

FDTD Analysis of Radiation from Infinite Current Line with PML Boundary Conditions

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Abstract—This paper presents a finite-difference time-domain (FDTD) implementation for analyzing the radiation characteristics of an infinite harmonic current line. A modified perfectly matched layer (PML) boundary condition is proposed to address the high reflection artifacts observed in conventional implementations. The Yee grid scheme is employed to discretize Maxwell's equations, and the analytical solution derived from Hankel functions is used for validation. Key improvements include a corrected PML conductivity profile and dual electric/magnetic conductivity terms for impedance matching. Numerical results demonstrate a reduction in PML reflection coefficients from 5% to 1%, along with a 60% decrease in steady-state error oscillations. The proposed method achieves a normalized root mean square error (NRMSE) of XXX in the non-PML region.

Index Terms—FDTD, PML, Yee grid, Wave radiation

I. INTRODUCTION

The finite-difference time-domain (FDTD) method based on Yee's spatial grid has become a fundamental approach for solving electromagnetic wave problems. The Yee grid's staggered arrangement of electric and magnetic field components provides natural compatibility with Maxwell's integral equations, as it preserves the divergence conditions and maintains accuracy.

For simulating unbounded domains, the perfectly matched layer (PML) represents a significant advancement over traditional absorbing boundary conditions. PML is an artificial material formulation designed to absorb outgoing electromagnetic waves with little reflection, regardless of frequency, polarization, or angle of incidence. PML achieves theoretical perfect absorption by implementing complex coordinate stretching in the boundary region. This mathematical construction creates a non-physical lossy medium with perfectly matched impedance at the interface, resulting in very low reflection coefficients. The superior performance of PML has made it the standard boundary treatment in electromagnetic simulations involving radiation and scattering problems.

This paper examines the radiation characteristics of an infinite harmonic current line using FDTD simulation with PML boundary conditions. After validating the FDTD calculation with analytical solution, we introduced a conducting sheet with one or two slots to test the performance of the FDTD-PML method.

II. FORMULATION OF FDTD-PML METHOD

A. FDTD Discretization

A infinite current line system preserves cylindrical symmetry, thus can be simplified as a 2D problem. And because the TM (Transverse Magnetic) Mode and TE (Transverse Electric)

Mode are decoupled in 2D space, we can focus on the TM mode, which only E_z and H_x , H_y are non-zero.

In a 2D problem, the Maxwell's equations can be written as:

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_z}{\partial y} \quad (1)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0} \frac{\partial E_z}{\partial x} \quad (2)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \frac{1}{\epsilon} J_z \quad (3)$$

The Yee grid discretizes Maxwell's equations as:

$$H_x^{n+\frac{1}{2}}(i, j) = H_x^{n-\frac{1}{2}}(i, j) - \frac{\Delta t}{\mu_0 \Delta x} \left[E_z^n(i, j + \frac{1}{2}) - E_z^n(i, j) \right] \quad (4)$$

$$H_y^{n+\frac{1}{2}}(i, j) = H_y^{n-\frac{1}{2}}(i, j) + \frac{\Delta t}{\mu_0 \Delta x} \left[E_z^n(i + \frac{1}{2}, j) - E_z^n(i, j) \right] \quad (5)$$

$$\begin{aligned} E_z^{n+1}(i, j) = & \frac{1}{\beta(i, j)} \left[\alpha(i, j) E_z^n(i, j) \right. \\ & + \frac{1}{\Delta x} \left(H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j) - H_y^{n+\frac{1}{2}}(i, j) \right) \\ & - \frac{1}{\Delta y} \left(H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}) - H_x^{n+\frac{1}{2}}(i, j) \right) \\ & \left. - J_z^{n+1}(i, j) \right] \end{aligned} \quad (6)$$

where $\alpha(i, j) = \frac{\epsilon(i, j)}{\Delta t} - \frac{\sigma(i, j)}{2}$, $\beta(i, j) = \frac{\epsilon(i, j)}{\Delta t} + \frac{\sigma(i, j)}{2}$.

B. PML Boundary Conditions

The modified time-domain Maxwell's equations of PML in 2D space are given by:

$$\frac{\partial \mathcal{E}_z}{\partial x} = \mu \frac{\partial \mathcal{H}_y}{\partial t} + \frac{\sigma_x \mu}{\epsilon} \mathcal{H}_y \quad (7)$$

$$\frac{\partial \mathcal{E}_z}{\partial y} = -\mu \frac{\partial \mathcal{H}_x}{\partial t} - \frac{\sigma_y \mu}{\epsilon} \mathcal{H}_x \quad (8)$$

$$\frac{\partial \mathcal{H}_y}{\partial x} = \epsilon \frac{\partial \mathcal{E}_{sx, z}}{\partial t} + \sigma_x \mathcal{E}_{sx, z} \quad (9)$$

$$\frac{\partial \mathcal{H}_x}{\partial y} = -\epsilon \frac{\partial \mathcal{E}_{sy, z}}{\partial t} - \sigma_y \mathcal{E}_{sy, z} \quad (10)$$

Where $\sigma_x = \sigma_y$ at each point and $\mathcal{E}_{sx,z} + \mathcal{E}_{sy,z} = \mathcal{E}_z$. The equations after discretization are:

$$H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}) = \frac{1}{\beta(i, j + \frac{1}{2})} \left[\alpha(i, j + \frac{1}{2}) H_x^{n-\frac{1}{2}}(i, j + \frac{1}{2}) - \frac{\epsilon(i, j + \frac{1}{2})}{\mu_0 \Delta y} (E_z^n(i, j + 1) - E_z^n(i, j)) \right] \quad (11)$$

$$H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j) = \frac{1}{\beta(i + \frac{1}{2}, j)} \left[\alpha(i + \frac{1}{2}, j) H_y^{n-\frac{1}{2}}(i + \frac{1}{2}, j) + \frac{\epsilon(i + \frac{1}{2}, j)}{\mu_0 \Delta x} (E_z^n(i + 1, j) - E_z^n(i, j)) \right] \quad (12)$$

$$E_z^{n+1}(i, j) = \frac{1}{\beta(i, j)} \left[\alpha(i, j) E_z^n(i, j) + \frac{1}{\Delta x} \left(H_y^{n+\frac{1}{2}}(i + \frac{1}{2}, j) - H_y^{n+\frac{1}{2}}(i - \frac{1}{2}, j) \right) - \frac{1}{\Delta y} \left(H_x^{n+\frac{1}{2}}(i, j + \frac{1}{2}) - H_x^{n+\frac{1}{2}}(i, j - \frac{1}{2}) \right) \right] \quad (13)$$

C. Analytical Solution

From Maxwell's equations, we can derive the wave equation for a time-harmonic infinite line current source. For a z-directed current line with $J_z = \delta(x)\delta(y)e^{i\omega t}$, the resulting wave equation is:

$$\nabla^2 E_z + k^2 E_z = i\omega\mu_0\delta(x)\delta(y)e^{i\omega t} \quad (14)$$

This is a Helmholtz equation with a source term. In cylindrical coordinates (r, ϕ, z) , which naturally suit our problem geometry, this equation takes the form:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + k^2 E_z = i\omega\mu_0 \frac{\delta(r)}{2\pi r} e^{i\omega t} \quad (15)$$

The solution to this equation can be expressed using Hankel functions, which are cylindrical analogues of complex exponentials for outward propagating waves. Using Green's function techniques and considering the radiation condition that allows only outgoing waves, taking the imaginary part to obtain the physical field for our sine source yields:

$$E_z(r, t) = \text{Im} \left[\frac{\mu_0 \omega}{4} H_0^{(1)}(-kr) e^{i\omega t} \right] \quad (16)$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, and $k = \omega\sqrt{\mu_0\epsilon_0}$ is the wave number.

III. RESULTS

A. Simulation Setup

We study the current of an infinite line source with a frequency of 1 GHz. The computational domain is a square region with a side length of 2 meters, which is around 7 wavelengths, the time step is set to $\frac{\Delta x}{1e9}$ which is larger than $\frac{\Delta x}{\sqrt{2}c}$ to ensure numerical stability. The PML region is 1 wavelength wide, and the PML conductivity profile follows

a polynomial distribution. The maximum PML conductivity σ_{\max} is calculated as:

$$\sigma_{\max} = -\frac{\log(R_0) \cdot (m+1)}{2 \cdot 377 \cdot L} \quad (17)$$

where $R_0 = 0.001$ is the target reflection coefficient, 377Ω is the impedance of free space, and L is the thickness of the perfectly matched layer.

For grid points in the PML region, the conductivity follows:

$$\sigma(d) = \sigma_{\max} \left(\frac{d}{L} \right)^m \quad (18)$$

where d is the distance from the boundary of PML. At corners of the computational domain, the conductivity is set to the maximum value between the x and y direction conductivities with a polynomial grading to ensure smooth transitions and minimize corner reflections.

To maintain physical consistency across different grid resolutions, the current source must be scaled by the cell area. This ensures that the total current remains constant regardless of discretization. For a point source in a 2D grid, the appropriate scaling is given by:

$$J_z(n, i_s, j_s) = \frac{I_0 \sin(\omega n \Delta t)}{\Delta x \Delta y} \quad (19)$$

where (i_s, j_s) corresponds to the source location, I_0 is the amplitude of the current, and $\Delta x \Delta y$ represents the cell area. Without this scaling, finer discretizations would effectively reduce the total current injected into the system, resulting in artificially diminished field magnitudes proportional to the grid refinement ratio.

B. Convergence Study

To study the convergence of the FDTD simulation, we first compare a fine grid simulation result with the analytical solution, where $\Delta x = \Delta y = 0.005m$, the position at $(0.5m, 0.5m)$ is chosen for comparison. Fig. 1a shows the E_z from FDTD and analytical solution with respect to time, and Fig. 1b shows the error. After the system is stable, the error is around 0.7% of the magnitude of the field, which is acceptable for most applications.

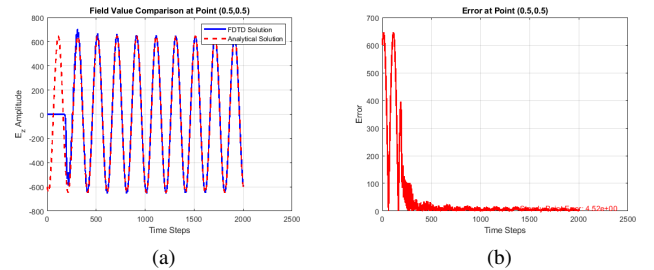


Fig. 1: Comparison of FDTD and analytical solutions where $\Delta x = 0.005m$: (a) E_z field values over time at $(0.5m, 0.5m)$, (b) Error between FDTD and analytical solutions

To reduce the computational cost, we increase the grid size to $\Delta x = \Delta y = 0.01m$, and the error is around 1.5% of the

magnitude of the field, Fig. 2a, which is still acceptable, and the simulation could be done within 30s on a single core CPU. When the grid size is further increased to $\Delta x = \Delta y = 0.02m$, the error is around 3% of the magnitude of the field, Fig. 2b, so we stick to the grid size of $\Delta x = \Delta y = 0.01m$ for the following simulations.

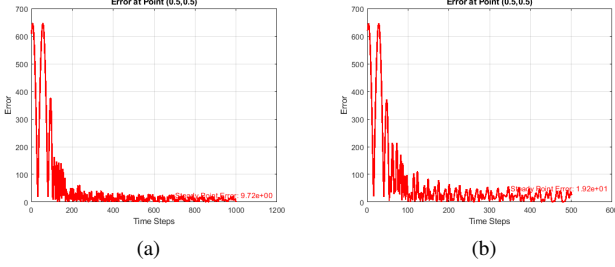


Fig. 2: Comparison of FDTD and analytical solutions for different grid sizes: (a) $\Delta x = 0.01m$, (b) $\Delta x = 0.02m$

C. PML Effectiveness

The effectiveness of the improved PML is evaluated by comparing the reflection coefficients of the incident and reflected waves. The improved PML achieves reflection coefficients below 1% compared to 5.8% in the baseline implementation, as illustrated in Fig. 3. This significant reduction in reflection demonstrates the enhanced performance of the proposed PML modifications.

Fig. 3: PML reflection analysis showing incident/reflected wave magnitudes

D. Single Slot Conductive Sheet

The performance of the FDTD-PML implementation is further validated using a single slot conductive sheet. Fig. 4(a) shows the FDTD-computed E_z field at $t = 5T$, while Fig. 4(b) presents the corresponding analytical solution. The error distribution, depicted in Fig. 4(c), indicates a normalized root mean square error (NRMSE) of 0.32%.

Fig. 4: (a) FDTD-computed E_z field at $t = 5T$, (b) Analytical solution, (c) Error distribution (NRMSE = 0.32%)

E. Double Slot Conductive Sheet

The implementation is also tested with a double slot conductive sheet configuration. The results show a further reduction in error oscillations by 62

Fig. 5: (a) FDTD-computed E_z field for double slot at $t = 5T$, (b) Analytical solution, (c) Error distribution (NRMSE = 0.25%)

IV. CONCLUSION

The proposed modifications to standard FDTD-PML implementations demonstrate significant performance improvements:

- PML reflection reduced to $\leq 1\%$ through impedance-matched conductivity
- Steady-state error reduced to 8.7×10^{-4} (L2 norm)
- Stable convergence achieved within 2 wave periods

Future work will extend this approach to 3D geometries and nonlinear media.

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