11.7 Maximum and Minimum Values

A function of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). In the opposite case when $f(x, y) ge f(a, b)$, then $f(a, b)$ is a local minimum.

If the inequalities hold for all points (x, y) in the domain of f or $\{(x, y)|(x, y) \in D\}$, then f has an absolute maximum or minimum at (a, b).

Theorem 12.2
If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ & $f_y(a, b) = 0$.

A point (a, b) is called a critical point (or stationary point), c, of f if $f_x(a, b) = 0$ & $f_y(a, b) = 0$

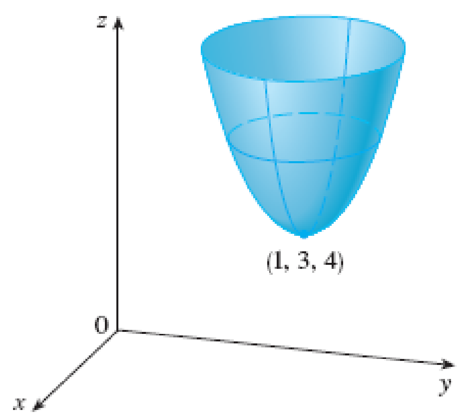
Ex 1
Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Then

$$f_x = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 at the point $(1, 3)$. So the only critical point is $(1, 3)$. By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

Since $(x - 1)^2 \geq 0$ & $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x & y, therefore $f(x, y) \geq 4$ for all values of x & y. Then not only is $f(1, 3) = 4$ is a local minimum, it is also an absolute minimum. This can be verified by the graph below.



Ex 2
Find the extreme values of $f(x, y) = y^2 - x^2$.

$$f_x = -2x \qquad f_y = 2y$$

The only critical point is $(0, 0)$. Notice that the points on the x axis where $y = 0$.

$$f(x, y) = -x^2 < 0 \ (x \neq 0)$$

The same can be said for points on the y axis where $x = 0$

$$f(x, y) = y^2 > 0 \ (x \neq 0)$$

Since every disk with the center $(0, 0)$ contains points where f takes on both positive and negative values. Then $f(0, 0) = 0$ can't be an extreme value so f does not have any extremas.

Second Derivatives Test
Suppouse the second partial derivatives of f are continuous on a disk with center (a, b) and supppouse that $f_x(a, b) = 0$ & $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f.) Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

a) If $D > 0$ & $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum. b) If $D > 0$ & $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum. c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Note 1
In case c, the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).

Note 2
If $D = 0$, the test gives no information therefore f could have either a local maximum or minimum at (a, b). There is also the case that (a, b) could be a saddle point of f.

Note 3
D could also be written as a determinant

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Ex 3
Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

$$f_x = 4x^3 - 4y \qquad f_y = 4y^3 - 4x$$

We now have the critical points, next we set them to zero to obtain the equations

$$x^3 - y = 0 \qquad y^3 - x = 0$$

To solve we substitute $y = x^3$ from the first equation into the second one.

$$(x^3)^3 - x = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1)$$
$$x(x^2 - 1)(x^2 + 1)(x^4 + 1) = x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1)$$
$$x = 0, 1, -1, \ x \in \mathbb{R}$$

We now have the real roots $x = 0, 1,$ & -1. Now we can find the three crtical points

$$x^3 = y$$

$$-1 = -1 \to (-1, -1) \qquad 0 = 0 \to (0, 0) \qquad 1 = 1 \to (1, 1)$$

We now calculate the second partial derivaties and $D(x, y)$

$$f_{xx} = 24x^2 \qquad f_{yy} = 24y^2 \qquad f_{xy} = -4$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

$$D(-1, -1) = 144(-1)^2(-1)^2 - 16 = 128 \qquad f_{xx}(-1, -1) = 12(-1)^2 = 12$$
$$D(0, 0) = 144(0)^2(0)^2 - 16 = 16$$
$$D(1, 1) = 144(1)^2(1)^2 - 16 = 128 \qquad f_{xx}(1, 1) = 12(1)^2 = 12$$

$D(-1, 1) > 0$ & $f_{xx}(-1, -1) = 12 > 0$, then $f(-1, -1)$ Since $D(0, 0) = -16 < 0$, then the origin $(0, 0)$ is a saddle point and f has no extrema values at the origin.

Ex 4
Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$

$$x + 2y + z = 4 \to z = 4 - x - 2y$$

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2} \to d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$$

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0 \qquad f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

$$4x + 4y - 14 = 0 \to y = -x + \frac{14}{4}$$
$$4x + 10y - 24 = 0 \to 4x + 10(-x + \frac{14}{4}) - 24 = 0$$
$$4x - 10x + \frac{140}{4} - \frac{96}{4} \to -6x + \frac{44}{4} = 0$$
$$-6x = -11$$
$$x = \frac{11}{6}$$

$$y = -x + \frac{14}{4} \to y = -\frac{11}{6} + \frac{14}{4}$$
$$y = \frac{40}{24} \to \frac{5}{3}$$

$$c = (x, y) = (\frac{11}{6}, \frac{5}{3})$$

$$f_{xx} = 4 \qquad f_{yy} = 10 \qquad f_{xy} = 4$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$
$$D(\frac{11}{6}, \frac{5}{3}) = 4 \cdot 10 - 16 = 24 > 0 \qquad f_{xx} = 4 > 0$$

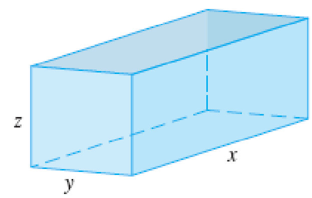
Now because $D = 24 > 0$ & $f_{xx} = 4 > 0$, then the critical point $c = (\frac{11}{6}, \frac{5}{3})$, then the critical point c is a local minimum. Intuitively, because there is only one critical point, then the local minimum must be an absolute minimum.

$$d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2} = \sqrt{(\frac{11}{6} - \frac{6}{6})^2 + (\frac{5}{3})^2 + (\frac{36}{6} - \frac{11}{6} - \frac{10}{3})^2}$$

$$d = \sqrt{(\frac{5}{6})^2 + (\frac{5}{3})^2 + (\frac{5}{6})^2} = \boxed{\frac{5\sqrt{6}}{6}}$$

Ex 5

A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.



$$V = xyz \qquad x = \text{length}, y = \text{width}, z = \text{height}$$

The 3 areas composing of an open lid box can be represented as so

$$A_1 = xz \qquad A_2 = yz \qquad A_3 = yx$$

Since the cardboard was made from $12m^2$ of cardboard, the surface area is $12m^2$, we get the equation

$$2A_1 + 2A_2 + yx = 12 \rightarrow 2xz + 2yz + yx = 12$$

By solving for z we get the equation

$$2xz + 2yz + xy = 12 \rightarrow 2xz + 2yz = 12xy - xy$$

$$2z(x+y) = 12 - xy \rightarrow z = \frac{12xy - xy}{2(x+y)}$$

$$V = xyz \rightarrow V = xy\frac{12 - xy}{2(x+y)} \rightarrow \frac{12xy - x^2y^2}{2(x+y)}$$

The reason being that z becomes a function of x & y so $z = f(x, y)$. We now compute the partial derivatives

$$\frac{\partial V}{\partial x} = f_x = \frac{y^2(12 - 2xy - x^2)}{2(x+y)^2} \qquad \frac{\partial V}{\partial y} = f_y = \frac{x^2(12 - 2xy - y^2)}{2(x+y)^2}$$

If V is a maximum, then the critical points $\frac{\partial V}{\partial x} = 0$ & $\frac{\partial V}{\partial y} = 0$, but $x = 0$ & $y = 0$ makes the equation $V = xyz \rightarrow 0$. So we must solve the equations

$$12 - 2xy - x^2 = 0 \qquad 12 - 2xy - y^2 = 0$$

$$x^2 = 12 - 2xy \qquad y^2 = 12 - 2xy$$

These equations imply that $x^2 = y^2 \to |x| = |y|$, indicating that not only $x = y$ but $\{x, y \in \mathbb{R}\}$.

$$y = x$$
$$x^2 = 12 - 2xy \to x^2 = 12 - 2x^2 \to 3x^2 = 12$$
$$x^2 = 4 \to x^2 = 2$$

$$x = y$$
$$y^2 = 12 - 2xy \to y^2 = 12 - 2yy \to 3y^2 = 12$$
$$y^2 = 4 \to y = 2$$

$$x = 2, y = 2, z = \frac{12xy - x^2y^2}{2(x+y)}$$
$$z = \frac{12xy - x^2y^2}{2(x+y)} \to \frac{12(4) - 16}{2(4)} = 1$$

$$c = (x, y, z) = (2, 2, 4)$$

From the physical nature of the problem, (x, y, z) can be argued to be an absolute maximum volume occuring at the critical point, $c = (2, 2, 1)$ of V. Then

$$V = xyz \to 2 \cdot 2 \cdot 1 = 4m^3$$