Stats 270, Homework 2

Due date: January 28

1. Let $\{X_n\}$ be a homogeneous Markov chain with state space E and transition matrix $\mathbf{P}$. Define $Y_n = (X_n, X_{n+1})$. The process $\{Y_n\}$ is also a homogeneous Markov chain with a state space $F = \{(i_0, i_1) \in E^2 : p_{i_0 i_1} > 0\}$. $\{Y_n\}$ is called a snake chain based on $\{X_n\}$.

   (a) Derive the general entry of the transition matrix of $\{Y_n\}$.

   (b) Show that if $\{X_n\}$ is irreducible, then so is $\{Y_n\}$.

   (c) Show that if $\{X_n\}$ has a stationary distribution $\boldsymbol{\pi}$, then $\{Y_n\}$ also has a stationary distribution. Express the general entry of this stationary distribution in terms of $\boldsymbol{\pi}$ and $\mathbf{P}$.

2. Prove that an irreducible homogeneous Markov chain on a finite state space is positive recurrent.
   Hint: Start by proving recurrence (don't worry about positive for now) of the Markov chain. Try to accomplish this using a proof by contradiction. Then use the fact that a recurrent Markov chain has an invariant measure to complete the proof.

3. It is possible to extend the Wright-Fisher model of genetic drift to include possibility of mutation between the two allelic types. Let $2m$ be the population size and $\{X_n\}$ be the number of A alleles in the population. We define mutation probabilities $u = \Pr(a \to A) > 0$ and $v = \Pr(A \to a) > 0$. Assuming that after sampling with replacement from the previous generation, each gene mutates to an opposite type with the corresponding probability, it is easy to show that one-step transition probabilities of $\{X_n\}$ remain the binomial form,

$$p_{ij} = \binom{2m}{j} p_i^j (1 - p_i)^{2m-j},$$

   where

$$p_i = \frac{i}{2m}(1 - v) + \left(1 - \frac{i}{2m}\right) u.$$

   (a) Argue that $\{X_n\}$ has a *unique* stationary distribution $\boldsymbol{\pi} = (\pi_0, \dots, \pi_{2m})$.

   (b) Derive the following formula for the stationary mean

$$\mu \overset{\text{def}}{=} \sum_{i=0}^{2m} i\pi_i = \frac{2mu}{u + v}.$$

   Hint: Do not try to find a formula for $\boldsymbol{\pi}$. Instead, use the following representation of the mean vector $\boldsymbol{\mu} = \boldsymbol{\pi}\mathbf{x}$, where $\mathbf{x}^T = (0, 1, \dots, 2m)$ (column vector).

   (c) Check numerically the validity of the above formula using simulations and the ergodic theorem. Use $2m = 10$, $u = 0.3$, and $v = 0.1$. Provide your code and numerical results.