

## 10.3 The Dot Product

### Def

If  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$  then the dot product of  $\vec{a}$  and  $\vec{b}$  is the number  $\vec{a} \cdot \vec{b}$  given by  $\vec{a} \cdot \vec{b}$  given by  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ . While for 2 dimensional vectors,  $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1b_1 + a_2b_2$

### Ex 1. Find the dot product

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 6 - 4 = \boxed{2}$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = -1(6) + 7(2) + 4(-\frac{1}{2}) = \boxed{6}$$

$$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k}) = 1(0) + 2(2) + (-3)(-1) = \boxed{7}$$

### Properties

If  $\vec{a}, \vec{b}$  and  $\vec{c}$  are 3-dimensional vectors, and  $c$  is a scalar, then

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$$

$$\vec{0} \cdot \vec{a} = 0$$

### Angle Between 2 Vectors

$0 \leq \theta \leq \pi$ . If  $\vec{a}$  and  $\vec{b}$  are parallel, then  $\theta = 0$  or  $\theta = \pi$

### Theorem

If  $\theta$  is the angle between the vectors  $\vec{a}$  and  $\vec{b}$ , then

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos \theta$$

### Ex 2

If the vectors  $\vec{a}$  and  $\vec{b}$  have lengths 4 and 6, and the angle between them is  $\frac{\pi}{3}$ , find  $\vec{a} \cdot \vec{b}$ .

$$|\vec{a}| = 4, |\vec{b}| = 6$$

$$\vec{a} \cdot \vec{b} = 4(6) \cos \frac{\pi}{3} = 24(\frac{1}{2}) = \boxed{12}$$

If  $\theta = 0$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$  since  $\cos 0 = 1$ .

If  $\theta = \pi$ , then  $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$  since  $\cos \pi = -1$ .

### Projections

$$\vec{b} = \vec{PR}$$

$$\vec{a} = \vec{PR}$$

$\vec{PR} = \vec{PS} + \vec{SR}$  Vector Projection of  $\vec{b}$  onto  $\vec{a}$  is  $\text{proj}_{\vec{a}} \vec{b}$

Scalar projection  $\vec{b}$  onto  $\vec{a}$  or "component of  $\vec{b}$  along  $\vec{a}$ " (  $\text{comp}_{\vec{a}}\vec{b}$  )

Since  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ,  $\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

**Ex 5**

Find the scalar and vector projects of  $\vec{b} = \langle 1, 1, 2 \rangle$  onto  $\vec{a} = \langle -2, 3, 1 \rangle$

$$\text{comp}_{\vec{a}}\vec{b} = \frac{-2(1) + 3(1) + 1(2)}{\sqrt{(-2)^2 + (3)^2 + (1)^2}} = \boxed{\frac{3}{\sqrt{14}}}$$

**Calculating Work**

The work done by a constant force  $f$  in moving an object through a distance  $d$  is  $W = FD$ .

Suppose the constant force is a vector  $\vec{F}$  pointing in a direction different from the displacement vector  $\vec{D}$ .

If the force moves the object from points  $P \rightarrow Q$ , then

$$W = (|\vec{F}| \cos \theta)$$

$$W = |\vec{F}|\vec{D} \cos \theta$$

$$W = \vec{F} \cdot \vec{D}$$

**Ex 6**

A force is given by a vector  $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$  and moves a particle from the point  $P(2, 1, 0)$  to the point  $Q(4, 6, 2)$ , find the work done.

$$\begin{aligned}\vec{D} &= \vec{PQ} = \langle 4 - 2, 6 - 1, 2 - 0 \rangle \\ &= \langle 2, 5, 2 \rangle\end{aligned}$$

$$\begin{aligned}W &= \vec{F} \cdot \vec{D} = 3(2) + 4(5) + 5(2) \\ &= 6 + 20 + 10 \\ &= \boxed{30}\end{aligned}$$