2.2 The Inverse of A Matrix

Definition
An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that $AC = I$ & $CA = I$. Where $I = I_n$ the $n \times n$ identity matrix.

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [Ie_1, Ie_2] \qquad I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [Ie_1, Ie_2, Ie_3]$$

$$I = I_n = [Ie_1, Ie_2, ...Ie_n]$$

Algebraic Representation
$(CA)x = Ix = x$, with this we can say that I is essentially "1".

$$AA^{-1} = A^{-1}A = I$$

Ex 1
If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$, then in the context of the definition $AC = I$ & $CA = I, C$ is the inverted matrix A^{-1}.

Theorem 4
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If the determinant $|A|, ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Ex 2

Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$|A| = 3(6) - 4(5) = -2$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}$$

Theorem 5

If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

based on the fact that $A^{-1}Ax = Ix = x$.

Ex 4

Use the inverse of the matrix A to solve the system

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

This system is in the format $Ax = b$, so

$$x = A^{-1}b = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Theorem 6

A) If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

B) If A & B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of A & B in the reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}$$

C) If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}.

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces $A \to I_n$, also transforms $I_n \to A^{-1}$.

Recall that the inverse of A is A^{-1} such that $AA^{-1} = A^{-1}A = I$.

An Algorithm for Finding A^{-1}

By placing A & I side by side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$, then row operations on this matrix will be applied to both A & I. Then according to Theorem 7, this means that there are row operations that transforms $A \to I_n$ &$I_n \to A^{-1}$ or else A is not an invertible matrix.

Ex 7

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$AI = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Since $A \to I_n$, then A is invertible, giving us

$$A^{-1} = \begin{bmatrix} \frac{-9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$