

## 2.2 The Inverse of A Matrix

### Definition

An  $n \times n$  matrix  $A$  is said to be invertible if there is an  $n \times n$  matrix  $C$  such that  $AC = I$  &  $CA = I$ . Where  $I = I_n$  the  $n \times n$  identity matrix.

$$I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [Ie_1, Ie_2] \quad I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [Ie_1, Ie_2, Ie_3]$$

$$I = I_n = [Ie_1, Ie_2, \dots, Ie_n]$$

### Algebraic Representation

$(CA)x = Ix = x$ , with this we can say that  $I$  is essentially "1".

$$AA^{-1} = A^{-1}A = I$$

### Ex 1

If  $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$  and  $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ , then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus  $C = A^{-1}$

### Theorem 4

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $ad - bc = 0$ , then  $A$  is not invertible.

**Ex 2**

Find the inverse of  $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$\text{det} A = 3(6) - 4(5) = -2$$

$$A^{-1} = -\frac{1}{2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}$$

**Theorem**

If  $A$  is an invertible  $n \times n$  matrix, then for each  $b$  in  $\mathbb{R}^n$ , the equation  $Ax = b$  has the unique solution  $x = A^{-1}b$ .

**Ex 4**

Use the inverse of the matrix  $A$  to solve the system

$$3x_1 + 4x_2 = 3$$

$$5x_1 + 6x_2 = 7$$

This system is in the format  $Ax = b$ , so

$$x = A^{-1}b = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

**Theorem**

A) If  $A$  is an invertible matrix, then  $A^{-1}$  is invertible and

$$(A^{-1})^{-1} = A$$

B) If  $A$  &  $B$  are  $n \times n$  invertible matrices, then so is  $AB$ , and the inverse of  $AB$  is the product of the inverses  $A$  &  $B$  in the reverse order.

$$(AB)^{-1} = B^{-1}A^{-1}$$

C) If  $A$  is an invertible matrix, then so is  $A^t$ , and the inverse of  $A^t$  is the transpose of  $A^{-1}$ .

$$(A^T)^{-1} = (A^{-1})^T$$

**Theorem**

An  $n \times n$  matrix  $A$  is invertible if and only if  $A$  is row equivalent to  $I_n$ , and in this case, any sequence of elementary row operations that reduces  $A \rightarrow I_n$ , also transforms  $I_n \rightarrow A^{-1}$ .

Recall that the inverse of  $A$  is  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = I$

**Ex 2**

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$