10.3 The Dot Product

Definition
If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_{1,2}, b_3 >$ then the dot product of $\vec{a}$ and $\vec{b}$ is the number $\vec{a} \cdot \vec{b}$ given by $\vec{a} \cdot \vec{b}$ given by $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. While for 2 dimensional vectors, $< a_1, a_2 > \cdot < b_1, b_2 >= a_1 b_1 + a_2 b_2$

Ex 1. Find the dot product

$$< 2, 4 > \cdot < 3, -1 >= 2(3) + 4(-1) = 6 - 4 = \boxed{2}$$
$$< -1, 7, 4 > \cdot < 6, 2, -\frac{1}{2} >= -1(6) + 7(2) + 4(-\frac{1}{2}) = \boxed{6}$$
$$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k}) = 1(0) + 2(2) + (-3)(-1) = \boxed{7}$$

Properties
If $\vec{a}, \vec{b}$ and $\vec{c}$ are 3-dimensional vectors, and c is a scalar, then

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$
$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$
$$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$$
$$\vec{0} \cdot \vec{a} = 0$$

Angle Between 2 Vectors
$0 \leq \theta \leq \pi$. If $\vec{a}$ and $\vec{b}$ are parallel, then $\theta = 0$ or $\theta = \pi$

Theorem
If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

Ex 2
If the vectors $\vec{a}$ and $\vec{b}$ have lengths 4 and 6, and the angle between them is $\frac{\pi}{3}$, find $\vec{a} \cdot \vec{b}$.

$$|\vec{a}| = 4, |\vec{b}| = 6$$
$$\vec{a} \cdot \vec{b} = 4(6)\cos\frac{\pi}{3} = 24(\frac{1}{2}) = \boxed{12}$$

Ex 3

Find the angle between the vectors $\vec{a} =< 2, 2, -1 > \& \vec{b} =< 5, -3, 2 >$.

$$\vec{a} \cdot \vec{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

$$|\vec{a}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{(5)^2 + (-3)^2 + (2)^2} = \sqrt{38}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{2}{3\sqrt{38}}$$

$$\theta = \cos^{-1} \frac{2}{3\sqrt{38}} \approx \boxed{1.46}$$

Definition
Two nonzero vectors $\vec{a}$ & $\vec{b}$ are called <u>perpendicular</u> or <u>orthogonal</u> if the angle between them is $\theta = \frac{\pi}{2}$.

Ex 4
Show that $2\vec{i} + 2\vec{j} - \vec{k}$ is perpendicular to $5\vec{i} - 4\vec{j} + 2\vec{k}$.

$$(2\vec{i} + 2\vec{j} - \vec{k}) \cdot (5\vec{i} - 4\vec{4} + 2\vec{k})$$
$$2(5) + 2(-4) + (-1)(2)$$
$$10 - 8 - 2 = 0$$

Thus, the vectors are perpendicular.

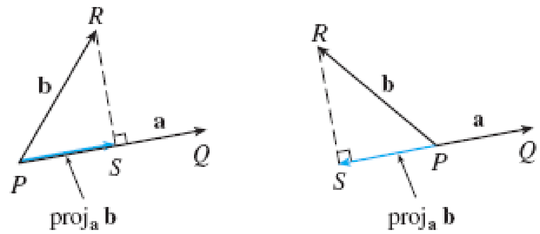
Intepretation of Dot Product

$$0 \leq \theta < \frac{\pi}{2}$$
$$\vec{a} \cdot \vec{b} > 0, \text{since} \cos\theta > 0$$
$$\theta = \frac{\pi}{2}$$
$$\vec{a} \cdot \vec{b} = 0, \text{since} \cos\frac{pi}{2} = 0$$

Theorem
Two vectors $\vec{a}$ & $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$ since $\cos 0 = 1$.
If $\theta = 0$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$ since $\cos 0 = -1$.

Projections
Suppouse we have two vectors, a & b. The projection of b onto another vector, a can be thought of the shadow of vector b that overlaps vector a from vertices P & S or vector $\vec{PS}$. So then the vector $\vec{PS}$ is the vector projection of b onto a or denoted as $\text{comp}_a b$.



There is another type of projection called a scalar projection, which is the signed magnitude of $\text{comp}_a b$.

$$\text{Scalar Projection of } b \text{ onto } a$$
$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

$$\text{Vector Projection of } b \text{ onto } a$$
$$\text{proj}_a b = \frac{a \cdot b}{|a|} \cdot \frac{a}{|a|}$$

Suppose we have two vectors $\vec{a}$ & $\vec{b}$. The vector projection of $\vec{b}$ onto $\vec{a}$ is $\text{proj}_{\vec{a}}\vec{b}$. While the scalar projetion of $\vec{b}$ onto $\vec{a}$ is $\text{comp}_{\vec{a}}\vec{b}$. Also note that the composition formula is derived from the dot product formula like so

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = |\vec{b}| \cos\theta$$

Ex 5
Find the scalar and vector projects of $\vec{b} =< 1, 1, 2 >$ onto $\vec{a} =< -2, 3, 1 >$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{-2(1) + 3(1) + 1(2)}{\sqrt{(-2)^2 + (3)^2 + (1)^2}} = \boxed{\frac{3}{\sqrt{14}}}$$

Calculating Work
The work done bt a constant force f in moving an object through a distance d is $W = FD$.
Suppouse the constant force is a vector $\vec{F}$ pointing in a direction different from the displacement vector $\vec{D}$.

If the force moves the object from points $P \to Q$ along a straight line ($\theta = 0,\ cos(\theta) = 1$), then the work can be calculated like so

$$W = (|\vec{F}| \cos\theta)$$
$$W = |\vec{F}|\vec{D} \cos\theta$$
$$W = \vec{F} \cdot \vec{D}$$

Ex 6
A force is given by a vector $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point

$Q(4,6,2)$, find the work done.

$$\vec{D} = \vec{PQ} =< 4-2, 6-1, 2-0 >$$
$$=< 2, 5, 2 >$$

$$W = \vec{F} \cdot \vec{D} = 3(2) + 4(5) + 5(2)$$
$$= 6 + 20 + 10$$
$$= \boxed{30}$$