5.4 Eigenvectors and Linear Transformations

Definition
Let V be a vector space. An eigenvector of a linear transformation $T : V \to V$ is a nonzero vector in V suc that $T(x) = \lambda x$ for some scalar λ. A scalar λ is an eigenvalue of T if there is a nontrivial solution to $T(x) = \lambda x$. Such an x is known as an eigenvector corresponding to λ.

The Matrix of a Linear Transformation
Let V be an n dimensional vector space and let T be any linear trasnformation from V to V. To associate a matrix with T, choose any basis $\mathcal{B}$ for V. Given any vector x, the coordinate vector $\left[x\right]_{\mathcal{B}}$ is in $\mathbb{R}^n$, as is the coordinate vector of its image, $\left[T(x)\right]_{\mathcal{B}}$.

Let $\{b_1, ..., b_n\}$ be the basis $\mathcal{B}$ for V. If $x = r_1 b_1 + ... + r_n b_n$, then

$$\left[x\right]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

And

$$T(x) = T(r_1 b_1 + ... + r_n b_n) = r_1 T(b_1) + ... + r_n T(b_n)$$

because T is linear. Since the coordinate mapping from B to $\mathbb{R}^n$ is linear, we get

$$T(x) = T(r_1 b_1 + ... + r_n b_n) = r_1 T(b_1) + ... + r_n T(b_n)$$
$$\left[T(x)\right]_{\mathcal{B}} = r_1 \left[T(b_1)\right]_{\mathcal{B}} + ... + r_n \left[T(b_n)\right]_{\mathcal{B}}$$

Since $\mathcal{B}$ coordinate vectors are in $\mathbb{R}^n$, the vector equation can be written as a matrix equation, namely

$$\left[T(x)\right]_{\mathcal{B}} = M \left[x\right]_{\mathcal{B}}$$

where

$$M = \left[\left[T(b_1)\right]_{\mathcal{B}} \quad \left[T(b_2)\right]_{\mathcal{B}} \quad ... \quad \left[T(b_n)\right]_{\mathcal{B}}\right]$$

The matrix M is a matrix representation of T, called the matrix for T relative to the basis $\mathcal{B}$ and denoted by $\left[T\right]_{\mathcal{B}}$.

Ex 1
Suppouse $\mathcal{B} = \{b_1, b_2\}$ is a basis for V. Let $T : V \to V$ be a linear transformation with the property that

$$T(b_1) = 3b_1 - 2b_2 \qquad T(b_2) = 4b_1 + 7b_2$$

Find the matrix M for T relative to $\mathcal{B}$.

$$\left[T(b_1)\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \left[T(b_2)\right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \end{bmatrix}$$

Ex 2
The mapping $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

T can be regonized as the differentiation operator

a) Find the $\mathcal{B}$ matrix for T, when B is the basis $\{1, t, t^2\}$

Firstly, compute the images of the basis vectors

$$T(1) = 0$$
$$T(t) = 1$$
$$T(t^2) = 2t$$

So

$$\left[T(1)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \left[T(t)\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \left[T(t^2)\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left[T\right]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

b) Verify that $\left[T(p)\right]_{\mathcal{B}} = \left[T\right]_{\mathcal{B}} \left[p\right]_{\mathcal{B}}$ for each p in $\mathbb{P}_2$.

For a general $p(t) = a_0 + a_1 t + a_2 t$

$$\left[T(p)\right]_{\mathcal{B}} = \left[a_1 + 2a_2 t\right]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ 2a_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \left[T\right]_{\mathcal{B}} \left[p\right]_{\mathcal{B}}$$

Theorem 5.8 Diaognal Matrix Representation
Suppouse $A = PDP^{-1}$,where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$ matrix for the transformation $x \mapsto Ax$.

Ex 4
Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that the $\mathcal{B}$ matrix for T is a diagonal matrix.

We know that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

The columns of P are b_1 & b_2, which are the eigenvectors of A. D is the $\mathcal{B}$ matrix for T when $\mathcal{B} = \{b_1, b_2\}$.