5.7 Differential Equations

Introduction
Given an unknown equation y

Suppouse $y' = Ay$, can we solve for y?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y)2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

$$\begin{aligned} ay_1 + by_2 &= y_1' \\ cy_1 + dy_2 &= y_2' \end{aligned}$$

Consider the differential equation,

$$Dy = y'$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

We can see that

$$\begin{aligned} \lambda_1 y_1 &= y_1' \\ \lambda_2 y_2 &= y_2' \end{aligned}$$

Now that this is decoupled, we can get

$$\begin{aligned} \lambda_1 y_1 = y_1' \rightarrow y_1 = e^{\lambda_1 x} + C_1 \\ \lambda_2 y_2 = y_2' \rightarrow y_2 = e^{\lambda_2 x} + C_2 \end{aligned}$$

Ex 1
Solve $x' = Ax$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{aligned} x_1(t) &= e^{3t} + C_1 \\ x_2(t) &= e^{-5t} + C_2 \end{aligned}$$

Ex 2

Solve $x' = Ax$, where $A = \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Firstly, find the eigenvalues of A and solve $\det(A - \lambda I) = 0$ in order to diagonalize A.

$$\begin{bmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{bmatrix} = 0$$

So

$$(\lambda + 1.5)(\lambda + 1) - 0.5 = 0$$
$$\lambda^2 + 2.5\lambda + 1 = 0$$

$$\lambda_1 = -0.5 \qquad \lambda_2 = -2$$

Then we solve for the eigenvectors v_1 & v_2

$$\lambda_1 = -0.5 \begin{bmatrix} -1 & 0.5 & 0 \\ 1 & -0.5 & 0 \end{bmatrix}$$
$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = -2$$
$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The general solution of $x' = \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix} x$ is

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$
$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-\frac{1}{2}t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$