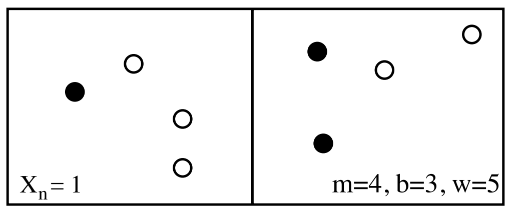
Stats 270, Homework 1

Due date: January 21

1. In the Bernoulli-Laplace model, we start with two boxes and m particles in each box. Among the $2m$ particles there are b black particles and w white particles and $b < w$. At each time step, one particle is selected uniformly at random from each box, and the two particles are exchanged. If we let X_n be the number of black particles in the first box, then $\{X_n\}$ is a Markov chain.



   (a) What is the state-space of $\{X_n\}$?

   (b) Write down one-step transition probabilities of $\{X_n\}$.

   (c) Is $\{X_n\}$ irreducible?

2. Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space Ω and transition probability matrix $\mathbf{P}$. Let τ be the first time n for which $X_n \neq X_0$, where $\tau = +\infty$ if $X_n = X_0$ for all $n \geq 0$. Express $\mathrm{E}[\tau \mid X_0 = i]$ in terms of p_{ii}.

3. Let $\{X_n\}_{n\geq 0}$ be a homogeneous Markov chain with state space $\Omega = \{1, 2, 3, 4\}$ and transition matrix
$$\mathbf{P} = \begin{pmatrix} 0.2 & 0.3 & 0.5 & 0 \\ 0 & 0.2 & 0.3 & 0.5 \\ 0.5 & 0 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0 & 0.2 \end{pmatrix}.$$
Use the first step analysis to compute the probability that when starting from state 1, the chain hits state 3 before it hits state 4?

4. Write a routine to simulate realizations of the gambler's ruin chain $\{X_n\}$ with probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, $p + q = 1$. The routine should stop simulations as soon as you hit one of the absorbing states. Your input will consist of an initial state i, state space size $N = a + b$, and probability of increasing gambler's fortune p. The routine should return a vector of Markov chain states until absorption.

(a) Provide the source code in any computer language of your choice and output of your routine in the form of 5 random realizations of the Markov chain for input parameters $N = 10$, $i = 3$, and $p = 0.27$.

(b) Use your simulation routine to estimate the probability of reaching the largest state $N = 10$ starting at state 5, $u(5, p)$, for probabilities $p_{i,i+1} = p = 0.1, 0.2, \ldots, 0.9$. Turn in a graph with estimated $u(5, p)$ plotted against p. In your graph, include values $u(5, p)$ computed using the formulae that we derived in class.