10.3 The Dot Product

Def
If $\vec{a} =< a_1, a_2, a_3 >$ and $\vec{b} =< b_{1,2}, b_3 >$ then the dot product of $\vec{a}$ and $\vec{b}$ is the number $\vec{a}\cdot\vec{b}$ given by $\vec{a}\cdot\vec{b}$ given by $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$. While for 2 dimensional vectors, $< a_1, a_2 > \cdot < b_1, b_2 >= a_1b_1 + a_2b_2$

Ex 1. Find the dot product

$$< 2, 4 > \cdot < 3, -1 >= 2(3) + 4(-1) = 6 - 4 = \boxed{2}$$
$$< -1, 7, 4 > \cdot < 6, 2, -\frac{1}{2} >= -1(6) + 7(2) + 4(-\frac{1}{2}) = \boxed{6}$$
$$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k}) = 1(0) + 2(2) + (-3)(-1) = \boxed{7}$$

Properties
If $\vec{a}$,$\vec{b}$ and $\vec{c}$ are 3-dimensional vectors, and c is a scalar, then

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$
$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$
$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$
$$(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$$
$$\vec{0}\cdot\vec{a} = 0$$

Angle Between 2 Vectors
$0 \le \theta \le \pi$. If $\vec{a}$ and $\vec{b}$ are parallel, then $\theta = 0$ or $\theta = \pi$

Theorem
If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$

Ex 2
If the vectors $\vec{a}$ and $\vec{b}$ have lengths 4 and 6, and the angle between them is $\frac{\pi}{3}$, find $\vec{a}\cdot\vec{b}$.

$$|\vec{a}| = 4, |\vec{b}| = 6$$
$$\vec{a}\cdot\vec{b} = 4(6)\cos\frac{\pi}{3} = 24(\frac{1}{2}) = \boxed{12}$$

If $\theta = 0$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$ since $\cos 0 = 1$.
If $\theta = 0$, then $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$ since $\cos 0 = -1$.
Projections
$\vec{b} = \vec{PR}$
$\vec{a} = \vec{PR}$
$\vec{PR} = \vec{PS} + \vec{SR}$ Vector Projection of $\vec{b}$ onto $\vec{a}$ is $\text{proj}_{\vec{a}}\vec{b}$

Scalar projection $\vec{b}$ onto $\vec{a}$ or "component of $\vec{b}$ along $\vec{a}$ " ($\text{comp}_{\vec{a}}\vec{b}$)
Since $\vec{a} \cdot \vec{b} = 1|\vec{a}||\vec{b}| \cos\theta$, comp $_{\vec{a}}\vec{b} = |\vec{b}| \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$
Ex 5
Find the scalar and vector projects of $\vec{b} =< 1, 1, 2 >$ onto $\vec{a} =< -2, 3, 1 >$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{-2(1) + 3(1) + 1(2)}{\sqrt{(-2)^2 + (3)^2 + (1)^2}} = \boxed{\frac{3}{\sqrt{14}}}$$

Calculating Work
The work done bt a constant force f in moving an object through a distance d is $W = FD$.
Suppouse the constant force is a vector $\vec{F}$ pointing in a direction different from the displacement vector $\vec{D}$.
If the force moves the object from points $P \to Q$, then

$$W = (|\vec{F}| \cos\theta)$$
$$W = |\vec{F}|\vec{D} \cos\theta$$
$$W = \vec{F} \cdot \vec{D}$$

Ex 6
A force is given by a vector $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$, find the work done.

$$\vec{D} = P\vec{Q} =< 4 - 2, 6 - 1, 2 - 0 >$$
$$=< 2, 5, 2 >$$

$$W = \vec{F} \cdot \vec{D} = 3(2) + 4(5) + 5(2)$$
$$= 6 + 20 + 10$$
$$= \boxed{30}$$