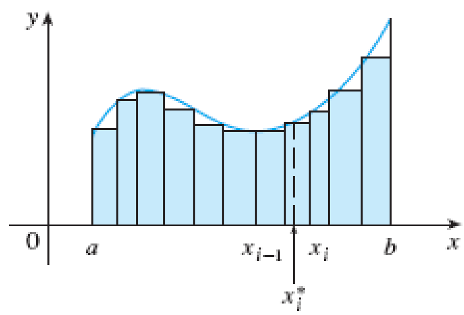
12.1 Double Integrals Over Rectangles

Introduction
In the same manner that integrating single variable functions, we obtained the area. We will be computing double integrals to find the volume of a solid.

Review of the Definite Integral
If $f(x)$ is defined for $a \leq x \leq b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ with length $\Delta x_i = x_i - x_{i-1}$ and choosing sample points x_i^* in these subintervals, visualized by the figure below.



The Riemann Sum equation form is obtained

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Then by taking the limit of such sums as the hugest of the lengths approach 0 to obtain the definite integral of f from $a \to b$

$$\int_a^b f(x)\ dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

In the special case where $f(x) \geq 0$, the Riemann sum can be intepreted as the sum of the areas of the approximating rectangles. Meaning that $\int_a^b f(x)\ dx$ represents the area under the curve $y = f(x)$ from $a \to b$.

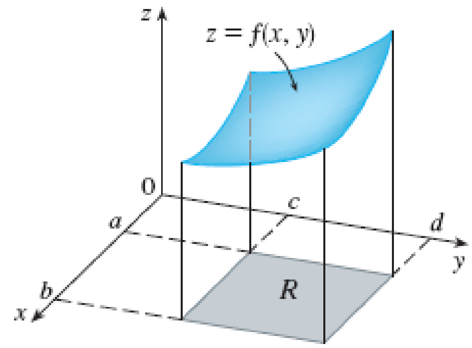
Volumes and Double Integrals
Then in a similar manner, a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

And by suppousing that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f and under the graph of f, that is

$$S = \{(x, y, z) \in \mathbb{R}^2 | 0 \leq z \leq f(x, y), (x, y) \in \mathbb{R}\}$$

Where our goal is to find the volume of S



To do so, we must take a partition P of R into sub rectangles. Done by dividing the intervals $[a,b]$ & $[c,d]$.

$$a = x_0 < x_1 < ... < x_{i-1} < x_i < x_m = bc = y_0 < y_1 < ... < y_{i-1} < y_i < y_n = d$$

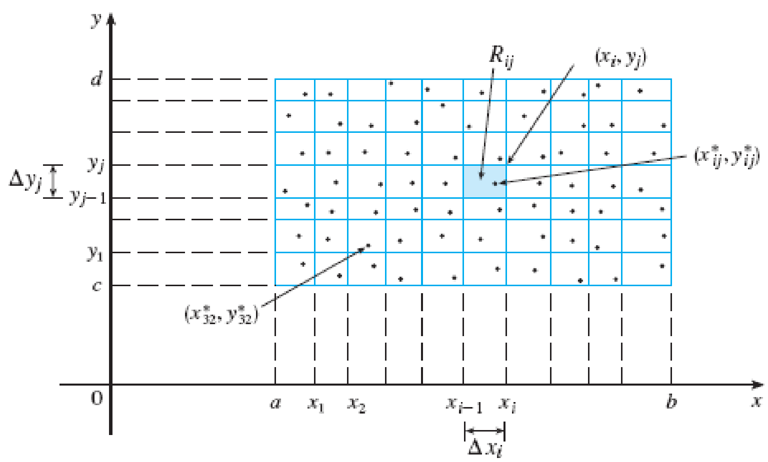
By drawing lines parallel to the coordinate axes through these partition points like the figure below

We form the subrectangles

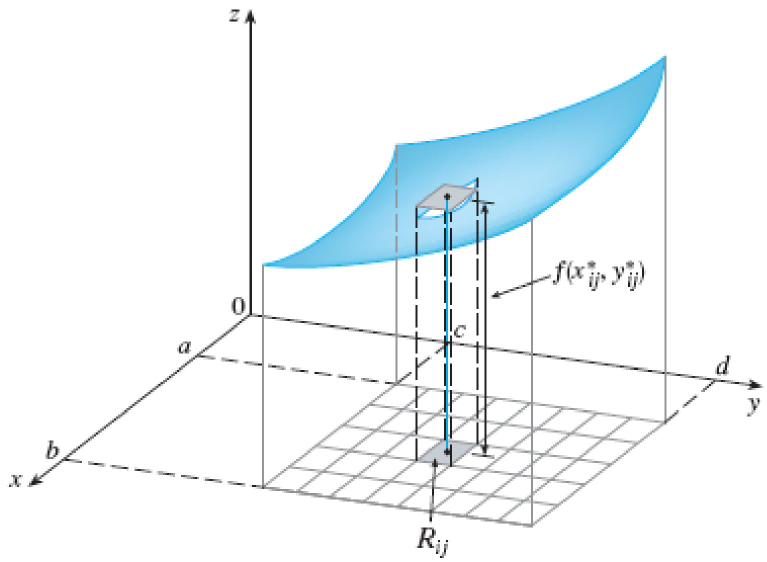
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x,y)|x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

for $i = 1, ..., m$ & $j = 1, ..., n$. There are mn of these subrectangles and they cover R. By letting $\Delta x_i = x_i - x_{i-1}$ & $\Delta y_j = y_j - y_{j-1}$, then we obtain the equation for the area of R_{ij}

$$\Delta A_{ij} = \Delta x_i \Delta y_j$$



By choosing a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin column or rectangular box. This column has base R_{ij} & height $f(x_{ij}^*, y_{ij}^*)$ as shown below.



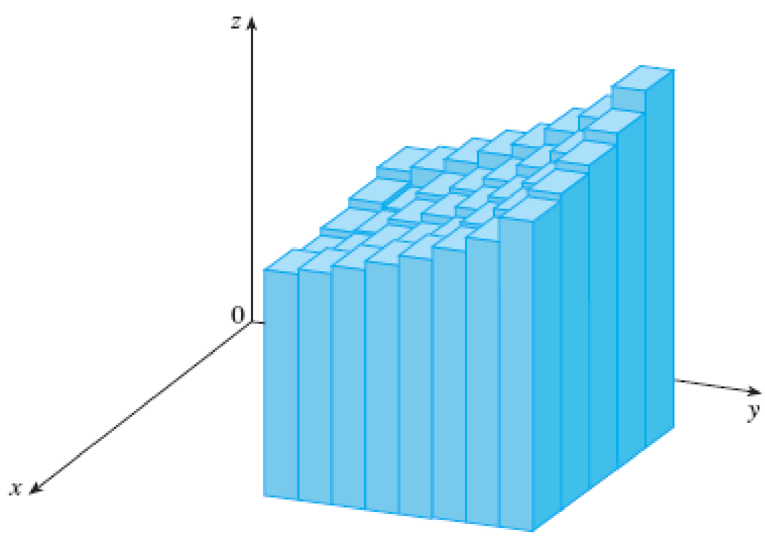
The volume of this box is the height of the box times the area of the base rectangle.

$$f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

By following this procedure for all the rectangles and summing up the volumes of the corresponding boxes, we get an approximation to the total volume of S

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

The intuition behind this double Riemann sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subtectangle and summing up the results. Visualized by the figure below



The smaller the subrectangles become, the better the approximation becomes so we can take the limit of the Riemann sum to obtain

$$V = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

Def
The double integral of f over the rectangle R is

$$\iint_R f(x,y)\ dA = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

if this limit exists.

Now if f is integrable over R, then the partitions P are regular, having the same dimensions and therefor the same area, $\delta A = \Delta x \Delta y$. In this case by letting $m \to \infty$ & $y \to \infty$. The sample point x_{ij}^*, y_{ij}^* can be any point. However for simplicity sake, the sample point will be the upper right-hand corner of R_{ij}, namely (x_i, y_j).

$$\iint_R f(x,y)\ dA = \lim_{m,n \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$

By definition, the volume of our solid can be written as a double integral.

If $f(x,y) \leq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is given by the equation

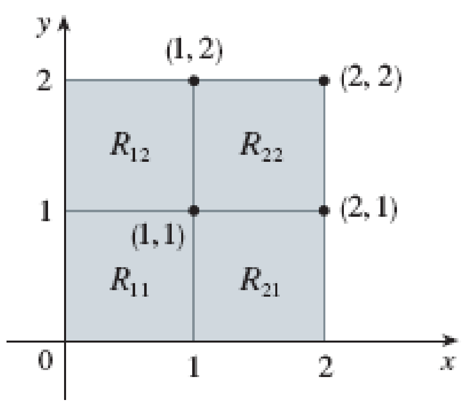
$$V = \iint_R f(x,y)\ dA$$

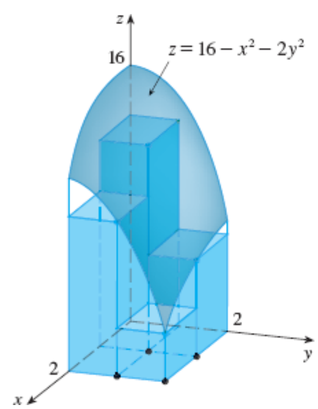
Ex 1
Estimate the volume of the solid that lies above the square $R = [0,2]\ times[0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}.

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_i) \Delta A$$

$$f(1,1)\Delta A + f(1,2)\Delta + f(2,1)\Delta A + f(2,2)\Delta A$$

$$13(1) + 7(1) + 10(1) + 4(1) = \boxed{34}$$





Ex 2

If $R = \{(x,y)| -1 \le x \le 1, -2 \le y \le 2\}$, evaluate the integral $\iint_R \sqrt{1-x^2}\ dA$

$$\iint_R \sqrt{1-x^2}\ dA \to \int_{-2}^{2}\int_{-1}^{1} \sqrt{1-x^2}\ dydx$$

$$\int_{-2}^{2}\left[\int_{-1}^{1} \sqrt{1-x^2}\ dx\right]dy \to \int_{-2}^{2}\Bigg]\int_{-1}^{1} \sqrt{1-\sin^2\theta}\cos\theta\ d\theta\Bigg]dy$$

$$x \to \frac{1}{1}\sin\theta,\ dx \to \cos\theta$$

$$\int_{-2}^{2}\left[\int_{-1}^{1} \sqrt{\cos^2\theta}\cos\theta\ d\theta\right]dy$$

$$\int_{-2}^{2}\left[\int_{-1}^{1} \cos^2\theta\ d\theta\right]dy$$

$$\int_{-2}^{2}\left[\int_{-1}^{1} \frac{\theta}{2} + \frac{\sin 2\theta}{2}\ d\theta]\right]dy$$

$$x = \sin\theta \to \sin^{-1}x = \theta, \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(-1) = -\frac{\pi}{2},\ \theta = \pm\frac{\pi}{2}$$

$$\int_{-2}^{2}\left[\frac{\pi}{2} + \frac{\sin 2\theta}{2}\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} dy = \int_{-2}^{2}\frac{\pi}{2}\ dy = \frac{\pi}{2}y\Bigg|_{y=-2}^{y=2} = \boxed{2\pi}$$

The Midpoint Rule

$$\iint_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A$$

Where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$

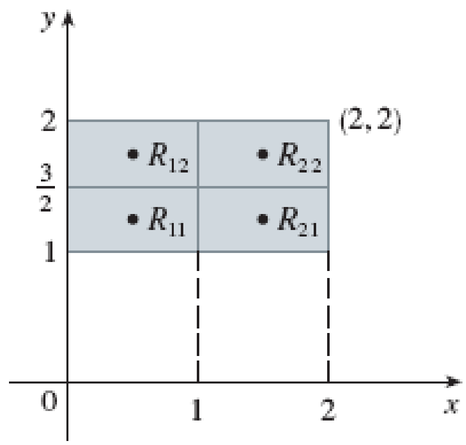
Ex 3
Use the midpoint rule with $m = n = 2$ to estimate the value of $\iint_R (x - 3y^2)\ dA$, where $R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$.

$$\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}, \overline{y}_2 = \frac{7}{4}$$

$$\Delta A = \frac{1}{2}$$

$$\iint_R (x - 3y^2)\ dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\Delta A$$

$$f(\overline{x}_1, \overline{y}_1)\Delta A + f(\overline{x}_2, \overline{y}_1)\Delta A + f(\overline{x}_1, \overline{y}_2)\Delta A + f(\overline{x}_2, \overline{y}_2)\Delta A = -\frac{95}{8} = \boxed{-11.875}$$



Iterated Integrals

$$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \Big[\int_c^d f(x,y)\ dy \Big] dx$$

Meaning that we first integrate with respect to y from $c \to d$ and then integrate once again but this time with respect to x and from $a \to b$.

The inverse also follows like so

$$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \Big[\int_a^b f(x,y)\ dx \Big] dy$$

Ex 4
Evaluate the following iterate integrals

A) $\int_0^3 \int_1^2 x^2 y\ dydx$

$$\int_0^3 \int_1^2 x^2 y\ dydx \to \int_0^3 \Big[\int_1^2 x^2 y\ dydx \Big]\ dx$$
$$\int_0^3 x^2 \frac{y^2}{2} \Big|_1^2\ dx \to \int_0^3 \frac{3}{2} x^2\ dx \to \frac{x^3}{2} \Big|_0^3 \to \boxed{\frac{27}{2}}$$

B) $\int_1^2 \int_0^3 x^2 y\ dxdy$

$$\int_1^2 \int_0^3 dx\ dy \to \int_1^2 \Big[\int_0^3 x^2 y\ dx \Big]\ dy$$
$$\int_1^2 \frac{x^3}{3} y^2 \Big|_0^3\ dy \to \int_1^2 9y\ dy \to 9y \Big|_1^2 \to \boxed{\frac{27}{2}}$$

Fubini's Theorem
If f is continuous on the rectangle $R = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$$

Ex 6

Evaluate $\iint_R y\sin(xy)\ dA$, where $R = [1,2]\times[0,\pi]$.

$$\iint_R y\sin(xy)\ dA \to \int_0^\pi \int_1^2 y\sin(xy)\ dxdy = \int_0^\pi \Big[-\cos(xy)\Big]_1^2\ dy$$
$$\int_0^\pi (-\cos(2y)+\cos y)\ dy = -\frac{1}{2}\sin(2y)+\sin y\Big|_0^\pi$$

Note that we can integrate in any order as the same answer will always be produced, however the order of integration also determines the difficulty of the integral as well.

$$\iint_R y\sin(xy)\ dA \to \int_1^2 \int_0^\pi y\sin(xy)\ dydx$$

If we were to integrate with respect to y first, the integral becomes much harder as it involves integration by parts twice. Therefore, we should integrate in whichever order is easiest.

There are special cases where $f(x,y)$ can be factored as the product of a function of x only and a function of y only. The double integral of f can be written in a simplier form. To exemplify, suppouse that $f(x,y) = g(x)h(y)$ and $R = [a,b]\times[c,d]$. Fubini's Theorem gives

$$\iint_R f(x,y)\ dA \to \int_c^d \int_a^b g(x)h(y)\ dxdy = \int_c^d \Big[\int_a^b g(x)h(y)\ dx\Big]dy$$

In the inner integral y is a constant, so $h(y)$ is a constant, allowing the equation to be written as so

$$\int_c^d \Big[\int_a^b g(x)h(y)\ dxdy = \int_c^d \Big[h(y)\int_a^b g(x)\ dx\Big]dy = \int_a^b g(x)\ dx \int_c^d h(y)\ dy$$

Since $\int_a^b g(x)\ dx$ is a constant, so the double integral of f can be written as the product of two single integrals.

Ex 8
Evaluate $\iint_R \sin x\cos y\ dA$, where $R = [0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$

$$\iint_R \sin x\cos y\ dA = \int_0^{\frac{\pi}{2}} \sin x\ dx \int_0^{\frac{\pi}{2}} \cos y\ dy$$
$$-\cos x\Big|_0^{\frac{\pi}{2}} \cdot \sin y\Big|_0^{\frac{\pi}{2}} = 1\cdot 1 = \boxed{1}$$

Properties of Double Integrals

$$\iint_R f(x,y)+g(x,y)\ dA = \iint_R f(x,y)\ dA + \iint g(x,y)\ dA$$
$$\iint_R cf(x,y)\ dA = c\iint_R f(x,y)\ dA \qquad \text{where } c \text{ is a constant}$$