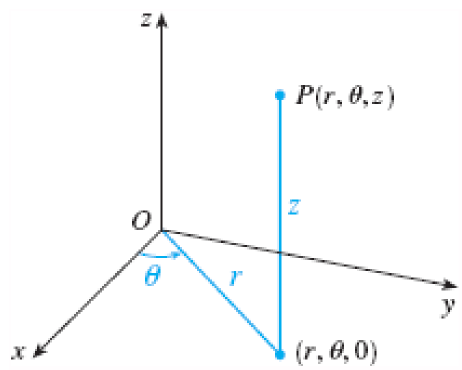
12.6 Triple Integrals in Cylindrical Coordinates

Introduction
In three dimensional dimensions, there is a coordinate system known as cylindrical coordinates, similar to polar coordinates. Some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates
In cylindrical coordiante systems, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r & θ are polar coordinates of the projection P onto the xy plane and z is the directed distance from the xy plane to P.

Below is a diagram showing the cylindrical coordinates of a point P



To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

While to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Ex 1
A) Convert the cylindrical coordinates $(2, \frac{2\pi}{3}, 1)$ to rectangular coordinates

$$x = 2\cos\frac{2\pi}{3} = -1$$
$$y = 2\sin\frac{2\pi}{3} = \sqrt{3}$$
$$z = 1$$

$$\boxed{(2, \frac{2\pi}{3}) \to (-1, 3, 1)}$$

B) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

$$\boxed{(3, -3, -7) \to (3\sqrt{2}, \frac{7\pi}{4} + 2n\pi, -7)}$$

Because of θ not being a constant and rather a function, there are multiple sets or triples of cylindrical coordinates.

Evaluating Triple Integrals with Cylindrical Coordinates

$$\iiint_E f(x, y, z)\ dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r\ dzdrd\theta$$

I is optimal to use this formula when E is a solid region easily described in cylindrical coordinates and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$

Ex 2

A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E

In cylindrical coordinates,

$$E = \{(r, \theta, z) | 0 \le \theta \le 2\pi, 0 \le r \le 1, 1 - r^2 \le z \le 4\}$$

Since density at (x, y, z) is proportional to the distance from the z axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant

$$m = \iiint_E K\sqrt{x^2 + y^2}\ dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)r\ dzdrd\theta = \boxed{\frac{12\pi K}{5}}$$

Ex 3

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} x^2+y^2\ dzdydx$

$$E = \{(x,y,z)|-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 2\}$$
$$\rightarrow$$
$$E = \{(r,\theta,z)|0 \le x \le 2\pi, 0 \le y \le 2, r \le z \le 2\}$$

$$\iiint\limits_E x^2+y^2\ dV$$
$$\int_0^{2\pi}\int_0^2\int_r^2 r^2 r\ dzdrd\theta$$
$$\int_0^{2\pi} d\theta \int_0^2 2r^3 - r^4\ dr$$
$$2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 = \boxed{\frac{16}{5}\pi}$$