problem 1

$$p(\theta) = Be(\theta;\alpha,\beta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad 0\leq\theta\leq 1$$

$B(\alpha,\beta)$ is normalization constant

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad z>0$$

$$B(\alpha,\beta) = \int_0^1 \theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta, \quad \Gamma(z+1) = z\Gamma(z)$$

① $E[\theta] = \int_0^1 \theta p(\theta)d\theta$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 \theta\,\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 \theta^{\alpha}(1-\theta)^{\beta-1}$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}$$

$$= \frac{\alpha\cancel{\Gamma(\alpha)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}\frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}}$$

$$= \frac{\alpha}{\alpha+\beta} \quad \text{is the mean}$$

②

to get mode is to get point derivative $=0$

$$\frac{\partial p(\theta)}{\partial\theta} = \frac{1}{B(\alpha,\beta)}\left[(\alpha-1)\theta^{\alpha-2}(1-\theta)^{\beta-1} + (\beta-1)\theta^{\alpha-1}(1-\theta)^{\beta-2}\right]$$

$$= \frac{1}{B(\alpha,\beta)}\left[\theta^{\alpha-2}(1-\theta)^{\beta-2}\right]\left[(\alpha-1)(1-\theta) - (\beta-1)\theta\right]$$

$$\overset{\text{want}}{=} 0$$

$\theta = 0, 1$ is trival

suppose $\theta \neq 0, 1$

then $0 = (\alpha-1)(1-\theta) - (\beta-1)\theta$

$$\alpha-1+(1-\alpha)\theta-(\beta-1)\theta=0$$

$$\theta=\frac{1-\alpha}{2-\alpha-\beta} \quad \text{is the mode}$$

Problem 2 $\quad D=\{x_1,\cdots,x_N\}$ where $x_i\in\{1,\cdots,K\}$

$\theta_k=P(x_i=k),\ 1\le k\le K\ ,\ \sum_{k=1}^{K}\theta_k=1$

Dirichelet prior for the parameters θ

where the prior has parameters $\alpha_1,\cdots,\alpha_K$ $\ \alpha_k>0$

①

$$\begin{aligned}P(\theta\mid D)&\propto P(D\mid\theta)P(\theta)\\&=\prod_{i=1}^{N}P(x_i\mid\theta)P(\theta)\\&=\prod_{i=1}^{K}\theta_i^{r_i}\prod_{i=1}^{K}\theta^{\alpha_i-1}\\&=\prod_{i=1}^{K}\theta_i^{r_i+\alpha_i-1}\\&\stackrel{\text{let}}{=}\prod_{i=1}^{K}\theta_i^{\alpha_i'-1}\end{aligned}$$

r_i is the time θ_i appears in D

where $\alpha_i'=r_i+\alpha_i$

thus posterior distribution on $\theta_1,\cdots,\theta_K$ is Dirchlet for parameter $\{\alpha_1',\alpha_2',\cdots\alpha_K'\}$

② by using maximum likelihood on posterior

$$\log(P(\theta\mid D))=\sum_{i=1}^{K}(\alpha_i'-1)\log\theta_i=(*)$$

adding Lagrange operator

$$(*)=\sum_{i=1}^{K}(\alpha_i'-1)\log\theta_i+\lambda(1-\sum_{i=1}^{K}\theta_i) \quad \text{since } \sum_{i=1}^{K}\theta_i=1$$

$$0 \overset{\text{want}}{=} \frac{\partial \ell(*)}{\partial \theta_i} = \frac{\alpha_i' - 1}{\theta_i} - \lambda \quad \text{for } \forall i \in \{1, \dots, K\}$$

thus $\theta_i = \frac{\alpha_i' - 1}{\lambda}$

$$1 = \sum_{i=1}^{K} \theta_i = \frac{1}{\lambda} \sum_{i=1}^{K} (\alpha_i' - 1)$$

thus
$$\lambda = \sum_{i=1}^{K} \alpha_i' - K$$
$$= \sum_{i=1}^{K} (r_i + \alpha_i) - K$$
$$= N + \sum_{i=1}^{K} \alpha_i - K$$

thus
$$\theta_k = \frac{r_k + \alpha_k - 1}{N - K + \sum_{i=1}^{K} \alpha_i} \qquad 1 \le k \le K$$

Problem 3

$D = \{x_1, \dots, x_N\}$ with real-valued scalar x_i

$$x_i \sim N(\mu, \sigma^2)$$
$$\mu \sim N(\mu_0, s^2)$$
$$\mu | D \sim N(\mu_N, \sigma_N^2)$$

NTS: $\mu_N = \gamma \hat{\mu}_{ml} + (1 - \gamma) \mu_0$

$$\frac{1}{\sigma_N^2} = \frac{N}{\sigma^2} + \frac{1}{s^2}$$

where $\gamma = \frac{N s^2}{N s^2 + \sigma^2}$

As show in class $\hat{\mu}_{ml} = \frac{1}{N} \sum x_i$

$$P(\mu | D) \propto P(D | \mu) P(\theta)$$
$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{(\mu - \mu_0)^2}{2 s^2}}$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \frac{1}{\sqrt{2\pi s^2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{N} (x - \mu)^2 - \frac{1}{2 s^2} (\mu - \mu_0)^2}$$

$$p(\mu|D) \propto e^{\underbrace{-\left(\mu^2\left(\frac{N}{2\sigma^2} + \frac{1}{2s^2}\right) - \mu\left(\frac{\Sigma x_i}{\sigma^2} + \frac{\mu_0}{s^2}\right) + \frac{1}{2\sigma^2}\Sigma x_i^2 + \frac{1}{2s^2}\mu_0^2\right)}_{(*)}}$$

let $$\mu_n = -\frac{-\left(\frac{\Sigma x_i}{\sigma^2} + \frac{\mu_0}{s^2}\right)}{2\left(\frac{N}{2\sigma^2} + \frac{1}{2s^2}\right)} = \frac{s^2 \Sigma x_i + \sigma^2 \mu_0}{Ns^2 + \sigma^2}$$

$$= \frac{s^2 \Sigma x_i}{Ns^2 + \sigma^2} + \frac{\sigma^2 \mu_0}{Ns^2 + \sigma^2}$$

$$= \frac{Ns^2}{Ns^2 + \sigma^2} \frac{\Sigma x_i}{N} + \frac{\sigma^2}{Ns^2 + \sigma^2} \mu_0$$

$$= \gamma \hat{\mu}_{ml} + (1-\gamma)\mu_0$$

$$ax^2 - bx + c \Rightarrow a\left(x^2 - \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

$$= a\left(x - \frac{b}{2a}\right)^2 - \frac{ab^2}{4a^2} + c$$

$$= a\left(x - \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

let $$\frac{1}{2\sigma_n^2} = a = \frac{N}{2\sigma^2} + \frac{1}{2s^2}$$

then $$\frac{1}{\sigma_n^2} = \frac{N}{\sigma^2} + \frac{1}{s^2}$$

then $$p(\mu|D) \propto e^{-\frac{(\mu - \mu_n)^2}{2\sigma_n^2}} \cdot e^{\text{const}}$$

$$p(\mu|D) \propto e^{-\frac{(\mu - \mu_n)^2}{2\sigma_n^2}}$$

thus $$p(\mu|D) \propto N(\mu_n, \sigma_n^2)$$

where $\mu_n = \gamma \hat{\mu}_{ml} + (1-\gamma)\mu_0$

$\frac{1}{\sigma_n^2} = \frac{n}{\sigma^2} + \frac{1}{s^2}$, $\gamma = \frac{ns^2}{ns^2 + \sigma^2}$

Problem 4

$D = \{x_1, \cdots, x_N\}$ $\quad x_i \in \mathbb{R}$, $x_i \geq 0$, $1 \leq i \leq N$ $\quad \alpha, \beta > 0$

$p(x|\theta) = \theta e^{-\theta x}$

Gamma prior for θ $\quad p(\theta|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$

$\Gamma(z+1) = z\Gamma(z)$, $\Gamma(1) = 1$

①

$$p(\theta|D) \propto p(D|\theta)\, p(\theta)$$

$$= \prod_{i=1}^{N} \theta e^{-\theta x_i} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{N+\alpha-1} e^{-(\beta+\Sigma x_i)\theta}$$

$$p(\theta|D) \propto \theta^{N+\alpha-1} e^{-(\beta+\Sigma x_i)\theta}$$

$$p(\theta|D) \sim \text{Gamma}(N+\alpha, \beta+\Sigma x_i)$$

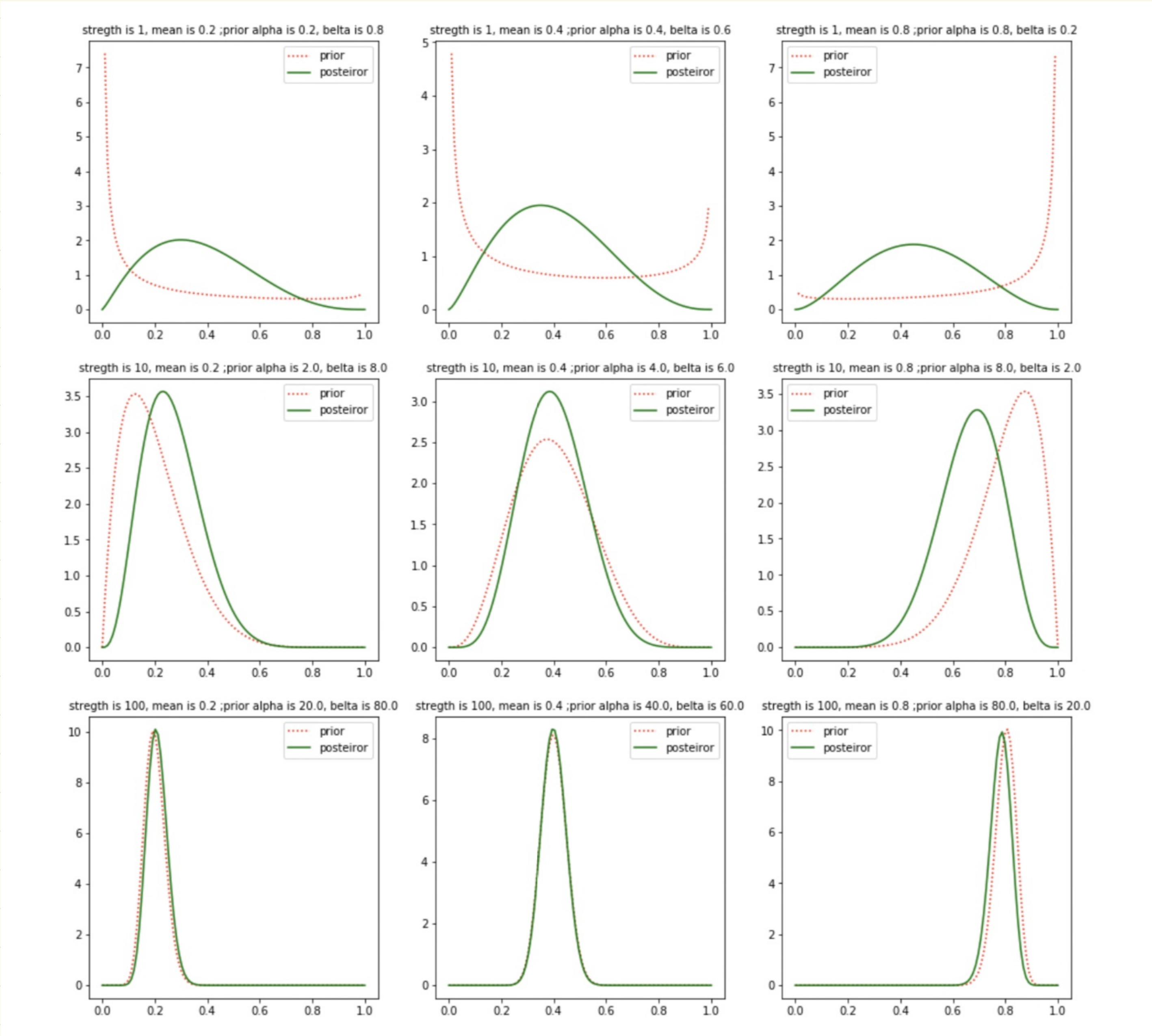
②

$$E[\theta] = \frac{N+\alpha}{\beta+\Sigma x_i}$$

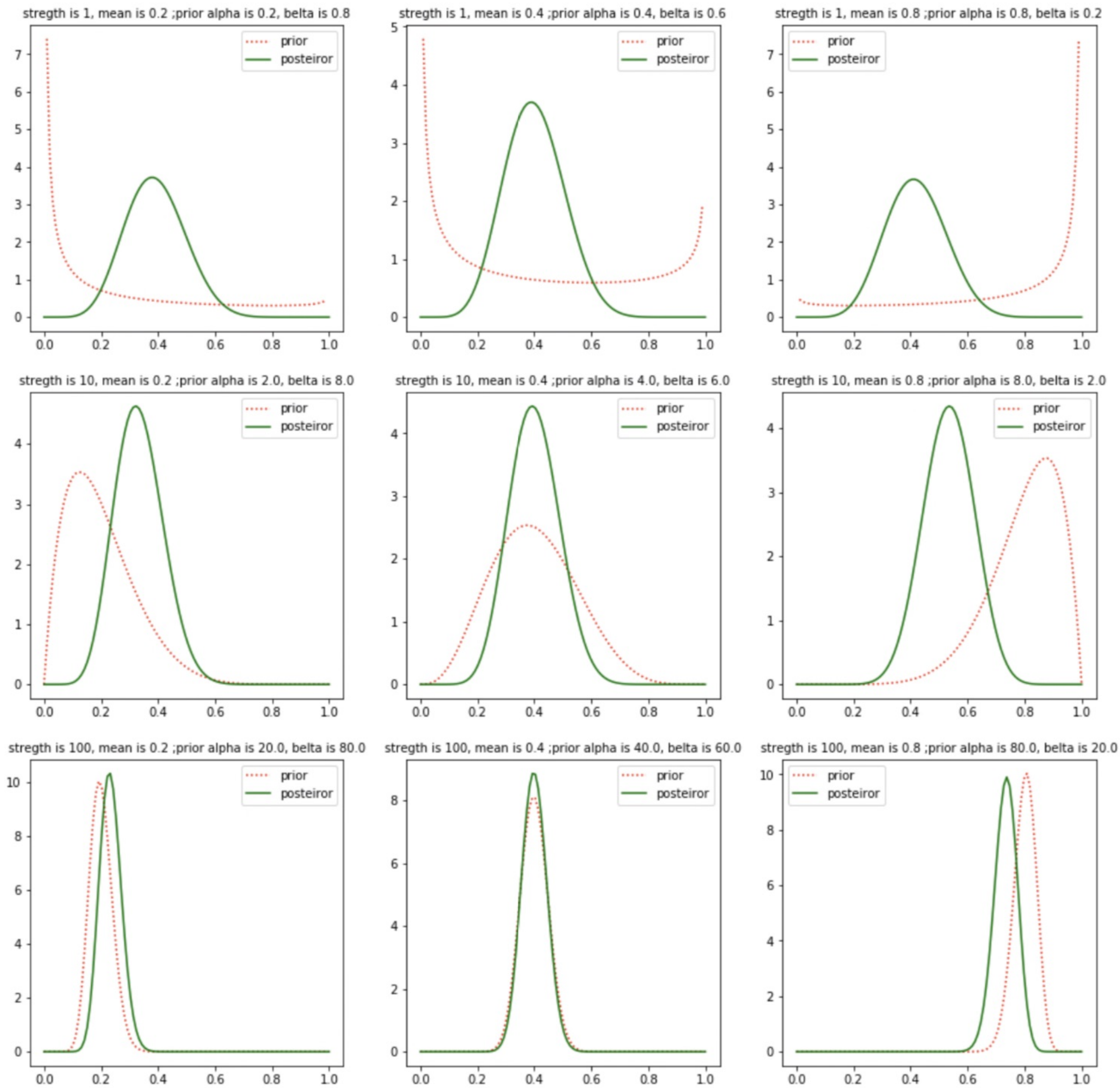
$$\text{mode } \theta = \frac{N+\alpha-1}{\beta+\Sigma x_i}$$

Problem 5

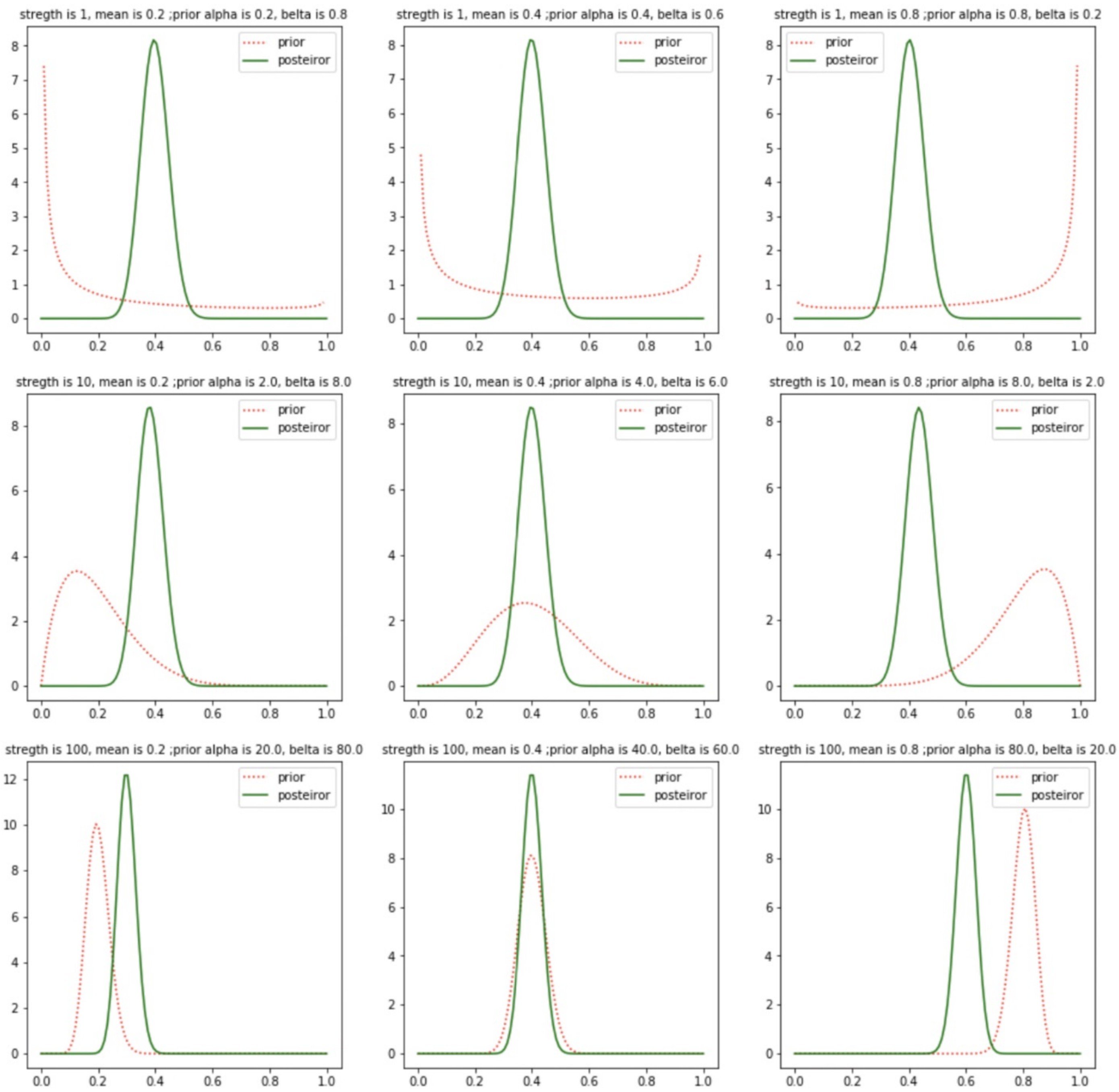
$n=5 \qquad r=2$



$n = 20 \quad r = 8$

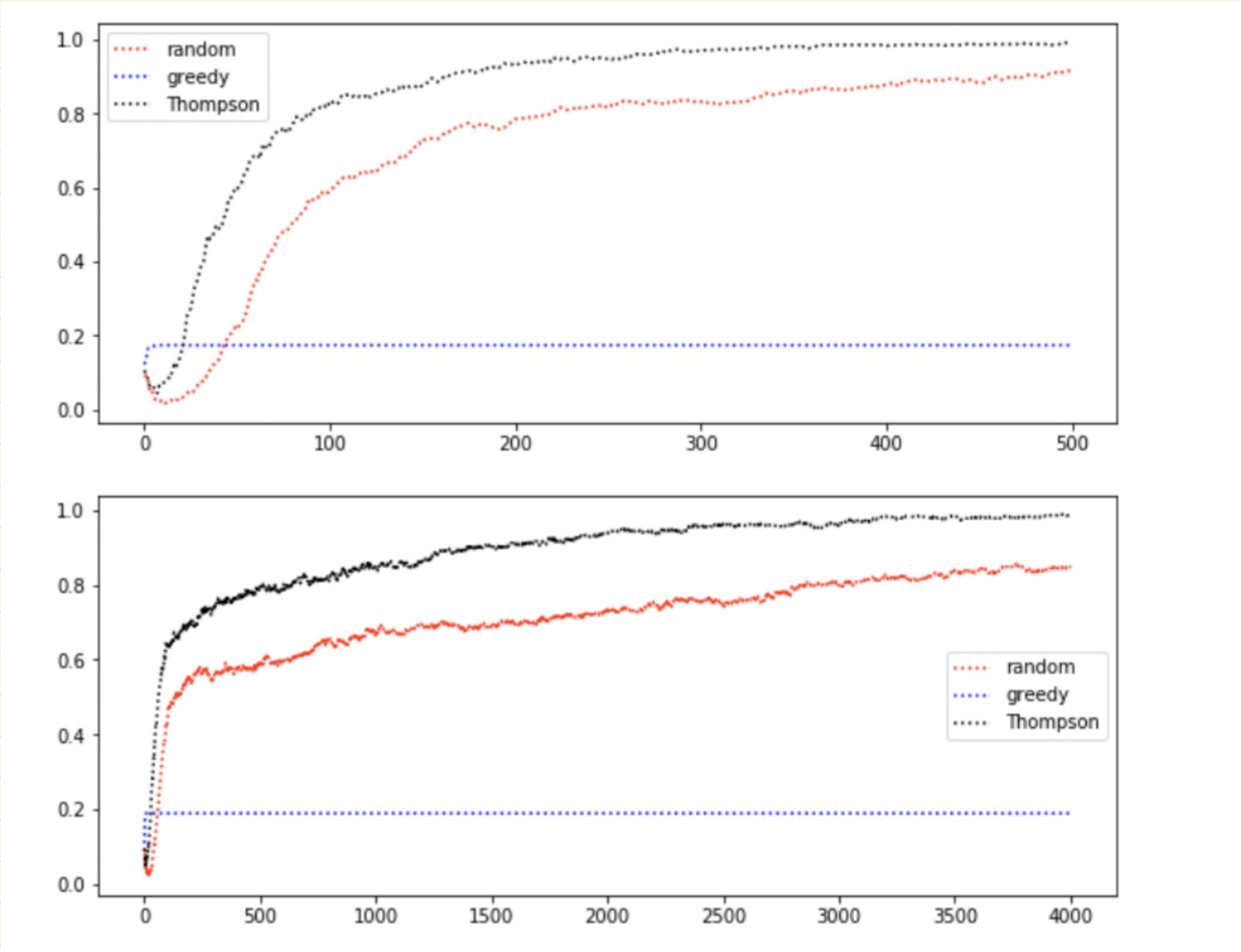

stregth is 1, mean is 0.2 ;prior alpha is 0.2, belta is 0.8
stregth is 1, mean is 0.4 ;prior alpha is 0.4, belta is 0.6
stregth is 1, mean is 0.8 ;prior alpha is 0.8, belta is 0.2
stregth is 10, mean is 0.2 ;prior alpha is 2.0, belta is 8.0
stregth is 10, mean is 0.4 ;prior alpha is 4.0, belta is 6.0
stregth is 10, mean is 0.8 ;prior alpha is 8.0, belta is 2.0
stregth is 100, mean is 0.2 ;prior alpha is 20.0, belta is 80.0
stregth is 100, mean is 0.4 ;prior alpha is 40.0, belta is 60.0
stregth is 100, mean is 0.8 ;prior alpha is 80.0, belta is 20.0
prior
posteiror

$n=100 \qquad r=40$



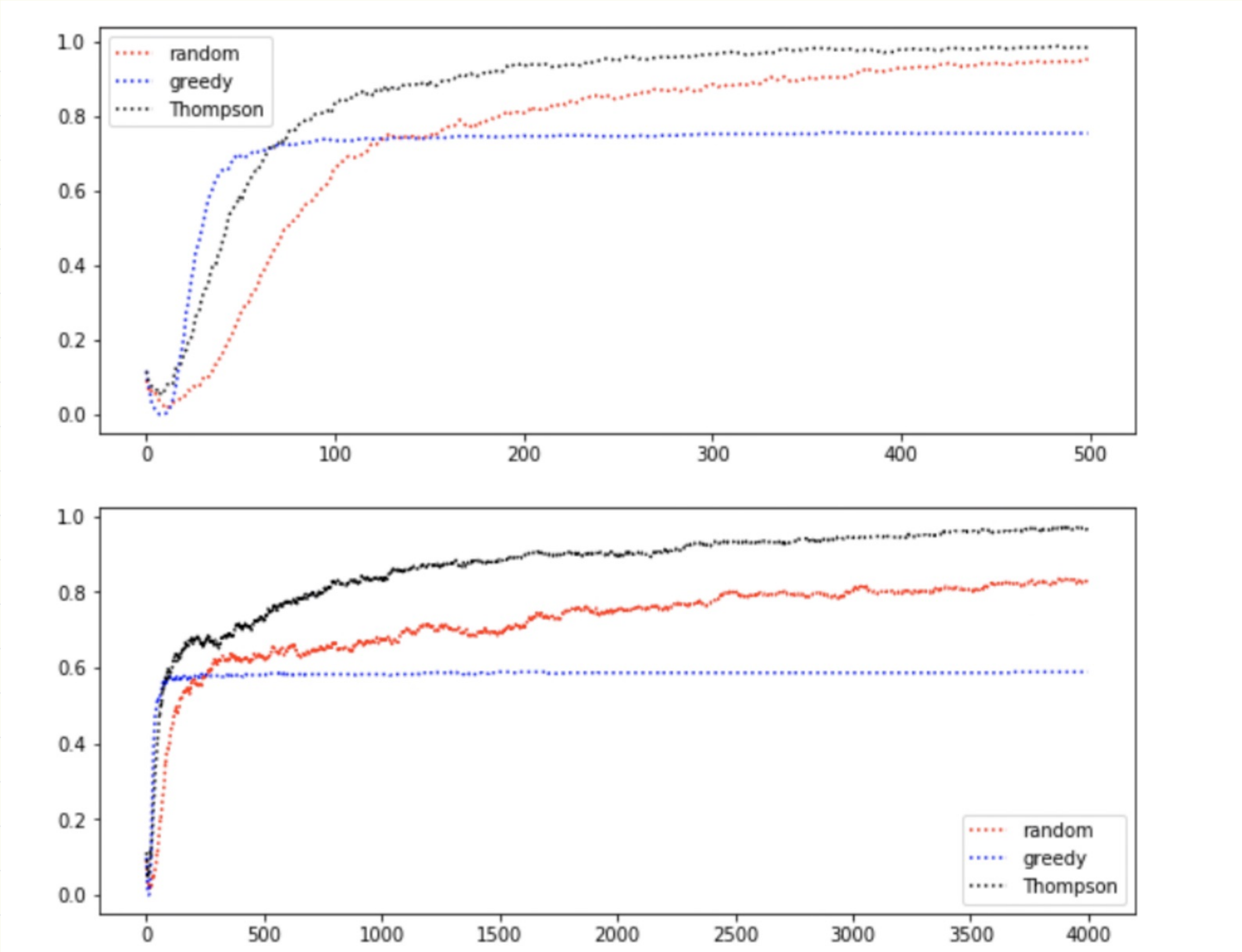
from those plots, I learned that the posterior mode will locate at some place between prior mode and r/n. when the strengh is high relative to n and r, the posetrior distribution will be more like prior distribution. On the contrary, it will be more like new data information.

Problem 6



first, consider it's no need for greedy to explore all the arm. then we can see that Thompson converges faster that random to fraction 1. This is because that by using a pdf for pulling arm. It will gives more probabity to pull the one with higher expectation than random. While Thompson also gives some small possibility to pull other arms for exploration.

second, for greedy since 8 of the 10 arms has probability 0.5. It is very possible for greedy algorithm to stack on one of them. other than giving more more exploration chance for the other arms with higher expectation win. the the probability is aroud 0.2



if let greedy to exploration all the arm once before choosing to most likehood arm. The greedy algorithm will do better as above shows.