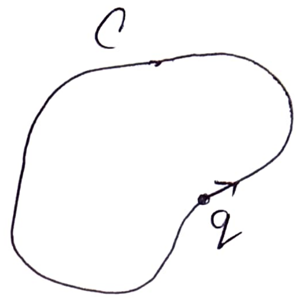
Lecture 15

~~Motional emf~~

Revisiting electromotive force (emf)



Consider a closed path C, where we are moving a charge q along that closed path. We define the emf as,

$$\epsilon = \oint \frac{\vec{F}\cdot d\vec{s}}{q}$$

So, emf is basically the line integral of the force along a closed path C per unit charge. The unit of emf is again volt. It looks exactly like what we defined in circuits, but in that case the force was purely electric, or Coulomb force. Now we are making the definition a bit broad, allowing the force to be anything, that can move the charge around a closed loop.

Motional emf

Consider a loop of wire moving to the right with a velocity $\vec{v}$. Let's first consider a constant magnetic field coming out of the page in some portion of the loop.

Now, lets calculate the emf by carrying out the line integral over the loop. The wire loop is a conductor. The ~~ne~~ charge q on the loop will feel a magnetic force, given by,

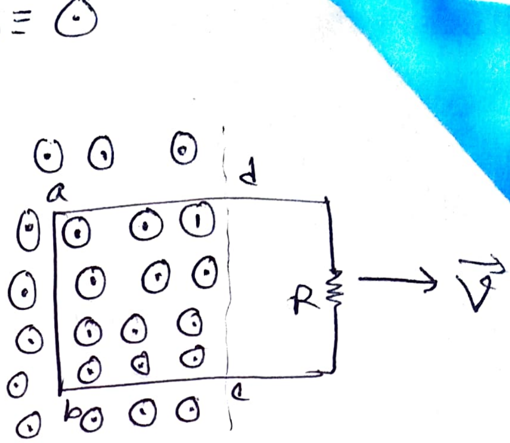


Figure :1

$$\vec{F}_{mag} = q\vec{V}\times\vec{B}$$

Now,

$$\epsilon = \oint \frac{\vec{F}_{mag}\cdot d\vec{s}}{q} = \int_b^a \frac{\vec{F}_{mag}\cdot d\vec{s}}{q} + \int_a^d \frac{\vec{F}_{mag}\cdot d\vec{s}}{q} + \int_b^{\cancel{d}\,c} \frac{\vec{F}_{mag}\cdot d\vec{s}}{q}$$

Here, $\vec{F}_{mag} = qV\hat{i}\times B\hat{k} = qVB(-\hat{j})$

~~$\vec{F}_{mag,\,ad} =$~~

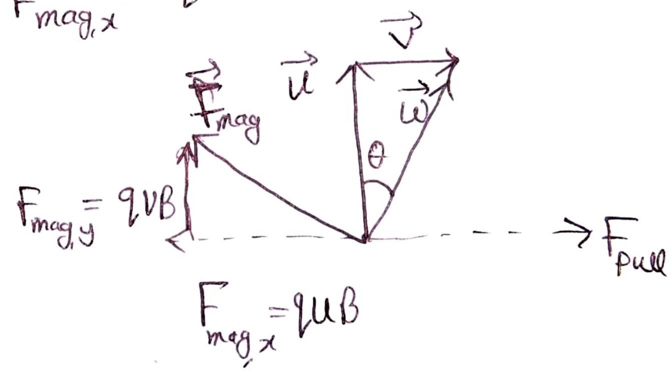
$$\therefore \epsilon = \frac{1}{q}\left[\int_b^a qVB(-\hat{j})\cdot dy\,\hat{j} + \int_a^d \underbrace{\cancel{qVB(-\hat{j})\cdot dx\,\hat{i}}}_{0} + \int_b^c \underbrace{\cancel{qVB(-\hat{j})\cdot dx\,\hat{i}}}_{0}\right]$$

$$\therefore \epsilon = -\frac{qVB}{q}(a-b) = -VBh$$, where h is the width of the loop.

In words, the magnetic force on ab will drive a current through the loop since ad and cb has no contribution (because the force is perpendicular to the wire, charges can't move along the wire). ~~F~~ And we get an emf around the loop, the light bulb (resistor R) will light up.

Now, something is fishy here. There is certainly work done here. Otherwise the light bulb wouldn't lit up. Now, the question is, is magnetic force doing the work? But we know, it simply can't. Actually, the person who is pulling the loop is doing all the work. How? Let's see.

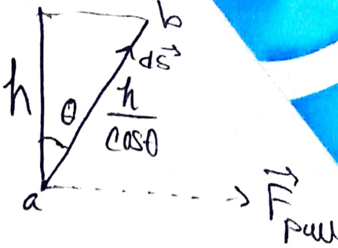
With the current flowing, the free charges moving in ab is moving with a velocity, say $\vec{u}$. But the whole set up is moving with velocity $\vec{v}$ to the right. So, the charges are really moving with a velocity $\vec{w} = \vec{u} + \vec{v}$. Accordingly the magnetic force has a component of $F_{mag,x} = quB$ to the left side of the wire. To counterbalance this, the person must pull towards the right with the force $F_{pull} = quB$. The work done by this force is given by,



$$\int \vec{F}_{pull} \cdot d\vec{s}$$

The particle actually moves in the direction of $\vec{w}$. So, while the particle moves ~~in~~ from a to b, its not really moving through a distance h, but

$\frac{h}{\cos\theta}$, as shown in the figure



$$\therefore \int \vec{F}_{pull} \cdot \vec{ds} = \int_a^b q u B \hat{i} \cdot \vec{ds}$$

$$= \int_a^b q u B \, ds \cos(90^\circ - \theta) = q u B \sin\theta \, (b-a)$$

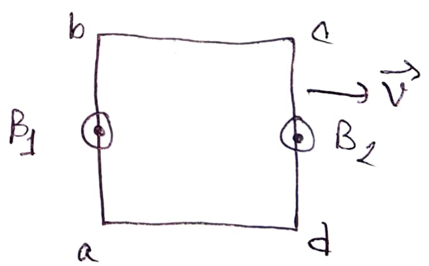
$$= q u B \sin\theta \cdot \frac{h}{\cos\theta} = q u B h \tan\theta = q u B h \cdot \frac{v}{u}$$

$$= q v B h$$

So, work done per unit charg by F_{pull} is $= \frac{q v B h}{q} = v B h$
$= \mathcal{E}$

So, $\vec{F}_{mag}$ contributes to the emf, but not to the work and $\vec{F}_{pull}$ contributes to the work.

We can also generalize the motional emf for any magnetic field. We can add complexity here by stating different magnetic field B_1 and B_2 (both pointing out of the page) at two different sides of the loop. At any instant of time, the ~~force~~ line integral is,

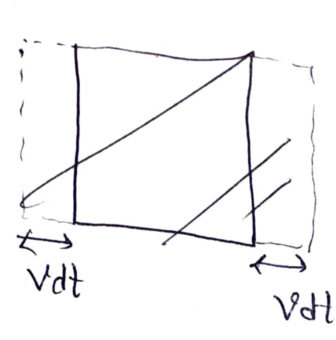


$$\epsilon = \oint \frac{\vec{F} \cdot \vec{ds}}{q} = -\cancel{q} v B_1 h + \cancel{q} v B_2 h$$

$$= -\cancel{q} v (B_2 - B_1) h \quad \text{—(1)}$$

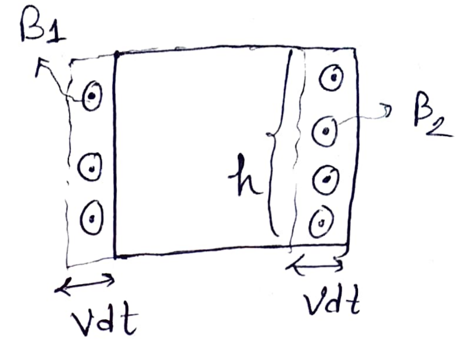
Now, the magnetic flux through a surface is given by, like electric flux,

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A}$$



$\cdots\cdots \rightarrow$ Loop at time t

$\longrightarrow$ Loop at time $t+dt$



Within the time interval ~~to~~ dt, the loop moves a distance of vdt. In this time, the loop gains a flux,

$$\phi_+ = \iint \vec{B}\cdot d\vec{A} = B_2 \iint dA = B_2 h v dt$$

Flux lost in this time, $\phi_- = \iint \vec{B}\cdot d\vec{A} = B_1 \iint dA = B_1 h v dt$

$$\therefore d\phi = v(B_2 - B_1) h dt \Rightarrow \frac{d\phi}{dt} = v(B_2 - B_1) h \quad \text{—(ii)}$$

Combining (i) and (ii), we can write,

$$\boxed{\mathcal{E} = -\frac{d\phi}{dt}}$$

This is the flux rule for motional emf.

There is a sign ambiguity in the emf and flux. Which way around are you supposed to integrate? And what is positive direction of $d\vec{A}$? If your right hand figers curl around the loop, then your thumb indicates the $d\vec{A}$. If emf is negative, then current will just flow in opposite direction of your loop direction.

There is another view on the sign and direction. The current flowing through the loop will create a

magnetic field. of its own. Now, Lenz's law states, the direction of current around a closed loop is such that, the magnetic flux it generates opposes the change in magnetic flux that was creating the current in the first place. If the flux decreases, like shown in figure 1, then the current through th loop must be 
anticlockwise, so that the flux increases. So, the minus sign in $\mathcal{E} = -\frac{d\Phi}{dt}$ is really Lenz's law.

Faraday's law: universal law of induction

We will start with three experiment, carried out by Faraday in 1831.

(i) A loop of wire was moved toward right with speed v, which is in table 2. On table 1, there is a coil where a constant current is flowing, generating a

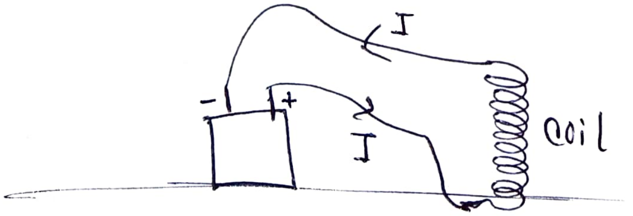


Table 1

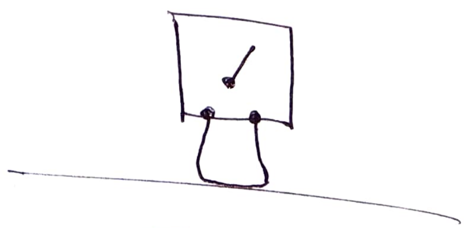


Table 2

constant magnetic field. We see a deflection on the galvanometer. This is not surprising to us, since we already know it is due to the motional emf.

(ii) Table 2 is fixed, but table 1 is moving to the left with speed v. Again we see a deflection on the galvanometer. This might be surprising because the loop of wire is not moving, hence it can't feel the magnetic force, so there is no motional emf. What is driving the current?

(iii) Both table 1 and table 2 are fixed, but the current on the coil is changing, hence the magnetic field on the loop on table 2 is also changing. Now, you also observe a deflection. How is the current produced?

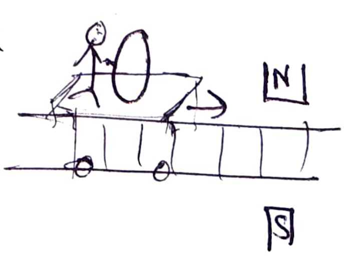
The first question that we ask is, are experiment (ii) and (iii) equivalent? If the magnetic field B in experiment (iii) was changed in such a manner that it is equal to the change as done in experiment (ii), then an isolated observer on table 2 will have no clue that these two are distinct experiments. He will observe (and he does observe) the same deflection. So, experiment (ii) and (iii) are equivalent.

Now, back to experiment (ii). There is a relative m[otion]
between the table ① and ②, where ① is moving.
Loop on ~~coil~~ table ② surely can't feel any
magnetic force. If there is an emf, it has to
come from electric force. So, Faraday had this
brilliant idea, that, "a changing magnetic field induces
an electric field". Experiment (ii) was found also to
be equivalent to experiment (i), creating an emf,

$$\mathcal{E} = -\frac{d\phi}{dt}$$

~~Since the origin is electric,~~

In classical electrodynamics, this ~~was simp~~ result, that
the effect is same, whether table ① or ② is
moving, was a remarkable coincidence. There are two
totally different observation in two different experiments,
yet giving the same result. Consider, a car with a loop
of wire is moving towards a magnet that looks like
shown in the picture. An emf will produce, as the car
moves towards the magnet. The reason
obviously is motional emf. But from
the perspective of the observer



on train, the loop is not moving, rather the magnetic
field through it is changing. The reason must be electri[c]

Which one is true? Is the observer on the train has a wrong interpretation even though his answer is wright? This ambiguity was taken by Einstein. He showed, the analysis of each observer is as valid as other one. He even wrote on the first page of his 1905 paper introducing special theory of relativity about this coincidance, called then the reciprocity / reciprocal action in electrodynamics.

Anyways, Faraday had no idea about SR; it was his ingenious idea that an electric field will be induced in the loop when there is a change in magnetic flux

$$\therefore \mathcal{E} = -\frac{d\phi}{dt}$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{s} = -\frac{d\phi}{dt} \qquad\qquad \phi = \iint_S \vec{B}\cdot d\vec{A}$$

$$\Rightarrow \oint_{\partial S} \vec{E}\cdot d\vec{s} = -\frac{d}{dt}\iint_S \vec{B}\cdot d\vec{A} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

$$\Rightarrow \iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

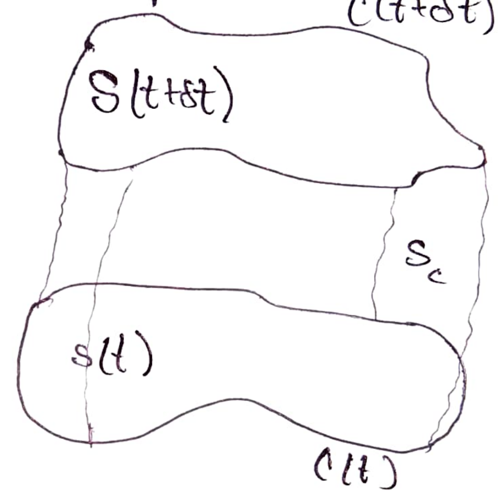
$$\boxed{\therefore \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}} \qquad \boxed{\cancel{\oint \vec{E}\cdot d\vec{A} = -\frac{d\phi}{dt} - \iint_S \vec{B}}}$$

$$\boxed{\oint \vec{E}\cdot d\vec{s} = -\frac{d\phi}{dt} = \iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}}$$

This is Faraday's law in differential and integral form.

The general proof of Faraday's low of induction

We consider a situation where the magnetic field is time dependent, and the closed loop C also changes with time (it can move, or change shape). We consider the loop $C(t)$ is moving, as shown in the figure. At time t, the loop is $C(t)$ and the enclosed surface is $S(t)$. At some later time δt, the loop is now $C(t+\delta t)$, the surface is $S(t+\delta t)$ enclosing the loop.



We want to calculate the change in flux through the surface enclosed by the loop between t and $t+\delta t$.

Now, there can be two contributions. One, because $\vec{B}$ is changing, another, because C is changing.

Now, $\delta\phi = \phi(t+\delta t) - \phi(t)$

$$= \iint_{S(t+\delta t)} \vec{B}(t+\delta t) \cdot d\vec{A} - \iint_{S(t)} \vec{B}(t) \cdot d\vec{A}$$

Both $d\vec{A}$'s are taken upward

Now, take the closed surface made out of $S(t+\delta t)$, $S(t)$ and the curved infinitesimal surface S_c between them. Since

Since $\vec{\nabla} \cdot \vec{B} = 0$, then,

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

$$\therefore \iint_{S(t+\delta t)} \vec{B}(t+\delta t) \cdot d\vec{A} - \iint_{S(t)} \vec{B}(t+\delta t) \cdot d\vec{A} + \iint_{S_c} \vec{B}(t) \cdot d\vec{A} = 0$$

It could be written as $\vec{B}(t+\delta t)$. But S_c is already proportional to δt, so the integral will be dominated by $\vec{B}(t)$.

$$\iint_{S_c} \vec{B}(t+\delta t) \cdot d\vec{A} = \iint_{S_c} \vec{B}(t) \cdot d\vec{A} + \underbrace{\iint_{S_c} \frac{\partial \vec{B}}{\partial t} \delta t \cdot d\vec{A}}_{\boxed{\text{the order of } (\delta t)^2}} + \cdots$$

One positive and one negative because, being a closed surface, the area vectors of $S(t+\delta t)$ and $S(t)$ must be opposite.

$$\therefore \delta\phi = \iint_{S(t)} \vec{B}(t+\delta t) \cdot d\vec{A} - \iint_{S_c} \vec{B}(t) \cdot d\vec{A} - \iint_{S(t)} \vec{B}(t) \cdot d\vec{A}.$$

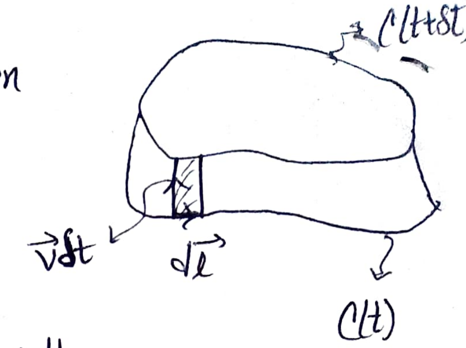
Now, let's do a Taylor expansion of $\vec{B}(t+\delta t)$, which will be given by,

$$\vec{B}(t+\delta t) = \vec{B}(t) + (\delta t) \left.\frac{\partial \vec{B}}{\partial t}\right|_{t=t} + (\delta t)^2 \left.\frac{\partial^2 \vec{B}}{\partial t^2}\right|_{t=t} + \cdots$$

$$\therefore \delta\phi = \cancel{\iint_{S(t)} \vec{B}(t) \cdot d\vec{A}} + \delta t \iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A} + O(\delta t^2) - \iint_{S_c} \vec{B}(t) \cdot d\vec{A} - \cancel{\iint_{S(t)} \vec{B}(t) \cdot d\vec{A}}$$

$$\therefore \delta\phi = \delta t \iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A} - \iint_{S_c} \vec{B}(t) \cdot d\vec{A} + O(\delta t^2)$$

Now, $d\vec{A}$ of S_c can be written as,



$$d\vec{A} = d\vec{l} \times \vec{v}\,\delta t$$

since S_c can be divided into rectangles having side length dl and $v\,dt$.

$$\therefore \delta\Phi = \delta t \iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A} - \iint_{S_c} \vec{B}(t) \cdot (d\vec{l} \times \vec{v}\,\delta t)$$

But a scalar tripple product can always be written as,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$\therefore \delta\Phi = \delta t \iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A} - \delta t \oint_{C(t)} (\vec{v} \times \vec{B}(t)) \cdot d\vec{l}$$

$$\therefore \cancel{\frac{d\Phi}{dt} = \lim_{\delta t \to 0}} \quad \therefore \frac{d\Phi}{dt} = \lim_{\delta t \to 0} \left[\iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A} - \oint_{C(t)} (\vec{v} \times \vec{B}(t)) \cdot d\vec{l} \right]$$

Now, $$\frac{d\Phi}{dt} = -\epsilon = -\iint_{C(t)} (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\therefore \oint_{C(t)} \vec{E} \cdot d\vec{l} = -\iint_{S(t)} \frac{\partial \vec{B}(t)}{\partial t} \cdot d\vec{A}$$

$$\boxed{\therefore \int_{C(t)} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_{S(t)} \vec{B}(t) \cdot d\vec{A}}$$

Yea!