Class 1. Linear Algebra

Advanced Econometrics I

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Problem 1

Let vectors be given $a = (3, -4, 12)^T$, $b = (7, 4, 3)^T$. Find: a) $2a$, b) $a + b$, c) $2a - 3b$, d) scalar product of a and b, e) length of the vector a.

Problem 2

Matrices are given:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & 4 \\ 1 & 6 \\ 0 & 5 \end{pmatrix}.$$

Find a) $3A$, b) $3A + 5B$, c) AC, d) CA.

Problem 3

Matrix X is given:

$$X = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 5 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

(a) Write X^T – the transpose of X;

(b) Find $A = X^T X$ and $B = XX^T$;

(c) Make sure that the inverse of A is $A^{-1} = (X^T X)^{-1} = \begin{pmatrix} 1.7 & -0.47 \\ -0.47 & 0.147 \end{pmatrix}$;

(d) Find traces of A and B;

(e) Find ranks of X^T, X, A, A^{-1} and B. Which of them is a full rank matrix?

(f) Find the determinant of matrix A;

(g) Write and solve the characteristic equation of A as $det(A - \lambda I) = 0$.

Problem 4

Matrices are given:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 1 \\ 2 & 4 & 5 \end{pmatrix}.$$

Find a) $detA$, b) $detB$, c) A^{-1}, d) B^{-1}, e) trace of matrix B.

Problem 5

Find X^TX if a) $X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$, b) $X = \begin{pmatrix} 1 & x_{12} & x_{13} \\ \vdots & \vdots & \vdots \\ 1 & x_{n2} & x_{n3} \end{pmatrix}$, c) $X = \begin{pmatrix} 1 & x_{12} & \dots & x_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n2} & \dots & x_{nk} \end{pmatrix}$.

Problem 6

Prove that the following matrices are symmetrical (X is a full rank matrix):
a) X^TX, b) $P(X) = X(X^TX)^{-1}X^T$, c)$\pi = ii^T/i^Ti$, where i is a column vector.

Problem 7

Find rank of the matrices: a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, b) $C = \begin{pmatrix} 1 & x_{12} & \dots & x_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n2} & \dots & x_{nk} \end{pmatrix}$.

Problem 8*

Prove that if rank of matrix X is k, then X^TX has a rank k as well.

Problem 9

Find eigenvalues and eigenvectors of matrices: a) $A = \begin{pmatrix} 1.3 & -0.1 \\ 0.8 & 0.4 \end{pmatrix}$, b) $B = \begin{pmatrix} 3 & -2 \\ 0.5 & 1 \end{pmatrix}$, c) $C = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Problem 10*

Prove that determinant of symmetrical matrices equals to the production of its eigenvalues.

Problem 11*

Prove that matrix X^TX is a non-negative definite matrix.

Problem 12

Consider the system of equations. Represent it in a matrix notation $Rz = q$, where $z = (x, y)^T$. What are the expressions for matrix R and vector q?

$$\begin{cases} 3x + 4y = 8 \\ 6x - 5y = -2 \end{cases}.$$

Problem 13

Write the system of equations, represented in the following matrix. Represent it in a matrix notation $Rz = q$, where $z = (x, y)^T$.

$$A = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 8 \\ 4 & 5 & 1 & 10 \\ 0 & 4 & 6 & 7 \end{array}\right)$$

Problem 14

Find the derivative of the function S with respect to b. (Verbeek ch. 2.1.4.)

$$S = Y^TY - b^TX^TY - Y^TXb + b^TX^TXb,$$

where Y is a $(n \times 1)$ vector, X is a $(n \times k)$ matrix and b is a vector of size $(k \times 1)$.

Matrix differentiation hints & excercises

Show that for vectors c and x the derivative over vector x is[1]

$$\frac{\partial c^T x}{\partial x} = c.$$

Prove that for a vectorial function Ax the derivative over vector x is

$$\frac{\partial Ax}{\partial x} = A^T.$$

Prove that for a symmetric matrix A it holds that

$$\frac{\partial x^T Ax}{\partial x} = 2Ax.$$

Should-know Issues in Matrix Algebra

A seminar on the main topics is not enough.You need to know the following topics:

- a vector
- a matrix: square, symmetric, diagonal, identity, scalar, transpose
- linear combinations of matrices
- matrix multiplication, inverse matrix
- an idempotent matrix
- the determinant of a matrix
- singular matrix
- the trace of a matrix
- the rank of a matrix
- an eigenvector and eigenvalue, characteristic equation
- the quadratic form and definite matrices.

[1] In this course we use a denominator layout.