11.3 Derivatives

Notations
If $z = f(x, y)$,

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} f(x,y) = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{d}{y} f(x,y)$$
$$\frac{\partial dz}{\partial dy},\ f_2,\ ,\ D_2 f,\ D_y f$$

Rules for Finding Partial Derivatives
1) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2) To find f_y, regard y as a constant and differentiate $f(x, y)$ with respect to y.

Ex 1
If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ & $f_y(2, 1)$.

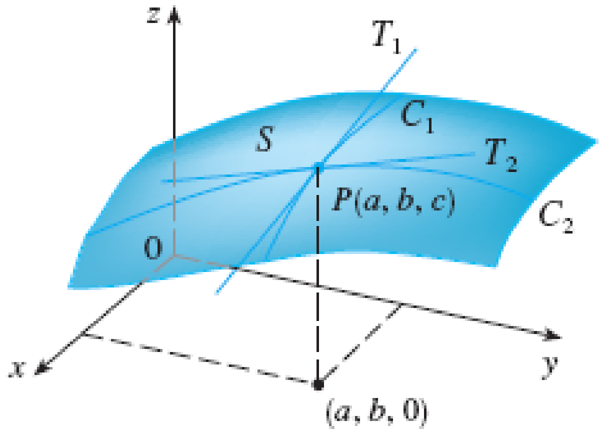
$$f_x(x,y) = \frac{d}{dx}x^3 + \frac{d}{dx}(x^2y^3) - \frac{d}{dx}2y^2$$

$$3x^2 + y^3 \cdot \frac{d}{dx}x^2 - 0 \to 3x^2 + 2xy^3$$

$$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 \to \boxed{16}$$

$$f_y(x,y) = \frac{d}{dy}x^3 + \frac{d}{dy}(x^2y^3) - \frac{d}{dy}2y^2.$$

Interpretations of Partial Derivatives
A geometric interpretation of partial derivatives can be started with regarding that the equation $f = f(x, y)$ represetns a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, our attention is restricted to the curve C_1 in which the vertible plane $y = b$ intersects S. In other words, C_1 of S in the plane $y = b$. Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 & C_2 pass through the point P.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 & C_2.

The curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve C_2 is the graph of then function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

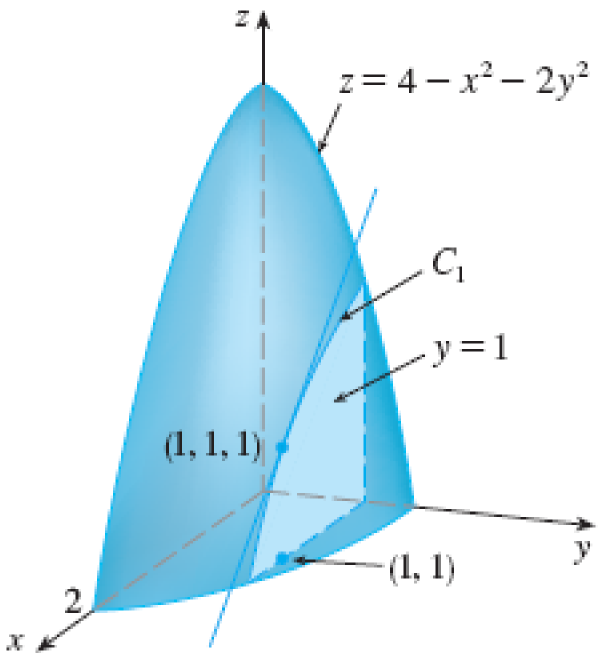
Thus the partial derivatives of $f_x(a, b)$ & $f_y(a, b)$ can be interpreted geometricaly as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 & C_2p of S in the planes $y = b$ & $x = a$.

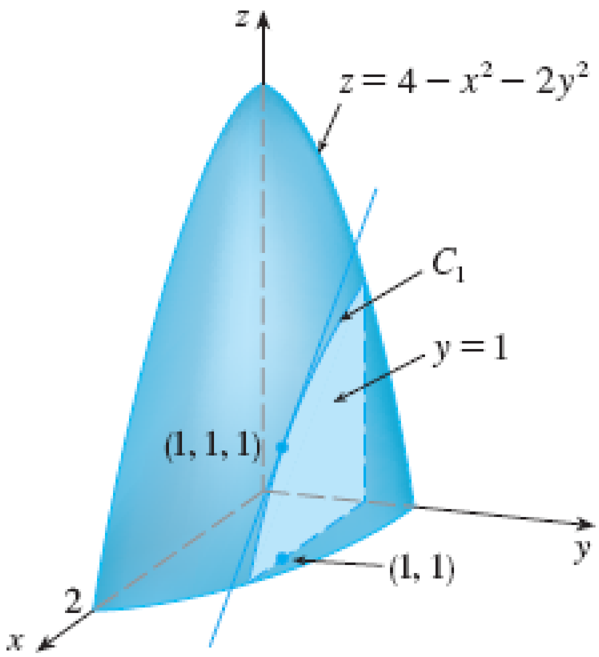
Partial derivatives can also be interpreted as rates of change. If $z = f(x, y)$, then $\frac{\partial z}{\partial x}$ represents the rate of change of z with respect to x when y is fixed.

Ex 2
If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ & $f_y(1, 1)$ and intepret these numbers as slopes.

$$\begin{aligned} f_x(x, y) &= -2x \qquad & f_y(x, y) &= -4y \\ f_x(1, 1) &= -2 \qquad & f_y(1, 1) &= -4 \end{aligned}$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ & the verticle plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$, labeled C_1 (Top Figure) Then the slope of the tangent line to this parabola at the point $(1, 1, 1) is f_x(1, 1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2, x = 1$ and the slope of the tangent line at $(1, 1, 1)$ is $f_y(1, 1) = 4$ (Bottom Figure).





Ex 4

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x & y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$

$$\frac{\partial z}{\partial x}$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy = 0$$

$$\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xv}$$

$$\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$