CZ2003: Tutorial 9

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Assoc Prof Alexei Sourin - SS3

Pang Yu Shao
U1721680D

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Problem 1

The following VRML code defines a Transform node:

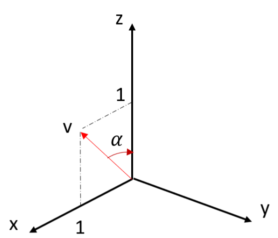
```
Transform {
        rotation     1 0 1   3.1415926
        scale        3 3 3
        translation  3 2 1
children [...]
}
```

Assuming a column represented position vector, write in a proper order the individual matrices composing this transformation. The final matrix is not required.

Solution
Order for VRML Transform node: 1) Scale, 2) Rotation, 3) Translation
Scaling:

$$S = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation:
First, Align vector $V = (1, 0, 1)$ to z-axis



Angle between V to Z-axis (α):
$\alpha = \pi/2 - tan^{-}1(1)$
$\alpha = \pi/4$
Therefore, first rotate about y-axis by $-\alpha = -\pi/4$
Then, since the rotation axis is aligned with z-axis, rotate about z-axis by π
Finally, undo the first preprocessing step by rotating about y-axis by $\pi/4$

$$R = \begin{bmatrix} cos(\pi/4) & 0 & sin(\pi/4) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\pi/4) & 0 & cos(\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\pi) & -sin(\pi) & 0 & 0 \\ sin(\pi) & cos(\pi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(-\pi/4) & 0 & sin(-\pi/4) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(-\pi/4) & 0 & cos(-\pi/4) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation:

Problem 1 continued on next page...

$$T = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, final transformation matrix (M):
$M = TRS$

$$M = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2

A unit cube with vertices at points (0,0,0), (0,1,0), (1,1,0), (1,0,0), (0,0,1), ..., (1,0,1) is transformed into a prism with the respective vertices at points (-1,0,0), (-1,2,0), (2,0,0), (2,-2,0), (0,0,2), ..., (3,-2,2) by an affine transformation matrix. Find a single matrix representing this transformation.

Solution

1.) Choose 4 non-coplanar points:
Before transformation: (0,0,0), (0,1,0), (1,1,0), (1,0,1)
After transformation: (-1,0,0), (-1,2,0), (2,0,0), (3,-2,2)
Transformation Matrix: M

$$M = \begin{bmatrix} a & b & c & l \\ d & e & f & m \\ g & h & i & n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} a & b & c & l \\ d & e & f & m \\ g & h & i & n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\mathbf{l = -1, m = 0, n = 0}$

$$\begin{bmatrix} a & b & c & -1 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$b - 1 = -1$

$\mathbf{b = 0, e = 2, h = 0}$

$$\begin{bmatrix} a & 0 & c & -1 \\ d & 2 & f & 0 \\ g & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$(a - 1 = 2),\ (d + 2 = 0),\ (g = 0)$

$\mathbf{a = 3, d = -2, g = 0}$

$$\begin{bmatrix} 3 & 0 & c & -1 \\ -2 & 2 & f & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

$(3 + c - 1 = 3),\ (-2 + f = -2),\ (i = 2)$

$\mathbf{c = 1, f = 0, i = 2}$

$$\therefore \mathbf{M} = \begin{bmatrix} 3 & 0 & 1 & -1 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3

Assuming a column represented position vector, write in a proper order individual matrices implementing the transformation of reflection about a straight line defined parametrically by $x = 1 - t$, $y = 0$, $z = 2t$, $t \in (-\infty, \infty)$. The final matrix is not required.

Solution

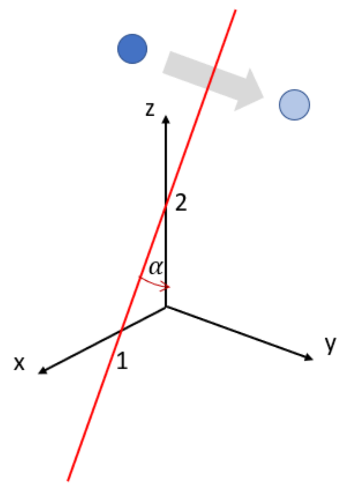
To get straight line, get for t = 1, 0 and -1

t=-1: $(2, 0, -2)$

t=0: $(1, 0, 0)$

t=1: $(0, 0, 2)$

By visualizing the reflection, we get the following figure:



Where $\alpha = tan^{-1}(1/2) = 0.463$

Therefore, the following steps are required:

1) Translate by -1 in x-axis such that the reflection axis passes through the origin

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Rotate about y-axis for an angle of α such that the reflection axis is aligned with z-axis

$$R_2 = \begin{bmatrix} cos(0.463) & 0 & sin(0.463) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(0.463) & 0 & cos(0.463) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0 & 0.447 & 0 \\ 0 & 1 & 0 & 0 \\ -0.447 & 0 & 0.894 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3) Perform reflection about z-axis

$$R_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) Rotate about y-axis for an angle of $-\alpha$ (undo step 2)

Problem 3 continued on next page...

$$R_4 = \begin{bmatrix} cos(-0.463) & 0 & sin(-0.463) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(-0.463) & 0 & cos(-0.463) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.894 & 0 & -0.447 & 0 \\ 0 & 1 & 0 & 0 \\ 0.447 & 0 & 0.894 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5) Translate by 1 in x-axis (undo step 1)

$$T_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, final transformation matrix (M):
$M = T_5 R_4 R_3 R_2 T_1$

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.894 & 0 & -0.447 & 0 \\ 0 & 1 & 0 & 0 \\ 0.447 & 0 & 0.894 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.894 & 0 & 0.447 & 0 \\ 0 & 1 & 0 & 0 \\ -0.447 & 0 & 0.894 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 4

A semi-circle on the ZX plane in shown in Fig.4 (left). It undergoes a sweeping by a full rotation about the Y-axis and a translation along the Y-axis by 2 units simultaneously, which provides a surface shown in Fig.4(right). Utilizing transformation matrices. derive a parametric representation of the surface.

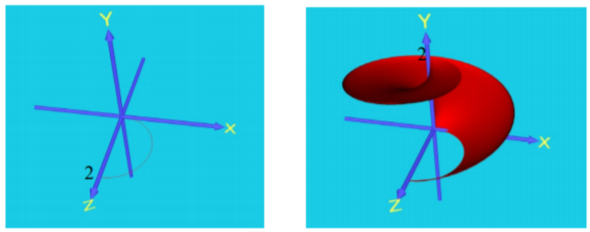


Fig.4

Solution

Step 1: define parametric equations for the semi-circle curve.

$$\begin{aligned} x_0(\alpha) &= sin(\alpha) \\ y_0(\alpha) &= 0 \\ z_0(\alpha) &= 1 + cos(\alpha) \\ alpha &\in [0, \pi] \end{aligned}$$

Step 2: Multiply the coordinates by 3D rotation matrix and figure out range of rotation angle

$$\begin{bmatrix} x(\alpha,\beta) \\ y(\alpha,\beta) \\ z(\alpha,\beta) \\ 1 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\beta) & 0 & cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0(\alpha) \\ y_0(\alpha) \\ z_0(\alpha) \\ 1 \end{bmatrix}$$

$$\begin{aligned} x(\alpha,\beta) &= cos(\beta)x_0(\alpha) + sin(\beta)z_0(\alpha) \\ y(\alpha,\beta) &= y_0(\alpha) \\ z(\alpha,\beta) &= -sin(\beta)x_0(\alpha) + cos(\beta)z_0(\alpha) \\ \beta &\in [0, 2\pi] \end{aligned}$$

Step 3: Handle translational sweeping

$$\begin{aligned} y(\alpha,\beta) &= y_0(\alpha) + f(\beta) \\ f(\beta) &= A + B\beta \\ f(0) &= A \\ &= 0 \\ \therefore A &= 0 \\ f(2\pi) &= 2\pi B \\ &= 2 \\ \therefore B &= 1/\pi \end{aligned}$$

$$\therefore y(\alpha,\beta) = y_0(\alpha) + \beta/\pi$$

$$x(\alpha, \beta) = cos(\beta)sin(\alpha) + sin(\beta)(1 + cos(\alpha))$$
$$y(\alpha, \beta) = \beta/\pi$$
$$z(\alpha, \beta) = -sin(\beta)sin(\alpha) + cos(\beta)(1 + cos(\alpha))$$
$$alpha \in [0, \pi] \quad \beta \in [0, 2\pi]$$

Step 4: Rename α, β to u, v, $u, v \in [0, 1]$

$$u = \alpha/\pi$$
$$v = \beta/2\pi$$

Therefore,

$$x(u, v) = cos(2\pi v)sin(\pi u) + sin(2\pi v)(1 + cos(\pi u))$$
$$y(u, v) = 2v$$
$$z(u, v) = -sin(2\pi v)sin(\pi u) + cos(2\pi v)(1 + cos(\pi u))$$
$$u, v \in [0, 1]$$