# 5.2 The Characteristic Equation

When solving for scalars $\lambda$ such that the matrix equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. Meaning that by the Invertible Matrix Theorem, we find that this problem is equivalent to finding all $\lambda$ such that matrix $A - \lambda I$ is not invertible.

The matrix fails to be invertible precisely when its determinant is zero. So the eigenvalues of $A$ are the solutions of the equation

$$\det(A - \lambda I) = 0$$

**Theorem 5.3 Properties of Determinants**
Let $A$ & $B$ be $n \times n$ matrices.

a) $A$ is invertible if and only if det $A \neq 0$

b) det $AB =$ (det $A$ )(det $B$ )

c) det $A^T =$ det $A$

d) If $A$ is triangular, then det $A$ is the product of the entries on the main diagonal of $A$.

e) A row replacement operation on $A$ does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

**The Invertible Matrix THeorem**
Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if

r) The number 0 is not an eigenvalue of $A$.

**The Characteristic Equation**
The scalar equation det $(A_\lambda I) = 0$ is called the characteristic equation of $A$.

A scalar $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

**Ex 1**
Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A_\lambda I) = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

So the matrix $A$ has the eigenvalues $5, 3, 1$, with 5 having multiplicity 2.