1-(a) What is the size of vector w and y? (10pt)

w : $d+1$

y : n

1-(b) What is the size of matrix A? Write A. (10pt)

$$A\underset{\sim}{w} = \underset{\sim}{y}$$

$(d+1) \times 1 \quad n \times 1$

$\therefore$ size of A : $n \times (d+1)$

1-(c) Let $d+1=n$, then, A becomes a square matrix. Compute the determinant of A. (40pt in total, Derivation: 30pt, Answer: 10pt)

$$\det A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2-x_1 & x_2^2-x_1x_2 & \cdots & x_2^{n-1}-x_1x_2^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n-x_1 & x_n^2-x_1x_n & \cdots & x_n^{n-1}-x_1x_n^{n-2} \end{bmatrix} = \begin{bmatrix} (x_2-x_1) & x_2(x_2-x_1) & \cdots & x_2^{n-2}(x_2-x_1) \\ (x_3-x_1) & x_3(x_3-x_1) & \cdots & x_3^{n-2}(x_3-x_1) \\ (x_4-x_1) & x_4(x_4-x_1) & \cdots & x_4^{n-2}(x_4-x_1) \\ \vdots & & & \\ (x_n-x_1) & x_n(x_n-x_1) & \cdots & x_n^{n-2}(x_n-x_1) \end{bmatrix}$$

이때 square matrix 이므로 $\begin{bmatrix} Ax_1 \\ Bx_2 \\ \vdots \\ m\,x_n \end{bmatrix} = A \times B \times \cdots \times m \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ 임을 이용하면

$$(x_2-x_1)(x_3-x_1)\cdots(x_n-x_1)\begin{bmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & & & \vdots \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}$$

위의 과정을 반복하여 최종적으로

$\sim \begin{bmatrix} 1 & x_{n-1} \\ 1 & x_n \end{bmatrix}$ 이 나오면 $\begin{bmatrix} 1 & x_{n-1} \\ 1 & x_n \end{bmatrix}$ 의 determinant가 $(x_n - x_{n-1})$ 임을 이용해

결론적으로 $\prod_{1 \le i < j \le n} (x_j - x_i)$ 값이 도출된다

1-(d) What is the condition that makes the determinant of A non-zero? (10pt)

The condition that makes the determinant of A non-zero is all x_i $(1 \leq i \leq n)$ are different.

1-(e) Assume that the determinant of A is non-zero, then, what is the solution of linear equation, $A\mathrm{w} = \mathrm{y}$, with respect to w? (10pt)

A는 square 한 invertible matrix 이다 $(\det A \neq 0)$

$\therefore$ A^{-1}이 존재하므로 $A^{-1}A = I$ 임을 이용하자

$$\underbrace{A^{-1}A}_{I} w = A^{-1} y$$

$$\therefore \; w = A^{-1} y$$

2. (20pt)

Suppose that $n > d + 1$. Then, we cannot compute the inverse of A since A is not a square matrix. In this case, how can we solve the linear equation $A\mathbf{w} = \mathbf{y}$?

* Use singular value Decomposition (SVD)

$n > d+1$ so matrix A is rectangular matrix

suppose U: $n \times n$ orthogonal matrix $(AA^T = U(\Sigma\Sigma^T)U^T)$

Σ: $n \times (d+1)$ diagonal matrix

V^T: $(d+1) \times (d+1)$ orthogonal matrix $(A^TA = V(\Sigma^T\Sigma)V^T)$

* U is orthogonal $\rightarrow$ $UU^T = U^TU = I$ $\therefore U^{-1} = U^T$

* $A = U\Sigma V^T$

$$
\begin{aligned}
AA^T &= U\Sigma V^T (U\Sigma V^T)^T \\
&= U\Sigma V^T V \Sigma^T U^T \\
&= U\Sigma\Sigma^T U^T \\
&= U(\Sigma\Sigma^T)U^T = U(\Sigma\Sigma^T)U^{-1}
\end{aligned}
$$

$\therefore$ AA^T를 고유값 분해하면 $U(\Sigma\Sigma^T)U^{-1}$ 이 된다

$\therefore$ U는 AA^T의 고유벡터로 구성된 행렬 (linearly independent)

$\therefore$ $Aw = y$

$w = V\Sigma^{-1}U^Ty$ 로 구한다