11.6 Differential Derivatives and the Gradient Vector

$D_{\vec{u}}f(x_0, y_0), \vec{u}$ is a unit vector

Def
The directional derivative of f at (x_0, y_0) is the direction of a unit vector $\hat{u} =< a, b >$ is

$$D_{\vec{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h_0, y_0 + h_0 - f(x_0, y_0))}{h}$$

if this limit exists.

Theorem
If f is a differentiable function of x & y, then

$$D_{\vec{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b \qquad \vec{u} =< a, b > \ \& \ \vec{u} = \text{unit vector}$$

Note
If $\vec{u}$ makes angle θ with the positive $x-$axis, then $\vec{u} = \cos\theta, \sin\theta$.

Ex 1
Find $D_{\vec{u}} = f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ & $\hat{u}$ is the unit vector given by angle $\theta = \frac{\pi}{6}$. What is $D_{\vec{u}}f(1, 2)$?

$$a = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad b = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$f_x(x, y) = 3x^2 - 3y \qquad f_y(x, y) = -3x + 8y$$

$$\boxed{D_u f(x, y) = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3 + 8y)\frac{1}{2}}$$

$$D_u f(x, y) = (3(1)^{2-3(2)})\frac{\sqrt{3}}{2} + (-3(1) + 8(2))\frac{1}{2}$$

$$D_u f(x, y) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$$

$$\boxed{D_u f(x, y) = \frac{-3\sqrt{3} + 13}{2}}$$

The Gradient Vector

$$D_u = f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$< f_x(x,y), f_y(x,y) > \cdot < a, b >$$

$$D_u f(x,y) = \nabla f(x,y) \cdot \hat{u} \qquad \nabla f(x,y) = \text{gradient vector of } f$$

Ex 2
Find the direcitonal derivative of the function $f(x,y) = x^2y^3 - 4y$ of the point $(2,-1)$ in the direction of the vector $\vec{v} = 2\vec{i} + 5\vec{j}$

$$\nabla f(x,y) =< 2xy^3, 3x^2y^2 - 4 >$$

$$\nabla f(2,-1) =< 2(2)(-1)^3, 3(2)^2(-1)^2 - 4 >$$

$$\nabla f(2,-1) =< -4, 8 >$$

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{< 2, 5 >}{\sqrt{29}}$$

$$D_u f(2,-1) = \nabla f(x,y) \cdot \hat{u}$$

$$D_u f(2,-1) =< -4, 8 > \cdot < \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} >$$

$$\boxed{D_{\vec{u}} f(2,-1) = \frac{32}{\sqrt{29}}}$$

Functions of 3 Variables
For a function f of 3 variables, the gradient vector ∇f is

$$\nabla f(x,y,z) =< f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) >$$

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\hat{u} =< a, b, c >$ is

$$D_u f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_)) \cdot \hat{u}$$

Ex 4A
If $f(x,y,z) = x \sin(yz)$, find ∇f

$$f_x(x,y,z) = \sin y, z \qquad f_y(x,y,z) = x \cdot \cos yz \cdot z = xz \cos yz \qquad f_z(x,y,z) = x \cdot \cos yz \cdot y = xy \cos(yz)$$

$$\boxed{\nabla f =< \sin(yz), xz \cos(yz), xy \cos yz >}$$

Ex 4B

Find the directional derivative of f at $(1,3,0)$ in the direction of $\vec{v} =< 1,2,-1 >$

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{< 1,2,-1 >}{\sqrt{6}} =< \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} >$$

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot \hat{u}$$

$$D_{\vec{u}} f(x,y,z) = \frac{\sin(yz)}{\sqrt{6}} + \frac{2xz\cos(yz)}{\sqrt{6}} - \frac{xy\cos(yz)}{\sqrt{6}}$$

$$D_u f(1,3,0) = \frac{\sin 0}{\sqrt{6}} + 0 - \frac{3\cos 0}{\sqrt{6}}$$

$$\boxed{D_u f(1,3,0) = -\frac{3}{\sqrt{6}}}$$

Maximizing the Directional Derivative

Supppouse that we have a multi-variable function f and we consider all possible directional derivatives of f at a given point. These give the rate of changes of f in all possible directions. From there, we can ask the following questions, "In which of these directions does f change the fastest" and "What is the maximum rate of change?". These questions can be answered by the folllowing theorem.

Theorem 15

Suppouse f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(x)$ is $|\nabla f(x)|$ (the magnitude of $\nabla f(x)$) and it occurs when u has the same direction as the gradient vector $\nabla f(x)$.

Ex 5A

If $f(x,y) = xe^y$, find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

$$\nabla f(x,y) =< f_x, f_y >=< e^y, xe^y >$$

$$\nabla(2,0) =< 1,2 >$$

$$u = \vec{PQ} =< -1.5, 2 >, \hat{u} =< -\frac{3}{5}, \frac{4}{5} >$$

$$D_{\hat{u}} = \nabla f(2,0) \cdot \hat{u}$$

$$D_{\hat{u}} =< 1,2 > \cdot < -\frac{3}{5}, \frac{4}{5} >$$

$$\boxed{D_{\hat{u}} = 1}$$

Ex 5B

According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2,0) =< 1,2 >$. The maximum rate of change is

$$|\nabla f(2,0)| = |< 1,2 >| = \boxed{\sqrt{5}}$$

Ex 6

Suppouse that the temperature at a point (x,y,z) in space is given by $T(x,y,z) = \frac{80}{1+x^2+2y^2+3z^2}$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

$$\nabla T = \frac{\partial T}{\partial x} i + \frac{\partial T}{\partial y} j + \frac{\partial T}{\partial z} k$$

$$\nabla T = -\frac{160x}{(1 + x^2 + 2y^2 + 3z^2)^2} i + \frac{320y}{(1 + x^2 + 2y^2 + 3z^2)^2} - \frac{480z}{(1 + x^2 + 2y^2 + 3z^2)^2}$$

$$\nabla T = \frac{160}{(1 + x^2 + 2y^2 + 3z^2)^2}(-xi + 2yj - 3zk)$$

At the point $(1, 1, -2)$ the gradient vector is

$$\nabla T(1, 1, -2) = \frac{160}{256}(-i - 2j + 6k) = \frac{5}{8}(-i - 2j + 6k)$$

$$u = -i - 2j + 6k, \hat{u} = \frac{-1, -2, 6}{\sqrt{41}}$$

$$|\nabla T(1, 1, -2)| = \frac{5}{8}| < -\frac{1}{\sqrt{41}}, -\frac{2}{\sqrt{41}}, \frac{6}{\sqrt{41}} > | =$$