Note on the c theorem

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In this note, we review the c theorem in various dimensions and the insight the quantum information theory can shed light on it.

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I. INTRODUCTION

For renormalization group flow, we can start from a UV fixed point which is a conformal field theory. By adding the relevant deformation term to the action, it can flow to the IR fixed point. The RG can only flow from UV to IR . It is interesting if we could assign some height function to this RG flow and the monitonic property of this height function means the direction of RG flow. In 2d CFT, Zamolodchikov has found some strongest form of the function which decreasing along the RG flow moreover it generates the potential for the RG flow. It is called c theorem because in the fixed point , Zamolodchikov's c function reduce to the central charge of CFT at the fixed point. In higher even dimension, there are also various candidates as the c function but no strongest form. In three dimension, it is conjectured that the free energy may serve as the c function, which is called F theorem. The entanglement entropy can also serves as the c function and it is an interesting ongoing research direction.

For the $\mathcal{C}$ function, it has several forms from weak to strong.

1. (weak version)The fixed point $\mathcal{C}$ decrease under the RG flow. $\mathcal{C}_{UV} > \mathcal{C}_{IR}$

2. (strong version) $\mathcal{C}$ is decreasing along the entire RG flow, we can compute the $\mathcal{C}$ value away from CFT

3. (strongest version) $\mathcal{C}$ is the potential generates the RG flows

$$\beta_i(g) = G_{ij}(g)\frac{\partial \mathcal{C}}{\partial g_j} \tag{1}$$

II. C THEOREM IN 2D CFT

In two-dimensional renormalizable QFTs, there exists Zamolodchikov's c-function $c(g_i, \mu)$ that depends on a set of dimensionless coupling constants $\{g_i\}$ and the energy scale $\mu(= \Lambda\, e^{-t})$, satisfying the following properties:

1. It takes the same value as the central charge c of the CFT corresponding to each fixed point of RG flows:

$$c(g_i, \mu)|_{\mathrm{CFT}} = c\ . \tag{2}$$

2. It monotonically decreases along any RG flow:

$$\frac{d\,c(g_i,\mu)}{dt} = -\mu \frac{d\,c(g_i,\mu)}{d\mu} \leq 0\ . \tag{3}$$

3. It is stationary only at the fixed points:

$$\left.\frac{\partial\,c(g_i,\mu)}{\partial g_i}\right|_{\mathrm{CFT}} = 0\ . \tag{4}$$

Hence, Zamolodchikov's c-theorem falls into the strongest version in the terminology introduced in the previous section The Zamolodchikov's c-function can be built explicitly from the two-point correlation functions of the stress-energy tensor in two dimensions, and the monotonicity follows from the reflection positivity of the correlator.

Apart from this c function, there are also entropic c function in 2d CFT. We change R to trigger the RG flow rather than μ . The function $c_E(R)$ should have following properties: first it should coincides with the central charge c in the fixed point

$$c_E(R)|_{CFT} = c$$

and second it is the monotonically decreasing function under RG flow

$$c'_E(R) \leqslant 0$$

In two dimension, the entanglement entropy takes the form

$$S(R)|_{CFT} = \frac{c}{3} log \frac{R}{\epsilon}$$

, the function satisfy the first condition is

$$c_E(R) = 3RS'(R) \tag{5}$$

So we prove the second condition, which was done by Casini Recalling the following picture, ABC is of length R, B is of length r. AB and BC is of length $\sqrt{rR}$ which can be shown using Minkowski metric. We can apply the strong subadditivity of entropy

$$S(R) + S(r) \leqslant 2S(\sqrt{rR}) \tag{6}$$

In the limit $r \to R$, the strong subadditivity reduces to

$$\frac{c'_E(R)}{3} = S'(R) + RS''(R) \leqslant 0 \tag{7}$$

It needs to be said that the proof only use the entropy inequatlity and does not refer to the conformal symmetry so it satisfies for any quantum field theory. But it is different to the Zamolodchikov c function and whether there is strongest form of c theorem remains open.

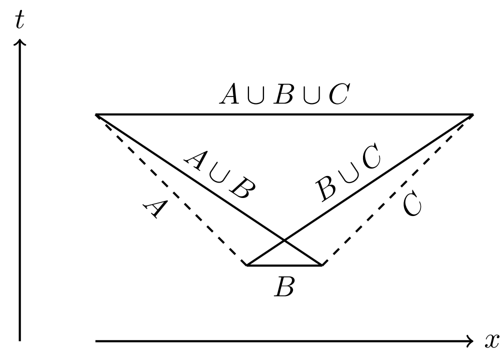

FIG. 1. A proof of the entropic c-theorem. Two intervals A and C are on the light rays $t = \pm x$ while an interval B is on a time slice $t = 0$.

III. F THEOREM IN 3D

It is interesting to look for the counterpart of $\mathcal{C}$ function beyond $d = 2$, there are various choices

1. Coefficient of the thermal free energy density $f_{thermal}$ at finite temperature defined by $f_{Thermal} \sim C_{thermal} T^d$

2. Coefficient C_T for the two point function of the stress energy tensor

3. Central charge A and B for the conformal anomaly. But we should note that only the even dimension has conformal anomaly, so this can not define the c function in odd dimension

In three dimension, however we see the three possibility all can not be suitable.

The first possibility is studied by Subir Sachdev by investigating three dimensional O(N) vector model in large N limit. It is a counter-example to this case because

$$\mathcal{C}_{Thermal}(critical) \leqslant \mathcal{C}_{Thermal}(goldstone) \tag{8}$$

Goldstone phase is in the IR due to the spontaneous symmetry breaking.

The second option is ruled out by Nishioka and Yonekura using the $\mathcal{N} = 2$ supersymmetric field theory,where two point function can be calculated using supersymmetric localization technique. They find this as a counter example.

Alternatively, a proposal was made in a different form from the holographic viewpoint based on the extremization principle of the supersymmetric partition function, which is now known as the F-theorem,

In three-dimensional QFTs, there exists a function $\mathcal{F}(g_i, \mu)$ on the theory space satisfying the following properties:

1. It takes the same value as the sphere free energy F of the CFT corresponding to each fixed point of RG flows,
$$\mathcal{F}(g_i, \mu)|_{\rm CFT} = F \ , \tag{9}$$
where $F = -\log|Z[S^3]|$ and $Z[S^3]$ is the (renormalized) Euclidean partition function on S^3.

2. It is a monotonically decreasing function under any RG flow,
$$\frac{dF}{dt} = -\mu\frac{dF}{d\mu} \leqslant 0 \ . \tag{10}$$

This statement can be regarded as a variant of the third option (c) since we can extract the type A central charge of the conformal anomaly from the sphere partition function $A \propto \log Z[S^d]$ in even dimensions.

There are also some entropic counterpart in for the F theorem. $\mathcal{F}$ function should satisfy the following

1. $\mathcal{F}(R)$ coincides with the sphere free energy F at fixed point of RG flow
$$\mathcal{F}(R)|_{CFT} = F \tag{11}$$

2. $\mathcal{F}(R)$ is the monotonically decreasing function $F'(R) \leqslant 0$

Recall the relation between the sphere free energy F and the entanglement entropy $S(R)$
$$S(R)|_{CFT} = \alpha\frac{2\pi R}{\epsilon} - F \tag{12}$$
In three dimension, the entanglement entropy is UV divergent and the leading order divergence is proportional to $O(1/\epsilon)$ by area law. In order to regularize, Hong Liu and Mezei introduce the renormalized entanglement entropy
$$\mathcal{F}(R) = (R\partial_R - 1)S(R) \tag{13}$$

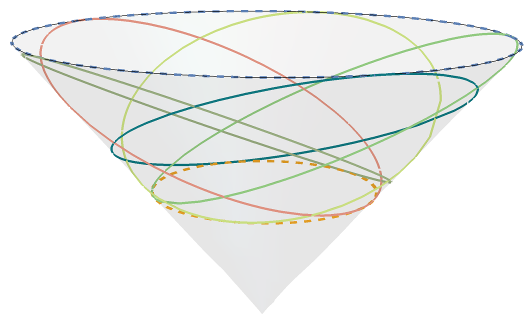

FIG. 2. $N = 5$ boosted disks of radii $\sqrt{rR}$ touching on the null cone in gray color. The dashed curves are circles of radii R and r at constant time slices $t = R$ and r, respectively.

Which serves as an $\mathcal{F}$ function. Casini and Huerta proves the monotonicity using the strong subadditivity of the entropy in Lorentz invariant field theory.

By repeatedly using the strong subadditivity of the entropy

$$S_{\cup_i X_i} + S_{\cup_{\{i,j\}} X_{ij}} + S_{\cup_{\{i,j,k\}} X_{ijk}} + \cdots + S_{\cap_i X_i} \leqslant \sum_i S_{X_i} \tag{14}$$

By dividing by N of the both sides

$$\frac{1}{\pi}\int_0^\pi d\theta\, S\left(\frac{2rR}{R+r-(R-r)\cos\theta}\right) \leqslant S\left(\sqrt{rR}\right) \ . \tag{15}$$

Letting $R = r - \epsilon$, one can find $S''(R) \leqslant 0$ at order $O(\epsilon^2)$

$$\mathcal{F}'(R) = RS''(R) \leqslant 0 \tag{16}$$

So we find the F function is monotonic under the RG flow.

IV. A THEOREM IN 4D

In 4d , the a function arising from conformal anomaly can be used $a_{UV} > a_{IR}$

For $d = 4$, the conformal anomaly reads

$$\langle T^\mu_\mu \rangle = -aE_4 + cW^2 \tag{17}$$

Where the Euler density is

$$E_4 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{18}$$

The weyl tensor is

$$W^2 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \tag{19}$$

a is decreasing and it is proposed as a function, while for c we can find some counterexample of it. The proof is given through the scattering of the dilaton field , the unitarity of the scattering amplitude is essential for the proof.

We expect that there may be similar proof in the $d = 4$ case, there are some obstacles to the higher dimensions.

1. Strong subadditivity becomes trivial due to logarithmic divergence coming from the singular surface

2. Wiggly sphere and round sphere has a finite mismatch in taking number N of the boosted sphere to infinity

A regularized measure is defined to overcome this difficulty

$$\Delta S_T(R) = S_T(R) - S_{UV}(R) \tag{20}$$

Define the new entanglement measure

$$\mathcal{S}_T(R) = [R\partial_R - (d-2)]\Delta S_T(R) \tag{21}$$

$\Delta S_T(R)$ also satisfies the strong subadditivity because of the Markov property of the CFT vacuum. Moreover, this definition removes the UV divergence and the finite mismatch.

Under the same argument, we can prove

$$\mathcal{S}'_T(R) \leqslant 0 \tag{22}$$

We can see the implication of this expression, for 2d and 3d , it is simple to check that the inequality reduces to the weak form of c-theorem and F theorem, while the entanglement entropy across the sphere in 4d CFT is

$$S(R)|_{CFT} = a\frac{4\pi R^2}{\epsilon^2} - alog\frac{R}{\epsilon} \tag{23}$$

Evaluate this in the fixed point

$$\mathcal{S}'_{IR}(R) = 2(a_{IR} - a_{UV})\frac{1}{R} \leqslant 0 \tag{24}$$

It reduces to the weak version of the a theorem.

For dimension $d > 4$, there are conjectured generalized F theorem reads

$$\tilde{F} = sin(\frac{\pi d}{2}) logZ[S^d] \tag{25}$$

$\tilde{F}_{UV} \geqslant \tilde{F}_{IR}$. By conformal map, there are relation between S^{d-2} entanglement entropy and S^d partition function

$$S_1 = logZ[S^d] \tag{26}$$

So we have

$$\tilde{F}_{CFT} = sin(\frac{\pi d}{2}) S_{CFT}(R) \tag{27}$$

for some dimension such as $2 < d < 4$, this is true, but for some dimension $4 < d < 6$, the result is opposite $\tilde{F}_{IR} \geqslant \tilde{F}_{UV}$

V. HOW TO SEE C THEOREM FROM HOLOGRAPHIC SIDE

In this section, there is another perspective towards intepreting c theorem from the holographic gravity side. Take the Einstein gravity for example

$$I = \frac{1}{2l_p^{d-1}} \int d^{d+1}x\sqrt{-g}(R + \mathcal{L}_{matter}) \tag{28}$$

The matter theory has various stationary point and at each point it is an AdS space and thus a critical point. There is a RG flow between different critical point.

$$ds^2 = e^{2A(r)}(-dt^2 + dx_{d-1}^2) + dr^2 \tag{29}$$

Define the a function

$$a(r) = \frac{\pi^{d/2}}{\Gamma(d/2)(l_p^{d-1}A'(r)^{d-1})}$$

, which reduces to the a charge for CFT case and

$$a'(r) = -\frac{(d-1)\pi^{d/2}}{\Gamma(d/2)(l_p^{d-1}A'(r)^d)} A''(r) = -\frac{\pi^{d/2}}{\Gamma(d/2)(l_p^{d-1}A'(r)^d)}(T_t^t - T_r^r) \tag{30}$$

By the null energy condition $a'(r) \geqslant 0$, so the function $a(r)$ decrease from UV to IR and act as an c function for the RG flow. But in Einstein gravity, $a = c$, so we can not distinguish between a and c, we can add higher derivative terms to know what is the correct choice in 4d CFT.

We can add the Gauss-Bonnet term and Quasi-topological term

$$I = \frac{1}{2l_p^3}\int d^5x\sqrt{-g}[\frac{12\alpha^2}{L^2} + R + \frac{\lambda}{2}L^2\xi_4 + \frac{7\mu}{4}L^4 Z_5] \tag{31}$$

Where ξ_4 is the Gauss-Bonnet term, Z_5 is the quasi-topological term.

Curvature of AdS vacuum $\tilde{L}$ is related to the L by $\tilde{L}^2 = L^2/f_\infty$, f_∞ is given by the following equation

$$\alpha^2 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^2 = 0 \tag{32}$$

The a charge and c charge can be determined

$$a = \pi^2\tilde{L}^3/l_p^3(1 - 6\lambda f_\infty + 9\mu f_\infty^2) \tag{33}$$

$$c = \pi^2\tilde{L}^3/l_p^3(1 - 2\lambda f_\infty - 3\mu f_\infty^2) \tag{34}$$

There is a simple extensions beyond CFT, which reduces to the central charge at fixed point. Flow function is

$$a(r) = \frac{\pi^2}{l_p^3 A'(r)^3}(1 - 6\lambda L^2 A'(r)^2 + 9\mu L^4 A'(r)^2) \tag{35}$$

$$c(r) = \frac{\pi^2}{l_p^3 A'(r)^3}(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^2) \tag{36}$$

$$a'(r) = -\frac{\pi^2}{l_p^3 A'(r)^4}(T_t^t - T_r^r) \geqslant 0 \tag{37}$$

This is 4d case, and it can be easily generalized to the higher even dimensional case. We must incorporate the odd dimensional case and also what this quantity corresponds to in the dual CFT.

If a CFT is placed on the hyperbolic space, ρ_E is negative. If it is heated up to $\rho_E = 0$, the entropy density is

$$s = \frac{2\pi}{\pi^{d/2}}\Gamma(d/2)\frac{a_d*}{L^{d-1}} \tag{38}$$

Bulk geometry is pure AdS

$$ds^2 = \frac{dr^2}{(\frac{r^2}{\tilde{L}^2} - 1)} - (\frac{r^2}{L^2} - 1)dt^2 + r^2 d\Sigma_2^{d-1} \tag{39}$$

The hyperbolic foilation divides the boudary into two halves, and the entropy is the entanglement entropy of this two halves. It is divergent as a function of the UV cutoff $\delta = L^2/\rho_{max}$.

The leading divergence is not universal because it depends on the choice of the regularization. The universal term is

$$S_{univ} = (-)^{d/2-1} 4a_d^* log(L/\delta)(even) \tag{40}$$

$$S_{univ} = (-)^{(d-1)/2} 2\pi a_d^* (odd) \tag{41}$$

Hence the universal term of EE is the c function of the dual CFT for odd and even dimension.