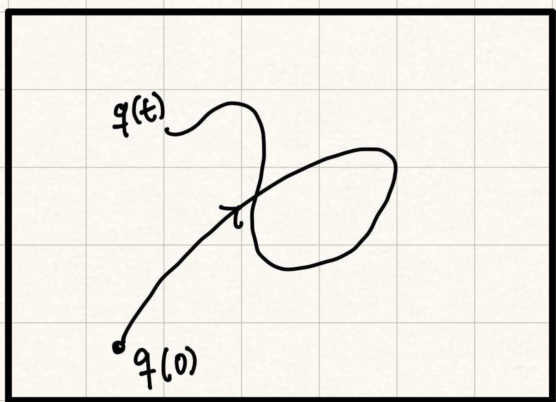
Classical Mechanics

Lecture #9

Hamiltonian Formulation of Mechanics:

In the Lagrangian formulation of classical mechanics the configuration space is the fundamental arena in which the dynamics take place. But a path through the configuration space does not tell the dynamics readily.



One also needs to specify $\dot{q}_i$, the generalized velocities. The Euler-Lagrange equations:

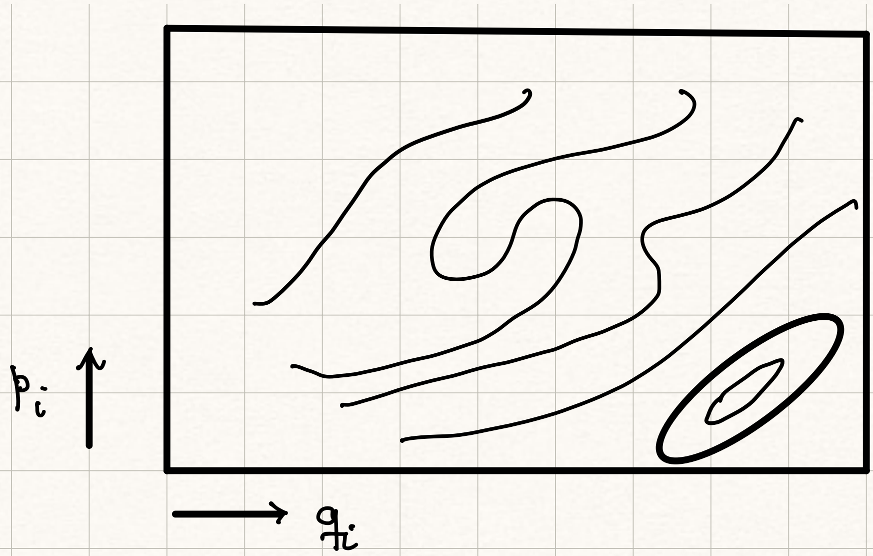
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1, \ldots, 3N$$

are second order in time-derivatives. So their solutions require that we specify $q_i(0)$ and $\dot{q}_i(0)$.

There is another formulation of classical mechanics in which the generalized coordinates and the generalized momenta (another name for the canonically conjugate momenta) are treated on an equal footing. In this formulation the dynamics of the system is described by a space which is 2N dimensional with coordinates given by $q_i(t)$ & $p_i(t)$, where $p_i(t)$ is given by

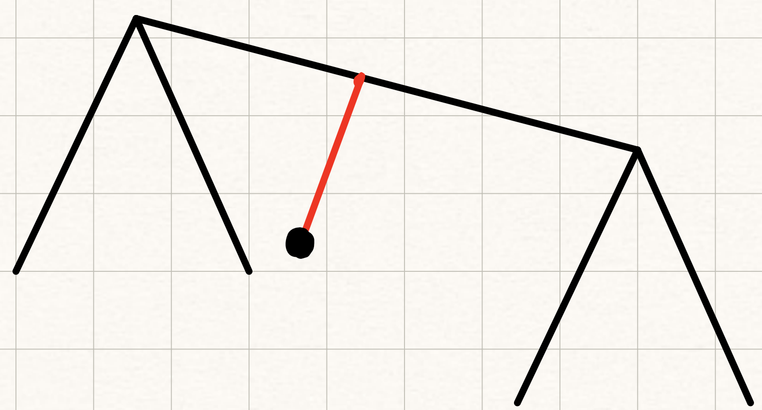
$$p_i(t) = \frac{\partial L}{\partial \dot{q}_i}$$

This space is called the phase space and a physical trajectories in this space are described by non-self-intersecting curves, unless the motion of the system is periodic in which case the trajectories are closed loops.



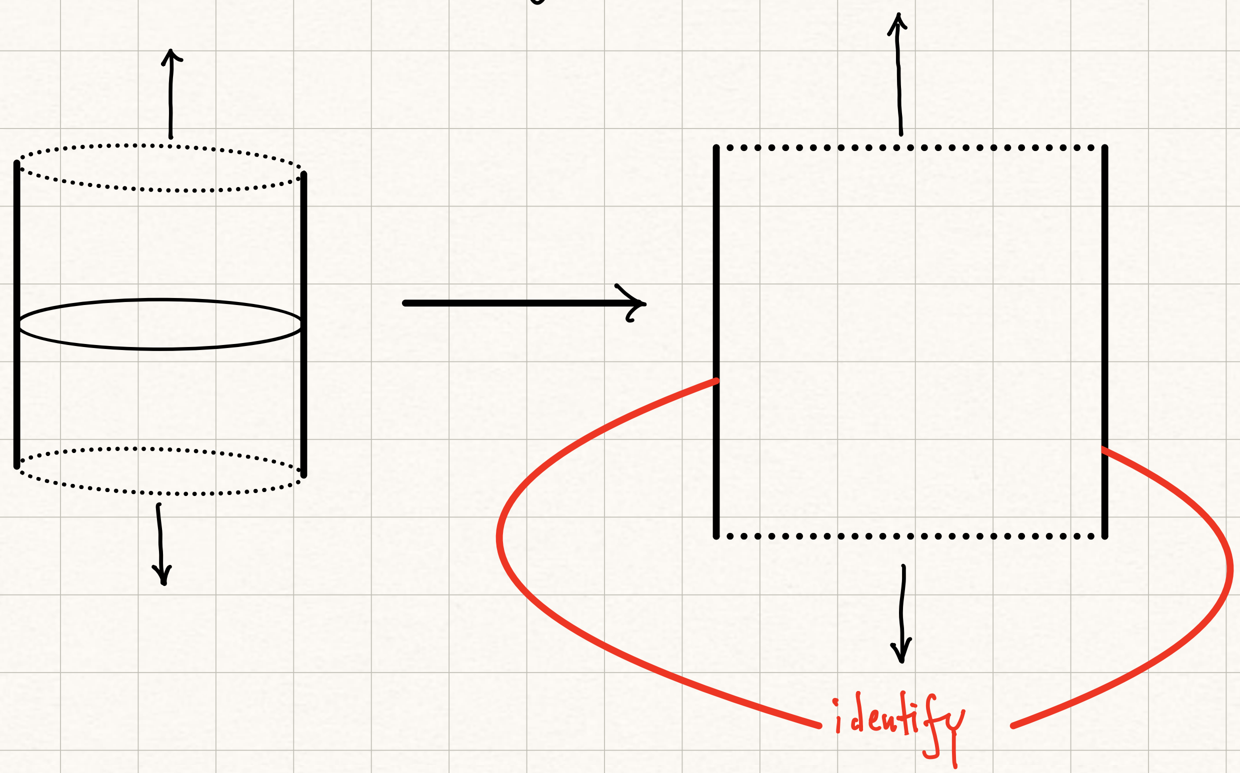
An example of a phase space.

An Example:



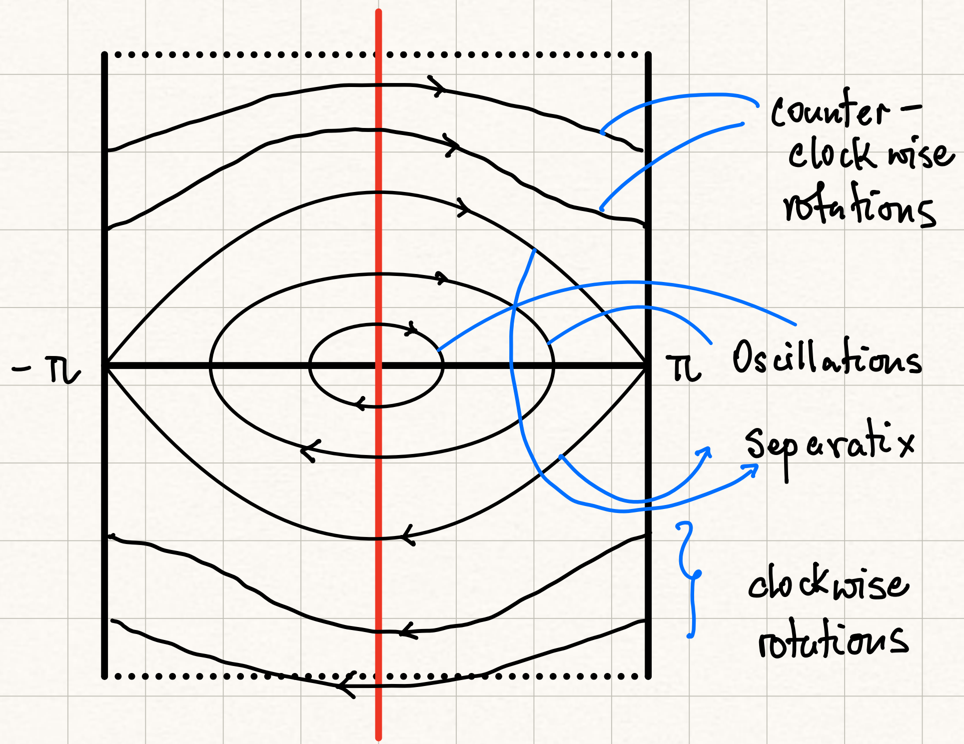
Consider a simple harmonic oscillator which is made out of a light rigid rod and a massive bob and is free to swing through a full rotation. Then the generalized coordinate is θ

and the angular momentum is the generalized momentum p_θ. The phase space is $S^1 \times \mathbb{R}^1$, where $\theta \in S^1$ and $p_\theta \in \mathbb{R}^1$.



$S^1 \times \mathbb{R}^1$ can be represented as an interval $\times\ \mathbb{R}^1$ with the opposite edges of the interval identified.

The trajectories are of the form shown below:



We see three kinds of trajectories:

1. Oscillations — Orbits are closed but are simply connected

2. Rotations — Orbits are closed but are not simply connected.

3. Separatixes: This is a trajectory that separates the two kinds of trajectories. It begins with the bob in the upright position and then falling

through a complete rotation back to the upright position.

From the Lagrangian to the Hamiltonian

Just as the Lagrangian determines the dynamics in the Lagrangian formulation via the Euler-Lagrange equations, in the Hamitonian formulation the dynamics is determined by the Hamiltonian:

$$\boxed{H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)}$$

This transformation is called the Legendre transformation and it ensures that if the Lagrangian depends on $q_i(t)$ and $\dot{q}_i(t)$, the Hamitonian depends on $q_i(t)$ and $p_i(t)$. The Legendre transformation is a reversible transformation.

An aside on Legendre Transformation:

The Legendre transformation is an extremely useful trick. It allows us to change the independent variable of a function in a way such that no information is lost in the process.

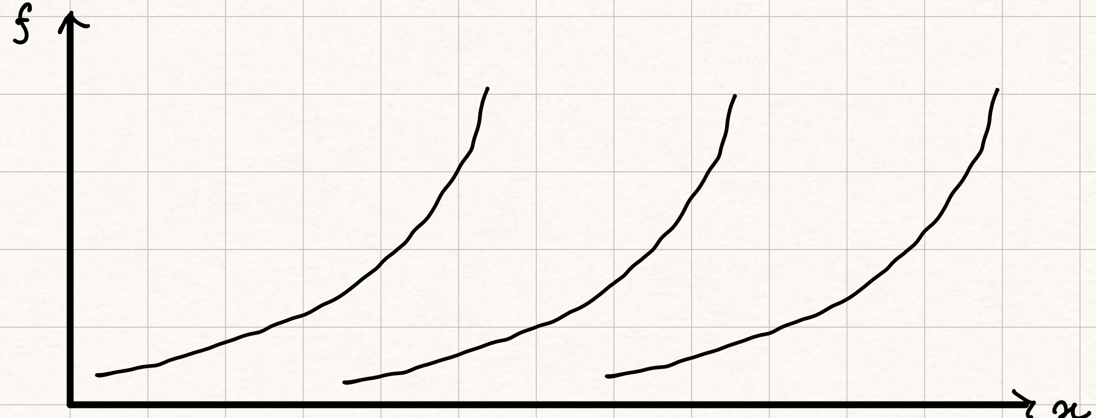
In thermodynamics, statistical mechanics, and quantum field theory, one works with a potential or a generating function(al) that is appropriate to the setting. The Legendre transformation allows one to switch between different settings.

The Legendre transformation in a simple setting:

Let us consider a function $f(x)$ where x is the independent variable. Now if we chose to change to $u = \frac{df}{dx}$ as the independent variable what quantity should we choose to as the dependent variable so that we can reconstruct the function $f(x)$?

(a) (b)

If we take f to be the dependent variable then we see that we can not reconstruct $f(x)$:



(c) Many functions give rise to the graph (b).

But if, in addition to u we specify the intercept g then we can reconstruct f. In fact, the equation for the tangent at x is

$$f = ux + g$$

And so $g = f - ux$ is the function of u which allows us to reconstruct $f(x)$.

Application to functions of two variables:

For the case at hand we change our independent variables from $(q_i, \dot{q}_i)$ to (q_i, p_i) and the dependent variable from L to $\mathcal{H}$:

$$\mathcal{H} = p_i \dot{q}_i - L \qquad (+)$$

Our claim is that $\mathcal{H}$ is a function of q_i and p_i (and perhaps also t):

$$\mathcal{H} \overset{?}{=} \mathcal{H}(q_i, p_i, t)$$

If the claim is true then we should have

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

From (†) we get:

$$dH = \dot{q}_i \, dp_i + \cancel{p_i \, d\dot{q}_i} - \frac{\partial L}{\partial q_i} dq_i$$
$$- \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

But we know that $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and so

$$dH = \dot{q}_i \, dp_i + \left(-\frac{\partial L}{\partial q_i}\right) dq_i + \left(-\frac{\partial L}{\partial t}\right) dt$$

Which has the correct form. We then identify

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{—} \ 1$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i} \quad \text{—} \ 2$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad \text{—} \ 3$$

If we consider $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and take its

time derivative $\dot{p}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$

Then by E-L equations we get:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \quad \text{——} \quad (\#)$$

we can then use equation ② and write

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

[Applying $L = p_i \dot{q}_i - H$ to (#) also leads to the same conclusion.]

Thus we arrive at Hamilton's equations:

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{ 2N first order equations.}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Examples:

1. A particle in a potential:

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - V(\vec{r})$$

What would be the Hamiltonian for this system? To find out we first compute the generalized momenta:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

and so $\quad H = \vec{p} \cdot \dot{\vec{r}} - L$

$$= \frac{\vec{p} \cdot \vec{p}}{m} - \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

$$H = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

And Hamilton's equations are:

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m}$$

Definition of momentum in Newtonian Phys

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -\vec{\nabla} V$$ Newton's Second Law

2. A charged particle in an electromagnetic field:

Recall that the motion of a charged particle in an electromagnetic field $(\vec{E}, \vec{B})$ is given by the Lorentz force law:

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

In electromagnetism, one can introduce a scalar potential $\varphi(\vec{x}, t)$ and a vector potential $\vec{A}(\vec{x}, t)$ by:

$$\vec{E}(\vec{x}, t) = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$$

and $\vec{B} = \vec{\nabla} \times \vec{A}$

The Lagrangian can then be expressed as

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - q\left(\varphi - \dot{\vec{r}} \cdot \vec{A}\right)$$

Recall that in the case of a charged particle Newton's third law in its simplest form breaks down. That is because some of the momentum is carried by the particle's presence in the field:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

From this we get that

$$\dot{\vec{r}} = \frac{\vec{p} - q\bar{A}}{m}$$

And so the Hamiltonian is

$$\mathcal{H} = \vec{p} \cdot \dot{\vec{r}} - L$$

$$= \vec{p} \cdot \left(\frac{\vec{p} - q\bar{A}}{m}\right) - \left[\frac{1}{2m}\left(\bar{p} - q\bar{A}\right)^2 - q\phi + \frac{q}{m}\left(\vec{p} - q\bar{A}\right) \cdot \bar{A}\right]$$

$$\boxed{\mathcal{H} = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi}$$

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p} - q\vec{A}}{m}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -q\vec{\nabla}\phi + \frac{q}{m}\sum_{i=1}^{3}\left[(p_i - qA_i)\right]\vec{\nabla}A_i$$

Conservation laws:

One of the great advantages of the Lagrangian formulation is the link between symmetries of the Lagrangian and conservation laws.

Here we mention how some of these conservation laws arise from properties of the Hamiltonian.

1. No explicit time dependence of the Hamiltonian leads to the conservation of energy.

If $\frac{\partial H}{\partial t} = 0$ then

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \underbrace{\cancel{\frac{\partial H}{\partial t}}}_{=0}$$

$$= -\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i \quad \text{[using Hamilton's equations]}$$

$$= 0.$$

2. If a coordinate q_α is ignorable then

$$\frac{\partial L}{\partial q_\alpha} = 0.$$

But it also means $\frac{\partial H}{\partial q_\alpha} = 0$

And so $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} = 0$

$\Rightarrow \quad p_\alpha = \text{Constant}.$