# 3.3 Cramer's Rule, Volume, and Linear Transformations

**Cramer's Rule**

Cramer's rule is integral in a variety of theoretical calculations. An example would be that it can be used to study how the solution of $Ax = b$ is affected by changes in the entries of $b$. However, the formula is inefficient for hand calculations on matrices larger than $3 \times 3$ matrices.

For any $n \times n$ matrix $A$ and any $b$ in $\mathbb{R}^n$, let $A_i(b)$ be the matrix obtained from $A$ by replacing column $i$ by the vector $b$.

$$A_i(b) = \begin{bmatrix} a_1 & ... & b & ... & a_n \end{bmatrix}$$

**Theorem 3.7 Cramer's Rule**

Let $A$ be an invertible $n \times n$ matrix. For any $b$ in $\mathbb{R}^n$, the unique solutions $x$ of $Ax = b$ has entries given by

$$x_i = \frac{\det\ A_i(b)}{\det\ A}, \qquad i = 1, 2, ...n$$

**Ex 1**

Use Cramer's rule to solve the system. As $\det\ A = 2$, the system has a unique solution.

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \qquad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \qquad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det\ A_1(b)}{\det\ A} = \frac{40}{2} = 20, \qquad x_2 = \frac{\det\ A_2(b)}{\det\ A} = \frac{54}{2} = 27$$

$$\boxed{x_1 = 20, x_2 = 27}$$

**Ex 2**

Consider the following system in which $s$ is an unspecified parameter. Determine the values of $s$ for which the system has a unique solution, and use Cramer's rule to describe the solution.

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + sx_2 = 1$$

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \qquad A_1(b) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, qquad A_2(b) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

$$\det\ A = 3s^2 - 12 = 3(s+2)(s-2)$$

$$x_1 = \frac{\det\ A_1(b)}{\det\ A} = \frac{4s+2}{3(s+2)(s-2)}$$
$$x_2 = \frac{\det\ A_2(b)}{\det\ A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}$$

**A Formula for $A^{-1}$**

Cramer's rule leads easily to a gernal formula for the inverse of an $n \times n$ matrix $A$. The $j$th column of $A^{-1}$ is a vector $x$ that satisfies

$$Ax = e_j$$

where $e_j$ is the $j$th column of the identity matrix, and the $i$th entry of $x$ is the $(i,j)$ entry of $A^{-1}$. By $Cramer'srule$,

$$\{(i,j) \text{ entry of } A^{-1}\ \} = x_i = \frac{\det\ A_i(e_j)}{\det\ A}$$

Recall that $A_{ij}$ denotes the submatrix of $A$ formed by deleting row $j$ and column $i$. A cofactor expansion down column $i$ of $A_i(e_j)$ shows that

$$\det\ A_i(e_j) = (-1)^{i+j} \det\ A_{ji} = C_{ji}$$

where $C_{ji}$ is a cofactor of $A$. So the $(i,j)$ entry of $A^{-1}$ is the cofactor $C_{ji}$ divided by det $A$.

$$A^{-1} = \frac{1}{\det\ A} = \begin{bmatrix} C_{11} & C_{21} & ... & C_{n1} \\ C_{12} & C_{22} & ... & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & ... & C_{nn} \end{bmatrix}$$

The matrix of cofactors on the right side is known as the adjugate or classical adjoint of $A$, denoted by adj $A$.

**Theorem 3.8 An Inverse Formula**

Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det\ A} \text{adj } A$$

**Ex 3**

Find the invese of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$$C_{11} = +\begin{vmatrix} 4 & -2 \\ 3 & -2 \end{vmatrix} = -2 \qquad C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} = 3 \qquad C_{13} = +\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} = 5$$

$$C_{21} = +\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = 14 \qquad C_{22} = -\begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} = -7 \qquad C_{23} = +\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 \qquad C_{33} = +\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -3$$

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}$$

Though det$A$ could be computed directly, but utilizing (adj $A$)$A$ = det $A$ is much quicker.

$$\begin{aligned} A^{-1} &= \frac{1}{\det A}\text{adj } A \\ (\det A)A^{-1} &= \text{adj } A \\ (\det A)A^{-1}a &= \text{adj } A \\ (\det A)I &= \text{adj } A \end{aligned}$$

Thus

$$(\text{adj } A)A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I$$

Since (adj $A = 14I$), then the determinant is 14. Thus

$$A^{-1} = \frac{1}{\det A}\text{adj } A$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{bmatrix}}$$

**Theorem 9**
If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of $A$ is $|\det A|$. If $A$ were to be a $3 \times 3$ matrix, the volume would be $|\det A|$.

**Ex 4**
Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1),$ & $(6,4)$.

Translate the parallelogram to one that has the origin as a vertex. For example, subtract $(-2,-2)$ from each of the four vertices. Which results in $(0,0), (2,5), (6,1),$ & $(8,6)$. This parallelogram is determined by the columns of

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} \ |\det A| = |-28| = \boxed{28}$$

**Linear Transformations**
Determinants can be used to describe an important geometric property of linear transformations in the plane and in $\mathbb{R}^3$.

**Theorem 10**
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\text{det } A| \cdot \{\text{area of } S\}$$

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\text{det } A| \cdot \{\text{volume of } S\}$$