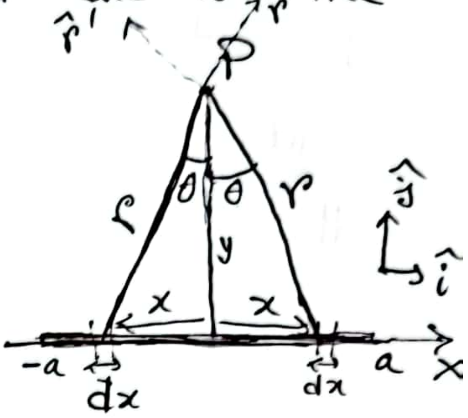
Lecture 3

Electric field for finite line charge distribution

We have talked about charge distributions. Let's now calculate some electric field due to the charge distributions.

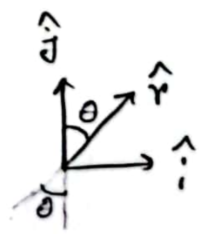


Let's say, we want to calculate the electric field due to a line charge distribution at a point P which is at a height of y directly from the midpoint of the charge distribution. Consider a very small, infinitesimal length element dx at a distance of x from the midpoint. If the infinitesimal length dx contain a charge of dq, then the electric field at point P is given by,

$$d\vec{E}(P) = K\frac{dq}{r^2}\,\hat{r}$$

Now, using the symmetry, we have another infinitesimal length dx, at the same distance x on the positive side of x. The electric field is given by,

$$d\vec{E}'(P) = K\frac{dq}{r^2}\,\hat{r}'$$



Now, $r = \sqrt{x^2+y^2}$ and $\hat{r} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

~~Similarly~~ Similarly, $\hat{r}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$$\therefore d\vec{E}_{tot}(P) = K\frac{dq}{r^2}\left[\hat{r}+\hat{r}'\right] = 2K\frac{dq}{r^2}\cos\theta\,\hat{j}$$

Now, to find the total electric field for the whole wire, we can jut integrate over one side of the wire. Now, if the line charge density is λ, then,

$$\lambda = \frac{dq}{dx} \Rightarrow dq = \lambda dx$$

$$\therefore \vec{E}(P) = \int_0^a 2K\frac{\lambda dx}{(\sqrt{x^2+y^2})^2}\cos\theta\,\hat{j} = 2K\lambda\int_0^a \frac{y}{(x^2+y^2)^{3/2}}dx\,\hat{j}$$

Here, $\tan\theta = \frac{x}{y} \Rightarrow x = y\tan\theta$ with $\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$

$$\therefore dx = y\sec^2\theta\, d\theta$$

when $x=0$, $\theta = \tan^{-1}\left(\frac{0}{y}\right) = 0$

$x = a$, $\theta = \tan^{-1}\left(\frac{a}{y}\right)$

$$\therefore \vec{E} = \int_0^{\tan^{-1}\frac{a}{y}} 2K\lambda y \cdot \frac{y\sec^2\theta\, d\theta}{\{y^2(1+\tan^2\theta)\}^{3/2}} = \frac{2K\lambda}{y}\int_0^{\tan^{-1}\frac{a}{y}} \frac{y^2\sec^2\theta}{y^3\sec^3\theta}d\theta\,\hat{j}$$

$$= \frac{2K\lambda}{y}\int_0^{\tan^{-1}\frac{a}{y}} \cos\theta\, d\theta\,\hat{j} = \frac{2K\lambda}{y}\left[\sin\theta\right]_0^{\tan^{-1}\frac{a}{y}}$$

$$= \frac{2K\lambda}{y}\cdot\left[\sin.\tan^{-1}\frac{a}{y} - 0\right]\hat{j}$$

$$= \frac{2K\lambda}{y}\sin.\sin^{-1}\frac{a}{\sqrt{a^2+y^2}}\,\hat{j}$$

$$\therefore \vec{E} = K\lambda \cdot \frac{2a}{y\sqrt{a^2+y^2}} \hat{j}$$

Now, $\lambda = \frac{q}{2a} \Rightarrow 2a = \frac{q}{\lambda}$

$$\boxed{\therefore \vec{E} = K \frac{q}{y\sqrt{a^2+y^2}} \hat{j}}$$

In the limit $a \gg y$, that is $a \to \infty$,

$$\vec{E} = K \frac{q}{y\,a\sqrt{1+\left(\frac{y}{a}\right)^2}} \hat{j}$$

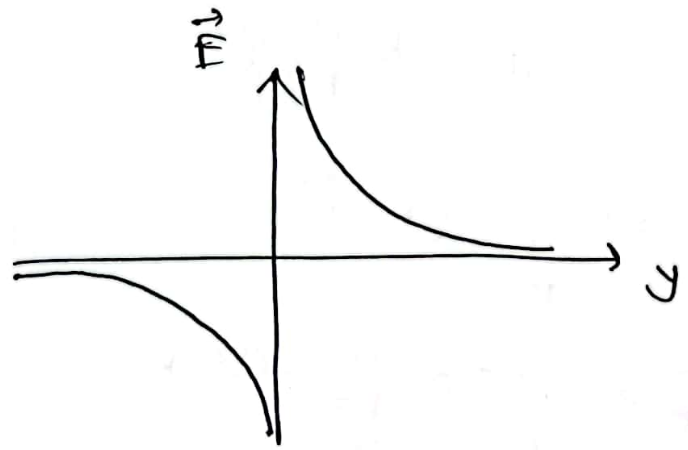
$$\therefore \vec{E} = K \frac{q}{ya} \hat{j}$$

$$\boxed{\therefore \vec{E} = 2K \frac{\lambda}{\cancel{a}y} \hat{j}}$$ with the substitution $q = \lambda * 2a$.

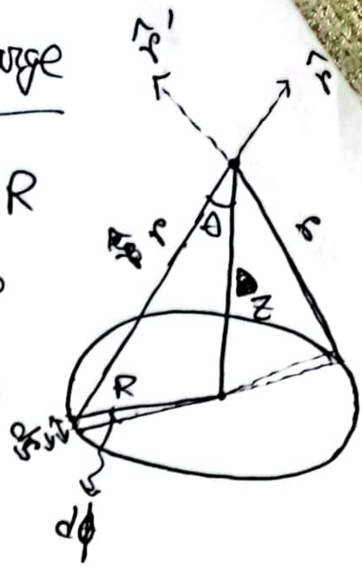
So, for an infinite wire, $E \propto \frac{1}{y}$.

However, for $y \gg a$, $\vec{E} = K \frac{q}{y\,y\sqrt{1+\left(\frac{a}{y}\right)^2}} \hat{j} = K\frac{q}{y^2}\hat{j}$

and thats the regular expression of field line for a point charge.



Electric field due to a ring of charge



A non-conducting ring of radius R is shown with uniform charge density λ and a total charge Q, which is lying, say in the XY plane. We want to find the electric field at P, located at a distance of z from the center of the ring along its symmetry axis.

Now, $d\vec{S} = R\,d\phi\,\hat{\phi}$

$\therefore dS = R\,d\phi$

and $\lambda = \dfrac{Q}{2\pi R}$

$\lambda = \dfrac{dq}{dS}$

$\Rightarrow dq = \lambda R\,d\phi$

$$\therefore d\vec{E} = K\frac{dq}{r^2}\hat{r}$$

By symmetry arguments, we have a similar ~~lengg~~ length of charge in the opposite direction, that gives the contribution,

$$d\vec{E} = K\frac{dq}{r^2}\hat{r}'$$

$$\therefore d\vec{E} = 2K\frac{dq}{r^2}\left[\hat{r}+\hat{r}'\right] = 2K\frac{dq}{r^2}\cos\theta\,\hat{k}$$

$$\therefore \vec{E} = \int_0^{\pi} 2K\frac{dq}{r^2}\cos\theta\,\hat{k}$$

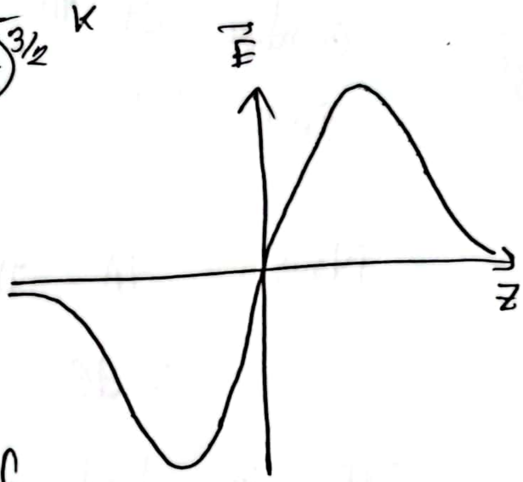
$$= \int_0^{\pi} 2k \cdot \frac{\lambda R\, d\phi}{r^2} \cdot \frac{z}{r}\, \hat{k} = \int_0^{\pi} \frac{2k\lambda R z}{r^3}\, d\phi\, \hat{k}$$

$$= \frac{2k\lambda R z}{r^3} \int_0^{\pi} d\phi\, \hat{k}$$

$$= \cancel{2} k \cdot \frac{Q}{\cancel{2\pi R}} \cdot \cancel{R} z \cdot \frac{1}{(R^2+z^2)^{3/2}} \cdot \cancel{\pi}\, \hat{k}$$

$$\cancel{= kq} \quad = kQ \frac{z}{(R^2+z^2)^{3/2}}\, \hat{k}$$

$$\therefore \quad \vec{E} = kQ \frac{z}{(R^2+z^2)^{3/2}}\, \hat{k}$$



So, at $z=0$, $\vec{E} = \vec{0}$ and it makes perfect ~~sense if~~ sense if you think about the symmetry of the problem.

If $R \gg z$, then,

$$\vec{E} = kQ \frac{z}{R^3}\, \hat{k}$$

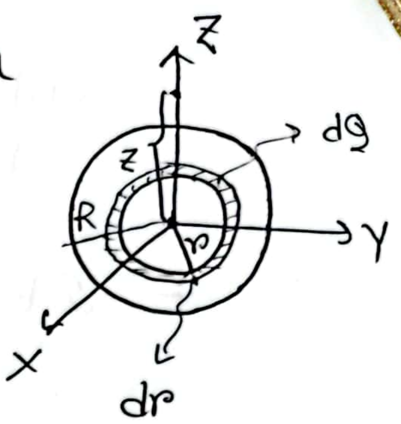
If $z \gg R$, then,

$$\vec{E} = \frac{kQ}{z^2}\, \hat{k}$$

and it reduces down to a point charge, that makes perfect sense.

Electric field due to a charged disc

A uniformly charged disc with radius R and a total charge Q lies on XY plane. We want to find the electric field at point P, along the Z axis, that passes through the center of the disc. The surface charge density is given by,



$$d\sigma = \frac{dQ}{dA} = \frac{Q}{\pi R^2}$$

Now,
$$dA = \pi (r+dr)^2 - \pi r^2 = 2\pi r\,dr$$

$$\therefore dQ = \sigma\,dA = 2\pi r \sigma\,dr$$

The electric field for this infinitesimal charge element will be given by,

$$\cancel{d\vec{E} = K\frac{dQ}{z^2}}$$

$$d\vec{E} = K\frac{dQ\;z}{\sqrt{(r^2+z^2)}^{3/2}}\hat{k}$$ from our previous calculations.

$$\therefore d\vec{E} = K\frac{2\pi r \sigma z\,dr}{(r^2+z^2)^{3/2}}\hat{k}$$

Total electric field due to the whole disc is,

$$\vec{E} = \int_0^R 2\pi K\sigma z \cdot \frac{r}{(r^2+z^2)^{3/2}}\,dr\;\hat{k}$$

$$\therefore \vec{E} = 2\pi K z \sigma \int_{|z|}^{\sqrt{R^2+z^2}} \frac{r}{u^3} \cdot \frac{u\,du}{r} \hat{k}$$

Let,
$\sqrt{r^2+z^2} = u$
$\Rightarrow r^2+z^2 = u^2$
$\Rightarrow 2r = 2u \frac{du}{dr}$
$\Rightarrow 2r\,dr = 2u\,du$
$\therefore dr = \frac{u}{r} du$
$r=0,\ u = \sqrt{0^2+z^2} = |z|$
$r=R,\ u = \sqrt{R^2+z^2}$

$$= 2\pi K z \sigma \int_{|z|}^{\sqrt{R^2+z^2}} \frac{1}{u^2} du \ \hat{k}$$

$$= -2\pi K z \sigma \left. \frac{1}{u} \right|_{|z|}^{\sqrt{R^2+z^2}} \hat{k}$$

$$= -2\pi K z \sigma \left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{|z|} \right] \hat{k}$$

$$\therefore \vec{E} = \frac{1}{4\pi\epsilon_0} \cdot 2\pi z \cdot \sigma \left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{k}$$

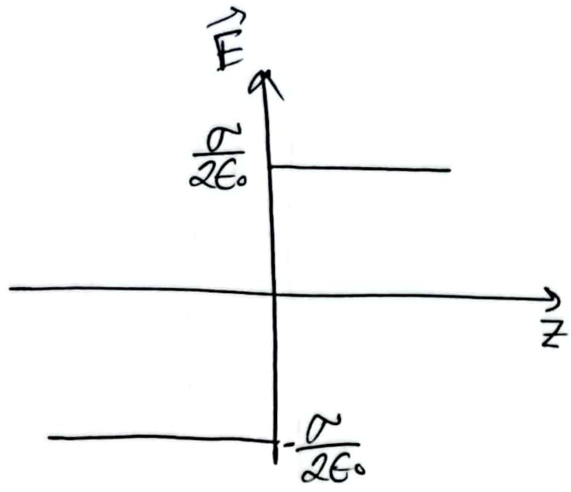
$$\Rightarrow \vec{E} = \frac{\sigma}{2\epsilon_0} z \left[\frac{1}{|z|} - \frac{1}{|z| \sqrt{\left(\frac{R}{z}\right)^2+1}} \right] \hat{k}$$

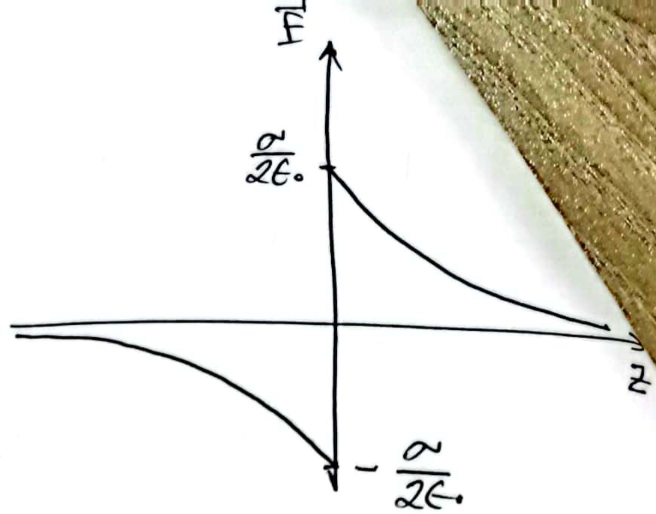
$$\therefore \vec{E} = \begin{cases} \frac{\sigma}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{\left(\frac{R}{z}\right)^2+1}} \right] \hat{k} & ; \text{ if } z>0 \\ -\frac{\sigma}{2\epsilon_0} \left[1 - \frac{1}{\sqrt{\left(\frac{R}{z}\right)^2+1}} \right] \hat{k} & ; \text{ if } z<0 \end{cases}$$

If the sheet is very large, in the limit $R \gg z$,

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{k} \quad ; z>0$$

$$\vec{E} = -\frac{\sigma}{2\epsilon_0} \hat{k} \quad ; z<0$$

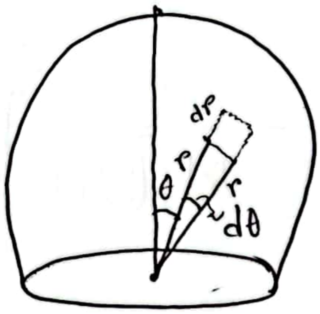




For $R \to \infty$

So, there is a discontinuity about $z = 0$.

Field due to a solid hemisphere with uniform charge density ρ at the center of the hemisphere

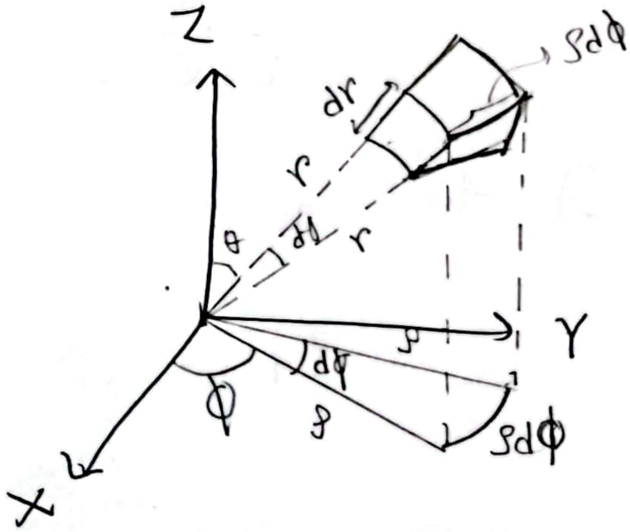


We can consider an infinitesimal volume element in the spherical polar coordinate.

$$dV = r\,d\theta \times dr \times \rho\,d\phi$$

$$= r\,d\theta\,dr\,r\sin\theta\,d\phi$$

$$= r^2 \sin\theta\,dr\,d\theta\,d\phi$$



Now, $$dq = \rho\, dV$$
$$= \rho r^2 \sin\theta\, dr\, d\theta\, d\phi$$

The electric field due to this small element of charge is given by,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\rho r^2 \sin\theta\, dr\, d\theta\, d\phi}{r^2}\hat{r}$$

But it becomes clumsy to calculate the electric field for the whole hemisphere owing to this direction dependency. But we have a overriding advantage. If we consider the hemisphere to be consisted of rings of charges, then we already know that the horizontal components for the whole ring cancels out and only the vertical component survives. So, we might only consider the integration over the vertical component, given by,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{\rho r^2 \sin\theta\, dr\, d\theta\, d\phi}{r^2} \cos\theta\, (-\hat{k})$$

$$\therefore \vec{E} = (-\hat{k}) \frac{1}{4\pi\epsilon_0} \int_{r=0}^{R} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \rho \sin\theta\, dr\, d\theta\, d\phi \cos\theta$$

$$= \cancel{(-\hat{k}) \frac{1}{4\pi\epsilon_0} \cdot \rho R \cdot \frac{\sin 2\theta}{2} \frac{1}{2} \int_{\theta=0}^{\pi/2} \sin 2\theta\, d\theta \int_0^{2\pi} d\phi}$$

$$= \cancel{(-\hat{k}) \frac{1}{4\pi\epsilon_0} \rho R \frac{1}{2} [\theta}$$

$$= (-\hat{k}) \frac{1}{4\pi\epsilon_0} \rho \int_0^R dr \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

Let, $\sin\theta = u$ $\qquad \theta = 0, \; u = 0$

$\Rightarrow \frac{du}{d\theta} = \cos\theta$ $\qquad \theta = \pi/2, \; u = 1$

$\Rightarrow d\theta = \frac{du}{\cos\theta}$

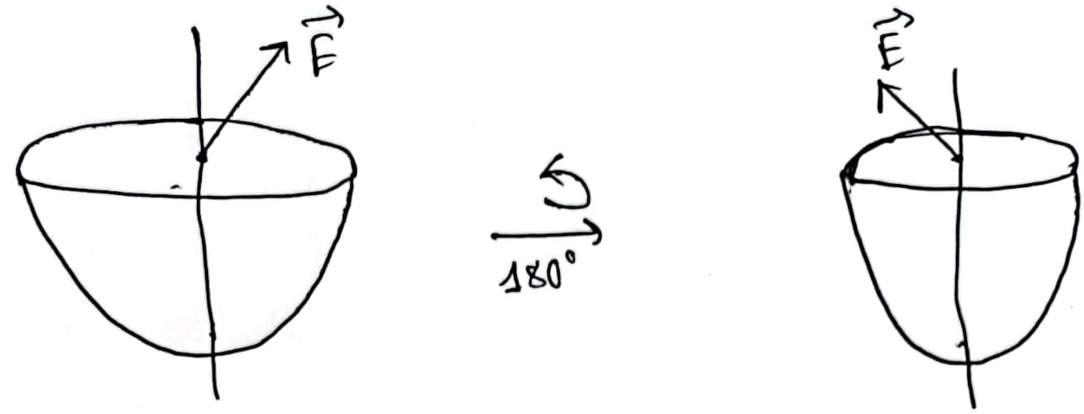
$$= (-\hat{k}) \frac{1}{4\pi\epsilon_0} \rho \times R \times 2\pi \int_0^1 u \, du$$

$$= (-\hat{k}) \frac{1}{4\pi\epsilon_0} 2\pi R\rho \cdot \frac{u^2}{2} \Big|_0^1$$

$$= \frac{\rho R}{4\epsilon_0} (-\hat{k})$$

$$\boxed{\therefore \vec{E} = \frac{\rho R}{4\epsilon_0} (-\hat{k})}$$

Now, you could also argue from symmetry why the electric field must be vertically down. Let's say it points at ~~ang ang~~ an angle with the vertical axis. If we now rotate the hemisphere by $180°$, then the electric field changes direction. But its sill the same hemisphere ~~orientag~~ orientation, nothing has changed. So the eledectric field can not change its direction. So, it must be vertically alligned.

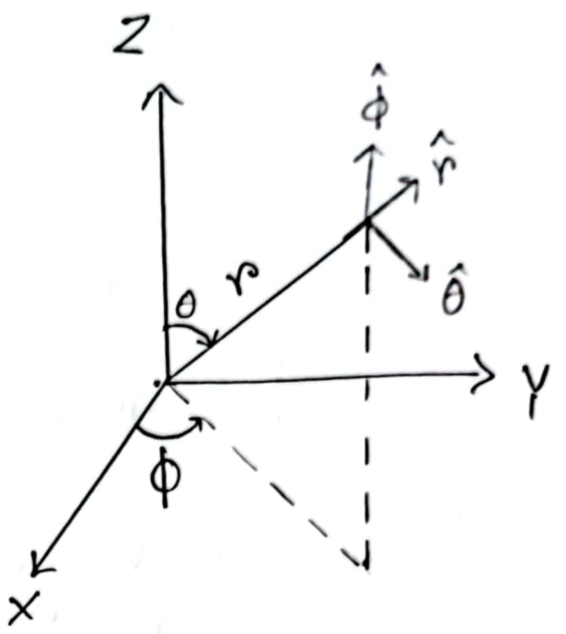


Important catch

If we consider a point charge, then the electric field varies as $\frac{1}{r^2}$ and exactly at the point charge, the field blows up. So, it makes no sense to think about field at a point charge. But a continuous charge distribution doesn't have this problem. If the charge distribution is nowhere infinite, then we can always find finite electric field at each and every point even within the distribution. This is because, the r^2 term in the volume element cancels the $\frac{1}{r^2}$ term in the numerator in Coulomb's law. So, as long as ρ is finite, the field will not blow up even at $r=0$, even in the interior or boundary of a charge distribution.

Spherical polar coordinate

r = distance from the origin
θ = polar angle
ϕ = azimuth angle



$x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$

The unit vectors are $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$.

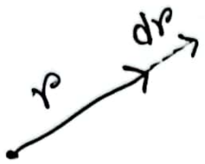
Any vector can be written in this coordinate system,

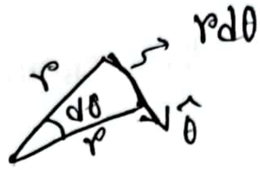
$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$$

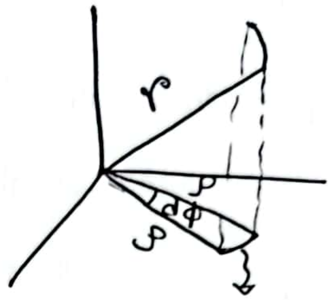
$$\hat{r} = \sin\theta \cos\phi \, \hat{i} + \sin\theta \sin\phi \, \hat{j} + \cos\theta \, \hat{k}$$
$$\hat{\theta} = \cos\theta \cos\phi \, \hat{i} + \cos\theta \sin\phi \, \hat{j} - \sin\theta \, \hat{k}$$
$$\hat{\phi} = -\sin\phi \, \hat{i} + \cos\phi \, \hat{j}$$

So, the unit vectors are in fact functions of θ and ϕ. That's why one should not take the unit vectors outside the integration.

Infinitesimal displacement







$$\therefore\ d\vec{s} = \overbrace{dr}^{dS_r}\,\hat{r} + \overbrace{r\,d\theta}^{dS_\theta}\,\hat{\theta} + \overbrace{r\sin\theta\, d\phi}^{dS_\phi}\,\hat{\phi}$$

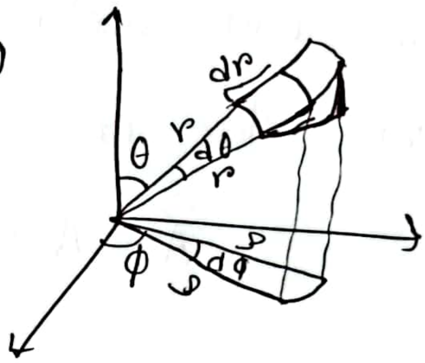
$$\rho\, d\phi = r\sin\theta\, d\phi$$

Infinitesimal volume element

$$d\tau = dS_r \times dS_\theta \times dS_\phi$$

$$= dr \times r\,d\theta \times r\sin\theta\, d\phi$$

$$= r^2 \sin\theta\, dr\, d\theta\, d\phi$$



Range: $r = 0 \to \infty$

$\theta = -\pi/2 \to \pi/2$

$\phi = 0 \to 2\pi$

Infinitesimal area

If r is a constant, such as on the surface of a sphere,

$$dA = dl_\theta\, dl_\phi = r\,d\theta \times r\sin\theta\, d\phi = r^2\sin\theta\, d\theta\, d\phi$$

$$d\vec{A} = r^2 \sin\theta\, d\theta\, d\phi\ \hat{r}$$

If the area is on the xy plane, then θ = constant.

$$\therefore\ dA = dS_r\, dS_\phi = \cancel{r\,dr\,d\phi}\ r\sin\theta\, dr\, d\phi$$

But since its in xy plane, $\theta = \pi/2$, $d\vec{A} = r\,dr\,d\phi\ \hat{\theta}$

$\therefore\ dA = r\,dr\,d\phi$ and