Classical Mechanics

Lecture #1

What is classical mechanics?

Befor the discovery of quantum mechanics, there was only mechanics. Thus classical mechanics is that part of mechanics which excludes quantum mechanics. Roughly speaking, classical mechanics describes the physics of particles and bodies which are much larger than atoms and molecules. However the demarcation between the two realms is not always clear.

In this course we shall mainly be concerned with reformulating Newton's laws in more sophisticated and elegant forms. But it turns out these laws break down not only when the particles are very small but also when the velocity of the particles are great, i.e. close to the speed of light. Infact, light not only breaks Newtonian physics, it also breaks the space-time in which Newtonian physics lives.

But this is only noticeable when speeds of the particles approach the speed of light. So, mechanics also involves the introduction of the special theory of relativity. Since this mechanics is also not quantum, we must include it in classical mechanics. Infact it is encumbent upon special relativity to yield Newtonian mechanics in the limit $c \to \infty$ (taking the speed of light to infinity).

The special Theory of relativity introduces a new geometry of space-time known as Minkowski space-time.

Review of Newtonian Mechanics:

Let $\vec{r} = (x, y, z)$ be the position of a particle which is defined to have insignificant dimensions but has non-zero mass. Then Newton's second law states:

$$\boxed{\vec{F}(\vec{r}, \dot{\vec{r}}) = \dot{\vec{p}}}$$

where $\vec{F}$ is the net force on the particle and $\vec{p} = m\dot{\vec{r}}$ is its momentum. We are usually able to identify the sources that contribute to the net force: electromagnetic field, gravitational field, contact forces etc. If $\vec{F}_{net} = 0$ then we get $\vec{p}$ is a constant. A reference frame in which $\vec{p}$ is constant in the absence a net force due to 'real' forces is called an intertial frame. This is Newton's 1st law.

Examples of non-inertial frames: 1. A ref. frame attached to a rotating body. 2. A reference frame attached to any accelerating body. Infact $2 \Rightarrow 1$.

Galilean Relativity:

Newton's law is invariant (means doesn't change its form) under the following continuous transformations:

1. 3 Rotations $\vec{r} \to \vec{r} = O\vec{r}$ where O is a 3×3 orthogonal matrix and $\det O = 1$.

2. Space translation in 3-directions: $\vec{r} \longrightarrow \vec{r} + \vec{a}$ where $\vec{a}$ is constant displacement vector.

3. Translation in the time direction: $t \to t + c$ for c a real number.

4. Velocity boosts in 3-directions: $\dot{\vec{r}} \to \dot{\vec{r}} + \vec{v}$ where $\vec{v}$ is a constant velocity vector.

These 10 transformations can be combined in any way imaginable and we will still get a symmetry of Newton's laws. These transformations together form a closed set under composition and therefore they form a group, the Galilean group.

The Galilean group can be thought of as capturing the symmetries of Newtonian space-time. Just as different spaces can have different symmetries, space-times can also have different symmetries.

Examples: What are the symmetries of:

1. a Sphere $SO(3)$
2. An Euclidean plane $SO(2) \ltimes \mathbb{R}^2$
3. A Lobachevsky plane (2-d hyperboloid) $SO(1,2)$

Torque and Angular momentum:

These concepts are important for rotating bodies and equilibrium of extended objects. If we define torque on a particle as:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

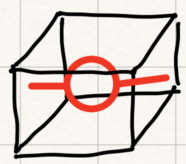
and the angular momentum as $\vec{l} = \vec{r} \times \vec{p}$ Then Newton's law gives us:

$$\boxed{\vec{\tau} = \dot{\vec{l}}}$$

This looks very similar to Newton's law and is a simple consequence of it. Note $\vec{\tau} = 0$ doesn't mean $\vec{F} = 0$. $\vec{\tau}$ & $\vec{l}$ are measured with respect to a single point.

Analogous to $\vec{F} = 0 \Rightarrow \vec{p} \sim$ constant we also have $\vec{\tau} = 0 \Rightarrow$

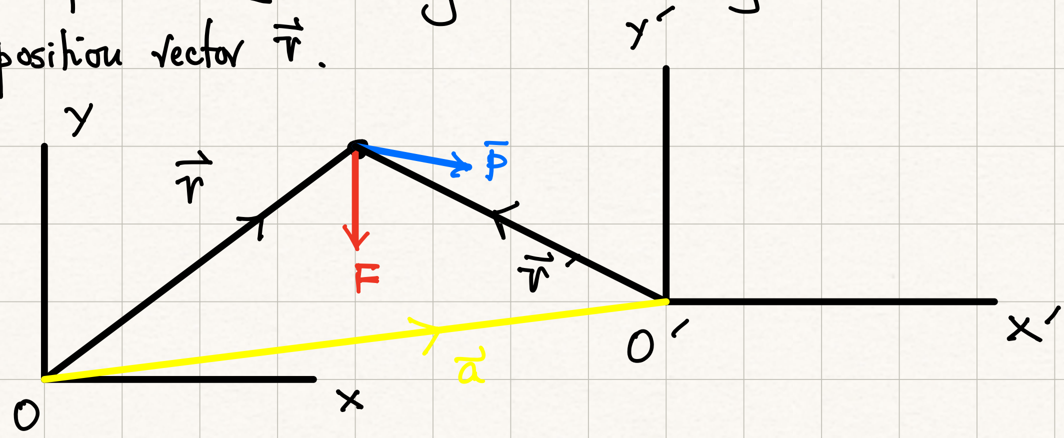
$\vec{l} \sim$ constant. Conservation of angular momentum.

<u>Example:</u> Suppose there is a rotating object (say a motorized rotor) in an  opaque box.

Suppose the box is sound proof and the rotor is so smooth that it produces no vibrations.

Is there a way to test whether the motor is on without opening the box?

* Note that the definition of $\vec{\tau}$ and $\vec{l}$ involve the origin of our reference system since they both involve the position vector $\vec{r}$.



$\vec{\tau} = \vec{r} \times \vec{F}$ $\qquad$ $\vec{\tau}' = \vec{r}' \times \vec{F}$

$\otimes$ $\qquad$ $\odot$

$\vec{l} = \vec{r} \times \vec{p}$ $\qquad$ $\vec{l}' = \vec{r}' \times \vec{p}$

$\otimes$ $\qquad$ $\odot$

We see from the above example that $\vec{\tau}$ & $\vec{\ell}$ are different in different inertial reference frames. What does it imply for the law:

$$\vec{\tau} = \dot{\vec{\ell}} \quad ?$$

To see how this law would look like in the primed ref. frame we assume the two frames are related by a Galilean translation vector:

$$\vec{r}(t) = \vec{r}\,'(t) + \vec{a} \quad , \qquad \vec{a} \text{ is a constant}$$

Then $\quad \vec{\tau} = \vec{r}\,' \times \vec{F} + \vec{a} \times \vec{F} = \vec{\tau}\,' + \vec{a} \times \vec{F}$

and $\quad \vec{\ell} = \vec{\ell}\,' + \vec{a} \times \vec{p}$

Thus in the new reference frame we get:

$$\vec{\tau}\,' + \vec{a} \times \vec{F} = \dot{\vec{\ell}}\,' + \vec{a} \times \dot{\vec{p}}$$

$$\Rightarrow \qquad \vec{\tau}\,' = \dot{\vec{\ell}}\,' + \vec{a} \times \underbrace{(\dot{\vec{p}} - \vec{F})}_{=0}$$

$$\Rightarrow \quad \vec{\tau}\,' = \dot{\vec{\ell}}\,'$$

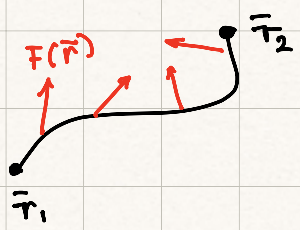
Thus we see, even though $\vec{\tau}$ & $\vec{\ell}$ depend on the reference frame the equation relating them is independent of the ref. frame used as long as it is inertial.

Energy and Work

The Work-Kinetic Energy Theorem:

Now suppose that the mass m of our particle remains constant and we do work on the particle by applying force on it.

$$W_{\vec{F}} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$



Note that the displacement of the particle need not be exclusively due to the force whose work is being calculated; The could be other forces involved in moving the particle around.

If the motion of the particle is exclusively due to the force $\vec{F}$, then according to Newton's law the work done by the force is

$$W_{\vec{F}} = m \int_{\vec{r}_1}^{\vec{r}_2} \ddot{\vec{r}} \cdot d\vec{r}$$

But $\vec{r} = \vec{r}(t)$ since it is the position of the particle

We can write

$$d\vec{r} = \frac{d\vec{r}}{dt}\,dt = \dot{\vec{r}}\,dt$$

Then
$$W = m\int_{t_1}^{t_2} \ddot{\vec{r}}\cdot\dot{\vec{r}}\,dt = \frac{m}{2}\int_{t_1}^{t_2} \frac{d}{dt}(\dot{\vec{r}}\cdot\dot{\vec{r}})\,dt = \frac{m}{2}\int_{\dot{\vec{r}}_1^2}^{\dot{\vec{r}}_2^2} d(\dot{\vec{r}}^2)$$

$$\equiv |\dot{\vec{r}}|^2$$

$$W = \frac{1}{2}m\dot{\vec{r}}_2^{\,2} - \frac{1}{2}m\dot{\vec{r}}_1^{\,2}$$

$$\boxed{W = T(t_2) - T(t_1)}$$ $\rightsquigarrow$ Work-Kinetic Energy Theorem

where $T(t) = \frac{1}{2}m\dot{\vec{r}}(t)^2$ is called the kinetic energy.

Conservative forces:

If the work done by a force when the particle is moved around in a closed loop is zero then we call that force conservative:

$$\oint \vec{F}\cdot d\vec{r} = 0$$

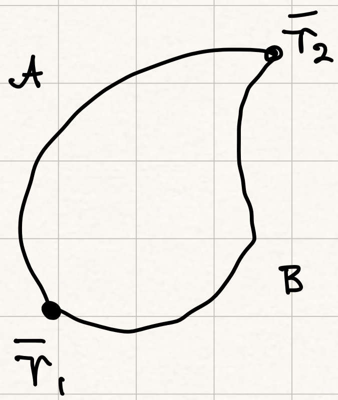
If this is the case we can introduce a scalar function V, associated with the conservative force such that:

$$\vec{F} = -\vec{\nabla}V(\vec{r})$$

Note that $V(\vec{r})$ is defined up to an arbitrary constant: $V(\vec{r})$ & $V(\vec{r}) + c$ gives rise to the same $\vec{F}$. Then we have

$$\oint (\vec{\nabla}V)\cdot d\vec{r} = 0$$

This means we can associate to each point in space a unique value of $V(\vec{r})$ and, as the particle is moved by the force from point $\vec{r}_1$ to $\vec{r}_2$, the change in the potential energy is **independent** of the path it takes.



$$\oint \vec{F}\cdot d\vec{r} = 0$$

conservative

$$= \underbrace{\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}}_{\text{Path A}} + \underbrace{\int_{\vec{r}_2}^{\vec{r}_1} \vec{F}\cdot d\vec{r}}_{\text{Path 2}}$$

$$\Rightarrow \boxed{\underbrace{\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}}_{\text{Path A}} = \underbrace{\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{r}}_{\text{Path B}}} \longrightarrow$$

Path Indepence of Conservative forces.

And so applying the work-energy theorem:

$$-V(\vec{r}_2) + V(\vec{r}_1) = T(t_2) - T(t_1)$$

$$\Rightarrow \quad V(\vec{r}_2(t_2)) + T(t_2) = V(r_1(t_1)) + T(t_1) \equiv E$$

Thus we see that the sole action of a conservative force The first integral of motion $E = V(t) + T(t)$ is conserved.

$E \rightarrow$ Total mechanical energy.