Stats105_Hw2

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Problem 1

(1)

Note that

$$x_{0.975} - \mu = \sigma * Z_{0.975}$$

Where need to satisfy $\mu \sim N(\mu_0, \tau_0)$

$$\tau = 1/\sigma^2$$

$$Z_{0.975} = qnorm(0.975, 1)$$

```
mu0  = 40

sig0 = (42-mu0) / 1.96
print(sig0^2)
```

```
## [1] 1.041233
```

```
tau0 = 1/sig0^2
print(tau0)
```

```
## [1] 0.9604
```

conjugate Normal prior $\mu \sim N(40, \tau_0 = 0.9603647)$

(2)

```
lst = c(38.6,42.4,57.5,40.5,51.7,67.1,33.4,60.9,64.1,40.1,40.7,6.4)
mu1 = sum(lst)/length(lst)
tau1 = 1/100
print(mu1)
```

```
## [1] 45.28333
```

```
print(tau1)
```

```
## [1] 0.01
```

$$\mu_1 \sim N(45.28333, 0.01)$$

Since nomal prior, normal likelihood gives conjugate distribution.

Then, posterior is

$$N(\frac{\tau_0}{n\tau_1 + \tau_0}\mu_0 + \frac{n\tau_1}{n\tau_1 + \tau_0}\mu_1, \frac{1}{n\tau_1 + \tau_0})$$

```
n   = length(lst)
mu3 = tau0*mu0/(n*tau1+tau0) + n*tau1*mu1/(n*tau1+tau0)
std2 = 1/(n*tau1+tau0)
print(n)
```

```
## [1] 12
```

```
print(mu3)
```

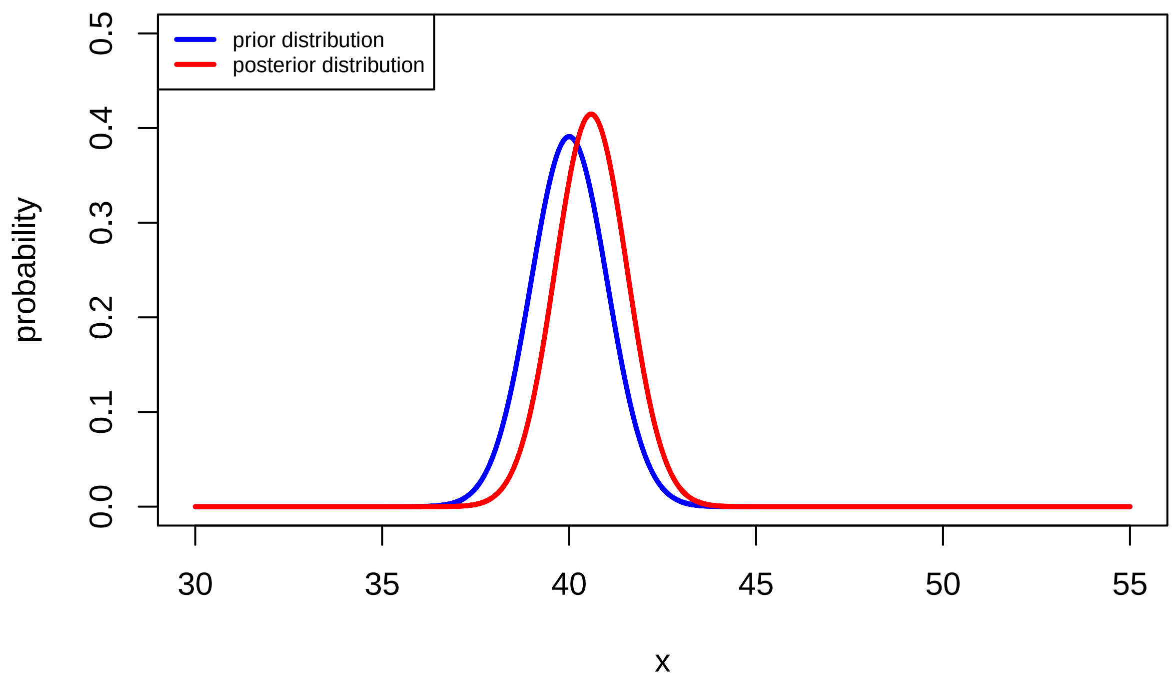
```
## [1] 40.58682
```

$$N_{post}(40.58684, 0.9256134)$$

The posterior's mean also follows a normal distriution, the relative contribution is to increase the mean by a bit and reduces its std by a little bit.

(3)

```
x_lst = seq(30,55,by=0.001)
plot(x_lst,dnorm(x_lst,mu0,(1/tau0)^0.5),type='l',col='blue',xlab='x',ylab='probability',ylim=c(0,.5),lw
lines(x_lst,dnorm(x_lst,mu3,(std2)^0.5),type='l',col='red',xlab='x',ylab='probability',ylim=c(0,.5),lwd
legend('topleft',col=c('blue','red'),legend= c('prior distribution','posterior distribution'),lwd =c(2.
```



The posterior mean larger than that of pior. The new distribution is like a shift of prior to right and the mean gets a bit larger.

(4)

```r
print(qnorm(c(0.1,0.9),mu3,(std2)^0.5))
```

```
## [1] 39.35387 41.81976
```

$$P_{post}(\mu < 39.35387|Data) = 0.1$$
$$P_{post}(\mu > 41.81976|Data) = 0.9$$

(5)

Note that $log\mu$ has mo definition if $\mu \le 0$ and that since $p(\mu|x) \sim e^{-x^2}$ which converges much faster that $\log \mu$ when x gets larger. Thus $log\mu P(\mu|x) -> 0$ as $x -> infinity$

Thus,

$$E[\log \mu] = \int_{0+}^{Inf} \log \mu \cdot p(\mu|x)dx \approx \int_{qnorm(0.0000001,mu_{post},1/tau_{post})}^{qnorm(0.9999999,mu_{post},1/tau_{post})} \log \mu \cdot p(\mu|x)dx \approx 3.703161$$

```r
interva = qnorm(c(0.0000001,0.9999999),mu3,std2^0.5)
fun <- function(x) log(x) * dnorm(x,mu3,std2^0.5)
Q <- integrate(fun,interva[1],interva[2])
print(Q)
```

```
## 3.703161 with absolute error < 2e-09
```

Problem 2

(1)

$$\lambda \sim Gamma(a, b)$$
$$Y|\lambda \sim poisson(N\lambda)$$

$$\lambda_{posterior}|Y \sim Gamma(a + Y, b + N)$$

Liklihood,

$$f(Y|\lambda) = \frac{e^{-N\lambda} \cdot (N\lambda)^y}{y!}$$

Pior,

$$f(\lambda) = \frac{b^a}{\gamma(a)}\lambda^{a-1}e^{-b\lambda}$$

$$f(\lambda|y) = \frac{f(Y|\lambda)f(\lambda)}{f(y)}$$

$$f(\lambda|y) \propto e^{-\lambda N}(N\lambda)^y \cdot \lambda^{a-1}e^{-b\lambda} \propto \lambda^{y+a-1}e^{-(\lambda+b)\lambda}$$

$$\lambda \sim Gamma(y + a, N + b)$$

Assum2 $Y_i \sim i.i.dPoisson(N\lambda)$

$$f(Y_1, Y_2 ... Y_n|\lambda) = f(Y_1|\lambda) \cdot f(Y_2|\lambda) \cdots f(Y_n|\lambda)$$

$$f(\lambda|Y_1,Y_2...Y_n) = \frac{f(Y_1|\lambda)\cdot f(Y_2|\lambda)\cdots f(Y_n|\lambda)f(\lambda)}{f(Y_1,Y_2...Y_n)}$$

$$f(\lambda|Y_1,Y_2...Y_n) \propto e^{-\lambda N}(N\lambda)^{y_1}\cdot e^{-\lambda N}(N\lambda)^{y_2}\cdots e^{-\lambda N}(N\lambda)^{y_n}\cdot \lambda^{a-1}e^{-b\lambda}$$
$$\propto e^{-(nN+b)\lambda}(N\lambda)^{a+\sum_i^n y_i}$$

$$\lambda_{posterior}|Y \sim Gamma(a+\sum_i^n Y_i, b+nN)$$

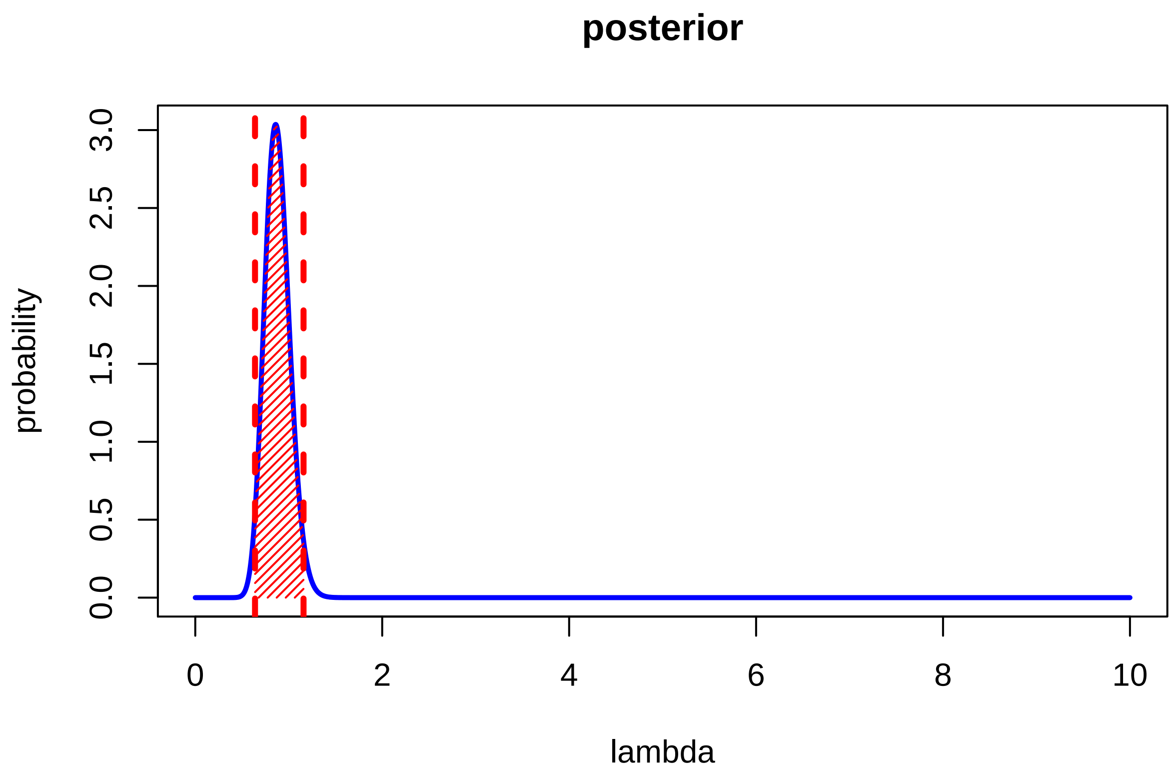
```
Y_lst = c(rep(1,12),rep(2,6),rep(10,2))
print(sum(Y_lst))
```

```
## [1] 44
```

(2)

$$\lambda_{posterior}|Y \sim Gamma(44.01, 50.01)$$

```
a = 0.01
b = 0.01
N = 50
ap =  sum(Y_lst)+a
bp =  (N+b)
x_lst = seq(0,10,by=0.001)
y_lst = dgamma(x_lst,shape=ap,scale=1/bp)
plot(x_lst,y_lst,type='l',col='blue',xlab='lambda',ylab='probability',lwd = 2.5,main='posterior')
inter = qgamma(c(0.025,0.975), shape=ap, scale =1/bp)
I= x_lst>qgamma(.025,shape=ap,scale=1/bp)
x_lstnew = x_lst[I==TRUE]
I= x_lstnew <qgamma(.975,shape=ap,scale=1/bp)
y_lstnew = dgamma(x_lstnew,shape=ap,scale=1/bp)
polygon(c(x_lstnew[I],rev(x_lstnew[I])),c(y_lstnew[I],
+ rep(0,sum(I))),col="red",density=30,border=NA)
abline(v = inter[1], col="red", lwd=3, lty=2)
abline(v = inter[2], col="red", lwd=3, lty=2)
```




```
print(inter)
```

```
## [1] 0.6394519 1.1584123
```

```
mean1 =  ap/bp
print(mean1)
```

```
## [1] 0.880024
```

The 95% credible interval is (0.6394519,1.1584123) and mean is 0.880024

```
a = 0.01
b = 0.01
N = 50
ap =  sum(Y_lst)+a
bp =  (N+b)
x_lst = seq(0,10,by=0.001)
y_lst = dgamma(x_lst,shape=ap,scale=1/bp)
plot(x_lst,y_lst,type='l',col='blue',xlab='lambda',ylab='probability',lwd = 2.5)
inter = qgamma(c(0.025,0.975), shape=ap, scale =1/bp)
print(inter)
```

```
## [1] 0.6394519 1.1584123
```

```
I= x_lst>qgamma(.025,shape=ap,scale=1/bp)
x_lstnew = x_lst[I==TRUE]
I= x_lstnew <qgamma(.975,shape=ap,scale=1/bp)
y_lstnew = dgamma(x_lstnew,shape=ap,scale=1/bp)
mean1 =  ap/bp
```

```
print(mean1)
```

```
## [1] 0.880024
```

```
abline(v = mean1, col="blue", lwd=1.2, lty=2)

a = 1
b = 1
N = 50
ap =  sum(Y_lst)+a
bp =  (N+b)
x_lst = seq(0,10,by=0.001)
y_lst = dgamma(x_lst,shape=ap,scale=1/bp)
lines(x_lst,y_lst,type='l',col='red',xlab='lambda',ylab='probability',lwd = 2.5)
inter2 = qgamma(c(0.025,0.975), shape=ap, scale =1/bp)
print(inter2)
```

```
## [1] 0.6435943 1.1581950
```

```
I= x_lst>qgamma(.025,shape=ap,scale=1/bp)
x_lstnew = x_lst[I==TRUE]
I= x_lstnew <qgamma(.975,shape=ap,scale=1/bp)
y_lstnew = dgamma(x_lstnew,shape=ap,scale=1/bp)
mean2 =  ap/bp
abline(v = mean1, col="black", lwd=1.2, lty=2)
print(mean2)
```

```
## [1] 0.8823529
```

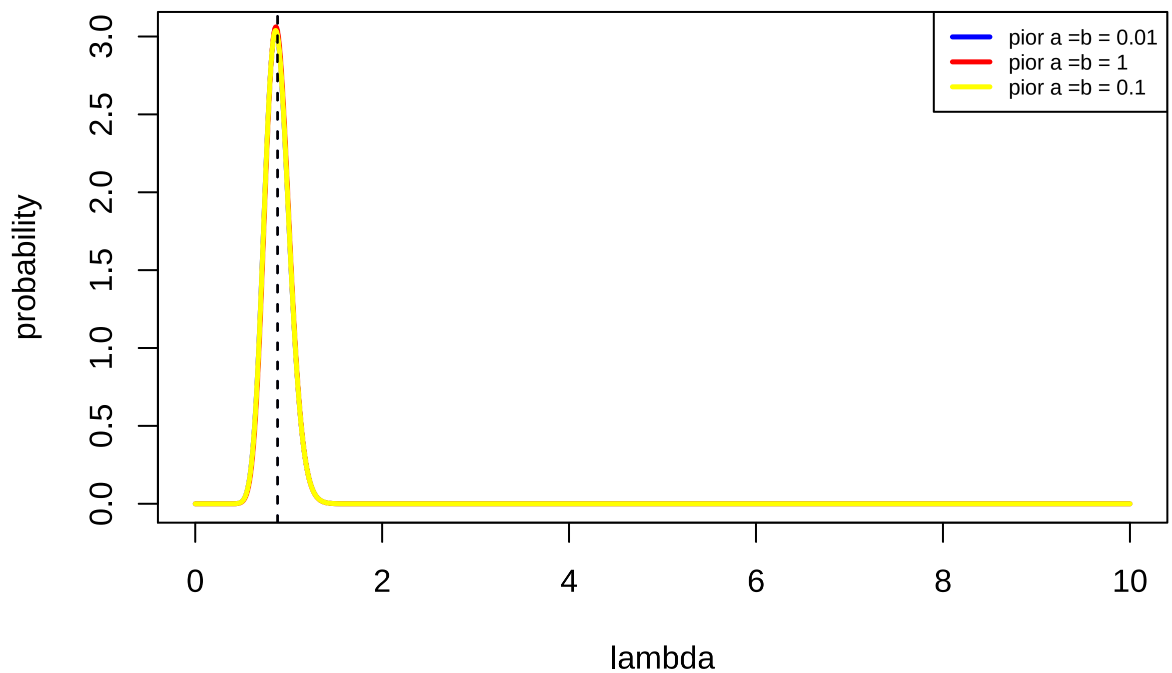
```
a = 0.1
b = 0.1
N = 50
ap =  sum(Y_lst)+a
bp =  (N+b)
x_lst = seq(0,10,by=0.001)
y_lst = dgamma(x_lst,shape=ap,scale=1/bp)
lines(x_lst,y_lst,type='l',col='yellow',xlab='lambda',ylab='probability',lwd = 2.5)
inter3 = qgamma(c(0.025,0.975), shape=ap, scale =1/bp)
print(inter3)
```

```
## [1] 0.6398339 1.1583935
```

```
I= x_lst>qgamma(.025,shape=ap,scale=1/bp)
x_lstnew = x_lst[I==TRUE]
I= x_lstnew <qgamma(.975,shape=ap,scale=1/bp)
y_lstnew = dgamma(x_lstnew,shape=ap,scale=1/bp)
mean3 =  ap/bp
abline(v = mean3, col="black", lwd=1.2, lty=2)
print(mean3)
```

```
## [1] 0.8802395
```

```r
legend('topright',col=c('blue','red','yellow'),legend= c('pior a =b = 0.01','pior a =b = 1','pior a =b =
```




```r
qtls = matrix(NA,3,3)
qtls[1,] = c(inter[1],inter[2],mean1)
qtls[2,] = c(inter3[1],inter3[2],mean3)
qtls[3,] = c(inter2[1],inter2[2],mean2)
df = data.frame(qtls)
colnames(df) = c('p(lambda<x)=0.025','p(lambda<x)=0.975','mean')
row.names(df) = c('pior gamma(0.01,0.01)','pior gamma(0.1,0.1)','pior gamma(1,1)')
knitr::kable(df, format = "markdown")
```

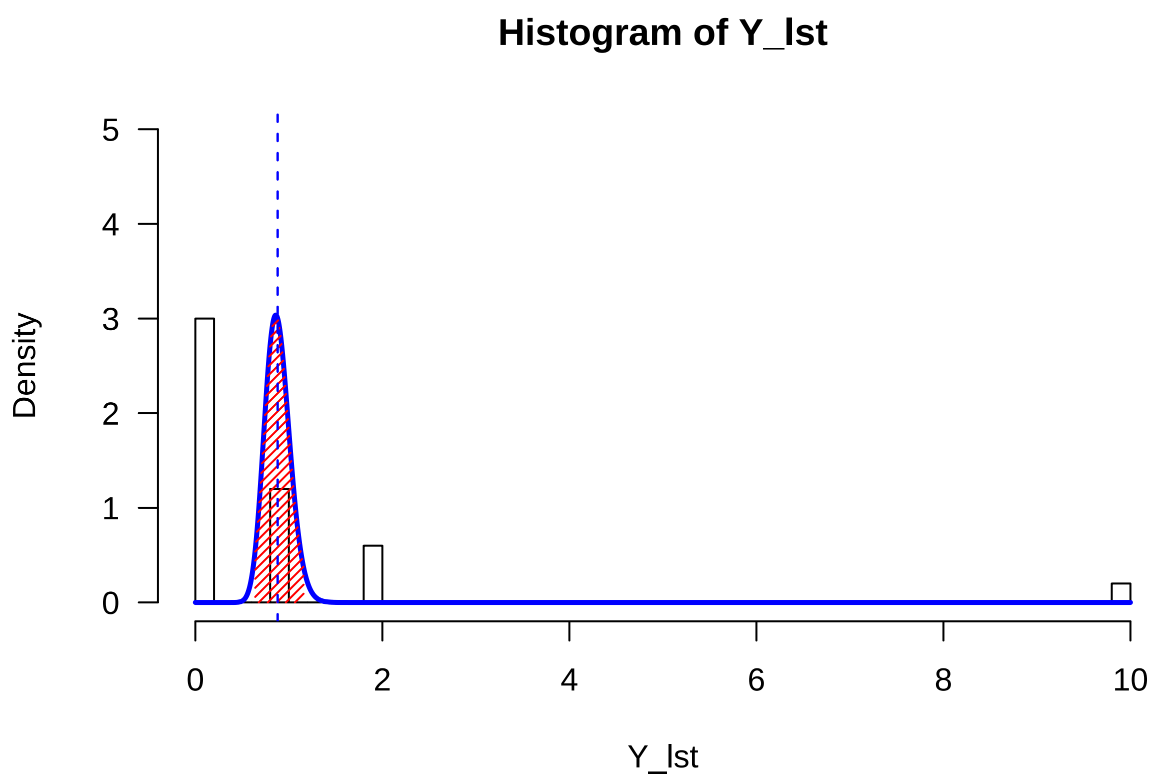
	p(lambda<x)=0.025	p(lambda<x)=0.975	mean
pior gamma(0.01,0.01)	0.6394519	1.158412	0.8800240
pior gamma(0.1,0.1)	0.6398339	1.158394	0.8802395
pior gamma(1,1)	0.6435943	1.158195	0.8823529

The credible interval 95% and the mean are pretty similar for Gamma(0.1,0.1) and Gamma(1,1), Gamma(1,1). The sensitvity for this change of pior is low.

(4)

```r
Y_lst = c(rep(1,12),rep(2,6),rep(10,2),rep(0,30))
a = 0.01
b = 0.01
```

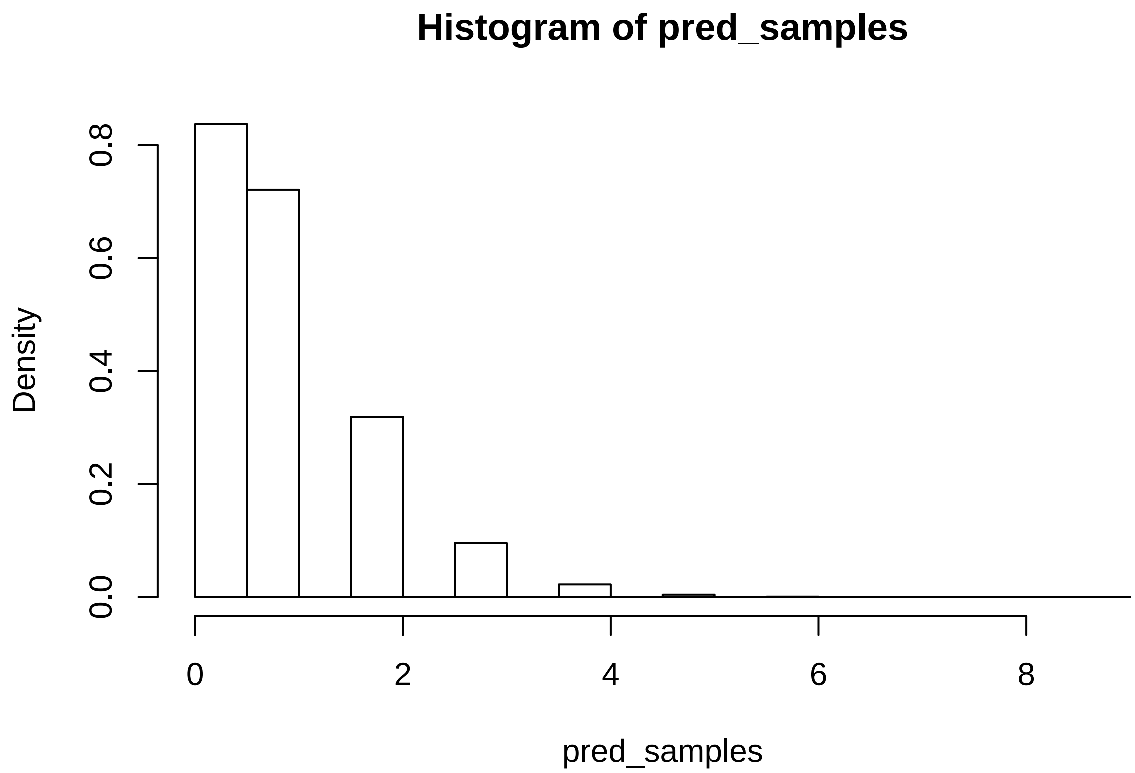
```r
N = 50
ap =  sum(Y_lst)+a
bp =  (N+b)
x_lst = seq(0,10,by=0.001)
y_lst = dgamma(x_lst,shape=ap,scale=1/bp)
hist(Y_lst,breaks=40,freq=FALSE,ylim = c(0,5),xlim=c(0,10),las=1)
lines(x_lst,y_lst,type='l',col='blue',xlab='lambda',ylab='probability',lwd = 2.5)
inter = qgamma(c(0.025,0.975), shape=ap, scale =1/bp)
I= x_lst>qgamma(.025,shape=ap,scale=1/bp)
x_lstnew = x_lst[I==TRUE]
I= x_lstnew <qgamma(.975,shape=ap,scale=1/bp)
y_lstnew = dgamma(x_lstnew,shape=ap,scale=1/bp)
polygon(c(x_lstnew[I],rev(x_lstnew[I])),c(y_lstnew[I],+ rep(0,sum(I))),col="red",density=30,border=NA)
mean1 =  ap/bp
abline(v = mean1, col="blue", lwd=1.2, lty=2)
```



The posterior distribution dosen't represent the high ASE patient well since it says the pobability denssity is pretty low at 10 side effects and 0 side effects.

(5)

```r
post_samples = rgamma(1000000, shape=ap, scale=1/bp)
pred_samples = sapply(post_samples, rpois, n=1)
hist(pred_samples, breaks = 20, freq = FALSE)
```




```
p = sum(pred_samples>1) / length(pred_samples)
print(p)
```

```
## [1] 0.220902
```

The posterior probablity that has higher side effect than one is around 22%. The result is not sensitvity to prior since the posterior possion gamma distribution is not sensitive to the prior.

Problem 3

From https://en.wikipedia.org/wiki/Gamma_distribution, properties Median calculation

$$100 = var = \frac{a}{b^2} => a = 100b^2 => b = \frac{\sqrt{a}}{10}$$

$$\mu = \frac{a}{b} = 10\sqrt{a}$$

$$75 = median \approx \mu\frac{3a-0.8}{3a+0.2} = 10\sqrt{a}\frac{3a-0.8}{3a+0.2}$$

```
fun <- function (x) 10*(x)^.5*(3*x-0.8)/(3*x+0.2)-75
a <- uniroot(fun, c(0, 100))$root
b = sqrt(a)/10
print(a)
```

```
## [1] 56.91393
```

```r
print(b)
```

```
## [1] 0.7544132
```

$$a = 56.9139, b = 0.7544132$$

```r
ratt = 1-pgamma(75,a,b)
```

$$P(\lambda > \lambda_0) = 0.5000023$$