2.8 Subspaces of $\mathbb{R}^n$

Def
A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has the three properties:
a) The zero vector is in H.
b) For each u & v in H, the sum $u + v$ is in H.
c) For each u in H and each scalar c, the vector cu is in H.

A subspace is closed under addition and scalar multiplication.

Ex 1
If v_1 & v_2 are in $\mathbb{R}^n$ & $H = \text{Span}\{v_1, v_2\}$, then H is a subspace of $\mathbb{R}^n$. TO verify this statement, note that the zero vector is in H because $0v_1 + 0v_2$ is a linear combination of v_1 & v_2. Take two arbitrary vectors in H, like so

$$u = s_1v_1 + s_2v_2 \qquad v = t_1v_1 + t_2v_2$$

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

Which shows that $u + v$ is a li near combination of v_1 & v_2 and hence is in H. Also note that for any scalar c, the vector cu is in H, because $cu = c(s_1v_1 + s_2v_2) = (cs_1)v_1 + (cs_2)v_2$.

If v_1 is not zero and if v_2 is a multiple of v_1, then v_1 & $v_2$simply span a line through the origin. So a line through the origin is another example of a subspace.

Ex 2
A line not through the origin is not a subspace as it does not contain the origin as show in figure below.

Column Space and Null Space of a Matrix
Subspaces of $\mathbb{R}^n$ usually occur in applications and theory in one of two ways. In both cases, the subspace can be related to a matrix.

Definition
The column space of a matrix A is the set Col A of all linear combinations of the columns of A.

If $A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$, with the columns in $\mathbb{R}^m$. The example below shows that the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$. Note that Col $A = \mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$. Otherwise Col A is only part of $\mathbb{R}^m$

Ex 4

Determine whether b is in the column space of A.

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The vector b is a linear combination of the columns of A if and only if the equation $Ax = b$ has a solution. Upon row reducing the augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$, we can say that $Ax = b$ is consistent and that b is indeed is in Col A.

Def
The null space of a matrix A is the set Nul A of all solutions of the homogeneous equation $Ax = 0$.

When A contains n columns, the solutions to $Ax = 0$ belong to $\mathbb{R}^n$ and so the null space of A is a subset of $\mathbb{R}^n$. In fact Nul A has the properties of a subspace of $\mathbb{R}^n$.

Theorem 12
The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Basis for a Subspace
Because a subspace will typically contain an infinite number of vectors, some problems involving a subspace are handled best by working with a small finite set of vectors that span the subspace. The smaller the set, the better.

Def
A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set that spans H.

Ex 5
The columns of an invertible $n \times n$ matrix form a basis for all of $\mathbb{R}^n$ because they are linearly independent and span $\mathbb{R}^n$ by the Invertible Matrix Theorem. An example would be the $n \times n$ identity matrix. Its columns are denoted by $e_1, ..., e_n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ..., e_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ n \end{bmatrix}$$

The set $\{e_1, ..., e_2\}$ is called the standard basis for $\mathbb{R}^n$.

The next example will show the standard procedure ofr writing the solution set of $Ax = 0$ in parametric vector form actually identifies a basis for Nul A.

Ex 6

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

First write the solution of $Ax = 0$ in parametric vector form.

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_2 = \text{free} \\ x_3 = -2x_4 + 2x_5 \\ x_4 = \text{free} \\ x_5 = \text{free} \end{cases} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} \rightarrow x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{x_2 u + x_4 v + x_5 w}$$

Ex 7

Find a basis for the column space of the matrix

$$\begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote the columns of B by $b_1, ..., b_n$ and note that $v_3 = -3b_1 + 2b_2$ & $b_4 = 5b_1 - b_2$. The fact that b_3 & b_4 are combinations of the pivot columns means that any combination of $b_1, ..., b_5$ is actually a combination of just $b_1, b_2,$ & b_5. If v is any vector in col B,

$$v = c_1 b_1 + c_2 b_2 + c_3 b_3 + c_4 b_4 + c_5 b_5$$

By substituting for b_3 & b_4, v can be written in the form

$$v = c_1 b_1 = c_2 b_2 + c_3(-3b_1 + 2b_2) + c_4(5b_1 - 1b_2) + c_5 b_5$$

The equation v is a linear combination of b_1, b_2 & b_5. So $\{b_1, b_2, b_5\}$ spans Col B. b_1, b_2 & b_5 are also linearly independent because they are columns from an identity matrix. So the pivot columns of B form a basis for

Col B.

$$\{b_1, b_2, b_5\} \text{ spans Col } B$$

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 & \\ 0 & 1 & 2 & -1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \qquad b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, b_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Linear dependence relations among the columns of a general matrix A can be expressed in the form $Ax = 0$ for some x. If some columns are not involved in a dependence relation, then the corresponding entries in x are zero.

Though when A is row reduced to echelon form B, the columns are changed but the equations $Ax = 0$ & $Bx = 0$ still have the same set of solutions. That is the columns of A have exactly the same linear dependence relationships as the columns of B.

Ex 8
It can be verified that the matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix to the matrix B in Example 7. Find a basis for Col A. Since row operations do not affect linear dependence relations, we should have

$$b_3 = -3b_1 + 2a_2 \to a_3 = -3a_1 + 2a_2 \qquad b_4 = 5b_1 - b_1 = a_4 = 5a_1 - a_2$$

Theorem 13
The pivot columns of a matrix A form a basis for the column space of A.

Note that the columns of an echelon form B are often not in the column space of A. Hence be careful to use pivot columns of A itself for the basis for Col A.

Determining if A Vector is in Nul A
Given an arbitrary matrix A and vector $\vec{u}$, determine if $\vec{u}$ is in Nul A. Recall that Nul A is the set of all solutions of the homogeneous equation $Ax = 0$. Meaning that if $\vec{u}$ is indeed in Nul A, $\vec{u}$ must be a solution to the homogeneous equation $Ax = 0$.

Given a matrix A of size 3×3 and a vector u in $\mathbb{R}^3$

$$\vec{u} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad Ax = 0 \to A\vec{u} = 0$$

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$