CS 274 A HW#1

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Problem 1:

① $LHS = E[aX + bY]$

$= \iint p(x,y)\,[ax+by]\,dx\,dy$

$= a\iint p(x,y)\, x\, dy\, dx + b\iint p(x,y)\, y\, dx\, dy$

$RHS = aE[X] + bE[Y]$

$= a\int p(x)\, x\, dx + b\int p(y)\, y\, dy$

Remark: by law of total probability. note X, Y can take continous

$\int p(x,y)\, dy = p(x)$

$\int p(x,y)\, dx = p(y)$

thus $RHS = a\int \int p(x,y)\, dy\; x\, dx + b\int\int p(x,y)\, dx\; y\, dy$

$= a\int\int p(x,y)\, x\, dy\, dx + b\iint p(x,y)\, y\, dx\, dy$

$= LHS$

thus $E[aX + bY] = aE[X] + bE[Y]$

② Remark

(*) $var(X) = E[(X-\mu_x)^2] = E[X^2 - 2E[X]X + \mu_x^2]$

$\overset{\text{by ①}}{=} E[X^2] - 2E[XE[X]] + E[\mu_x^2]$

$= E[X^2] + (E[X])^2 - 2E[XE[X]]$

$= E[X^2] + E[X]^2 - 2E[X]E[X] = E[X^2] - (E[X])^2$

Recall let X, Y be two real-valued random variables

(**) then $E[XY] = \iint p(x,y)\, xy\, dx\, dy$ since X, Y independent

$= \iint p(x)\, p(y)\, xy\, dx\, dy$

$= \int p(x)\, x\, dx \cdot \int p(y)\, y\, dy$

$= E[X]\, E[Y]$

method (I)

$$\mathrm{Var}(aX+bY) \overset{\text{by }(*)}{=} E[(aX+bY)^2] - (E[aX+bY])^2$$

$$\overset{\text{by }①}{=} E[a^2X^2 + 2abXY + b^2Y^2] - (aE[X]+bE[Y])^2$$

$$\overset{\text{by }①}{=} a^2E[X^2] + 2abE[XY] + b^2E[Y^2] - a^2(E[X])^2 - b^2(E[Y])^2 - 2abE[X]E[Y]$$

since X, Y independent by $(**)$ $\quad 2abE[XY] - 2abE[X]E[Y]$

$$= 2ab(E[XY] - E[XY]) = 0$$

thus $\mathrm{var}(aX+bY) = a^2[E[X^2] - (E[X])^2] + b^2[E[Y^2] - E[Y]^2]$

$$= a^2\,\mathrm{var}[X] + b^2\,\mathrm{var}[Y]$$

method (II)

$$\mathrm{var}(aX+bY) = \iint p(x,y)\,(ax+by-a\mu_x-b\mu_y)^2\,dx\,dy$$

since $X\,Y$ independent $p(x,y)=p(x)\,p(y)$

$$= \iint p(x)p(y)\,[(ax-a\mu_x) + (by-b\mu_y)]^2\,dx\,dy$$

$$= a^2\int p(x)(x-\mu_x)^2 \int p(y)\,dy\,dx$$

$$+ b^2\int p(y)(y-\mu_y)^2 \int p(x)\,dx\,dy$$

$$+ \iint p(x)\,p(y)\;2\cdot[abxy - abx\mu_y - ab\mu_x y + ab\mu_x\mu_y]\,dx\,dy$$

$$= a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y)$$

$$+ 2ab\int xp(x)\,dx \int yp(y)\,dy - 2ab\int xp(x)\,dx\;\mu_y\int p(y)\,dy$$

$$- 2ab\mu_x\int p(x)\,dx\int yp(y)\,dy + 2ab\;\mu_x\int p(x)\,dx\;\mu_y\int p(y)\,dy$$

note $\int y\,p(y)\,dy = \mu_y$

$\int x\,p(x)\,dx = \mu_x$

$\int p(y)\,dy = \int p(x)\,dx = 1$

$$= a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y)$$

$$+ 2ab\mu_x\mu_y - 2ab\mu_x\mu_y - 2ab\mu_x\mu_y + 2ab\mu_x\mu_y$$

$$= a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y)$$

$\blacksquare$

Problem 2

① $E[X] = \int p(x)\, x\, dx = \int_0^a p(x)\, x\, dx + \int_a^b p(x)\, x\, dx + \int_b^\infty p(x)\, x\, dx$

$= \int_0^a 0 \cdot x\, dx + \int_a^b \frac{1}{b-a} x\, dx + \int_b^\infty 0\, x\, dx$

$= \frac{1}{b-a} \quad \frac{1}{2}x^2 \Big|_a^b$

$= \frac{1}{b-a} \cdot \frac{1}{2} (b^2 - a^2)$

$= \frac{b+a}{2}$

② $var(X) = \int p(x)\,(x-\mu_x)^2\, dx = \int_0^a 0 \cdot (x-\mu_x)^2 dx + \int_a^b \frac{1}{b-a} (x-\mu_x)^2 dx + \int_b^\infty 0\, (x-\mu_x)^2 dx$

$= \frac{1}{b-a} \quad \frac{1}{3}(x-\mu_x)^3 \Big|_a^b$

$= \frac{1}{b-a} \cdot \frac{1}{3} \cdot \left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3 \right]$

$= \frac{1}{24} \frac{1}{b-a} \left[(b-a)^2 (b-a-a+b) \right]$ note $b-a>0$

$= \frac{1}{24} (b-a)(2b-2a)$

$= \frac{1}{12} (b-a)^2$

Problem 3

let $a_n = (1-\theta)^{n-1}$

(1.) $\sum_{X=1}^{\infty} P(X) = \sum_{X=1}^{\infty} (1-\theta)^{X-1}\theta = \theta \sum_{X=1}^{\infty} (1-\theta)^{X-1}$; then $\frac{a_{n+1}}{a_n} = \frac{(1-\theta)^{(X-1+1)}}{(1-\theta)^{(X-1)}} = (1-\theta)$.

by summation formula of geometric progression where $0 < 1-\theta < 1$

$= \theta \frac{1-(1-\theta)^n}{1-(1-\theta)}$ where $n \to \infty$, $(1-\theta)^n \to 0$

$= 1$.

$a_1 = (1-\theta)^0 \theta = \theta$

(2) $E[X] = \sum_1^{\infty} x\, p(x)$

$= p(1) + 2p(2) + 3p(3) + \cdots\cdots$

$= (p(1) + p(2) + p(3) + \cdots\cdots) + (p(2) + p(3) + \cdots) + (p(3) + \cdots) + \cdots$

$= P(X \geq 1) + P(X \geq 2) + \cdots\cdots$

Note $P(X \geq i) = (1-\theta)^{i-1}$ $i \in \mathbb{Z}^*$

thus $E[X] = \sum_{i=1}^{n} (1-\theta)^{i-1}$ since $0 < 1-\theta < 1$

$= \frac{1-(1-\theta)^n}{1-(1-\theta)}$ $n \to \infty$

$= \frac{1}{\theta}$

(3) $Var(X) = E[(X-\mu_X)^2] \overset{\text{by Problem 1 (2)(*)}}{=} E[X^2] - (E[X])^2$

let $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

then $f'(x) = \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$

$f''(x) = \sum_{k=2}^{\infty} k(k-1) x^{k-2} = \frac{2}{(1-x)^3}$

note that $E[X^2] = \theta \sum_{k=1}^{\infty} k^2 (1-\theta)^{k-1}$

$= \theta(1-\theta) \sum_{k=1}^{\infty} k(k-1)(1-\theta)^{k-2} + \theta \sum_{k=1}^{\infty} k(1-\theta)^{k-1}$

$= \theta(1-\theta) \frac{2}{\theta^3} + \frac{\theta}{\theta^2} = \frac{2-2\theta}{\theta^2} + \frac{1}{\theta} = \frac{2}{\theta^2} - \frac{1}{\theta}$

$$Var(X) = E[X^2] - (E[X])^2$$
$$= \frac{2}{\theta^2} - \frac{1}{\theta} - \frac{1}{\theta^2} = \frac{1}{\theta^2} - \frac{1}{\theta}$$

Problem 4

1 when independent random feature values $\{X_1, \cdots, X_n\}$ are added, their properly normalized sum tends toward a normal distribution, even if the orignal feature value X_i themselves are not normally distributed

2. see last page of this homework

3. As n gets larger, the figure will be more like normal distribution bell-like shape. As n gets larger the variance decreases which is agree with the fact that $var = \frac{\sigma^2}{n}$ where σ is a finite variance of the feature of uniform distrib

4. see last page of this homework

Problem 5

$$p(x)=\sum_{k=1}^{K}\alpha_k P_k(x;\theta_k) \quad \text{where } \sum_{k=1}^{K}\alpha_k=1,\ 0\le\alpha_k\le 1$$

each $P_k(x;\theta_k)$ is pdf $\quad \theta_k=\{\mu_k,\sigma_k^2\}$

(1)

sum to 1: $\int p(x)\,dx = \int \sum_{k=1}^{K} a_k P_k(x;\theta_k)\,dx$

$$= \sum_{k=1}^{K} a_k \int P_k(x;\theta_k)\,dx = \sum_{k=1}^{K} a_k \cdot 1 = 1$$

since each $P_k(x;\theta_k)$ is pdf

then $\int P_k(x;\theta_k)\,dx = 1$

nonegative since each $P_k(x;\theta_k)$ is pdf then $P_k(x;\theta_k)\ge 0$ for $\forall x$

since $a_k \in [0,1]$ for $\forall k$

then $\forall k$, $\alpha_k P_k(x;\theta_k)\ge 0$ then $p(x)=\sum_{k=1}^{K}\alpha_k P_k(x;\theta_k)\ge 0$

(2) (a) $\mu = E[X] = \int x\,p(x)\,dx = \int x \sum_{k=1}^{K}\alpha_k P_k(x;\theta_k)\,dx$

$$= \sum_{k=1}^{K}\alpha_k \int x\, P_k(x;\theta_k)\,dx$$

$$= \sum_{k=1}^{K}\alpha_k \mu_k$$

(b) $E[X^2] = \int x^2 p(x)\,dx = \sum_{k=1}^{K}\alpha_k \int x^2 P_k(x;\theta_k)\,dx$

$$= \sum_{k=1}^{K}\alpha_k \left[\mathrm{var}_k[X] + (E_k[X])^2 \right]$$

$$= \sum_{k=1}^{K}\alpha_k (\sigma_k^2 + \mu_k^2)$$

$$(E[X])^2 = \left(\sum_{k=1}^{K}\alpha_k\mu_k\right)^2$$

$$\sigma^2 = \mathrm{var}(X) = E[X^2]-(E[X])^2 = \sum_{k=1}^{K}\alpha_k(\sigma_k^2+\mu_k^2) - \left(\sum_{k=1}^{K}\alpha_k\mu_k\right)^2$$

Problem 6

$$\Sigma = \begin{pmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

① $$0 < C = p(\underline{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^t \Sigma^{-1}(\underline{x}-\underline{\mu})}$$

let $K = (2\pi)^{d/2}|\Sigma|^{1/2} > 0$

then α $$Ck = e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^t \Sigma^{-1}(\underline{x}-\underline{\mu})}$$

$$-2\log(Ck) = (\underline{x}-\underline{\mu})^t \Sigma^{-1}(\underline{x}-\underline{\mu}) = (x_1-\mu_1, x_2-\mu_2)\frac{\begin{pmatrix} \sigma_2^2 & -r\sigma_1\sigma_2 \\ -r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}}{|\Sigma|}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$0 < -2|\Sigma|\log(Ck) = (a \ \ b)\begin{pmatrix} \sigma_2^2 & -r\sigma_1\sigma_2 \\ -r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$ where let $a = x_1-\mu_1$, $b = x_2-\mu_2$

$$= (a\sigma_2^2 - br\sigma_1\sigma_2, \ -ar\sigma_1\sigma_2 + b\sigma_1^2)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= a^2\sigma_2^2 - abr\sigma_1\sigma_2 - abr\sigma_1\sigma_2 + b^2\sigma_1^2$$

$$= \sigma_2^2(x_1-\mu_1)^2 - 2\sigma_1\sigma_2 r(x_1-\mu_1)(x_2-\mu_2) + \sigma_1^2(x_2-\mu_2)^2 = (**)$$

by https://en.wikipedia.org/wiki/Ellipse

the $(**)$ is the ellipse $\sigma_2^2(x_1-\mu_1) + \sigma_1^2(x_2-\mu_2)^2 = -2|\Sigma|\log(Ck) > 0$

rotate $\arctan\left(\frac{1}{-2\sigma_1\sigma_2 r}\left(\sigma_1^2-\sigma_2^2-\sqrt{(\sigma_2^2-\sigma_1^2)^2+(2\sigma_1\sigma_2 r)^2}\right)\right)$ counter clock wise

② see last page

problem 7

① for $i=1$: floor $(N/2)$

U_1 vector = Uniform(0,1) · ones(1, floor(d/2))

U_2 vector = Uniform(0,1) · ones(1, floor(d/2))

for $i=1$: length(U_1vector)

$Z[2i, 2i+1]$ = box-muller (U_1 vector(i), U_2 vector(i)) %which give 2·floor(N/2) normal distribution number N(0,1)

end

A = cholesky (Σ)

$\underline{z}$ = $z(1:N)$

$\underline{x}$, = $A\underline{z} + \mu$

output $\underline{x}$

② get box-muller involves multiplication $O(d^2)$

cholesky involves geting eigenvalues matrix $O(d^3)$

geting $\underline{x}$ need matrix mutitiply for N values $O(Nd)$'

$O(Nd^2 + d^3)$

③ since $\underline{z}$ is multi normal distribution and $A\underline{z} + \mu$ is linear transformation then $\underline{x}$ is also normal distribution.

$$E[X] \underset{\text{by Problem ①}}{=\!=} A\,E[\underline{z}] + \mu = A\cdot 0 + \mu = \mu$$

$$Var[X] = E[(X-\mu)^2] = E[(A\underline{z})^2] = E[A z z^t A^t] = E[A]\cdot E[zz^t]\cdot E[A^t]$$

note $E[zz^t] = \begin{pmatrix} E[t_1^2] & E[t_1 t_2] & & \\ E[t_2 t_1] & \ddots & & \\ & & \ddots & E[t_{n-1} t_n] \\ & & E[t_n t_{n-1}] & E[t_n^2] \end{pmatrix}$

note $E[t_i^2] = Var(t_i) + E[t_i]^2$
$= 1 + 0 = 1$

independent
$E[t_i t_j] = E[t_i]\, E[t_j] = 0\cdot 0 = 0$
$i \neq j$

thus $E[zz^t] = I$

thus $Var[X] = E[A]\, I\, E[A^t] = E[AA^t] = \Sigma$

Problem 8

① $(\Rightarrow)$ $$P(C=1|\underline{x}) = \frac{1}{1+e^{(-d_0-d^t\underline{x})}} \iff \log\frac{P(C=1|\underline{x})}{P(C=0|\underline{x})} \text{ is linear in } \underline{x}$$

$$P(C=0|\underline{x}) = 1 - P(C=1|\underline{x}) = \frac{e^{(-d_0-d^t\underline{x})}}{1+e^{(-d_0-d^t\underline{x})}}$$

then $$\log\frac{P(C=1|\underline{x})}{P(C=0|\underline{x})} = \log\frac{1/(1+e^{(-d_0-d^t\underline{x})})}{e^{(-d_0-d^t\underline{x})}/(1+e^{(-d_0-d^t\underline{x})})} = \log e^{d_0+d^t\underline{x}}$$
$$= d_0+d^t\underline{x}$$

$$P(C=1|\underline{x}) = \frac{1}{1+e^{(-d_0-d^t\underline{x})}} \Leftarrow \log\frac{P(C=1|\underline{x})}{P(C=0|\underline{x})} \text{ is linear in } \underline{x}$$

$(\Leftarrow)$

$$P(C=1|\underline{x}) = \frac{P(\underline{x}|C=1)\,P(C=1)}{P(\underline{x}|C=1)\,P(C=1) + P(\underline{x}|C=0)\,P(C=0)}$$

$$= \frac{1}{1 + \frac{P(\underline{x}|C=0)\,P(C=0)}{P(\underline{x}|C=1)\,P(C=1)}}$$

$$= \frac{1}{1+\frac{P(C=0)}{P(C=1)}\,e^{d_1+d^t x}}$$

WLOG let $e^{d_0} = \frac{P(C=0)}{P(C=1)}e^{d_1}$

$$= \frac{1}{1+e^{d_0+d^t x}}$$

② let $\Sigma^{-1} = (k_{ij}) := \begin{pmatrix} k_{11} & k_{12} & & \\ k_{21} & & \ddots & \\ & & & \\ & & & k_{nn} \end{pmatrix}$

$$(X-\mu_2)^t = (X^1-\mu_2^1, X^2-\mu_2^2, \cdots X^n-\mu_2^n)$$

$$(X-\mu_2) = \begin{pmatrix} X^1-\mu_2^1 \\ \vdots \\ X^n-\mu_2^n \end{pmatrix}$$

$$(X-\mu_2)^t \Sigma^{-1} = \left(\sum_{i=1}^n k_{i1}(X^i-\mu_2^i), \cdots\cdots, \sum_{i=1}^n k_{in}(X^i-\mu_2^i) \right)$$

$$① = (X-\mu_2)^t \Sigma^{-1}(X-\mu_2) = \sum_{j=1}^n \left\{ (X^j-\mu_2^j) \sum_{i=1}^n k_{ij}(X^i-\mu_2^i) \right\}$$

$$= \sum_{j=1}^n (X^j-\mu_2^j)\left(\sum_{i=1}^n k_{ij}X^i - \sum_{i=1}^n k_{ij}\mu_2^i \right)$$

$$= \sum_{j=1}^n X^j \sum_{i=1}^n k_{ij}X^i - \sum_{j=1}^n X^j \sum_{i=1}^n k_{ij}\mu_2^i - \sum_{j=1}^n \mu_2^j \sum_{i=1}^n k_{ij}X^i + \sum_{j=1}^n \mu_2^j \sum_{i=1}^n k_{ij}\mu_2^i$$

$$\downarrow$$

$$- \sum_{j=1}^n \sum_{i=1}^n k_{ij}\mu_2^i X^j$$

$$\downarrow$$

$$- \sum_{i=1}^n \mu_2^i \sum_{j=1}^n k_{ij}X^j$$

$$= X^t\Sigma^{-1}X - \mu_2^t\Sigma^{-1}X - \mu_2^t\Sigma^{-1}X + \mu_2^t\Sigma^{-1}\mu_2$$

$$② = (X-\mu_1)^t\Sigma^{-1}(X-\mu_2) = X^t\Sigma^{-1}X - \mu_1^t\Sigma^{-1}X - \mu_1^t\Sigma^{-1}X + \mu_1^t\Sigma^{-1}\mu_1$$

$$① - ② = -2\mu_2^t\Sigma^{-1}X - 2\mu_1\Sigma^{-1}X + \mu_1^t\Sigma^{-1}\mu_1 - \mu_2^t\Sigma^{-1}\mu_2$$

$$= (-2\mu_2^t\Sigma^{-1} + 2\mu_1\Sigma^{-1})X + \mu_1^t\Sigma^{-1}\mu_1 - \mu_2^t\Sigma^{-1}\mu_2$$

thus $$p(x|C=1) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-1/2 (x-\mu_1)^t \Sigma^{-1} (x-\mu_1)}$$

$$p(x|C=0) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-1/2 (x-\mu_2)^t \Sigma^{-1} (x-\mu_2)}$$

$$\frac{P(C=1|x)}{P(C=0|x)} = \frac{p(x|C=1)\, P(C=1)}{p(x|C=0)\, P(C=0)}$$

thus $$\log \frac{P(C=1|x)\ P(C=1)}{P(C=0|x)\ P(C=0)} = \log(e^{-1/2 (x-\mu_1)^t \Sigma^{-1} (x-\mu_1)}) - \log(e^{-1/2 (x-\mu_2)^t \Sigma^{-1} (x-\mu_2)})$$

$$\log \frac{P(C=1|x)}{P(C=0|x)} = -\frac{1}{2} [(x-\mu_1)^t \Sigma^{-1} (x-\mu_1) - (x-\mu_2)^t \Sigma^{-1} (x-\mu_2)] + \log(\frac{P(C=1)}{P(C=0)})$$

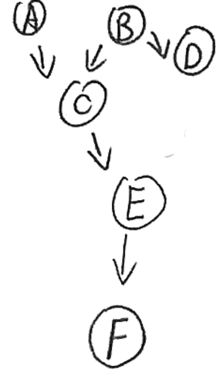
by page before

$$= \underbrace{(\mu_2^t \Sigma^{-1} - \mu_1 \Sigma^{-1})}_{\text{let} = \alpha^t} x \underbrace{- \frac{1}{2} \mu_1^t \Sigma^{-1} \mu_1 + \frac{1}{2} \mu_2^t \Sigma^{-1} \mu_2 + \log(\frac{P(C=1)}{P(C=0)})}_{\text{let} = \alpha_0}$$

$$= \alpha^t x + \alpha_0$$

by ① $P(C=0|x)$ is in the form of a logistic function

Problem 9



(1)

$$P(a,b,c,d,e,f) = P(f|e)\ P(e|c)\ P(d|b)\ P(c|a,b)\ P(b)\ P(a)$$

(2) for a, b (need K-1)

for c need $(K-1)(K^2)$ since 2 parent

for d 1 parent $(K-1)K$

for e 1 parent $(K-1)K$

for f 1 parents $(K-1)K$

parameter $= (K-1)K^2 + 3(K-1)K + 2(K-1)$

(3) $K^6 - 1$

(4)
$$
\begin{aligned}
P(f_K|a^*,d^*) &= \sum_{a,b,c,d,e} P(a,b,c,d,e,f|a^*,d^*)\\
&= \sum_{a,b,c,d,e} P(f|e)\,P(e|c)\,P(d^*|b)\,P(c|a^*,b)\,P(b)\,P(a^*)\\
&= P(a^*) \sum_{b,c,d,e} P(f|e)\,P(e|c)\,P(d^*|b)\,P(c|a^*,b)\,P(b)\\
&= P(a^*) \sum_{b,c,e} P(f|e)\,P(e|c)\,P(c|a^*,b)\ g_1(b) && \text{where } g_1(b) \equiv P(b)\,P(a^*\\
&= P(a^*) \sum_{c,e} P(f|e)\,P(e|c) \sum_{b} P(c|a^*,b)\,g_1(b)\\
&= P(a^*) \sum_{c,e} P(f|e)\,P(e|c)\ g_2(c) && \text{where } g_2(c) = \sum_b P(c|a^*,b)\,g\\
&= P(a^*) \sum_{e} P(f|e) \sum_{c} P(e|c)\,g_2(c)\\
&= P(a^*) \sum_{e} P(f|e)\ g_3(c) && \text{where } g_3(c) = \sum_c P(e|c)\,g_2(c)\\
&= P(a^*)\ g_4(e) && \text{where } g_4(e) = \sum_e P(f|e)\,g_3(e
\end{aligned}
$$

5. grahic model $K^2+K^2+K^2+K^2 = 4K^2$

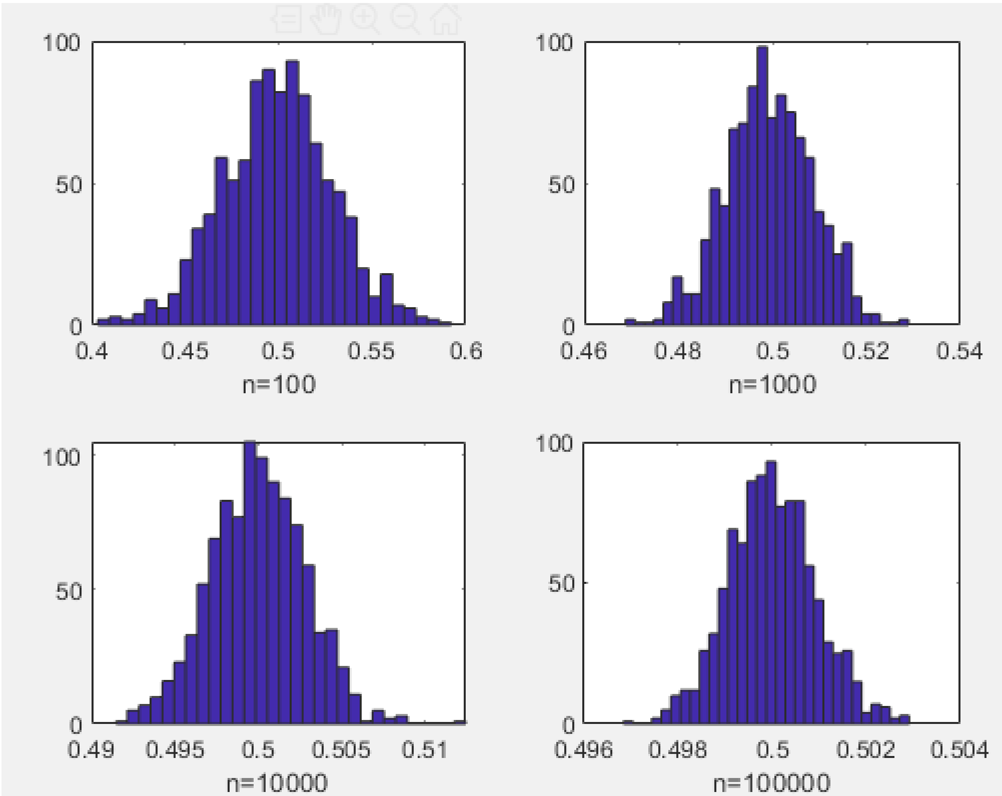
$O(nk^2)$ where n is unobserved nodes

satured moder $O(k^n)$

6. $O(k^3)$ since $\sum_{a,b} P(c|a,b)\, P(a)\, P(b)\, P(d^*|b)$

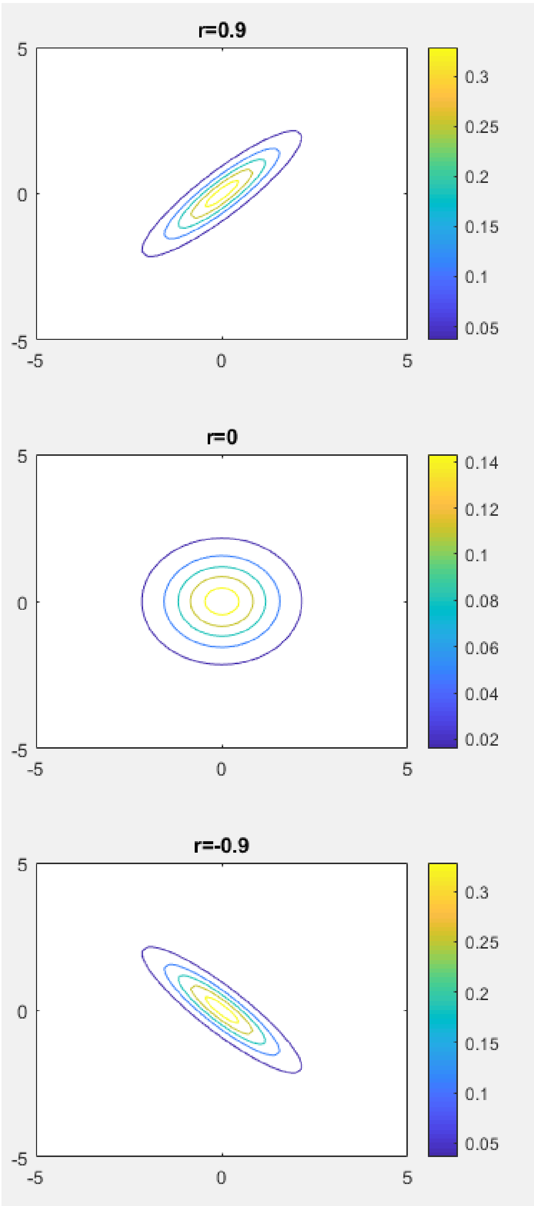
will have $(k+1)K^2 = k^3$ complexity

Problem 4




```
>> cs274ahw1prob4
          n = 100          n = 1000          n= 10000             n= 100000

mean_lst =

   0.500315260996849   0.499742398791919   0.499985623765814   0.500038984282360


var_lst =

   0.029361103601407   0.009081463005706   0.002908322176448   0.000879601183466


error_mean =

   1.0e-03 *

   0.315260996848643   0.257601208080827   0.014376234185931   0.038984282359600


error_var =

   0.029352770268074   0.009081379672373   0.002908321343114   0.000879601175132
```

Problem 6



The below are the P values respect to the isocontour.

	yellow	light green	deep green	light blue	deep blue
Figure 1	0.328613832616992	0.255588536479883	0.182563240342773	0.109537944205664	0.036512648068555
Figure 2	0.143239448782706	0.111408460164327	0.079577471545948	0.047746482927569	0.015915494309190
Figure 3	0.328613832616992	0.255588536479883	0.182563240342773	0.109537944205664	0.036512648068555