

10.5 Equations of Line and Planes

A line in the xy plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using point-slope form.

Vector Equation

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \langle x, y, z \rangle \rightarrow \text{a point on the line}$$

$$\vec{v} = \text{parallel to line } L$$

If $\vec{v} = \langle a, b, c \rangle$, then $t\vec{v} = \langle ta, tb, tc \rangle$ and

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Thus,

$$x = x_0 + at$$

$$y = y_0 + bt \rightarrow \text{Parametric Equation}$$

$$z = z_0 + ct$$

Note

If a line L passes through the tips of position vectors \vec{r}_0 & \vec{r}_1 , then the vectors $\vec{r}_1 - \vec{r}_0$ is parallel to L and

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

$$\vec{r} = \vec{r}_0 + t\vec{r}_1 - t\vec{r}_0$$

$$\vec{r} = \vec{r}_0 - t\vec{r}_0 + t\vec{r}_1$$

$$\vec{r} = (1 - t)\vec{r}_0 + t\vec{r}_1, \quad 0 \leq t \leq 1$$

The equation above is the vector equation of line segment from $\vec{r}_0 \rightarrow \vec{r}_1$.

Def

2 lines are skew if they do not intersect and are not parallel.

Ex 3

How do we know if two lines are parallel or not? We can use parametric equations as they are able to show if

2 lines are skew lines. The reasoning behind this is these skew lines can be thought of the vector \vec{v} that said lines are parallel to. Intuitively, the skew lines can be represented by \vec{v} from $r = r_0 + t\vec{v}$.

$$r = r_0 + t\vec{v}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\begin{array}{llll} L_1, & x = 1 + t & y = -2 + 3t, & z = 4 - t \\ L_2, & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

$$L_1 = \langle 1, 3, -1 \rangle \quad L_2 = \langle 2, 1, 4 \rangle$$

$\vec{v} \neq \vec{v}_2$ hence these lines are not parallel. Next we are to determine if these lines intersect or not.

$$1 + t = 2s \quad -2 + 3t = 3 + s \quad 4 - t = -3 + 4s$$

$$t = 2s - 1 \quad 3t = 5 + s \quad 7 = 4s + t$$

$$\text{Plug in } t = 2s - 1, \quad 3t = 5 + s \rightarrow 6s - 3 = 5 + s \rightarrow s = \frac{8}{5}$$

$$t = 2s - 1 \rightarrow t = \frac{11}{5}$$

$$7 = 4s + t \rightarrow 7 \neq \frac{32}{5} + \frac{11}{5}$$

Because the third equation fails to be satisfied, these lines are not intersecting nor parallel due to failing the previous test. Hence, these are valid skew lines.

Equations of Planes

A plane cannot be described by a mere point and direction like a line. Thus a vector that is parallel to the plane will not be able to give us the "direction" of a plane. However, if the vector were to be perpendicular, the "direction" of the plane will be given.

A plane can be determined by a point $P_0(x_0, y_0, z_0)$ in the plane and an orthogonal vector \vec{n} . This orthogonal vector \vec{n} is to be called a normal vector. Given an arbitrary point $P(x, y, z)$ and r_0 & r_1 be the position vectors of P_0 & P .

So now the vector $\vec{r} - \vec{r}_0$ be represented by $P_0\vec{P}$. The normal vector \vec{n} is orthogonal to every vector in the given plane especially to the vector $\vec{r} - \vec{r}_0$. Due to the fact that two vectors are orthogonal if their dot product is zero, we have

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\vec{n} \cdot \vec{r} - \vec{n} \cdot \vec{r}_0 = 0$$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

This is the vector equation of our plane. By writing $\vec{n} = \langle a, b, c \rangle, \vec{r} = \langle x, y, z \rangle, \& \vec{r}_0 = \langle x_0, y_0, z_0 \rangle$. We

can obtain a scalar equation of the plane by transforming the vector equation like so

$$n \cdot (r - r_0) = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Ex 4

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $n = \langle 2, 3, 4 \rangle$ (why no arrow?). Find the intercepts and sketch the plane.

$$v = \langle a, b, c \rangle \rightarrow \langle 2, 3, 4 \rangle$$

$$r_0 = \langle x_0, y_0, z_0 \rangle \rightarrow \langle 2, 4, -1 \rangle$$

$$n \cdot (r - r_0) = 0 \rightarrow \langle 2, 3, 4 \rangle \cdot \langle x - 2, y - 4, z + 1 \rangle = 0$$

$$2x + 3y + 4z = 12.$$

Intercepts

$$x, y = z = 0$$

$$y, x = z = 0$$

$$z, x = y = 0$$

$$x = (6, 0, 0) \quad y = (0, 4, 0) \quad z = (0, 0, 3)$$

There is another way to write the equation of a plane

$$ax + by + cz + d = 0, \quad d = -(ax_0 + by_0 + cz_0)$$

This equation is called a linear equation in x, y, z .

Ex 5

Find the equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, & $R(5, 2, 0)$.

$$\vec{PQ} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\vec{i}(8 - (-4)) - \vec{j}(-4 - 16) + \vec{k}(-2 - (-16)) \rightarrow 12\vec{i} + 20\vec{j} + 14\vec{k}$$

$$\vec{i} = (x - 1), \quad \vec{j} = (y - 3), \quad z - 2 = 0$$

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0 \rightarrow \boxed{6x + 10y + 7z - 50}$$

Parallel Planes

If normal vectors of 2 planes are parallel to each other, then those 2 planes are parallel. However if 2 planes are not parallel, then there exists an acute angle between the normal vector of those two planes.

Ex 6

Find the angle between the planes $x + y + z = 1$ & $x - 2y + 3z = 1$. Then find the symmetric equations of the line of intersection L of these two planes.

$$\vec{n} = -(1, 1, 1)$$