MH1100 Revision Summary

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September 7, 2025

1 Functions

Function

A function f is a rule that assigns each element x in a set D (domain) to exactly one element in R. Note different values of $x_i \in D$ may map to the same value $f(x_i)$ in R. Hence to check if a function is well defined, we can use a vertical line test.

Domain

The domain of a function f, commonly referred to as D is the set of all possible values of x that the function accepts. The range, commonly referred to as R_f refers to the set of possible values of $f(x)$ for all $x \in D$.

$$R_f = \{f(x) \in \mathbb{R} \mid x \in D_f\}$$

Odd and Even Functions

A function is even $\Leftrightarrow f(-x) = f(x)$ and odd $\Leftrightarrow f(-x) = -f(x)$.

Theorem

Any function $f(x)$ can be written as a sum of an odd and an even function.

Proof

Let $f(x) = E(x) + O(x)$ where E, O are the even and odd functions respectively.

$$f(x) = E(x) + O(x) \quad (1)$$
$$f(-x) = E(-x) + O(-x) \quad (2)$$
$$f(-x) = E(x) - O(x) \quad (3)$$

Solving, we get that

$$E(x) = \frac{f(x) + f(-x)}{2}, \quad O(x) = \frac{f(x) - f(-x)}{2}$$

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2 Limits

2.1 Limit Laws

If f, g are continuous at $x = a$ (limit exists), then we have the following hold true:

1. $\lim(f \pm g) = \lim f \pm \lim g$
2. $\lim \frac{f}{g} = \frac{\lim f}{\lim g}$
3. $\lim fg = \lim f \times \lim g$
4. $\lim f \circ g = \lim f \circ \lim g$. The implication here is that composition of continuous functions is continuous. Prove using epsilon delta (if it comes out just cry).

2.2 Squeeze Theorem

Theorem

If functions f, g, h are such that

$$f \leq g \leq h$$

Then we have that

$$\lim f \leq \lim g \leq \lim h$$

Specifically, if $\lim f = \lim h$, then $\lim g = \lim f = \lim h$.

2.3 Important Inequalities

1. $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$
2. $x^n - a^n = (x - a)\left(a^{n-1} + a^{n-2}x + a^{n-3}x^2 + \cdots + ax^{n-2} + x^{n-1}\right)$

2.4 Epsilon–Delta Definition of a Limit

Definition

We say that $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Proof Strategy

To prove a limit using the ϵ–δ definition:

1. Begin with $|f(x) - L|$ and manipulate it into a form involving $|x - a|$.
2. If additional terms appear, bound them appropriately (often by restricting $|x - a| < 1$).
3. Choose δ in terms of ϵ so the inequality is satisfied.

2.4.1 Basic Examples

Example

Example 1. $\lim_{x\to a} x = a$.

Proof

We want $|x-a|<\epsilon$. Choosing $\delta=\epsilon$ works:

$$|x-a|<\delta \quad\Rightarrow\quad |f(x)-a| = |x-a| < \epsilon.$$

Example

Example 2. $\lim_{x\to a} x^2 = a^2$.

Proof

We compute

$$|x^2-a^2| = |x-a||x+a|.$$

If $|x-a|<1$, then $|x+a| \le |x-a| + 2|a| < 1+2|a|$. Choose

$$\delta = \min\left(1, \frac{\epsilon}{1+2|a|}\right).$$

Then $|x-a|<\delta \implies |x^2-a^2|<\epsilon$.

2.4.2 Advanced Examples

Example

Example. $\lim_{x\to 0} x^{2019}(1+\sin^2(2020x)) = 0$.

Proof

We bound

$$|x^{2019}(1+\sin^2(2020x))| \le 2|x|^{2019}.$$

Choose $\delta = \sqrt[2019]{\epsilon/2}$ and the proof follows.

Remark

Infinite limits can be defined analogously: $\lim_{x\to a} f(x) = \infty$ means for every $N>0$, there exists $\delta>0$ such that $0<|x-a|<\delta \Rightarrow f(x) > N$.

2.5 Limit Laws with ϵ–δ Proofs

2.5.1 Sum Rule

Theorem

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} \big(f(x) + g(x)\big) = L + M.$$

Proof

Given $\epsilon > 0$, since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \;\Rightarrow\; |f(x) - L| < \tfrac{\epsilon}{2}.$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \;\Rightarrow\; |g(x) - M| < \tfrac{\epsilon}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x) + g(x) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus the result follows.

2.5.2 Product Rule

Theorem

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof

We write

$$f(x)g(x) - LM = (f(x) - L)g(x) + L(g(x) - M).$$

Take $\epsilon > 0$.
Since $\lim_{x\to a} g(x) = M$, there exists $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \;\Rightarrow\; |g(x) - M| < 1.$$

This implies $|g(x)| \leq |M| + 1$ when $|x-a| < \delta_1$.
Now, since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \;\Rightarrow\; |f(x) - L| < \frac{\epsilon}{2(|M|+1)}.$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists $\delta_3 > 0$ such that

$$0 < |x-a| < \delta_3 \;\Rightarrow\; |g(x) - M| < \frac{\epsilon}{2(1+|L|)}.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then for $0 < |x-a| < \delta$,

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\epsilon}{2(|M|+1)}(|M|+1) + |L|\frac{\epsilon}{2(1+|L|)} \\ &\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus the product rule holds.

2.5.3 Quotient Rule

Theorem

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof

We want

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| < \epsilon.$$

Rewrite:

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \left|\frac{Mf(x) - Lg(x)}{Mg(x)}\right| = \frac{1}{|M||g(x)|}\,|M(f(x) - L) + L(M - g(x))|.$$

Since $\lim_{x\to a} g(x) = M$, there exists $\delta_1 > 0$ such that for $|x - a| < \delta_1$,

$$|g(x) - M| < \frac{|M|}{2}.$$

Thus $|g(x)| \geq |M| - |g(x) - M| > \frac{|M|}{2}$, so $\frac{1}{|g(x)|} \leq \frac{2}{|M|}$.
Now, since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, there exist $\delta_2, \delta_3 > 0$ such that

$$|f(x) - L| < \frac{\epsilon|M|}{4(|M| + |L|)}, \quad |g(x) - M| < \frac{\epsilon|M|}{4(|M| + |L|)}.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then for $0 < |x - a| < \delta$,

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\leq \frac{2}{|M|^2}\Big(|M||f(x) - L| + |L||g(x) - M|\Big) \\
&< \frac{2}{|M|^2}\Big(|M| \cdot \tfrac{\epsilon|M|}{4(|M|+|L|)} + |L| \cdot \tfrac{\epsilon|M|}{4(|M|+|L|)}\Big) \\
&= \frac{2}{|M|^2} \cdot \frac{\epsilon|M|(|M| + |L|)}{4(|M| + |L|)} \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Thus the quotient rule holds.

2.5.4 Composition Rule

Theorem

If $\lim_{x\to a} f(x) = L$ and g is continuous at L, then

$$\lim_{x\to a} g(f(x)) = g(L).$$

Proof

Given $\epsilon > 0$, since g is continuous at L, there exists $\eta > 0$ such that

$$|y - L| < \eta \quad \Rightarrow \quad |g(y) - g(L)| < \epsilon.$$

Since $\lim_{x\to a} f(x) = L$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \ \Rightarrow \ |f(x) - L| < \eta.$$

Then

$$|g(f(x)) - g(L)| < \epsilon$$

for all $0 < |x - a| < \delta$. Hence the composition rule holds.

2.6 L'Hôpital's Rule

Remark

Important: Before applying the rule, you must first *prove differentiability* of the functions involved on an open interval containing the point of interest (except possibly at the point itself).

Suppose f and g are real-valued functions defined on an open interval I containing a, with f and g differentiable on $I \setminus \{a\}$, and with $g'(x) \neq 0$ for all $x \in I \setminus \{a\}$. If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \text{or} \quad \lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty,$$

and if the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (finite or infinite), then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Checking Differentiability Before Applying L'Hôpital

To justify using L'Hôpital's Rule:

- Confirm that f and g are differentiable on an open interval around a (except possibly at a itself).
- Verify that $g'(x) \neq 0$ in this interval.
- Establish the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ directly from the limits of f and g.

Example

$$\lim_{x\to 0^+} \frac{\ln x}{x}.$$

First, note that $\ln x$ and x are differentiable on $(0, \infty)$, and $x > 0$ ensures $g'(x) = 1 \neq 0$. Since $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} x = 0$, this is an $\frac{-\infty}{0^+}$ form. Applying L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{x} = \lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{1} = \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

3 Continuity

Continuity

A function is continuous at a point $x = a \Leftrightarrow$

1. $\lim_{x\to a} f(x)$ exists
2. The limit is equal to $f(a)$

3.1 Intermediate Value Theorem

Theorem

If a function $f(x)$ is continuous in an interval $[a, b]$, then there exists $f(c)$ where $c \in (a, b)$ where $f(c)$ is between $f(a)$ and $f(b)$.

Roots finding

Given $f(a) = -4$, $f(b) = 5$, there exists $x \in (a, b)$ such that $f(x) = 0$ by IVT.

4 Differentiation

Differentiable $\Rightarrow$ Continuity $\Rightarrow$ Limit exists.

Proof

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) = f'(x_0) \cdot 0 = 0$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$$

Formal Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example

Prove that $\frac{x}{x+1} < \ln(x+1) < x$

Proof

First consider $f(x) = \ln(x+1) - x$, note that

1. $f(0) = 0$
2. $f'(x) = \frac{1}{x+1} - 1 = \begin{cases} > 0 & x < 0 \\ < 0 & x > 0 \end{cases}$

Hence we prove that $\ln(x+1) < x$. A similar method is used for the other half of the inequality and is left as an exercise for the reader.

4.1 Linear Approximation

Before applying linear (or tangent line) approximation, it is essential to verify that f is differentiable at the point of approximation.

Theorem

If f is differentiable at a, then for x near a we may approximate

$$f(x) \approx f(a) + f'(a)(x - a).$$

Remarks

- Differentiability at a implies continuity at a, so no separate continuity check is needed.
- The quality of the approximation depends on the size of $(x-a)$ and higher-order derivatives of f.

Example

For $f(x) = \sqrt{x}$ at $a = 4$,

$$f(4) = 2, \quad f'(x) = \tfrac{1}{2\sqrt{x}}, \quad f'(4) = \tfrac{1}{4}.$$

Thus,

$$\sqrt{x} \approx 2 + \tfrac{1}{4}(x - 4).$$

4.2 Extreme Value Theorem

Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some $c, d \in [a, b]$.

Conditions to Check

- Verify that f is continuous on the entire closed interval $[a, b]$.
- Discontinuities or open intervals invalidate the theorem.

Example

$f(x) = x^2$ on $[-1, 2]$ is continuous.

$$f(-1) = 1, \quad f(2) = 4, \quad f(0) = 0.$$

Hence, $\min f = 0$ at $x = 0$, $\max f = 4$ at $x = 2$.

4.3 Rolle's Theorem

Theorem

Let f be a function such that:

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b),
3. $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Conditions to Check

- Continuity on the closed interval $[a, b]$.
- Differentiability on the open interval (a, b).
- Equal endpoint values: $f(a) = f(b)$.

Example

For $f(x) = \cos x$ on $[0, 2\pi]$, $f(0) = f(2\pi) = 1$. By Rolle's theorem, there exists c with $f'(c) = -\sin(c) = 0$, i.e. $c = \pi$.

4.4 Mean Value Theorem

Theorem

Let f be a function such that:

1. f is continuous on $[a, b]$,
2. f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Conditions to Check

- Continuity on $[a, b]$.
- Differentiability on (a, b).

Example

For $f(x) = x^2$ on $[1, 3]$,

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Since $f'(x) = 2x$, we need $2c = 4$, so $c = 2$ satisfies the theorem.