4.2

Nul SPaces, Column Spaces, Row Spaces and Linear Transformations

Given an $m \times n$ matrix $A = (a_{ij})$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Def

Null Space of A is $NulA = \{x \in \mathbb{R}^n LAx = 0\}$. The Column Space of A is $ColA = \text{Span}\{a_1, a_2, ..., a_n\}$. The Row Space of A is $RowA = \text{Span}\{r_1, r_2, ...r_m\}$.

Ex 3

Find a spanning set for the null space of the matrix. As in solve $Ax = 0$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 4 \end{bmatrix}$$

Solution, find all x that satisfies $Ax = 0$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & 4 & -0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x must be in $\mathbb{R}^n$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Ex 4

let $w = \{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R}\}$. Find a matrix A, such that $w = \text{Col } A$.

$$w = \{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbb{R}\} = \{a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R}\} = \text{Span}\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \}$$

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

We now use the vectors in the spanning set as the columns of A as shown above. Then $w = ColA$

Theorem 4 1.4
Recall that the columns of A span $\mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for each b. This can be restated like so.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$.

Based on Theorem 1 4.1, If $v_1, ..., v_p$ are in a vector space V, then Span$\{v_1, ..., v_p\}$ is a subspace of V.

Ex 6
Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

A) If the column space of A is a subspace of $\mathbb{R}^k$, what is k? The columns of A each have three entries so Col A is a subspace of $\mathbb{R}^k$, where

B) If the null space of A is a subspace of $\mathbb{R}^k$, what is k? A vector x such that $Ax = 0$ must have four entries so Nul A is a subspace of $\mathbb{R}^k$, where $k = 4$.

To extend our understanding...

Ex 8
Given the matrix A from the Example 6 and the following vectors

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

Answer the following questions.

A) Determine if u is in Nul A. Could u be in Col A?

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since u is not a solution to $Ax = 0$, u is not in the Nul A. Also with four entries, u could not be in Col A as Col A is a subspace of $\mathbb{R}^3$

B) Determine if v is in Col A. Could v be in Nul A?

$$\begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

The equation $Ax = v$ is consistent so v is in Col A. However v cannot be in Nul A as v only has three entries and Nul A is a subspace of $\mathbb{R}^4$.