1.4 The Matrix Equation

Introduction
A fundamental idea in Linear Algebra is to view a linear combination of vectors as the product of a matrix and a vector.

Definition
If A is an $m \times n$ matrix, with columns $a_1, ...a_n$, and if x is in $\mathbb{B}^n$, then the product of A & x, denoted by Ax, is the linear combination of the columns of A using the corresponding entries in x as weights.

$$Ax = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + ... + x_n a_n$$

Note that Ax is defined only if the number of columns of A, n, equals the number of entries in x.

Ex 1

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Ex 2
For $v_1, v_2, v_3 \in \mathbb{R}^m$, write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector.

$$3v_1 - 5v_2 + 7v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = Ax$$

In section 1.3, we were taught to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Which can also be rewritten in the $Ax = b$ form like so

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

The form $Ax = b$ is called a matrix equation.

Theorem
If an $m \times n$ matrix, A, with columns $a_1, ..., a_n$ and if $b \in \mathbb{R}^m$

$$Ax = b$$

will have the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + ... + x_na_n = b$$

which will also have the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & ... & a_n & b \end{bmatrix}$$

In Section 1.3, there was an existence question, "Is b in Span $\{a_1, ..., a_n\}$?" Or sometimes written as "Is $Ax = b$ consistent?" The question that we will now ask is an extension of the aforementioned existence question, whether the equation $Ax = b$, $\{b|b \in \mathbb{R}^m\}$ holds true or not.

Ex 3
Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible $b1, b2, b3$?

$$\begin{bmatrix} 1 & 3 & 4b & b1 \\ -4 & 2 & -6 & b2 \\ -3 & -2 & -7 & b3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 4 & b1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_1 - \frac{1}{2}b_2 + b3 \end{bmatrix}$$

The equation $Ax = b$ is not consistent for every b because there exists values of b that will make $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$. Hence the entries in b must satisfy

$$b_1 - \frac{1}{2}b_2 + b_3 = 0$$

Which is also the equation of a plane through the origin in $\mathbb{R}^3$. Hence the plane is the set of all linear combinations of the three columns belong to A

To extend, the equation $Ax = b$ failed to be consistent for all values of b because the echelon form of A has a row of zeros.

Theorem
Let A be an $m \times n$ matrix. The following statements are logically equivalent meaning that they are all either true statements otherwise they are all false.

A) For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution.
B) Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.
C) The columns of A span $\mathbb{R}^m$.
D) A has a pivot position in every row.