3.2 Properties of Determinants

Introduction
Determinants change when row operations are performed.

Theorem 3.3 Row Operations
Let A be a square matrix.

a) If a multiple of one row of A is added to another row to produce a matrix B, then det $B =$ det A.

b) If two rows of A are interchanged to produce B, then det $B = -$det A.

c) If one row of A is multiplied by k to produce B, then det $B = k$ det A.

Ex 1
Compute det A, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

The strategy is to reduce A to echelon form and use the fact that the deteerminant of a triangular matrix is the product of its diagonal entries. First, conduct row replacement as they do not change the determinant.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{bmatrix}$$

Then interchange rows 2 & 3, which reverses the sign of the determinant.

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)(-5) = \boxed{15}$$

Ex 2
Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Since we want a 1 in the first row, we could interchange rows 1 & 4. However we could also factor out 2 from the top row, then proceed with row replacements.

$$\det A = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & 2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1) = \boxed{-36}$$

Theorem 3.4
A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 3.5
If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Theorem 3.6 Multiplicative Property
If A & B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.