Limits and Continuity

Introduction
The limit of multi-variable functions is similar to the limit of a functin of a single variable.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

This notation indicates that the values of $f(x,y)$ approach the number L as the point (x,y) approaches (a,b) along any path that is within the domain of f.

Ex 1
Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

$$\lim_{(x,y)\to(x,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{x^2}{x^2} \qquad \lim_{(x,y)\to(0,y)} = \frac{-y^2}{y^2} = \frac{-y^2}{y^2} = -1$$

$$\lim_{(x,y)\to(x,0)} \frac{x^2-y^2}{x^2+y^2} \neq \lim_{(x,y)\to(x,0)} \frac{x^2-y^2}{x^2+y^2}$$

Since f has two different limits along two different lines, the given limit does not exist.

Ex 2
If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

If $y=0$, then $f(x,0) = \frac{0}{x^2} = 0$. Therefore $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the x axis. If $x=0$, then $f(0,y) = \frac{0}{y^2} = 0$. Therefore $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the y axis.

$$\lim_{(x,y)\to x,0} \frac{xy}{x^2+y^2} = \frac{0}{x^2} = 0 \qquad \lim_{(x,y)\to 0,y} \frac{xy}{x^2+y^2} = \frac{0}{y^2} = 0$$

$$\lim_{(x,y)\to x,0} \frac{xy}{x^2+y^2} \neq \lim_{(x,y)\to 0,y} \frac{xy}{x^2+y^2}$$

Although f have identical limits along two different lines, we have still yet to show that the given limit is 0. Let's approach $(0,0)$ along another line such as $y=x, x =\neq 0$.

$$\lim_{(x,y)\to x,x} \frac{xy}{x^2+y^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

$$\boxed{\therefore f(x,y)\to \frac{1}{2} \text{ as } (x,y)\to(0,0) \text{ along } y=x}$$