Stats 270, Homework 5

Due date: March 12

1. (a) Simulate $(\mathbf{x}_{1:1000}, \mathbf{y}_{1:1000})$ from a hidden Markov model of the occasional dishonest casino example. Assume that 1=fair die, 2=loaded die and use transition probabilities

$$\mathbf{P} = \begin{pmatrix} 0.98 & 0.02 \\ 0.05 & 0.95 \end{pmatrix},$$

emission probabilities

$$\mathbf{E} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{2} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \end{pmatrix},$$

and initial state $\boldsymbol{\mu} = (\frac{1}{2}, \frac{1}{2})$.

   (b) Implement the backward and forward algorithms for the occasional dishonest casino example. Run these algorithms on the observed states generated in part (a) and use the marginal hidden state reconstruction. Report the inferred hidden states and the simulated true hidden states in a figure with x-axis representing observation indices from 1 to 1000 and y-axis representing hidden state labels, 1=fair die, 2=loaded die.

   (c) (extra credit) Pretend you don't know the initial distribution, transition probabilities, and emission probabilities corresponding to the loaded die. In other words, you only know the first row of the emission probability matrix.

      i. Implement the Baum-Welch algorithm to estimate all unknown parameters.

      ii. Implement the Gibbs algorithm discussed in class that jointly approximates the posterior distribution of hidden states and model parameters.

2. Let X_i, $i = 1, \ldots, n$, be independent exponential random variables with intensities $\lambda_i > 0$, $i = 1, \ldots, n$ respectively. Let $Z = \min(X_1, \ldots, X_n)$ and $J = \arg\min_j X_j$. J is well defined, because the probability that two or more X_i attain the same value is 0. Show that Z and J are independent and that marginally $Z \sim \text{Exp}(\sum_{i=1}^n \lambda_i)$, $\Pr(J = k) = \lambda_k / \sum_{i=1}^n \lambda_i$.
Hint: Start with $\Pr(J = k, Z \geq t)$ and rewrite the event in question in terms of $X_1, \ldots, X_n$. After you obtain the joint distribution, find the marginals via summation/integration.

3. Consider an m-state stable and conservative continuous-time Markov chain X_t. Let $\mathbf{e}(t) = (e_1(t), \ldots, e_m(t))^T$, where

$$e_i(t) = \mathrm{E}(X_t \mid X_0 = i).$$

Derive a differential equation for $\mathbf{e}(t)$. What initial condition does $\mathbf{e}(t)$ satisfy?

4. A household consists of 4 individuals. Let X_t be the number of individuals infected with a flu. We assume that **each pair** of individuals "bump" into each other at rate λ. We further assume that the disease is transmitted with probability 1 when an infected individual meets an uninfected individual. Assume that when everyone in the household is healthy, one of the members becomes infected at rate α and that each infected individual recovers at rate μ independently of each other. We would like to model $\{X_t\}$ as a continuous-time homogeneous Markov chain.

   (a) Write down the infinitesimal generator for $\{X_t\}$.

   (b) Assume $\mu = 0$ and $X_0 = 1$. What is the mean time until all members of the household become infected?