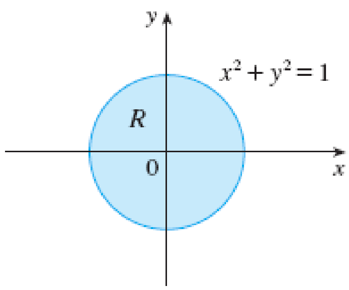
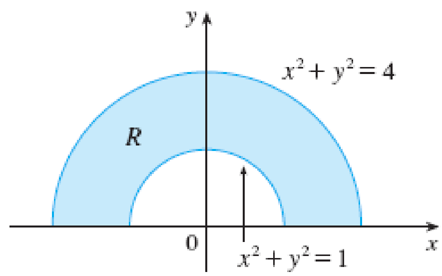
12.3 Double Integrals in Polar Coordinates

Introduction
Suppose we want to evaluate a double integral $\iint_R f(x,y)\,dA$, where R is one of the regions shown below. In either case the description of R in a traditonal coordiante system is complicated. However, R is easily described by using polar coordinates.

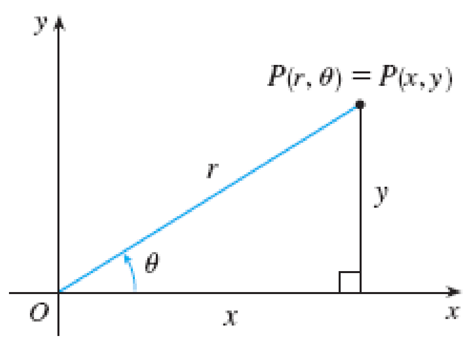




(a) $R = \{(r,\theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r,\theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Recall from the figure below that the polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) by the equations

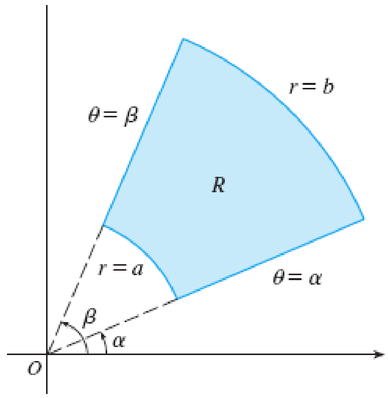
$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

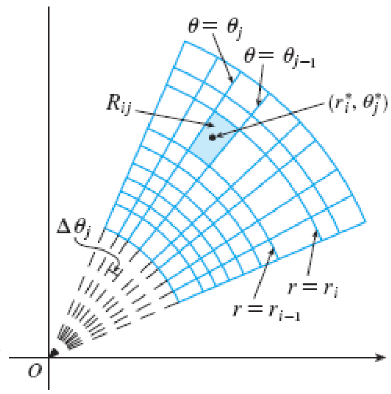


The regions in the first figure are special cases of a polar rectangle

$$R = \{(r,\theta) | a \leq r \leq, \alpha \leq \theta \leq \beta\}$$

shown in the figure below.





The area of R_ij is

$$r_i^* \Delta r_i \Delta\theta_j$$

The rectangular coordinates of the center of R_{ij} are $(r_i^* \cos\theta_j^*), r_i^* \sin\theta_j^*$, so a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r_i \Delta\theta_j$$

If we write $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$, the Riemann sum becomes

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)$$

Which is a Riemann sum for the double integral

$$\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\ drd\theta$$

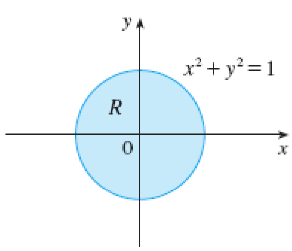
Which is also

$$\int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta) r\ drd\theta$$

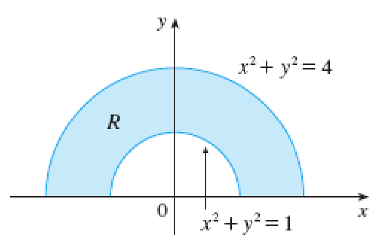
Ex 1
Evaluate $\iint_R 3x + 4y^2\ dA$, where R is the region in the upper halfplane bounded by the circles $x^2 + y^2 = 1$ & $x^2 + y^2 = 4$

The region R can be described as

$$R = \{(x,y)|y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

R is the half right shown on the figure above (right side). In polar coordinates it is given by $1 \le r \le 2$ & $0 \le \theta \le \pi$.

Therefore

$$\iint_R 3x + 4y^2 \; dA = \int_0^{\pi} \int_1^2 3(r\cos\theta) + 4(r^2 \sin^2\theta) r \; dr d\theta$$

$$\int_0^{\pi} \left[\int_1^2 3r^2 \cos\theta + 4r^3 \sin^2\theta \; dr \right] d\theta$$

$$\int_0^{\pi} \left[r^3 \cos\theta + r^4 \sin^2\theta \right]_{r=1}^{r=2} d\theta$$

$$\int_0^{\pi} 7\cos\theta + 15\sin^2\theta \; d\theta$$

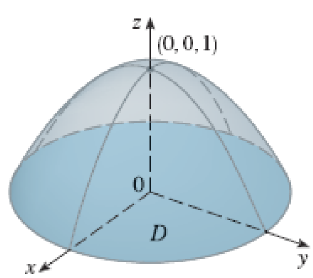
$$\int_0^{\pi} 7\cos\theta + \frac{15}{2}(1 - \cos 2\theta) \; d\theta$$

$$7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \Big|_0^{\pi} = \boxed{\frac{15\pi}{2}}$$

Ex 2
Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

By setting $z = 0$ in the equation $z = 1 - x^2 - y^2$, we get the equation $x^2 + y^2 = 1$. Meaning that the plane intersects the circle $x^2 + y^2 = 1$ formed by the paraboloid. The solid lies under the paraboloid and above the circular disk D given by $x^2 + y^2 \le 1$.

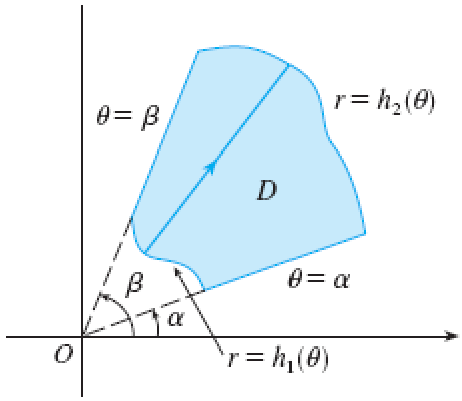
In terms of polar coordinates D is given by $0 \le r \le 1$ & $0 \le \theta \le 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, derived from the Pythagorean Theorem $x^2 + y^2 = r^2$.



$$V = \iint_D 1 - x^2 - y^2 \; dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r \; dr d\theta$$

$$\int_0^{2\pi} d\theta \int_0^1 r - r^3 \; dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \boxed{\frac{\pi}{2}}$$

What has been done so far can be extended to more complicated region types shown in the figure below.



Theorem
If f is continuous on a polar region of the form

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\ drd\theta$$

Ex 3
Find the volume of the solid that lies under the parabolid $z = x^2 + y^2$, above the xy plane and inside the cylinder $x^2 + y^2 = 2x$.

Since $x = r\cos\theta, x^2 + y^2 = 2x \to r^2 = 2x \to r^2 = 2r\cos\theta \to r = 2\cos\theta$.

$$D = \{(r, \theta) | -\frac{\pi}{2} \leq \theta, \frac{\pi}{2}, 0 \leq r \leq 2\cos\theta\}$$

Then we get

$$V = \iint_D x^2 + y^2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 r\ drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\ d\theta = \boxed{\frac{3\pi}{2}}$$