10.5 Equations of Line and Planes

A line in the xy plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using point-slope form.

Vector Equation

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = < x, y, z > \rightarrow \text{a point on the line}$$

$$\vec{v} = \text{parallel to line } L$$

If $\vec{v} = < a, b, c >$, then $t\vec{v} = < ta, tb, tc >$ and

$$< x, y, z > = < x_0, y_0, z_0 > + < ta, tb, tc >$$

$$< x, y, z > = < x_0 + ta, y_0 + tb, z_0 + tc >$$

Thus,

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \ \rightarrow \ \text{Parametric Equation} \\ z &= z_0 + ct \end{aligned}$$

Note

If a line L passes through the tips of position vectors $\vec{r}_0$ & $\vec{r}_1$, then the vectors $\vec{r}_1 - \vec{r}_0$ is parallel to L and

$$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$$

$$\vec{r} = \vec{r}_0 + t\vec{r}_1 - t\vec{r}_0$$

$$\vec{r} = \vec{r}_0 - t\vec{r}_0 + t\vec{r}_1$$

$$\vec{r} = (1 - t)\vec{r}_0 + t\vec{r}_1, \ 0 \leq t \leq 1$$

The equation above is the vector equation of line segment from $\vec{r}_0 \rightarrow \vec{r}_1$.

Def

2 lines are skew if they do not intersect and are not parallel.

Ex 3

How do we know if two lines are parallel or not? We can use parametric equations as they are able to show if

2 lines are skew lines. The reasoning behind this is these skew lines can be thought of the vector $\vec{v}$ that said lines are parallel to. Intuitionally, the skew lines can be represented by $\vec{v}$ from $r = r_0 + t\vec{v}$.

$$r = r_0 + t\vec{v}$$

$$< x, y, z > = < x_0, y_0, z_0 > + < ta, tb, tc >$$

$$< x, y, z > = < x_0 + ta, y_0 + tb, z_0 + tc >$$

$$\begin{array}{cccc} L_1, & x = 1 + t & y = -2 + 3t, & z = 4 - t \\ L_2, & x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

$$L_1 = < 1, 3, -1 > \qquad L_2 = < 2, 1, 4 >$$

$\vec{v} \neq \vec{v}_2$ hence these lines are not parallel. Next we are to determine if these lines intersect or not.

$$1 + t = 2s \qquad -2 + 3t = 3 + s \qquad 4 - t = -3 + 4s$$

$$t = 2s - 1, 3t = 5 + s, 7 = 4s + t$$

$$3t = 5 + s \rightarrow 6s - 3 = 5 + s \rightarrow s = \frac{8}{5}$$

$$t = 2s - 1 \rightarrow t = \frac{11}{5} 7 = 4s + t \rightarrow 7 \neq \frac{44}{11} + \frac{8}{5}$$

Because the third equation fails to be satisfied, these lines are not intersecting nor parallel due to failing the previous test. Hence, these are valid skew lines.

Equations of Planes

A plane cannot be described by a mere point and direction like a line. Thus a vector that is parallel to the plane will not be able to give us the "direction" of a plane. However, if the vector were to be perpendicular, the "direction" of the plane will be given.

A plane can be determined by a point $P_0(x_0, y_0, z_0)$ in the plane and an orthogonal vector $\vec{n}$. This orthogonal vector $\vec{n}$ is to be called a normal vector. Given an arbitrary point $P(x, y, z)$ and r_0 & r_1 be the position vectors of P_0 & P.

So now the vector $\vec{r} - \vec{r_0}$ be represented by $\vec{P_0 P}$. The normal vector $\vec{n}$ is orthogonal to every vector in the given plane especially to the vector $\vec{r} - \vec{r_0}$. Due to the fact that two vectors are orthogonal if their dot product is zero, we have

$$n \cdot (r - r_0) = 0$$

$$n \cdot r - n \cdot r_0 = 0$$

$$n \cdot r = n \cdot r_0$$

This is the vector equation of our plane. By writing $n = < a, b, c >, r = < x, y, z >,$ & $r_0 = < x_0, y_0, z_0 >$. We can obtain a scalar equation of the plane by transforming the vector equation like so

$$n \cdot (r - r_0) = 0$$

$$< a, b, c > \cdot < x - x_0, y - y_0, z - z_0 >= 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Ex 4
Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $n =< 2, 3, 4 >$ (why no arrow?). Find the intercepts and sketch the plane.

$$v =< a, b, c >\to< 2, 3, 4 >$$

$$r_0 =< x_0, y_0, z_0 >\to< 2, 4, -1 >$$

$$n \cdot (r - r_0) = 0 \to< 2, 3, 4 > \cdot < x - 2, y - 4, z + 1 >= 0$$

$$2x + 3y + 4Z = 12.$$

Intercepts

$$x,\ y = z = 0$$
$$y,\ x = z = 0$$
$$z,\ x = y = 0$$

$$x = (6, 0, 0) \qquad y = (0, 4, 0) \qquad z = (0, 03)$$

There is another way to write the equation of a plane

$$ax + by + cz + d = 0,\ d = -(ax_0 + by_0 + cz_0)$$

This equation is called a linear equation in x, y, z.

Ex 5
Find the equation of the plane that passes through the points $P(1, 3, 2), Q(3, -1, 6),\ \&\ R(5, 2, 0)$.

$$\vec{PQ} =< 2, -4, 4 >$$
$$\vec{PR} =< 4, -1, -2 >$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\vec{i}(8 - (-4)) - \vec{j}(-4 - 16) + \vec{k}(-2 - (-16)) \to 12\vec{i} + 20\vec{j} + 14\vec{k}$$

$$\vec{i} = (x - 1),\ \vec{j} = (y - 3), z \vec{-} 2 = 0$$

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0 \to \boxed{6x + 10y + 7z - 50}$$

Parallel Planes

If normal vectors of 2 planes are parallel to each other, then those 2 planes are parallel. However if 2 planes are not parallel, then there exists an acute angle betweenthe normal vector of those two planes.

Ex 6

Find the angle between the planes $x + y + z = 1$ & $x - 2y + 3z = 1$. Then find the symmetric equations of the line of intersection L of these two planes.

$$\vec{n} = -(1, 1, 1)$$