where A_m corresponds to middle oscillation, A_f corresponds to fast oscillation and A_s corresponds to slow oscillation, corresponding to the frequencies. There are total six undetermined constants, which can be found by six initial conditions — three position and three velocities. Now,

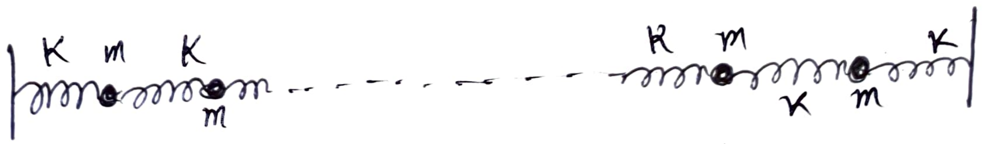
$$x_1 = A_m \cos(\sqrt{2}\,\omega_0 t + \phi_m) + A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

$$x_2 = -\sqrt{2} A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + \sqrt{2} A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

$$x_3 = -A_m \cos(\sqrt{2}\,\omega_0 t + \phi_m) + A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

N masses

Now, we are ready to derive the result for N masses. We will take equal masses and equal spring constants. So, all our masses are m and spring constants are K, and they are connected with each other, with the two remote springs connected to two fixed wall.



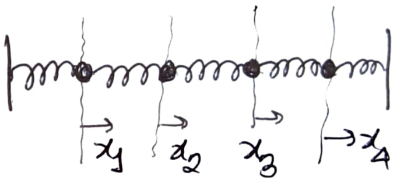
The displacements of the masses relative to the equilibrium positions are given by $x_1, x_2, \ldots, x_N$.
The displacement of the boundary walls will be denoted by x_0 and x_{N+1}, and since they are fixed, so,

$$x_0 = x_{N+1} = 0.$$

Now, for three masses, we have seen –

$$m\ddot{x}_1 = -Kx_1 - K(x_1 - x_2)$$
$$m\ddot{x}_2 = -K(x_2 - x_1) - K(x_2 - x_3)$$
$$m\ddot{x}_3 = -K(x_3 - x_2) - Kx_3$$

For four masses, it would have been –



Check

$$\begin{cases} m\ddot{x}_1 = -Kx_1 - K(x_1 - x_2) \\ m\ddot{x}_2 = -K(x_2 - x_1) - K(x_2 - x_3) \\ m\ddot{x}_3 = -K(x_3 - x_2) - K(x_3 - x_4) \\ m\ddot{x}_4 = -K(x_4 - x_3) - Kx_4 \end{cases}$$

We can then generalize for the n^{th} mass as –

$$m\ddot{x}_n = -K(x_n - x_{n-1}) - K(x_n - x_{n+1})$$

$$\boxed{\therefore m\ddot{x}_n = Kx_{n-1} - 2Kx_n + Kx_{n+1}} \quad \text{--- (1)}$$

We want to write the matrix equation, $M\ddot{X} = -KX$

Let's look at how the K-matrix looks like.

For three masses,
$$K = \begin{pmatrix} 2K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & 2K \end{pmatrix}$$

For four masses,
$$K = \begin{pmatrix} 2K & -K & 0 & 0 \\ -K & 2K & -K & 0 \\ 0 & -K & 2K & -K \\ 0 & 0 & -K & 2K \end{pmatrix}$$

$$m\ddot{x}_1 = -2Kx_1 + Kx_2$$
$$m\ddot{x}_2 = Kx_1 - 2Kx_2 + Kx_3$$
$$m\ddot{x}_3 = Kx_2 - 2Kx_3 + Kx_4$$
$$m\ddot{x}_4 = +Kx_3 - 2Kx_4$$

So, for the middle masses (without two boundary masses) the entries are always in the pattern of $-K \quad 2K \quad -K$ and the other entries are simply zero. For N masses,

$$K = \begin{pmatrix} \cdots & -K & 2K & -K & \cdots & & \\ & \cdots & -K & 2K & -K & \cdots & \\ & & \cdots & -K & 2K & -K & \cdots \end{pmatrix}$$

for $(n-1)^{th}$, n^{th} and $(n+1)^{th}$ masses. ~~having the non-zero elements~~.

$$X = \begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} m & 0 & 0 & \cdots \\ 0 & m & 0 & \cdots \\ 0 & 0 & m & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$= mI$$

$$\therefore \; \boxed{m\, I_{N\times N}\, \ddot{X}_{N\times 1} = -K_{N\times N}\, X_{N\times 1}} \longrightarrow (ii)$$

This is our familiar equation. But we do not have to stress about this. We again are going to guess the solution of the form,

$$X = \begin{pmatrix} \vdots \\ A_{n-1} \\ A_n \\ A_{n+1} \\ \vdots \end{pmatrix} e^{i\omega t}$$

Equation (ii) will will simply then become,

$$\left(K - \omega^2 m I\right) X = 0$$

We can go on solving this using the deter-minent method, by setting,

$$\det\left[K - \omega^2 m I\right] = 0$$

However, for large N, this is going to take some ~~vallian~~ valiant effort. If you are like Captain America, and you can do this all day, you can go on with solving with determinent method for large N. But, although being a fan of Cap, I am going to skip this method and try something new, that would save much time.

We plug our solution into the equation of motion (i), that is —

$$m\ddot{x}_n = Kx_{n-1} - 2Kx_n + Kx_{n+1}$$

Plugging in, $x_n = A_n e^{i\omega t}$ we get,

$$m(i\omega)^2 A_n e^{i\omega t} = (KA_{n-1} - 2KA_n + KA_{n+1})e^{i\omega t}$$

$$\Rightarrow -m\omega^2 A_n = K(A_{n-1} - 2A_n + A_{n+1})$$

$$\Rightarrow -\omega^2 A_n = \omega_0^2 (A_{n-1} - 2A_n + A_{n+1}) \quad \text{with } \omega_0 = \sqrt{\tfrac{K}{m}}$$

$$\Rightarrow (2\omega_0^2 - \omega^2)A_n = \omega_0^2 (A_{n-1} + A_{n+1})$$

$$\therefore \boxed{\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}} \longleftarrow \text{(III)}$$

This equation must hold for any value of $n = 1$ to N. So, we have N number of equations here. But, how do we get a solution?

There is a catch here. For a particular normal mode, that is, for a particular value of ω, ~~fixed~~ the right hand side is a constant, that is, independent of n. So, the left hand side should also be independent of n, and hence, the ratio $\frac{A_{n-1} + A_{n+1}}{A_n}$ is independent of n.

So, we look for such values of $A_1, A_2, \ldots, A_N$ where this ratio is the same for all n.

Now, if you ~~or~~ are given three adjacend A's, say A_4, A_5, A_6; then, you can calculate any value of A using the recursive method.

$$A_n = A_{n-1} + A_{n+1} \rightarrow \begin{aligned} A_6 &= A_5 + A_7 \\ A_7 &= A_6 + A_8 \\ A_8 &= A_7 + A_9 \end{aligned}$$

and so on.

Alternatively, if you are given two adjacent A's and the value of ω, that will also suffice, since you can then find another value of A from here and start the recursive method.