10.4 The Cross Product

Determinant of Order 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Ex 1

$$\begin{bmatrix} 2 & 1 \\ -6 & 4 \end{bmatrix} = 2(4) - 1(-6) = \boxed{14}$$

Determinant of Order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_3 \end{bmatrix}$$

Ex 2

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{bmatrix}$$

$$1 \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} - 2 \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 & 0 \\ -5 & 4 \end{bmatrix}$$

$$1(0(2) - 1(4)) - 2(3(2) - 1(3(2) - 1(-5))) + (-1)(3(4) - 0(-5))$$

$$1(-4) - 2() - 1(12) \rightarrow \boxed{-38}$$

Definition

If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_1, b_2, 2_3 >$, then the <u>cross product</u> of $\vec{a}$ & $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Ex 3

If $\vec{a} =< 1, 3, 4 >$ & $\vec{b} =< 2, 7, -5 >$ find $\vec{a} \times \vec{b}$.

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{bmatrix}$$

$$\vec{i} \begin{bmatrix} 3 & 4 \\ 7 & -5 \end{bmatrix} - \vec{j} \begin{bmatrix} 1 & 4 \\ 2 & -5 \end{bmatrix} + \vec{k} \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

$$\vec{i}(-15 - 28) - \vec{j}(-5 - 8) + \vec{k}(7 - 6)$$

$$\boxed{-43\vec{i} + 13\vec{j} + \vec{k}}$$

Example 4
Show that if $\vec{a} =< a_1, a_2, a_3 >$, then $\vec{a} \times \vec{b} = 0$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \vec{i} \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \vec{j} \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \vec{k} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_1 \end{bmatrix}$$

$$\vec{a}(a_2a_3 - a_3b_2) - \vec{j}(a_1a_3 - a_3a_1) + \vec{k}(a_1a_2a_2a_1)$$

$$\vec{0} + \vec{0} + \vec{0} = \boxed{\vec{0}}$$

Theorem
The vector $\vec{a} \times \vec{b}$ is orthogonal to both vectors $\vec{a}$ & $\vec{b}$.

Proof
Let $\vec{a} =< a_1, a_2, a_3 >$ & $\vec{b} =< b_1, b_2, b_3 >$. Now,

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \vec{i} \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - \vec{j} \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + \vec{k} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_1 \end{bmatrix}$$

$$= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 - a_3b_1 + a_2a_3b_1 + ab_1 + a_3a_1b_2 - a_3a_2b_1 = 0$$

By similar computations, $(\vec{a} \cdot \vec{b}) \cdot \vec{b} = 0$. Therefore, $\vec{a} \cdot \vec{b}$ is orthogonal to both $\vec{a}$ & $\vec{b}$.

Theorem
If θ is the angle between $\vec{a}$ & $\vec{b}$ (so $0 \leq \theta \leq \pi$), then

$$|\vec{a} \cdot \vec{b}| = \vec{a}\vec{b} \sin\theta$$

Corollory
Two nonzero vectors $\vec{a}$ & $\vec{b}$ are parallel if and only if $\vec{a} \cdot \vec{b} = 0$.

Interpretation of Cross Product
Area of parallelogram = base $\cdot$ height $= |\vec{a}||\vec{b}|\sin\theta| = |\vec{a} \times \vec{b}|$, the magnitude of the cross product $\vec{a} \times \vec{b}$.

Ex 5
Find a vector perpendicular to the plane that passes through the points $P(1,4,6), Q(-2,5-1),$ & $R(1,-1,1)$.

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Ex 6
Find the area of the triangle with vectices $P(1,4,6), Q(-2,5-1),$ & $R(1,-1,1)$.

$$\text{Area} = \frac{1}{2}|\vec{PQ} \cdot \vec{PR}|$$

$$\frac{1}{2}\sqrt{(-40)^2 + (-15)^2 + (15)^2}$$

$$\frac{1}{2}\sqrt{2050}$$

$$\frac{1}{2}\sqrt{25(82)}$$

$$\boxed{\frac{5}{2}\sqrt{82}}$$

Theorem
If $\vec{a}, \vec{b}$ & $\vec{c}$ are vectors and c is a scalar, then

$$1)\ \vec{a} \cdot \vec{b} = -\vec{b} \cdot \vec{a}$$

$$2)\ (c\vec{a})$$

$$3)$$

$$4)$$

$$5)$$

$$6)\ \vec{a} \cdot (\vec{b} \cdot \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Triple Products

$$\vec{a} \cdot (\vec{b} \cdot \vec{c}) = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_2 & b_2 & b_3 \\ c_1 & c_2 & c_2 \end{bmatrix}$$

$$V = (\text{Area of Base})(\text{Height})$$

$$|\vec{b} \cdot \vec{c}| \cdot |\vec{a}| \cdot |\cos\theta|$$

$$|\vec{a} \cdot (\vec{b} \cdot \vec{c})|$$

Note
If $\vec{a} \cdot (\vec{b} \cdot \vec{c}) = 0$, then the vectors are coplanar (lie on the same plane).

Ex 7
If $\vec{a} =< 1, 4, -7 >, \vec{b} =< 2, -1, 4 >$ & $\vec{c} =< 0, -9, 18 >$, then

$$\vec{a} \cdot (\vec{b} \cdot \vec{c}) = \begin{bmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{bmatrix}$$

$$1(-18 - (-36)) - 4(36 - 0) - 7(-18 - 0)$$

$$18 - 144 + 126$$

$$0$$

Thus, $\vec{a}, \vec{b},$ & $\vec{c}$ are coplanar.

Torque
Consider a force $\vec{F}$ acting on a rigid body at a point given by a position vector $\vec{r}$. The torque τ (relative to the origin) is

$$\vec{\tau} = \vec{r} \cdot \vec{F}$$

and measures the tendency of the body to rotate about the origin. THe magnitude of the torque vector is

$$|\vec{\tau}| = |\vec{r} \cdot \vec{F}| = |\vec{r}| \cdot |\vec{F}| \cdot \sin\theta$$

where θ is the angle between $\vec{r}$ & $\vec{F}$.

Ex 8
A bolt is tightened by applying a $40N$ force to a $0.25m$ wrench. Find the magnitude of the torque about the center of the bolt.

$$|\tau| = 0.25(40)\sin 75^o$$

$$\boxed{9.66N \cdot m}$$