# Autocorrelation

$y_t = \beta_1 + \beta_2 x_t + u_t$ , $cov(u_i, u_j) \neq 0$ , $i \neq j$

↳ Gauss-Markov is ruined.

**1-st order autocorr.**

$u_t = \rho u_{t-1} + \varepsilon_t$

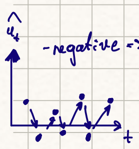
$-1 < \rho < 1$

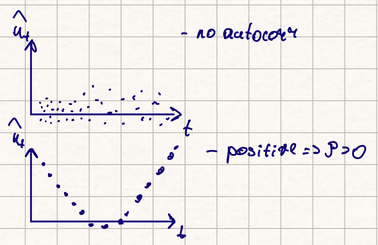
**2-nd order autocorr**

$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t$

* How to test?

**1. Graphs**




=> Estimators are not efficient!

**② Durbin-Watson test.** $e_t = \hat{\varepsilon}_t$

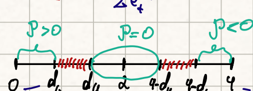
exclud const!

$$\boxed{d = \frac{\sum_2^T (e_t - e_{t-1})^2}{\sum_1^T e_t^2}} \sim DW \text{ distribution}(n, k)$$

$\rho = 1 \Rightarrow e_t = 1 \cdot e_{t-1} + \varepsilon_t \Rightarrow e_t \approx e_{t-1} \Rightarrow e_t - e_{t-1} \approx 0 \Rightarrow \boxed{d \approx 0}$

$\rho = -1 \Rightarrow e_t = -1 \cdot e_{t-1} + \varepsilon_t \Rightarrow e_t \approx -e_{t-1} \Rightarrow e_t - \underbrace{e_{t-1}}_{-e_t} \approx 2e_t \Rightarrow d = \frac{\sum (2e_t)^2}{\sum e_t^2} \approx 4$

$\rho = 0 \Rightarrow \boxed{d} = \frac{\sum (e_t - e_{t-1})^2}{\sum e_t^2} = \frac{\sum e_t^2 - 2\underbrace{\sum e_t e_{t-1}}_{\approx 0} + \sum e_{t-1}^2}{\sum e_t^2} = \frac{\sum e_t^2 + \sum e_{t-1}^2}{\sum e_t^2} \approx \boxed{2}$




$$\boxed{d} = \frac{\sum_2^T (e_t - e_{t-1})^2}{\sum_{t=1}^T e_t^2} = \frac{\sum_2^T e_t^2 - 2\sum_2^T e_t e_{t-1} + \sum_2^T e_{t-1}^2}{\sum_1^T e_t^2} \approx \frac{2\sum_2^T e_t^2 - 2\sum_2^T e_t e_{t-1}}{\sum_1^T e_t^2} = \frac{2\sum_2^T e_t^2 - 2\hat{\rho}\sum_2^T e_{t-1}^2}{\sum_1^T e_t^2} =$$

$\sum_2^T e_t e_{t-1} = \hat{\rho} \cdot e_{t-1}$

$$\frac{\cancel{2\sum e_t^2}(1-\hat{\rho})}{\cancel{\sum e_t^2}} = 2(1-\hat{\rho})$$

$= \boxed{2(1-\hat{\rho})}$

**Limitations:**

1. Only 1-st order autocorr.
2. There should be a constant!
3. No lags of $y_{t-1}$ in regression.

$H_0: cov(\varepsilon_i, \varepsilon_j) = 0, \ i \neq j$

$H_1: cov(\varepsilon_i, \varepsilon_j) \neq 0, \ i \neq j$

**③ Asymptotic test (h-Durbin).** T is large: $T \to \infty$

$h = \hat{\rho}\sqrt{T} \sim N(0,1)$

$\hat{\rho} = (1 - 0.5d) \cdot \sqrt{T}$

1. 1 lag variable $y_{t-1}$!
2. Maybe without const
3. Only 1-st order.

**④ Breusch-Godfrey Test.**

1. Original model.
2. $e_t = \lambda_0 + \rho e_{t-1} \rightarrow R^2 \Rightarrow \boxed{BG = R^2(T-1) \sim \chi^2(1)}$

order of autocorr.

**How to repair?**

* St. errors in Newey-West form
* Cochrane-Orcutt Method

Prais-Winsten.

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**Problem 1**

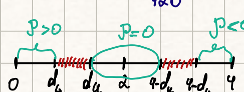
$RSS = 120$, $\sum_{t=2}^{100} e_t e_{t-1} = -50$ , $\sum_{t=1}^{99} e_t^2 = 116$; $\sum_{t=2}^{100} e_t^2 = 119$.

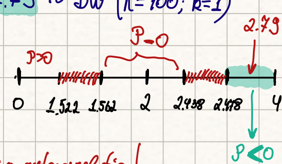
$$\boxed{d = \frac{\sum_2^T (e_t - e_{t-1})^2}{\sum_1^T e_t^2}} = \frac{\overbrace{\sum_2^{100} e_t^2}^{119} - 2 \cdot \overbrace{\sum_2^{100} e_t e_{t-1}}^{-50} + \overbrace{\sum_2^{100} e_{t-1}^2}^{116 \, *}}{\underbrace{\sum_1^{100} e_t^2}_{RSS}} \ (=)$$

$$* \ \sum_{t=2}^{100} e_{t-1}^2 = e_1^2 + e_2^2 + e_3^2 + \ldots e_{99}^2$$
$$\sum_{t=1}^{99} e_t^2 = e_1^2 + e_2^2 + \ldots + e_{99}^2$$

the same! $\Rightarrow \sum_2^{100} e_{t-1}^2 = \sum_{t=1}^{99} e_t^2 = 116$

$$(=) \ \frac{119 - 2 \cdot (-50) + 116}{120} = 2.79 \sim DW(n = 100; k = 1)$$




=> $H_0$ is rejected. There is a negative autocorrelation!

$d = 2 \cdot (1 - \hat{\rho}) \Rightarrow 1 - \hat{\rho} = \frac{d}{2} \Rightarrow \hat{\rho} = 1 - \frac{1}{2}d = 1 - \frac{1}{2} \cdot 2.79 \approx -0.395$

**Problem 2.**

$y_t = \alpha + \beta_1 x_t + \varepsilon_t \rightarrow \hat{\varepsilon}_t$.

$\Rightarrow \hat{\varepsilon}_t = \lambda_0 + \rho_1 \cdot \hat{\varepsilon}_{t-1} + \rho_2 \cdot \hat{\varepsilon}_{t-2} + u_t \rightarrow R^2$

$\Rightarrow BG = R^2(T-1) \sim \chi^2(2)$ $R^2 = 0.8$

$\Rightarrow BG = 0.8(101-1) = 80 > \chi^2_{cr}$

=> $H_0$ is rejected

=> Autocorrelation!