CZ2003: Tutorial 8

Due on March 16, 2021 at 10:30am

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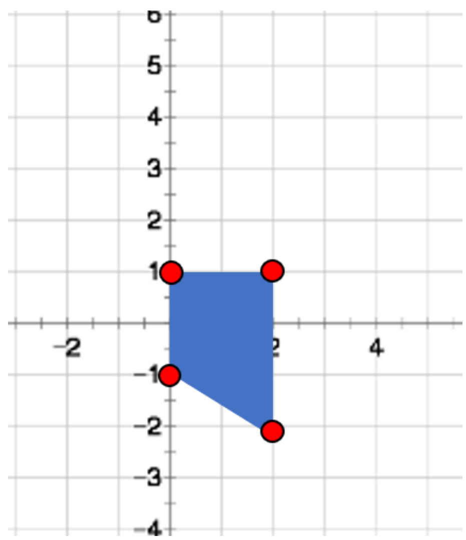
Problem 1

A quadrilateral on the XY plane is defined by four corner points whose homogenous coordinates are $(0, 2, -2)$, $(2, -2, 1)$, $(6, 3, 3)$, $(0, 0.5, 0.5)$. Analyze whether the quadrilateral a square, a rectangle, or a trapezium.

Solution
First, normalize the homogenous coordinates such that the last parameter is 1.
$(0, 2, -2) \to (0, -1, 1)$
$(2, -2, 1) \to (2, -2, 1)$
$(6, 3, 3) \to (2, 1, 1)$
$(0, 0.5, 0.5) \to (0, 1, 1)$
Plotting the 4 corners on a graph, we get:



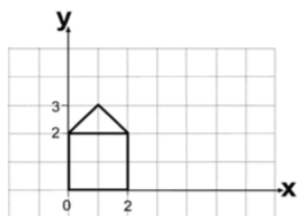
Therefore, it is a **trapezium**.

Problem 2

Figure Q2 shows a 2D house model and a 2D affine transformation matrix **M**.

i Apply **M** to the house and sketch the transformed model. Label the coordinates of all vertices on your sketched figure.

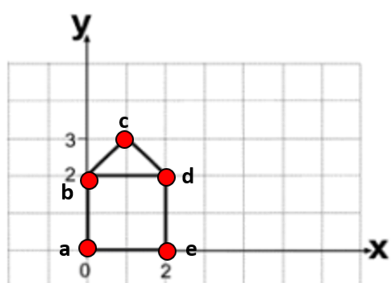
ii This matrix **M** represents a composition of several basic transformations. Write the matrix for each basic transformation and describe the order of these transformations. Note that translation, rotation and scaling are basic transformations.



$$\mathbf{M} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure Q2

Solution
Part i: First, we label the vertices on the original figure and obtain their homogenous coordinates:



$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad d = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad e = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

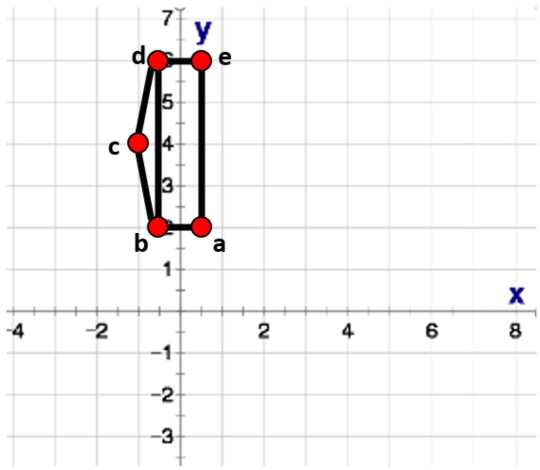
Multiplying **M** to all vertices,

$$a = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 2 \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \quad d = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 6 \\ 1 \end{bmatrix}$$

$$e = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 6 \\ 1 \end{bmatrix}$$

Sketching the resulting house using the new locations of the vertices, we get the following:



Part ii: We can visually deduce the basic transformations and the order they were performed:

1. Rotate CCW about origin for 90 degrees / 0.5π
2. Scale in Y axis by 2 times, X axis by 0.5 times.
3. Translate by y: +2, x: +0.5

We get the matrices for the basic transformations described above,

$$R = \begin{bmatrix} cos(0.5\pi) & -sin(0.5\pi) & 0 \\ sin(0.5\pi) & cos(0.5\pi) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we try to apply the transformation matrices in reverse order (i.e., TSR):

$$SR = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TSR = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.5 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{M}$$

As we see that the resultant affine matrix is the same as **M**, we conclude that the transformation matrices and the order they are applied are correct.

Problem 3

A 2D geometric object is scaled with respect to the point with coordinates $(1, 1)$ in the x-coordinate by 5 times and the y-coordinate by 3 times. Then the object is rotated about the origin by $90°$ in the clockwise direction. Finally, the object is reflected through the x-axis. Write in a proper order the matrices constituting this transformation.

Solution
Steps:

1. Translate by x: -1, y:-1 ($\mathbf{T_1}$)
2. Scale by: x: 5, y:3 ($\mathbf{S_1}$)
3. Translate by x: 1, y: 1 ($\mathbf{T_2}$)
4. Rotate by $90°$ clockwise ($\mathbf{R_1}$)
5. Reflect by x-axis ($\mathbf{R_2}$)

Proper order of multiplying matrices for the transformation: $X' = R_2 R_1 T_2 S_1 T_1 X$

Problem 4

Let **R** be a 2D rotation about the origin and **T** a 2D translation. Do **RT** and **TR** define the same composite transformation? Justify your answer mathematically

Solution

For rotation,

$$R = \begin{bmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $a = cos(\alpha)$, $b = sin(\alpha)$

$$R = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For translation,

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$RT = \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -b & at_x - bt_y + 1 \\ b & a & bt_x + at_y + 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, RT and TR do not define the same transformations.