4.5 The Dimension of a Vector Space

Theorem 4.10
If a vector space V has a basis $\mathcal{B} = \{b_1, ..., b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof
Let $\{u1, ..., u_p\}$ be a set in V with more than n vectors. The coordinate vectors $[u_1]_{\mathcal{B}}, ..., [u_p]_{\mathcal{B}}$ form a linearly dependent set in $\mathbb{R}^n$, because there are more vectors (p) than entries (n) in each vector. So there exists scalars $c_1, ..., c_p$, not all zero such that

$$c_1 [u_1]_{\mathcal{B}} + ... + c_p [u_p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since the coordinate mapping is a linear transformation,

$$[c_1 u_1 + ... + c_p u_p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The zero vector displays the n weights needed to build the vectors $c_1 u_1 + ... + c_p u_p$ from the basis vectors in $\mathcal{B}$. That is, $c_1 u_1 + ... + c_p u_p = 0b_1 + ... + 0b_n = 0$. Then since c_i are not all zero, $\{u_1, ..., u_p\}$ is linearly dependent.

Meaning that Theorem 4.10 implies that if a vector space V has a basis $\mathcal{B} = \{b_1, ..., b_n\}$, then each linearly independent set in V has no more than n vectors.

Theorem 4.11
If a vector space V has a basis of n vectors, then every basis of V must consistent of exactly n vectors.

Definition
If a vector space V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Ex 1
The standard basis for $\mathbb{R}^n$ contains n vectors, so dim $\mathbb{R}^n = n$. The standard polynomial basis $\{1, t, t^2\}$ shows that dim $\mathbb{P}_2 = 3$. In general, dim $\mathbb{P}_n = n + 1$. The space $\mathbb{P}$ of all polynomials is infinite-dimensional.

Ex 2

Let $H = \text{Span }\{v_1, v_2\}$, where $v_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. A basis for H is $\{v_1, v_2\}$, since v_1 & v_2 are not multiples and hence are linearly independent. Thus dim $H = 2$.

Ex 3

Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

H is the set of all linear combinations of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

We can see that v_3 is a multiple of v_2. So by the Spanning Set Theorem, v_3 can be discarded and we would stil have a set that spans H. The 3 other vectors are linearly independent so $\{v_1, v_2, v_4\}$ is linearly independent and hence it is a basis for H. Thus

$$\boxed{\text{dim } H = 3}$$

Subspaces of a Finite-Dimensional Space

The next theorem is a natural counterpart to the Spanning Set Theorem.

Theorem 4.12

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\text{dim } H \leq \text{dim } V$$

Theorem 4.13 The Basis Theorem

Let V be a p-dimensional vector space, $b \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

The Dimensions of Nul A, Col A, & Row A

Since the dimensions of the null space and column space of an $m \times n$ matrix are referred to frequently, they have specific names:

Definition

The rank of an $m \times n$ matrix A is the dimension of the column space and the nullity of A is the dimension of the null space.

The rank of an $m \times n$ matrix A is the number of pivot columns and the nullity of A is the number of free variables. Since the dimension of the row space is the number of pivot rows, it is also equal to the rank of A.

Theorem 4.14 The Rank Theorem
The dimensions of the column space and the null space of an $m \times n$ matrix A satisfy the equation

$$\text{rank } A + \text{nullity } A = \text{number of columns in } A$$

Ex 5
Find the nullity and rank of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$Ax = 0, \text{ after row reduction} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 01 & 2 & -2 & 0 & \\ 0 & 00 & 0 & 0 & 0 & \end{bmatrix}$$

There are three free variables,ll.run(2133) .$x_2, x_4,$ & x_5. Hence the nullity of A is 3. Also the rank of A is 2 because A has two pivot columns.

Ex 6
a) If A is a 7×9 matrix with nullity 2, what is the rank of a?

Since A has 9 columns, rank A + 2 =9, rank $A = 7$.

b) Could a 6×9 matrix have nullity 2?

The columns of the matrix are vectors in $\mathbb{R}^6$, thus the dimension of the columns cannot exceed 6, that is the rank cannot exceed 6. So 6 + nulllity $A = 9$ to be true, the nullity has to be 3.

The Invertible Matrix Theorem (continued)
Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m) The columns of A form a basis of $\mathbb{R}^n$. n) Col $A = \mathbb{R}^n$ o) rank $A = n$ p) nullity $A = 0$ q) Nul $A = \{0\}$