

5.4 Eigenvectors and Linear Transformations

Definition

Let V be a vector space. An eigenvector of a linear transformation $T : V \rightarrow V$ is a nonzero vector in V such that $T(x) = \lambda x$ for some scalar λ . A scalar λ is an eigenvalue of T if there is a nontrivial solution to $T(x) = \lambda x$. Such an x is known as an eigenvector corresponding to λ .

The Matrix of a Linear Transformation

Let V be an n dimensional vector space and let T be any linear transformation from V to V . To associate a matrix with T , choose any basis \mathcal{B} for V . Given any vector x , the coordinate vector $[x]_{\mathcal{B}}$ is in \mathbb{R}^n , as is the coordinate vector of its image, $[T(x)]_{\mathcal{B}}$.

Let $\{b_1, \dots, b_n\}$ be the basis \mathcal{B} for V . If $x = r_1 b_1 + \dots + r_n b_n$, then

$$[x]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

And

$$T(x) = T(r_1 b_1 + \dots + r_n b_n) = r_1 T(b_1) + \dots + r_n T(b_n)$$

because T is linear. Since the coordinate mapping from B to \mathbb{R}^n is linear, we get

$$\begin{aligned} T(x) &= T(r_1 b_1 + \dots + r_n b_n) = r_1 T(b_1) + \dots + r_n T(b_n) \\ [T(x)]_{\mathcal{B}} &= r_1 [T(b_1)]_{\mathcal{B}} + \dots + r_n [T(b_n)]_{\mathcal{B}} \end{aligned}$$

Since \mathcal{B} coordinate vectors are in \mathbb{R}^n , the vector equation can be written as a matrix equation, namely

$$[T(x)]_{\mathcal{B}} = M [x]_{\mathcal{B}}$$

where

$$M = \begin{bmatrix} [T(b_1)]_{\mathcal{B}} & [T(b_2)]_{\mathcal{B}} & \dots & [T(b_n)]_{\mathcal{B}} \end{bmatrix}$$

The matrix M is a matrix representation of T , called the matrix for T relative to the basis \mathcal{B} and denoted by $[T]_{\mathcal{B}}$.

Ex 1

Suppose $\mathcal{B} = \{b_1, b_2\}$ is a basis for V . Let $T : V \rightarrow V$ be a linear transformation with the property that

$$T(b_1) = 3b_1 - 2b_2 \quad T(b_2) = 4b_1 + 7b_2$$

Find the matrix M for T relative to \mathcal{B} .

$$[T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad [T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix}$$

Ex 2

The mapping $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ defined by

$$T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$$

T can be regonized as the differentiation operator

a) Find the \mathcal{B} matrix for T , when B is the basis $\{1, t, t^2\}$

Firstly, compute the images of the basis vectors

$$T(1) = 0$$

$$T(t) = 1$$

$$T(t^2) = 2t$$

So

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

b) Verify that $[T(p)]_{\mathcal{B}} = [T]_{\mathcal{B}} [p]_{\mathcal{B}}$ for each p in \mathbb{P}_2 .

For a general $p(t) = a_0 + a_1t + a_2t^2$

$$[T(p)]_{\mathcal{B}} = [a_1 + 2a_2t]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = [T]_{\mathcal{B}} [p]_{\mathcal{B}}$$

Theorem 5.8 Diaognal Matrix Representation

Suppouse $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If \mathcal{B} is the basis for \mathbb{R}^n formed from the columns of P , then D is the \mathcal{B} matrix for the transformation $x \mapsto Ax$.