10.7 Vector Functions and Space Curves

A vector-valued function or vector function is a function whose domain is a seet of real numbers and whose range is a set of vectors. If $f(t), g(t)$ & $h(t)$ are the component vector $r(t)$, then $f, g,$ & h are real-valued functions called the component functions of r. Due to this we can write

$$r(t) =< f(t), g(t), h(t) >= f(t)i + g(t)j + h(t)k$$

Ex 1
If

$$r(t) =< t^3, ln(3-t), \sqrt{t} >$$

then the component functions are

$$f(t) = t^3 \qquad g(t) = ln(3-t) \qquad h(t) = \sqrt{t}$$

The domain of r are all values of t for which the expression for $r(t)$ is defined. We can see that $r(t)$ is defined when

$$3 - t > 0 \ \& \ t \geq 0 \rightarrow \{t|0 \leq t < 3\} \rightarrow [0, 3)$$

Limit of Vector Functions
The limit of a vector function r is defined by taking the limits of its component functions.

Given the vector function, $r(t) =< f(t), g(t), h(t) >$, we can take the limit of it like so

$$\lim_{t\to a} r(t) =< \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)$$

provided that the limits of the component functions exist.

If $\lim_{t\to a} r(t) = \mathcal{L}$, this definition is equal to saying that the length and direction of the vector $r(t)$ approach the length and direction of the vector $\mathcal{L}$.

Ex 2 Find $\lim_{t\to 0} r(t)$, where $r(t) = (1+t^3)i + te^{-t}j + \frac{\sin t}{t}k$

$$\lim_{t\to 0} r(t) = \big[\lim_{t\to 0}(1+t^3)\big]i + \big[\lim_{t\to 0} te^{-t}\big]j + \big[\lim_{t\to 0}\frac{\sin t}{t}\big]k \rightarrow \boxed{i+k}$$

A vector function r is continuous at a if

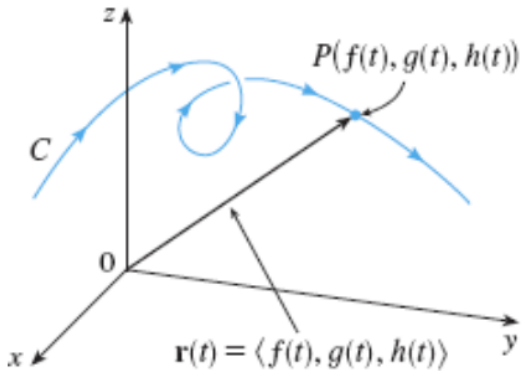
$$\lim_{t\to a} r(t) = r(a)$$

Due to this then r is continuous at a if and only if its component functions f, g & h are continuous at a.

There is a connection between continuous vector functions and space curves. Suppousing that f, g & h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I, is called a space curve. Imagine C as being traced out by a moving particle whose position at time t is $< f(t), g(t), h(t) >$. Then it can be said that $r(t)$ is the position vector of the point $P(f(t), g(t), h(t))$.



C is traced out by the tip of a moving position vector $r(t)$.

Ex 5
Find a vector equation and parametric equations for the line segment that joins the point $P(1,3,-2)$ to the point $Q(2,-1,3)$.

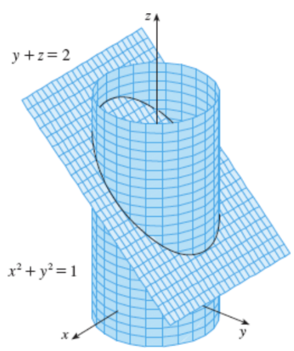
$$r(t) = (1-t)r_0 + tr_1 \ 0$$

$$r(t) = (1-t) < 1,3,-2 > +t < 2,-1,3 > \quad 0 \leq t \leq 1$$

$$r(t) =< t+1, -4t+3, 5t-2 >$$

$$x = t+1 \qquad y = -4t+3 \qquad z = 5t-2 \ 0 \leq t \leq 1$$

Ex 6
Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 + 1$ and the plane $y + z$=2



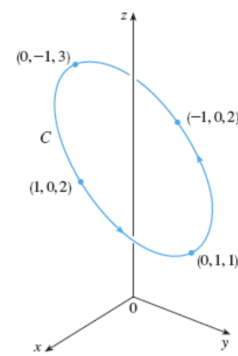


Figure 5 (LHS) shows the intersection of the plane and cylinder. While Figure 6 (RHS) shows the curve of intersection C, which is an eclipse.

The projection of C onto the xy plane is the circle $x^2 + y^{2=1}$, $z = 0$. So we can rewrite the equation like so

$$x^2 + y^2 = 1, \ z = 0 \rightarrow \sin^2 t + \cos^2 t = 1$$

$$x = \sin t \qquad y = \cos t \qquad 0 \leq t \leq 2\pi$$

$$y + z = 2 \rightarrow z = 2 - y \rightarrow z = 2 - \cos t$$

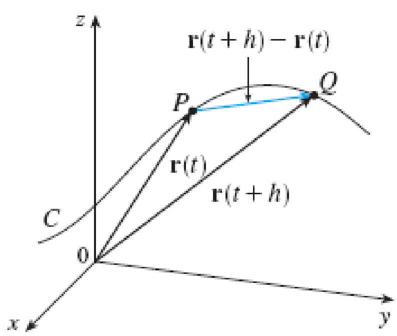
$$C : x = \sin t \qquad y = \cos t z = 2 - \cos t$$

$$r(t) = \sin t i + \cos t k + (2 - \cos t)k \qquad 0 \leq t \leq 2\pi$$

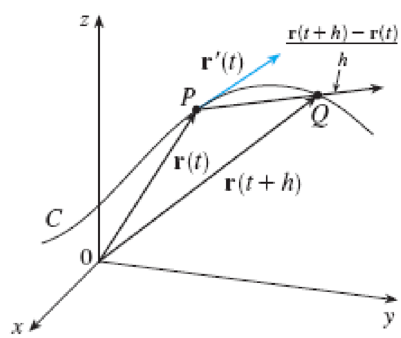
Derivatives
The derivative r' of a vector function r is defined in much the same way as for real-valued functions:

$$\frac{dr}{dt} = r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}$$

if this limit exists. The geometric signifance of this is shown below.



(a) The secant vector



(b) The tangent vector

There exists a tangent vector, $r'(t)$, to the curve defined by r at the point P, provided that $r'(t)$ & $r'(t) \neq 0$. The tangent line to C at P is defined to be the line through P parallel to the tangent vector $r'(t)$. There is also the unit tangent vector

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

The reason that the tangent vector $r'(t)$ at the point P is parallel to its tangent line is because the derivative gives us the instantaneous rate of change at a point. By finding the derivative of a vector equation at a point P, we are finding the instant rate of change as a vector. Since the tangent vector is the slope of the tangent line, the tangent vector is parallel to the direction of the tangent line.

$$r(t) = r_0 + tv \rightarrow \gamma(t) = P + t \cdot r'(t)$$

Theorem
If $r(t) =< f(t), g(t), h(t) >= f(t)i + g(t)j + h(t)k$, where $f, g,$ & h are differentiable functions, then

$$r'(t) =< f'(t), g'(t), h'(t) >= f'(t)i + g'(t)j + h'(t)k$$

Ex 8A
Find the derivative of $r(t) = (1 + t^3)i + te^{-t}j + \sin 2tk$

$$\boxed{r'(t) = 3t^2 + (1 - t)e^{-t} + 2\cos 2tk}$$

Ex 8B

Find the unit tangent vector at the point where $t = 0$

$$r'(0) \rightarrow j + 2k >$$

$$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{j+2k}{\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}j + \frac{2}{\sqrt{5}}k}$$

Ex 10

Find parametric equations of the tangent line to the helix, $x = 2\cos t, y = \sin t,$ & $z = t$, with parametric equations at the point $(0, 1, \frac{\pi}{2})$.

$$r(t) =< 2\cos t, \sin t, t >$$

$$r'(t) =< -2\sin t, \cos t, 1 >$$

$$t = \frac{\pi}{2}$$

$$r'(\frac{\pi}{2}) =< -2, 0, 1 >$$

The tangent line is through $(0, 1, \frac{\pi}{2})$ parallel to the vector $< -2, 0, 1 >$ so

$$tv = t \cdot r'(\frac{\pi}{2}) = t < -2, 0, 1 >=< -2, 0, t >$$

$$x = -2t \qquad y = 1 \qquad z = t + \frac{pi}{2}$$

Differentiation Rules

Suppouse $\vec{u}$ & $\vec{v}$ are different vector functions, c is a scalar and $f$is a real valued function.

$$1)\ \frac{d}{dt}\big[\vec{u}(t) + \vec{v}(t)\big] = \vec{u'}(t) + \vec{v'}(t)$$

$$2)\ \frac{d}{dt}\big[c\vec{u}(t)\big] = c\vec{u'}(t)$$

$$3)\ \frac{d}{dt}\big[f(t) \cdot \vec{u}(t)\big] = f'(t) \cdot \vec{u}(t) + f(t) \cdot \vec{u'}(t)$$

$$4)\ \frac{d}{dt}\big[\vec{u}(t) \cdot \vec{v}(t)\big] = \vec{u'}(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v'}(t)$$

$$5)\ \frac{d}{dt}\big[\vec{u}(t) \times \vec{v}(t)\big] = \vec{u'}(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v'}(t)$$

$$6)\ \frac{d}{dt}\big[\vec{u}(f(t))\big] = f'(t) \cdot \vec{u'}(f(t))$$

Definite Integral

If $\vec{r}(t)$ is a continuous vector function with components

$$r(t) =< f(t), g(t), h(t) >$$

$$\int_a^b r(t)\ dt \rightarrow \Big(\int_a^b f(t)\ dt \Big) i + \Big(\int_a^b g(t)\ dt \Big) j + \Big(\int_a^b h(t)\ dt \Big) k$$

As shown, evaluating the integral of a vector function is done so by integrating each component function, $< x, y, z >$. The Fundamental Theorem of Calculus applies here as well as shown below.

$$\int_a^b r(t)\ dt = R(t)\Big]_a^b = R(b) - R(a)$$

Where R is an antiderivative of r, that is $R'(t) = r(t)$.

Ex 12
If $r(t) = 2\cos ti + \sin tj + 2tk$, then

$$\int r(t)\ dt = \Big(\int 2\cos t\ dt \Big) i + \Big(\int \sin t\ dt \Big) j + \Big(\int 2t\ dt \Big) k$$

$$\boxed{\big(2\sin t\big)i - \big(\cos t\big)j + 2k + C}$$

$$\int_0^{\frac{\pi}{2}} r(t)\ dt = \big[2\sin ti - \cos tj + t^2 k\big]_0^{\frac{\pi}{2}} = \boxed{2i + j + \frac{\pi^2}{4}k}$$