Finite, Countable, and Uncountable Sets

MH5100 Advanced Investigation in Calculus

Handout #2.1

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1 Functions and Mappings

We begin this section with a definition of the function concept.

Definition 2.1

Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a **function from** A **to** B (or a mapping of A into B). The set A is called the **domain** of f (we also say f is defined on A) and the elements $f(x)$ are called the **values** of f. The set of all values of f is called the **range** of f.

Definition 2.2

Let A and B be two sets and let f be a mapping of A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$; we call $f(E)$ the **image** of E under f, and in this notation $f(A)$ is the range of f, so $f(A) \subset B$. If $f(A) = B$, we say that f maps A **onto** B ("onto" is more specific than "into"). If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$, and if $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a **1-1 mapping** of A into B, equivalently $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ with $x_1, x_2 \in A$.

2 Cardinal Numbers and Equivalence

Definition 2.3

If there exists a 1-1 mapping of A onto B, we say that A and B can be put in **1-1 correspondence**, or that A and B have the **same cardinal number**, or briefly that A and B are **equivalent**, and we write $A \sim B$; this relation is an equivalence relation (reflexive, symmetric, and transitive).

Definition 2.4

For any positive integer n, let $N_n = \{1, 2, \ldots, n\}$ and let N be the set of all positive integers. For any set A, we say:

(a) A is **finite** if $A \sim N_n$ for some n (the empty set is also considered finite).

(b) A is **infinite** if A is not finite.

(c) A is **countable** if $A \sim N$.

(d) A is **uncountable** if A is neither finite nor countable.

(e) A is **at most countable** if A is finite or countable.

For two finite sets A and B, we have $A \sim B$ if and only if A and B contain the same number of elements; for infinite sets, the idea of "having the same number of elements" is vague, whereas 1-1 correspondence is clear.

3 Examples of Countable Sets

Example 2.5

Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ is countable: arrange $\mathbb{Z}$ and N as

$$\mathbb{Z} : 0, 1, -1, 2, -2, 3, -3, \ldots \quad \text{and} \quad N : 1, 2, 3, 4, 5, 6, 7, \ldots$$

An explicit bijection $f : N \to \mathbb{Z}$ is

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Remark 2.6

A finite set cannot be equivalent to one of its proper subsets, but this is possible for infinite sets, as shown by the example where N is a proper subset of $\mathbb{Z}$ and both are countable; indeed, one may replace Definition 2.4(b) by: A is infinite if A is equivalent to one of its proper subsets.

Definition 2.7

By a **sequence**, we mean a function f defined on N. If $f(n) = x_n$ for $n \in N$, we denote the sequence by $(x_n)_{n=1}^{\infty}$, or more briefly by (x_n), or sometimes by $x_1, x_2, x_3, \ldots$; the elements x_n are called the **terms** of the sequence, and if A is a set and $x_n \in A$ for all n, then (x_n) is a sequence in A. The terms of a sequence need not be distinct; every countable set is the range of a 1-1 function defined on N and can be arranged in a sequence of distinct terms.

4 Fundamental Theorems

Theorem 2.8

Every infinite subset of a countable set A is countable.

Proof

Suppose $E \subset A$ is infinite. Arrange the elements of A in a sequence (x_n) of distinct elements. Construct a sequence (n_k) by choosing n_1 as the smallest positive integer with $x_{n_1} \in E$, and given n_{k-1}, choose n_k as the smallest integer $> n_{k-1}$ such that $x_{n_k} \in E$. Then $f(k) = x_{n_k}$ gives a 1-1 correspondence between E and N.

5 Unions and Intersections

Definition 2.9

Let A and Ω be sets, and suppose that with each $\alpha \in A$ there is associated a subset $E_\alpha \subset \Omega$. The collection $\{E_\alpha\}$ denotes the set whose elements are the sets E_α. The **union** of the sets E_α is the set S such that $x \in S$ iff $x \in E_\alpha$ for at least one $\alpha \in A$, denoted

$$S = \bigcup_{\alpha \in A} E_\alpha. \tag{2.1}$$

If $A = \{1, 2, \ldots, n\}$, one writes

$$S = \bigcup_{m=1}^{n} E_m = E_1 \cup E_2 \cup \cdots \cup E_n. \tag{2.2, 2.3}$$

If $A = N$, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m. \tag{2.4}$$

The **intersection** of the sets E_α is the set P such that $x \in P$ iff $x \in E_\alpha$ for every $\alpha \in A$, denoted

$$P = \bigcap_{\alpha \in A} E_\alpha, \quad P = \bigcap_{m=1}^{n} E_m = E_1 \cap E_2 \cap \cdots \cap E_n, \quad P = \bigcap_{m=1}^{\infty} E_m. \tag{2.5–2.7}$$

If $A \cap B$ is empty, A and B are **disjoint**.

Example 2.10

(a) If $E_1 = \{1, 2, 3\}$ and $E_2 = \{2, 3, 4\}$, then $E_1 \cup E_2 = \{1, 2, 3, 4\}$ and $E_1 \cap E_2 = \{2, 3\}$.
(b) Let $A = \{x \in \mathbb{R} : 0 < x \leq 1\}$. For each $x \in A$, let $E_x = \{y \in \mathbb{R} : 0 < y < x\}$. Then:

(i) $E_x \subset E_z$ iff $0 < x \leq z \leq 1$.

(ii) $\displaystyle\bigcup_{x \in A} E_x = E_1$.

(iii) $\displaystyle\bigcap_{x \in A} E_x = \varnothing$.

Remark 2.11

Many properties of unions and intersections parallel sums and products. The commutative and associative laws are

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A, \tag{2.8}$$
$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C). \tag{2.9}$$

The distributive law holds:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{2.10}$$

Further relations:

$$A \subset A \cup B, \qquad A \cap B \subset A; \tag{2.11–2.12}$$

with $\varnothing$ denoting the empty set,

$$A \cup \varnothing = A, \qquad A \cap \varnothing = \varnothing; \tag{2.13}$$

and if $A \subset B$,

$$A \cup B = B, \qquad A \cap B = A. \tag{2.14}$$

6 Countable Unions

Theorem 2.12

Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof

Arrange each E_n as a sequence $(x_{nk})_{k\geq 1}$ and form the infinite array

$$\begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Enumerate along successive diagonals to obtain a sequence containing all elements of S; duplicates may occur if the E_n intersect, so S is equivalent to a subset of N and is at most countable, but since $E_1 \subset S$ and E_1 is infinite, S is infinite and hence countable.

Corollary

Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Then $T = \bigcup_{\alpha \in A} B_\alpha$ is at most countable (it is equivalent to a subset of $\bigcup_{n=1}^{\infty} E_n$ as above).

Theorem 2.13

Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \dots, a_n)$ with $a_k \in A$ (not necessarily distinct). Then B_n is countable.

Proof

Trivially $B_1 = A$ is countable. Suppose B_{n-1} is countable. Elements of B_n have the form (b, a) with $b \in B_{n-1}$ and $a \in A$. For each fixed b, the set $\{(b, a) : a \in A\}$ is countable, hence B_n is a countable union of countable sets and is therefore countable.

Corollary

The set of all rational numbers is countable.

Proof

Apply Theorem 2.13 with $n = 2$: every rational r is b/a with integers a, b; the set of pairs (a, b), hence the set of fractions b/a, is countable.

7 Uncountable Sets

Not all infinite sets are countable, as the next theorem shows.

Theorem 2.14

Let A be the set of all sequences whose elements are the digits 0 and 1. The set A is uncountable.

Proof

Let $E = \{s_1, s_2, s_3, \dots\} \subset A$ be countable. Construct a sequence s by letting the n-th digit of s be 0 if the n-th digit of s_n is 1, and 1 otherwise. Then s differs from every s_n in at least one place, so $s \notin E$ but $s \in A$; hence every countable subset of A is proper, and A is uncountable. This is Cantor's diagonal process.

8 Additional Topics and Historical Context

8.1 Well-Ordering and the Axiom of Choice

Well-Ordering Theorem (Zermelo 1904)

Every set can be well-ordered.

Historical Context

Ernst Zermelo introduced the Axiom of Choice in 1904 to prove the Well-Ordering Theorem. The Axiom of Choice states: For any collection of non-empty sets, there exists a choice function that selects one element from each set.
The equivalence AC $\Leftrightarrow$ Well-Ordering Theorem is fundamental:

- Zermelo's proof: AC implies every set has a well-ordering
- Converse: Well-ordering implies choice functions exist

This axiom became central to modern mathematics despite its non-constructive nature.

8.2 Banach-Tarski Paradox

Banach-Tarski Theorem

A solid ball can be decomposed into finitely many pieces and reassembled into two balls of the same size using only rigid motions (rotations and translations).

Remark

The paradox arises because the pieces are **non-Lebesgue-measurable sets** constructed using the Axiom of Choice. This demonstrates that our intuitive notion of "volume" breaks down for pathological sets that require choice functions for their construction.

8.3 Independence Results

Gödel and Cohen's Results

Gödel (1940): The Continuum Hypothesis is consistent with ZFC

- Used the constructible universe L
- Showed: ZFC $\nvdash \neg$CH

Cohen (1963): The Continuum Hypothesis is independent of ZFC

- Developed the forcing technique
- Showed: ZFC $\nvdash$ CH

This means the Continuum Hypothesis is undecidable within standard set theory, analogous to how the parallel postulate is independent of the other axioms of geometry.

9 Exercises

9.1 Exercise 2.1

Prove the empty set is a subset of every set.

9.2 Exercise 2.2

Prove the set of all algebraic numbers is countable.

9.3 Exercise 2.3

Prove there are real numbers which are not algebraic.

Hint: Use the fact that algebraic numbers are countable but $\mathbb{R}$ is uncountable.

9.4 Exercise 2.4

Is the set of all irrational real numbers countable?