12.5 Triple Integrals

Triple integrals are functions of three variables. The simplest case is where f is defined on a rectangular box

$$B = \{(x, yz) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

As always the region, in this case, B is divided into sub-boxes. Done by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ with lengths $\Delta x_i = x_i - x_{i-1}$, dividng $[c, d]$ into m subintervals with lengths $\Delta y_j = y_j - y_{j-1}$, and dividing $[r, s]$ into n subintervals with lengths $\Delta z_k = z_k - z_{k-1}$. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

As shown below. The sub-box B_{ijk} has volume $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$

Which can be used to form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}$$

where the sample point $x^*_{ijk}, y^*_{ijk}, z^*_{ijk}$ is in B_{ijk}. From there we can take the limit of the triple Riemann sum to define the triple integral

Definition
The triple integral of f over the box B is

$$\iiint\limits_B f(x, y, z)\ dV = \lim_{\max \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}$$

if this limit exists.

Though the sample point can be any point in the sub-box, our triple integral definition can be simplified by choosing (x_i, y_j, z_k) as the sample point and also choosing sub-boxes with the same dimensions, so that $\Delta V_{ijk} = \Delta V$

$$\iiint\limits_B f(x, y, z)\ dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\ \Delta V$$

Fubini's Theorem for Triple Integrals
If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint\limits_B f(x, y, z)\ dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\ dxdydz$$

Ex 1
Evaluate the triple integral $\iiint\limits_B xyz^2\ dV$, where B is the rectangular box given by

$$B = \{(x, y, z)|0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

$$\iiint\limits_B xyz^2\ dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\ dxdydz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dydz$$
$$\int_0^3 \int_{-1}^2 \frac{yz^2}{2}\ dydz = \int_0^3 \left]\frac{y^2z^2}{4}\right]_{y=-1}^{y=2} dz = \int_0^3 \frac{3z^2}{4}\ dz = \frac{z^3}{4}\Big|_0^3 = \boxed{\frac{27}{4}}$$

The triple integral can be defined over a general bounded region E in a three dimensional space (a solid). E is enclosed in a box B and a function F is defined to agree with f on E but is 0 for points in B that are outside E. By definition,

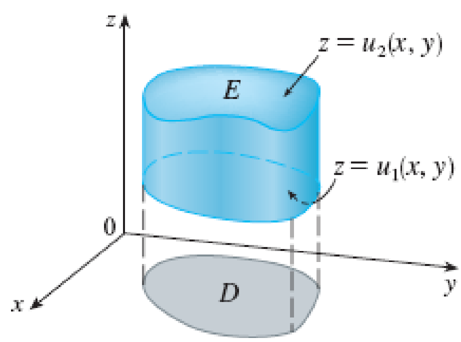
$$\iiint\limits_E f(x, y, z)\ dV = \iiint\limits_B F(x, y, z)\ dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth."

A solid region E is said to of Type I if it lies between the graphs of two continuous functions of x & y, that is,

$$E = \{(x, y, z)|(x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection E onto the xy plane as shown in the figure below. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$



If E is a Type 1 region, then we can calculate the volume of E like so

$$\iiint\limits_E f(x, y, z) = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\right] dA$$

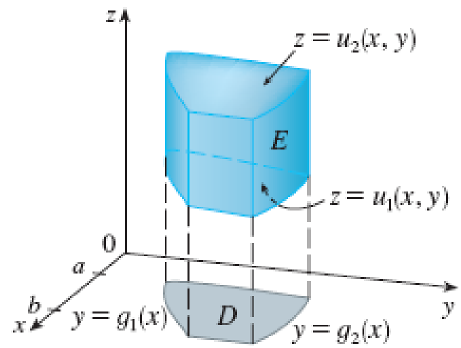
The meaning of the inner integral on the right side is that x & y are held fixed, and therefore $u_1(x, y)$ & $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular if the projection D of E onto the xy plane is a Type I plane region (as in the figure below), then

$$E = \{(x, y, z)|a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

Giving us

$$\iiint\limits_E f(x, y, z)\ dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dzdydx$$

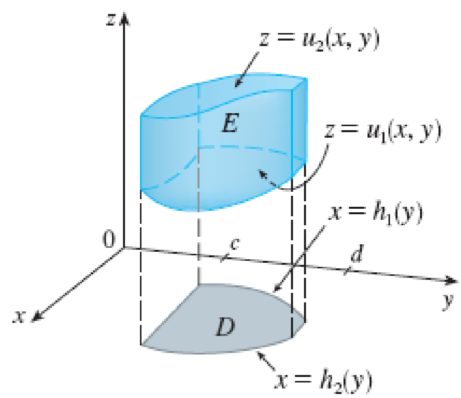


However if D is a Type II plane region like the figure below, then

$$E = \{(x, y, z) | c \leq y \leq d, h_1(y) \leq y \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$

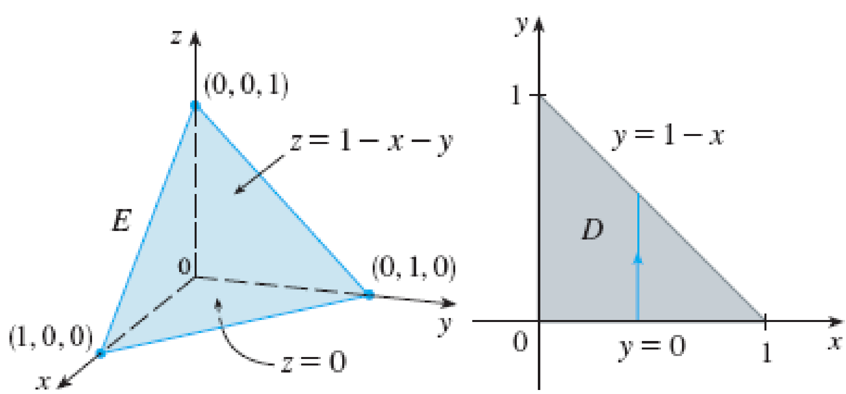
Which gives

$$\iiint\limits_E f(x, y, z)\ dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dzdydx$$



A Type I solid region with a Type II projection is shown in the figure above.

Ex 2
Evaluate $\iiint\limits_E z\ dV$, where E is the solid tetrahedron bounded by the four planes $x = 0, y = 0,$ & $x + y + z = 1$.



After sketching out the solid and region, we find that

$$E = \{(x, y, z) | 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$$

$$\iiint\limits_E z \; dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \; dzdydx$$

$$\int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dydx$$

$$\frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2 \; dydx$$

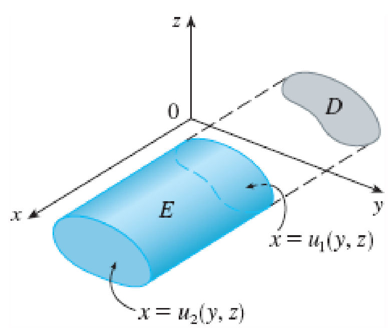
$$\frac{1}{2} \int_0^1 \left[- \frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$\frac{1}{6} \int_0^1 (1 - x)^3 \; dx = \frac{1}{6}\left[- \frac{(1 - x)^4}{4}\right]_0^1 = \boxed{\frac{1}{24}}$$

A solid region E is of Type II if it is of the form

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

Where D is the projection of E onto the yz plane. The back surface is $x = u_1(y, z)$, as shown below



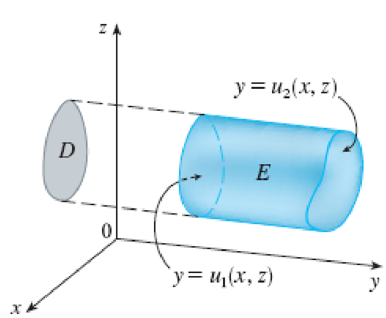
Which gives

$$\iiint\limits_E f(x, y, z) \; dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \; dx\right] dA$$

Finally, a Type III region is of the form

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

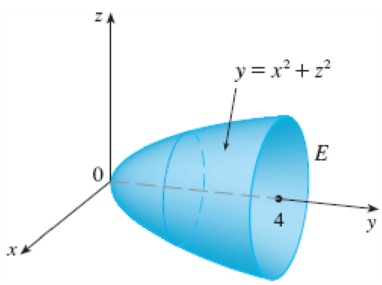
Where D is the projection of E onto the xz plane, $y = u_1(x, z)$ is the left surface and $y = u_2(x, z)$ is the right surface.

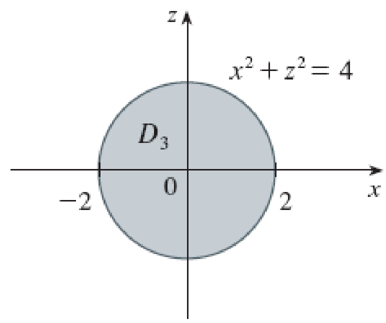


$$\iiint_E f(x,y,z)\ dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right] dA$$

In each of the equations for Type II and Type III regions, there may be two possible expressions for the integral depending on whether D is a Type I or Type II plane region.

Ex 3
Evaluate $\iiint_E \sqrt{x^2+z^2}\ dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.





We can view the solid E as a Type III region with its projection D being on the xz plane. That is the upper and lower bounds of y are $y = 4$ & $y = x^2 + z^2$ respectively.D is also the disk $x^2 + z^2 \leq 4$ based on our boundaries set for y. We can get the region

$$E = \{(x,z)| -2 \leq x \leq 2, -\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}, x^2+z^2 \leq y \leq 4\}$$

$$\iiint_E \sqrt{x^2+z^2}\ dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\ dy\right] dA$$

$$\iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\ dA$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\ dzdx$$

Due to the circular nature of the xz plane, we can also convert to polar coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\ dzdx = \int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{r^2}r\ drd\theta$$

$$\int_0^{2\pi}\int_0^2 4r^2 - r^4\ drd\theta$$

$$\int_0^{2\pi} d\theta \int_0^2 4r^2 - r^2\ dr = 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \boxed{\frac{128\pi}{15}}$$

Applications of Triple Integrals
If the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit voume, at any given point (x,y,z), then its mass is

$$m = \iiint_E \rho(x,y,z)\ dV$$

So its moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)\ dV \qquad M_{xz} = \iiint_E y\rho(x,y,z)\ dV \qquad M_{xy} = \iiint_E z\rho(x,y,z)\ dV$$

The center of mass is located at the point $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{M_{yz}}{m} \qquad \overline{y} = \frac{M_{xz}}{m} \qquad \overline{y} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass of the solid is called the centroid of E. The moments of inertia about the three coordinate axes are

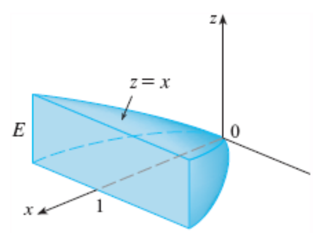
$$I_x = \iiint_E (y^2 + z^2)\rho(x,y,z)\ dV \qquad I_y = \iiint_E (x^2 + z^2)\rho(x,y,z)\ dV$$

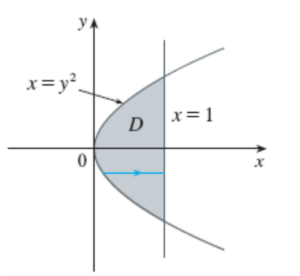
$$I_z = \iiint_E (x^2 + y^2)\rho(x,y,z)\ dV$$

The total electric charge on a solid object occupying a region E and having charge density $\rho(x,y,z)$ is

$$Q = \iiint_E \sigma(x,y,z)\ dV$$

Ex 6
Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0,\ \&\ x = 1$.





$$m = \iiint_E \rho\ dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} \rho\ dV = \frac{4\rho}{5}$$

Since E & ρ is symmetrical about the xz plane, we can sa that $M_{xz} = 0$ and therefore $\overline{y} = 0$.

$$M_{yz} = \iiint\limits_E x\rho \; dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} x\rho \; dzdxdy = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho \; dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z\rho \; dzdxdy = \frac{2\rho}{7}$$

$$(\overline{x}, \overline{y}, \overline{z}) = (\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}) = \boxed{(\frac{5}{7}), 0, \frac{5}{14}}$$