Lecture 7

Field of a shell via direct integration, for $\frac{1}{r^{2+\delta}}$ field

$$V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{s}$$

For a point charge, $V = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^{2+\delta}} \hat{r} \cdot dr\, \hat{r}$

$$= -Kq \int_{\infty}^{r} r^{-2-\delta} dr$$

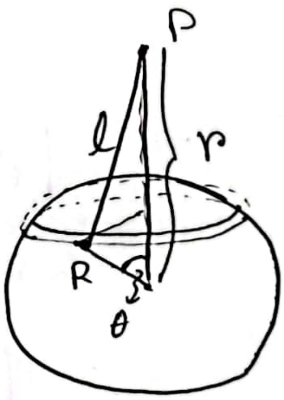
$$= -Kq \cdot \frac{r^{-1-\delta}}{-1-\delta} \Big|_{\infty}^{r}$$

$$= \frac{Kq}{1+\delta} \cdot \frac{1}{r^{1+\delta}} \Big|_{\infty}^{r}$$

$$\therefore V(r) = Kq \cdot \frac{1}{(1+\delta)\, r^{1+\delta}}$$

Now, let's calculate the potential due to a spherical non-conducting shell at a distace r from the center.

$$l^2 = R^2 + r^2 - 2Rr\cos\theta$$



The infinitesimal area of the ring can be written as, $dA = 2\pi R\sin\theta\,(R\,d\theta)$

$$\therefore dQ = \sigma\, 2\pi R\sin\theta\, R\,d\theta$$

Now, $dV(P) = K\frac{dq}{(1+\delta)\, l^{1+\delta}} = K\frac{\sigma\, 2\pi R^2 \sin\theta\, d\theta}{(1+\delta)\, l^{1+\delta}}$

$$= \frac{2\pi\sigma K R^2 \sin\theta\, d\theta}{(1+\delta)\,(R^2+r^2-2Rr\cos\theta)^{\frac{1+\delta}{2}}}$$

$$\therefore \quad \phi(r) = \int_{0}^{\pi/2} \frac{2\pi\sigma K R^2}{1+\delta} \frac{\sin\theta}{(R^2+r^2-2Rr\cos\theta)^{1+\delta/2}} d\theta$$

$$= \frac{2\pi\sigma K R^2}{1+\delta} \int_{R^2+r^2-2Rr}^{R^2+r^2+2Rr} \frac{1}{2Rr} \cdot u^{-\frac{1+\delta}{2}} du$$

$R^2+r^2-2Rr\cos\theta = u$

$\Rightarrow \frac{du}{d\theta} = +2Rr\sin\theta$

$\therefore d\theta = \frac{du}{2Rr\sin\theta}$

If, $\theta = 0$, $u = R^2+r^2-2Rr$

$\theta = \pi$, $u = R^2+r^2+2Rr$

$$= \frac{2\pi\sigma K R^2}{(1+\delta)2Rr} \frac{u^{\frac{1}{2}-\frac{\delta}{2}}}{\frac{1}{2}(1-\delta)} \Bigg|_{R^2+r^2-2Rr}^{R^2+r^2+2Rr}$$

$$= \frac{2\pi\sigma K R^2}{(1-\delta^2)Rr} \left[(R^2+r^2+2Rr)^{\left(\frac{1-\delta}{2}\right)} - (R^2+r^2-2Rr)^{\frac{1-\delta}{2}} \right]$$

$$\cancel{= \frac{2\pi\sigma K R^2}{(1-\delta^2)Rr} \left[\left(\sqrt{R^2+r^2+2Rr}\right)^{2\times\frac{1-\delta}{2}} - \right.}$$

$$= \frac{2\pi\sigma K R^2}{(1-\delta^2)Rr} \left[(R+r)^{2\times\frac{1-\delta}{2}} - (R-r)^{2\frac{1-\delta}{2}} \right]$$

$$= \frac{2\pi\sigma K R}{(1-\delta^2)r} \left[(R+r)^{1-\delta} - (R-r)^{1-\delta} \right]$$

$$\therefore \quad \phi(r) = \frac{K 2\pi \frac{Q}{4\pi R^2} \cdot R}{(1-\delta^2) r} \left[f(R+r) - f(R-r) \right]$$

$$\therefore \quad \phi(r) = \frac{KQ}{2(1-\delta^2) rR} \left[f(R+r) - f(R-r) \right] \quad \text{for } r<R$$

$$\phi(r) = \frac{KQ}{2(1-\delta^2) rR} \left[f(r+R) - f(r-R) \right] \quad \text{for } r>R$$

If $\delta = 0$, then $\phi(r) = \frac{KQ}{2rR} \cdot 2R = \frac{KQ}{r}$; $r > R$

$$\phi(r) = \frac{KQ}{2rR} \cdot 2r = \frac{KQ}{R} \; ; \; r < R$$

(b) The potential ~~at~~ on the surface of the shell of radius a, ignoring $(1-\delta^2)$ term,

$$\phi_{a,a} = \frac{KQ_a}{2a.a} \cdot \left[f(a+a) - f(a-a) \right] \quad \Big| \; a > b$$

Similarly the potential of the ~~sp~~ shell with radius b

$$\phi_{a,b} = \frac{KQ_b}{2ab} \left[f(a+b) - f(a-b) \right]$$

$$\therefore \phi_a = \frac{KQ_a}{2a^2} f(2a) + \frac{KQ_b}{2ab} \left[f(a+b) - f(a-b) \right] \quad \text{——(i)}$$

Similarly, $\phi_b = \frac{KQ_b}{2b^2} f(2b) + \frac{KQ_a}{2ab} \left[f(a+b) - f(a-b) \right]$ ——(ii)

(c) For equipotential,

$$\phi_a = \phi_b = \phi$$

$$\Rightarrow \cancel{\frac{KQ_a}{2a^2} f(2a) + \frac{KQ_b}{2ab} \left[f(a+b) - f(a-b) \right] = \frac{KQ_b}{2b^2} f(2b) + \frac{KQ_a}{2ab} \left[f(a+b) - f(a-b) \right]}$$

Multiplying (i) by $a\left[f(a+b) - f(a-b) \right]$ and (ii) by $b f(2a)$ and subtracting —

$$\phi \left[b f(2a) - a\left[f(a+b) - f(a-b) \right] \right] = \frac{KQ_b}{2b} \left[-\left[f(a+b) - f(a-b) \right]^2 + f(2a) f(2b) \right]$$

$$\therefore Q_b = \frac{2b\phi}{k} \frac{bf(2a) - a[f(a+b) - f(a-b)]}{f(2a)f(2b) - [f(a+b) - f(a-b)]^2}$$

If $\delta = 0$, then $f(x) = x^{1-\delta} = x$

$$\therefore Q_b = \frac{2b\phi}{k} \frac{2ab - a \times 2b}{\text{-- -- -- -- --}}$$

$$\boxed{\therefore Q_b = 0}$$

Work and energy in electrostatics

Say, you have a configuration of source charges, and you want to move a test charge Q from $\vec{a}$ to $\vec{b}$. We will have apply a force which is equal to the electrostatic force, but in opposite direction. In this way, you apply the minimum force. You could apply more force than the electrostatic force, but that would increase the velocity and kinetic energy. We are only interested in the minimum force needed.

$$\therefore W = \int_a^b \vec{F} \cdot d\vec{s} = \int_a^b -Q\vec{E} \cdot d\vec{s} = -Q \int_a^b \vec{E} \cdot d\vec{s}$$

$$= Q[V(b) - V(a)]$$

$$\therefore V(b) - V(a) = \frac{W}{Q}$$

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Now, if you want to bring Q from very far away to a point $\vec{r}$, then, the work you must do is–

$$W = Q\left[V(\vec{r}) - V(\infty)\right]$$

$$\therefore W = QV(\vec{r})$$

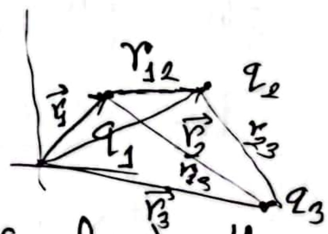
This is the work you will need, which is basically Q times the potential. This can also be quantified as the potential energy, as we will get this much work/energy back when the system is dismantled.

Energy of a point charge distribution

We want to calculate the work needed to assemble an entire collection of point charges. For the first charge q_1, there is no field and we do not need any work to to assemble it. For bringing q_2 at a position $\vec{r_2}$, where q_1 is at $\vec{r_1}$, ~~is given b~~ we need,

$$W_2 = q_2 V_1(\vec{r_2})$$

$$= q_2 \cdot \frac{Kq_1}{r_{12}}$$



For bringing a third charge (keeping q_1 and q_2 fixed), the work done will be given by,

$$W_3 = q_3 V_1(\vec{r_3}) + q_3 V_2(\vec{r_3})$$

$$= q_3 \frac{Kq_1}{r_{13}} + q_3 \frac{Kq_2}{r_{23}}$$

~~For~~ $\therefore$ Total work $= K\frac{q_1 q_2}{r_{12}} + K\frac{q_2 q_3}{r_{23}} + K\frac{q_3 q_1}{r_{31}}$

For a fourth charge q_4, we will have in total $^4C_2$ terms in our total work. In general,

$$W = K\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{r_{ij}}$$

$j>i$ is making sure we are not counting a pair twice.

If we count all the pairs twice, then –

$$W = \frac{1}{8\pi\epsilon_0}\cdot\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{r_{ij}}$$

$$\Rightarrow W = \frac{1}{2}\sum_{i=1}^{n} q_i\left[\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{r_{ij}}\right]$$

$$\boxed{\therefore W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r_i})} \quad \text{with} \quad V(\vec{r_i}) = \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{Kq_j}{r_{ij}}$$

The term in the parantheses is the potential at point $\vec{r_i}$ (where q_i is located) due to all other charges. This is the energy stored in a configuration.

The energy of a continuous charge distribution

$$W = \frac{1}{2}\int \rho V\, d\tau$$

We can write, $\vec{\nabla}\cdot\vec{E} = +\frac{\rho}{\epsilon_0}$

$$\therefore \rho = \epsilon_0\,\vec{\nabla}\cdot\vec{E}$$

$$\therefore \quad W = \frac{\epsilon_0}{2} \int (\vec{\nabla}.\vec{E})\, V\, d\tau$$

Now, in vector calculus, there is a vector identity,

$$\vec{\nabla}.(\vec{p}f) = f\, \vec{\nabla}.\vec{p} + \vec{\nabla}f.\vec{p}$$

where $\vec{p}$ is a vector field and f is a scalar function.

$$\therefore \quad W = \frac{\epsilon_0}{2} \int_V \left(\vec{\nabla}.(\vec{E}V) - \vec{\nabla}V.\vec{E} \right) d\tau$$

$$= \frac{\epsilon_0}{2} \left[-\int_V -\vec{E}.\vec{E}\, d\tau + \int_V \vec{\nabla}.(V\vec{E})\, d\tau \right]$$

$$= \frac{\epsilon_0}{2} \left[\int_V E^2 d\tau + \oint_{S=\partial V} V\vec{E}.d\vec{A} \right]$$

Now, consider the volume where we are integrating. Initially we started with, $W = \frac{1}{2}\int \rho V d\tau$. So, the integration was over the space where the charges are located. But, we could expand the boundary as large as we want, upto infinity, because in that region, $\rho = 0$ and it contributes nothing to the total integral. Now, as we increase the volume, the volume integral increases, while the surface integral must decrease to keep W constant. The volume integral increases since we are integrating over the whole space of the volume. So, the integral increases if we pick a larger volume. However, the surface integral is only computed on the boundary, which grows like r^2, where

$E \propto \frac{1}{r^2}$ and $V \propto \frac{1}{r}$. So, roughly speaking, the surface integral drops of as $\frac{1}{r}$ and if the volume is all space, then the surface integral goes to zero.

$$\boxed{\therefore \quad W = \frac{\epsilon_0}{2} \iiint_{\text{Whole space}} E^2 \, d\tau}$$

$$\iiint \frac{q^2}{(4\pi\epsilon_0)^2} \frac{1}{r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{q^2}{(4\pi\epsilon_0)^2} \times 4\pi \int_{R_1}^{R_2} \frac{1}{r^2} dr$$

$$= \frac{4\pi q^2}{16\pi^2\epsilon_0^2} \left[\frac{1}{R_1} - \frac{1}{R_2} \right]$$

If R_2 increases, the integral also increases.

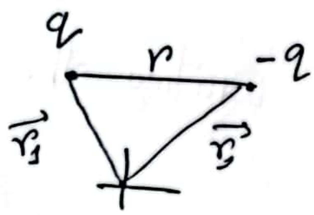
One could argue then that the electrostatic energy is stored in the electric field with energy density $U' = \frac{\epsilon_0}{2} E^2$.

Now, there is a particular inconsistency between two equations —

$$W = \frac{1}{2} \sum_i q_i V(\vec{r}_i)$$

$$W = \frac{\epsilon_0}{2} \iiint_{\text{All space}} E^2 d\tau$$

First equation tells you that energy can be both positive and negative. Say, for example, the energy of two equal and opposite charges at a ditance r apart is given by,



$$W = \frac{1}{2}\left[q_1 V(r_1) + q_2 V(r_2)\right]$$

$$= \frac{1}{2}\left[q \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{-q}{r} + (-q) \cdot \frac{q}{4\pi\epsilon_0 r}\right]$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2}$$

However, second equation says energy is always positive. Actually, they both are correct. In the first equation, we didn't take the contribution of energy to make the point charges. We merely calculated the work done for moving $-q$ to a distance r from q from infinity, where the charges were given readymade. But, ~~equation 1~~ second equation gives you energy of the whole configuration. But, there still is a problem with the second equation. If we calculate the total energy of a point charge, then,

$$W = \frac{\epsilon_0}{2} \iiint \frac{q^2}{(4\pi\epsilon_0)^2} \frac{1}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{\epsilon_0 q^2}{32\pi^2 \epsilon_0^2} \times 4\pi \int_0^\infty \frac{1}{r^2} dr$$

$$= \frac{q^2}{8\pi\epsilon_0} \left[-\frac{1}{r}\right]_0^\infty = \infty$$

But this doesn't make any sense, right? Where is the problem? The problem actually crept into one place. In the equation, $W = \frac{1}{2}\sum_i q_i V(\vec{r_i})$, $V(\vec{r_i})$ was the potential due to all the charges except i. But while writing the continuous form, we wrote,

$$W = \frac{1}{2} \iiint \rho \, V(\vec{r}) \, d\tau$$

where $V(\vec{r})$ is a continuous function, a full potential. For continuous distribution there is no problem, since the amount of charge right at $\vec{r}$ is vanishingly small and the contribution to potential is zero. But for point charges, we should use the first equation.

Energy of uniformly charged spherical shell

$$W = \frac{1}{2} \int \sigma V \, dA = \frac{1}{2} \int_{\text{Surface}} \sigma \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R} \cdot da$$

$$= \frac{q}{8\pi\epsilon_0 R} \cdot \int_{\text{surface}} \sigma \, da = \frac{q^2}{8\pi\epsilon_0 R}$$

In field method,

$$W = \frac{\epsilon_0}{2} \iiint_{\text{All space}} E^2 d\tau = \frac{\epsilon_0}{2} \left[\cancel{\int_{V,\,\text{inside}} E^2 d\tau} \to 0 + \int_{V,\,\text{outside}} E^2 d\tau \right]$$

$$= \int_{\text{outside}} \frac{q^2}{16\pi^2\epsilon_0^2} \cdot \frac{1}{r^4} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{q^2}{16\pi^2\epsilon_0^2} \times 4\pi \int_{r=R}^{r=\infty} \frac{1}{r^2} \, dr$$

$$= \frac{q^2}{8\pi\epsilon_0 R}$$

Energy for solid sphere of radius R and charge q

✓

ⓐ Using $W = \frac{1}{2}\iiint \rho V d\tau$.

ⓑ Field: $W = \frac{\epsilon_0}{2}\iiint_{\text{All space}} E^2 d\tau$

$$E_{in} = \frac{\rho}{3\epsilon_0} r \qquad E_{out} = \frac{\rho R^3}{3\epsilon_0 r^2}$$

$$\therefore \quad W = \frac{\epsilon_0}{2} 4\pi \int_{r=0}^{r=R} \frac{\rho^2}{9\epsilon_0^2} r^2 r^2 dr + \frac{\epsilon_0}{2} 4\pi \int_{r=R}^{r=\infty} \frac{\rho^2 R^6}{9\epsilon_0^2} \frac{1}{r^4} r^2 dr$$

$$= \frac{4\pi\epsilon_0\rho^2}{18\epsilon_0^2} \frac{R^5}{5} + \frac{4\pi\epsilon_0\rho^2 R^6}{18\epsilon_0^2}\left[\frac{1}{R}\right]$$

$$= \frac{4\pi\epsilon_0\rho^2}{18\epsilon_0^2}\left[\frac{R^5}{5} + R^5\right]$$

Superposition principle for energy

$$W_{net} = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int \left(E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2\right) d\tau$$

$$= W_1 + W_2 + \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d\tau$$

So, superposition principle doesn't hold.