# 5.2 The Characteristic Equation

**Introduction**
Useful information about the eigenvalues of a square matrix $A$ is encoded by a special scalar equation known as the characteristic equation of $A$.

We must find all scalars $\lambda$ such that the equation above has a nontrivial solution. By the Invertible Matrix Theorem, this problem is equivalent to finding all $\lambda$ such that the matrix $A - \lambda I$ is not invertible.

**Ex 1**
Find the egienvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

$$(A - \lambda I)x = 0$$

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

Then $\lambda = 3$ or $\lambda = -7$, so the eigenvalues of $A$ are 3 & $-7$.

**Ex 2**
Find the egienvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

$$(A - \lambda I)x = 0$$

We must find all scalars $\lambda$ such that the equation above has a nontrivial solution. By the Invertible Matrix Theorem, this problem is equivalent to finding all $\lambda$ such that the matrix $A - \lambda I$ is not invertible.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

The matrix fails to be invertible when its determinant is zero. So the eigenvalues of $A$ are the solutions to the equation

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$$

$$\begin{gathered} \det(A-\lambda I) = (2-\lambda)(-6-\lambda)-(3)(3) \\ -12+6\lambda-2\lambda+\lambda^2-9 \\ \lambda^2+4\lambda-21 \\ (\lambda-3)(\lambda+7) \end{gathered}$$

So if $\det(A-\lambda I)=0$, then $\lambda=3$ or $\lambda=-7$, then the eigenvalues of $A$ are 3 & $-7$.

**The Invertible Matrix Theorem (continued)**
Let $A$ be an $n\times n$ matrix. Then $A$ is invertible if and only if

r) The number 0 is not an eigenvalue of $A$.

**The Characteristic Equation**
A scalar $\lambda$ is an eigenvalue of an $n\times n$ matrix $A$ if and only if $\lambda$ satisfies the characteristic equation

$$\det(A-\lambda I)=0$$

The eigenvalue 5 is said to have multiplicity 2 because $(5-\lambda)$ occurs twice as a factor of the characteristic polynomial. The algebraic multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic equation.

**Theorem 5.4**
If $n\times n$ matrices $A$ & $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities)