Solutions to Exercises 2.1–2.4

Exercise 2.1. Prove that the empty set is a subset of every set.

Solution. *Let E be any set. By definition, $\varnothing \subseteq E$ means*

$$\forall x\,(\,x \in \varnothing \implies x \in E\,).$$

But $\forall x\,(x \notin \varnothing)$, so the implication is true for all x (vacuous truth), hence $\varnothing \subseteq E$.

Exercise 2.2. Prove that the set of all algebraic numbers is countable.

Solution. *A complex number z is algebraic if there exist integers $n \geq 1$ and $a_0, \ldots, a_n$, not all zero, such that*

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

For each integer $N \geq 2$, let A_N be the set of numbers satisfying at least one such equation with

$$n + |a_0| + |a_1| + \cdots + |a_n| = N.$$

For fixed N, only finitely many integer $(n, a_0, \ldots, a_n)$ satisfy this constraint; each corresponding polynomial has finitely many roots, so A_N is finite. Therefore the set of algebraic numbers

$$\mathcal{A} \;=\; \bigcup_{N=2}^{\infty} A_N$$

is a countable union of finite sets and hence countable.

Exercise 2.3. Prove that there exist real numbers which are not algebraic.

Solution. *From Exercise 2.2, the algebraic numbers form a countable set, while the set $\mathbb{R}$ of real numbers is uncountable. Hence $\mathbb{R} \setminus \mathcal{A} \neq \varnothing$, so there exist real numbers that are not algebraic (i.e., transcendental).*

Exercise 2.4. Is the set of irrational real numbers countable?

Solution. *No. If the irrationals $\mathbb{R} \setminus \mathbb{Q}$ were countable, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would be a countable union of countable sets and thus countable, contradicting the uncountability of $\mathbb{R}$. Therefore the irrationals are uncountable.*