COMMUTATIVE ALGEBRA

YU-SHENG LEE

Contents

1. Completion

1.1. Completion.

Definition 1.1. Let A be a ring, (A_n) be a decreasing filtration of ideals of A satisfying

$$A_0 = A, \quad A_{n+1} \subset A_n, \quad A_n A_m \subset A_{n+m}.$$

Let M be an A-module, (M_n) be a compatible decreasing filtration of submodules of M satisfying

$$M_0 = M, \quad M_{n+1} \subset M_n, \quad A_n M_m \subset M_{n+m}.$$

When $\mathfrak{q} \subset A$ is an ideal, the $\mathfrak{q}$-adic filtration is defined by $A_n = \mathfrak{q}^n A, M_n = \mathfrak{q}^n M$.

Definition 1.2. A ring A is said to be graded if

$$A = \oplus_{n \geq 0} A_n \text{ and } A_n A_m \subset A_{n+m}.$$

An A-module M is said to be compatibly graded if

$$M = \oplus_{n \geq 0} M_n \text{ and } A_n M_m \subset M_{n+m}.$$

Given a filtered ring A and a compatibly filtered A-module M, we can define the associated graded ring $\mathrm{gr}(A)$ and the associated compatibly graded $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$

$$\mathrm{gr}(A) = \oplus_{n \geq 0} A_n / A_{n+1}, \quad \mathrm{gr}(M) = \oplus_{n \geq 0} M_n / M_{n+1}.$$

Let A be a Noetherian ring and $\mathfrak{q} \subset A$ be an ideal. Consider the following property on (M_n).

$$(\mathfrak{q}\text{-good}) \qquad \begin{aligned} &\text{there exists a positive } n_0 \text{ such that } M_{n+k} = \mathfrak{q}^k M_n \text{ for all } n \geq n_0 \text{ and } k \geq 0 \\ \iff &\mathrm{gr}(M) \text{ is finite over the Noetherian ring } \mathrm{gr}(A). \end{aligned}$$

Proposition 1.3 (Artin-Rees)**.** *Let A be a Noetherian ring, M be an A-module, (M_n) be a $\mathfrak{q}$-good filtration, $N \subset M$ be a submodule, then the induced filtration $N_n := M_n \cap N$ is also $\mathfrak{q}$-good.*

Proposition 1.4 (Krull intersection)**.** *Let A be a Noetherian ring, M be a finite A-module, then*

$$x \in \bigcap_{n \geq 0} \mathfrak{q}^n M \iff \textit{there exists } d \in \mathfrak{q} \textit{ such that } dx = x.$$

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Proof. The submodule $N \coloneqq \cap_n \mathfrak{q}^n M$ satisfies $\mathfrak{q}N = N$ by the Artin-Rees lemma. Pick any generating set $\{x_1 \cdots x_n\}$, there exists $A = (a_{ij}), a_{ij} \in \mathfrak{q}$ such that $x_j = \sum a_{ij} x_i$, then we can pick $1 - d = \det(\mathbf{1}_n - A)$. □

Remark 1.5. When A is a local ring and M is finite, $M \to \hat{M}$ is injective by the above.

Note that $\hat{\mathfrak{q}} \subset \mathrm{rad}(\hat{A})$ since $1 + \mathfrak{q} \subset \hat{A}^{\times}$. Moreover, when A is Noetherian, the followings hold.

(a) The $\mathfrak{q}$-adic filtration is exact on finite A-modules.
(b) Let M be a finite A-module, then $M \otimes_A \hat{A} \to \hat{M}$ is an isomorphism.
(c) Let $I \subset A$ be an ideal and M be a finite A-module, then $I\hat{M} = \hat{IM} = \hat{I}\hat{M}$. In particular $\hat{A}$ is flat over A since $I \otimes \hat{A} \cong \hat{I} \cong I\hat{A}$ for any finitely generated ideal I.
(d) $M/\mathfrak{q}^n M = \hat{M}/\mathfrak{q}^n \hat{M} = \hat{M}/\hat{\mathfrak{q}}^n \hat{M}$.
(e) Let A be a local ring, then the completion $\hat{A}$ at the maximal ideal is faithfully-flat over A.

Proposition 1.6. *Suppose $A/\mathfrak{q}$ is Noetherian and $\mathfrak{q}$ is finitely-generated, then $\hat{A}$ is Noetherian.*

Proof. Use that $\mathrm{gr}^{\mathfrak{q}}(A)$ is Noetherian. Let I be an ideal of $\hat{A}$, then $\mathrm{gr}^{\mathfrak{q}}(I)$ is finitely generated, pick any generating set and lift which to an homomorphism between $\mathfrak{q}$-adically filtered $\hat{A}$-modules

$$u \colon \hat{A}^s \to I \text{ such that } \mathrm{gr}(u) \text{ is surjective.}$$

Then u is also surjective since $\hat{A}$ is complete and $I \subset \hat{A}$ is separated. □

1.2. **Support of a sheaf.**

Definition 1.7. Let X be a ringed space and F be an $\mathcal{O}_X$-module, the set of points $x \in X$ such that $F_x \neq 0$ is called the support of F and is denoted $\mathrm{Supp}(F)$.

(a) Given an exact sequence $0 \to F' \to F \to F'' \to 0$ of A-modules, then
$$\mathrm{Supp}(F) = \mathrm{Supp}(F') \cup \mathrm{Supp}(F'').$$
(b) Let M, M' be A-modules of finite type, then both $\mathrm{Supp}(M)$ and $\mathrm{Supp}(M')$ are closed and
$$\mathrm{Supp}(M \otimes_A M') = \mathrm{Supp}(M) \cap \mathrm{Supp}(M').$$
(c) Let M be a A-module of finite type and J be the annihilator, then $\mathrm{Supp}(M) = V(J)$.
(d) Let M be a A-module of finite type and I be an ideal, then $\mathrm{Supp}(M/IM) = \mathrm{Supp}(M) \cap V(I)$.
(e) Let $f \colon X \to Y$ be a morphism of schemes and F be an $\mathcal{O}_X$-module of finite type, then
$$\mathrm{Supp}(f^* F) = f^{-1}(\mathrm{Supp}(F)).$$

Proposition 1.8 (Weak Nullstellensatz)**.** *Let M be a finite A-module and $f \in A$. Then $f \colon M \to M$ is a nilpotent if and only if f lies in every prime of* $\mathrm{Supp}(M)$*.*

Proof. The map is a nilpotent if and only if $M_f = 0$, use $\mathrm{Supp}(M_f) = \mathrm{Supp}(M) \cap D(f)$. □

1.3. **Associated primes.**

Definition 1.9. A prime ideal $\mathfrak{p}$ is an associated prime for an A-module M if it is the annihilator of some $x \in M$. If X is a scheme and F is an $\mathcal{O}_X$-module, then $\mathrm{Ass}(F) = \{x \in X \mid \mathfrak{m}_x \in \mathrm{Ass}(F_x)\}$.

(a) Given $0 \to M' \to M \to M'' \to 0$, then $\mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.
(b) When A is Noetherian, an A-module M is nonzero if and only if $\mathrm{Ass}(M) = \emptyset$.
(c) When A is Noetherian, the set of zero divisors of M is the union of the associated primes of M.

(d) When A is Noetherian and M is a finite A-fmodule, there exists a filtration
$$M = M_0 \supset \cdots \supset M_n = 0 \text{ such that } M_i/M_{i+1} \cong A/\mathfrak{p}_i.$$
Therefore $\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$ is finite.

(e) When A is Noetherian, $\mathrm{Ass}_{S^{-1}A}(S^{-1}M) = \{p \in \mathrm{Ass}_A(M) \mid \mathfrak{p} \cap S = \emptyset\}$.

(f) When A is Noetherian, $\mathrm{Supp}(M) = \cup_{\mathfrak{p} \in \mathrm{Ass}(M)} V(\mathfrak{p})$.

1.4. **Characteristic functions.**

Definition 1.10. A composition series of an A-module M is a filtration
$$M = M_0 \supset \cdots \supset M_n = 0 \text{ such that } M_i/M_{i+1} \text{ are simple.}$$
Define $\ell_A(M) = n$, then $\ell(M) = \ell(M') + \ell(M'')$ given $0 \to M' \to M \to M'' \to 0$. When A is Noetherian, M is of finite length if and only if $\mathrm{Supp}(M)$ consists only of maximal ideals.

Proposition 1.11. *A ring A is Artinian if and only if A is Noetherian and every prime ideal is maximal.*

Proof. If A is Noetherian, by Noetherian induction every ideal contains a finite product of primes. Therefore $0 = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ and one obtain a composition series from which.

If A is Artinian, among finite product of maximal ideals of A pick a minimal $\mathfrak{m}$, so
$$\mathfrak{m}^2 = \mathfrak{m} \text{ and } \mathfrak{m} \subset \mathrm{rad}(A).$$
If $\mathfrak{m} \neq 0$, among ideals such that $I\mathfrak{m} \neq 0$ pick a minimal I. Then for $x \in I$, either

(1) $I \not\supsetneq Ax$ and hence $x\mathfrak{m} = 0$, or

(2) $I = Ax$, but then $I = 0$ by Nakayama lemma since $I\mathfrak{m} = I$ from the minimality of I.

Therefore $\mathfrak{m} = 0$, and consequently A is of finite length and every prime ideal of A is maximal. □

Let $\mathfrak{m}_1 \cdots \mathfrak{m}_n$ be the maximal ideal of an Artinian ring A, then

(a) $A \cong \prod_{i=1}^n A_{\mathfrak{m}_i}$.

(b) $A_{\mathfrak{m}_i} \to A/\mathfrak{m}_i^k$ is an isomorphism when k is sufficiently large.

Lemma 1.12. *Let H be a graded ring and M be a graded H-module of finite type. If H satisfies*

(a) H_0 is Artinian,

(b) H is an H_0-algebra generated by $x_1 \cdots x_r \in H_1$.

Then there exists a polynomial Q_M of degree $\leq r-1$ such that $Q_M(n) = \ell_{H_0}(M_n)$ for $n \gg 0$.

Proof. In general, let k_i be the degree of x_i, then the Poincare series $P(x) = \sum_{n=0}^\infty \ell_{H_0}(M_n)x^n$ is of the form
$$P(x) = \frac{f(x)}{\prod_{i=1}^r (1 - x^{k_i})}, \quad f(x) \text{ is polynomial.}$$
Thus $\ell(M_n)$, i.e. the coefficients of x^n in $P(x)$, is a polynomial of degree at most $r-1$ in the case when $k_i = 1$ for all i since
$$(1-x)^{-r} = \sum_{n \geq 0} \binom{n+r-1}{r-1} t^n.$$
□

Proposition 1.13. *Let A be a Noetherian ring and M be a finite A-module. Given a q-good filtration (M_n), then when M/qM has finite lengh,*

(a) there exists a polynomial $P_{(M_n)}$ of degree $\leq r$ such that $P_{(M_n)}(m) = \ell(M/M_m)$ for $m \gg 0$.

(b) The degree and leading coefficient of $P_{(M_n)}$ are independent of the choice of filtration.

(c) *Given* $0 \to M' \to M \to M'' \to 0$, *then* $P_{(M_n)} - P_{(M'_n)} - P_{(M''_n)}$ *has degree* $\le r-1$.

Proof. Let $I = \mathrm{Ann}(M), B = A/I$ and $\mathfrak{p} = \mathfrak{q} + I/I$. Then $H = \mathrm{gr}^{\mathfrak{p}}(B)$ acting on $\mathrm{gr}(M)$ satisfies the assumptions of the previous lemma since $\mathrm{Supp}(M/\mathfrak{q}M) = V(I) \cap V(\mathfrak{q})$ consists only of maximal primes. One then use

$$(1-x)\sum_{n=1}^{\infty} \ell(M/M_n)x^n = \sum_{n=0}^{\infty} \ell(M_n/M_{n+1})x^n.$$

Since (M_n) is $\mathfrak{q}$-good, there exists n_0 such that $M_{n+1} = \mathfrak{q}M_n$ for $n \ge n_0$. Thus when $n \gg 0$

$$\mathfrak{q}^{n+n_0}M \subset M_{n+n_0} = \mathfrak{q}^n M_{n_0} \subset M_n \Longrightarrow P_{(\mathfrak{q}^m M)}(n+n_0) \ge P_{(M_m)}(n+n_0) \ge P_{(\mathfrak{q}^m M)}(n) \ge P_{(M_m)}(n),$$

therefore the degree and leading coefficient do not depends on (M_n).

The last property then follows from the previous two and that

$$\ell(M/\mathfrak{q}^n M) = \ell(M''/\mathfrak{q}^n M'') + \ell(M'/M'_n), \quad M'_n := M' \cap \mathfrak{q}^n M.$$

□

2. Depth and Dimension

Let A be a Noetherian local ring with maximal ideal m and M be a finite A-module. Define

(a) $\dim(M) := \dim(\mathrm{Supp}(M))$.
(b) $d(M) := \deg(P_{\mathfrak{q}^n M})$ for an ideal of definition $\mathfrak{m}^n \subset \mathfrak{q} \subset \mathfrak{m}$.
(c) $s(M)$ is the minimal number $\{x_1, \cdots, x_n\} \subset \mathfrak{m}$ such that $M/(x_1, \cdots, x_n)M$ is of finite length.

Theorem 2.1. *The three values are equal to each other.*

Proof. To prove $\dim(M) \le d(M)$, pick $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p})$ and reduce to show

$$\dim(M) = \dim(A/\mathfrak{p}) \le d(A/\mathfrak{p}) \le d(M).$$

If $d(A/\mathfrak{p}) = 0$, then $A/\mathfrak{p}$ is of finite length and Artinian. If $d(A/\mathfrak{p}) > 0$, there is nothing to prove when $\dim(A/\mathfrak{p}) = 0$. And when there exists

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n, \quad n = \dim(A/\mathfrak{p}),$$

pick $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_0$, then $\dim(A/(\mathfrak{p}, x)) \ge n-1$ while $d(A/(\mathfrak{p}, x)) \le d(A) - 1$ since $0 \to A/\mathfrak{p} \xrightarrow{x} A/\mathfrak{p} \to A/(\mathfrak{p}, x) \to 0$ is exact, thus by induction on $d(M)$

$$\dim(A/\mathfrak{p}) - 1 \le \dim(A/(\mathfrak{p}, x)) \le d(A/(\mathfrak{p}, x)) \le d(A/\mathfrak{p}) - 1 \Longrightarrow \dim(A/\mathfrak{p}) \le d(A/\mathfrak{p}).$$

That $d(M) \le s(M)$ follows from the proof of Proposition 1.13.

To prove that $s(M) \le \dim(M)$. Observe that M is of finite length when $\dim(M) = 0$. If $\dim(M) \ge 0$, let $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_h\}$ be the set of associated primes such that $\dim(M) = \dim(A/\mathfrak{p}_i)$, then none of $\mathfrak{p}_i$ are maximal and there exists $x \in \mathfrak{m} \setminus \cup_i \mathfrak{p}_i$. For which $s(M/xM) + 1 \ge s(M)$ and $\dim(M/xM) \le \dim(M) - 1$, thus by induction on $\dim(M)$

$$s(M) \le s(M/xM) + 1 \le \dim(M/xM) + 1 \le \dim(M).$$

□

(1) $\dim(M/xM) \ge \dim(M) - 1$ for $x \in \mathfrak{m}$. The equality holds when x doesn not belong to any $\mathfrak{p} \in \mathrm{Supp}(M)$ such that $\dim(M) = \dim(A/\mathfrak{p})$. In particular when x is not a zero-divisor.
(2) $\dim_A(M) = \dim_{\hat{A}}(\hat{M})$.
(3) a prime ideal $\mathfrak{p}$ has height $\le n$ if and only there exists an ideal I generated by n elements such that $\mathfrak{p}$ is a minimal ideal of A/I (Krull's principal ideal theorem when $n = 1$).

Theorem 2.2 (Noether normalization lemma)**.** *Let k be a field, A be a finite type k-algebra, and $I_1 \subset \cdots \subset I_r$ be a sequence of proper ideals for A. Then there exists algebraically independent $y_1, \cdots, y_n$ such that*

(a) A is integral over $B = k[y_1, \cdots, y_n]$.
(b) For each $1 \leq i \leq r$, there exists $h(i) \geq 0$ such that $I_i \cap B$ is generated by $\{y_1, \cdots, y_{h(i)}\}$.
(c) when A is a domain and $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ is a maximal chain of prime ideals, $n = \operatorname{tr.deg}_k A$.
(d) if $\mathfrak{m}$ is a maximal ideal, then $A/\mathfrak{m}$ is algebraic over k (Nullstellensatz).

Let $A = k[x_1, \cdots, x_n]$. If $\{x_1, \cdots, x_r\}$ is an alg.independent set in $frac(A/\mathfrak{p})$, then $\mathfrak{p} \cap k[x_1, \cdots, x_r] = 0$. If furthermore it's a transcendental basis, let $S = k[x_1, \cdots, x_r] \setminus 0$, then $A_S/\mathfrak{p}A_S$ is a finite field extension over $k(x_1, \cdots, x_r)$.

(a) If $\mathfrak{p} \subset \mathfrak{q}$, then $\operatorname{tr.deg}_k(A/\mathfrak{p}) > \operatorname{tr.deg}_k(A/\mathfrak{q})$.
(b) If $\mathfrak{p} \subset \mathfrak{q}$ and $\operatorname{tr.deg}_k(A/\mathfrak{p}) \geq \operatorname{tr.deg}_k(A/\mathfrak{q}) + 2$, suppose

$$\{x_1, \cdots, x_r\} \text{ is a transcendental basis in } A/\mathfrak{q}, S = k[x_1, \cdots, x_r] \setminus 0$$
$$\{x_1, \cdots, x_r, x_{r+1}\} \text{ is alg.independent in } A/\mathfrak{p}, S' = k[x_1, \cdots, x_{r+1}] \setminus 0$$

then $\mathfrak{p}A_{S'} \subset A_{S'}$ is not maximal and $\mathfrak{m} \cap A_S$ is not maximal for any $\mathfrak{m} \subset A_{S'}$. Thus there exists $\mathfrak{p} \subset \mathfrak{p}' \subset \mathfrak{q}$.

We conclude that $\operatorname{tr.deg}_k(A/\mathfrak{p}) = \dim(A/\mathfrak{p}) = n - \operatorname{ht}(\mathfrak{p}) = \dim(A) - \operatorname{ht}(\mathfrak{p})$.

2.1. **Depth.**

Assume throughout the subsection that A is a Noetherian ring.

Definition 2.3. Let M be an A-module, a sequence $(x_1, \cdots, x_r)$ of elements in a is M-regular if

$$0 \to M_{i-1} \xrightarrow{x_i} M_{i-1}, \quad M_{i-1} := M/(x_1, \cdots, x_{i-1})M \text{ for all } 1 \leq i < r.$$

It is M-quasi-regular if the canonical surjection

$$\varphi_r \colon (M/JM)[T_1, \cdots, T_r] \to \operatorname{gr}_J(M), \quad J = (x_1, \cdots, x_r)$$

is an isomorphism. When $x_1, \cdots, x_r \in \operatorname{rad}(A)$, the sequence is M-regular if and only if it is M-quasi-regular. Let I be an ideal, define $\operatorname{depth}_I(M)$ to be the maximal length of M-regular sequence in I.

Lemma 2.4. *If $(x_1, \cdots, x_r)$ is a regular sequence, then $\operatorname{ht}(x_1, \cdots, x_r) = r$.*

Proof. When $r = 1$ and $\mathfrak{p}$ is a minimal prime containing (x_1), then $\operatorname{ht}(\mathfrak{p}) \leq 1$ by Krull's principal ideal theorem. Since $\mathfrak{p}$ contains a non-zero-divisor x_1, the equality hodls.

When $r > 1$, let $A' = A/(x_1)$ and $\mathfrak{p}' = \mathfrak{p}'/(x_1)$. By induction on r, $\operatorname{ht}(\mathfrak{p}') = r - 1$. Pick $\mathfrak{p}'_{r-1} \supset \cdots \supset \mathfrak{p}'_0 = \mathfrak{p}'$, which pulls back to $\mathfrak{p}_{r-1} \supset \cdots \supset \mathfrak{p}_0 = \mathfrak{p}$, then $\mathfrak{p}_{r-1}$ is a minimal prime containing (x_1) and hence $\operatorname{ht}(\mathfrak{p}_{r-1}) = 1$, therefore $\operatorname{ht}(\mathfrak{p}) = r$. □

Proposition 2.5. *Let A be a Noetherian ring and M be a finite A-module, the following are equivalent.*

(1) $\operatorname{Ext}^q(N, M) = 0$ for all $q < r$ and all finite A-modules N such that $\operatorname{Supp}(N) \subset V(I)$.
(2) $\operatorname{Ext}^q(N, M) = 0$ for all $q < r$ and some finite A-modules N such that $\operatorname{Supp}(N) = V(I)$.
(3) Given $x_1, \cdots, x_n \in I$ such that $(x_1, \cdots, x_n)$ is M-regular, there exists $x_{n+1}, \cdots, x_r \in I$ such that $(x_1, \cdots, x_r)$ is M-regular.
(4) There exists an M-regular sequence $x_1, \cdots, x_r \in I$.

In particular $\operatorname{depth}(A) = \operatorname{depth}(\hat{A})$ *since $\hat{A}$ is faithfully-flat over A and* $\operatorname{Ext}^i(A/\mathfrak{m}, M) \otimes_A \hat{A} \cong \operatorname{Ext}^1(\hat{A}/\hat{\mathfrak{m}}, \hat{M})$.

Proof. If $\operatorname{Hom}(N,M)=0$ and $\operatorname{Supp}(N)=V(I)$, then $I \not\subseteq \mathfrak{p}$ for any $\mathfrak{p}\in \operatorname{Ass}(M)$. Otherwise there exists a nonzero $N \to A/\mathfrak{p} \hookrightarrow M \neq 0$. Hence there exists $x_0 \in I \setminus \cup_{\mathfrak{p}\in\operatorname{Ass}(M)}\mathfrak{p}$, and therefore M-regular. Use the argument inductively on $M_n = M/(x_1,\cdots,x_n)M$ to show $(ii) \Longrightarrow (iii)$.

On the other hand, if x_0 is M-regular then

$$0 \to \operatorname{Hom}(N,M) \xrightarrow{x_0} \operatorname{Hom}(N,M).$$

But $x_0 \in I$ is a nilpotent on N if $\operatorname{Supp}(N) \subset V(I)$, and hence $\operatorname{Hom}(N,M)=0$. Use the argument inductively on r to show $(iv) \Longrightarrow (i)$. $\square$

Proposition 2.6. *Let A be a Noetherian local ring and M be a finite A-module, then* $\operatorname{depth}(M) \le \dim(A/\mathfrak{p})$ *for any* $\mathfrak{p}\in\operatorname{Ass}(M)$. *In particular* $\operatorname{depth}(M)\le \dim(M)$ *if* $M\neq 0$.

Proof. If $0<r\le \operatorname{depth}(M)$, pick an M-regular $x_0\in\mathfrak{m}$ and form $0\to M \xrightarrow{x_0} M \to M' \to 0$. Then $r-1\le \operatorname{depth}(M')$ and inductively $r-1\le\dim(A/\mathfrak{p}')$ for any $\mathfrak{p}'\in\operatorname{Ass}(M')$. Now, if $\mathfrak{p}\in\operatorname{Ass}(M)$, then

$$0\to \operatorname{Hom}(A/\mathfrak{p},M) \xrightarrow{x_0}\to \operatorname{Hom}(A/\mathfrak{p},M) \to \operatorname{Hom}(A/\mathfrak{p},M') = \operatorname{Hom}(A/\mathfrak{p}+Ax, M')$$

is exact. Thus Nakayama's lemma implies that $\operatorname{Hom}(A/\mathfrak{p}+Ax,M')\neq 0$, but then there exists $\mathfrak{p}'\in\operatorname{Ass}(M')\cap V(\mathfrak{p}+Ax)$, for which $r-1\le\dim(A/\mathfrak{p}')\le\dim(A/\mathfrak{p})-1$. $\square$

2.1.1. *Kozul complex.*

Definition 2.7. For $x\in A$, let $K(x).$ be the complex $\cdots 0\to A \xrightarrow{x} A \to 0\cdots$. The tensor $C.(x) \coloneqq C.\otimes K.(x)$ of which with any complex $C.$ of A-modules is defined as the total complex of

$$\begin{tikzcd}
\cdots \arrow[r] & C_p \arrow[r, "d"] \arrow[d, "(-1)^p x"] & C_{p-1} \arrow[r, "d"] \arrow[d, "(-1)^{p-1} x"] & \cdots \\
\cdots \arrow[r] & C_p \arrow[r, "d"] & C_{p-1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

In general, given a sequence $(x_1,\cdots,x_n)$, the Kozul complex is $K.(x_1,\cdots,x_n) \coloneqq K.(x_1)\otimes\cdots\otimes K.(x_n)$. In particular, $H_n(\underline{x},M)=M[\underline{x}]$ and $H_0(\underline{x},M)=M/(x_1,\cdots,x_n)M$.

Lemma 2.8. *If the sequence $(x_1,\cdots,x_n)$ is M-regular, then $H_0(x_1,\cdots,x_n,M)=M/(x_1,\cdots,x_n)M$ and $H_p(x_1,\cdots,x_n,M)=0$ for $p>0$. Conversely, when A is local and M is a finite A-module, $(x_1,\cdots,x_n)$ is M-regular if $H_1(x_1,\cdots,x_n,M)=0$.*

Proof. For any complex $C.$, the above construction gives an exact sequence of complexes $0\to C.\to C.(x)\to C.[-1]\to 0$ and an associated long exact sequence $\cdots\to H_p(C.)\to H_p(C.(x))\to H_{p-1}(C.)\to\cdots$.

Now, the lemma is trivial if $n=1$. If $n>1$ and $p>1$, then inductive

$$0=H_p(x_1,\cdots,x_{n-1},M)\to H_p(x_1,\cdots,x_n,M)\to H_{p-1}(x_1,\cdots,x_{n-1},M)=0$$

and $0\to H_1(x_1,\cdots,x_n,M)\to M/(x_1,\cdots,x_{n-1}) \xrightarrow{x_n} M/(x_1,\cdots,x_{n-1})$ when $p=1$.

Conversely, if $H_1(x_1,\cdots,x_n,M)=0$, then

$$H_1(x_1,\cdots,x_{n-1},M) \xrightarrow{x_n} H_1(x_1,\cdots,x_{n-1},M)\to 0\to M/(x_1,\cdots,x_{n-1}) \xrightarrow{x_n} M/(x_1,\cdots,x_{n-1}).$$

Therefore $H_1(x_1,\cdots,x_{n-1},M)=0$ by Nakayama's lemma and x_n is regular for $M/(x_1,\cdots,x_{n-1})$. Inductively, $(x_1,\cdots,x_n)$ is M-regular. $\square$

2.2. **Regular local rings.** Throughout the subsection, A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$.

Definition 2.9. Let A be a Noetherian local ring.

(a) A is regular if $\mathfrak{m}$ is generated by $r = \dim(A)$ elements.
(b) A is la local complete intersection ring if it is the quotient of a regular local ring by a regular sequence.
(c) A is Gorenstein if $\mathrm{Ext}^i(k, A) = 0$ when $i \neq \dim(A)$ and $\dim_k \mathrm{Ext}^i(k, A) = 1$ when $i = \dim(A)$.
(d) A is Cohen-Macaulay if $\mathrm{depth}(A) = \dim(A)$.

2.2.1. *Cohen-Macaulay.* In general, a finite A-module M is Cohen-Macaulay if $\dim(M) = \mathrm{depth}(M)$. then $\dim(M) = \mathrm{depth}(M) \leq \dim(A/\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{Ass}(M)$. Thus A/I is equi-dimensional since $\mathrm{Supp}(A/I) = \mathrm{Supp}(M) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p}$.

Lemma 2.10. *The followings hold when M or A is Cohen-Macaulay.*

(a) $\mathrm{depth}_{\mathfrak{p}}(M) = \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *when* $M_{\mathfrak{p}} \neq 0$.
(b) a sequence $(x_1, \cdots, x_r)$ *is regular if and only if* $\mathrm{ht}(x_1, \cdots, x_r) = r$.
(c) $\mathrm{ht}(I) + \dim(A/I) = \dim(A)$.

Proof. Observe that $\mathrm{depth}_{\mathfrak{p}}(M) \leq \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. If $\mathrm{depth}_{\mathfrak{p}}(M) = 0$, then $\mathfrak{p}$ is minimal in $\mathrm{Supp}(M)$ and hence $\dim(M_{\mathfrak{p}}) = 0$. If $\mathrm{depth}_{\mathfrak{p}}(M) > 0$, pick any non-zero-divisor $x \in \mathfrak{p}$, then $M' = M/xM$ is also Cohen-Macaulay. By induction on $\mathrm{depth}_{\mathfrak{p}}$

$$\mathrm{depth}_{\mathfrak{p}}(M) - 1 = \mathrm{depth}_{\mathfrak{p}}(M') = \dim(M'_{\mathfrak{p}}) = \dim(M_{\mathfrak{p}}) - 1.$$

If $r = \dim(A)$, then $(x_1, \cdots, x_r)$ is a system of parameters. In particular x_1 doesn't belong to any $\mathfrak{p} \in \mathrm{Ass}(A)$. Otherwise $(x_2, \cdots, x_r)$ is also a system of parameters for $A/\mathfrak{p}$ but $\dim(A) = \dim(A/\mathfrak{p})$. Therefore x_1 is regular and $A/(x_1)$ is also Cohen-Macaulay. Inductively, $(x_1, \cdots, x_r)$ is regular.

If $r < \dim(A)$, there exists $x_{r+1} \in \mathfrak{m}$ such that $x_{r+1} \notin \mathfrak{p}$ for any minmail prime $\mathfrak{p}$ containing $(x_1, \cdots, x_r)$. Then $\mathrm{ht}(x_1, \cdots, x_{r+1}) = r + 1$, and the sequence can be inductively extended to a system of parameters. Thus $(x_1, \cdots, x_r)$ is regular and $A/(x_1, \cdots, x_r)$ is Cohen-Macaulay. Therefore for any minimal prime $\mathfrak{p}$ containing $(x_1, \cdots, x_r)$, $\dim(A/\mathfrak{p}) = \mathrm{depth}(A) - r = \dim(A) - \mathrm{ht}(\mathfrak{p})$.

At last, since $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) = \mathrm{depth}_{\mathfrak{p}}(A)$, there exists a regular sequence $(x_1, \cdots, x_r), r = \mathrm{ht}(\mathfrak{p})$, contained in $\mathfrak{p}$. And the proof above shows that $\mathrm{ht}(\mathfrak{p}) + \dim(A/\mathfrak{p}) = \dim(A)$. □

(1) If x_0 is a non-zero-divisor, then A is Cohen-Macaulay if and only if $A/(x_0)$ is.
(2) If A is Cohen-Macaulay, so is $\hat{A}$.
(3) If A is Cohen-Macaulay, so is $A[[X]]$.
(4) If A is Cohen-Macaulay, so is $A_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$.

Theorem 2.11. *if A is CM, so is $A[[X]]$.*

Proof. Let $(x_1, \cdots, x_n)$ be a regular sequence, with $n = \dim(A)$, then it is also a system of parameters. Then $(X, x_1, \cdots, x_n)$ is both a regular sequence and a system of parameters for $A[[X]]$, so

$$n + 1 \leq \mathrm{depth}(A[[X]]) \leq \dim(A[[X]]) \leq n + 1.$$

□

Elements of $\mathfrak{m}$ that are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$ are called **regular parameters**. By Nakayama's lemma, $\mathfrak{m}$ is generated by regular parameters of size $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$, therefore $\dim(A) = s(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. The equality holds if and only if A is regular. When this is the case, let $(x_1, \cdots, x_r), r = \dim(A)$, be a set of regular parameters that generate $\mathfrak{m}$. Then the natural surjection

$$\phi \colon k[T_1, \cdots, T_r] \to \mathrm{gr}^{\mathfrak{m}}(A), \quad T_i \mapsto x_i$$

is an isomorphism. Otherwise let $I = \ker(\phi)$ and consider the exact sequence

$$0 \to I \to k[T_1, \cdots, T_r] \to \mathrm{gr}^{\mathfrak{m}}(A) \to 0$$

If there exists some nonzero $u_h \in I_h$, then $I_n \supset u_h(k[T_1, \cdots, T_r])_{n-h}$ for any $n \geq h$. Therefore the Hilbert polynomial for

$$\begin{aligned}\ell(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \dim_k(\mathrm{gr}^{\mathfrak{m}}(A))_n &= \dim(k[T_1, \cdots, T_r]_n) - \dim_k(I_n) \\ &\leq \dim(k[T]_n) - \dim(k[T]_{n-h}) = \binom{n+r-1}{r-1} - \binom{n-h+r-1}{r-1}\end{aligned}$$

has degree $\leq r-2$, which is a contradiction. Consequently $\mathrm{gr}^{\mathfrak{m}}(A)$ is a domain, and so therefore so is A since $\cap_n \mathfrak{m}^n = 0$ by Krull's intersection.

Proposition 2.12. *Let I be an ideal of a local Noetherian ring of dimension $r = \dim(A)$. The followings are equivalent.*

(a) A is regular and I is generated by s regular parameters.
(b) $B = A/I$ is regular of dimension $r - s$ and I is generated by s element
(c) A is regular and B is regular of dimsnion $r - s$

Proof. Consider the exact sequence

$$0 \to I + \mathfrak{m}^2/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0, \quad \mathfrak{n} := \mathfrak{m}/I.$$

If $r = \dim_k \mathfrak{m}/\mathfrak{m}^2$ and I is generated by s regular parameters, then

$$r - s \leq \dim(B) \leq \dim_k \mathfrak{n}/\mathfrak{n}^2 = r - s,$$

and B is therefore regular (the first inequality follows from Krull's principal ideal theorem).

If B is regular of dimension $r - s$ and I is generated by s element, then

$$r = \dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq r.$$

At last, if A, B are regular and $\dim(B) = r - s$, there exists s regular parameters among generators of $I/\mathfrak{m}I \twoheadrightarrow I/I \cap \mathfrak{m}^2 \cong I + \mathfrak{m}^2/\mathfrak{m}^2$. Let $I' \subset I$ be the ideal generated by them, then both I' and I are prime ideals of co-dimension $r - s$ and thus $I = I'$. □

By the above proof, when A is regular and $r = \dim(A)$, a set of regular parameters $(x_1, \cdots, x_r)$ is also a regular sequence since each $(x_1, \cdots, x_i), 1 \leq i \leq r$ is a prime ideal. In particular $\mathrm{depth}(A) = \dim(A)$.

Remark 2.13. Conversely, if the maximal ideal of a local Noetherian ring A is generated by a regular sequence, then $\dim(A) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq \mathrm{depth}(A) \leq \dim(A)$, and thus A is regular.

2.3. **Projective dimension.**

Definition 2.14. The projective dimension of M, denoted by $\mathrm{proj.dim}(M)$, is the infimum length of projetive resolutions $0 \to L_n \to \cdots \to L_0$. The following conditions are equivalent.

(a) $\mathrm{proj.dim}(M) \leq n$.
(b) $\mathrm{Ext}^i(M, N) = 0$ for all $i > n$.
(c) Given $0 \to R \to L_{n-1} \to \cdots \to L_0 \to M \to 0$ where L_i are projective, then R is also projective.

When A is Noetherian local and M is a finite A-module recall that

$$M \text{ is free} \iff M \text{ is prjective} \iff M \text{ is flat} \iff \mathrm{Tor}_1(M, k) = 0.$$

A resolution of M can be constructed as follows.

(1) Lift a basis of M and form $0 \to K_1 \to L_0 \to M \to 0$.

(2) Lift a basis of K_1 and form $0 \to K_2 \to L_1 \to K_1 \to 0$ and so on.

Such a resolution is a minimal free resolution in the sense that all L_i are free, $L_0 \otimes k \cong M \otimes k$, and that $\bar{d} = 0$. Any minimal free resolution is isomorphic. From the minimal resolution of M, one sees that

$$\operatorname{proj.dim}(M) = r \iff \operatorname{Tor}_{r+1}(k, M) = 0 \text{ but } \operatorname{Tor}_r(k, M) \neq 0.$$

Theorem 2.15 (Auslander-Buchsbaum)**.** *Let A be a Noetherian local ring. If $M \neq 0$ is a finite A-module and* $\operatorname{proj.dim}(M) < \infty$*, then* $\operatorname{proj.dim}(M) + \operatorname{depth}(M) = \operatorname{depth}(A)$.

Proof. Consider the first step $0 \to K_1 \to L_0 \to M \to 0$ of the minimal resolution of M. If $K_1 = 0$, then $\operatorname{proj.dim}(M) = 0$ and M is free, hence $\operatorname{depth}(M) = \operatorname{depth}(A)$. If $K_1 \neq 0$, then $\operatorname{proj.dim}(K_1) = \operatorname{proj.dim}(M) - 1 \geq 0$ and from the fact that $K_1 \to L_0$ is residually trivial the following is exact.

$$0 \to \operatorname{Ext}^i(k, L_0) \to \operatorname{Ext}^i(k, M) \to \operatorname{Ext}^{i+1}(k, K_1) \to 0$$

By induciton on the projective dimension, $\operatorname{depth}(K_1) = \operatorname{depth}(A) - \operatorname{proj.dim}(K_1) \leq \operatorname{depth}(A)$. Since $\operatorname{depth}(M) = \inf\{i \mid \operatorname{Ext}^i(k, M) \neq 0\}$, we see $\operatorname{Ext}^i(k, M) \cong \operatorname{Ext}^{i+1}(k, K_1)$ when $i < \operatorname{depth}(K_1)$ and therefore

$$\operatorname{depth}(M) = \operatorname{depth}(K_1) - 1 = \operatorname{depth}(A) - \operatorname{proj.dim}(K_1) - 1 = \operatorname{depth}(A) - \operatorname{proj.dim}(M).$$

□

Theorem 2.16 (Serre)**.** *Let A be a Noetherian loccal ring and $k = A/\mathfrak{m}$ be the residue field, then*

$$A \text{ is regular } \iff \operatorname{proj.dim}(k) = \dim(A) \iff \operatorname{proj.dim}(k) < \infty.$$

Proof. Write $r = \operatorname{proj.dim}(k)$, $s = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. If $\mathfrak{m} = \mathfrak{m}^2$, then $\mathfrak{m} = 0$ and $s = 0$, hence $r = 0$. If $\mathfrak{m} \neq \mathfrak{m}^2$, then $s > 0$. Note also that $\mathfrak{m} \notin \operatorname{Ass}(M)$, for an injection $k \hookrightarrow A$ would imply $\operatorname{Tor}_r(k, k) = 0$. Thus there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2 \cup \bigcup_{\in \operatorname{Ass}(A)} \mathfrak{p}$, which is regular and $\mathfrak{m}$-regular. Let $B = A/(x)$, then $\operatorname{Tor}_i(\mathfrak{m}, B) = 0$ for $i \geq 1$. Therefore if $L_* \to \mathfrak{m} \to 0$ is a projective resolution, then $L_* \otimes B \to \mathfrak{m} \otimes B \to 0$ is a projective resolution and

$$\operatorname{Ext}^i_A(\mathfrak{m}, N) = \operatorname{Ext}^i_B(\mathfrak{m}/x\mathfrak{m}, N) \text{ for all } B\text{-module } N.$$

Moreover, $\mathfrak{m}/xA$ is a direct summand of $\mathfrak{m}/x\mathfrak{m}$: Pick a minimal basis $x = x_1, \cdots, x_s$ of $\mathfrak{m}$ and set $\mathfrak{b} = (x_2, \cdots, x_s)$, then $\mathfrak{b} \cap (x) \subset x\mathfrak{m}$ and the composition $\mathfrak{m}/xA = (\mathfrak{b} + (x))/(x) \cong \mathfrak{b}/\mathfrak{b} \cap (x) \to \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/xA$ is the indentity. Therefore

$$\operatorname{proj.dim}_B(\mathfrak{m}/xA) \leq \operatorname{proj.dim}_B(\mathfrak{m}/x\mathfrak{m}) \leq \operatorname{proj.dim}(\mathfrak{m}) \leq r$$

Combine a resolution of $\mathfrak{m}/xA$ with $0 \to \mathfrak{m}/xA \to B \to k \to 0$, we see $\operatorname{proj.dim}_B(k) < \infty$ as well. Thus B is regular by induction on $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$. And A is also regular by Proposition 2.12 since x is regular. □

If $\mathfrak{p}$ is a prime ideal, there exists a projctive resolution $L_* \to A/\mathfrak{p} \to 0$ of finite length. Then $L_* \otimes A_\mathfrak{p} \to (A/\mathfrak{p}) \otimes A_\mathfrak{p} \to 0$ is a projective resolution of the residue field of $A_\mathfrak{p}$. Therefore $A_\mathfrak{p}$ is also regular.

(1) If x_0 is a non-zero-divisor, then A is regular if and only if $A/(x_0)$ is.
(2) If A is regular, so is $\hat{A}$.
(3) If A is regular, so is $A[[X]]$.
(4) If A is regular, so is $A_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$.

3. Spectran sequence

Lemma 3.1. *Let $T \colon C \to C'$ an $S \colon C' \to C''$ be two additive functors between abelian categories such that S is left exact and $R^qS(TQ) = 0$ for $q > 1$ if $Q \in C$ is injective. Then $E_2^{p,q} = R^qS(R^pT(A)) \Longrightarrow R^{p+q}(S \circ T)(A)$, in particular there is the exact sequence*

$$0 \to R^1S(T(A)) \to R^1(S \circ T)(A) \to S(R^1TA) \to R^2S(T(A)) \to R^2(S \circ T)(A).$$

Proof. Let $A \to Q^*$ be an injective resolution, there is an Cartan-Eillenberg resolution $TQ^* \to J^{**}$ such that

(a) Each $0 \to T(Q^p) \to \cdots \to J^{q,p} \to \cdots$ is an injective resolution.
(b) Each $0 \to Z^p(TQ^*) \to \cdots \to Z^p(J^{q,*}) \to \cdots$ is an injective resolution.
(c) Each $0 \to B^p(TQ^*) \to \cdots \to B^p(J^{q,*}) \to \cdots$ is an injective resolution.
(d) Each $0 \to H^p(TQ^*) \to \cdots \to H^p(J^{q,*}) \to \cdots$ is an injective resolution.

Consider the double complex below. Then $E_{1,I}^{q,p} = R^q S(TQ^p) = 0$ for $q > 0$ and $E_{2,I}^{0,p} = R^p(S \circ T)(A)$. On the other hand, $E_{1,II}^{p,q} = H^p(S(J^{q,*})) = S(H^p(J^{q,*}))$ since both $0 \to Z^{q,p} \to J^{q,p} \to B^{q,p+1} \to 0$ and $0 \to B^{q,p} \to Z^{q,p} \to H^{q,p} \to 0$ split. But $H^p(J^{q,*})$ is an injective resolution of $R^pT(A)$, so $E_{2,II}^{p,q} \cong R^q S(R^pT(A))$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S(TQ^0) & \longrightarrow & \cdots & \longrightarrow & S(TQ^p) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S(J^{0,0}) & \longrightarrow & \cdots & \longrightarrow & S(L^{0,p}) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \ddots & & \vdots & & \ddots \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & S(L^{q,0}) & \longrightarrow & \cdots & \longrightarrow & S(L^{q,p}) & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \ddots & & \vdots & & \ddots
\end{array}
$$

□

- Let X be a ringed space and F, G are $\mathcal{O}_X$-modules, then $H^p(X, \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(F,G)) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(F,G)$.
- Let $f\colon X \to Y$ be a morphism of ringed space, then $H^p(Y, R^q f_* F) \Longrightarrow H^{p+q}(X, F)$.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: yushglee@umich.edu