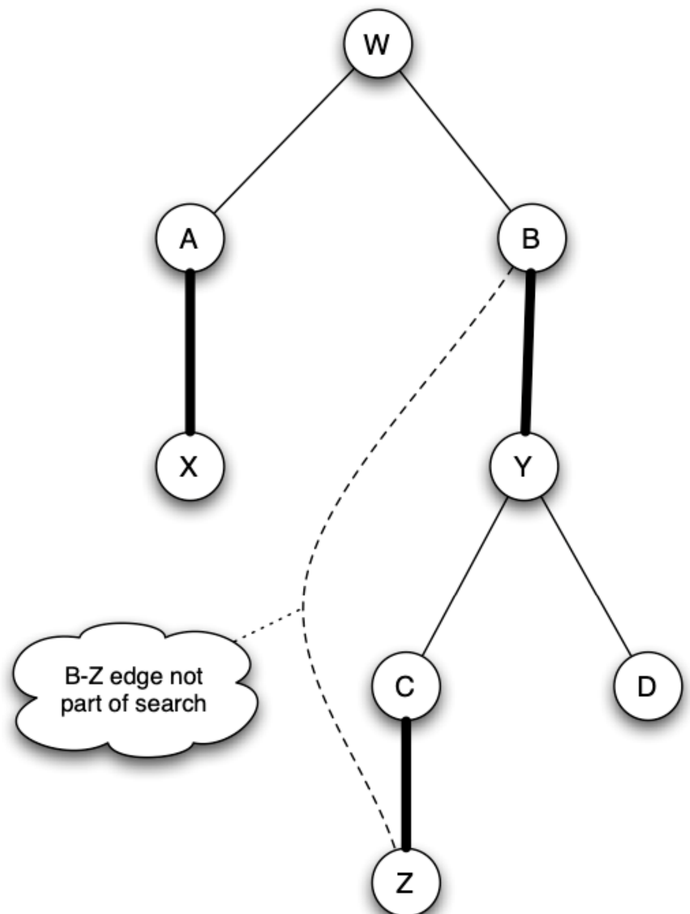
Finding the augmenting path is not easy



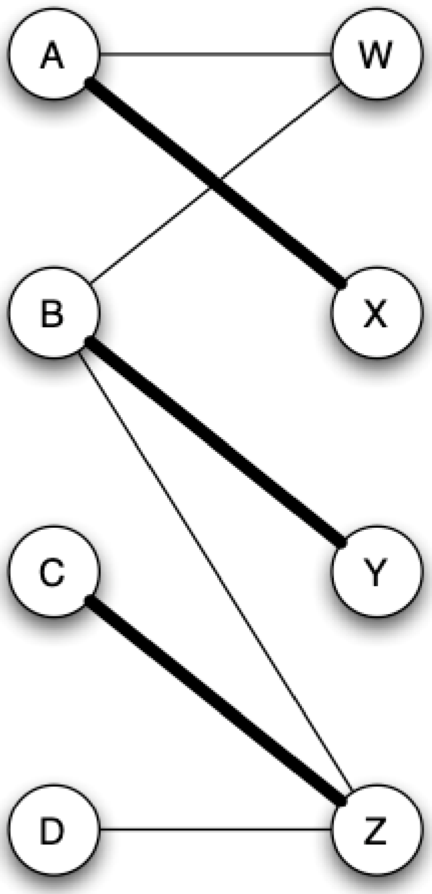
- We need an algorithm to find it.

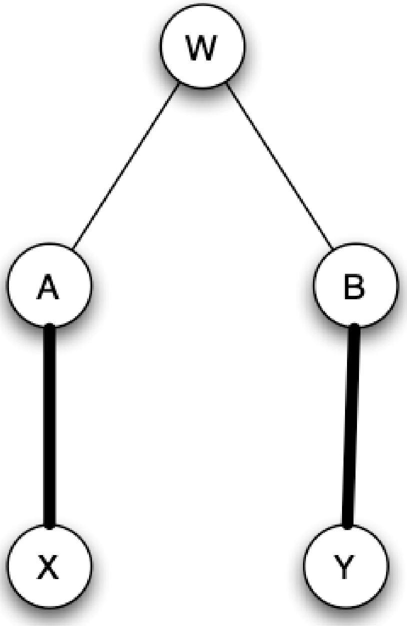
Alternating BFS





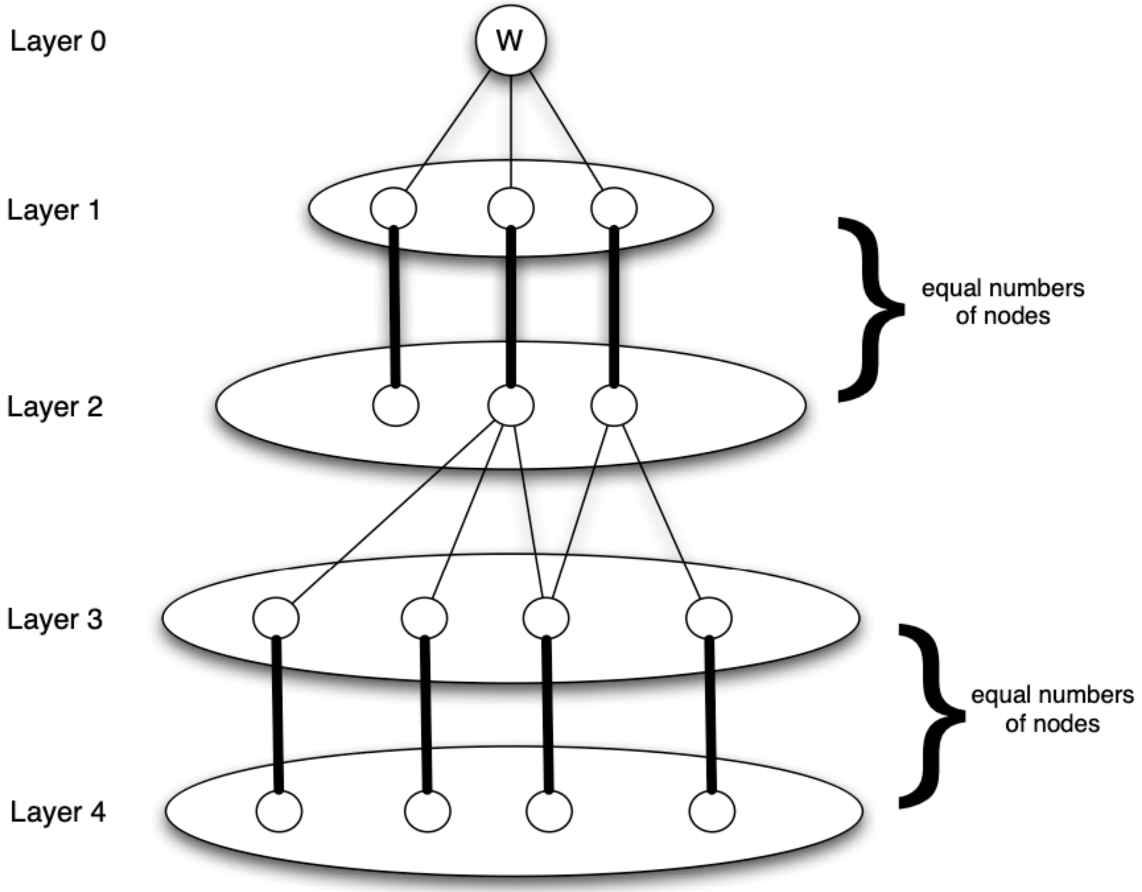
Identifying the constricted set





Constricted set: The set of nodes in all even layers at the end of a failed alternating BFS

Identifying the constricted set



Constricted set: The set of nodes in all even layers at the end of a failed alternating BFS

Back to Matching Theorem

W: any unmatched node on the right-hand side
Run alternating BFS on current matching, then, either there is an augmenting path beginning at W, or there is a constricted set containing W

- Matching Theorem: If a bipartite graph (with equal numbers of nodes on the left and right) has no perfect matching, then it must contain a constricted set.
  - If no perfect matching, only maximum matching exists
  - from unmatched nodes, no way to enlarge it further.
  - Constricted set identified.

Matching based on valuations

- Not only buying or not buying, there can be different valuation on a single object
  - The same object has different values for different people
  - The same person has different values for different objects
- How to do matching with valuations?

Marriage Model

- **Participants**
  - Set of men M, with typical man $m \in M$
  - Set of women W, with typical woman $w \in W$.
  - One-to-one matching: each man can be matched to one woman, and vice-versa.
- **Preferences**
  - Each man has strict preferences over women, and vice versa.
  - A woman w is *acceptable* to m if m prefers w to being unmatched.

Marriage Model: Matching

- A ***matching*** is a set of pairs *(m,w)* such that each individual has one partner.
  - If the match includes *(m,m)* then m is unmatched.
- A matching is ***stable*** if
  - Every individual is matched with an acceptable partner.
  - There is no man-woman pair, each of whom would prefer to match with each other rather than their assigned partner.
- If such a pair exists, they are a ***blocking pair*** and the match is ***unstable***.

Example 1

- Two men *m,m'* and two women *w,w'*

- *m* prefers *w* to *w'*
- *m'* prefers *w'* to *w*
- *w* prefers *m* to *m'*
- *w'* prefers *m'* to *m*

- Possible match: *(m,w')* and *(m',w)*
- Unique stable match: *(m,w)* and *(m',w')*

Example 2

- Two men *m,m'* and two women *w,w'*

- *m* prefers *w* to *w'*
- *m'* prefers *w'* to *w*
- *w* prefers *m'* to *m*
- *w'* prefers *m* to *m'*

- Two stable matches *{(m,w),(m',w')}* and *{(m,w'),(m',w)}*
- First match is better for the men, second for the women.

- *Is there always a stable match? How to find one?*

Deferred Acceptance

- Men and women rank all potential partners
- **Algorithm**
  - Each man proposes to highest woman on his list
  - Women make a “tentative match” based on their preferred offer, and reject other offers, or all if none are acceptable.
  - Each rejected man removes woman from his list, and makes a new offer.
  - Continue until no more rejections or offers, at which point implement tentative matches.
- This is the “man-proposing” version of the algorithm; there is also a “woman proposing” version.

Example

- Preferences of men and women

m1: w1 > w2 > w3	w1: m2 > m3 > m1
m2: w3 > w2 > w1	w2: m2 > m3 > m1
m3: w3 > w1 > w2	w3: m2 > m1 > m3

- Find a stable matching.

Stable matchings exist

Theorem. The outcome of the DA algorithm is a stable one-to-one matching (so a stable match exists).

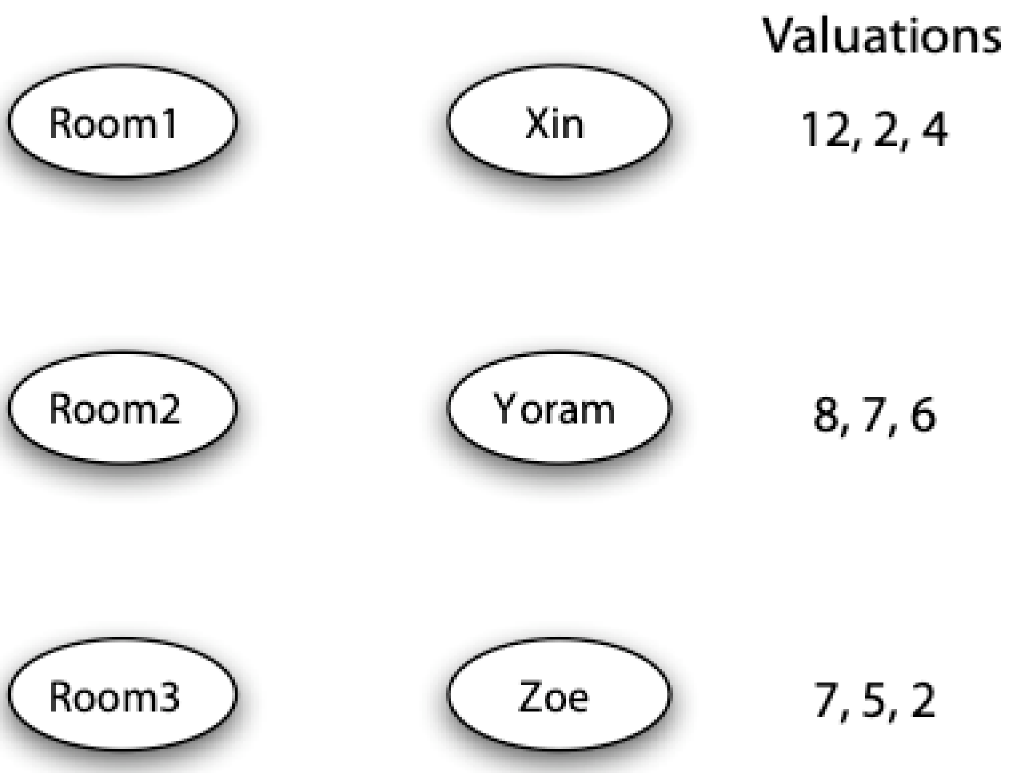
Proof.

- Algorithm must end in a finite number of rounds.
- Suppose *m, w* are matched, but *m* prefers *w'*.
  - At some point, *m* proposed to *w'* and was rejected.
  - At that point, *w'* preferred her tentative match to *m*.
  - As algorithm goes forward, *w'* can only do better.
  - So *w'* prefers her final match to *m.*
- Therefore, there are *NO BLOCKING PAIRS.*

Further analysis on DA: truthfulness, pareto-optimality, etc could be done.

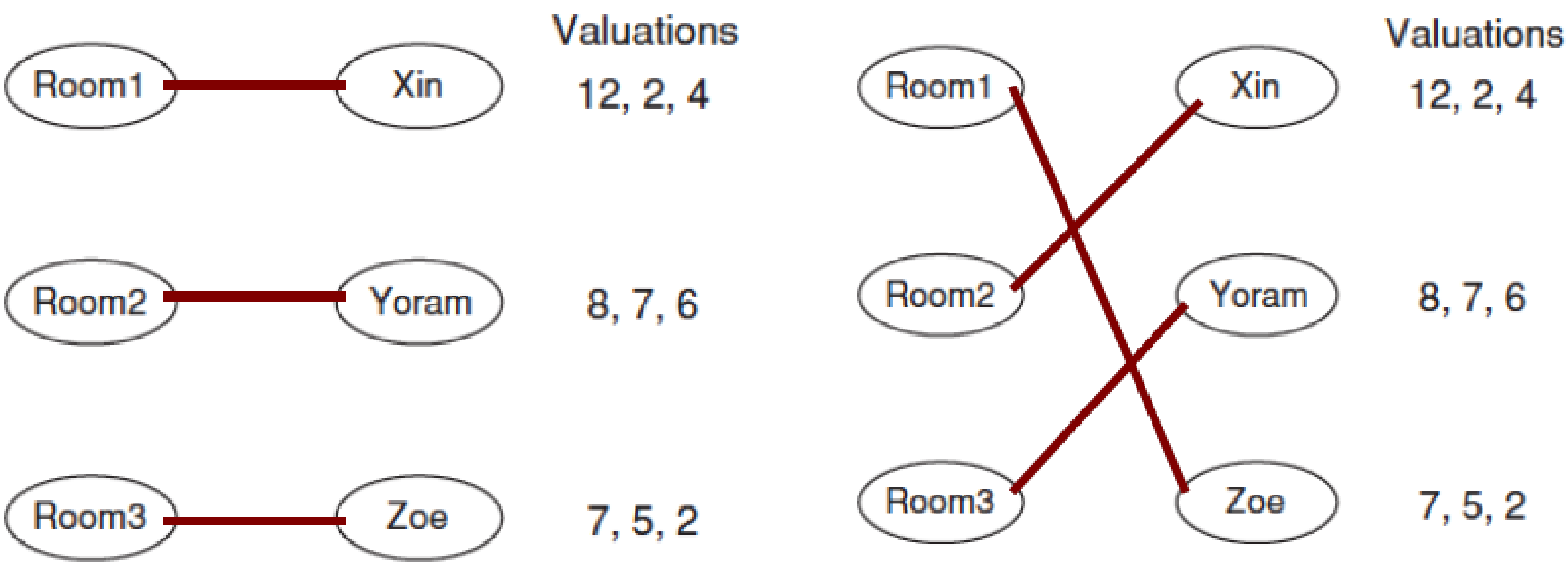
Matching Market

- Valuation (different on different objects)



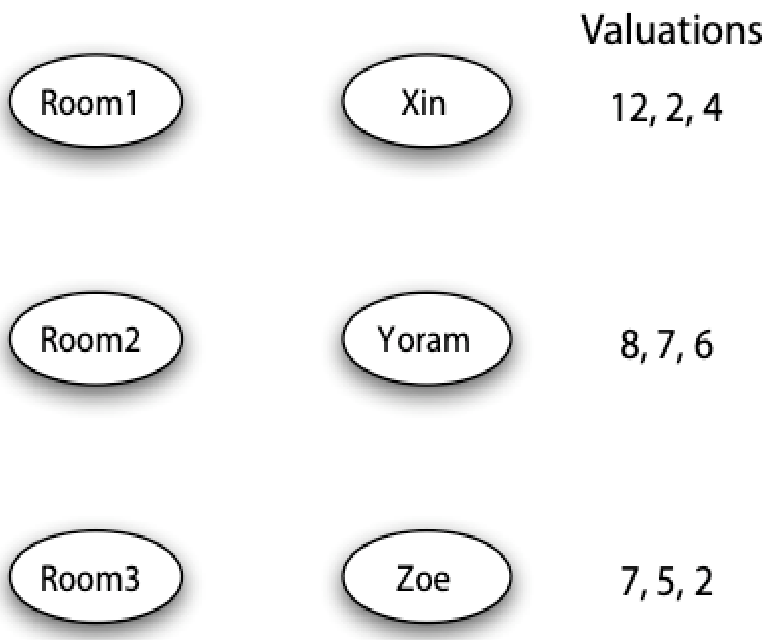
- Matching always exists
- Which is good?

Which is better?



Quality evaluation

- We care whether there is a matching that everybody will be satisfied (i.e., happy)

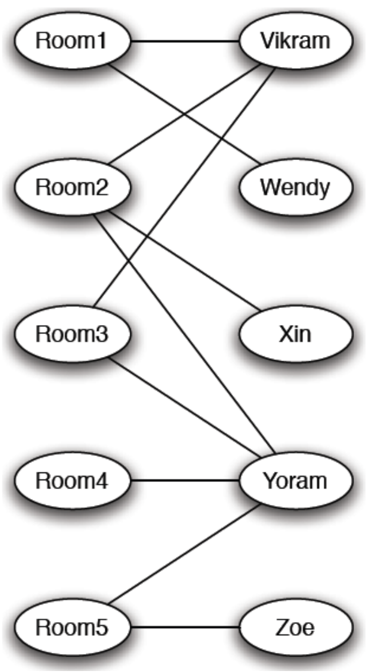


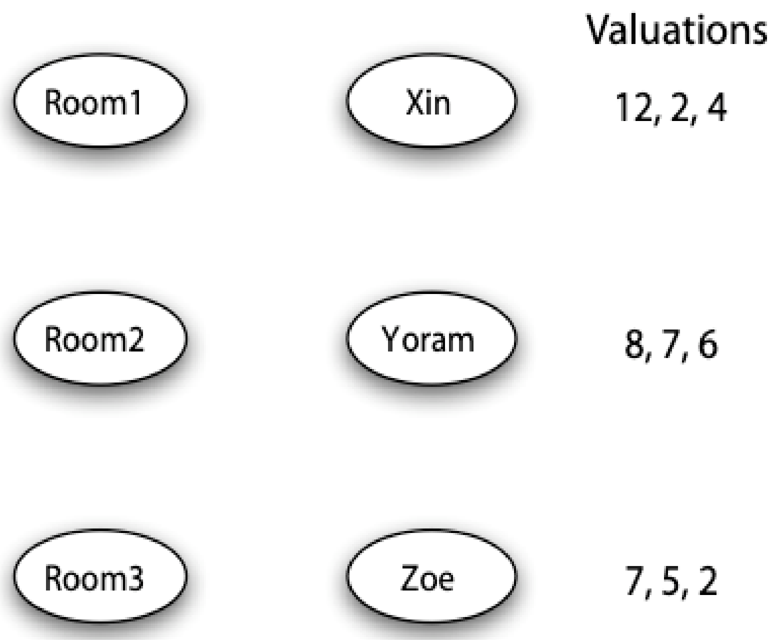
Quality of an assignment: sum of individual's valuation for what they get

Optimal assignment: assignment with the maximum quality

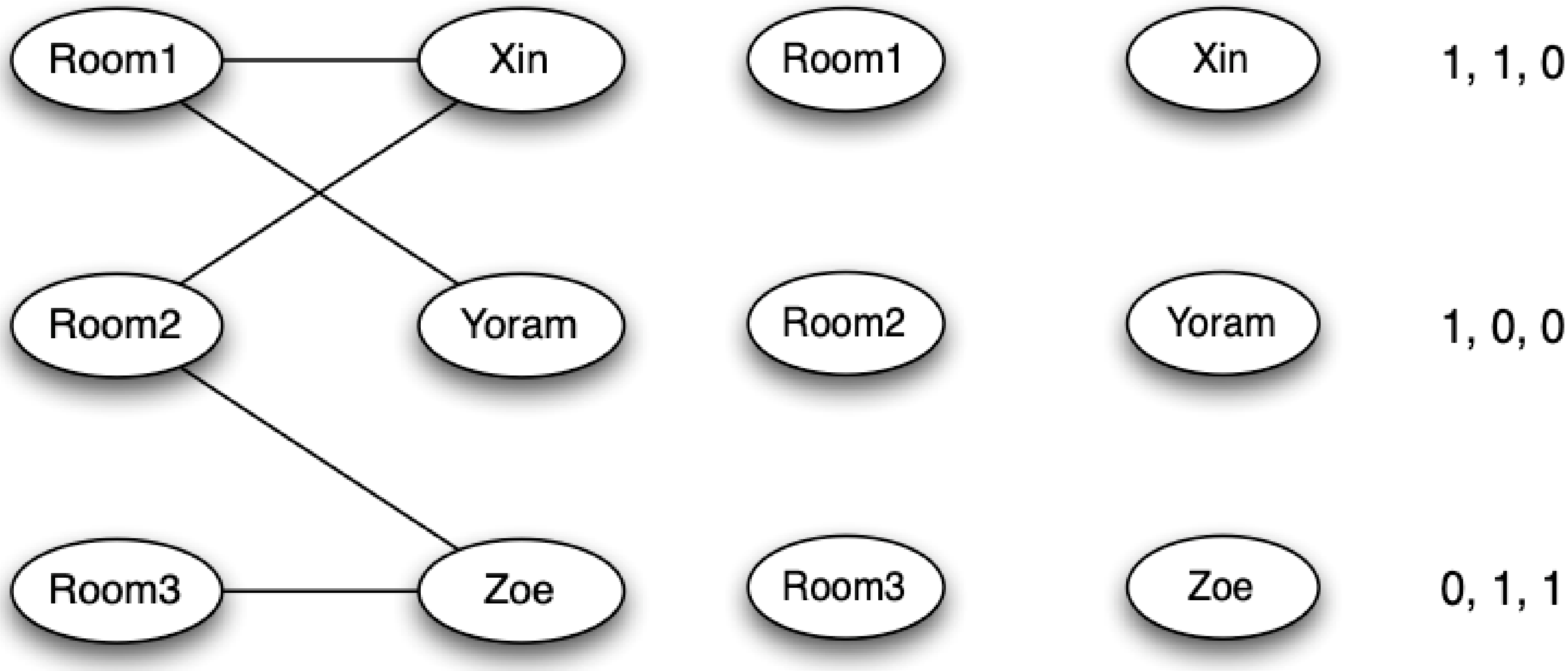
There can be different ways to measure optimality

Relationship between two cases



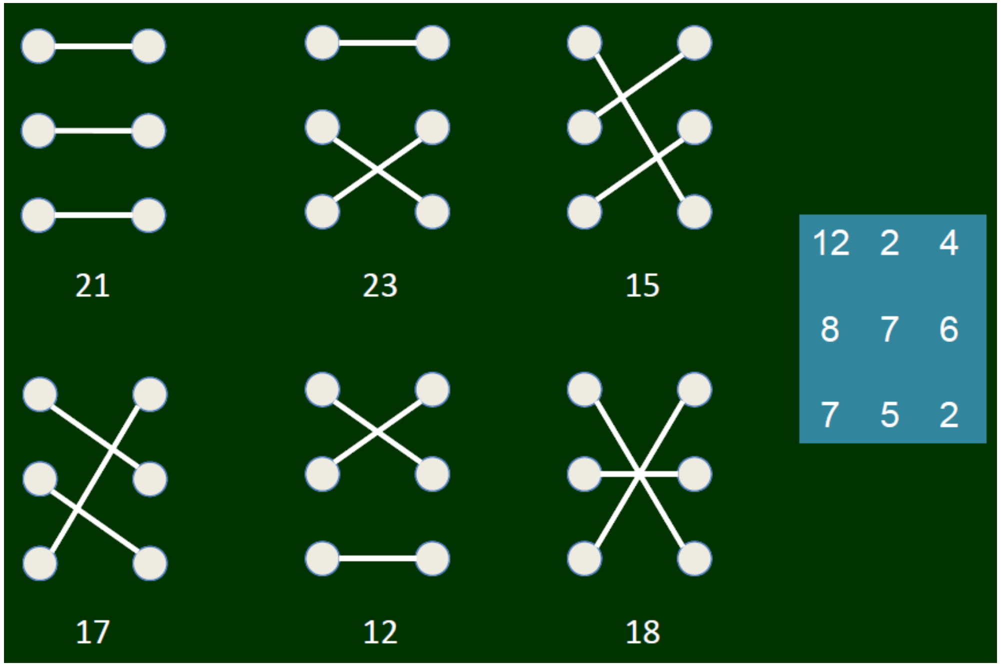


Describes the basic case as well



How to find the optimal assignment?

- Sellers, buyers, valuation (different on different objects)
- Exhaustive search?
  - Time complexity n!



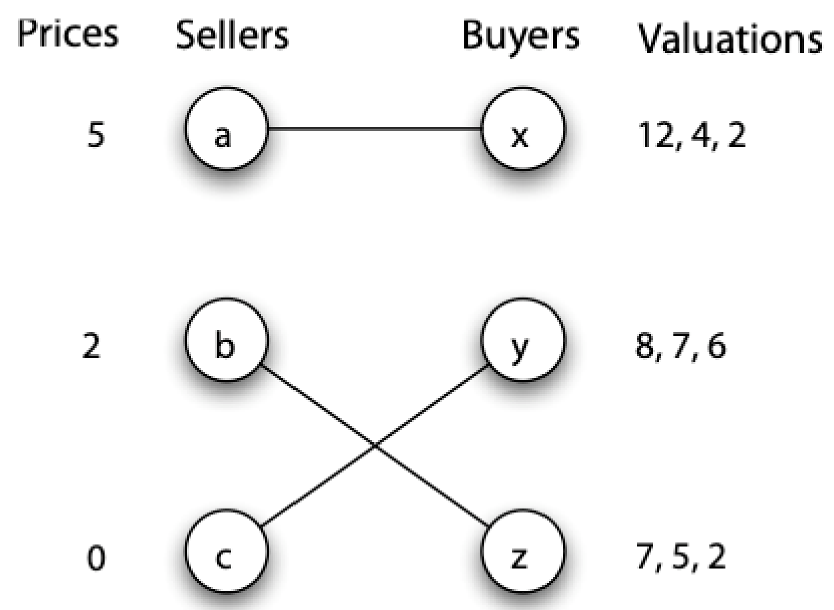
Matching Market Framework

- Sellers, buyers, valuation (different on different objects)
- We care whether there is a matching that everybody will be satisfied (i.e., happy)
- What kind of mechanism will provide such matching?
- What if we do not have a centralized coordinator?

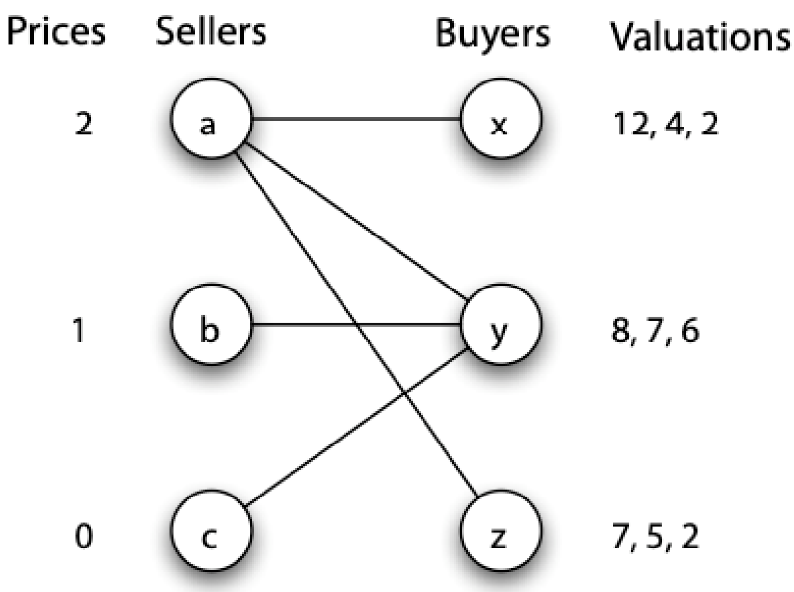
Elements overview

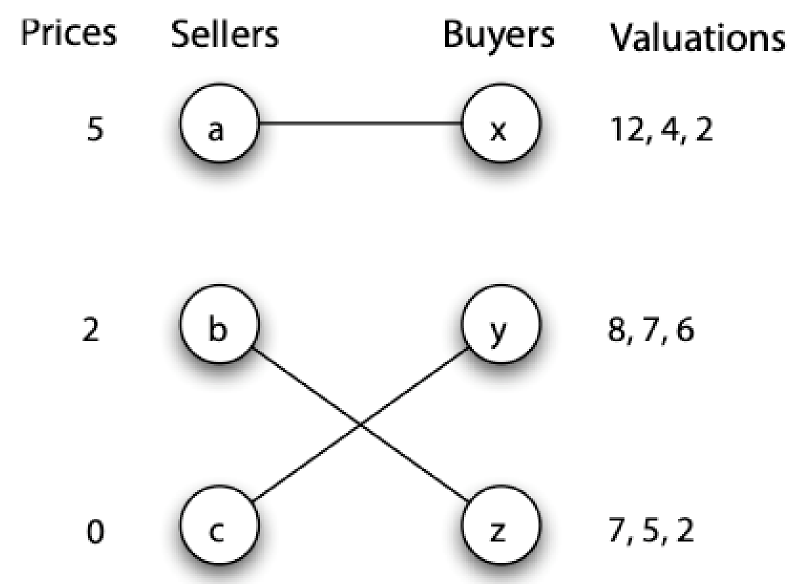
- What if we do not have a centralized coordinator?
  - Replace the central coordinator by a pricing scheme
- Each seller offer a price pi
- Buyer payoff: valuation minus the price
- Preferred sellers: the seller or sellers that maximize the payoff for buyer j
  - Preferred seller graph

Preferred seller graph

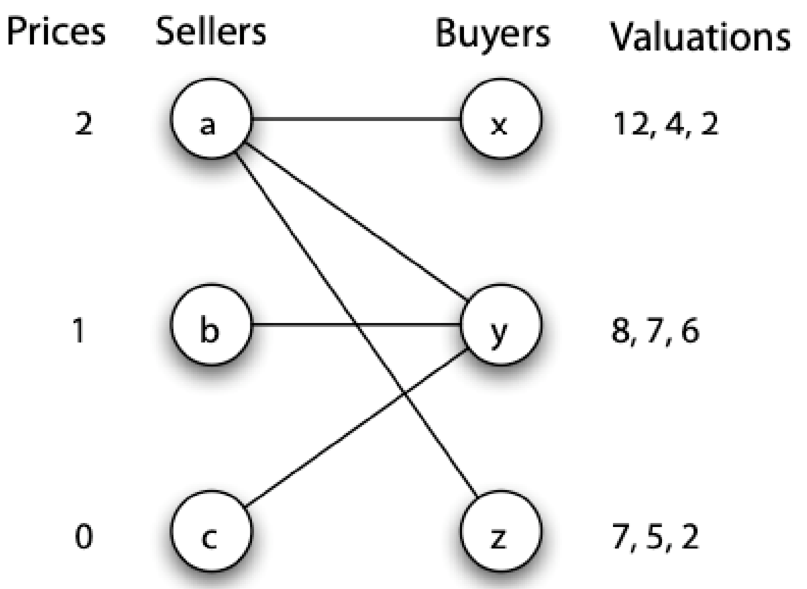


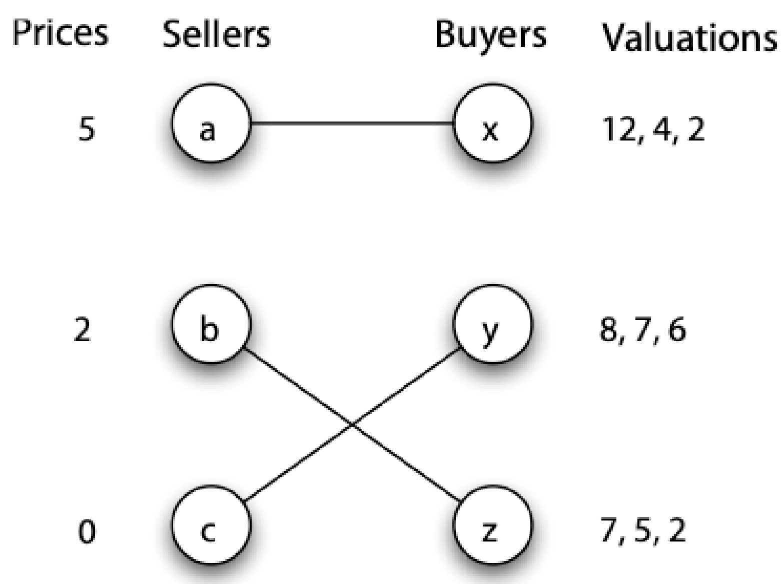
Preferred seller graph





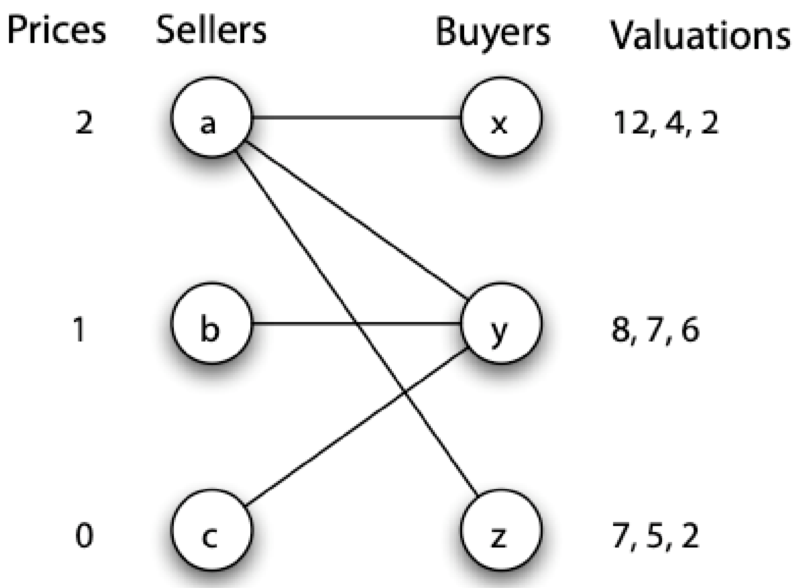
Market clearing price

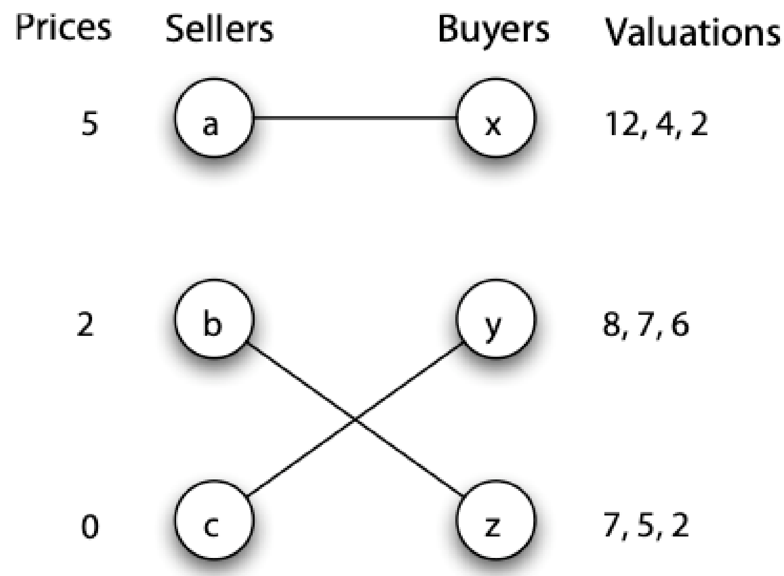




A set of price is **market clearing price** if the resulting preferred-seller graph has a perfect matching.

Market clearing price





A set of price is **market clearing price** if the resulting preferred-seller graph has a perfect matching.

- Coordination over tie-breaking allowed
- Multiple sets of market clearing price could exist

Optimality of the Market-clearing Prices

Optimality of the Market-clearing Prices: For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum total valuation of any assignment of sellers to buyers

Reason:
M: perfect matching
Total Payoff of buyers in M = Total Valuation of buyers in M – Sum of all prices

Optimality of the Market-clearing Prices (v2): For any set of market-clearing prices, a perfect matching in the resulting preferred-seller graph has the maximum sum of payoffs of all sellers to buyers

Existence of Market-clearing prices

Existence of Market-clearing prices: For any set of buyer valuations, there exists a set of market-clearing price.

- Why it always exists?
  - Construct a procedure that stops only when market-clearing prices are found
  - This procedure has limited rounds.

How to find a set of Market-Clearing Prices?

- Two parts: price probing(increase) and price reduction
- Procedure:
  1. At the start of each round, a set of prices, with the smallest one equal to 0
  2. Construct the preferred-seller graph and check whether there is a perfect matching
  3. Stops when perfect matching exists, output current prices
  4. If not, identify a constricted set of buyers, S and their neighbors N(S)
  5. Each seller in N(S) simultaneously raises his price by one unit
  6. Reduce price to guarantee the smallest price equal to 0.
  7. Using the updated price to start a new round

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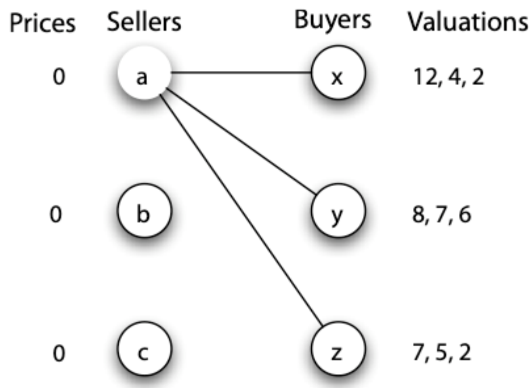
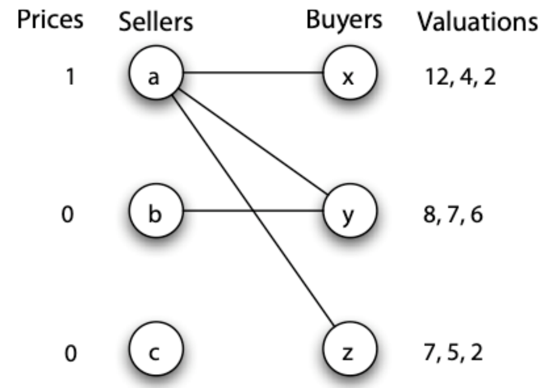
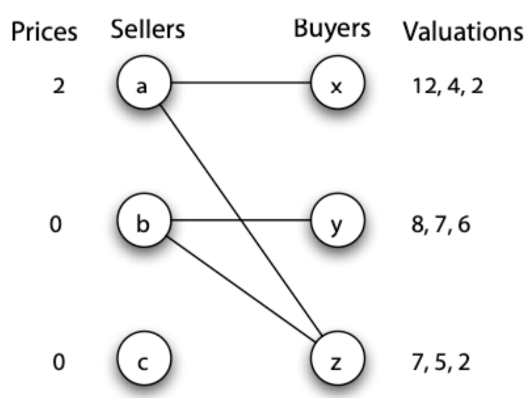


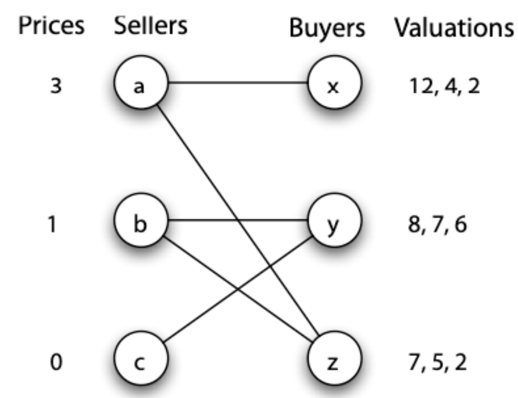
(a) *Start of first round*



(b) *Start of second round*



(c) *Start of third round*



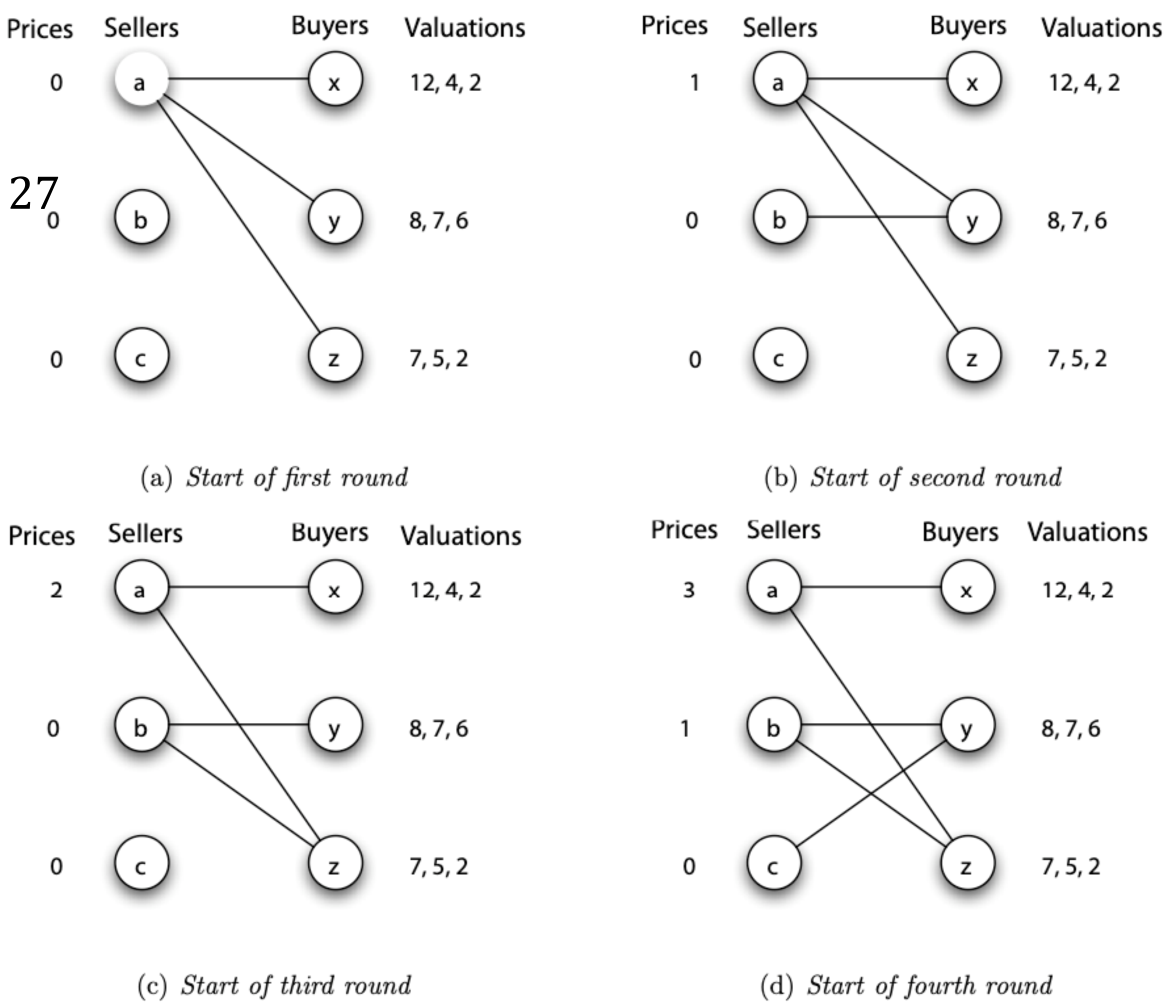
(d) *Start of fourth round*

Existence of the market clearing price

- This procedure stops after finite rounds.
- Potential energy:
  - Potential energy of a buyer: maximum payoff under current prices
  - Potential energy of a seller: current price
  - Potential energy of the auction: sum of potentials of all buyers and sellers
- Potential energy at the begin: P≥ 0
- Potential energy at the start of each round at least zero
- Potential energy only changes when the prices change
  - Price reduction: no change
- Price probing (S: constricted buyer set):
  - Each seller potential goes up by one unit
  - Each buyer potential goes down by one unit
  - But $S \geq N(S)$

Existence of the market clearing price

$P_s: 0$
$P_b: 12 + 8 + 7 = 27$
$P_t: 27$



(a) *Start of first round*

(b) *Start of second round*

(c) *Start of third round*

(d) *Start of fourth round*

$P_s: 1$
$P_b: 11+7+6=24$
$P_t: 25$

$P_s: 2$
$P_b: 10+7+5=22$
$P_t: 24$

$P_s: 4$
$P_b: 9+6+4=19$
$P_t: 23$

Relationship with auction?

Summary

- Perfect matching v.s. constricted set
- Alternating BFS -> perfect matching
  - Augmented path || constricted set
- Existence of the market clearing price
- Optimality of the market clearing price