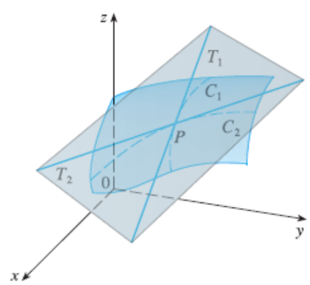
11.4 tangent Planes and Linear Approximations

Introduction
In single-variable calculus, one of the most important concepts is linear approximation. As we zoom into toward a point on the graph of a differentiable function, said graph becomes indistinguishable from its tangent line. Hence why being able to approximate the function by a linear function is so important. This applies in three dimensions as well, where we zoom towards the point of a surface that is the graph of a differentiable function of two variables, the surface looks more like its tangent plane. Said function can be approximated by a linear function of two variables. This concept can be extended to functions of two or more variables as well.

Tangent Planes
Suppouse a surface S with an equation $z = f(x, y)$, where f has continuous first partial derivatives.

Let $P(x_0, y_0, z_0)$ be a point on S. Then we have two curves, C_1 & C_2 obtained from intersecting the verticle planes $y = y_0$ & $x = x_0$. The point P lies on both curves C_1 & C_2. T_1 & T_2 are the tangent lines to its curves C_1 & C_2 at the point P.

Then the tangent plane to the surface S at the point P is the plane containing both tangent lines T_1 & T_2.



Recall the equation for a plane passing through the point $P(x_0, y_0, z_0$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

We can achieve an equation for finding $z - z_0$ by dividing by C

$$\frac{A(x - x_0) + B(y - y_0) + C(z - z_0)}{C} = \frac{0}{C}$$

$$z - z_0 = -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0)$$

Further on we can let $a = -\frac{A}{C}$ & $b = -\frac{B}{C}$ to get

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

Now we can also say that the intersection with the plane $y = y_0$ must be the tangent line T_1

$$z - z_0 = a(x - x_0) + b(y_0 - y_0) \to z - z_0 = a(x - x_0)$$

This can be recognized as a point slope form equation with slope a. From Section 11.3, we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

The same can also be done to achieve a point slope form equation with b as the slope. That being said the intersection with the plane $x = x_0$ must be the tangent line T_2.

$$z - z_0 = a(x_0 - x_0) + b(y - y_0) \to z - z_0 = b(y - y_0)$$

Based on our previous statement for the slope of the T_1, we can say that the slope of the tangent T_2 is $f_y(x_0, y_0)$

Now suppouse that f has continuous partial derivatives. The equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Ex 1
Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$

$$f(x, y) = 2x^2 + y^2$$

$$\begin{aligned} f_x(x,y) &= 4x \qquad & f_y(x,y) &= 2y \\ f_x(1,1) &= 4 \qquad & f_y(1,1) &= 2 \end{aligned}$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 3 = 4(x - 1) + 2(y - 1) \to \boxed{4x + 2y - 3}$$

Linear Approximations
In Example 1, we found an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$. Therefore the linear function of two variables

$$L(x, y) = 4x + 2y - 3$$

is a good approximation to $f(x, y)$ when x, y is near $(1, 1)$. The function L is called the linearization of f at $(1, 1)$ and the approximation $f(x, y) \approx 4x + 2y - 3$ is known as the linear approximation or tangent plane approximation of f at $(1, 1)$.

To exemplify, we can approximate $f(1.1, 0.95)$ by plugging in the point $(1.1, 0.95)$ into the $L(x, y)$

$$f(1.1, 0.95) = 3.3225 f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

As shown $L(x, y)$ is fairly close, but if we were to use a point farther away from $(1, 1)$, the accuracy falls off.

$$f(2, 3) = 17 \qquad L(2, 3) = 11$$

Knowing the equation of a tangent plane to the graph of a function f of two variables at the point $a, b, f(a, b)$, we can rewrite the equation like so

$$z - z_0 = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$z = z_0 + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

If f_x & f_y are both continuous, the linear function whpse graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is the linearization of f at (a, b) and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the linear approximation or the tangent plane of f at (a, b).

Theorem
If the partial derivatives f_x & f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Ex 2
Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then f is differentiable at $(1.1, -0.1)$.

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2e^{xy}$$
$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

$$f_x \ \& \ f_y = \text{continuous functions so} f \text{is differntiable.}$$

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$
$$1 + 1(x - 1) + 1(y - 0) = x + y$$

$$\boxed{xe^{xy} \approx x + y}$$
$$f(1.1, -0.1) \approx 1.1 - 0.1 = \boxed{1}$$

Differentials
For a differentiable single variable function, $y = f(x)$, the differential dx is defined to be an independent variable. Meaning that $dx = \mathbb{R}$. So the differential of y can be defined as

$$dy = f'(x)\ dx$$

Now in the case of double variable differentiable functions, $z = f(x, y)$, the differentials dx & dy are defined as independent variables also. So the differential dz, which is also known as the total differential, is defined like so

$$dz = f_x(x, y)\ dx + f_y(x, y)\ dy = \frac{\alpha z}{\alpha x} dx + \frac{\alpha z}{\alpha y}\ dy$$

Ex 3A
If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

$$dz = f_x(a, b)\ dx + f_y(a, b)\ dx$$

$$\boxed{dz = (2x + 3y)\ dx + (3x - 2y)\ dy}$$

Ex 3B
If x changes from to $2 \to 2.05$ and y changes from $3 \to 2.96$, compare the values of Δz & dz.

$$x = 2,\ y = 3 \qquad dx = \Delta x = 0.05,\ dy = \Delta y = -0.04$$

$$dz = [2(2) + 3(y)]\ \Delta x + [3(2) - 2(3)]\Delta y = \boxed{0.65}$$

Ex 4

The base radius and height of a right circular cone are measured as $10cm$ & $25cm$, respectively with a $0.1cm$ margin of error in each measurements. Use differentials to estimate the maximum error in the calculated volume of the cone.

$$V = \frac{\pi r^2 h}{3}$$

$$dV = f_r(a,b)\ dr + f_h(a,b)\ dy$$

$$dV = \frac{2\pi r h}{3}\ dr + \frac{\pi r^2}{3}\ dy$$

$$|\Delta r| \leq 0.1 \qquad \Delta h| \leq 0.1$$

$$dr = 0.1,\ dh = 0.1$$

$$dV = 0.1(\frac{500\pi}{3}) + (0.1)\frac{100\pi}{3} = \boxed{20\pi}$$

The maximum error in the calculated volumne is $20\pi cm^3 \approx 63cm^3$