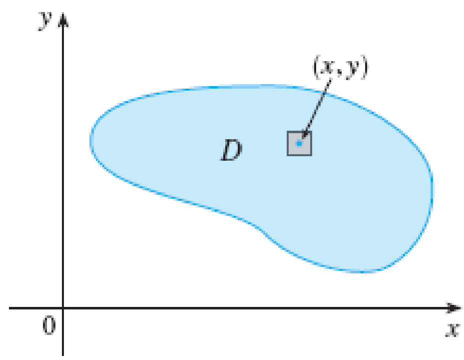
12.4 Application of Double Integrals

Introduction
Imagine a lamina with variable density and suppouse said lamina occupies a region D of the region xy plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $Q(x, y)$, where Q is a continuous function on D. This means that

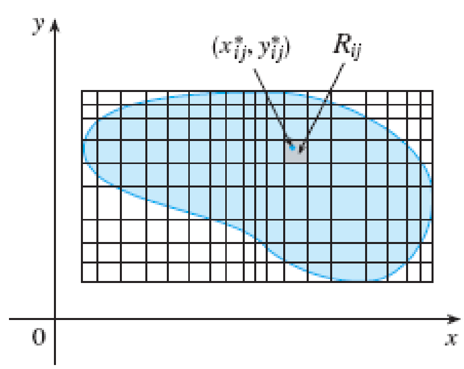
$$Q(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where Δm & ΔA are the mass and area of a small ectangle that contains (x, y) and the limit is taken as the dimensions of the rectangle approach 0.



To find the total mass m of the lamina, a rectangle R that contains D is divided into subrectangles R_{ij} and consider $Q(x, y)$ to be 0 outside D. By choosing a point (x_{ij}^*, y_{ij}^*) in R_{ij}, then the mass of the part of the lamina occupying R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$, where ΔA_{ij} is the area of R_{ij}. By adding all such masses, an approximation of the total mass is obtained.

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$



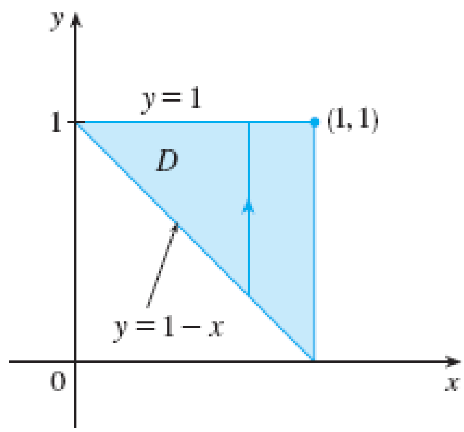
By taking the finer partitions using smaller rectangles, the total mass m of the lamina is obtained through the limit of our summation.

$$m = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D \rho(x,y)\ dA$$

Other types of density are treated in the same manner by physicists. AN example would be an electric charge distrubuted over a region D and the charge desnity (in units of charge per unit area)is given by $\sigma(x,y)$ at a point (x,y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x,y)\ dA$$

Ex 1
Charge is distributed over the triangular region D in the figure below so that the cahrge density at (x,y) is $\rho(x,y) = xy$, measured in coulombs per square meter C/m^2. Find the total charge.



$$Q = \iint_D \rho(x,y)\ dA = \int_0^1 \int_{1-x}^1 xy\ dydx$$
$$\int_0^1 [x\frac{y^2}{2}]_{y=1-x}^{y=1}\ dx = \int_0^1 \frac{x}{2}[1^2 - (1-x)^2]\ dx$$
$$\frac{1}{2}\int_0^1 2x^2 - x^3\ dx = \frac{1}{2}[\frac{2x^3}{3} - \frac{x^4}{4}]_0^1 = boxed\frac{5}{24}$$