11.5 Chain Rule

Introduction
There are several versions of the Chain Rule for multi variable functions. Each of these versions give a rule for differentiating a composite functions.

Chain Rule Case 11
$z = f(x, y)$ with $x, y = g(t), h(t)$, meaning that x & y are both functions of t. Then z is a differentiable function of t. Assume that f_x & f_y are continuous so f is differentiable.

$$z = f(x, y), \ x = g(t), \ y = h(t)$$

$$\frac{dz}{dt} = z_x \frac{dx}{dx} + z_y \frac{dy}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Ex 1
If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ & $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

$$\frac{\partial z}{\partial x} = 2xy + 3y^4, \ \frac{dx}{dt} = 2\cos 2t \qquad \frac{\partial z}{\partial x} = x^2 + 12xy^3, \ \frac{dy}{i} = -\sin t$$

$$\frac{dz}{dt} = (2xy + 3y^3)(2\cos 2t) + (x^2 + 12x^3)(-\sin t)$$

$$t = 0, \ x = \sin 0 = 0, \ y = \cos 0 = 1$$

$$\frac{dz}{dt}\Big|_{t=0} = (0 + 3)(2 \cdot 1) + (0 + 0)(-\sin 0) = \boxed{6}$$

Ex 2
$PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is $300K$ and increasing

at a rate of $0.1K/s$ and the volume is $100L$ and ijncreasing at a rate of $0.2L/s$.

$$T = 300, \ \frac{dT}{dt} = 0.1, \ V = 100, \ \frac{dV}{dt} = 0.2$$

$$P = 8.31\frac{T}{V}$$

$$\frac{dP}{dt} = P_T\frac{dT}{dt} + P_v\frac{dV}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt}$$

$$\frac{\partial}{\partial T} = 8.31\frac{T}{V}, \ \frac{dT}{dt} = 0.1 \qquad \frac{\partial P}{\partial V} = -2\frac{8.31T}{v^2}, \ \frac{dV}{dt} = 0.2$$

$$\frac{dP}{dt} = 0.1(\frac{8.31}{100}) - 0.2(\frac{8.31(300)}{100^2}) = \boxed{-0.04155}$$

The pressure is decreasing at a rate of about $0.042kPa/s$

Chain Rule Case 2

Suppouse that $z = f(x, y)$ is a differentiable function of x & y, where $x = g(s, t)$ & $y = h(s, t)$ are differentiable functions of s & t. Then

$$\frac{\partial z}{\partial s} = z_x x_s + z_y y_s = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = z_x x_t + z_y y_t = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Ex 3

If $z = e^x \sin y$, where $x = st^2$ & $y = s^2t$, find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} := \qquad \frac{\partial z}{\partial x} = e^x \sin y, \ \frac{\partial x}{\partial s} = t^2, \ \frac{\partial z}{\partial y} = e^x \cos y, \ \frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial z}{\partial s} = t^2e^x \sin y) + 2ste^x \cos y$$

$$\boxed{\frac{\partial z}{\partial s} = t^2e^{st^2} \sin (s^2t) + 2ste^{st^2} \cos (s^2t)}$$

$$\frac{\partial z}{\partial s} := \qquad \frac{\partial z}{\partial x} = e^x \sin y, \ \frac{\partial x}{\partial t} = 2ts, \ \frac{\partial z}{\partial y} = e^x \cos y, \ \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial z}{\partial s} = 2ste^x \sin y + s^2e^x \cos y$$

$$\boxed{\frac{\partial z}{\partial s} = 2ste^{st^2} \sin (st^2) + s^2e^{st^2} \cos (s^2t)}$$