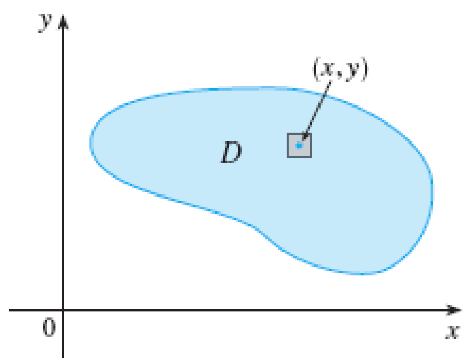
12.4 Application of Double Integrals

Introduction
Imagine a lamina with variable density and suppouse said lamina occupies a region D of the region xy plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $Q(x, y)$, where Q is a continuous function on D. This means that

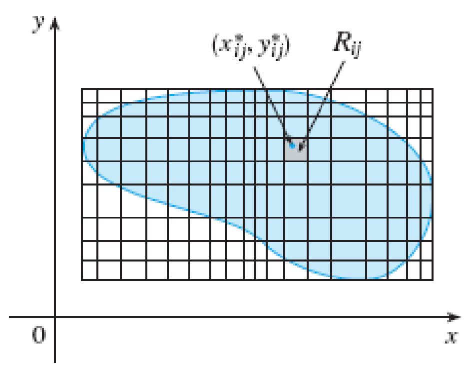
$$Q(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm & ΔA are the mass and area of a small ectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0.



To find the total mass m of the lamina, a rectangle R that contains D is divided into subrectangles R_{ij} and consider $Q(x, y)$ to be 0 outide D. By choosing a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina occupying R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$, where ΔA_{ij} is the area of R_{ij}. By adding all such masses, an approximation of the total mass is obtained.

$$m \approx \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$



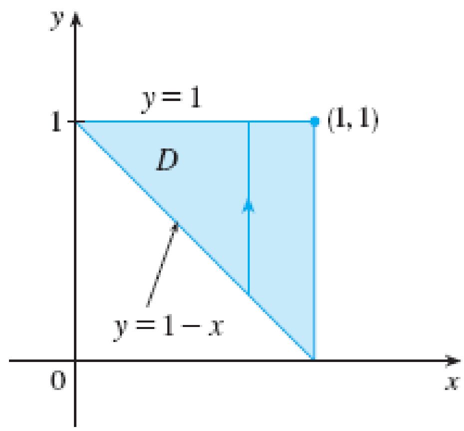
By taking the finer partitions using smaller rectangles, the total mass m of the lamina is obtained through the limit of our summation.

$$m = \lim_{\max\ \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D \rho(x,y)\ dA$$

Other types of density are treated in the same manner by physicists. AN example would be an electric charge distrubuted over a region D and the charge desnity (in units of charge per unit area)is given by $\sigma(x,y)$ at a point (x,y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x,y)\ dA$$

Ex 1
Charge is distributed over the triangular region D in the figure below so that the cahrge density at (x,y) is $\rho(x,y) = xy$, measured in coulombs per square meter C/m^2. Find the total charge.



$$Q = \iint_D \rho(x,y)\ dA = \int_0^1 \int_{1-x}^{1} xy\ dydx$$

$$\int_0^1 [x\frac{y^2}{2}]_{y=1-x}^{y=1}\ dx = \int_0^1 \frac{x}{2}[1^2 - (1-x)^2]\ dx$$

$$\frac{1}{2}\int_0^1 2x^2 - x^3\ dx = \frac{1}{2}[\frac{2x^3}{3} - \frac{x^4}{4}]_0^1 = \boxed{\frac{5}{24}C}$$

Moments and Centers of Mass
Consider a lamina with variable density. That is suppouse the lamina occupies a region D and has density function $\rho(x,y)$. The mass of R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$, so the moment of R_{ij} can be approximated with respect to the x axis by

$$[\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}]y_{ij}^*$$

The moment of the entire lamina about the x axis is obtained from

$$M_x = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D y\rho(x,y)\ dA$$

Similarly, the moment about the y axis is

$$M_y = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D x\rho(x,y)\ dA$$

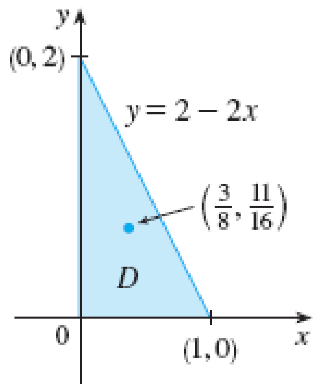
The center of mass is defined as $(\overline{x}, \overline{y})$ so that $m\overline{x} = M_y$ & $m\overline{y} = M_x$. Obtained from

$$m = \iint_D \rho(x,y)\ dA$$

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\ dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\ dA$$

Ex 2

Find the mass and center of mass of a triangular lamina with verticles $(0,0), (1,0),$ & $(0,2)$ if the density function is $\rho(x,y) = 1 + 3x + y$. The triangle's hypotenuse is calculated as $y = 2 - 2x$.



$$m = \iint_D\ dA = \int_0^1 \int_0^{2-2x} 1 + 3x + y\ dydx$$

$$\int_0^1 \left[y + 3xy + \frac{y^2}{2}\right]_{y=0}^{y=2-2x}\ dx$$

$$4\int_0^1 (1 - x^2)\ dx = 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3}$$

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x,y)\ dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} x + 3x^2 + xy\ dydx$$

$$\frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2}\right]_{y=0}^{y=2-2x}\ dx = \frac{3}{2}\int_0^1 x - x^3\ dx$$

$$\frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x,y)\ dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} y + 3xy + y^2\ dydx$$

$$\frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=2-2x}\ dx = \frac{1}{4}\int_0^1 7 - 9x - 3x^2 + 5x^3\ dx$$

$$\frac{1}{4}\left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

$$\boxed{(\overline{x}, \overline{y}) = (\frac{3}{8}, \frac{11}{16})}$$

Ex 3
The density at any point on a semicircular lamina is proportational to the distance from the center of the circle. Find the center of mass of the lamina.

The lamina's density at a any point on the semicircle, $x^2 + y^2 = a^2$ is is proportional to $K\sqrt{x^2 + y^2}$. Where K is some constant and $\sqrt{x^2 + y^2}$ is the distance formula. So $\rho(x, y) = K\sqrt{x^2 + y^2}$ however we must consider the nature of the problem suggesting that we should use spherical coordiantes.

Consider that $\sqrt{x^2 + y^2} = r$, so $K\sqrt{x^2 + y^2} = Kr$, thus $\rho(x, y) = Kr$. The semicircle has a max height of a starting from 0, while it goes from 0 to π on the x axis. So the region D is given $0 \le r \le a, 0 \le \theta \le \pi$. Thus the mass of the lamina is

$$m = \iint_D \rho(x, y)\ dA = \int_\pi^0 \int_0^a (Kr)r\ drd\theta = \int_0^\pi \int_0^a Kr^2\ drd\theta$$
$$K \int_0^\pi d\theta \int_0^a r^2\ dr = K\pi \frac{r^3}{3}\bigg|_0^a = \frac{K\pi a^3}{3}$$

Since the lamina and density function are symmetric with respect to the y axis, so the center of mass must lie on the y axis, that is $\overline{x} = 0$. Recall that $y = r\sin\theta$ The y is given by

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x, y)\ dA = \frac{3}{K\pi a^3}\int_0^\pi \int_0^a r\sin\theta(Kr)r\ drd\theta$$
$$\frac{3}{\pi a^3}\int_0^\pi \sin\theta\ d\theta \int_0^a r^3\ dr = \frac{3}{\pi a^3}\Big[-\cos\theta\Big]_0^\pi \Big[\frac{r^4}{4}\Big]_0^4$$
$$\frac{3}{\pi a^3}\frac{2a^4}{4} = \frac{3a}{2\pi}$$

$$\boxed{(\overline{x}, \overline{y}) = (0, \frac{3a}{2\pi})}$$

Moment of Inertia
The moment of inertia also known as the moment of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. This concept is extended to a lamina with density function $\rho(x, y)$ and occupying a region D.

The moment of inertia about the x axis

$$I_x = \lim_{\max\Delta x_{ij}, \Delta y_j \to 0} \sum_{i=1}^m \sum_{j=1}^n (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} = \iint_D y^2 \rho(x, y)\ dA$$

Similarly, the moment of inertia about the y axis is

$$I_y = \lim_{\max\Delta x_i, \Delta y_j \to 0} (x_{ij}^*, y_{ij}^*)\Delta A_{ij} = \iint_D x^2 \rho(x, y)\ dA$$

It is also of interest to consider the moment of inertia about the origin, also called the polar moment of inertia

$$I_0 = \lim_{\max\Delta x_i, \Delta y_j \to 0} \sum_{i=1}^m \sum_{j=1}^n \Big[(x_{ij}^*)^2 + (y_{ij}^*)^2\Big]\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \iint_D (x^2 + y^2)\rho(x, y)\ dA$$

Note that $I_0 = I_x + I_y$

Find the moments of inertia $I_x, I_y,$ & I_0 of a homogeneous disk D with density $\rho(x,y) = \rho$, center the origin, and radius a

The boundary is the circle $x^2 + y^2 = a^2$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a$. Recall that $x^2 + y^2 = r^2$

$$I_0 = \iint_D (x^2 + y^2)\rho \; dA = \rho \int_0^{2\pi} \int_0^a r^2 r \; dr d\theta$$
$$\rho \int_0^{2\pi} d\theta \int_0^a r^3 \; dr = 2\pi\rho \left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{2}$$

Based on the symmetrical nature of the problem, $I_x = I_y$ so we can say that $I_0 = I_x + I_y \rightarrow \frac{I_0}{2} = I_x = I_y = \frac{\pi\rho a^4}{4}$

Notice that the mass of the disk is

$$m = \text{density} \times \text{area} = \rho(\pi a^2)$$

So the moment of inertia of the disk about the origin can be written as

$$I_0 = \frac{\pi\rho a^4}{2} = \frac{1}{2}(\rho\pi a^2)a^2 = \frac{1}{2}ma^2$$

So by increasing the mass or radius of the disk, in turn the moment of inertia is increased.

The radius of gyration of a lamina about an axis is the number R usc that

$$mR^2 = I$$

where m is the mass of the lamina and I is the moment of inertia about the given axis. Meaning that if the mass of the lamina were to be concentrated at a distance R from the axis, then the moment of inertia of this "point mass" would be the same as the moment of inertia of the lamina.

So

$$m\overline{y}^2 = I_x \qquad m\overline{x}^2 = I_y$$

Thus $(\overline{x}, \overline{y})$ is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes.

Ex 5
Find the radius of gyration about the x axis of the disk in Example 4.

$$m = \rho\pi a^2 \qquad m\overline{y}^2 = I_x \rightarrow \overline{y}^2 = \frac{I_x}{m}$$
$$\overline{y}^2 = \frac{\frac{\pi\rho a^4}{4}}{\rho\pi a^2} = \frac{a^2}{4}$$
$$\boxed{\overline{y} = \frac{a^2}{2}}$$

Therefore the radius of gyration about the x axis is $\overline{y} = \frac{1}{2}a$, half the radius of the disk.