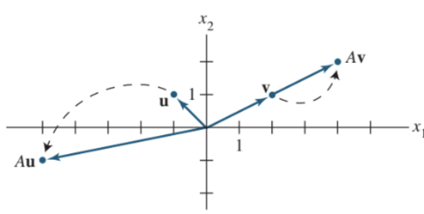
5.1 Eigenvectors and Eigenvalues

Introduction
Although a transformation $x \mapsto Ax$ may move vectors in various directions, it often happens that there are special vectors on which the action of A is very simple

Ex 1
Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of u & v under multiplication by A is shown in the figure below. It turns out that $Av = 2v$, so A "stretches" or dilates v.



Definition
An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is known as an eigenvalue of A if x is a nontrivial solution of $Ax = \lambda x$. Such an x is called an eigenvector corresponding to λ.

Ex 2
Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, & $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are u & eigenvectors of A.

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$$
$$Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

u is an eigenvector corresponding to an eigenvalue (-4), but v is not an eigenvector of A, because Av is not a scalar of v.

Ex 3
Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

$$Ax = 7x$$

The scalar 7 is an eigenvalue of A if and only if the equation has a nontrivial solution.

$$Ax = 7x$$
$$Ax - 7x = 0$$
$$(A - 7)x = 0$$
$$(A - 7I)x = 0$$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of $A - 7I$ are linearly dependent so $A - 7I$ has nontrivial solutions. The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Each vector of this form with a nonzero x_2 is an eigenvector corresponding to $\lambda = 7$.

Do note that that although row reduction can be used to find eigenvectors, it cannot be used to find eigenvalues.

λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. The null space of $A - \lambda I$ of $(A - \lambda I)x = 0$ is the set of all solutions to $(A - \lambda I)x = 0$. So the set is a subspace of $\mathbb{R}^n$ and is called the eigenspace of A corresponding to λ. Meaning the eigenspace consists of the zero vector and all the eigenjvectors corresponding to λ.

Ex 4

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 01 & 6 \\ 2 & 01 & 6 \\ 2 & 01 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$(A - 2I)x = 0$ has free variables, so 2 is indeed an eigenvalue of A. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So the basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$