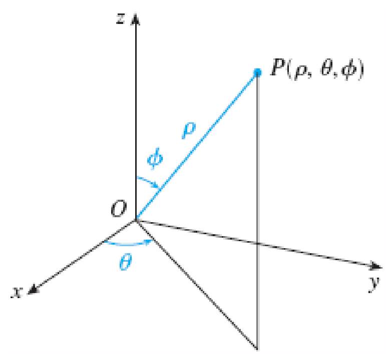
12.7 Triple Integrals in Spherical Coordinates

Introduction
Another useful coordinate system in three dimensions is the spherical coordinate system, simplifying evaluating triple integrals over regions bounded by spheres or cones.

Spherical Coordinates
The spherical coordinates (ρ, θ, ϕ) of a point P is shown below. $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z axis and the line segment OP.



Note that

$$\rho \geq 0 \qquad 0 \leq \phi\pi$$

To convert from spherical to rectangular coordinates

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

The distance formula shows that

$$\rho^2 = x^2 + y^2 + z^2$$

This equation is used to convert from rectangular to spherical coordinates

Ex 1
Convert $(2, \frac{\pi}{4}, \frac{\pi}{3})$ from its spherical coordinate form to rectangular coordinates

$$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \sqrt{\frac{3}{2}}$$

$$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 1$$

$$\boxed{(2, \frac{\pi}{4}, \frac{\pi}{3}) \to (\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1)}$$

Ex 2

Convert the rectangular coordinate point $(0, 2\sqrt{3}, -2)$ into its spherical coordinate form

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$
$$\cos\phi = \frac{z}{\rho} = -\frac{1}{2} \qquad \phi = \frac{2\pi}{3}$$
$$\cos\theta = \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2}$$
$$\boxed{(0, 2\sqrt{3}, -2) \to (4, \frac{\pi}{2}, \frac{2\pi}{3})}$$

Evaluating Triple Integrals with Spherical Coordinates

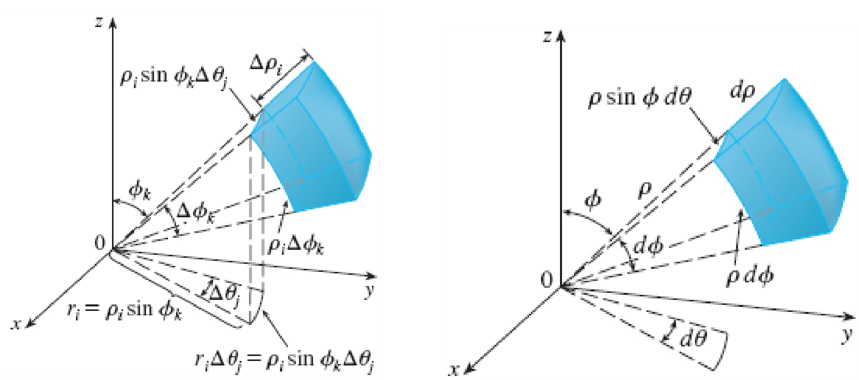
In spherical coordinate systems, the counterpart of a rectangular box is a spherical wedge

$$E = \{(\phi, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

where $a \geq 0, \beta - \alpha \leq 2\pi,\ \&\ d - c \leq 2\pi.$

Though we typically divide solids into small boxes, using smaller spherical wedges yield the same result. We divide E into these smaller partitions E_{ijk} by means of spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.

E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho_i, \rho_i\Delta\phi_k,\ \&\ \rho_i \sin\phi_k \Delta\theta_j$, shown in the figure below.



$$\iiint_E f(x, y, z)\ dV \to \int_c^d \int_\beta^\alpha \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho d\theta d\phi$$

where E is a spherical wedge given by

$$E = \{(\phi, \theta, \phi) a \leq \rho \leq b, \alpha \leq \theta\beta, c \leq \phi \leq d\}$$

Which can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) | \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$$

Usually spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Ex 3

Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}$, where B is the unit ball

$$B = \{(x, y, z)|x^2 + y^2 + z^2\}$$

$$\iiint\limits_B e^{(x^2+y^2+x^2)^{\frac{3}{2}}}\ dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^3}\rho^2 \sin\phi\ d\rho d\theta d\phi = \boxed{\frac{4\pi(e-1)}{3}}$$