CZ2003: Tutorial 1

Due on January 19, 2021 at 10:30am

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18/01/2021

Problem 1

A straight line is defined by equation $y = 3x + 4$ in Cartesian coordinate system XY

i Define this straight line in polar coordinates r, a as an explicit function $r = f(a)$

ii Specify the domain for the polar coordinate a in both radians and degrees for this straight line.

Solution
Part i
Recall the conversion between x, y to r

- $x = rcos(a)$
- $y = rsin(a)$

$$\begin{aligned} y &= 3x + 4 \\ rsin(a) &= 3rcos(a) + 4 \\ rsin(a) - 3rcos(a) &= 4 \\ r(sin(a) - 3cos(a)) &= 4 \\ \mathbf{r} &= \mathbf{4/(sin(a) - 3cos(a))} \end{aligned}$$

Part ii
Domain of a is not continuous as r will be undefined when denominator equates to 0:

$$\begin{aligned} sin(a) - 3cos(a) &= 0 \\ sin(a) &= 3cos(a) \\ tan(a) &= 3 \\ a &= tan^{-1}(3) \\ &= 1.249 \ or \ \pi + 1.249 \\ &= 1.249 \ or \ 4.391 \ (radians) \\ &= 71.57^\circ \ or \ 251.56^\circ \ (degrees) \end{aligned}$$

- *Radians*: $a \in (1.249, 4.391)$
- *Degrees*: $a \in (71.57^\circ, 251.56^\circ)$

Problem 2

i Define in polar coordinates $r = f(a)$ the origin-centred circle with radius R. Specify the domain for the polar coordinate a

ii Define in polar coordinates $r = f(a)$ a circle with radius R and the centre at the Cartesian coordinates $(R, 0)$. Specify the domain for the polar coordinate a

Solution

Part i

For a circle centred at the origin with radius R, the circle can be defined in polar co-ordinates by:

$$r = R, a \in [0, 2\pi)$$

Part ii

For a circle centred at $(R, 0)$ with radius R, the circle can be defined in cartesian co-ordinates using the following equation:

$$(x - h)^2 + (y - k)^2 = r^2$$

where,

- h is the x-coordinate of centre of circle
- k is the y-coordinate of centre of circle
- r is the radius of the circle

In this case we have:

$$(x - R)^2 + (y)^2 = R^2$$

Substituting

- $x = rcos(a)$
- $y = rsin(a)$

We get:

$$(rcos(a) - R)^2 + (rsin(a))^2 = R^2$$
$$r^2cos^2(a) - 2R * rcos(a) + R^2 + r^2sin^2(a) = R^2$$
$$r^2cos^2(a) - 2R * rcos(a) + r^2sin^2(a) = 0$$
$$r^2cos^2(a) + r^2sin^2(a) = 2R * rcos(a)$$
$$r(sin^2(a) + cos^2(a)) = 2Rcos(a)$$

Since $sin^2(a) + cos^2(a) = 1$,

$$r = 2Rcos(a)$$

Thus, for a circle centred at $(R, 0)$ with radius R, the circle can be defined in polar co-ordinates by:

$$r = 2Rcos(a), a \in [-0.5\pi, 0.5\pi]$$

Problem 3

With reference to Figure 1, write formulas deriving Cartesian coordinates x, y, z, from the cylindrical r, α, h and spherical coordinates r, α, β. Notice that the axes layout is different in the two cases.

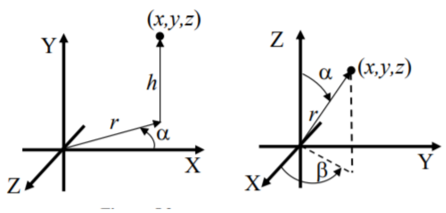


Figure 1: Reference figures for converting between Cartesian to cylindrical and spherical coordinates

Solution

Converting from Cylindrical to Cartesian Coordinates

At first glance, we can easily to derive y from y:

$$\mathbf{y = h}$$

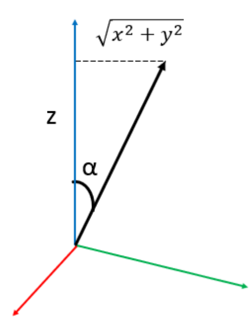
Next, x and z can be derived from r and α like converting from 2-dimensional cartesian coordinates to polar coordinates.

However, it is worth noting that in this case, z-axis is in place of the typical y-axis and the z-axis is also flipped. Therefore, we arrive at the following derivations:

$$\mathbf{x = rcos}(\alpha)$$
$$\mathbf{y = h}$$
$$\mathbf{z = -rsin}(\alpha)$$

Converting from Spherical to Cartesian Coordinates

For converting from spherical coordinates to cartesian coordinates, We look at the projection of R on the XY plane. Let this projection be **p**



We can obtain the following equations:

$$\mathbf{z = rcos}(\alpha)$$
$$p = rsin(\alpha)$$

With the projection of p, we can then get the following

$$x = pcos(\beta)$$
$$y = psin(\beta)$$

Which allows us to arrive at the derivations:

$$\mathbf{x} = \mathbf{rsin}(\alpha)\mathbf{cos}(\beta)$$
$$\mathbf{y} = \mathbf{rsin}(\alpha)\mathbf{sin}(\beta)$$
$$\mathbf{z} = \mathbf{rcos}(\alpha)$$

Problem 4

i With reference to Figure 2, calculate coordinates (numbers) of the unit (magnitude is equal to 1) normal vector N.

ii What are the coordinates of the unit normal vector to the opposite side of the triangle?

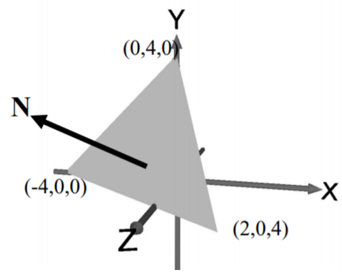
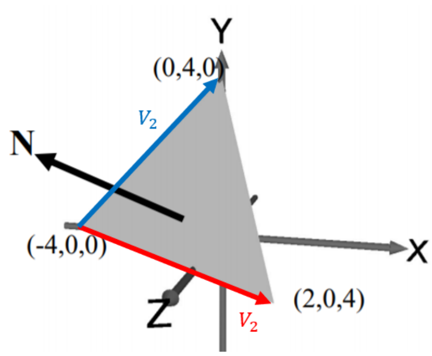


Figure 2: Reference figure for finding Unit Normal Vector from surface

Solution
Part i
To find the normal vector N, we define two other vectors, v_1 and v_2:



The vector Normal to the surface can then be found by obtaining the cross product of the two vectors, $N = v_1 \times v_2$

$$\begin{aligned} v1 &= [(2-(-4))\ (0-0)\ (4-0)] \\ &= [6\ 0\ 4] \\ v2 &= [(0-(-4))\ (4-0)\ (0-0)] \\ &= [4\ 4\ 0] \end{aligned}$$

$$\begin{aligned} N &= v1 \times v2 \\ &= [(0*0)-(4*4)\ (4*4)-(6*0)\ (6*4)-(0*4)] \\ &= [-16\ 16\ 24] \end{aligned}$$

$$\begin{aligned} ||N|| &= \sqrt{(-16)^2+16^2+24^2} \\ &= 8\sqrt{17} \end{aligned}$$

$$\begin{aligned} Unit\ Normal\ Vector, N_n &= \left[\frac{-16}{8\sqrt{17}}\ \frac{16}{8\sqrt{17}}\ \frac{24}{8\sqrt{17}}\right] \\ &= \left[\frac{-2}{\sqrt{17}}\ \frac{2}{\sqrt{17}}\ \frac{3}{\sqrt{17}}\right] \end{aligned}$$

Part ii

To get coordinates of Unit Normal Vector to the opposite side of the triangle, we can apply a scale of -1 to the vector obtained in **Part i**.

$$\begin{aligned} (-1)(N_n) &= \left[-1*\frac{-2}{\sqrt{17}}\ \ -1*\frac{2}{\sqrt{17}}\ \ -1*\frac{3}{\sqrt{17}}\right] \\ &= \left[\frac{2}{\sqrt{17}}\ \frac{-2}{\sqrt{17}}\ \frac{-3}{\sqrt{17}}\right] \end{aligned}$$