3.1 Introduction to Determinants

Recall that a 2×2 matrix is invertible if and only if its determinant is nonzero. The same applies for other $n \times n$ matrixes.

1×1 **Matrix**

$$A = \begin{bmatrix} a_{11} \end{bmatrix}$$
$$\det A = a_{11}$$

3×3 **Matrix**

$$\Delta = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Delta = a_{11}\det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

For brevity, we can write

$$\Delta = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A .

Take the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

Then the submatrix A_{32} would be

$$A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Through this, we can obtain a recursive definition of a determinant. When $n = 3$, det A is defined using determinants of the 2×2 submatrices A_{1j}. When $n = 4$, $\det A$ An $n \times n$ determinant is determined by determinants of $(n-1) \times (n-1)$ submatrices.

Definition
For $n \geq 2$, the determinant of an $n \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is the sum of n terms of the form $\pm a_{1j}$ detA_{1j} with alternating plus and minus signs.

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + ... + (-1)^{1+n} a_{1_n}\det A_{1n}$$
$$\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

Ex 1
Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$
$$\Delta = 1(0-2) - 5(0-0) + 0(-4-0) = \boxed{-2}$$

For the next theorem, it is convenient to write the definition of det$A$ in a slightly different form. Given $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, the (i, j) cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j}\det A_{ij}$$

Then
$$\det A = a_{11}C_{11} + a_{12}C_{12} + ... + a_{1n}C_{1n}$$

This is known as the cofactor expansion across the first row of A.

Theorem 3.1
The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

As for the cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$

Use a cofactor expansion across the third row to compute detA, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$
$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$(-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}\det A_{32} + (-1)^{3+3}a_{33}\det A_{33}$$
$$0(5-(-1)) - (-2)(-1-0) + 0(4-10) = 0 + 2(-1) + 0 = \boxed{-2}$$

Theorem 3.2
If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A.