12.8 Change of Variables in Multiple Integrals

In one dimensional calculus, change of variable (substitution) is often used to simplfy an integral. By reversing the roles of x & u, the Substitution Rule can be written as

$$\int_a^b f(x)\ dx = \int_c^d f(g(u))g'(u)\ du$$

where $x = g(u), a = g(c),\ \&\ b = g(d)$. Another way to express the Substitution Rule is

$$\int_a^b f(x)\ dx = \int_c^d f(x(u))\ \frac{dx}{gu} dun$$

A change of variables is also useful in double integrals, an example would be conversion to polar coordinates. Where the new variables r & θ are related to the old variables x & y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

The change of variables formula can be written as

$$\iint\limits_E f(x,y)\ dA = \iint\limits_S f(r\cos\theta, r\sin\theta)r\ drd\theta$$

where S is the region in the $r\theta$ plane that corresponds to the region R in the xy plane.

Consider a change of variables that is given by a transformation T from the uv plane to the xy plane

$$T(u,v) = (x,y)$$

where x & y are related to u & v by the equations

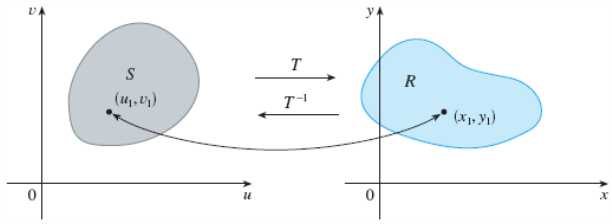
$$x = g(u,v) \qquad y = h(u,v)$$

or alternatively

$$x = x(u,v) \qquad y = y(u,v)$$

It is usually assumed that T is a C^1 transformation, meaning that g & h have continuous first-order partial derivatives.

T is a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is known as the image of the point (u_1, v_1). T is called one-to-one if no two points have the same image.



The figure above shows the effect of a transformation T on a region S in the uv plane. T transforms S into a region R in the xy plane called the image of S. The region R consists of images of all points in S.

The inverse transformation T^{-1} exists if T is one-to-one. To clarify, T^{-1} is the transformation from the xy plane to the uv plane and it may be possible to solve for u & v in terms of x & y in terms of the equations below

$$x = g(u, v) \qquad y = h(u, v)$$

So

$$u = G(x, y) \qquad v = H(x, y)$$

Ex 1

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \Big| 0 \le u \le 1, 0 \le v \le 1\}$

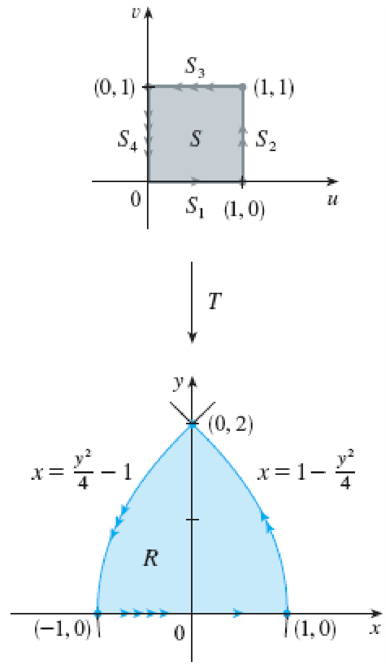
$$S_1, v = 0 \ (0 \le u \le 1)$$
$$x = u^2 \qquad y = 2u(0) = 0, \ (0 \le x \le 1)$$

$$S_2, u = 0 \ (0 \le v \le 1)$$
$$x = -v^2 \qquad y = 2(0)v = 0 \ (-1 \le x \le 0)$$

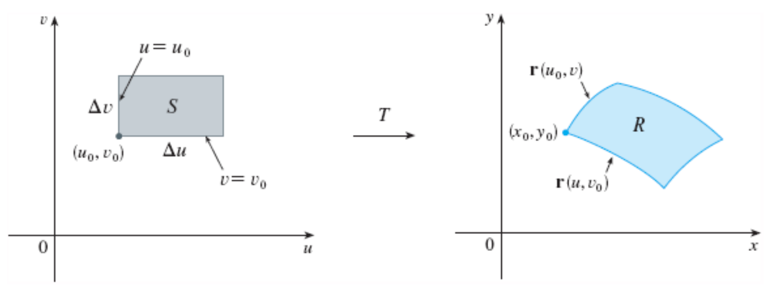
$$S_3, v = 1 \ (0 \le u \le 1)$$
$$x = u^2 - 1 \qquad y = 2u(1) = 2u \to u = \frac{y}{2} \qquad x = \frac{y^2}{4} - 1 \ (-1 \le x \le 0)$$

$$S_4, u = 1 \ (0 \le v \le 1)$$
$$x = 1 - v^2 \qquad y = 2(1)v = 2v \to v = \frac{y}{2} \qquad x = 1 - \frac{y^2}{4} \ (0 \le x \le 1)$$

So we have $y = 0 \ (0 \le x \le 1), y = 0 \ (-1 \le x \le 0), x = \frac{y^2}{4} - 1 \ (-1 \le x \le 0), x = 1 - \frac{y^2}{4} \ (0 \le x \le 1)$, giving us the image of S which we call R.



Effects of Change of Variable on Double Integrals
A small rectangle S in the uv plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu & Δv, shown in the figure below



The image of S is a region R in the xy plane, one of whose boundary points is $x_0, y_0 = T(u_0, v_0)$. The vector

$$r(u, v) = g(u, v)i + h(u, v)j$$

is the position vector of the image of the point (u, v). The equation representing the lower side of S is $v = v_0$, whose image curve is given by the vector function $r(u, v_0)$. The tangent vector at x_0, y_0 to this image curve is

$$r_u = g_u(u_0, v_0)i + h_u(u_0, v_0)j = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial eqRu}$$

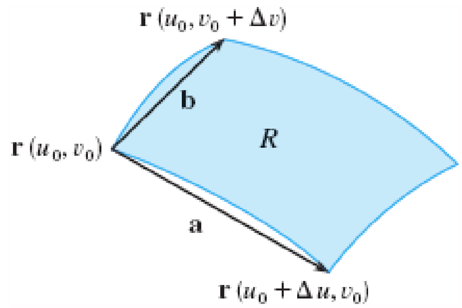
While the tangent vector at x_0, y_0 to the image curve of the left side of S, namely $u = u_0$ is

$$r_v = g_v(u_0, v_0)i + h_v(u_0, v_0)j = \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v}eqj$$

The image region $R = T(S)$ can be approximated by the secant vectors

$$a = r(u_0 + \Delta u, v_0) - r(u_0, v_0) \qquad b = r(u_0, v_0 + \Delta v) - r(u_0, v_0)$$

demonstrated in the diagram below



But

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u}$$

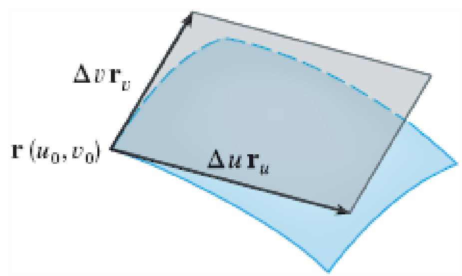
and so

$$r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u r_u$$

Similarly

$$r(u_0, v_0 = \Delta v) - r(u_0, v_0) \approx \Delta u r_v$$

Meaning R can be approximated by a paralellogram determined by the vectors $\Delta u r_u$ & $\Delta v r_v$ (see figure below).



Therefore the area of R can be approximated by the area of this paralleoogram,

$$\left|(\Delta u r_u) \times (\Delta v r_v)\right| = \left|r_u \times r_v\right| \Delta u \Delta v$$

Computing the cross product gives

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} k$$

This determinant is called the Jacobian of the transformation and has a special notation

Definition
The Jacobian of the transformation T given by $x = g(u, v)$ & $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

With this notation, $r(u_0, v_0 = \Delta v) - r(u_0, v_0) \approx \Delta u r_v$ can be used to give an approximation to the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v$$

where the Jacobian is evaluated at (u_0, v_0)

Next, S in the uv plane is divided into rectangles S_{ij} and their images in the xy plane is known as R_{ij}.

By approximating to each R_{ij}, the double integral of f over R is

$$\iint\limits_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\ \Delta A$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i), h(u_i, v_j))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u \Delta v$$

where the Jacobian is evaluated at $(u_i, v)j$. We can see this double sum is a Riemann sum for the integral

$$\iint\limits_S f(g(u,v), h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv$$

Theorem 9)Change of Variables in a Double Integral
Suppouse T, a C^1 transformation, whose Jacobian is nonzero and maps a region S in the uv plane onto a region R in the xy plane. Suppouse that f is continuous on R and that R & S are Type I or Type II plane regions. Suppouse that T is one-to-one, except perhaps on S's boundaries. Then

$$\iint\limits_R f(x,y)\ dA = \iint\limits_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv$$

Meaning that we change from an integral in x & y to an integral in u & v by expressing x & y as functions of u & v and writing

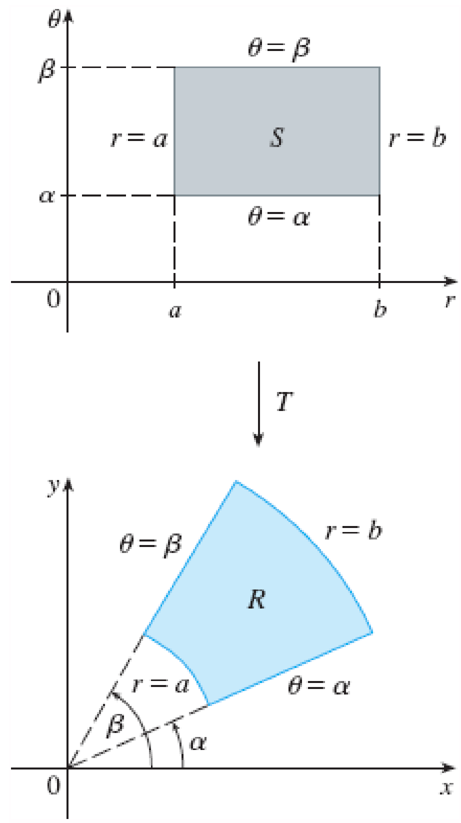
$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv$$

The transformation T from the $r\theta$ to the xy plane is given by

$$x = g(r,\theta) = r\cos\theta \qquad y = h(r,\theta) = r\sin\theta$$

The geometry of the transformation is shown below. T maps a rectangle in the $r\theta$ plane to a polar rectangle in the xy plane. The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$



So Theorem 9 gives

$$\iint_R f(x,y)\ dxdy = \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\ drd\theta$$

$$\int_\beta^\alpha \int_a^b f(r\cos\theta, r\sin\theta)r\ drd\theta$$

Ex 2

Use the change of variables $x = u^2 - v^2$ & $y = 2uv$ to evaluate the integral $\iint_R y\ dA$, where R is the region bounded by the x axis and the parabolas $y^2 = 4 - 4x$ & $y^2 = 4x, y \geq 0$.

Previously, we worked with $x = u^2 - v^2$ & $y = 2uv$ and $T(S) = R$, where S is the square $\left|0,1\right| \times \left|0,1\right|$. Evaluating the integral for S is much simpler than for R. First, compute the jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

So

$$\iint_R y\ dA = \iint_S 2uv\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dA = \int_0^1 \int_0^1 (2uv)4(u^2+v^2)\ dudv = \boxed{2}$$

Ex 3

Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}$, where R is the trapezoidal region with vertices $(1,0), (2,0), (0,-2),$ & $(0,-1)$.

$$u = x + y \qquad v = x - y$$

We then solve for x & y

$$u + v = (x + y) + (x - y)$$
$$2x = u + v$$
$$x = \frac{1}{2}(u + v)$$

$$u - v = (x + y) - (x - y)$$
$$2y = u - v$$
$$y = \frac{1}{2}(u - v)$$

Now compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

The sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

By using the equations x & y that are in terms of u & v, we obtain

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

So

$$S = \{(u,v) \Big| 1 \leq v \leq 2, -v \leq u \leq v\}$$

Which gives

$$\iint\limits_R e^{\frac{x+y}{x-y}}\ dA = \iint\limits_S e^{\frac{u}{v}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv$$
$$\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} \frac{1}{2}\ dudv = \boxed{\frac{3}{4}(e - e^{-1})}$$

Triple Integrals

Let T be a transformation that maps a region S in uvw space onto a region R in xyz space by means of the equations

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

The Jacobian of T is the 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint\limits_R f(x,y,z)\ dV$$

$$\rightarrow$$

$$\iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ dudvdw$$

Ex 4
Derive the formula for triple integration in spherical coordinates

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Compute the Jacobian

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = -rho^2 \sin\phi$$

Since $0 \le \phi \le \pi$, $\sin\phi \ge 0$. Therefore

$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \rho^2 \sin\phi$$

$$\iiint\limits_R f(x,y,z)\ dV$$

$$\rightarrow$$

$$\boxed{\iiint\limits_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\ d\rho d\theta d\phi}$$