4.3 Linearly Independent Sets; Bases

An indexed set of vectors $\{v_1, ..., v_p\}$ in V is said to be linearly independent if the linear equation

$$c_1v_1 + c_2v_2 + ... + c_pv_p = 0$$

has only the trivial solution.

THe set $\{v_1, ..., v_p\}$ is said to be linearly dependent if (1) has a nontrivial solution, that is, if there are some weights $c_1, ..., c_p$, not all zero, such that (1) holds. In such a case, (1) is called a linear dependence relation among $v_1, ..., v_p$.

Theorem 4.4
An indexed set $\{v_1, ..., v_p\}$ of two or more vectors, with $v_1 \neq 0$ is linearly dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors $v_1, ..., v_{j-1}$.

A main difference between linear dependence in $\mathbb{R}^n$ and in a general vector space is that when the vectors are not $n - tuples$, the homogeneous equation (1) usually cannot be written as a system of n linear equations. Meaning the vectors cannot be made into the columns of a matrix A in order to study the equation $Ax = 0$. The definition of linear independence and Theorem 4.4 must be used.

Ex 1
Let $p_1(t) = 1, P_2(t) = t$ & $p_3(t) = 4 - t$. Then $\{p_1, p_2, p_3\}$ is linearly dependent in $\mathbb{P}$ because $p_3 = 4p_1 - p_2$.

Def
Let H be a subspace of a vector space V. A set of vectors $\mathcal{B}$ in V is a basis for H if

(i) $\mathcal{B}$ is linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span } \mathcal{B}$$

The definition of a basis applies to the case when $H = V$, because any vector space is a subsapce of itself. Thus a basis of V is a linearly independent set that spans V.

Ex 3
Let A be an invertible $n \times n$ matrix. Then the columns of A form a basis for $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.

Ex 5

Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, & $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ has three pivot positions thus A is invertible. So the columns of A form a basis for $\mathbb{R}^3$.

The Spanning Set Theorem
A basis is an "efficient" spanning set that contains no unnecessary vectors. A basis can be constructed from a spanning set by discarding unneeded vectors.

Ex 7
Let

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}, \ \& \ H = \text{Span}\ \{v_1, v_2, v_3\}$$

Note that $v_3 = 5v_1 + 3v_2$, and show that Span $\{v_1, v_2, v_3\}$ = Span $\{v_1, v_2\}$. Then find a basis for the subspace H.

Every vector in Span $\{v_1, v_2\}$ belongs to H because

$$c_1v_1 + c_2v_2 = c_1v_1 + c_2v_2 + 0v_3$$

Now let x be any vector in H, $x = c_1v_1 + c_2v_2 + c_3v_3$. Since $v_3 = 5v_1 + 3v_2$, we may substitute

$$x = c_1v_1 + c_2v_2 + c_3(5v_1 + 3v_2)$$
$$(c_1 + 5c_3)v_1 + (c_2 + 3c_3)v_2$$

This x is in Span $\{v_1, v_2\}$, so every vector in H already belongs to Span $\{v_1, v_2\}$. Thus H and Span $\{v_1, v_2\}$ are actually the same set of vectors.

The Spanning Set Theorem
Let $S = \{v_1, ..., v_p\}$ be a set in a vector space V, and let $H =$ Span $\{v_1, ..., v_p\}$

a. If one of the vectors in S such as v_k is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.

b. If $H \neq \{0\}$, some subset of S is a basis for H.

Ex 8

Find a basis for Col B, where

$$B = \begin{bmatrix} b_1 & b_2 & \dots & b_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Each nonpivot column of B is a linear combination of the pivot columns. In fact, $v_2 = 4b_1$ & $b_4 = 2b_1 - b_3$. By the Spanning Set Theorem, we may discard v_2 & b_4. By the Spanning Set Theorem, we may discard b_2 & b_4 & $\{b_1, b_3, b_5\}$ will still span Col B. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since $b_1 \neq 0$ and no vector in S is a linear combination of the vectors that precide it, S is linearly independent. Thus S is a basis for Col B.

Theorem 4.6
The pivot columns of a matrix A form a basis for Col A.