Simple and Interesting Problems on Normals to Parabolas

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Problems

Problem 1. Let l be a normal chord (the part of a normal line that lies inside the parabola) of the parabola $P : y = x^2$.

(1) Find the minimum value of the length of l.

(2) Find the minimum value of the area of the region enclosed by P and l.

(3) Find the minimum value of the volume swept by the region enclosed by P and l when rotated once around the axis l.

Problem 2. Draw three normal lines from point $Q(a, b)$ to the parabola $P : y = x^2$, and let their feet be A, B, C.

(1) Illustrate the range of existence of Q.

(2) Show that the four points A, B, C, O lie on the same circle.

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Solutions

Problem 1.

(1) The tangent vector at point $A(t, t^2)$ on P is $(1, 2t)$, so the equation of the normal at A is

$(x-t, y-t^2)\cdot(1,2t)=0$

$\Leftrightarrow\ x+2ty-2t^3-t=0$

Let B be the other intersection point with P. When $t=0$, B does not exist, so $t\neq 0$. Substituting $y=x^2$ into the above equation,

$$2tx^2+x-2t^3-t=0$$
$$\Leftrightarrow\ (x-t)(2tx+2t^2+1)=0$$
$$\Leftrightarrow\ x=t,\ -t-\frac{1}{2t}\ \ (\because t\neq 0)$$

Thus, $B\left(-t-\frac{1}{2t}, t^2+1+\frac{1}{4t^2}\right)$. Therefore,

$$AB^2=\left\{\left(-t-\frac{1}{2t}\right)-t\right\}^2+\left\{\left(t^2+1+\frac{1}{4t^2}\right)-t^2\right\}^2$$
$$=4t^2+\frac{3}{4t^2}+\frac{1}{16t^4}+3$$

Let $T=t^2$. Since $t\neq 0$, we have $T>0$, and

$$AB^2=4T+\frac{3}{4T}+\frac{1}{16T^2}+3$$

Let this be $f(T)$. Then

$$f'(T)=4-\frac{3}{4T^2}-\frac{1}{8T^3}=\frac{1}{8T^3}(2T-1)(4T-1)^2$$

Therefore, the table of variation for $f(T)$ when $T>0$ is as follows.

T	(0)	$\cdots$	$\frac{1}{4}$	$\cdots$	$\frac{1}{2}$	$\cdots$
$f'(T)$		$-$	0	$-$	0	$+$
$f(T)$		$\searrow$		$\searrow$		$\nearrow$

Therefore, $f(T)$ attains its minimum value $\frac{27}{4}$ when $T=\frac{1}{2}$. Hence, the minimum value of the length of l is $\boldsymbol{\frac{3\sqrt{3}}{2}}$.

(2) From (1), $t\neq 0$, and by symmetry, it suffices to consider $t>0$. Therefore, the area of the region enclosed by P and l is

$$-\int_{-t-\frac{1}{2t}}^{t}\left(x+t+\frac{1}{2t}\right)(x-t)\,dx$$
$$=\frac{\left\{t-\left(-t-\frac{1}{2t}\right)\right\}^3}{6}\quad(\because 1/6\text{ formula})$$
$$=\frac{\left(2t+\frac{1}{2t}\right)^3}{6}$$
$$\geqq\frac{\left(2\sqrt{2t\cdot\frac{1}{2t}}\right)^3}{6}=\frac{4}{3}\quad(\because\text{AM-GM inequality})$$

Therefore, the required minimum value is $\boldsymbol{\frac{4}{3}}$ when $t=\frac{1}{2}$.

(3) As in (2), it suffices to consider $t>0$. The distance from point $C(u,u^2)$ $\left(-t-\frac{1}{2t}\leqq u\leqq t\right)$ on P to l is $\frac{u+2tu^2-2t^3-t}{\sqrt{1+4t^2}}$ by the point-to-line distance formula. Therefore, since the slope of the normal is $-\frac{1}{2t}$, the volume swept by the part $u\leqq x\leqq u+\Delta u$ of the region enclosed by P and l when rotated can be approximated by the volume of a cylinder with height $\Delta u\cdot\sqrt{1+\frac{1}{4t^2}}$ and radius $\frac{u+2tu^2-2t^3-t}{\sqrt{1+4t^2}}$. Therefore, the swept volume is

$$\int_{-t-\frac{1}{2t}}^{t}\pi\left(\frac{u+2tu^2-2t^3-t}{\sqrt{1+4t^2}}\right)^2\cdot\sqrt{1+\frac{1}{4t^2}}\cdot du$$
$$=\frac{\pi}{2t\sqrt{1+4t^2}}\int_{-t-\frac{1}{2t}}^{t}(u+2tu^2-2t^3-t)^2du$$
$$=\frac{\pi}{2t\sqrt{1+4t^2}}\int_{-t-\frac{1}{2t}}^{t}(u-t)^2(2tu+2t^2+1)^2du$$
$$=\frac{2\pi t}{\sqrt{1+4t^2}}\int_{-t-\frac{1}{2t}}^{t}(u-t)^2\left(u+t+\frac{1}{2t}\right)^2du$$
$$=\frac{2\pi t}{\sqrt{1+4t^2}}\cdot\frac{1}{30}\left\{t-\left(-t-\frac{1}{2t}\right)\right\}^5\quad(\because 1/30\text{ formula})$$
$$=\frac{\pi t}{15\sqrt{1+4t^2}}\cdot\left(2t+\frac{1}{2t}\right)^5$$
$$=\frac{\pi}{15\sqrt{2}}\cdot t^{\frac{1}{2}}\left(2t+\frac{1}{2t}\right)^{\frac{9}{2}}$$
$$=\frac{\pi}{15\sqrt{2}}\sqrt{t\left(2t+\frac{1}{2t}\right)^9}$$

Let $g(t)=t\left(2t+\frac{1}{2t}\right)^9$ and find its minimum value. Using the product rule for differentiation,

$$g'(t)=\left(2t+\frac{1}{2t}\right)^9+t\cdot 9\left(2t+\frac{1}{2t}\right)^8\cdot\left(2-\frac{1}{2t^2}\right)$$
$$=\left(2t+\frac{1}{2t}\right)^8\left(20t-\frac{4}{t}\right)$$
$$=\left(2t+\frac{1}{2t}\right)^8\frac{4}{t}\left(5t^2-1\right)$$

Therefore, the table of variation for $g(t)$ when $t>0$ is as follows.

t	(0)	$\cdots$	$\frac{1}{\sqrt{5}}$	$\cdots$
$g'(t)$		$-$	0	$+$
$g(t)$		$\searrow$		$\nearrow$

Therefore, $g(t)$ attains its minimum value when $t=\frac{1}{\sqrt{5}}$:

$$\frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}+\frac{\sqrt{5}}{2}\right)^9=\frac{1}{\sqrt{5}}\left(\frac{9}{2\sqrt{5}}\right)^9=\frac{9^9}{2^9 5^5}$$

Hence, the minimum value of the swept volume is

$$\frac{\pi}{15\sqrt{2}}\sqrt{\frac{9^9}{2^9 5^5}}=\boldsymbol{\frac{6561\sqrt{5}}{20000}\pi}.$$

Problem 2.

(1) Let the point of tangency be (t, t^2). The tangent vector at this point is $(1, 2t)$, so the equation of the normal is

$$(x - t, y - t^2) \cdot (1, 2t) = 0$$
$$\Leftrightarrow \quad x + 2ty - 2t^3 - t = 0$$

When this passes through $Q(a, b)$, substituting gives

$$-2t^3 + (2b - 1)t + a = 0$$

Let the left side be $f(t)$. Since the point of tangency and the normal correspond one-to-one on parabola P, we need to consider the condition for the cubic equation $f(t) = 0$ to have three distinct real solutions.

$$\begin{aligned} f'(t) &= -6t^2 + 2b - 1 \\ &= -6\left(t + \sqrt{\frac{2b-1}{6}}\right)\left(t - \sqrt{\frac{2b-1}{6}}\right) \end{aligned}$$

When $b \geqq \dfrac{1}{2}$, the table of variation for $f(t)$ is as follows.

t	$\cdots$	$-\sqrt{\dfrac{2b-1}{6}}$	$\cdots$	$\sqrt{\dfrac{2b-1}{6}}$	$\cdots$
$f'(t)$	$+$	0	$-$	0	$+$
$f(t)$	$\nearrow$		$\searrow$		$\nearrow$

Therefore, the condition for $f(t) = 0$ to have three distinct real solutions is

$$f\left(-\sqrt{\frac{2b-1}{6}}\right) f\left(\sqrt{\frac{2b-1}{6}}\right) < 0$$
$$\Leftrightarrow \quad \left\{a - \frac{\sqrt{6}}{9}(2b-1)^{\frac{3}{2}}\right\} \cdot \left\{a + \frac{\sqrt{6}}{9}(2b-1)^{\frac{3}{2}}\right\} < 0$$
$$\Leftrightarrow \quad a^2 - \frac{2}{27}(2b-1)^3 < 0$$
$$\Leftrightarrow \quad (2b-1)^3 > \frac{27}{2}a^2$$

Since $(RHS) \geqq 0$, we have $(LHS) > 0$. Taking the $\dfrac{1}{3}$ power of both sides,

$$2b - 1 > \frac{3}{\sqrt[3]{2}}a^{\frac{2}{3}}$$
$$\Leftrightarrow \quad b > \frac{3}{\sqrt[3]{16}}a^{\frac{2}{3}} + \frac{1}{2}$$

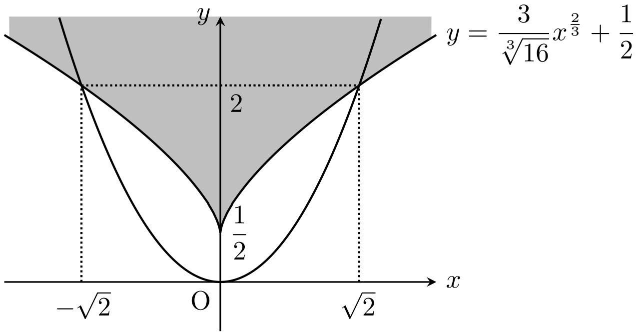
Therefore, let $g(x) = \dfrac{3}{\sqrt[3]{16}}x^{\frac{2}{3}} + \dfrac{1}{2}$. The range of existence of Q is the region above the graph of $y = g(x)$. Since $g(x) = g(-x)$, $g(x)$ is an even function. Considering $x > 0$,

$$g'(x) = \frac{2}{\sqrt[3]{16}}x^{-\frac{1}{3}} > 0$$
$$g''(x) = -\frac{2}{3\sqrt[3]{16}}x^{-\frac{4}{3}} < 0$$

Therefore, the table of variation for $g(x)$ is as follows, satisfying $b \geqq \dfrac{1}{2}$.

x	0	$\cdots$	∞
$g'(x)$		$+$	
$g''(x)$		$-$	
$g(x)$	$\frac{1}{2}$	$\curvearrowright$	∞

From the above, the range of existence of Q is the shaded region in the figure below. (Boundary not included)



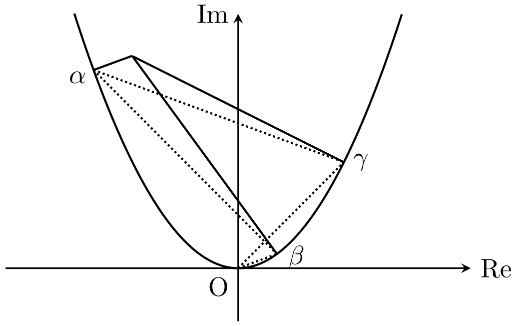
$\boxed{\text{Note}}$ The curve $y = g(x)$ is the **evolute** of P.

(2) Let the x-coordinates of A, B, C be t_1, t_2, t_3 respectively ($t_1 < t_2 < t_3$). Then t_1, t_2, t_3 are the three solutions of $f(t) = 0$. By Vieta's formulas,

$$t_1 + t_2 + t_3 = 0 \qquad \cdots\cdots\cdots\cdots\cdots \text{※}$$

In the complex plane with origin at O, let the complex numbers corresponding to A, B, C be α, β, γ respectively.

$$\alpha = t_1 + t_1^2 i, \quad \beta = t_2 + t_2^2 i, \quad \gamma = t_3 + t_3^2 i$$



When $\beta = 0$ or $\gamma = 0$, A, B, C, O lie on the same circle. For the case when $\beta \neq 0$ and $\gamma \neq 0$, let $z_1 = \dfrac{\gamma - \alpha}{\beta - \alpha}$ and $z_2 = \dfrac{\gamma - 0}{\beta - 0} = \dfrac{\gamma}{\beta}$. Since $z_2 \neq 0$,

$$z_1 = \frac{(t_3 + t_3^2 i) - (t_1 + t_1^2 i)}{(t_2 + t_2^2 i) - (t_1 + t_1^2 i)} = \frac{(t_3 - t_1)\{1 + (t_3 + t_1)i\}}{(t_2 - t_1)\{1 + (t_2 + t_1)i\}}$$
$$z_2 = \frac{t_3 + t_3^2 i}{t_2 + t_2^2 i}$$

From ※,

$$z_1 = \frac{(t_3 - t_1)(1 - t_2 i)}{(t_2 - t_1)(1 - t_3 i)}$$

Therefore, since $z_2 \neq 0$,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(t_3 - t_1)(1 - t_2 i)(t_2 + t_2^2 i)}{(t_2 - t_1)(1 - t_3 i)(t_3 + t_3^2 i)} \\ &= \frac{(t_3 - t_1)(t_2 + t_2^3)}{(t_2 - t_1)(t_3 + t_3^3)} \in \mathbb{R} \end{aligned}$$

Therefore, from the figure above,

$$\arg z_1 = \arg z_2 \quad \Leftrightarrow \quad \arg\left(\frac{\gamma - \alpha}{\beta - \alpha}\right) = \arg\left(\frac{\gamma - 0}{\beta - 0}\right)$$

By the converse of the inscribed angle theorem, A, B, C, O lie on the same circle. $\square$

