# Physics 110

## Chapter 1: Vectors

## Vector Products.

### Vector Products:

Several products between vectors can be defined. They are:

i) The scalar product or the dot product.

2) The cross-product.

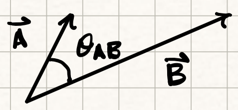
We discuss each in turn:

### The Dot/Scalar product:

The scalar product or the dot product between two vectors $\vec{A}$ & $\vec{B}$ is defined to be

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta_{AB}$$

where $\theta_{AB}$ is the angle between the two vectors.

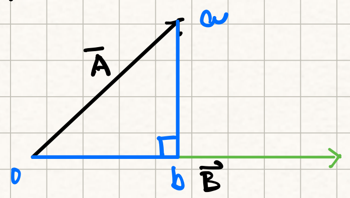



Comments:

1. The result of the dot product is a real number and **not** a vector.
2. The dot product is commutative:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

## Interpretation of the dot product:




To understand geometric interpretation of the dot product we consider the dot product between $\vec{A}$ and the unit vector in the direction of $\vec{B}$:

$$\vec{A} \cdot \hat{B} = \vec{A} \cdot \frac{\vec{B}}{|\vec{B}|} = \frac{|\vec{A}|\cancel{|\vec{B}|}\cos\theta_{AB}}{\cancel{|\vec{B}|}}$$

$$= |\vec{A}|\cos\theta_{AB}$$

Thus $\vec{A} \cdot \hat{B}$ is simply the projection (ob in the figure above) of $\vec{A}$ in the direction of $\bar{B}$. Then $\vec{A} \cdot \vec{B}$ is just that projection scaled by the magnitude $|\vec{B}|$ of the vector $\vec{B}$.

### Comments:

1. The dot product is defined in a way such that it is completely independent of any coordinate system.

2. For $\theta_{AB} = \frac{\pi}{2}$ the dot product between two vectors is zero when $\vec{A} \cdot \vec{B} = 0$, we say that $\vec{A}$ & $\vec{B}$ are orthogonal vectors.

3. For $\frac{3\pi}{2} > \theta_{AB} > \frac{\pi}{2}$, the dot product is negative.

4. The dot product of a vector with itself yields the square of its magnitude:

$$\vec{A} \cdot \vec{A} = |\vec{A}||\vec{A}| \cos\theta_{AA} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$$

## Direction Cosines

Let $\hat{n}$ be a unit vector in some direction. Then the dot product between a vector $\vec{A}$ & $\hat{n}$ is given by:

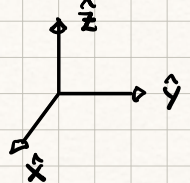
$$\vec{A} \cdot \hat{n} = |\vec{A}||\hat{n}| \cos\theta_{An}$$
$$= |\vec{A}| \cos\theta_{An}$$

And so $\cos\theta_{An} = \dfrac{\vec{A} \cdot \hat{n}}{|\vec{A}|}$

is called the direction cosine.

## The cartesian coordinate system

Suppose we have three unit vectors $\hat{x}, \hat{y}$ & $\hat{z}$ which are all mutually orthogonal to each other:




Thus $\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$

And $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$

These vectors define the cartesian coordinate system. For an arbitrary vector $\vec{A}$ we can define:

$$A_x := \vec{A} \cdot \hat{x}$$

$$A_y := \vec{A} \cdot \hat{y}$$

$$A_z := \vec{A} \cdot \hat{z}$$

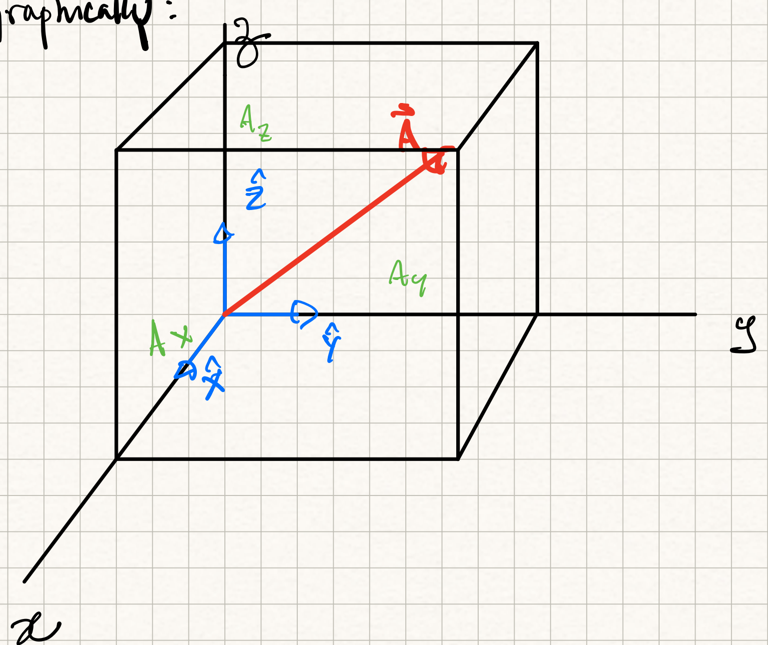
Then we can say $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

$A_x, A_y, A_z$ are known as the **components** of vector $\vec{A}$ in the Cartesian coordinate system.

The magnitude of $\vec{A}$ is then given by:

$$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Graphically:




In the Cartesian coordinate system

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

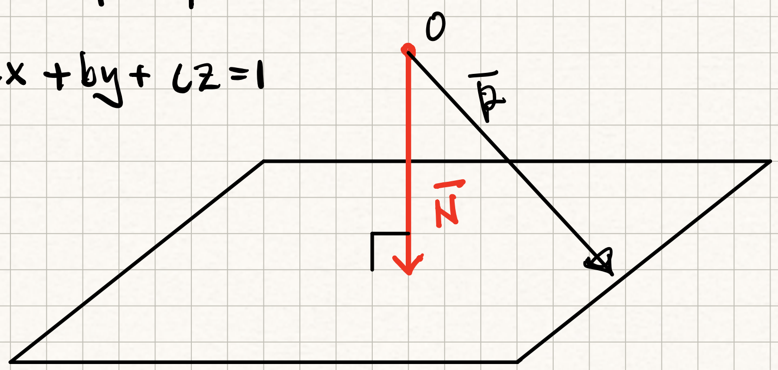
**Applications**

1. Law of Cosines: $(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{C} \cdot \vec{C}$

$$|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}| \cos\theta_{AB}$$

2. Equation of a plane:

$$ax + by + cz = 1$$




If $\vec{N}$ is a vector from O to the plane which is ortho-gonal to the plane and $\vec{R}$ is an arbitrary vector from O to the plane. Then

$$\vec{N} \cdot \vec{R} = |\vec{N}|\,|\vec{R}| \cos\theta_{NR} = |\vec{N}|\,|\vec{N}| = |\vec{N}|^2$$

Claim:

$\vec{N} \cdot \vec{R} = |\vec{N}|^2$ is the equation of a plane.

Proof:

$$\vec{N} \cdot \vec{R} = |\vec{N}|^2$$

$$\Rightarrow \quad N_x R_x + N_y R_y + N_z R_z = |\vec{N}|^2$$

$$\frac{N_x}{|\vec{N}|^2} R_x + \frac{N_y}{|\vec{N}|^2} R_y + \frac{N_z}{|\vec{N}|^2} R_z = 1$$

If O is the origin of the cartesian system then $R_x = x$, $R_y = y$, $R_z = z$ and with the identification

$$a = \frac{N_x}{|\vec{N}|^2}, \quad b = \frac{N_y}{|\vec{N}|^2}, \quad c = \frac{N_z}{|\vec{N}|^2} \quad \text{we get}$$

$$ax + by + cz = 1. \quad \blacksquare$$

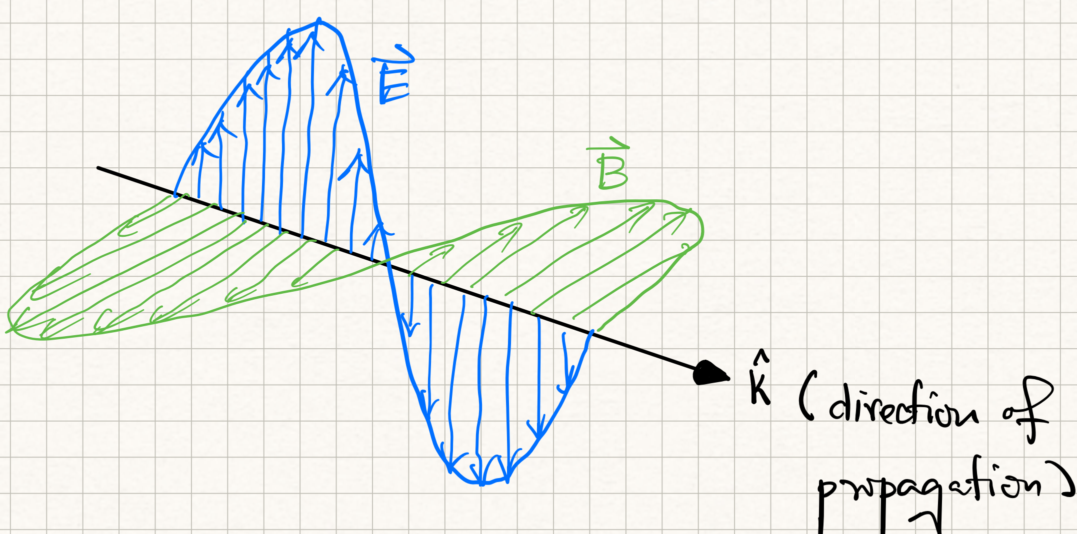
3. If the dot product between two vectors vanish they are orthogonal. In three dimensions there are only three independent direction. An electromagnetic wave, which is a solution to Maxwell's equation, consists of two fields — an electric field $\vec{E}$ & a magnetic field $\vec{B}$. These fields are orthogonal to each other:

$$\vec{B} \cdot \vec{E} = 0$$

But we know that electromagnetic waves are transverse waves and if $\hat{k}$ is the instantaneous direction of propagation we also have

$$\vec{E} \cdot \hat{k} = 0$$
$$\vec{B} \cdot \hat{k} = 0$$

Graphically:



## The Cross Product

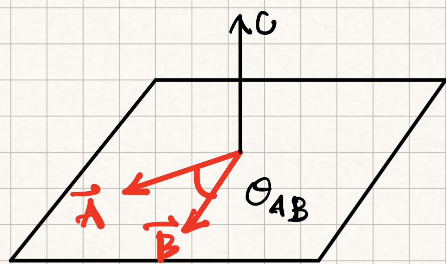
There is another product between vectors that yields as a result something that is very close to a vector. That is known as the cross product: If $\vec{A}$ & $\vec{B}$ are two vectors then the cross product is defined as:

$$\vec{C} = \vec{A} \times \vec{B}$$

where the magnitude of $\vec{C}$ is given by

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta_{AB}$$

and the direction is given by the right-hand-rule:




The direction $\vec{C}$ is given by curling the $\vec{A}$ vector with right hand towards the vector $\vec{B}$. $\vec{A}$ & $\vec{B}$ naturally define a plane. The direction orthogonal to this plane that is in the direction of the outstretched thumb gives the diretion of $\hat{C}$.

**Comments:**

1. The cross product is not commutative $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.
2. Like the dot product, the cross product distributes over vector addition:
$$\vec{C} \times (\vec{A} + \vec{B}) = (\vec{C} \times \vec{A}) + (\vec{C} \times \vec{B})$$
3. $|\vec{C}| = 0$ if $\vec{A}$ & $\vec{B}$ are parallel or antiparallel. i.e. $\theta_{AB} = 0$ or $\pi$.
4. The right-handed coordinate system is one in which x, y, and z directions are chosen so that $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ & $\hat{k} \times \hat{i} = \hat{j}$.

The cross-product in the Right-Handed Cartesian System:

Let $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$

$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

$$\vec{C} = \vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_y \, \hat{i} \times \hat{j} + A_x B_z \, \hat{i} \times \hat{k} + A_y B_x \, \hat{j} \times \hat{i} + A_y B_z \, \hat{j} \times \hat{k} + A_z B_x \, \hat{k} \times \hat{i}$$

$$+ A_z B_y \, \hat{k} \times \hat{j}$$

$$= (A_x B_y - A_y B_x) \hat{k} + (A_z B_x - A_x B_z) \hat{j} + (A_y B_z - A_z B_y) \hat{i}$$

$$= \hat{i} (A_y B_z - A_z B_y) + \hat{j} (A_z B_x - A_x B_z) + \hat{k} (A_x B_y - A_y B_x)$$

We can write this as

$$\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

**Comment:**

1. The cross product between two vectors yield a pseudo-vector. A vector naturally flips its direction under $x \to -x$, $y \to -y$ & $z \to -z$ (Parity transformation).

$$\vec{A} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k})$$

Under parity $\hat{i} \to -\hat{i}$, $\hat{j} \to -\hat{j}$, $\hat{k} \to -\hat{k}$ & so

$\vec{A} \to -\vec{A}$.

But if $\vec{C} = \vec{A} \times \vec{B}$ and under parity $\vec{A} \to -\vec{A}$ & $\vec{B} \to -\vec{B}$,

$\vec{C} \to \vec{C}$.

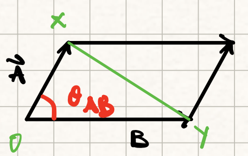
Such vectors are called pseudo-vectors or axial vectors.

Example of an axial / pseudo vector: $\vec{L} = \vec{r} \times \vec{p}$ (angular momentum).

Parity: Reflection in the xy plane + 180° rotation in the xy plane.

**Applications of cross product:**

1. The area of a parallelogram:




Area: $2 \times$ area of $\Delta$ Oxy

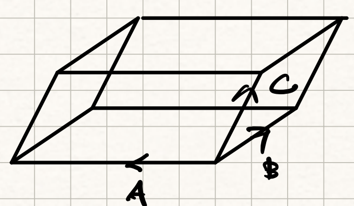
$$= 2 \times \frac{1}{2} |\vec{A}| \sin\theta_{AB} \, |\vec{B}|$$

$$= |\vec{A}||\vec{B}| \sin\theta_{AB}$$

$$= |\vec{A} \times \vec{B}|$$

Comment: In 3D we can associate a direction to a flat surface. But there is a choice since that surface has two faces. The cross-product gives us a natural orientation if $\vec{A}$ & $\vec{B}$ are specified. The orientation of the surface $\vec{A} \times \vec{B}$ is opposite of that of $\vec{B} \times \vec{A}$.

2. Volume of a parallelopiped




Area of the bottom $|\vec{A} \times \vec{B}|$


$$\text{Volume} = |\vec{A} \times \vec{B}| \, c = |\vec{A} \times \vec{B}| \, |\vec{C}| \cos\theta = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

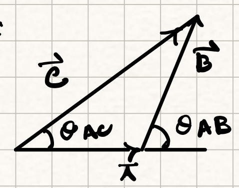
We could have chosen to start with the parallelogram defined by $\vec{B}$ & $\vec{C}$:

$$\text{Volume } (\vec{B} \times \vec{C}) \cdot \vec{A} = \vec{A} \cdot (\vec{B} \times \vec{C})$$

Thus we get the triple vector identity

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}).$$

3. Law of sines:




$$\vec{C} = \vec{A} + \vec{B}$$

$$\vec{A} \times \vec{C} = \vec{A} \times \vec{B}$$

$$\Rightarrow \; |\bar{A}||\bar{C}| \sin\theta_{AC} = |\bar{A}|\,|\bar{B}| \sin\theta_{AB}$$

$$\Rightarrow \; \frac{\sin\theta_{AC}}{|\bar{B}|} = \frac{\sin\theta_{AB}}{|C|}$$