1. Basic concepts about matrices. Operations on matrices. (Show examples)

1. Definition of a Matrix

A **matrix** is a rectangular array of numbers or expressions arranged in rows and columns, usually enclosed in brackets. It is written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij}: Element at the i^{th} row and j^{th} column.
- m: Number of rows.
- n: Number of columns.

Operations on Matrices

1. Matrix Addition

- Matrices can be added if they have the same dimensions ($m \times n$).
- Add corresponding elements.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Matrix Subtraction

Subtract corresponding elements.

Example:

$$A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar Multiplication

Multiply every element in the matrix by a scalar (constant).

Example:

$$k = 3, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$k \cdot A = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

2. Two and three order determinants. (Show examples)

Determinants of 2x2 and 3x3 Matrices

1. Determinant of a 2x2 Matrix

For a 2x2 square matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of A, denoted as $\det(A)$ or $|A|$, is calculated as:

$$\det(A) = ad - bc$$

Example

Given:

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

$$\det(A) = (3 \cdot 6) - (8 \cdot 4) = 18 - 32 = -14$$

2. Determinant of a 3x3 Matrix

For a 3x3 square matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant of A, denoted as $\det(A)$, is calculated using the formula:

$$\det(A) = a \cdot (ei - fh) - b \cdot (di - fg) + c \cdot (dh - eg)$$

Example

Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A) = 1 \cdot ((5 \cdot 9) - (6 \cdot 8)) - 2 \cdot ((4 \cdot 9) - (6 \cdot 7)) + 3 \cdot ((4 \cdot 8) - (5 \cdot 7))$$

3. A system of linear algebraic equations. Matrix form of a system of linear equations. (Show examples)

A: Coefficient matrix,

X: Column vector of variables,

B: Column vector of constants.

Example 1: A 2x2 System

Given the system:

$$2x + 3y = 8$$

$$x - y = -2$$

Matrix Form:

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

Where:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

4. Solving of the linear equations system with Kramer's rule. (Show examples)

Cramer's Rule is a method for solving a system of n linear equations with n unknowns. The system must have a unique solution, and the determinant of the coefficient matrix must be non-zero ($|A| \neq 0$).

General Form of the System

$$A \cdot X = B$$

Example 1: Solving a 2x2 System

System of Equations:

$$2x + 3y = 8$$

$$x - y = -2$$

Step 1: Write the Coefficient Matrix A, Variable Matrix X, and Constant Matrix B

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

Step 2: Calculate $\det(A)$

$$\det(A) = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = (2 \cdot -1) - (3 \cdot 1) = -2 - 3 = -5$$

Step 3: Compute $\det(A_1)$ and $\det(A_2)$

- Replace the 1st column of A with B to get A_1:

$$A_1 = \begin{bmatrix} 8 & 3 \\ -2 & -1 \end{bmatrix}, \quad \det(A_1) = \begin{vmatrix} 8 & 3 \\ -2 & -1 \end{vmatrix} = (8 \cdot -1) - (3 \cdot -2) = -8 + 6 = -2$$

- Replace the 2nd column of A with B to get A_2:

$$A_2 = \begin{bmatrix} 2 & 8 \\ 1 & -2 \end{bmatrix}, \quad \det(A_2) = \begin{vmatrix} 2 & 8 \\ 1 & -2 \end{vmatrix} = (2 \cdot -2) - (8 \cdot 1) = -4 - 8 = -12$$

Step 4: Solve for x and y

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-2}{-5} = \frac{2}{5}$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-12}{-5} = \frac{12}{5}$$

Solution:

$$x = \frac{2}{5}, \quad y = \frac{12}{5}$$

5. Inverse matrix and its existence. (Show examples)

Inverse of a Matrix

The **inverse** of a square matrix A is another matrix, denoted by A^{-1}, such that:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Where I is the identity matrix (a square matrix with 1's on the diagonal and 0's elsewhere).

Conditions for Existence of the Inverse

1. **Square Matrix:** The matrix must be square (same number of rows and columns).
2. **Non-zero Determinant:** The matrix must have a non-zero determinant. If $\det(A) = 0$, the matrix does not have an inverse and is called **singular**.

Example 1: Inverse of a 2x2 Matrix

Given a 2x2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse A^{-1} is calculated using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Conditions: For the inverse to exist, the determinant of A must be non-zero:

$$\det(A) = ad - bc$$

Example 1.1: Find the Inverse of a 2x2 Matrix

Given:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

1. **Calculate the Determinant:**

$$\det(A) = (4 \cdot 6) - (7 \cdot 2) = 24 - 14 = 10$$

Since $\det(A) \neq 0$, the inverse exists.

2. **Calculate the Inverse:**

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \cdot \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{6}{10} & \frac{-7}{10} \\ \frac{-2}{10} & \frac{4}{10} \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Thus, the inverse of A is:

$$A^{-1} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

6. Minor and cofactor of a matrix. (Show examples)

Minors and **cofactors** are two most important concepts in matrices as they help us to find inverse of a matrix (we will talk about this in the next lessons). **Minor** of an element in a matrix is defined as the determinant obtained by deleting the row and column in which that element stand. (M_{ij})

The product of the element a_{ij} of the determinant by the number $(-1)^{i+j}$ of the minor M_{ij} is called the **algebraic complement** of that element and is denoted as A_{ij}:

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}.}$$

For example: $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ 6 & 7 & -8 \end{pmatrix}$

Find the minor and algebraic complement of element a_{ij} of matrix A:

$$M_{21} = \begin{vmatrix} 1 & 2 \\ 7 & -8 \end{vmatrix} = -22; \quad A_{21} = (-1)^{2+1} M_{21} = 22.$$

4. **Minor of an Element:**

The *minor* of an element in a matrix is the determinant of the smaller matrix obtained by deleting the row and column that contains the element.

Example:

For 3*3 matrix:

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The minor of a_{11} (remove the first row and first column):

$$\mathrm{M}_{11} = det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = (5)(9) - (6)(8) = 45 - 48 = -3$$

Cofactor of an Element:

The *cofactor* of an element a_{ij} is denoted by A_{ij} and is related to its minor M_{ij} by:

$$\mathrm{A}_{ij} = (-1)^{i+j} \mathrm{M}_{ij}$$

The cofactor includes a sign adjustment based on the position *(i,j):*

- $(-1)^{i+j}$ alternates between +1 and −1:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Example (continued):

For a_{11}, the cofactor A_{11} is:

$$\mathrm{A}_{11} = (-1)^{1+1} \mathrm{M}_{11} = (+1)(-3) = -3$$

7. The rank of the matrix. (Show examples)

2. The rank of the matrix:
First step:Finding rank of a matrix by minor method.
If matrix is a square matrix find the determinant of matrix.If determinant matrix is none zero then the rank of matrix is equal order of matrix (ie if matrix is 3x3 rank of matrix is 3)
Second Step:If either determinant matrix is zero or matrix is a rectengular matrix.Then see whether there exist any minor of max possible order is none zero.If there exist such none zero minor the rank of matrix is equal order of that particular minor.
Third Step:
Repeat above step if all the minors are zero and then try to find a none zero minor of order that is one less than the order from above step.

Step 3: Check for Non-zero Minors of Order 2

Since no non-zero minors of order 3 were found, we now check for minors of order 2 by deleting two rows and two columns.

1. **Minor of Order 2 by deleting the first and second rows and the first and second columns:**

$$M_3 = \begin{vmatrix} 1 & 1 \\ 6 & 9 \end{vmatrix} = (1 \cdot 9) - (1 \cdot 6) = 9 - 6 = 3$$

Since this minor is non-zero, we find that the rank of the matrix is **2**.

8. Two and three order determinants. Basic properties of the determinant. (Show examples)

Matrices in linear algebra have a specific parameter so called **determinant**. Determinant can be found only for square matrices. Formulas used for 2x2 and 3x3 matrices will be different.

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$\begin{aligned}
\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} &= 2 \cdot \det\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix} - (-3) \cdot \det\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \\
&= 2[0-(-4)] + 3[10-(-1)] + 1[8-0] \\
&= 2(0+4) + 3(10+1) + 1(8) \\
&= 2(4) + 3(11) + 8 \\
&= 8 + 33 + 8 \\
&= 49 \checkmark
\end{aligned}$$

9. Solving the system of linear equations by the Gauss method. (Show examples)

The **Gaussian elimination method** is a process of solving a system of linear equations by transforming the system's augmented matrix into a form that is easier to solve. The goal is to convert the system into an upper triangular form (or row echelon form) and then use **back-substitution** to find the solution.

10. A system of linear algebraic equations. Matrix form of a system of linear equations. (Show examples)

A system of linear equations can be written in **matrix form**. The matrix form expresses the system of equations in a compact way, making it easier to solve using matrix operations.

Example: A System of Linear Equations

Consider the system of equations:

$$\begin{aligned} 2x + 3y - z &= 1 \\ 4x - y + 2z &= 2 \\ 3x + 2y + z &= 3 \end{aligned}$$

We can represent this system in matrix form as follows:

Step 1: Write the Coefficient Matrix $\mathbf{A}$

The matrix $\mathbf{A}$ is the matrix of coefficients from the left-hand side of the equations:

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Step 2: Write the Column Vector of Unknowns $\mathbf{x}$

The vector $\mathbf{x}$ is the column vector of unknowns x, y, and z:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Step 3: Write the Column Vector of Constants $\mathbf{b}$

The vector $\mathbf{b}$ is the column vector of constants from the right-hand side of the equations:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Step 4: Write the Matrix Form of the System

Now we can write the entire system in matrix form as:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

This is the matrix form of the system of linear equations.

11. The rank of the matrix. Calculation methods of rank. (Show examples)

Gaussian elemination

Example 1: Using Row Reduction

Matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Steps:

1. Perform Gaussian elimination:
   - Subtract $2 \times \text{row}_1$ from row_2, and $3 \times \text{row}_1$ from row_3:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Count the non-zero rows: There is **1 non-zero row**.

Rank: $\text{rank}(A) = 1$

Example 2: Using Determinants

Matrix:

$$B = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 6 & 9 & 3 \end{bmatrix}$$

Steps:

1. Compute the determinant of the 3×3 matrix:

$$\det(B) = 2(7 \cdot 3 - 2 \cdot 9) - 3(4 \cdot 3 - 6 \cdot 2) + 1(4 \cdot 9 - 6 \cdot 7) = 0$$

2. Check smaller submatrices:
   - Consider a 2×2 submatrix:

$$\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 2(7) - 3(4) = 2 \neq 0$$

3. Largest non-zero determinant is 2×2.

Rank: $\text{rank}(B) = 2$

12. Operations on matrices. (Show examples)

All the elements of a matrix can be multiplied and divided by some scalar number. Let's say the matrix A is given. We want to find the product of 2 and A:

$$2 \cdot A = 2 \cdot \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$$

It is possible to find the sum and the difference of two matrices which has **equal size.** When doing this we will perform element-wise operations.

$$\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 10 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 6 \end{bmatrix}$$

Some rules related to matrices:

1. $A + O = A$
2. $A + B = B + A$
3. $(A + B) + C = (A + C) + B$

Examples:

$$\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$$

$$\left(\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}\right) + \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} = \left(\begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\right) + \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}$$

A very important thing about matrices is **matrix multiplication.**

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 & 1 \cdot 2 + 2 \cdot 6 \\ 2 \cdot 3 + 5 \cdot 4 & 2 \cdot 2 + 5 \cdot 6 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 26 & 34 \end{bmatrix}$$

We can do matrix multiplication only in the case when **the number of the columns in the first matrix is the same as the number of rows in the second matrix.**

$$\boxed{\mathbf{(m, n) * (n, d) = (m, d)}}$$

The rule is to multiply the elements in the rows of a first matrix by the elements in the columns of a second matrix and write in the first entry of a new matrix,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1\text{x}10 + 2\text{x}20 + 3\text{x}30 & 1\text{x}11 + 2\text{x}21 + 3\text{x}31 \\ 4\text{x}10 + 5\text{x}20 + 6\text{x}30 & 4\text{x}11 + 5\text{x}21 + 6\text{x}31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

13. Transposing of a matrix.(Show examples)

When finding **transpose** of a matrix we replace the row of a matrix by its column:

A	A^T
$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$	$\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$
$[5]$	$[5]$
$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\\7 & 8\end{bmatrix}$	$\begin{bmatrix}1 & 3 & 5 & 7\\2 & 4 & 6 & 8\end{bmatrix}$
$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$	$\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}$

14. Types of matrices. (Show examples)

$$\mathbf{A} = \begin{bmatrix}3 & 8\\4 & 3\\6 & 5\end{bmatrix}$$

If all the elements of two matrices are equal ($A_{ij} = B_{ij}$) then they are said to be **equal**.

$$\mathbf{B} = \begin{bmatrix}3 & 8\\4 & 3\\6 & 5\end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}2\\3\\1\end{bmatrix}$$

A matrix which has just only one column (the size of Mx1) is called **column matrix.**

A matrix which has just only one row (the size of 1xN) is called **row matrix.**

$$\boldsymbol{B} = \begin{bmatrix}3 & 7 & 1\end{bmatrix}$$

A matrix in which the number of the rows is equal to the number of columns (n=m) is called **square matrix.**

$$\mathbf{H} = \begin{bmatrix}3 & 8 & 5\\4 & 3 & 3\\7 & 2 & 8\end{bmatrix}$$

The square matrices have something so called **diagonal of a matrix.**

$$\mathbf{H} = \begin{bmatrix}3 & 8 & 5\\4 & 2 & 3\\7 & 2 & 8\end{bmatrix}$$

Here 3, 2 and 8 are called **diagonal elements of a matrix H.**

The sum of a diagonal elements of a matrix is called **the trace of a matrix.**

$$\mathbf{H} = \begin{bmatrix}3 & 8 & 5\\4 & 2 & 3\\7 & 2 & 8\end{bmatrix}$$

Trace(H)=3+2+8=13

If the elements of a matrix below diagonal are all 0 it is said to be **lower triangle matrix.**

$$\mathbf{L} = \begin{bmatrix} 3 & 1 & 8 & 5 \\ 0 & 8 & 2 & 3 \\ 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If the elements of a matrix above diagonal are all 0 it is said to be **upper triangle matrix.**

$$\mathbf{U} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 8 & 0 & 0 \\ 2 & 5 & 9 & 0 \\ 4 & 7 & 2 & 1 \end{bmatrix}$$

If all the elements of a matrix except for diagonal elements are equal to zero matrix is called **diagonal matrix**.

$$\mathbf{H}_{33} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

If all diagonal elements of matrix are equal, the matrix is said to be **scalar matrix.**

$$\mathbf{H}_{33} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

If all diagonal elements of matrix are equal to 1, the matrix is called as **identity or unit matrix** and is shown with big letter i**.**

$$\mathbf{I}_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If all the elements of a matrix are equal to 0, the matrix is called as **zero matrix** and is shown with big O**.**

$$\mathbf{O}_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

15. Operations on matrices. Transposing of a matrix. (Show examples)

Operations on Matrices

1. Matrix Addition

- Matrices can be added if they have the same dimensions ($m \times n$).
- Add corresponding elements.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Matrix Subtraction

Subtract corresponding elements.

Example:

$$A - B = \begin{bmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Scalar Multiplication

Multiply every element in the matrix by a scalar (constant).

Example:

$$k = 3, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$k \cdot A = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

When finding **transpose** of a matrix we replace the row of a matrix by its column:

A	A^T
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
$[5]$	$[5]$
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$	$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

16. Algebraic complement of a minor. (Show examples)

The product of the element a_{ij} of the determinant by the number $(-1)^{i+j}$ of the minor M_{ij} is called the **algebraic complement** of that element and is denoted as A_{ij}:

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}.}$$

For example:
$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ 6 & 7 & -8 \end{pmatrix}$$

Find the minor and algebraic complement of element a_{ij} of matrix A:

$$M_{21} = \begin{vmatrix} 1 & 2 \\ 7 & -8 \end{vmatrix} = -22; \quad A_{21} = (-1)^{2+1} M_{21} = 22.$$

17. The determinant of the product of two matrices. (Show examples)

Determinants of 2x2 and 3x3 Matrices

1. Determinant of a 2x2 Matrix

For a 2x2 square matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of A, denoted as $\det(A)$ or $|A|$, is calculated as:

$$\det(A) = ad - bc$$

Example

Given:

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

$$\det(A) = (3 \cdot 6) - (8 \cdot 4) = 18 - 32 = -14$$

18. Calculation of inverse matrix. (Show examples)

$$A_2^{-1} = \frac{1}{\Delta(A_2)}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}, \qquad A_3^{-1} = \frac{1}{\Delta(A_3)}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

Problem: Find inverse of 2x2 matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3/2 & 1/2 \end{pmatrix}$$

19. Basic properties of the determinant. (Show examples)

Minors and **cofactors** are two most important concepts in matrices as they help us to find inverse of a matrix (we will talk about this in the next lessons). **Minor** of an element in a matrix is defined as the determinant obtained by deleting the row and column in which that element stand. (M_{ij})

The product of the element a_{ij} of the determinant by the number $(-1)^{i+j}$ of the minor M_{ij} is called the **algebraic complement** of that element and is denoted as A_{ij}:

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}.}$$

For example: $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & -4 & 5 \\ 6 & 7 & -8 \end{pmatrix}$

Find the minor and algebraic complement of element a_{ij} of matrix A:

$$M_{21} = \begin{vmatrix} 1 & 2 \\ 7 & -8 \end{vmatrix} = -22; \quad A_{21} = (-1)^{2+1} M_{21} = 22.$$

Matrix Example

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

Minor and Cofactor of $a_{11} = 4$

- Minor: Remove row 1, column 1.

$$M_{11} = 1$$

- Cofactor:

$$C_{11} = (-1)^{1+1} \cdot M_{11} = 1 \cdot 1 = 1$$

Minor and Cofactor of $a_{12} = 3$

- Minor: Remove row 1, column 2.

$$M_{12} = 2$$

- Cofactor:

$$C_{12} = (-1)^{1+2} \cdot M_{12} = -1 \cdot 2 = -2$$

Results:

- $M_{11} = 1, C_{11} = 1$
- $M_{12} = 2, C_{12} = -2$

20. System of three linear equations with three variables. (Show examples).

System of Equations

1. $x + 2y + 3z = 14$
2. $2x + 3y + z = 11$
3. $3x + y + 2z = 10$

Solving the System

We will solve this system using the substitution method, elimination method, or matrix method (Gaussian elimination). Let's solve it step-by-step using Gaussian elimination:

Step 1: Represent the System as a Matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 14 \\ 11 \\ 10 \end{bmatrix}$$

Step 2: Form the Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 2 & 3 & 1 & 11 \\ 3 & 1 & 2 & 10 \end{array}\right]$$

Step 3: Row Reduce the Matrix

Perform elementary row operations to simplify the matrix:

1. Subtract $2 \times R_1$ from R_2 and $3 \times R_1$ from R_3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -1 & -5 & -17 \\ 0 & -5 & -7 & -32 \end{array}\right]$$

2. Divide R_2 by -1 to make the pivot element 1:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 5 & 17 \\ 0 & -5 & -7 & -32 \end{array}\right]$$

3. Add $5 \times R_2$ to R_3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 18 & 53 \end{array}\right]$$

4. Divide R_3 by 18:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & 1 & 5 & 17 \\ 0 & 0 & 1 & 53/18 \end{array}\right]$$

Step 4: Back Substitution

Solve for z, y, and x:

1. From the third row:

$$z = \frac{53}{18}$$

2. Substitute $z = \frac{53}{18}$ into the second row:

$$y + 5z = 17 \longrightarrow y + 5 \cdot \frac{53}{18} = 17 \longrightarrow y = \frac{31}{18}$$

3. Substitute $y = \frac{31}{18}$ and $z = \frac{53}{18}$ into the first row:

$$x + 2y + 3z = 14 \longrightarrow x + 2 \cdot \frac{31}{18} + 3 \cdot \frac{53}{18} = 14 \longrightarrow x = \frac{37}{18}$$

Solution

$$x = \frac{37}{18}, \quad y = \frac{31}{18}, \quad z = \frac{53}{18}$$