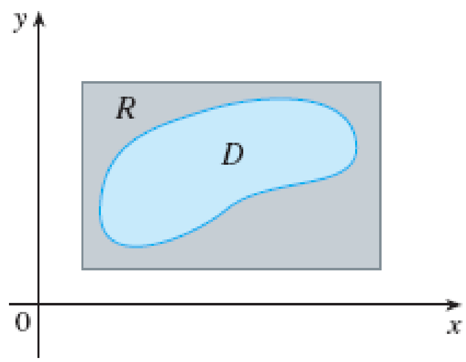
12.2 Double Integrals Over General Regions

When integrating a function f over regions D that on a more general shape as opposed to rectangles, we suppouse that D is a bounded region. Meaning D can be enclosed in a rectangular region R. A new function F with domain R can defined like so

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not } D \end{cases}$$



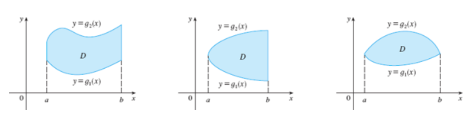
If the double integral of F exists over R, then the double integral of f over D is defined by

$$\iint_D f(x,y)\ dA = \iint_R F(x,y)\ dA$$

A plane region D is of type I if it lies between the graphs of two continuous functions of x, that is

$$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Examples of type I plane regions



In order to evaluate $\iint_D f(x,y)\ dA$ when D is a region of type I, we choose a rectangle $R = [a,b] \times [c,d]$ that contains D. Let F be the function

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not } D \end{cases}$$

Meaning F is 0 outside D or in a simpler context, F is 0 when $y < g_1(x)$ or $y > g_2(x)$. Then by Fubini's Theorem,

$$\iint_D f(x,y)\ dA = \iint_R F(x,y)\ dA = \int_a^b \int_c^d F(x,y)\ dydx$$

If f is continuous on a type I region D such that

$$D = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

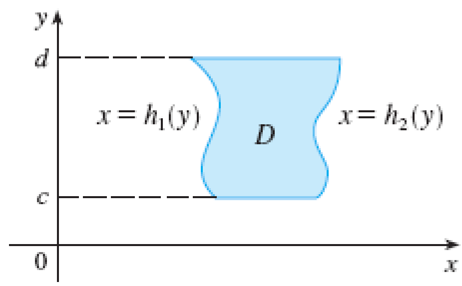
$$\iint_D f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$

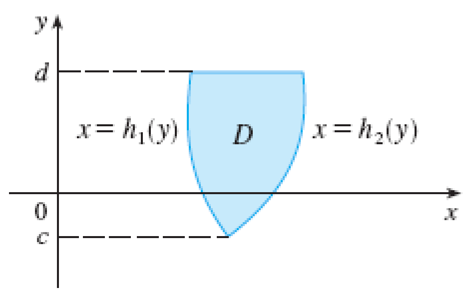
Observe the right hand integral is an iterated integral, however within the inner integral, x is regarded as a constant not only in $f(x,y)$, but also in the limits of integration, $g_1(x)$ & $g_2(x)$.

Consider plane regions of type II, which is expressed as

$$D = \{(x,y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Where h_1 & h_2 are continuous. Consider some examples of type II plane regions below.





Plane regions of type II can be integrated like below

$$\iint_D f(x,y)\ dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\ dxdy$$

Evaluate $\iint_D (x+2y)\ dA$ where D is the region bounded by the parabolas $y = 2x^2$ & $y = 1 + x^2$

The parabolas intersect when $2x^2 = 1 + x^2$ and by solving for x we get ± 1. D is also a type I region so we write

$$D = \{(x,y)| -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Where the upper bound is $1 + x^2$ and the lower bound is $2x^2$. This can be determined by plugging in an arbitrary value x that belongs to the set D.

$$\iint_D (x+2y)\ dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\ dydx = \int_{-1}^{1}\Big[\int_{2x^2}^{1+x^2}\ dy\Big]\ dx$$

$$\int_{-1}^{1}\Big[xy+y^2\Big]_{y=2x^2}^{y=1+x^2}\ dx = \int_{-1}^{1} x(1+x^2)+(1+x^2)^2-x(2x^2)-(2x^2)^2\ dx$$

$$\int_{-1}^{1} -3x^4-x^3+2x^2+x+1\ dx = -3\frac{x^5}{5}-\frac{x^4}{4}+2\frac{x^3}{3}+\frac{x^2}{2}+x\Big|_{-1}^{1} = \boxed{\frac{32}{15}}$$

Ex 2

Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$ and above the region D in the xy plane bounded by the two lines $y = 2x$ & $y = x^2$.

First find the type I region D

$$D = \{(x,y)|0 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

Since we are asked to find the volume under the parabolid $z = x^2+y^2$, that means $f(x,y) = z = x^2+y^2$

$$V = \iint_D (x^2+y^2)\ dA = \int_0^2\int_{x^2}^{2x} (x^2+y^2)\ dydx = \int_0^2 \Big[x^2y+\frac{y^3}{3}\Big]_{y=x^2}^{y=2x}\ dy$$

$$\int_0^2 x^2(2x)+\frac{(2x)^3}{3}-x^2x^2-\frac{(x^2)^3}{3}\ dx = \int_0^2 (-\frac{x^6}{3}-x^4+\frac{14x^3}{3})\ dx$$

$$-\frac{x^7}{21}-\frac{x^5}{5}+\frac{7x^4}{6}\Big|_0^2 = \boxed{\frac{216}{35}}$$

We can write type I regions as type II regions and vice versa. However, we should choose to integrate whichever region type is easiest.

Ex 3

Evaluate $\iint_D xy\ dA$, where D is bounded by the line $y = x^2-1$ and the parabola $y^2 = 2x+6$.

We should choose to evaluate the integral with D as a type II region because the type I region is much harder. Due to $y^2 = y = 2x+y \to y = \pm\sqrt{2x+6}$ being complex to setup, we use region type II like so.

$$D = \{(x,y)|-2 \leq y \leq 4, \frac{y^2}{2} \leq x \leq y+1\}$$

$$\iint_D xy\ dA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xydx\ dy = \int_{-2}^{4}\Big[\frac{x^2}{2}y\Big]_{x=\frac{y^2}{2}-3}^{x=y+1}\ dy$$

$$\frac{1}{2}\int_{-2}^{4} y\Big[(y+1)^2-(\frac{y^2}{2}-3)^2\Big]\ dy$$

$$\frac{1}{2}\int_{-2}^{4} -\frac{y^5}{4}+4y^3+2y^2-8y\ dy$$

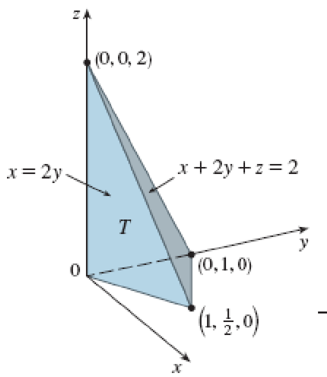
$$\frac{1}{2}\Big[-\frac{y^6}{24}+y^4+2\frac{y^3}{3}-4y^2\Big]_{-2}^{4} = \boxed{36}$$

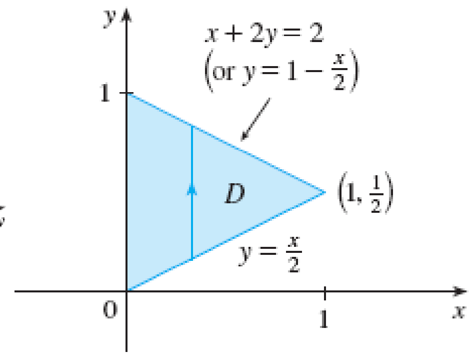
If D was expressed as a type I region, we would get the integral

$$\iint_D xy\ dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\ dydx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\ dydx$$

Ex 4
Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2, x = 2y, x = 0,\ \&\ z = 0$.

With these questions, we should draw two diagrams. One wil visualize the three-dimensional solid and another of the plane region D over which it lies.





Here we can see that the plane region D is bounded by the functions $y = 1 - \frac{x}{2}$ & $y = \frac{x}{2}$. $1 - \frac{x}{2}$ is obtained from $x + 2y = 2$ when $z = 0$. However a simpler way to derive these bounds is to plot the equations $x = 2y$ and connect a line from the x & y intercepts. The slope of the line gives us $y = 1 - \frac{x}{2}$ and we set that equal to $y = 1 - \frac{x}{2}$.

By then solving for where those two lines intersect, we will have drawn the plane region D.

The upper and lower bounds are $y = 1 - \frac{x}{2}$ & $y = \frac{x}{2}$ respectively. The region also starts from $x = 0$ and ends at $x = 1$. Giving the region

$$D = \{(x, y)|0 \le x \le 1, \frac{x}{2} \le y \le 1 - \frac{x}{2}\}$$

Therefore

$$V = \iint_D 2 - x - 2y\ dA = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} 2 - x - 2y\ dydx$$

$$\int_0^1 \left[\int_{\frac{x}{2}}^{1-\frac{x}{2}} 2 - x - 2y\ dy\right] dx = \int_0^1 \left[2y - xy - y^2\right]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} dx$$

$$\int_0^1 \left[2 - x - x(1 - \frac{x}{2}) - (1 - \frac{x}{2})^2 - x + \frac{x^2}{2} + \frac{x^2}{4}\right] dx$$

$$\int_0^1 x^2 - 2x + 1\ dx = \frac{x^3}{3} - x^2 + x\Big|_0^1 = \boxed{\frac{1}{3}}$$

Ex 5
Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\ dydx$.

Notice that the inner integral $\int \sin(y^2)\ dy$ is simply impossible in finite terms. So we must rewrite the integral into a simpler form. The first approach is to go backwards.

$$\int_0^1 \int_x^1 \sin(y^2)\ dydx \to \iint_D \sin(y^2)\ dA$$

Where the plane region is

$$D = \{(x, y)|0 \le x, x \le y \le 1\}$$

Which can be rewritten as

$$D = \{(x, y)|0 \le y \le 1, 0 \le x \le y\}$$

Converting the integral like so

$$\int_0^1 \int_x^1 \sin(y^2)\ dydx \to \iint_D \sin(y^2)\ dA \to \int_0^1 \int_0^y \sin(y^2)\ dxdy$$
$$\int_0^1 \left[\int_0^y \sin(y^2)\ dx\right] dy = \int_0^1 \left[x\sin(y^2)\right]_{x=0}^{x=y} dy$$
$$\int_0^1 y\sin\left(y^2\right)\ dy = -\frac{1}{2}\cos\left(y^2\right)\Big|_0^1$$
$$\frac{1}{2}(1-\cos 1) = \boxed{\frac{1-\cos 1}{2}}$$

Properties of Double Integrals

$$\iint_D [f(x,y)+g(x,y)]\ dA = \iint_D f(x,y)\ dA + \iint g(x,y)\ dA$$

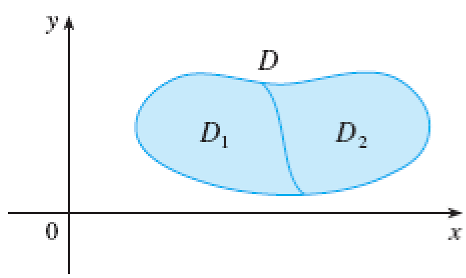
$$\iint_D cf(x,y)\ dA = c\iint f(x,y)\ dA$$

If $f(x,y) \geq g(x,y) \forall (x,y) \in D$, then

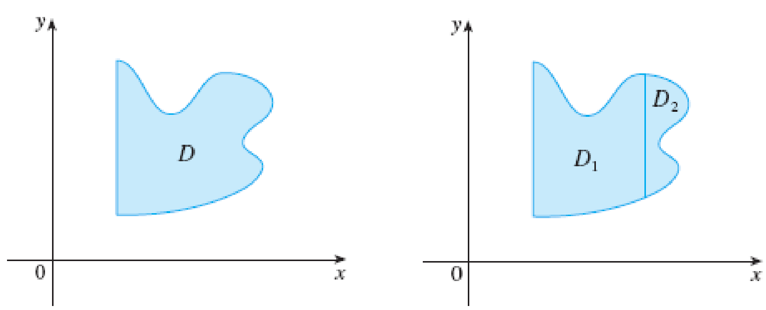
$$\iint_D f(x,y)\ dA \geq \iint g(x,y)\ dA$$

If $D = D_1 \cup D_2$, where D_1 & D_2 don't overlap except perhaps on their boundaries then

$$\iint f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA$$



The above integral can be used to evaluate double integrals over regions D that can be expressed as a union of regions of type I or type II. As shown below.



(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$,
D_1 is type I, D_2 is type II.

If we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D

$$\iint_D 1 \; dA = A(D)$$