Design and Analysis of Algorithms
Part I: Divide and Conquer
Lecture 4: The Polynomial Multiplication Problem and Quicksort Problem

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Outline

- Review to Divide-and-Conquer Paradigm
- Polynomial Multiplication Problem
  - Problem definition
  - A brute force algorithm
  - A first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

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Review to Divide-and-Conquer Paradigm

- Divide-and-conquer (D&C) is an important algorithm design paradigm.
  - **Divide**
    Dividing a given problem into two or more subproblems (ideally of approximately equal size)
  - **Conquer**
    Solving each subproblem (directly if small enough or recursively)
  - **Combine**
    Combining the solutions of the subproblems into a global solution

Review to Divide-and-Conquer Paradigm

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Lower Bound for Sorting (基于比较的排序下界)

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The Polynomial Multiplication Problem

Definition (Polynomial Multiplication Problem)

Given two polynomials

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\begin{aligned}
A(x) &= a_0 + a_1 x + \cdots + a_n x^n \\
B(x) &= b_0 + b_1 x + \cdots + b_m x^m
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Compute the product $A(x)B(x)$

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Example

$$\begin{aligned} A(x) &= 1 + 2x + 3x^2 \\ B(x) &= 3 + 2x + 2x^2 \\ A(x)B(x) &= 3 + 8x + 15x^2 + 10x^3 + 6x^4 \end{aligned}$$

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- Assume that the coefficients a_i and b_i are stored in arrays A[0…n] and B[0…m]
- Cost: number of scalar multiplications and additions

What do we need to compute exactly?

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- We need to calculate convolutions. This is a major problem in digital signal processing

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The First Divide-and-Conquer: Divide

Assume n is a power of 2

Define

$$A_0(x) = a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$$

$$A_1(\mathrm{x}) = \mathrm{a}_{\frac{\mathrm{n}}{2}} + \mathrm{a}_{\frac{\mathrm{n}}{2}+1}\mathrm{x} + \cdots + \mathrm{a}_{\mathrm{n}}\mathrm{x}^{\frac{\mathrm{n}}{2}}$$

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The original problem (of size n) is divided into 4 problems of input size n/2

Example

$$A(x) = 2 + 5x + 3x^2 + x^3 - x^4$$
$$B(x) = 1 + 2x + 2x^2 + 3x^3 + 6x^4$$
$$A(x)B(x) = 2 + 9x + 17x^2 + 23x^3 + 34x^4$$
$$+39x^5 + 19x^6 + 3\mathrm{x}^7 - 6\mathrm{x}^8$$
$$A_0(x) = 2 + 5x, A_1(x) = 3 + x - x^2$$
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$$A_0(x)B_1(x) = 4 + 16x + 27x^2 + 30x^3$$
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$$A_0(x)B_0(x) + \big(A_0(x)B_1(x) + A_1(x)B_0(x)\big)x^2 + A_1(x)B_1(x)x^4$$

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The First Divide-and-Conquer: Conquer

Conquer: Solve the four subproblems

- Compute $A_0(x)B_0(x), A_0(x)B_1(x), A_1(x)B_0(x), \mathrm{A}_1(\mathrm{x})\mathrm{B}_1(\mathrm{x})$

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Combine

- Add the following four polynomials

$$
\begin{aligned}
A_0(x)B_0(x) &+ A_0(x)B_1(x)x^{\frac{n}{2}} \\
&+ A_1(x)B_0(x)x^{\frac{n}{2}} \\
&+ A_1(x)B_1(x)x^{n}
\end{aligned}
$$

The First Divide-and-Conquer: Conquer

Conquer: Solve the four subproblems

- Compute
  $A_0(x)B_0(x), A_0(x)B_1(x), A_1(x)B_0(x), \mathrm{A}_1(\mathrm{x})\mathrm{B}_1(\mathrm{x})$
  by recursively calling the algorithm 4 times

Combine

- Add the following four polynomials

$$A_0(x)B_0(x) + A_0(x)B_1(x)x^{\frac{n}{2}} + A_1(x)B_0(x)x^{\frac{n}{2}} + A_1(x)B_1(x)x^n$$

- Takes O( ) operations

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Input: $A(x), B(x)$
Output: $A(x) \times B(x)$
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$$T(n) = \begin{cases} 4T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

Analysis of Running Time

Assume that n is a power of 2

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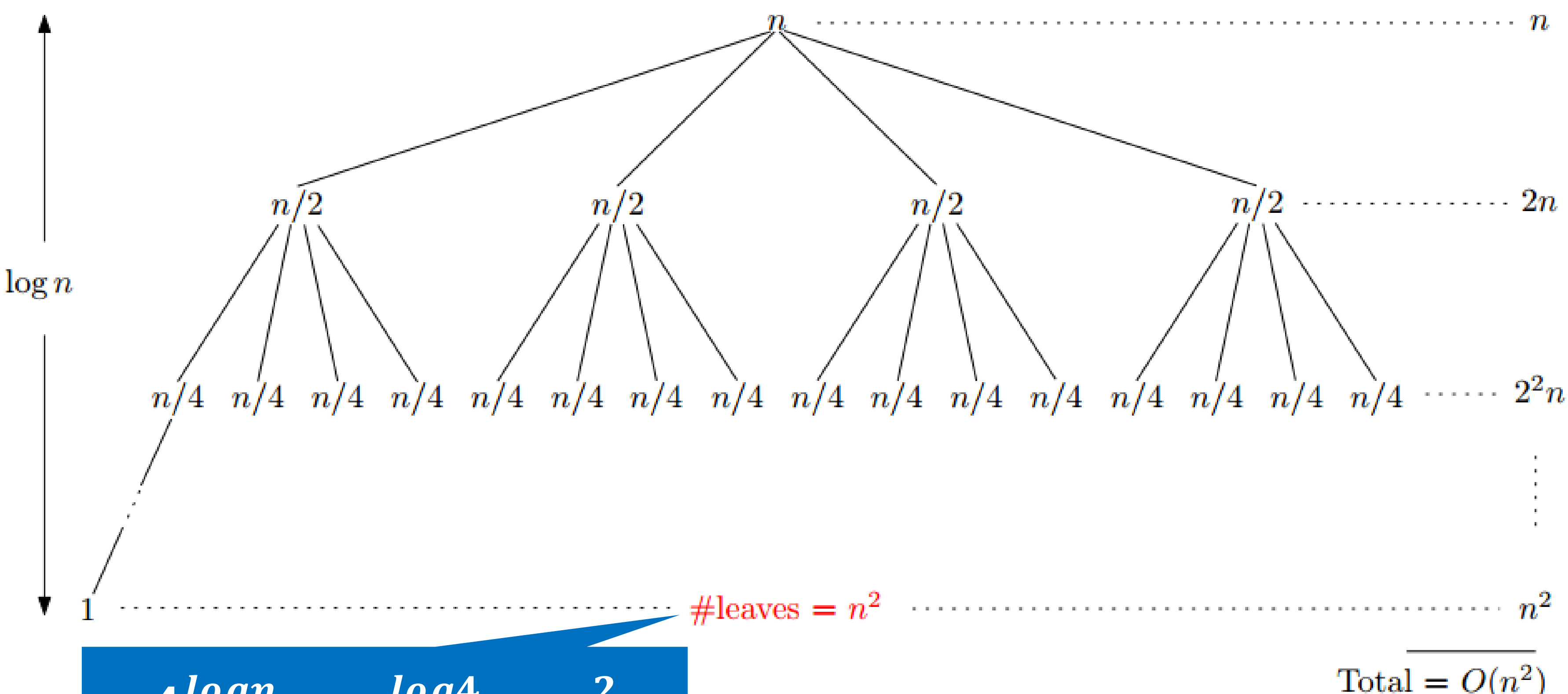

$$4^{log n} = n^{log 4} = n^2$$

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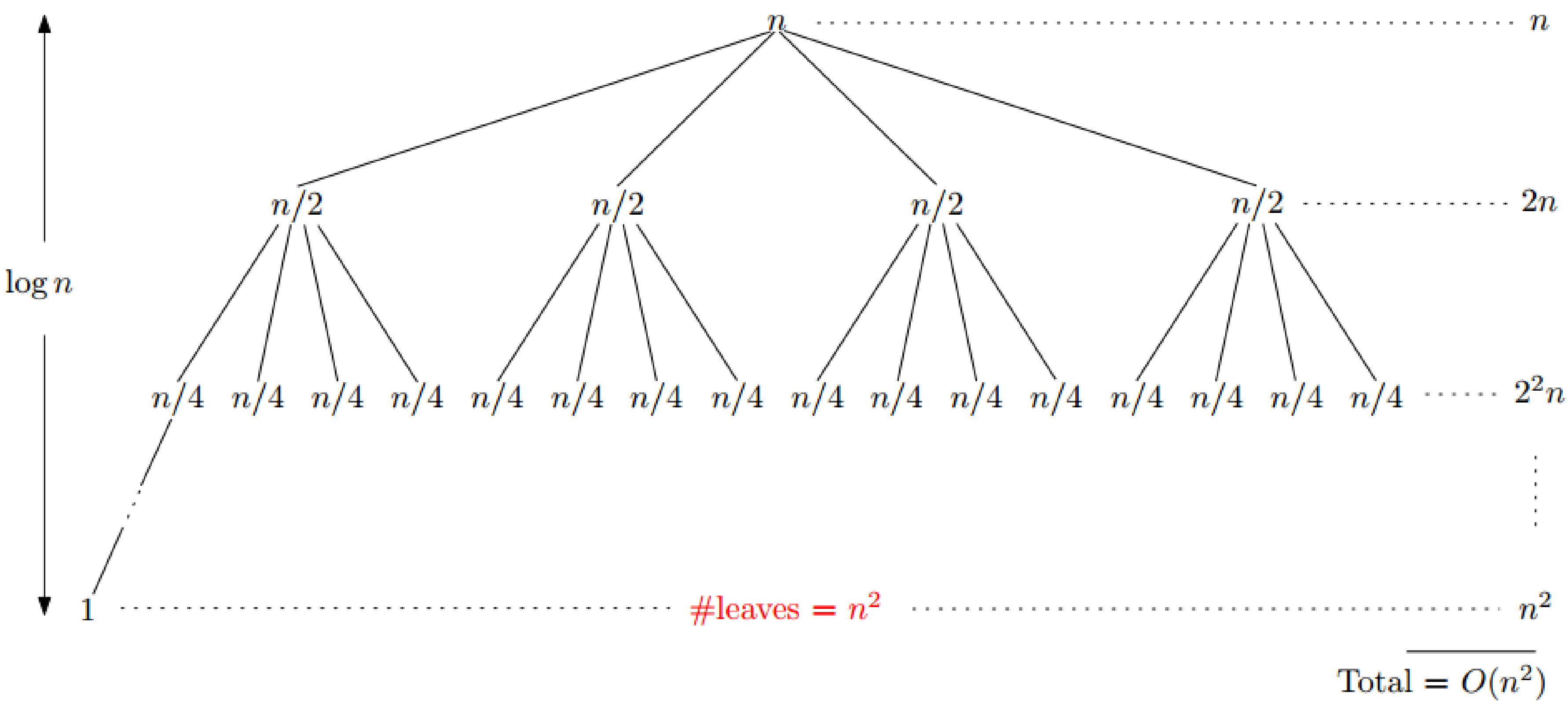

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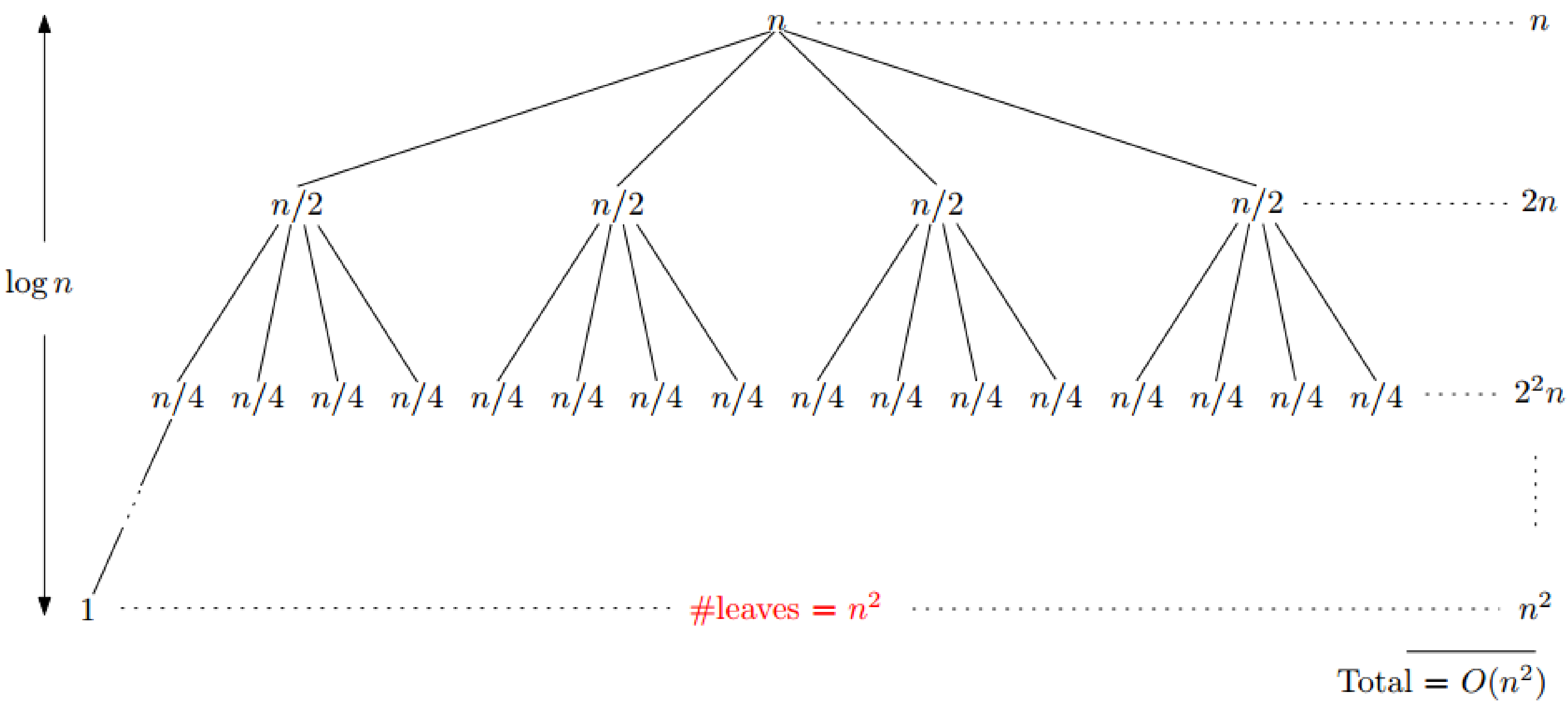

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Two Observations

Observation 1:

What we really need are the following 3 terms:

$$A_0B_0,\ A_0B_1 + A_1B_0,\ A_1B_1?$$

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The three terms can be obtained using only 3 multiplications:

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- $A_0B_1 + A_1B_0 = Y - U - Z$

The improved Divide-and-Conquer Algorithm

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Input: $A(x), B(x)$
Output: $A(x) \times B(x)$
$A_0(x) \leftarrow a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$;
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return $(U(x) + [Y(x) - U(x) - Z(x)]x^{\frac{n}{2}} + Z(x)x^{2\frac{n}{2}})$;$//O(n)$

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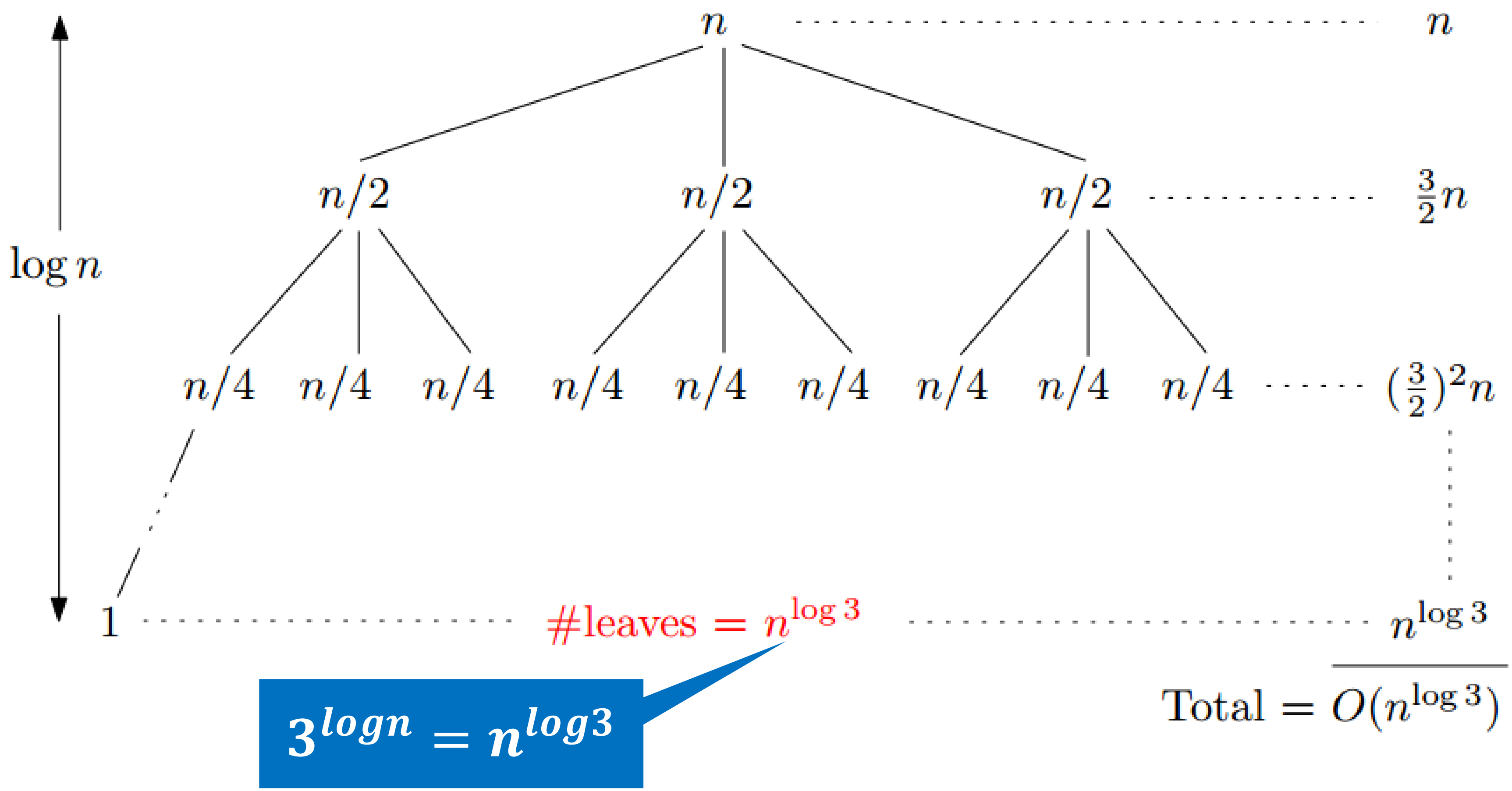

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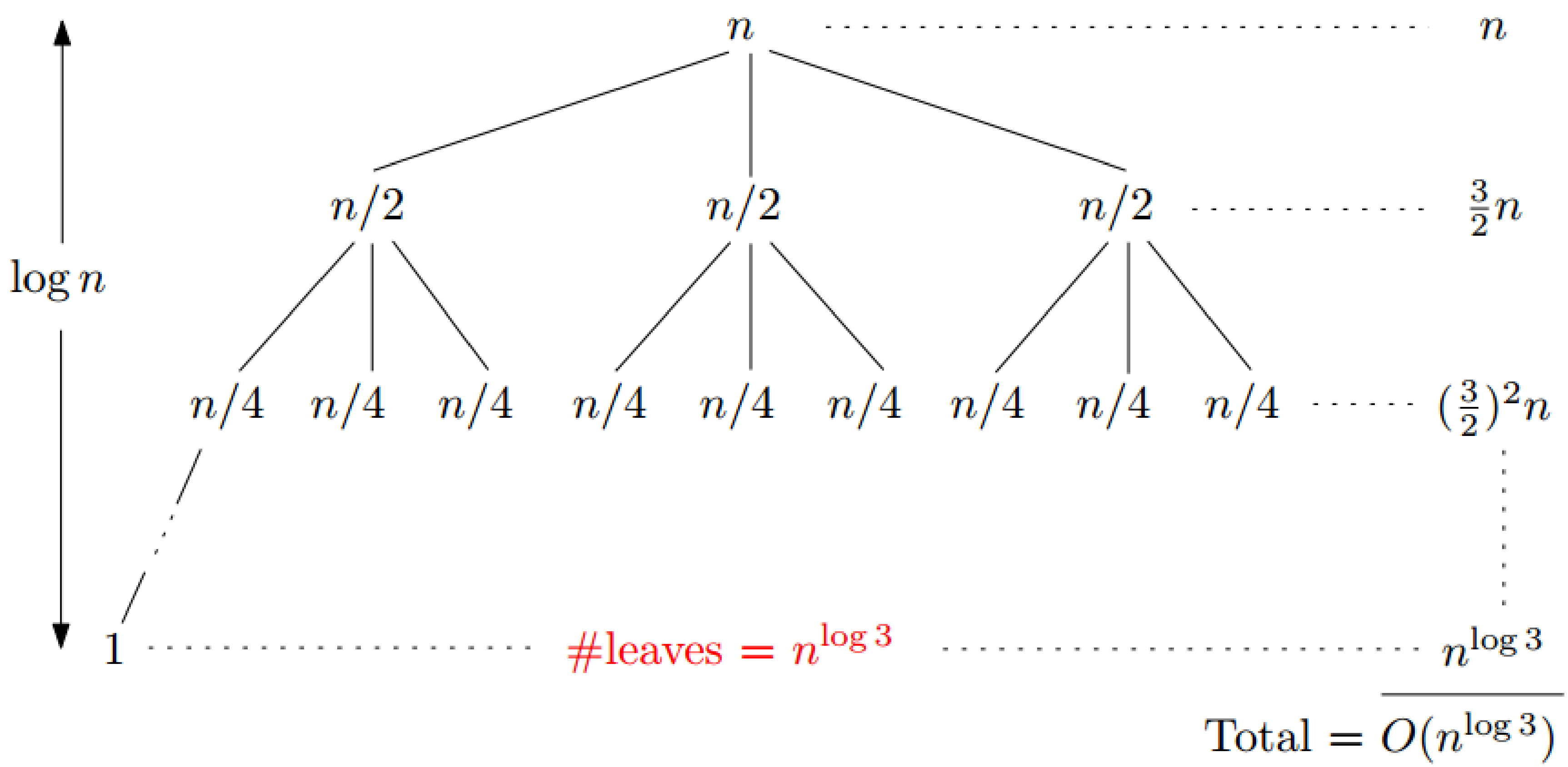

The second method is much better!

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  - It involves using the Fast Fourier Transform algorithm as a subroutine
  - The FFT is another classic divide-and-conquer algorithm(check Chapt 30 in CLRS if interested)
- The idea of using 3 multiplications instead of 4 is used in large-integer multiplications
  - A similar idea is the basis of the classic Strassen matrix multiplication algorithm (CLRS 4.2)

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  - Given: An array of numbers
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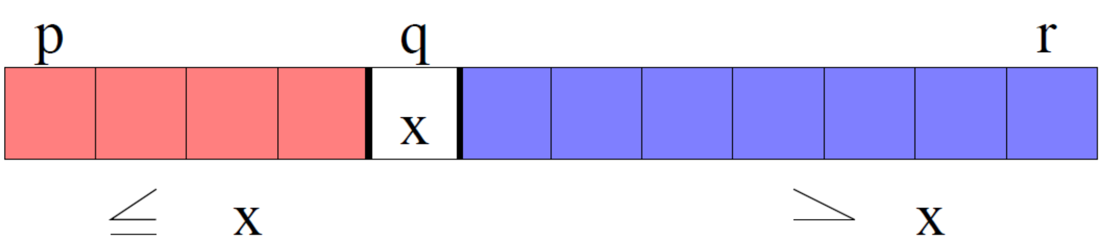

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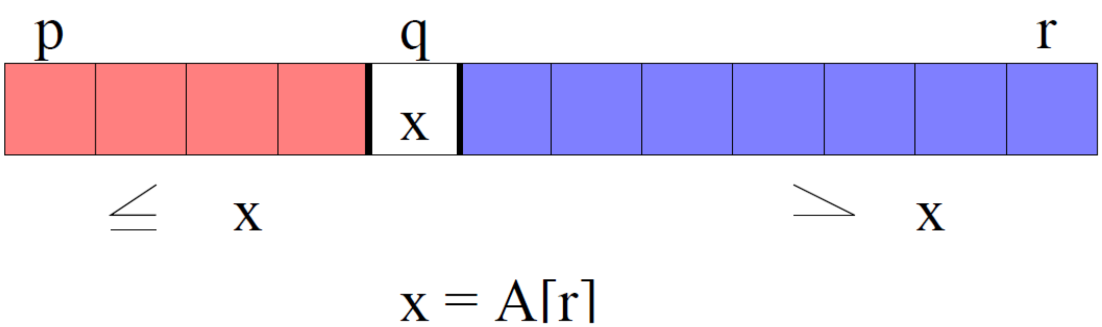

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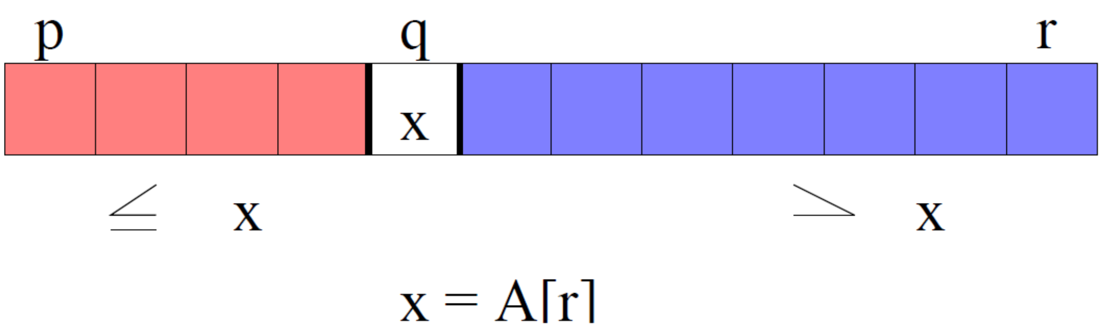

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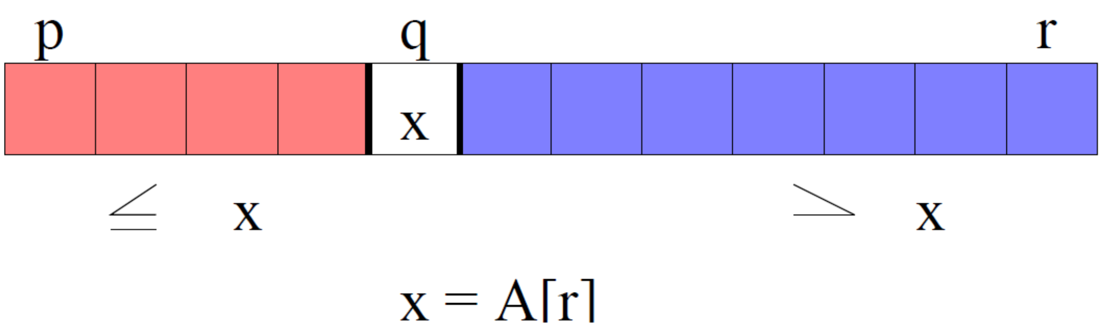

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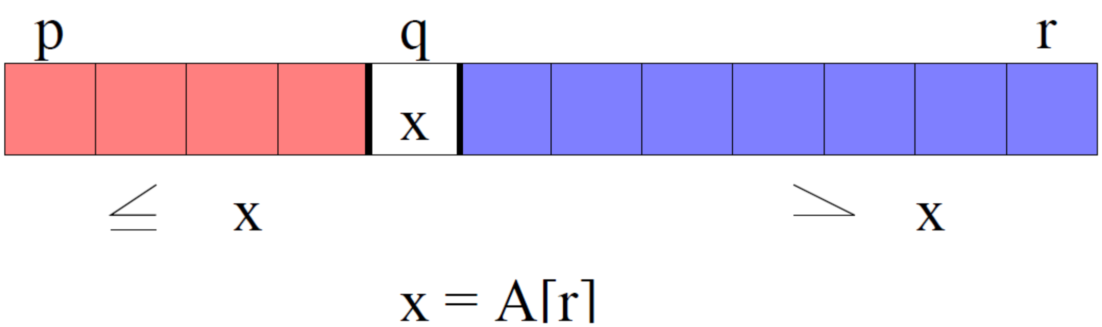

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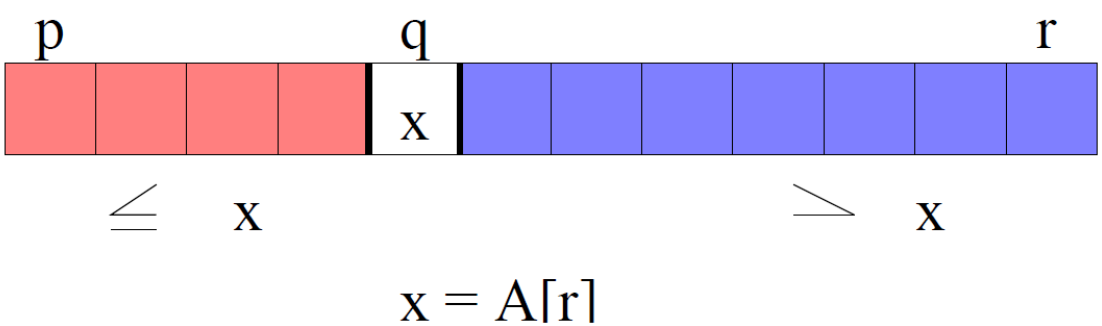

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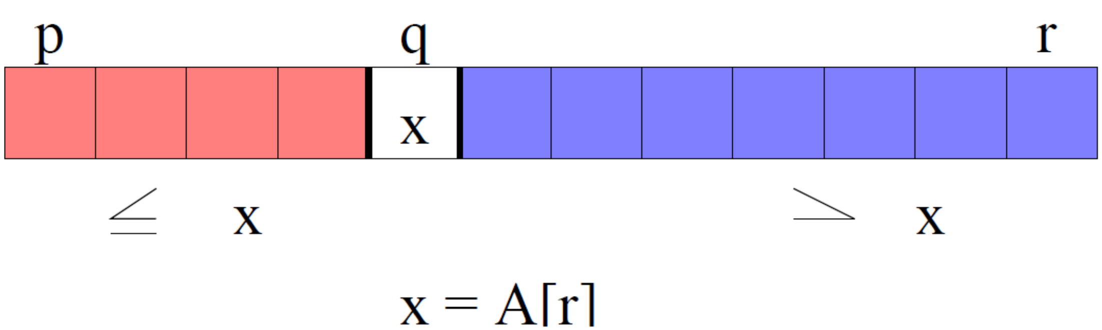

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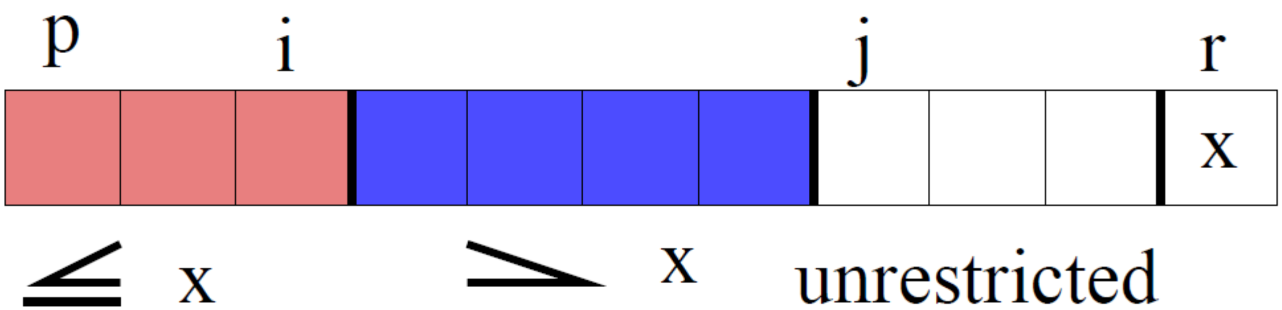

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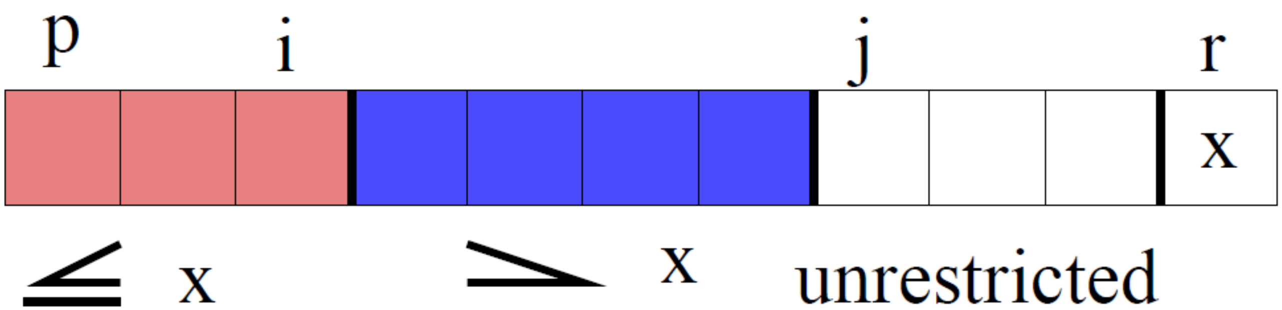

  - Initially (i, j) = (p–1, p)
  - Increase j by 1 each time to find a place for A[j]

    At the same time increase i when necessary
  - Stops when j = r

Partition

- One Iteration of the Procedure Partition
  - Increase j by 1 each time to find a place for A[j]

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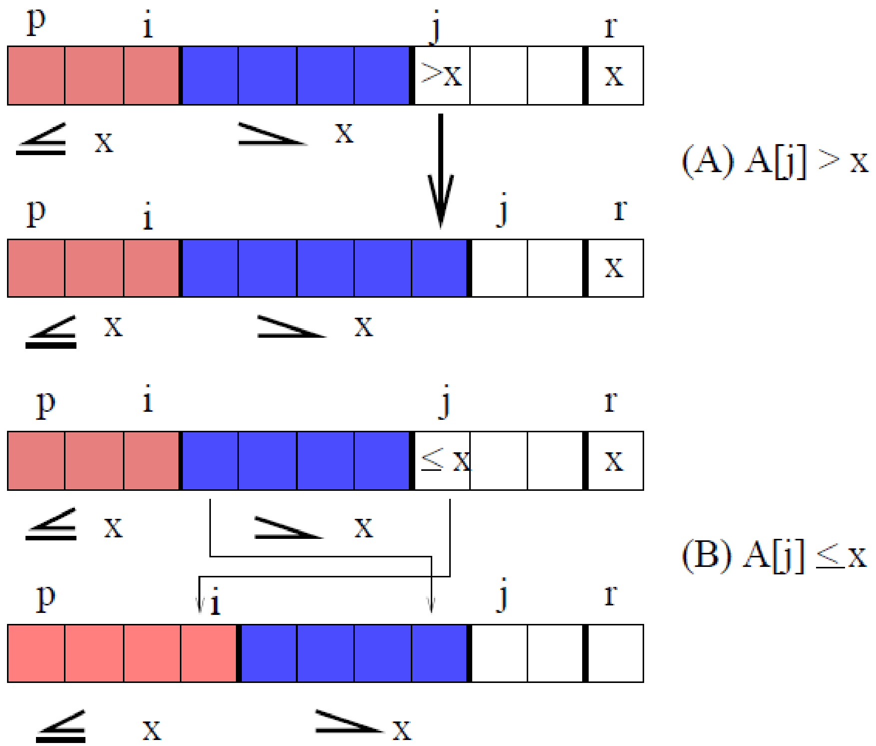

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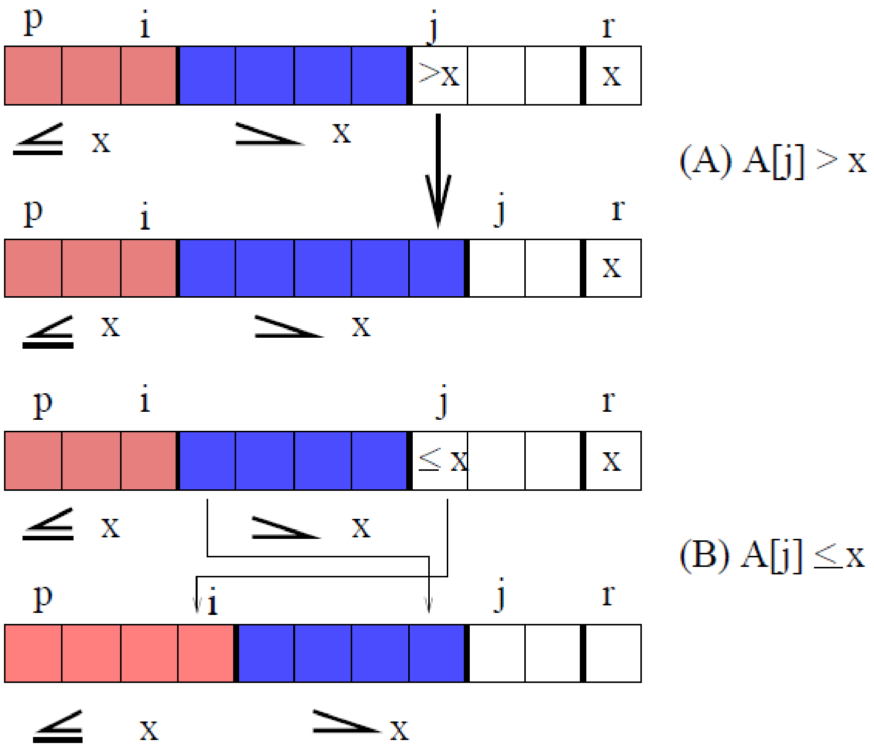

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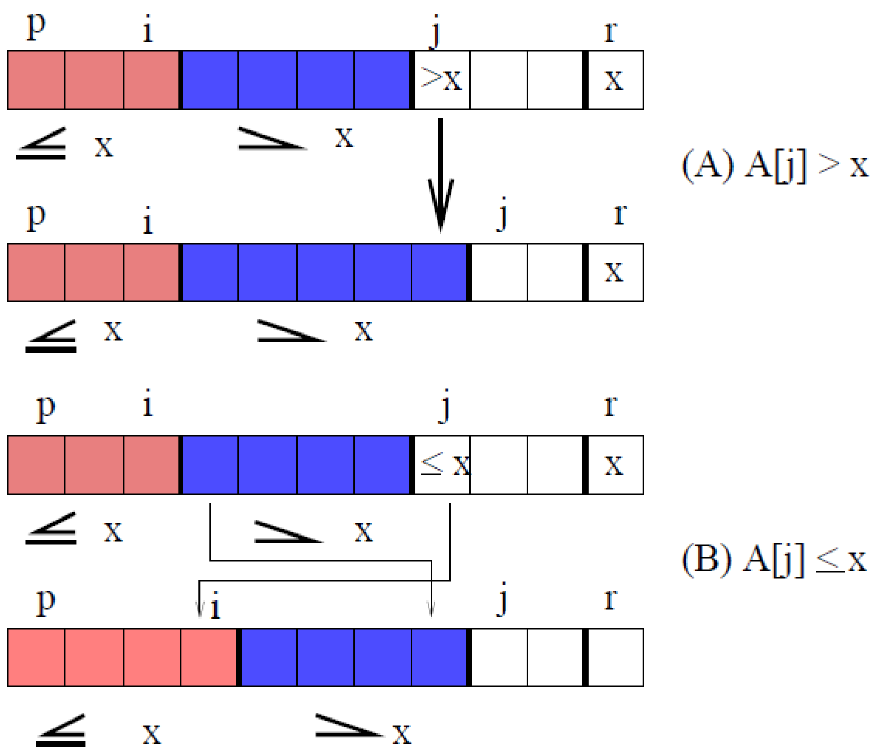
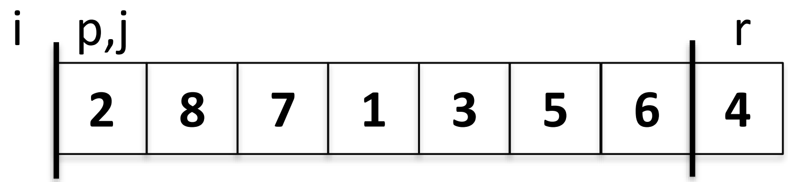


  - Case (A): Only increase j by 1
  - Case (B): i = i + 1; A[i] ↔ A[j]; j = j + 1.

Partition-Example

- The Operation of Partition(A, p, r)



Partition-Example

- The Operation of Partition(A, p, r)

i	p,j							r
	2	**8**	**7**	**1**	**3**	**5**	**6**	**4**

$$A[j] < A[r]$$

Partition-Example

- The Operation of Partition(A, p, r)

p,i	j						r
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$$i = i + 1, A[i] \leftrightarrow A[j], j = j + 1$$

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- The Operation of Partition(A, p, r)

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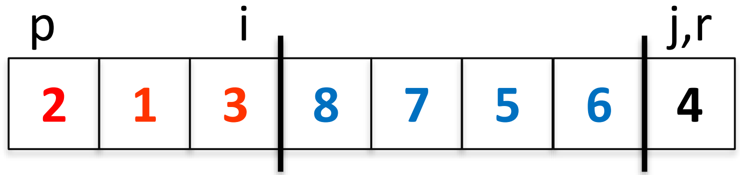

Partition-Example

- The Operation of Partition(A, p, r)

p		i					j,r
2	1	3	4	7	5	6	8

$$A[i+1] \leftrightarrow A[r]$$

Partition - Pseudocode

Partition(A, p, r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Index of the pivot after partition
$x \leftarrow A[r]$;//$A[r]$ is the pivot element

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- Running time is O(   )

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- Running time is O(r – p)

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- Running time is O(r – p)
  - linear in the length of the array A[p..r]

Quicksort

Quicksort(A,p,r)

Input: An array $\boldsymbol{A}$ waiting to be sorted,the range of index $\boldsymbol{p}$,$\boldsymbol{r}$
Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
    $q \leftarrow$Partition(A, p, r);
    Quicksort(A,   );
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end
return A;

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A Divide-and-Conquer Framework

Quicksort

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \le 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.

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A Divide-and-Conquer Framework

- If we could always partition the array into halves, then we have the recurrence $T(n) \leq 2T(n/2) + O(n)$, hence $T(n) = O(n \log n)$.
- However, if we always get unlucky with very unbalanced partitions, then $T(n) \leq T(n-1) + O(n)$, hence $T(n) = O(n^2)$.

Outline

- Review to Divide-and-Conquer Paradigm
- Polynomial Multiplication Problem
  - Problem definition
  - A brute force algorithm
  - A first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Randomized-Partition(A, p, r)

- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot *x* to partition the array A[p..r].

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  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot *x* to partition the array A[p..r].
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- Idea
  - In the algorithm Partition(A, p, r), A[r] is always used as the pivot x to partition the array A[p..r].
  - In the algorithm Randomized-Partition(A, p, r), we randomly choose an j , p ≤ j ≤ r , and use A[j] as pivot.
  - Idea is that if we choose randomly, then the chance that we get unlucky every time is extremely low.

Randomized-Partition(A, p, r)

- Pseudocode of Randomized-Partition
  - Let random(p, r) be a pseudorandom-number generator that returns a random number between p and r.

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Output: A random index in $[p..j]$

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  - We make use of the Randomized-Partition idea to develop a new version of quicksort.

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Output: Sorted array $\boldsymbol{A}$
if $p < r$ **then**
  $q \leftarrow$ $\boxed{\text{Randomized-Partition}(A, p, r);}$
  Randomized-Quicksort(A,   );
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Quicksort - Example

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Quicksort - Example

Divide

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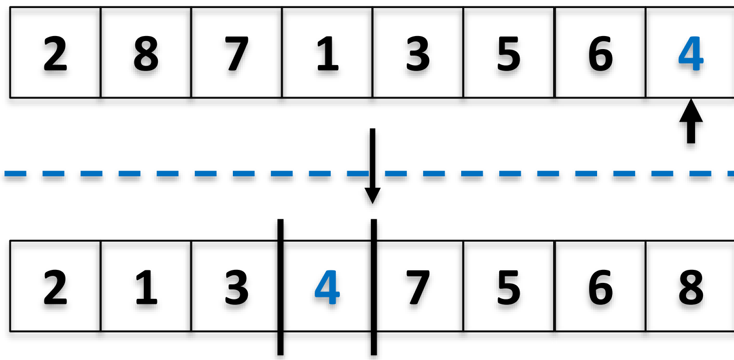

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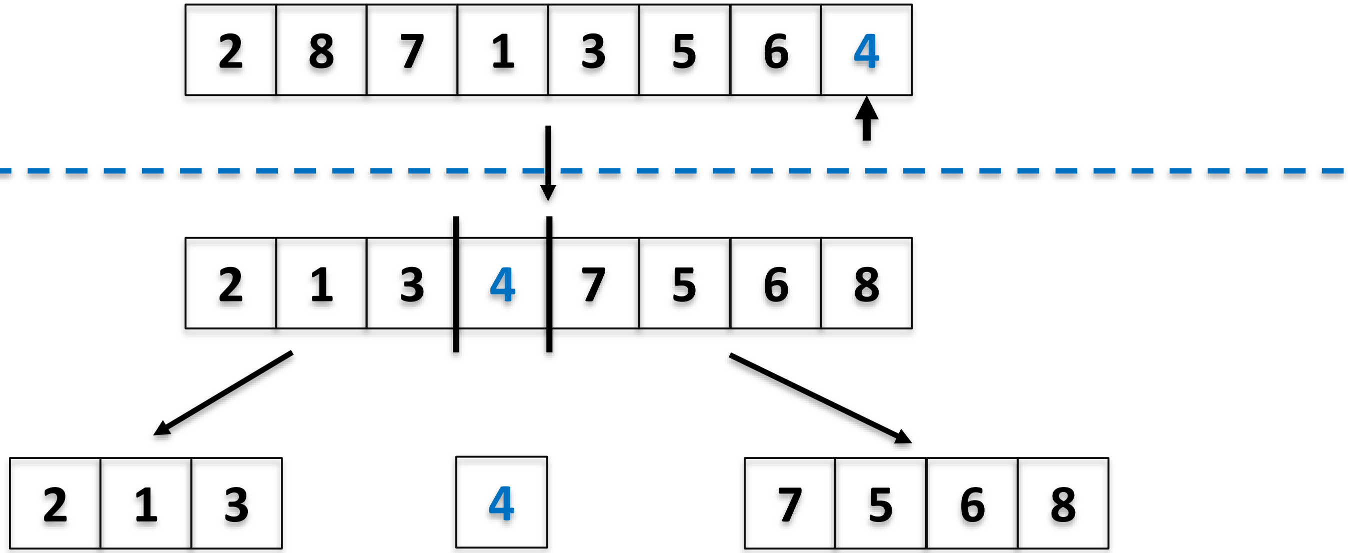

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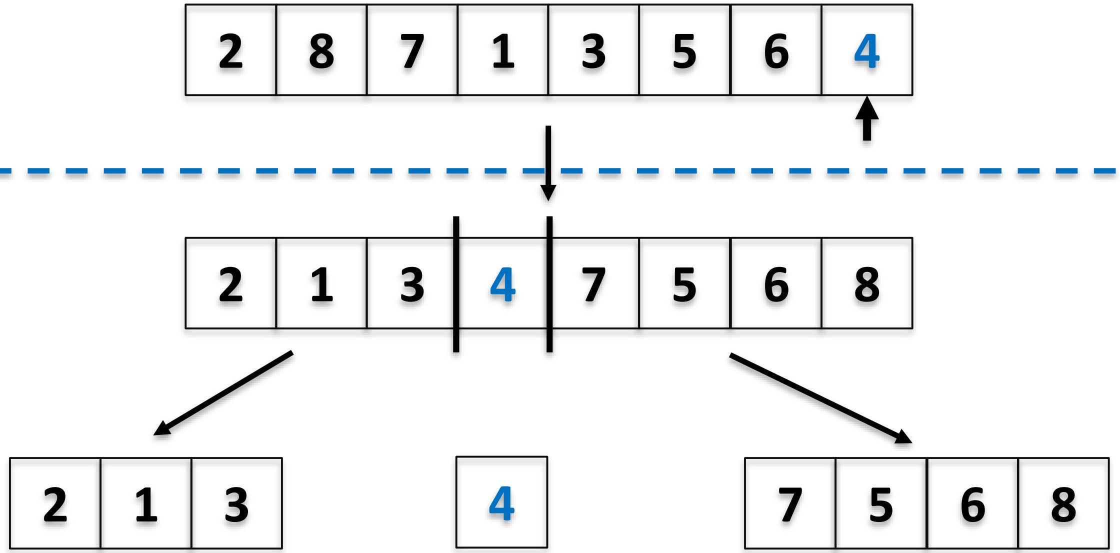

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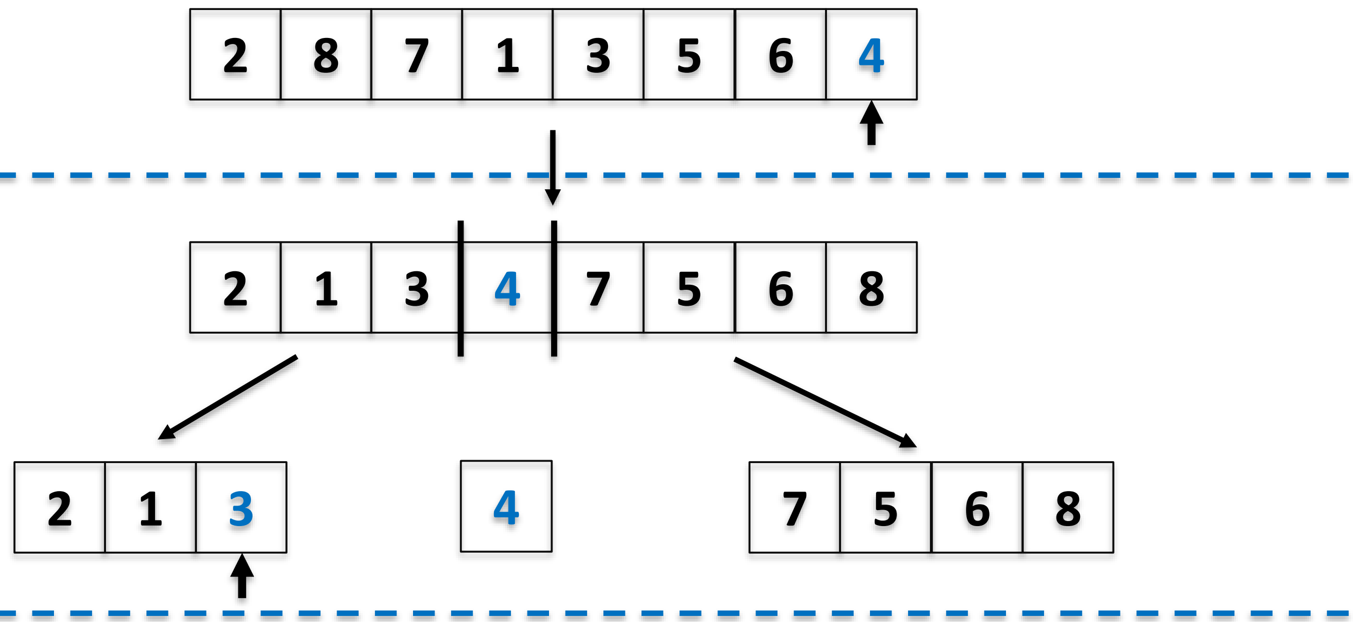

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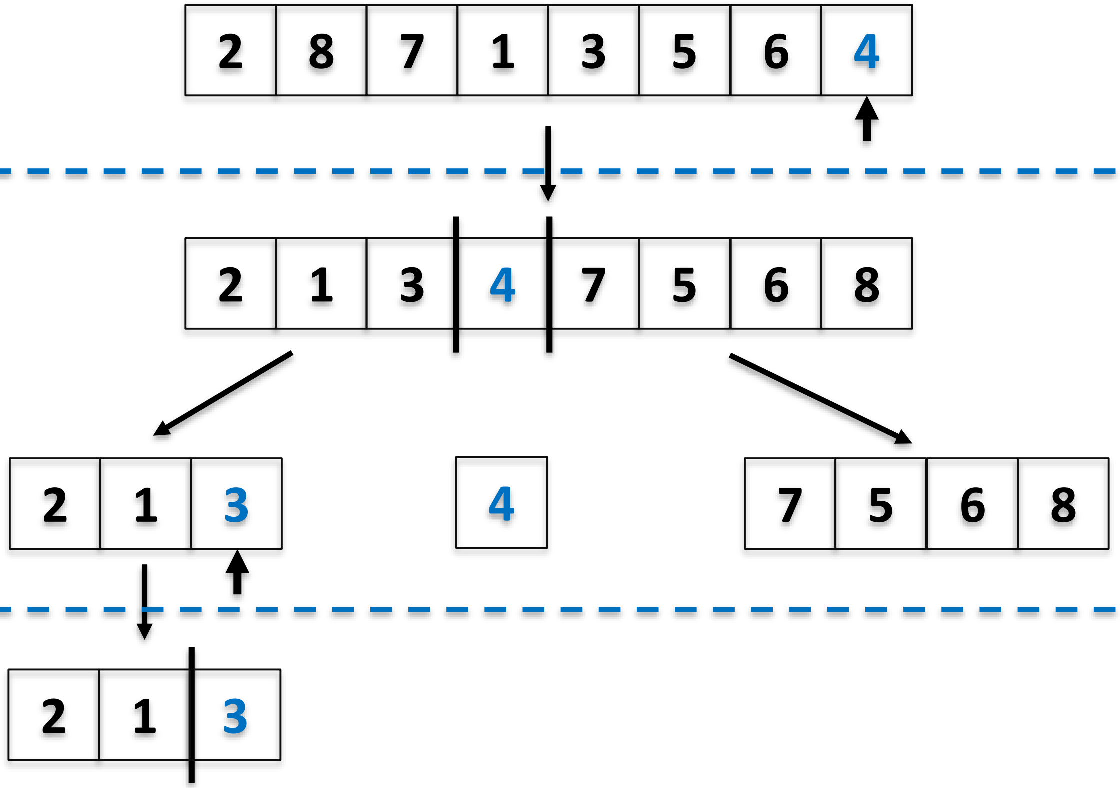

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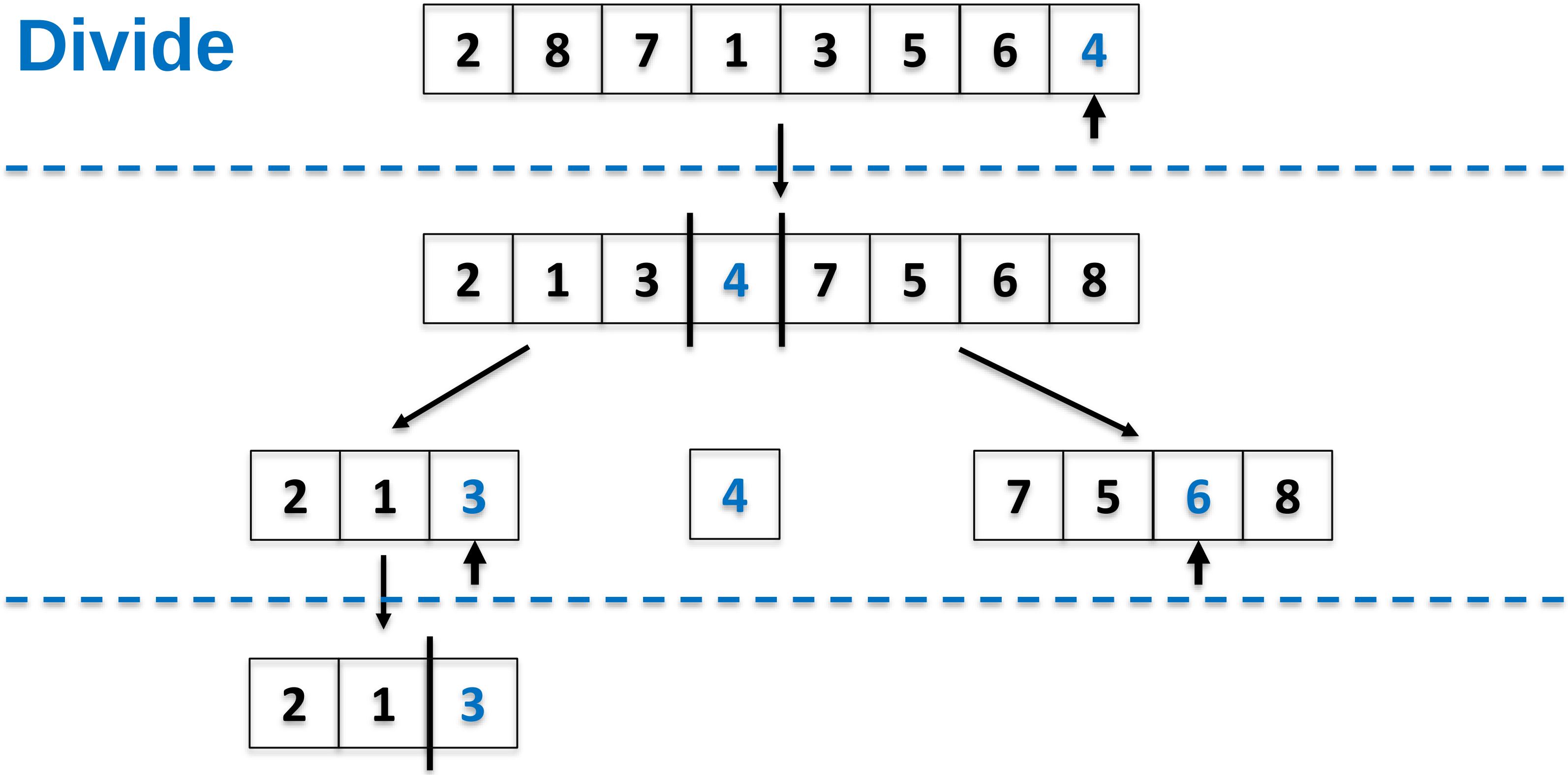

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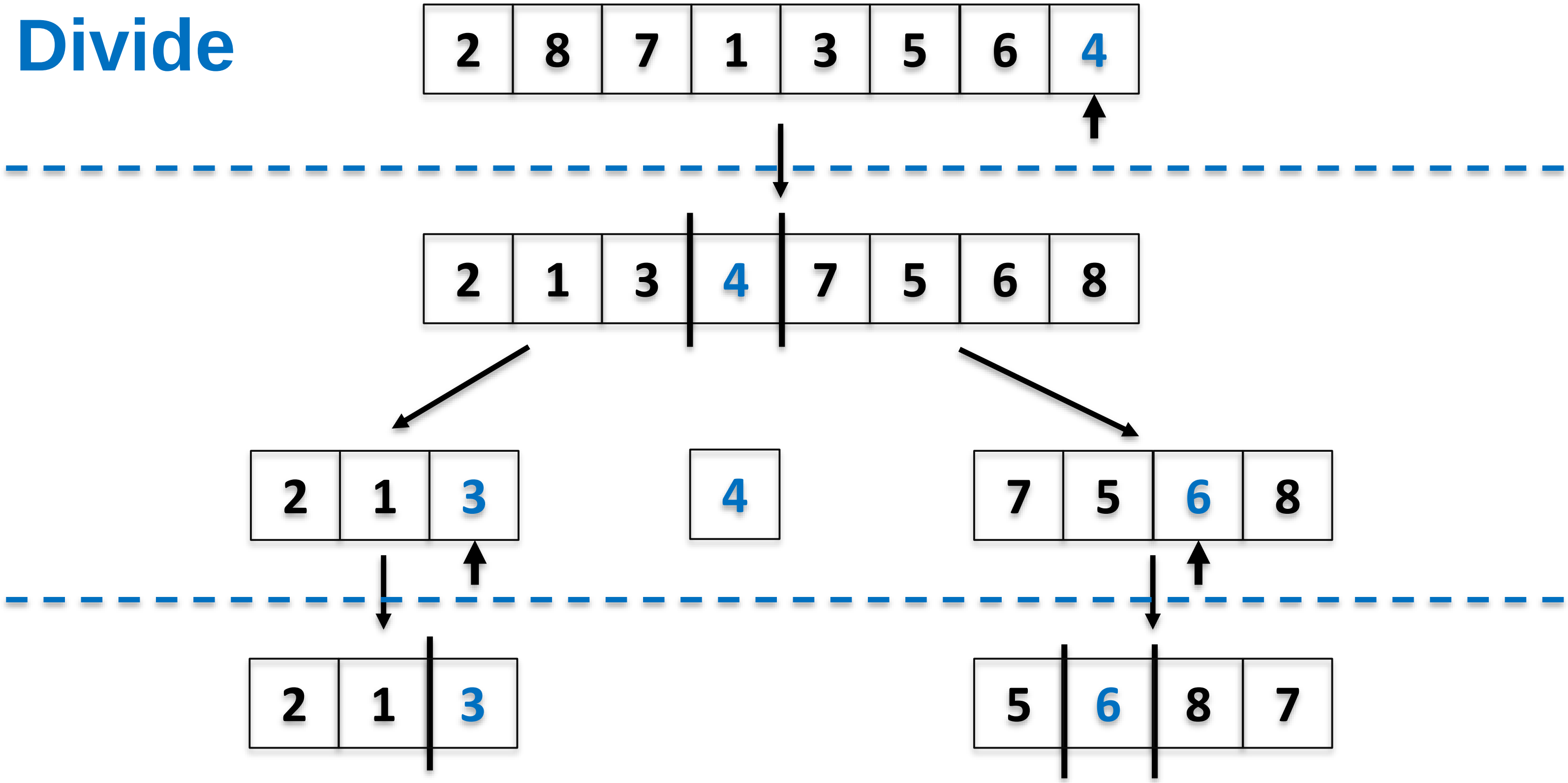

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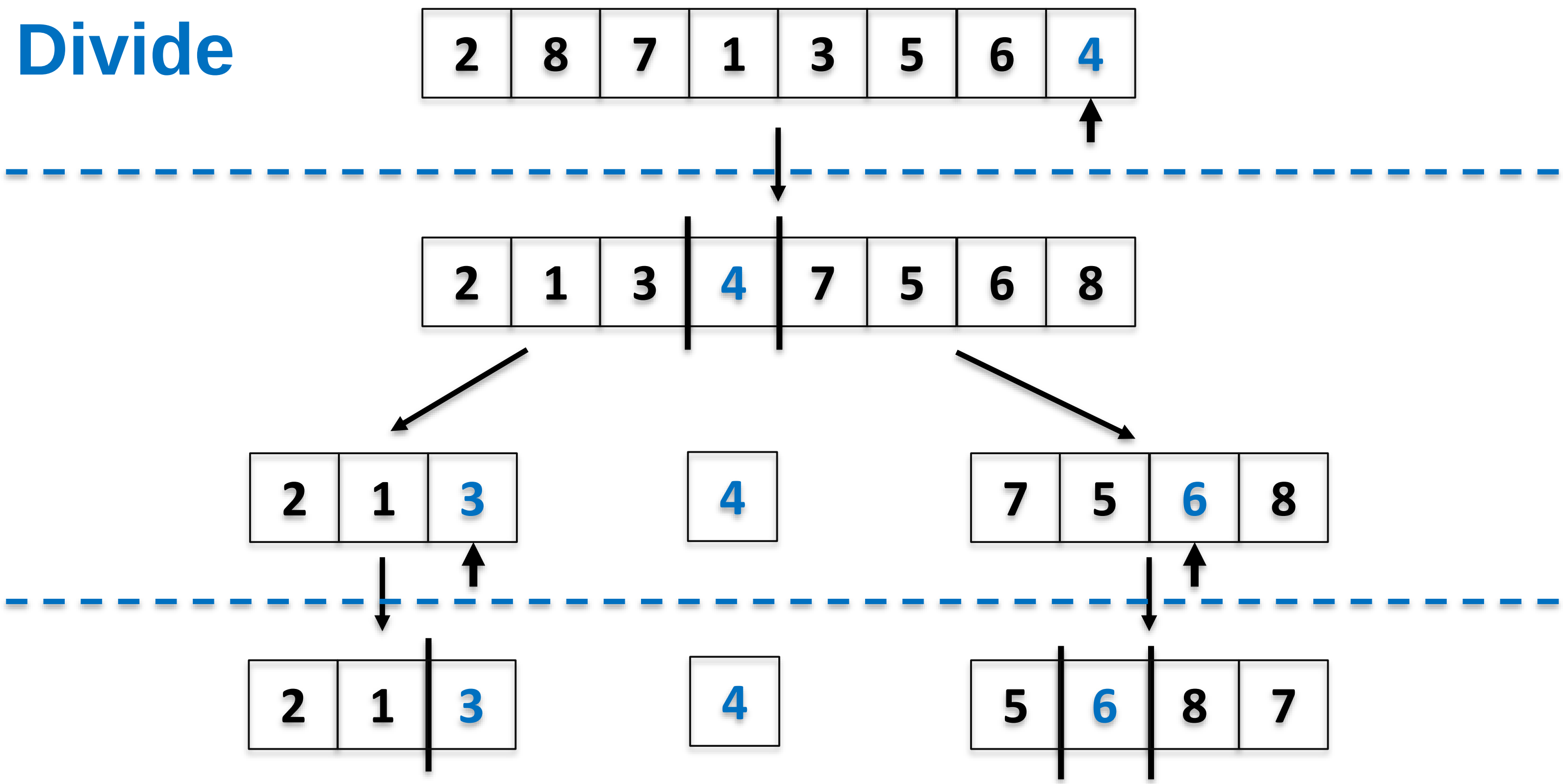

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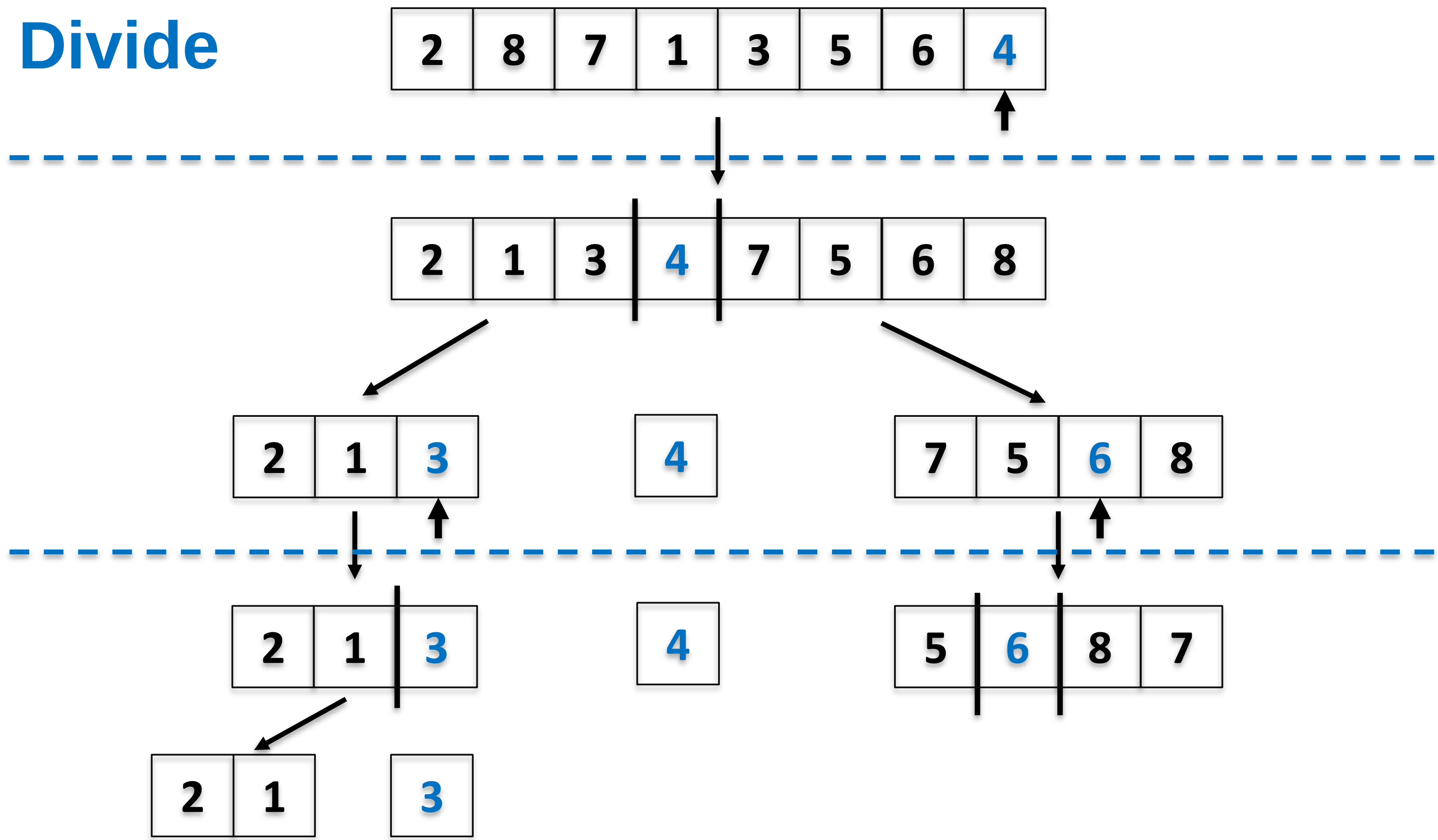

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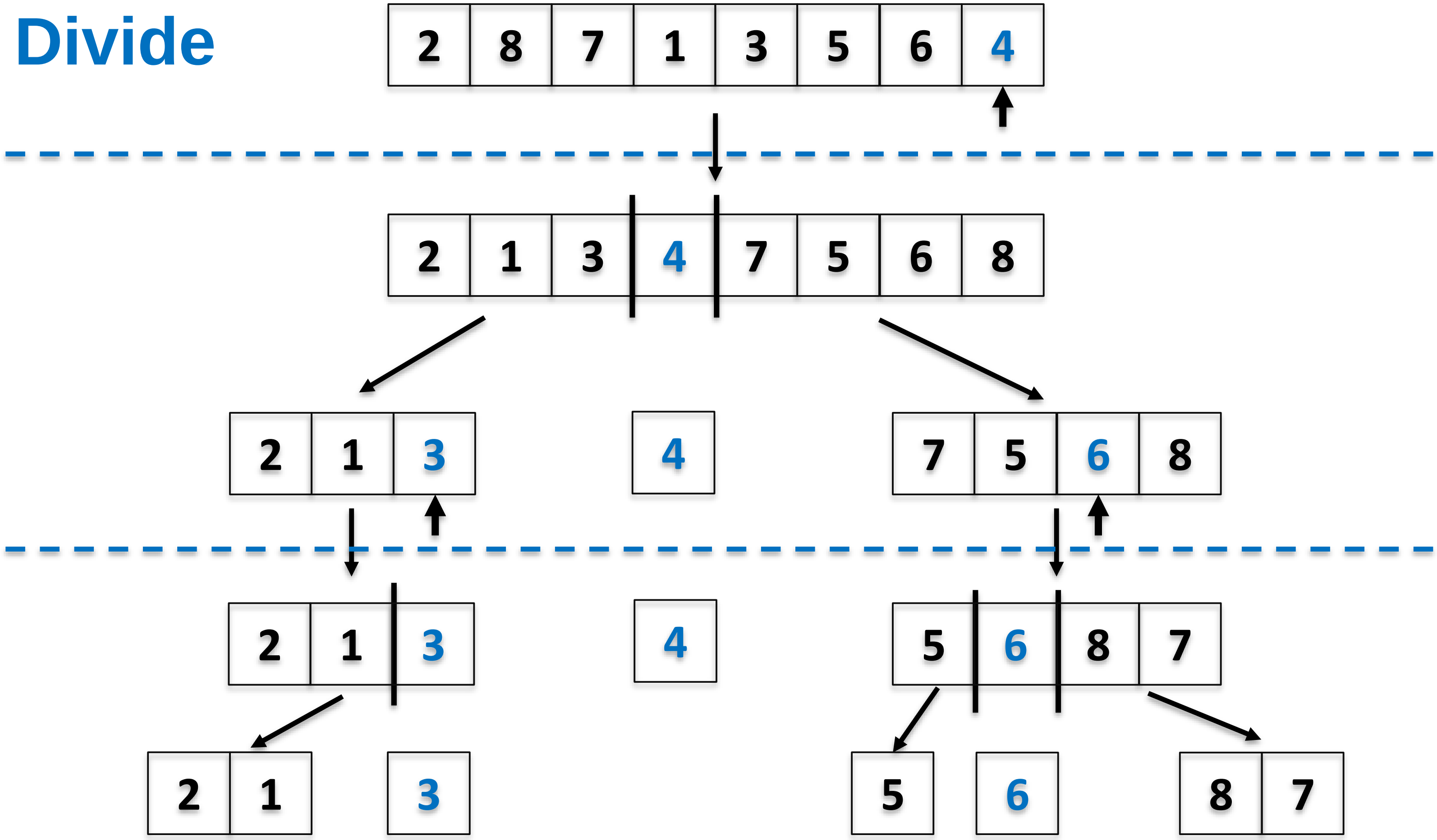

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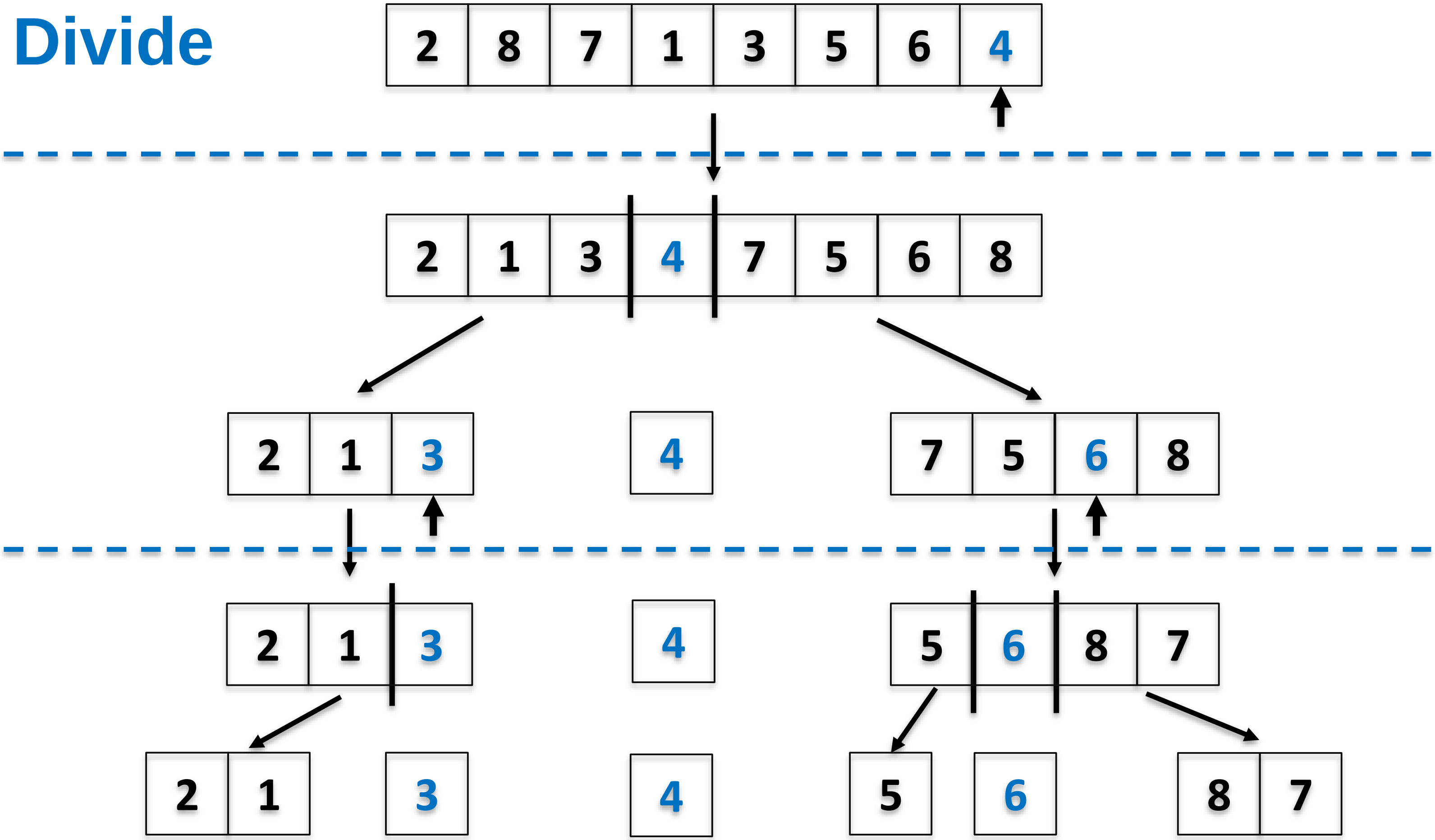

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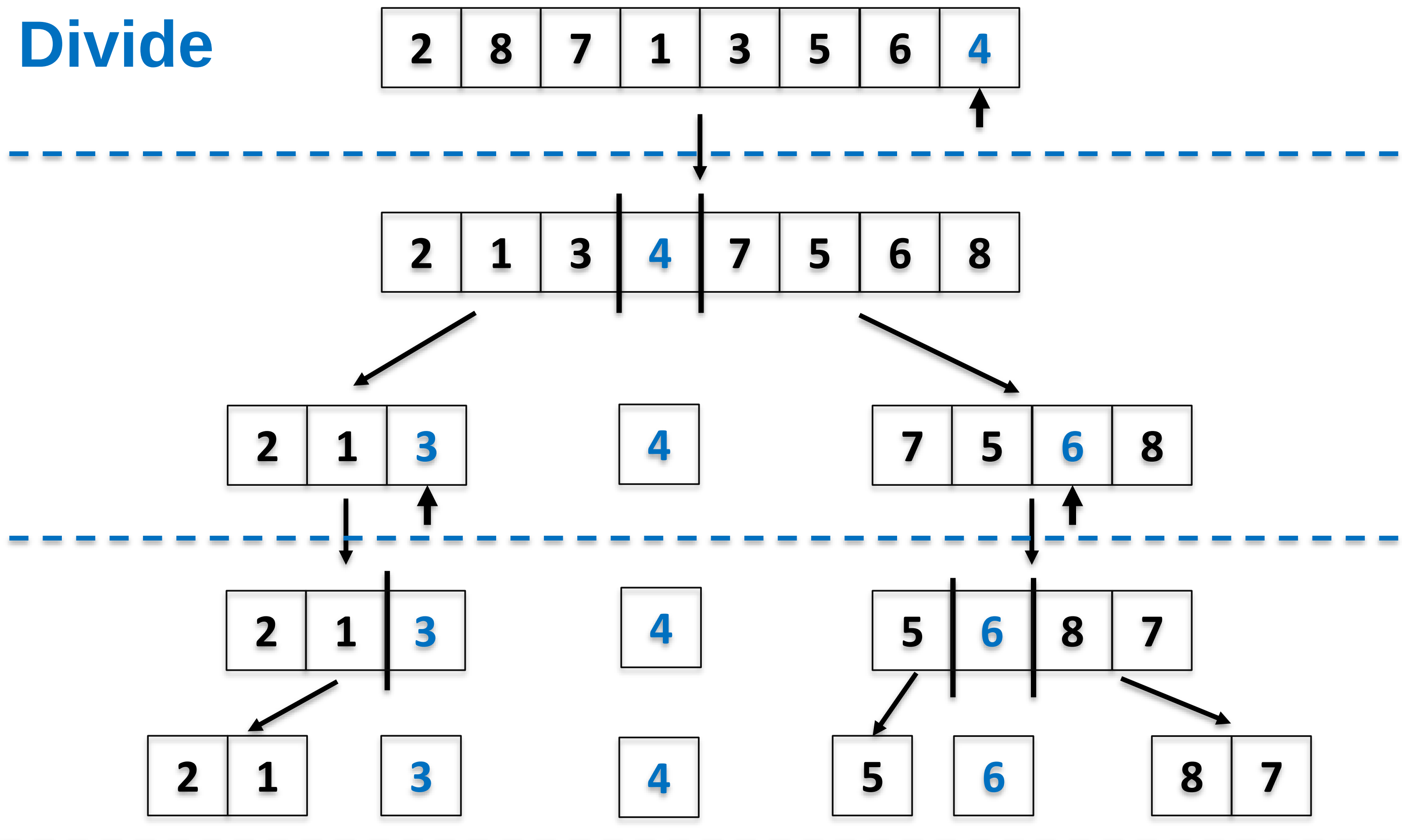

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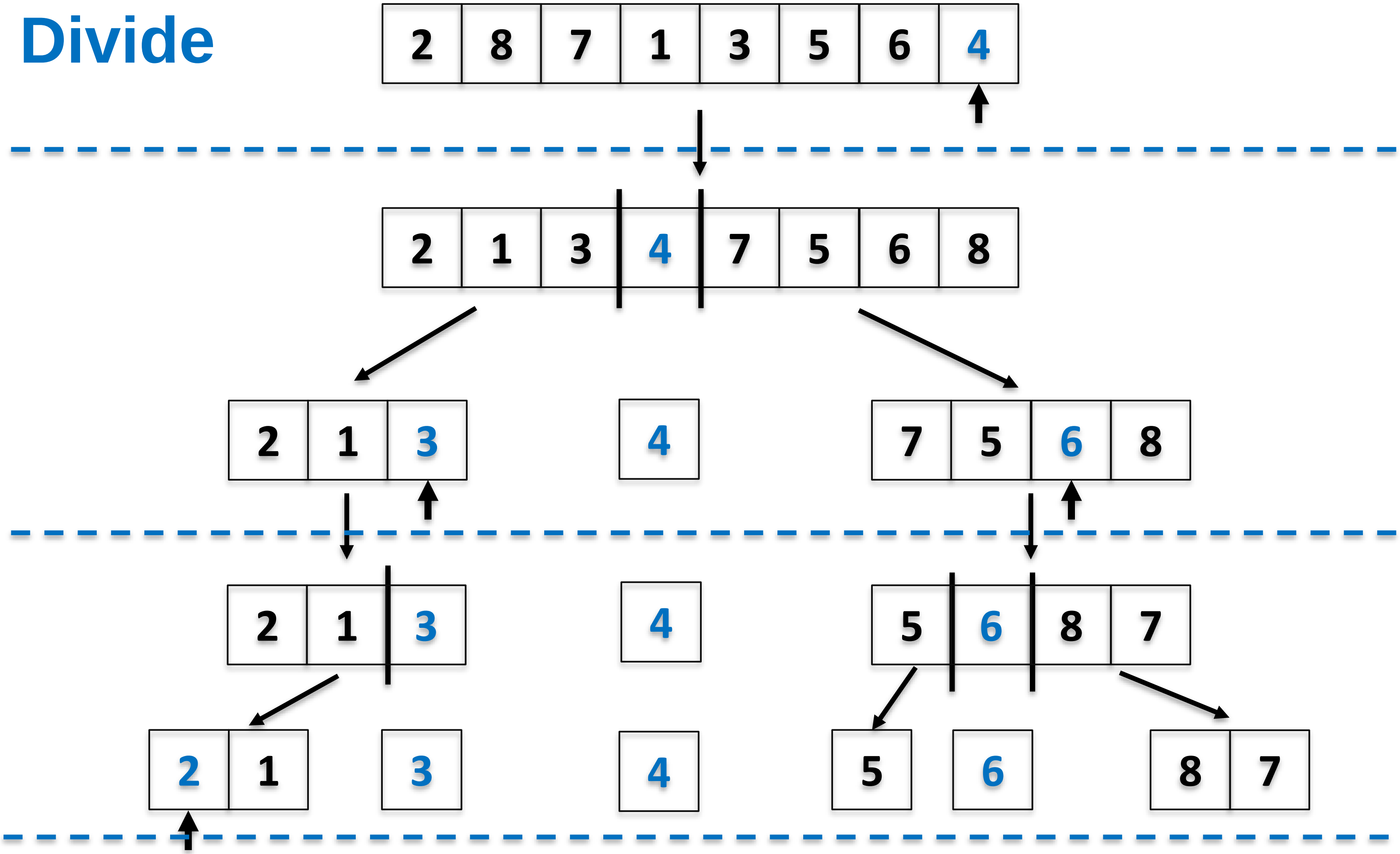

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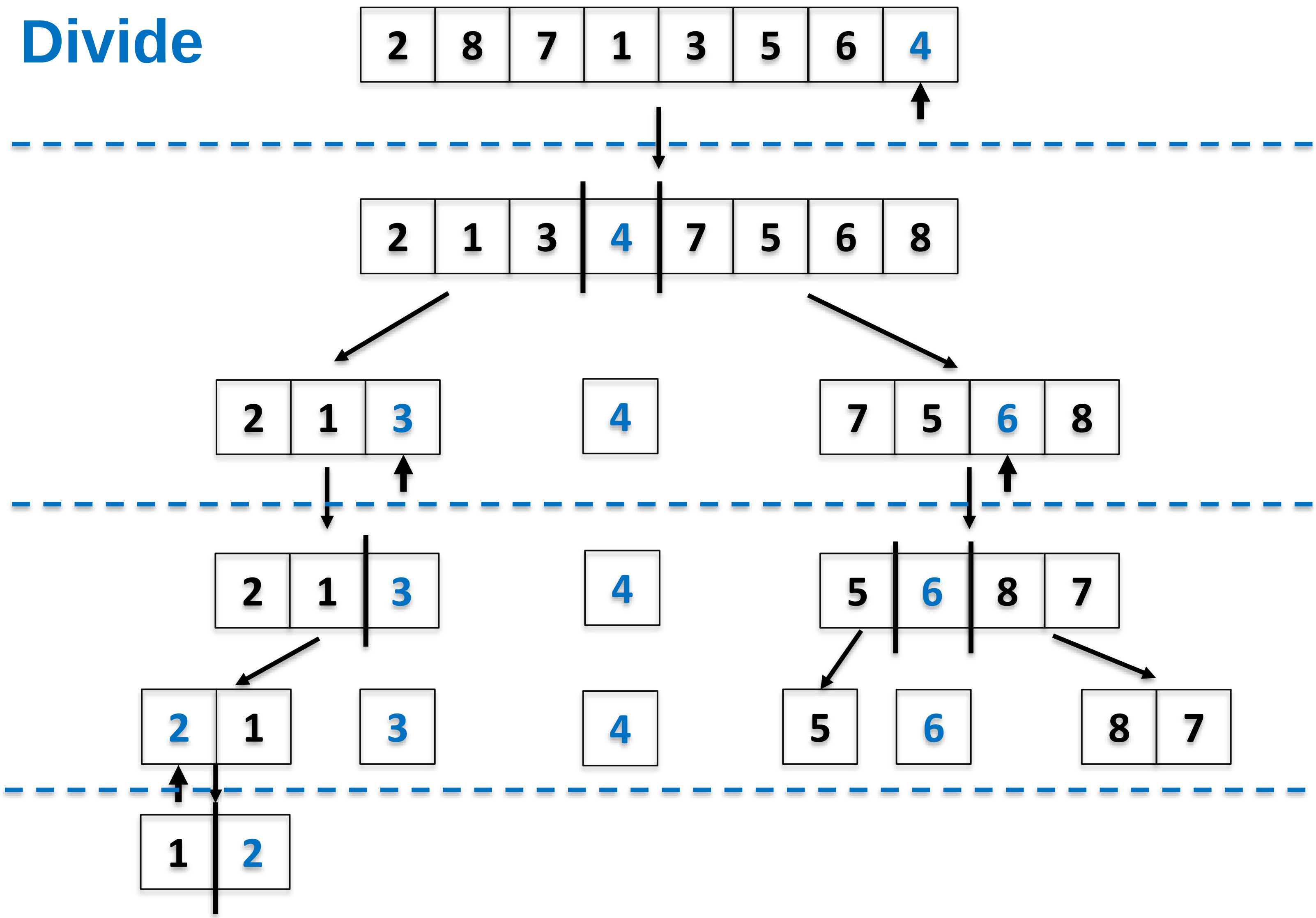

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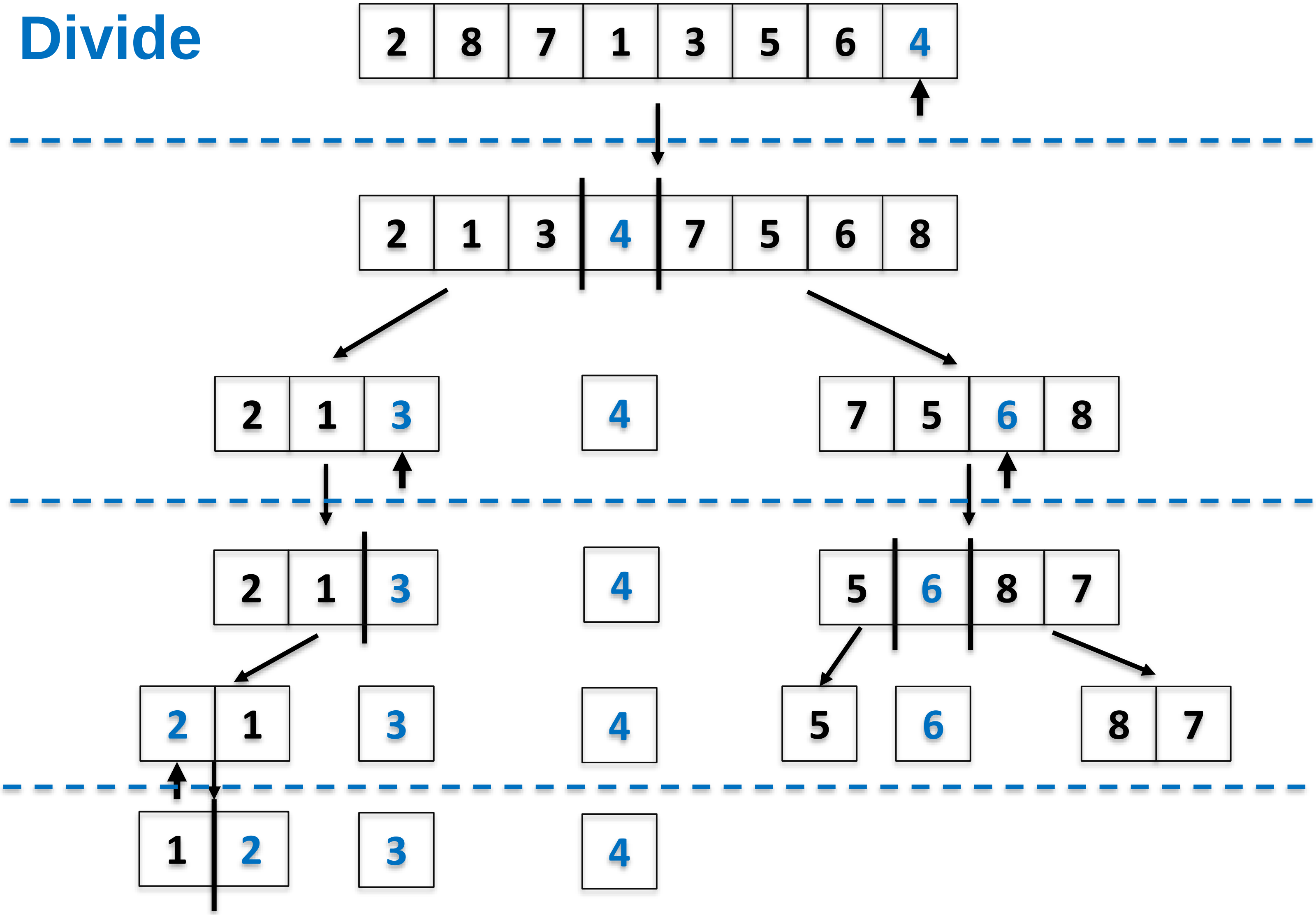

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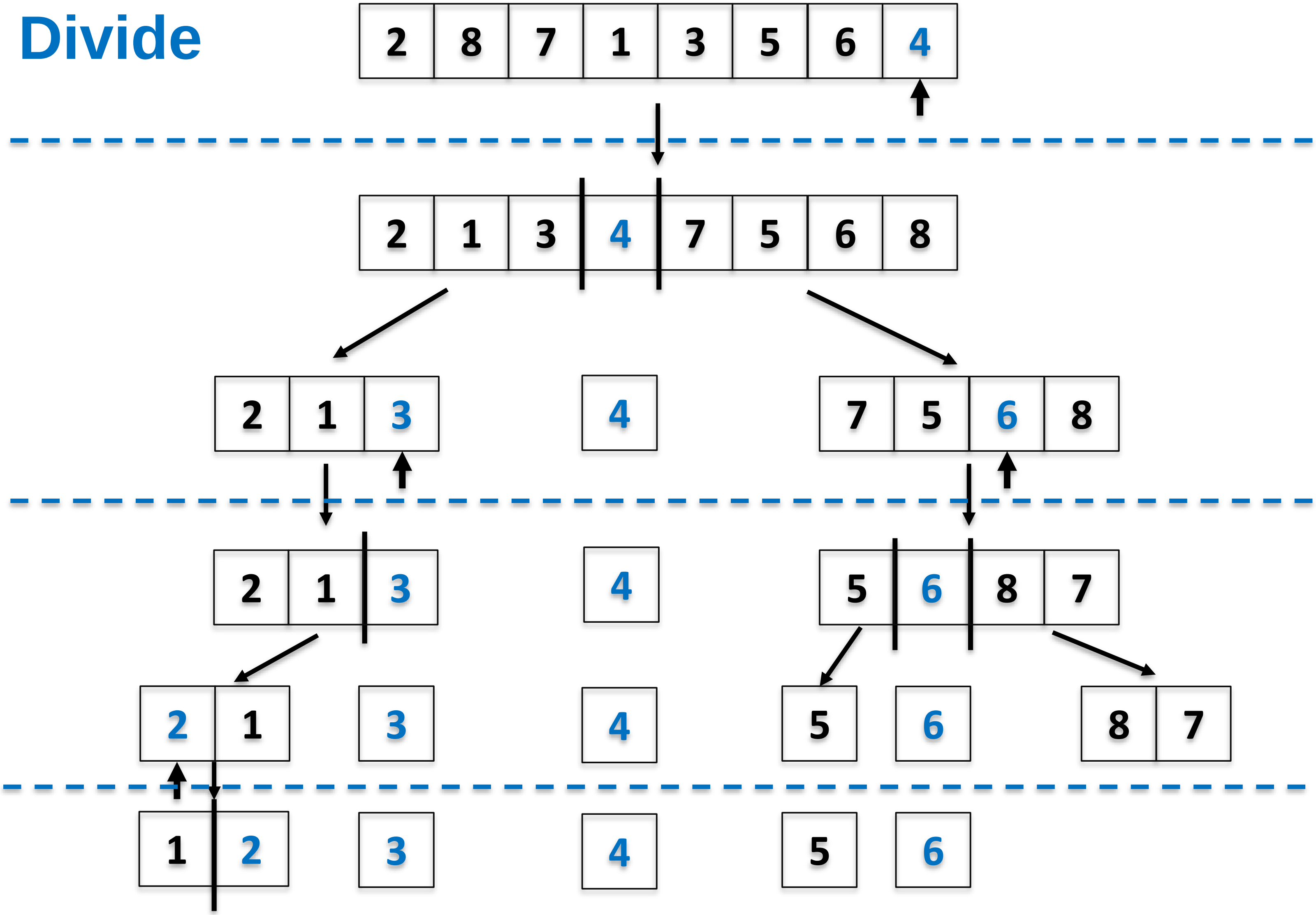

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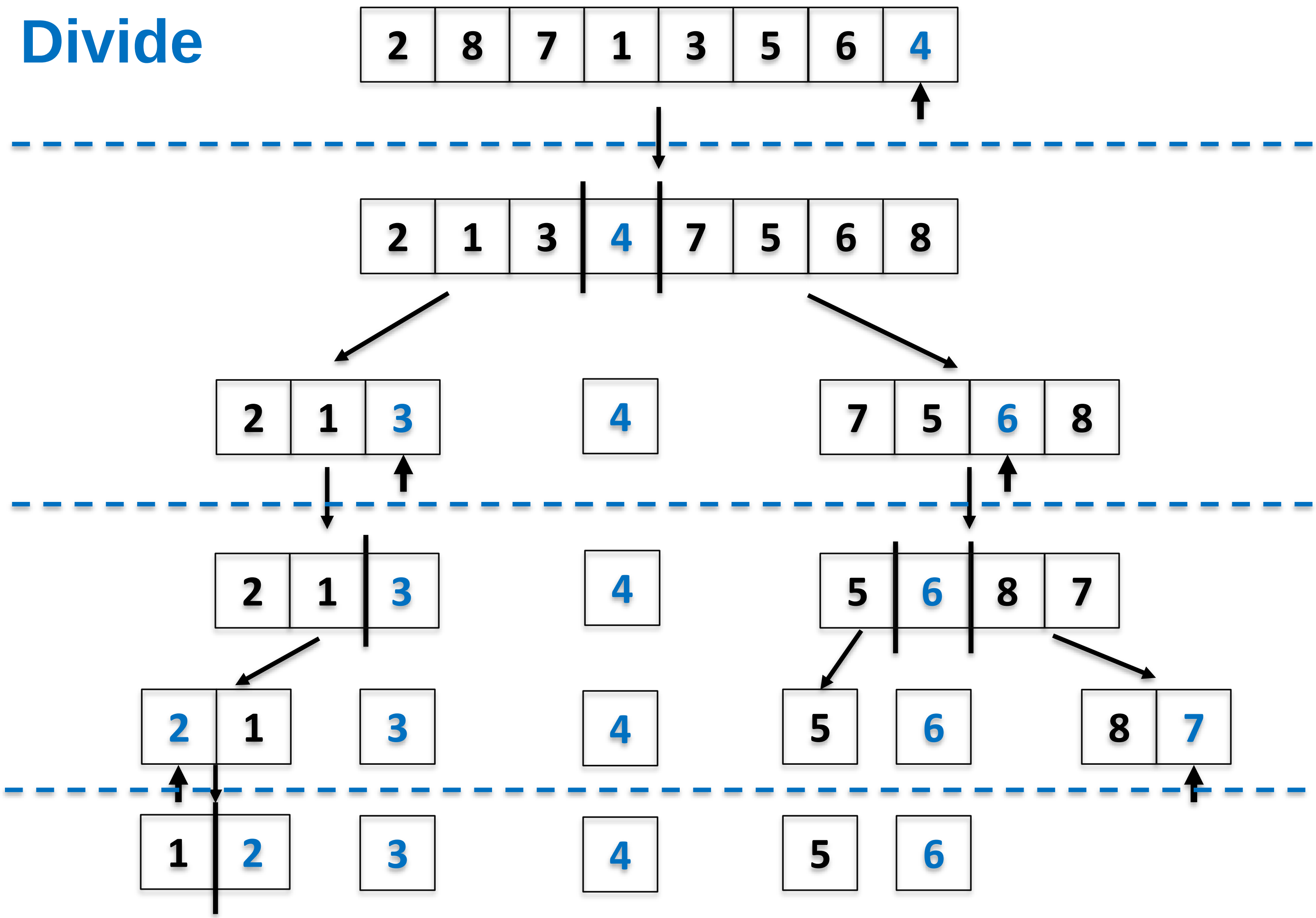

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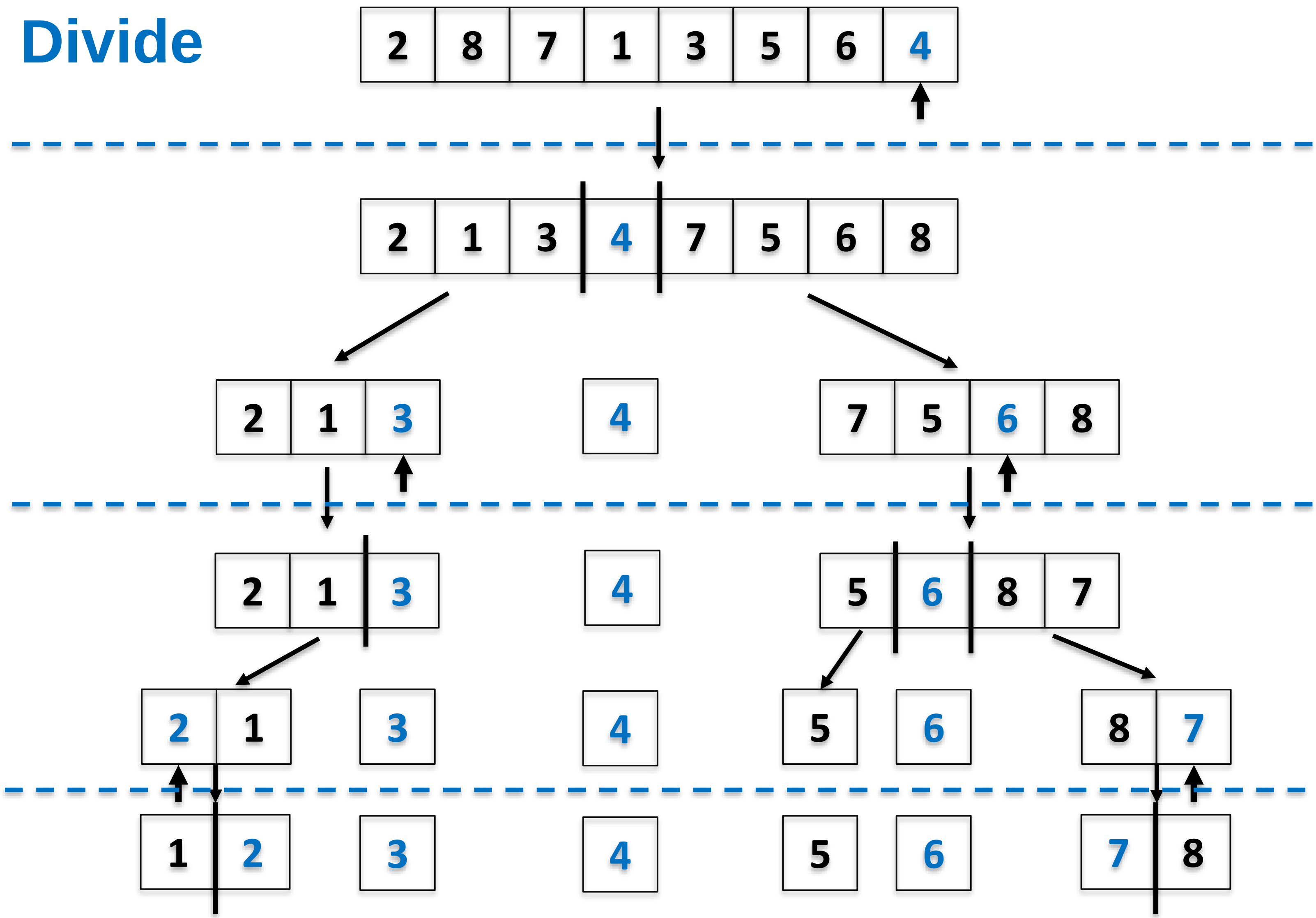

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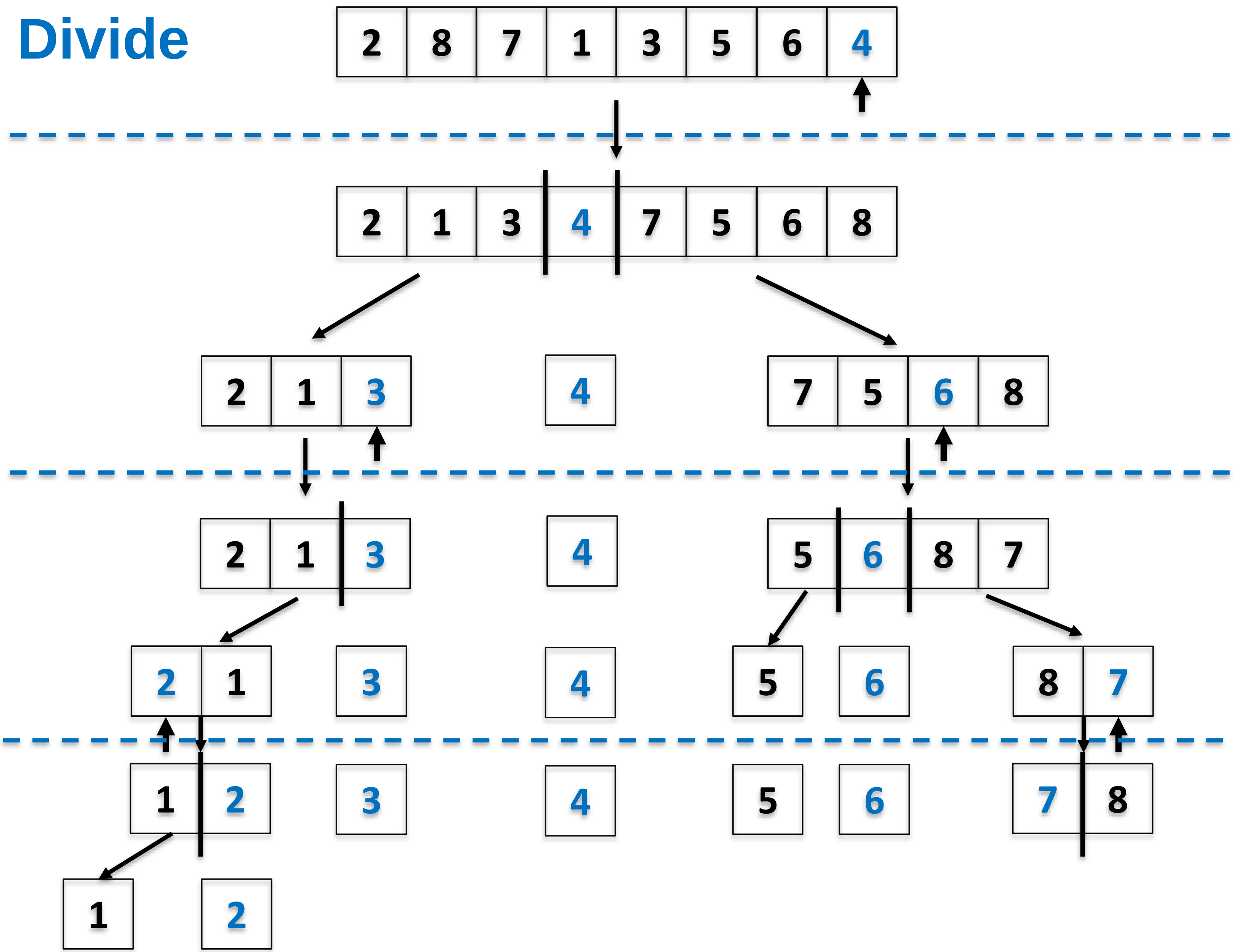
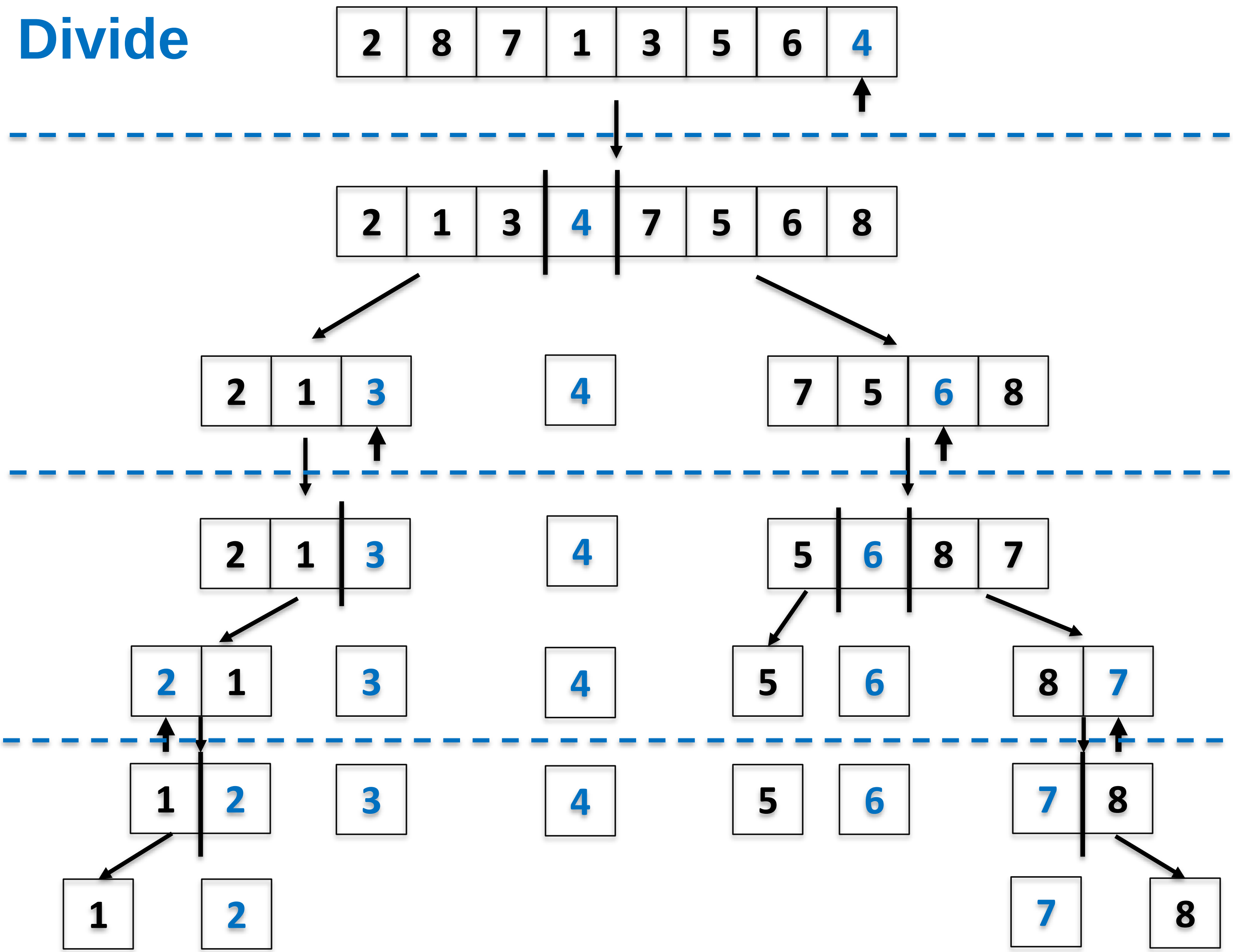


Quicksort - Example

Divide



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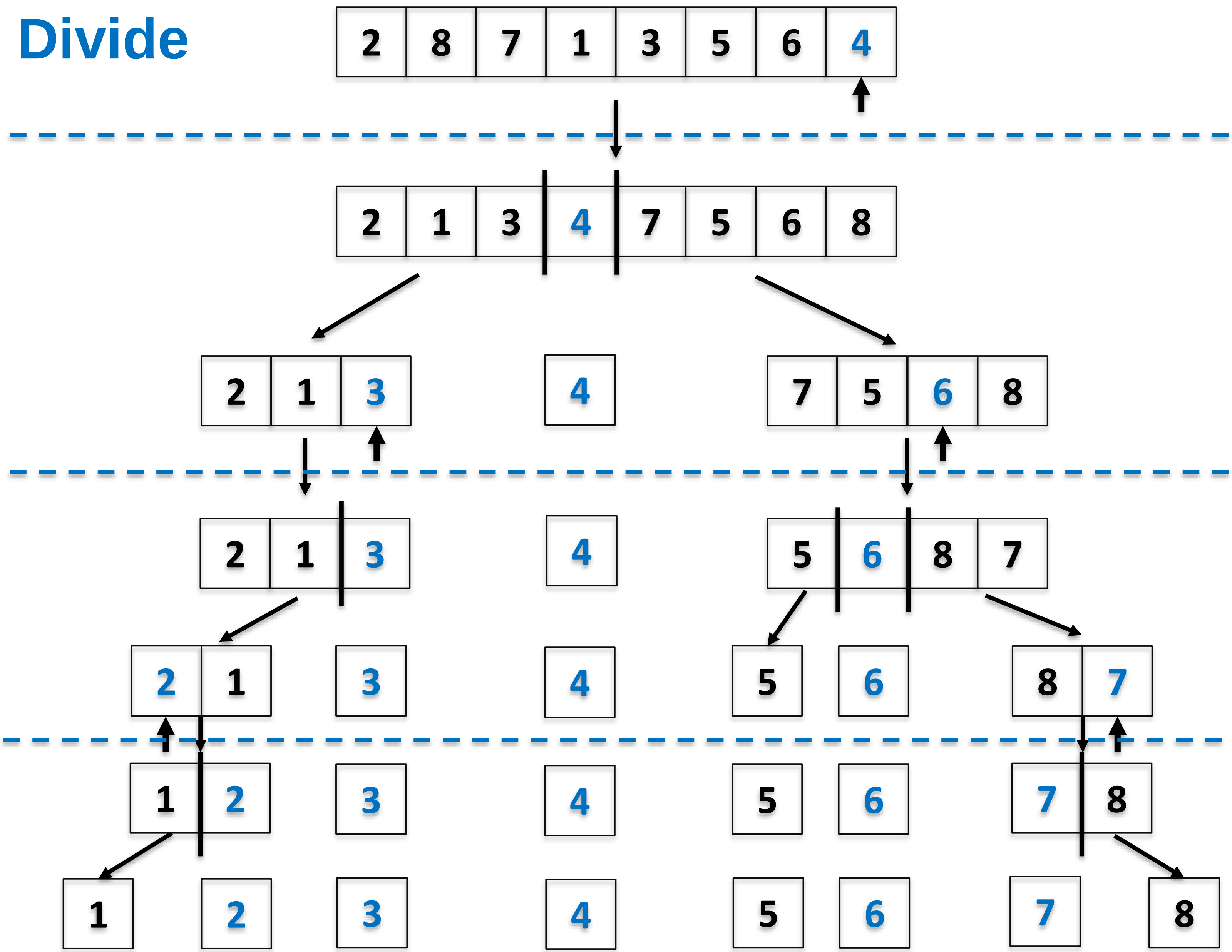


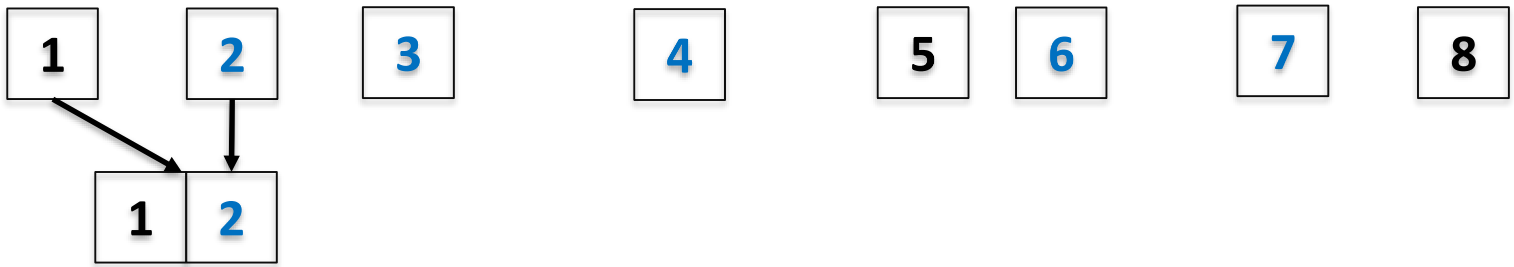
Quicksort - Example

Conquer

1	2	3	4	5	6	7	8

Quicksort - Example

Conquer



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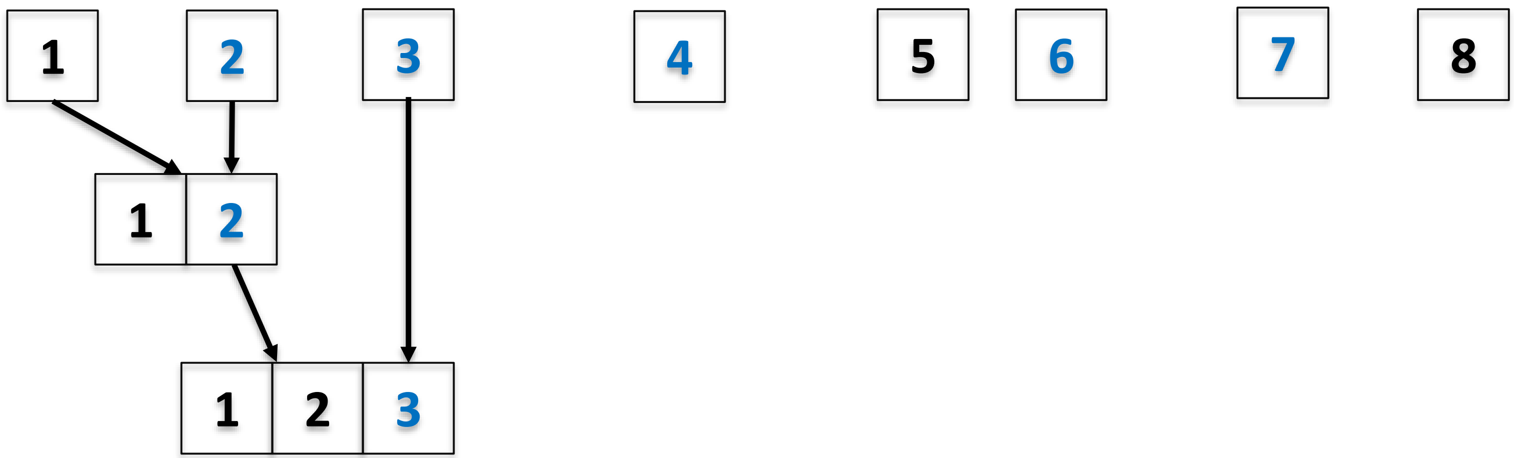

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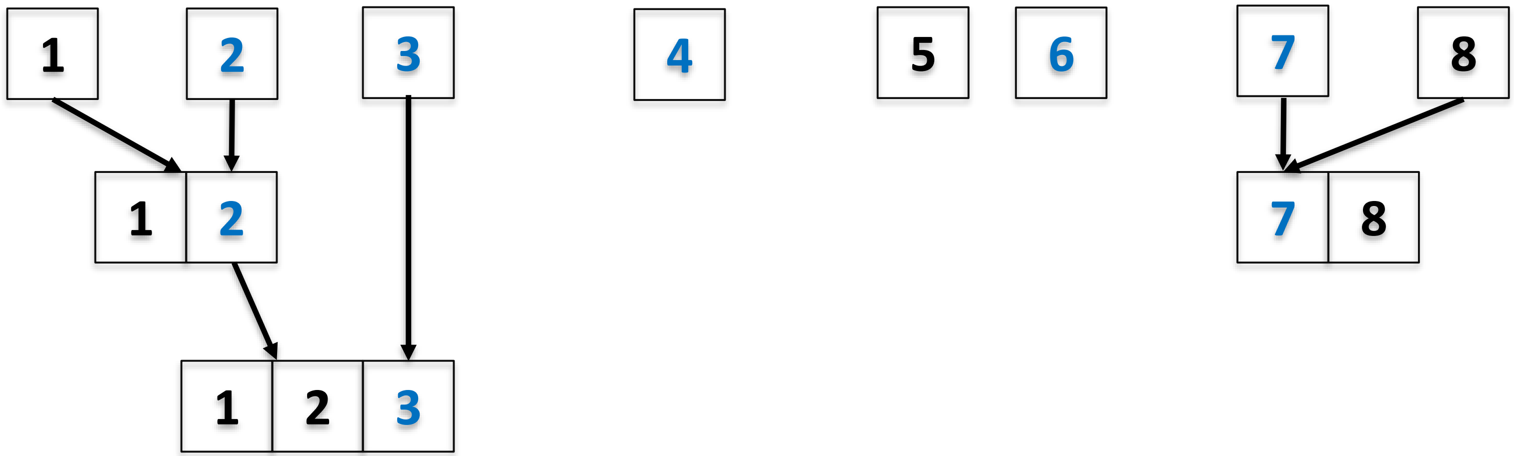

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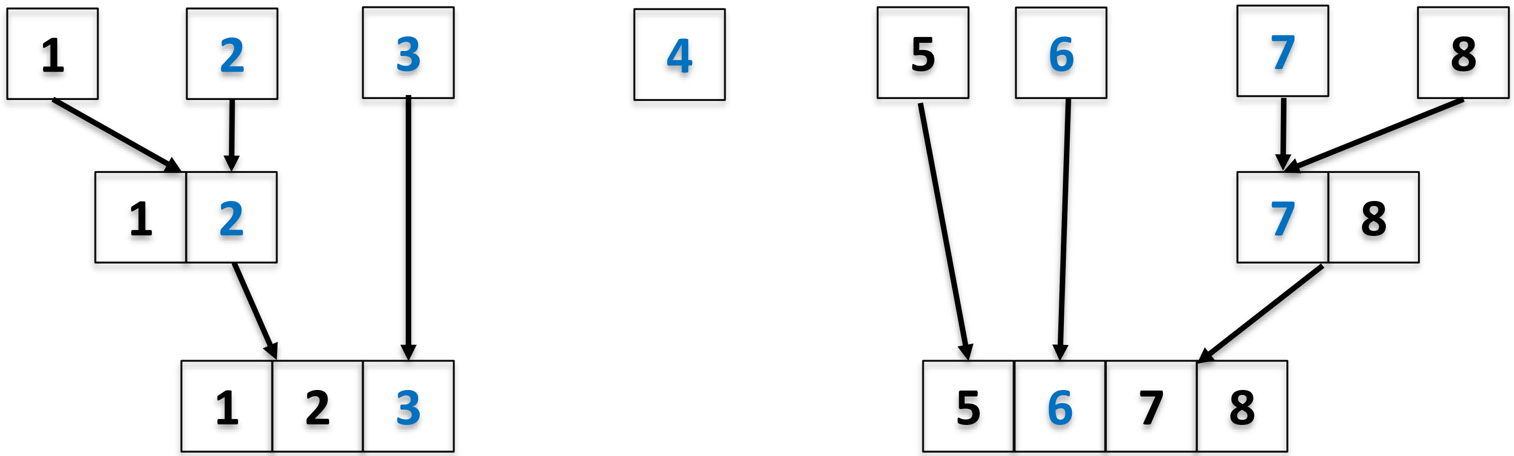

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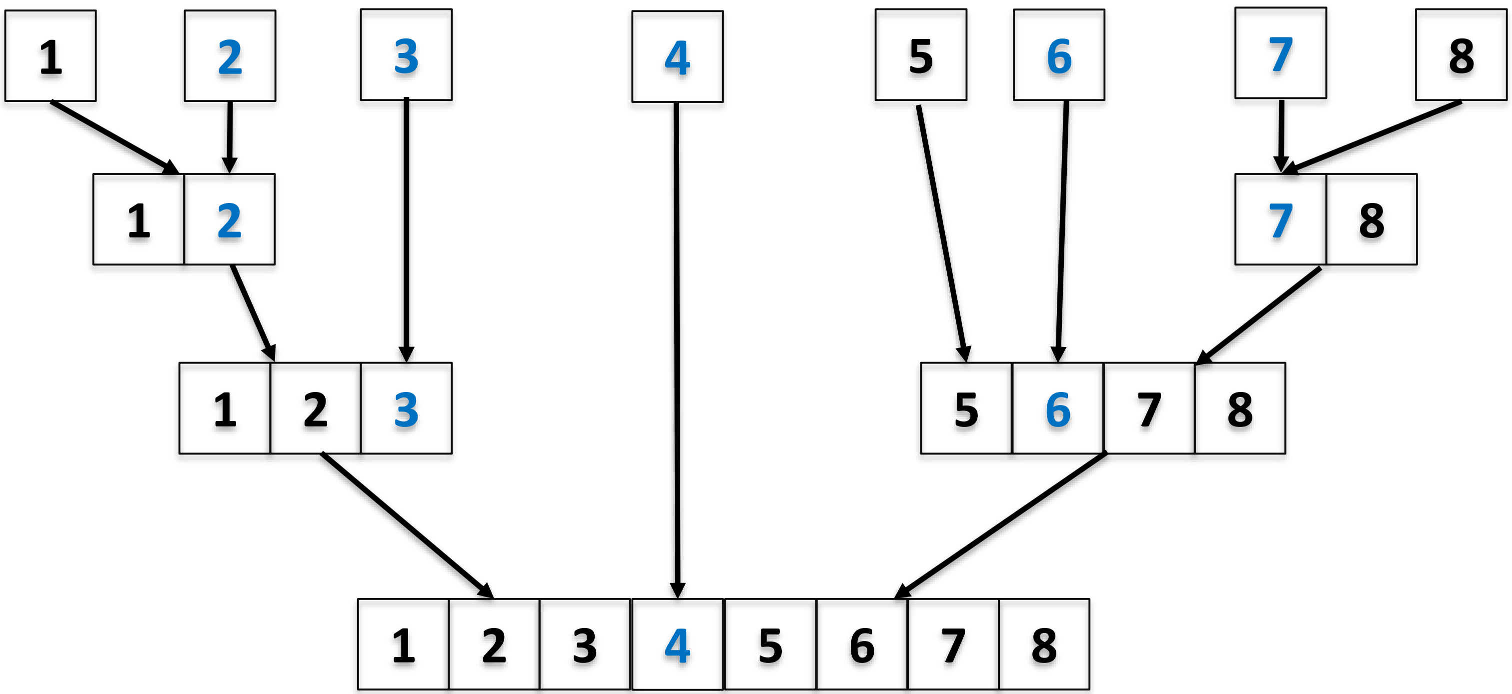

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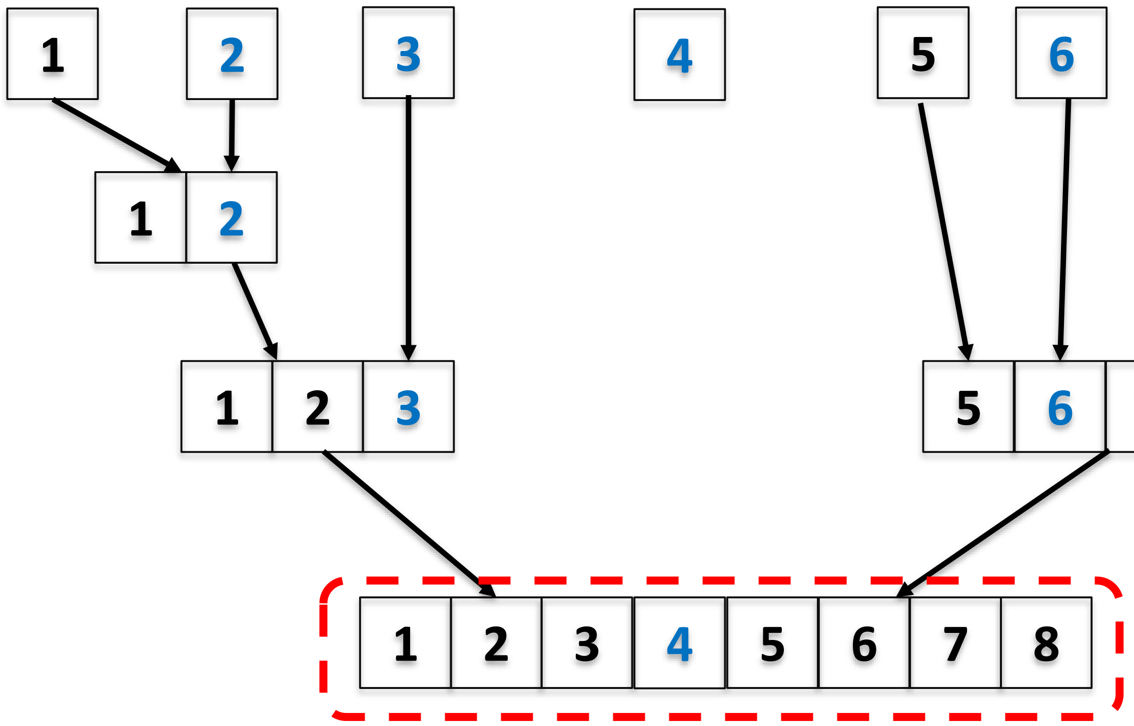

Outline

- Review to Divide-and-Conquer Paradigm
- Polynomial Multiplication Problem
  - Problem definition
  - A brute force algorithm
  - A first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Quicksort Problem
  - Basic partition
  - Randomized partition and randomized quicksort
  - Analysis of the randomized quicksort

Running Time of the Quicksort

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  - Worst case performance results only if the random number generator always produces the worst choice.

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Expected Case

- Two methods to analyze the expected running time of a divide-and-conquer randomized algorithm:
  - Old fashioned: Write our a recurrence on T(n), where T(n) is the expected running time of the algorithm on an input of size n, and solve it.
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  - New: Indicator variables.
    —— Simple and elegant, but needs practice to master.

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  - Elements in different partitions are never compared with each other in all operations

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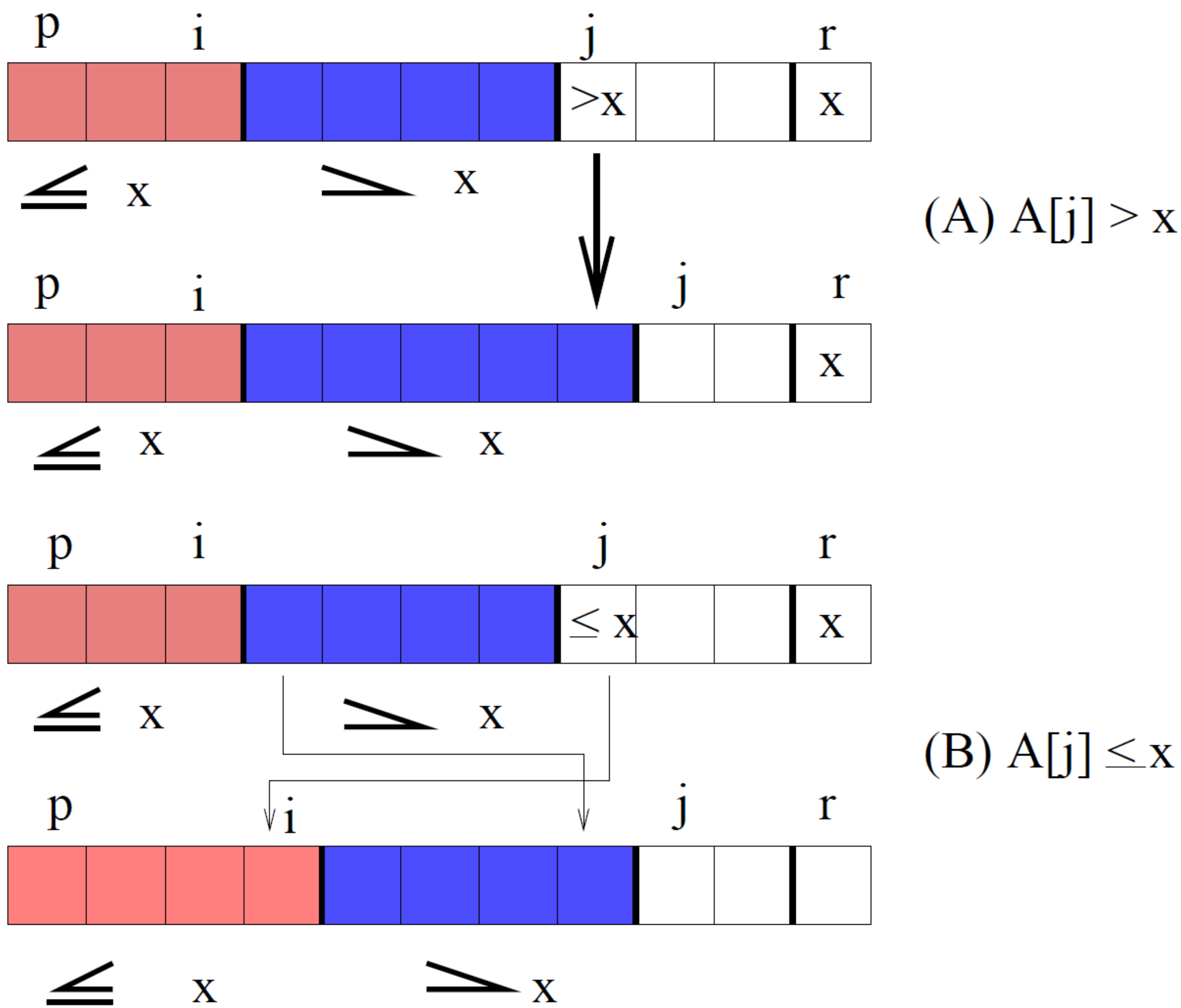

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- If the pivot is either z_i or z_j
  - z_i and z_j will be compared
- If the pivot is any element in Z_{ij} other than z_i or z_j
  - z_i and z_j are not compared with each other in all randomized-partition calls

How to Find $\Pr\{z_i \text{ is compared with } z_j\}$?

$$\Pr\{z_i\ is\ compared\ with\ z_j\}$$

$$=$$

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\end{aligned}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \le \log(n)$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \text{ is compared with } z_j\}$$
$$= \Pr\{z_i \text{ or } z_j \text{ is the first pivot chosen from } Z_{ij}\}$$
$$= \Pr\{z_i \text{ is the first pivot chosen from } Z_{ij}\}$$
$$+ \Pr\{z_j \text{ is the first pivot chosen from } Z_{ij}\}$$
$$= \frac{1}{j-i+1} + \frac{1}{j-i+1} = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

How to Find Pr{z_i is compared with z_j}?

$$\Pr\{z_i \textit{ is compared with } z_j\}$$
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$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \textit{ is compared with } z_j\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$
$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} = \sum_{i=1}^{n-1} O(\log n) = O(n \log n)$$

Note: $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n)$

Hence, the expected number of comparisons is $O(n \log n)$, which is the expected running time of Randomized-Quicksort


dank u
ju faleminderit
Tack
Asante
谢谢
Tak
mulţumesc
kiitos
Salamat!
Gracias
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
gam