5.7 Applications to Differential Equations

In many applied problems, there are several quantities varying continuously in time, and they are related by a system of differential equations,

$$\begin{aligned} x_1' &= a_{11}x_1 + ... + a_{1n}x_n \\ x_2' &= a_{21}x_1 + ... + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + ... + a_{nn}x_n \end{aligned}$$

Here $x_1, ...x_n$ are differentiable functions of t, with derivatives $x_1', ..., x_n'$, and the a_{ij} are constants. The crucial feature of such a system is that it is linear. To visualize, write the system as a matrix differential equation

$$x'(t) = Ax(t) \qquad (1)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad x'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix}$$

A solution to the equation (1) above is a vector-valued function that satisfies the equation for all t in some interval of real numbers, such as $t \geq 0$.

The equation is linear because both differentiation and multiplication of vectors are linear transformations. Thus, if u & v are solutions of $x' = Ax$, then $cu + dv$ is also a solution, because

$$\begin{aligned} (cu + dv)' = cu' + dv' \\ cAu + dAv = A(cu + dv) \end{aligned}$$

There always exists a fundamental set of solutions to (1). If A is $n \times n$, then there are n linearly independent functions in a fundamental set. With each solution of (1) being a unique linear combination of these n functions.

That is, a fundamental set of solutions is a basis for the set of all solutions of (1), and the solution set is an n-dimensional vector space of functions. If a vector x_0 is specified, then the initial value problem is to construct the unique function x such that $x' = Ax$ & $x(0) = x_0$.

When A is a diagonal matrix, the solutions of (1) can be produced by calculus. To exemplify, we consider

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad (2)$$

that is,

$$\begin{aligned} x_1'(t) &= 3x_1(t) \\ x_2'(t) &= -5x_2(t) \end{aligned} \qquad (3)$$

The system (2) is said to be decoupled because each derivative of a function depends on the function itself, rather than on a combination or coupling of both $x_1(t)$ & $x_2(t)$.

From calculus, the solutions of (3) are $x_1(t) = c_1 e^{3t}$ & $x_2(t) = c_2 e^{-5t}$, for any constants c_1 & c_2.

So each solution of equation (2) can be written in the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-5t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$$

Which suggests that for the general equations $x' = Ax$, a solution might be a linear combination of functions of the form

$$x(t) = ve^{\lambda t}$$

for some scalar λ and some fixed nonzero vector v. If $v = 0$, the function $x(t)$ is identically zero, satisfying $x' = Ax$. Observe that

$$\begin{aligned} x'(t) &= \lambda v e^{\lambda t} && \text{By calculus, since } v \text{ is a constant vector} \\ Ax(t) &= Av e^{\lambda t} && \text{Multiplying both sides by } A \end{aligned}$$

Since $e^{\lambda t}$ is never zero, $x'(t) = Ax(t)$ if and only if $\lambda v = Av$, that is, if and only if λ is an eigenvalue of A and v is a corresponding eigenvector.

So each eigenvalue-eigenvector pair provides a solution to $x' = Ax$. Such solutions are sometimes called eigenfunctions of the differential equation. Eigenfunctions are key to solving systems of differential equations.