11.5 Chain Rule

Introduction
There are several versions of the Chain Rule for multi variable functions. Each of these versions give a rule for differentiating a composite functions.

Chain Rule Case 1
$z = f(x, y)$ with $x, y = g(t), h(t)$, meaning that x & y are both functions of t. Then z is a differentiable function of t. Assume that f_x & f_y are continuous so f is differentiable.

$$z = f(x, y),\ x = g(t),\ y = h(t)$$

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ex 1
If $z = x^2y + 3xy^4$, where $x = \sin 2t$ & $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

$$\frac{\partial z}{\partial x} = 2xy + 3y^4,\ \frac{dx}{dt} = 2\cos 2t \qquad \frac{\partial z}{\partial x} = x^2 + 12xy^3,\ \frac{dy}{i} = -\sin t$$

$$\frac{dz}{dt} = (2xy + 3y^3)(2\cos 2t) + (x^2 + 12x^3)(-\sin t)$$

$$t = 0,\ x = \sin 0 = 0,\ y = \cos 0 = 1$$

$$\frac{dz}{dt}|_{t=0} = (0 + 3)(2 \cdot 1) + (0 + 0)(-\sin 0) = \boxed{6}$$

Ex 2
$PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is $300K$ and increasing

at a rate of $0.1K/s$ and the volume is $100L$ and increasing at a rate of $0.2L/s$.

$$T=300,\ \frac{dT}{dt}=0.1,\ V=100,\ \frac{dV}{dt}=0.2$$

$$P=8.31\frac{T}{V}$$

$$\frac{dP}{dt}=P_T\frac{dT}{dt}+P_v\frac{dV}{dt}=\frac{\partial P}{\partial T}\frac{dT}{dt}+\frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$\frac{\partial}{\partial T}=8.31\frac{T}{V},\ \frac{dT}{dt}=0.1 \qquad \frac{\partial P}{\partial V}=-2\frac{8.31T}{v^2},\ \frac{dV}{dt}=0.2$$

$$\frac{dP}{dt}=0.1(\frac{8.31}{100})-0.2(\frac{8.31(300)}{100^2})=\boxed{-0.04155}$$

The pressure is decreasing at a rate of about $0.042kPa/s$

Chain Rule Case 2

Suppouse that $z=f(x,y)$ is a differentiable function of x & y, where $x=g(s,t)$ & $y=h(s,t)$ are differentiable functions of s & t. Then

$$\frac{\partial z}{\partial s}=z_x x_s+z_y y_s=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t}=z_x x_t+z_y y_t=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex 3

If $z=e^x\sin y$, where $x=st^2$ & $y=s^2t$, find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s}:= \qquad \frac{\partial z}{\partial x}=e^x\sin y,\ \frac{\partial x}{\partial s}=t^2,\ \frac{\partial z}{\partial y}=e^x\cos y,\ \frac{\partial y}{\partial s}=2st$$

$$\frac{\partial z}{\partial s}=t^2e^x\sin y)+2ste^x\cos y$$

$$\boxed{\frac{\partial z}{\partial s}=t^2e^{st^2}\sin{(s^2t)}+2ste^{st^2}\cos{(s^2t)}}$$

$$\frac{\partial z}{\partial s}:= \qquad \frac{\partial z}{\partial x}=e^x\sin y,\ \frac{\partial x}{\partial t}=2ts,\ \frac{\partial z}{\partial y}=e^x\cos y,\ \frac{\partial y}{\partial t}=s^2$$

$$\frac{\partial z}{\partial s}=2ste^x\sin y+s^2e^x\cos y$$

$$\boxed{\frac{\partial z}{\partial s}=2ste^{st^2}\sin{(st^2)}+s^2e^{st^2}\cos{(s^2t)}}$$

General Version

Suppouse that u is a differentiable function of the n variables $x_1,x_2,...,x_n$ and each x_i is a differentiable function of the m variables $t_1,t_2,...,t_m$.

$$\frac{\partial u}{\partial t_i}=\frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_j}+...+\frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

for each $j = 1, 2, ..., m$.

Ex 4
If $u = x^4y + y^2z^3$, where $= rse^t, y = rs^2e^{-t},\ \&\ z = r^2s\sin t$, find $\frac{\partial u}{\partial s}$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\boxed{\frac{\partial u}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)}$$

Implicit Differentiation
Suppouse an equation of the form $F(x, y) = 0$ defines y implicity as a differentiable function of x.

If F is differentiable and $\frac{\partial F}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{Fx}{Fy}$$

Ex 5 Find y' if $x^3 + y^3 = 6xy$, $F(x, y) = x^3 + y^3 - 6xy = 0$

$$F_x = 3x^2 - 6y \qquad F_y = 3y^2 - 6x$$

$$y' = -\frac{3x^2 - 6y}{3y^2 - 6x} = \boxed{-\frac{x^2 - 2y}{y^2 - 2x}}$$

Suppouse z is given implicitly as $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. If F & f are differentiable and $\frac{\partial F}{\partial Z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial F_x}{\partial F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Ex 6
Find $\frac{dz}{dx}$ if $x^3 + y^3 + z^3 + 6xzy = 1$

$$F(x, y, z) = x^3 + y^3 + 6xy - 1 = 0$$

$$F_x = 3x^2 + 6yz \qquad F_z = 3z^2 + 6xy$$

$$\frac{dz}{dx} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = \boxed{-\frac{x^{2+2yz}}{z^2 + 2xy}}$$