Design and Analysis of Algorithms
Part II: Dynamic Programming
Lecture 5: 0-1 Knapsack and Rod Cutting Problems

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Outline

- Introduction to Part II

- 0-1 Knapsack Problem
  - Problem Definition
  - A Bruteforce Algorithm
  - A Dynamic Programming Algorithm
  - Analysis of DP Algorithm

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  - Recursively define the value of an optimal solution
  - Compute the value of an optimal solution (usually bottom-up)

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- In Part II, we will illustrate Dynamic Programming (DP) using several examples:
  - 0-1 Knapsack (0–1背包)
  - Rod-Cutting (钢条切割)
  - Chain Matrix Multiplication (矩阵链乘法)
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Question

How should we select the items?

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$$V[i,w] = \max(V[i-1,w], v_i + V[i-1,w-w_i])$$

Simple Recursion

KnapsackSR(i,w)

Input: Candidate item set $\{1, 2, ..., \boldsymbol{i}\}$, allowed maximum weight of items $\boldsymbol{w}$.

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- Why? We invoke the same function call too many times!

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$$
\begin{tikzcd}[column sep=tiny]
 & & & V(n,W) \arrow[dll, no head] \arrow[drr, no head] & & & \\
 & V(n-1,W) \arrow[dl, no head] \arrow[dr, no head] & & & & V(n-1,W-1) \arrow[dl, no head] \arrow[dr, no head] & \\
V(n-2,W) & & {\color{red}V(n-2,W-1)} & & {\color{red}V(n-2,W-1)} & & V(n-2,W-2)
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- The recursion is why we chose the problem space {V[i, w] : 1 ≤ i ≤ n; 0 ≤ w ≤ W} at the first place.

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Step 3: Bottom-up computation of V [i, w]

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$$V[i,w] = \max(V[i-1,w], v_i + V[i-1,w-w_i])$$

We compute and save V[i, w]($0 \le i \le n$, $0 \le w \le W$) in such an order that:

Developing a DP Algorithm for Knapsack

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Developing a DP Algorithm for Knapsack

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So, we fill the following table row by row and left to right.

V[i,w]	w=0	1	2	3	…	…	W	
i= 0	0	0	0	0	...	...	0	bottom
1								
2								
⋮								
n								

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$ **0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$										

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0

Intialization

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1											

w[i]>w

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0						

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5		6	3

$V[i]$

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0			0	0	0
1	0				0						

$V[i-1,w-w[i]]$

$V[i-1,w]$

$V[i]+V[i-1,w-w[i]]>V[i-1,w]$

$$V[i,w]=\max(V[i-1,w],v_i+V[i-1,w-w_i])$$

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10					

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10				

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i, w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10			

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10		

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0							

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40						

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40					

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = \mathbf{0}$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40				

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40		

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40					

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40					

$V[i] + V\left[i - 1, j - w[i]\right] < \boldsymbol{V}[\boldsymbol{i} - \mathbf{1}, \boldsymbol{j}]$

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40				

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40			

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40		

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0								

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50							

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50						

An Example of the Bottom-up Computation

i	1	2	3	4
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$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50					

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50				

An Example of the Bottom-up Computation

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90			

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90		

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

An Example of the Bottom-up Computation

i	1	2	3	4
v_i	10	40	30	50
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$V[i,w]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

Max value = 90

The Dynamic Programming Algorithm

Knapsack(v,w,n,W)

Input: $\boldsymbol{v}$ and $\boldsymbol{w}$ are values and weights of $\boldsymbol{n}$ items, $\boldsymbol{W}$ is the allowed maximum weight of items.

Output: Maximum value of any subset of items $\{1, 2, ..., n\}$ of weight at most W.

The Dynamic Programming Algorithm

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Let $V[0..n, 0..W]$ be a new 2-dimension array;

The Dynamic Programming Algorithm

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Output: Maximum value of any subset of items $\{1, 2, ..., n\}$ of weight at most W.

Let $V[0..n, 0..W]$ be a new 2-dimension array;

for $i \leftarrow 0$ *to* W **do**

  $V[0, i] \leftarrow 0$;

end

The Dynamic Programming Algorithm

Knapsack(v,w,n,W)

Input: $\boldsymbol{v}$ and $\boldsymbol{w}$ are values and weights of $\boldsymbol{n}$ items, $\boldsymbol{W}$ is the allowed maximum weight of items.

Output: Maximum value of any subset of items $\{1, 2, ..., n\}$ of weight at most W.

Let $V[0..n, 0..W]$ be a new 2-dimension array;

for $w = 0$ **to** W **do**
  $V[0, w] = 0$
end
for $i = 1$ **to** n **do**
  for $w = 0$ **to** W **do**
    if $w[i] \leq w$ **then**
      $V[i, w] = \max\{V[i-1, w], v[i] + V[i-1, w-w[i]]\}$
    else
      $V[i, w] = V[i-1, w]$
    end
  end
end
return $V[n, W]$

Constructing the Optimal Solution

- The algorithm for computing V [i, w] described in the previous slide does not record which subset of items gives the optimal solution.

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Constructing the Optimal Solution

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Constructing the Optimal Solution

- The algorithm for computing V [i, w] described in the previous slide does not record which subset of items gives the optimal solution.
- To compute the actual subset, we can add an auxiliary boolean array keep[i, w] which is
  - 1 if we choose item i in V [i, w] and
  - 0 otherwise.

Question

How do we use all the values $keep[i, w]$ to determine the subset T of items having the maximum value?

Constructing the Optimal Solution

If keep[n, W] is 1, then $n \in T$.

Constructing the Optimal Solution...

If keep[n, W] is 1, then $n \in T$.

We can now repeat this argument for keep[$n-1$, $W-w_n$].

Constructing the Optimal Solution

If keep[n, W] is 1, then $n \in T$.

We can now repeat this argument for keep[n−1, W−w_n].

If keep[n, W] is 0, then $n \notin T$.

Constructing the Optimal Solution

If keep[n, W] is 1, then $n \in T$.

We can now repeat this argument for keep[n−1, W−w_n].

If keep[n, W] is 0, then $n \notin T$.

We repeat the argument for keep[n−1, W].

Constructing the Optimal Solution

If keep[n, W] is 1, then $n \in T$.

We can now repeat this argument for keep[n−1, $W-w_n$].

If keep[n, W] is 0, then $n \notin T$.

We repeat the argument for keep[n−1, W].

- Therefore, the following partial program will output the elements of T:

Constructing the Optimal Solution

If keep[n, W] is 1, then $n \in T$.

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getResult(W,V)

Input: Allowed maximum weight $\boldsymbol{W}$, intermediate array from Knapsack $\boldsymbol{V}$

Output: Maximum value of any subset of items $\{1, 2, ..., n\}$ of weight at most W.

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for $i \leftarrow n$ *to* 1 **do**

  if $keep[i, K]$ *is equal to* 1 **then**

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    Output i;

    $K \leftarrow$

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If keep[n, W] is 1, then $n \in T$.

We can now repeat this argument for keep[n−1, W−w_n].

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We repeat the argument for keep[n−1, W].

- Therefore, the following partial program will output the elements of T:

getResult(W,V)

Input: Allowed maximum weight $\boldsymbol{W}$, intermediate array from Knapsack $\boldsymbol{V}$

Output: Maximum value of any subset of items $\{1, 2, ..., n\}$ of weight at most W.

$K \leftarrow W$;
for $i \leftarrow n$ *to* 1 **do**
  if *keep*$[i, K]$ *is equal to* 1 **then**
    Output i;
    $K \leftarrow K - w[i]$;
  end
end

The Complete Algorithm for the Knapsack Problem

Let $V[0..n, 0..W]$ and $keep[0..n, 0..W]$ be two new 2-dimension arrays;

The Complete Algorithm for the Knapsack Problem

```
Let V[0..n, 0..W] and keep[0..n, 0..W] be two new 2-dimension arrays;
for w = 0 to W do V[0, w] = 0;
for i = 1 to n do
    for w = 0 to W do
        if (w[i] ≤ w) and (v[i] + V[i − 1, w − w[i]] > V[i − 1, w]) then
            V[i, w] = v[i] + V[i − 1, w − w[i]];
            keep[i, w] = 1;
        else
            V[i,w]= V[i-1,w];
            keep[i, w] = 0;
        end
    end
end
K = W;
for i = n downto 1 do
    if keep[i, K] == 1 then
        output i;
        K= K-w[i];
    end
end
return V[n, W]
```

Example of Optimal Solution Construction

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

Example of Optimal Solution Construction

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

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i	1	2	3	4
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$											

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Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0

$keep$	0	1	2	3	4	5	6	7	8	9	10

Example of Optimal Solution Construction

i	**1**	**2**	**3**	**4**
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$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$	0	0	0	0	0	0	0	0	0	0	0

Intialization

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1											

$keep$	0	1	2	3	4	5	6	7	8	9	10
1											

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1											

w[i]>j

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i = 1$											

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i	1	2	3	4
v_i	10	40	30	50
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0						

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{i} = 1$	0	0	0	0	0						

Example of Optimal Solution Construction

i	**1**	**2**	**3**	**4**
v_i	10	40	30	50
w_i	5		6	3

W=10

$V[i]$

$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$	0	0	0	0	0	0			0	0	0
1	0				0						

$V[i-1,j-w[i]]$

$V[i-1,j]$

$V[i]+V[i-1,j-w[i]] > V[i-1,j]$

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=1$	0	0	0	0	0						

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i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10					

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1					

Example of Optimal Solution Construction

i	1	2	3	4
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10				

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$\mathbf{i = 1}$	0	0	0	0	0	1	1				

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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10			

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1			

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i	1	2	3	4
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10		

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1		

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$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	

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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1

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$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1

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i	**1**	**2**	**3**	**4**
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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0							

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0							

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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40						

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1						

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$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40					

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1					

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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40				

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1				

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$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40			

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1			

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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40		

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1		

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i	1	2	3	4
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = \mathbf{1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1

Example of Optimal Solution Construction

i	**1**	**2**	**3**	**4**
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W=10

$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3											

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3											

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i	**1**	**2**	**3**	**4**
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$\boldsymbol{i = 0}$	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{1}$	0	0	0	0	0	10	10	10	10	10	10
$\mathbf{2}$	0	0	0	0	40	40	40	40	40	50	50
$\mathbf{3}$	0	0	0	0	40	40					

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
$\mathbf{2}$	0	0	0	0	1	1	1	1	1	1	1
$\mathbf{3}$	0	0	0	0	0	0					

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i	1	2	3	4
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$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40					

$V[i] + V[i-1, j-w[i]] < V[i-1, j]$

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0					

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i	1	2	3	4
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$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40				

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i}=1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0				

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$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40			

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0			

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$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40		

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0		

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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i=1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4											

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4											

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0								

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0								

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50							

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1							

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50						

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1						

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50					

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i=1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1					

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
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W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50				

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1				

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90			

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1			

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90		

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1		

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	

$keep$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i=1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

Max value = 90

$keep$	0	1	2	3	4	5	6	7			
$i=1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathrm{i} = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

W=10   Item set = {}

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	$\boxed{1}$

W=7 Item set = {4,}

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

W=7 Item set = {4,}

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$i = 1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

W=7 Item set = {4,}

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i = 0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i = 1}$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

W=3   Item set = {4,2}

Example of Optimal Solution Construction

i	1	2	3	4
v_i	10	40	30	50
w_i	5	4	6	3

W=10

$V[i,j]$	0	1	2	3	4	5	6	7	8	9	10
$i=0$	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	10	10	10	10	10	10
2	0	0	0	0	40	40	40	40	40	50	50
3	0	0	0	0	40	40	40	40	40	50	70
4	0	0	0	50	50	50	50	90	90	90	90

$keep$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{i}=1$	0	0	0	0	0	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0	0	0	0	1
4	0	0	0	1	1	1	1	1	1	1	1

W=3 Item set = {4,2}

Outline

- Introduction to Part II
- 0-1 Knapsack Problem
  - Problem Definition
  - A Bruteforce Algorithm
  - A Dynamic Programming Algorithm
  - Analysis of DP Algorithm
- Rod Cutting Problem
  - Problem Definition
  - A Bruteforce Algorithm
  - A Dynamic Programming Algorithm

Dynamic Programming vs. Simple Recursion

- Dynamic programming saves us from having to recompute
- previously calculated subsolutions!
- The DP algorithm runs in O(   ) time.

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- The DP algorithm runs in $O(nW)$ time.
- The simple recursion algorithm runs in $\Omega(2^n)$ time.

Summary of DP

How to develop a dynamic programming?

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How to develop a dynamic programming? Four steps

- Structure: Analyze structure of an optimal solution, and thereby choose a space of subproblems (states).

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Notes:

- Steps 1 and 2 are related.
- Step 4 is not always necessary: we sometimes need only the optimal value.

Summary of DP

Dynamic programming (DP) vs Divide-and-Conquer (DC)

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    - Some subproblems might be solved many times.
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  - Solve the subproblems.
  - Combine the solutions to solve the original one.
- Differences:
  - DC:
    - Efficient when the subproblems are independent.
    - Not efficient when subproblems share subsubproblems.
    - Some subproblems might be solved many times.
  - DP:
    - Suitable when subproblems share subsubproblems.
    - Do each subproblem only once.

Summary of DP

Dynamic programming (DP) vs Divide-and-Conquer (DC)

- Commonalities:
  - Partition the problem into particular subproblems.
  - Solve the subproblems.
  - Combine the solutions to solve the original one.
- Differences:
  - DC:
    - Efficient when the subproblems are independent.
    - Not efficient when subproblems share subsubproblems.
    - Some subproblems might be solved many times.
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Outline

- Introduction to Part II

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- Rod Cutting Problem
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- Example: Consider n = 4 and $p_1=1$, $p_2=5$, $p_3=8$, $p_4=9$
  - If we do not cut the rod, we can earn $p_4 = 9$
  - If we cut it into 4 pieces of length 1 each, we can earn $4 \cdot p_1 = 4$

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- Example: Consider n = 4 and p_1=1, p_2=5, p_3=8, p_4=9
  - If we do not cut the rod, we can earn $p_4 = 9$
  - If we cut it into 4 pieces of length 1 each, we can earn $4 \cdot p_1 = 4$
  - If we cut it into 2 pieces of length 2 each, we can earn $2 \cdot p_2 = 10$

Rod Cutting

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  - There are more options, but the maximum revenue is 10

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  - If we cut it into 2 pieces of length 2 each, we can earn $2 \cdot p_2 = 10$
  - There are more options, but the maximum revenue is 10
- In general, we can cut the rod of length n in 2^{n-1} different ways, since we have an independent option of cutting, or not cutting, at distance i ($1 \le i \le n - 1$) from the left end

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Optimal Solution

- We can define the maximum revenue r_n in terms of optimal revenues for shorter rods

$$\mathrm{r}_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \dots, r_{n-1} + r_1)$$

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- Simpler definition

$$\mathrm{r}_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

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  - We cut a piece of length i, and a remainder of length n− i

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- Simpler definition

$$\mathrm{r}_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

  - We cut a piece of length i, and a remainder of length n− i
  - Only the remainder, and not the first piece, may be further divided

Recursive Top-down Implementation

Cut-Rod(p,n)

```
Input: Price list p, a rod of length n.
Output: Maximum revenue q.
if n is equal to 0 then
 |  return 0;
end
q ← −∞; for i ← 1 to n do
 |  q ←max(q, p[i]+Cut-Rod(p, n − i));
end
return q;
```

Recursive Top-down Implementation

Cut-Rod(p,n)

Input: Price list $\boldsymbol{p}$, a rod of length $\boldsymbol{n}$.
Output: Maximum revenue $\boldsymbol{q}$.
if n *is equal to* 0 **then**
  | **return** 0;
end
$q \leftarrow -\infty$; **for** $i \leftarrow 1$ *to* n **do**
  | $q \leftarrow \max(q, p[i] + \text{Cut-Rod}(p, n-i))$;
end
return q;

Cost

Recursive Top-down Implementation

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- T(n): the total number of calls made to Cut-Rod when called with rod length n

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Cost

- T(n): the total number of calls made to Cut-Rod when called with rod length n

$$T(n) = \begin{cases} 1 + \sum_{0 \le j \le n-1} T(j), & if\ n > 0 \\ 1, & if\ n = 0 \end{cases}$$

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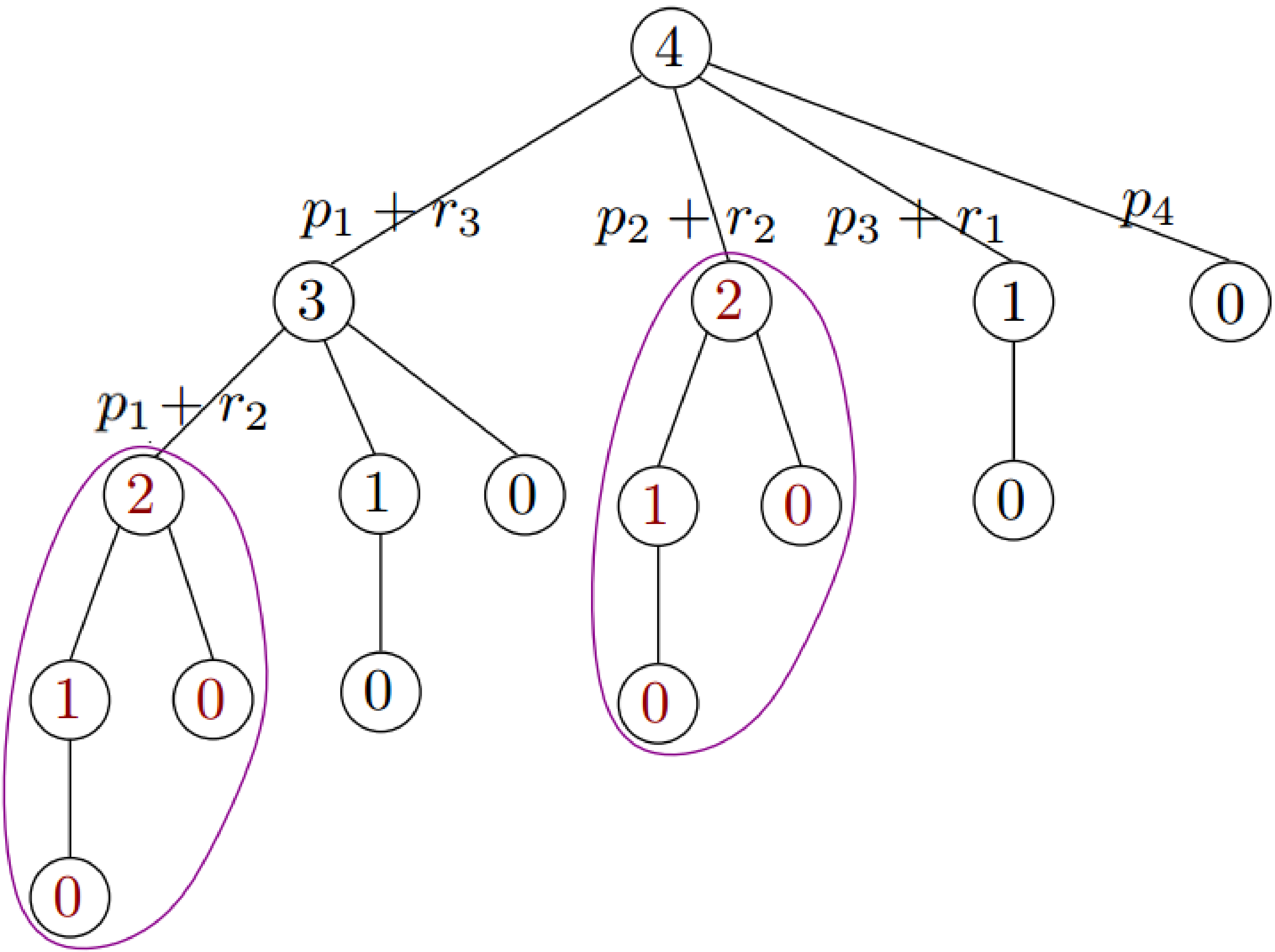
- By induction, we have $T(n)=2^n$

Explanation of Exponential Cost

- We solve the same subproblem many times



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Concept of DP

- When you solve a subproblem, store the solution
  - Next time you find the same subproblem, lookup the solution, instead of solving it again
  - Use space to save time

- Two main methodologies: top-down and bottom-up
  - Corresponding algorithms have the same asymptotic cost, but bottom-up is usually faster in practice

- Main idea of bottom-up DP
  - We sort the subproblems in size and solve the smallest subproblem first

DP Bottom-up Implementation

Bottom-Up-Cut-Rod(p,n)

```
Input: Price list p, a rod of length n.
Output: Maximum revenue q.
Let r[0..n] be a new array;// Array stores the computed optimal values.
r[0] ← 0;
for j ← 0 to n do
    // Consider problems in increasing order of size.
    q ← −∞;
    for i ← 1 to j do
        // To solve a problem of size j, we need to consider all
        decompositions into i and j − i.
        q ←max(q, p[i] + r[j − i]);
    end
    r[j] ← q;
end
return r[n];
```

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end
return r[n];
```

- Cost: $O(n^2)$

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    end
    r[j] ← q;
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- Cost: $O(n^2)$
  - The outer loop computes r[1], r[2],…, r[n] in this order

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  |   | decompositions into i and j − i.
  |   | q ←max(q, p[i] + r[j − i]);
  | end
  | r[j] ← q;
end
return r[n];
```

- Cost: $O(n^2)$
  - The outer loop computes r[1], r[2],…, r[n] in this order
  - To compute r[j], the inner loop uses all values r[0], r[1],…, r[j − 1] (i.e., r[j − i] for 1 ≤ i ≤ j)

Extended Implementation

Extended-Bottom-Up-Cut-Rod(p,n)

```
Input: Price list p, a rod of length n.
Output: Maximum revenue q and sizes of pieces.
Let r[0..n] and s[0..n] be two new arrays;
r[0] ← 0;
for j ← 0 to n do
    q ← −∞;
    for i ← 1 to j do
        // Solve problem of size j.
        if q < p[i] + r[j − i] then
            q ← p[i] + r[j − i];
            s[j] ← i;// Store the size of the first piece.
        end
    end
    r[j] ← q;
end
while n > 0 do
    Output s[n];
    n ← n − s[n];
end
```

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p}[\boldsymbol{k}]$	1	5	8	9	10	17	17	20	24	26

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$											
$s[k]$											

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0										
$s[k]$											

Intialization

Example of Optimal Solution Construction

n=10

k	1	2		4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

$p[i]$

j = 1

i	1
	1

$p[i] + r[j - i]$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0										
$s[k]$											

$r[j - i]$

Example of Optimal Solution Construction

n=10

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$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 1

i	1
	$\boxed{1}$

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0										
$s[k]$											

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j = 1

$\boldsymbol{i}$	**1**
	1

$\boldsymbol{max\{p[i] + r[j-i]\}}$

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1									
$s[k]$		1									

Example of Optimal Solution Construction

n=10

k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 2

i	**1**	**2**
	2	5

k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1									
$s[k]$		1									

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	2	5

$max\{p[i] + r[j-i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1									
$s[k]$		1									

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	1	2	3	4	5	6	7	8	9	10
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 2

$\boldsymbol{i}$	1	2
	2	5

$\boldsymbol{max\{p[i] + r[j-i]\}}$

$\boldsymbol{k}$	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5								
$s[k]$		1	2								

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 3

i	1	2	3
	6	6	8

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5								
$s[k]$		1	2								

Example of Optimal Solution Construction

n=10

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	6	6	8

$\boldsymbol{max\{p[i] + r[j-i]\}}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5								
$s[k]$		1	2								

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j = 3

i	1	2	3
	6	6	$\boxed{8}$

$max\{p[i] + r[j-i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8							
$s[k]$		1	2	3							

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 4

i	1	2	3	4
	9	10	9	9

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8							
$s[k]$		1	2	3							

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k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

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i	1	2	3	4
	9	10	9	

$max\{p[i] + r[j-i]\}$

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$r[k]$	0	1	5	8							
$s[k]$		1	2	3							

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k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 4

i	1	2	3	4
	9	10	9	

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10						
$s[k]$		1	2	3	2						

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 5

i	1	2	3	4	5
	11	13	13	10	10

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10						
$s[k]$		1	2	3	2						

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$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 5

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**
	11	13			

$\boldsymbol{max\{p[i]+r[j-i]\}}$

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10						
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n=10

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$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 5

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**
	11	13	1[illegible]		

$\boldsymbol{max\{p[i] + r[j-i]\}}$

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13					
$s[k]$		1	2	3	2	2					

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 6

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**	**6**
	14	15	16	14	11	17

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13					
$s[k]$		1	2	3	2	2					

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 6

i	1	2	3	4	5	6
	14	15	16	14	11	17

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13					
$s[k]$		1	2	3	2	2					

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 6

i	1	2	3	4	5	6
	14	15	16	14	11	17

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17				
$s[k]$		1	2	3	2	2	6				

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p}[\boldsymbol{k}]$	1	5	8	9	10	17	17	20	24	26

j = 7

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
	18	18	18	17	15	18	17

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13	17				
$s[k]$		1	2	3	2	2	6				

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 7

i	1	2	3	4	5	6	7
	$\boxed{18}$	18				[illegible]	17

$$max\{p[i] + r[j-i]\}$$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17				
$s[k]$		1	2	3	2	2	6				

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 7

i	1	2	3	4	5	6	7
	18	18				[illegible]	17

$$max\{p[i] + r[j - i]\}$$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18			
$s[k]$		1	2	3	2	2	6	1			

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 8

i	1	2	3	4	5	6	7	8
	19	22	21	19	18	22	18	20

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18			
$s[k]$		1	2	3	2	2	6	1			

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 8

i	1	2	3	4	5	6	7	8
	19	22	21					20

$\boldsymbol{max\{p[i] + r[j - i]\}}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18			
$s[k]$		1	2	3	2	2	6	1			

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 8

i	1	2	3	4	5	6	7	8
	19	22	21					20

$$\max\{p[i] + r[j-i]\}$$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22		
$s[k]$		1	2	3	2	2	6	1	2		

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 9

i	1	2	3	4	5	6	7	8	9
	23	23	25	22	20	25	22	21	24

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22		
$s[k]$		1	2	3	2	2	6	1	2		

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 9

i	1	2	3	4	5	6	7	8	9
	23	23	25	22					24

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22		
$s[k]$		1	2	3	2	2	6	1	2		

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 9

i	1	2	3	4	5	6	7	8	9
	23	23	25	22	[illegible]	[illegible]	[illegible]	[illegible]	24

$max\{p[i] + r[j - i]\}$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	
$s[k]$		1	2	3	2	2	6	1	2	3	

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 10

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	26	27	26	26	23	27	25	25	25	26

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13	17	18	22	25	
$s[k]$		1	2	3	2	2	6	1	2	3	

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 10

i	1	2	3	4	5	6	7	8	9	10
	26	$\boxed{27}$						25	25	26

$$max\{p[i] + r[j - i]\}$$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	
$s[k]$		1	2	3	2	2	6	1	2	3	

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 10

i	1	2	3	4	5	6	7	8	9	10
	26	27	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	25	25	26

$$max\{p[i] + r[j - i]\}$$

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 10

i	1	2	3	4	5	6	7	8	9	10
	26	27	26	26	23	27	25	25	25	26

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Max revenue = r[10] = 27

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 10

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	26	27	26	26	23	27	25	25	25	26

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Max revenue = r[10] = 27

Pieces of rods = { 2,

Example of Optimal Solution Construction

n=10

$\boldsymbol{k}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{p[k]}$	1	5	8	9	10	17	17	20	24	26

j = 10

$\boldsymbol{i}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	26	27	26	26	23	27	25	25	25	26

$\boldsymbol{k}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Max revenue = r[10] = 27

Pieces of rods = { 2, 2

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 10

i	1	2	3	4	5	6	7	8	9	10
	26	27	26	26	23	27	25	25	25	26

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Max revenue = r[10] = 27

Pieces of rods = { 2, 2, 6

Example of Optimal Solution Construction

n=10

k	1	2	3	4	5	6	7	8	9	10
$p[k]$	1	5	8	9	10	17	17	20	24	26

j = 10

i	1	2	3	4	5	6	7	8	9	10
	26	27	26	26	23	27	25	25	25	26

k	0	1	2	3	4	5	6	7	8	9	10
$r[k]$	0	1	5	8	10	13	17	18	22	25	27
$s[k]$		1	2	3	2	2	6	1	2	3	2

Max revenue = r[10] = 27

Pieces of rods = { 2, 2, 6}


dank u
ju faleminderit
Tack
谢谢
Tak
mulţumesc
Asante
kiitos
Gracias
Salamat!
Terima kasih
Aliquam
Merci
Dankie
Obrigado
ありがとう
köszönöm
grazie
Aliquam
Go raibh maith agat
děkuji
Thank you
gam