Chapter 1: Preliminary Algebra

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1 Polynomials and polynomial equations

Recall that a *polynomial* is an algebraic function which can be represented as,

$$f(x) = a_1 x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, \tag{1}$$

where $f(x)$ reads as *function of* x where x is the general symbol used to identify some unknown parameter, and $\{a_0, a_1, .., a_n\} \in \mathbf{R}$ are known coefficients. Commonly we refer to $f(x)$ as an $n^{th}-$degree polynomial, i.e. the highest power of x is n with $n > 0$ and $n \in \mathbb{Z}$. Thus, it follows from here that,

$$f(x) = a_1 x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0 = 0, \tag{2}$$

is a *polynomial equation* which is satisfied by particular values of x, called the *roots* of Eq.(2) or alternatively, the *zeros* of the function $f(x)$. In physical problems, we are generally interested in finding some or all the roots of an equation such as Eq.(2), α_k, that satisfy $f(\alpha_k) = 0$ where the index $k \in \mathbb{Z}$ and can take up any value from $1, 2, ..., n$. When α_k are *real*, they correspond to the points at which a graph of $f(x)$ intersects the $x-$axis, while for *imaginary* α_k no such graphical interpretation exists. Some general polynomial equations and their solutions are given below.

- Linear equation ($n = 1$): For the case of $n = 1$, Eq.(2) reduces to,

  $$a_1 x + a_0 = 0, \tag{3}$$

  with root $\alpha_1 = -a_0/a_1$.

- Quadratic equation ($n = 2$): For the case of $n = 2$, Eq.(2) reduces to,

  $$a_2 x^2 + a_1 x + a_0 = 0, \tag{4}$$

  which has two roots α_1 and α_2 given by,

  $$\alpha_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}. \tag{5}$$

  – If the value of the quantity under the radical (or square root) is positive, then both roots are real.

  – If, on the other hand, the value of the quantity is negative, then the roots form a *complex conjugate* pair of the form $p \pm iq$ where $p, q \in \mathbb{R}$.

  – If the quantity is zero, then the roots are said to be equal.

In cases of $n = 3$ or $n = 4$, we have *cubic* and *quartic* equations respectively. The solutions for Eq.(2) in cases when $n \geq 4$, are not very obvious and easy to calculate and as such we often resort to approximation methods.

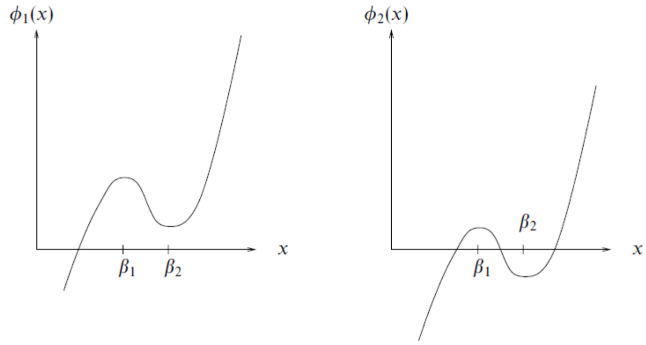

Figure 1: Two cubic functions with different number of real roots.

Example 1. Let us investigate what we can learn about the roots of the cubic equation

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0.$$

Note that: if $x \to +\infty$ then $g(x) \to +\infty$, while if $x \to -\infty$ then $g(x) \to -\infty$;

from which it follows intuitively that: $g(x) = 0$ must have at least one real root.

Next, we ask, "how many real roots *could* $g(x) = 0$ have?" To answer this question, we recall a fundamental theorem of algebra that is stated below without proof,

Theorem: *An nth−degree polynomial equation has exactly n roots.*

To decide how many real roots $g(x) = 0$ has, we implement the notions of the *derivative of a function* and a known result from calculus known as *Rolle's theorem.*

Derivative of $g(x)$: $\quad g'(x) = \dfrac{d}{dx}(4x^3 + 3x^2 - 6x - 1) = 12x^2 + 6x - 6.$

Rolle's theorem states if $f(x)$ has equal values at two different values of x then at some point between these two $x-$values its derivative is equal to zero; i.e. the tangent to the graph is parallel to the $x-$axis at that point. Thus, if $g(x) = 0$ has three real roots $\alpha_1, \alpha_2, \alpha_3$, then by Rolle's theorem, between α_1 and α_2 there must be a real value of x at which $g'(x) = 0$, which also holds true between α_2 and α_3. Thus a ***necessary*** but **not *sufficient*** condition for three real roots of $g(x) = 0$ is that $g'(x) = 0$ itself has two real roots.

Hence, returning to the original equation at hand,

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0$$

from which we can investigate the roots of,

$$g'(x) = 12x^2 + 6x - 6 = 0 \Rightarrow 2x^2 + x - 1 = 0$$

which yields,

$$\beta_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} \Rightarrow \beta_1 = \frac{1}{2} \text{ and } \beta_2 = -1.$$

From this we get,

$$g(\beta_1) = -\frac{11}{4} \text{ and } g(\beta_2) = 4 \ .$$

Since these are of opposite sign, it is indicative of three real roots of $g(x) = 0$. One of these roots lies in between $-1 < x < \frac{1}{2}$, one lies in the region $x < -1$ and the other lies in the region $x > \frac{1}{2}$. This is supported by Figure 1, which shows two functions $\phi_1(x)$ and $\phi_2(x)$, both with zero derivatives at the same values of $x = \beta_1, \beta_2$ but $\phi_1(x)$ has one real root but $\phi_2(x)$ has three real roots.

Example 2. A more general polynomial equation reads,

$$f(x) = x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0 \ .$$

Without having to solve this higher degree equation, we can make the following general arguments:

- $f(x) = 0$ is a $7^{th}-$degree polynomial and thus the number of real roots it can have is $1, 3, 5, \text{ or } 7$.
- $f(0) = -2$ while $f(+\infty) = +\infty$ which necessitates that there be at least 1 positive root.
- Now, investigating the roots of of $f'(x) = 0$, we have

  $$f'(x) = x(7x^5 + 30x^4 + 4x^2 - 3x + 2) = 0 \Rightarrow x = 0 \text{ is a root of } f(x) = 0.$$

  Furthermore,
  $$f''(x) = 42x^5 + 150x^4 + 12x^2 - 6x + 2 \Rightarrow f''(0) = +2$$
  suggests that $x = 0$ is a *minimum* of $f(x)$.
- To get a count of the number of *real*, positive roots of $f(x) = 0$ we can apply

  Descartes' rule of sign $-$ let the number of variations in the signs of the coefficients $a_n, a_{n-1}, .., a_0$ be v and the number of real positive roots be n_p. Then it must hold that,

  1. $n_p \leq v$,
  2. $v - n_p$ is an even integer.
  3. A negative zero of $f(x)$ is a positive zero of $f(-x)$.

  For the $f(x)$ at hand we have $v = 3$ so that $n_p = 0, 1, 2, \text{ or, } 3$. But, by Rule 2: $n_p = 1, \text{ or, } 3 \Rightarrow f(x) = 0$ has either 1 or 3 real *positive* roots. Investigating Rule 3: for $f(-x) = -x^7 + 5x^6 + x^4 + x^3 + x^2 - 2$ we have $v = 2 \Rightarrow n_p = 0, 1, \text{ or, } 2$ which after Rule 2 becomes $n_p = 0, \text{ or, } 2 \Rightarrow f(x) = 0$ has either 0 or 2 real *negative* roots.
- The remaining roots must then be complex conjugate pairs since the coefficients are all real.

1.1 Factorising Polynomials

A polynomial with r given distinct zeros α_k can be written as the product of factors containing these zeros,

$$\begin{aligned} f(x) &= a_n(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}...(x-\alpha_r)^{m_r} \\ &= a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \end{aligned} \tag{6}$$

with $m_1 + m_2 + ... + m_r = n$, the degree of the polynomial. With no loss of generality we can assume $m_k = 1$ for all k (i.e. all zeros are *simple*) and $r = n$. Sometimes, what we want to do to investigate or find the roots of $f(x) = 0$, is reverse the process such that if, say α_1, has been found by some method, we can write Eq.(6) as

$$f(x) = (x-\alpha_1)f_1(x), \tag{7}$$

where $f_1(x)$ is an $n-1-$degree polynomial. Now, let

$$f_1(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + ... + b_1 x + b_0 \tag{8}$$

and substitute into Eq.(7) which leads to

$$\begin{aligned} f(x) &= (x-\alpha_1)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + ... + b_1 x + b_0) \\ &= b_{n-1}x^n - \alpha_1 b_{n-1}x^{n-1} + b_{n-2}x^{n-1} - \alpha_1 b_{n-2}x^{n-2} + ... + b_1 x^2 - \alpha_1 b_1 x + b_0 x - \alpha_1 b_0 \end{aligned} \tag{9}$$

and comparing coefficients between Eq.(9) and the second line of Eq.(6) yields,

$$\begin{aligned} & b_{n-1} = a_n \\ & b_{n-2} - \alpha_1 b_{n-1} = a_{n-1} \\ & \vdots \\ & b_0 - \alpha_1 b_1 = a_1 \\ & -\alpha_1 b_0 = a_0. \end{aligned} \tag{10}$$

These can be solved successively and the final equation serves as a check; if it is not satisfied then either at least one mistake was made in the computation or α_1 is not a root of $f(x) = 0$.

Example 3. Determine, by inspection, the simple roots of the equation

$$f(x) = 3x^4 - x^3 - 10x^2 - 2x + 4 = 0$$

and hence, by factorisation, find the rest of its roots.

By simple observation we notice that $x = -1$ is a root of $f(x) = 0$ which means $(x+1)$ is a factor of the equation. Thus,

$$f(x) = (x+1)(b_3x^3 + b_2x^2 + b_1x + b_0)$$

where from Eq.(10) with $\alpha_1 = -1$, we get

$$b_3 = 3,$$

$$b_2 + b_3 = -1 \Rightarrow b_2 = -1 - 3 = -4,$$

$$b_1 + b_2 = -10 \Rightarrow b_1 = -10 - (-4) = -6,$$
$$b_0 + b_1 = -2 \Rightarrow b_0 = -2 - (-6) = 4$$

which leads to,

$$f(x) = (x+1)f_1(x) = (x+1)(3x^3 - 4x^2 - 6x + 4)\ .$$

Now, note that

$$f_1(x) = 0 \text{ when } x = 2 \Rightarrow (x-2) \text{ is a factor of } f_1(x)$$

and so it follows that,

$$f_1(x) = (x-2)f_2(x) = (x-2)(c_2x^2 + c_1x + c_0)$$

and similarly as before we get,

$$c_2 = 3,$$
$$c_1 - 2c_2 = -4 \Rightarrow c_1 = -4 + 2(3) = 2,$$
$$c_0 - 2c_1 = -6 \Rightarrow c_0 = -6 + 2(2) = -2$$

which leads to

$$f_2(x) = 3x^2 + 2x - 2$$

and the solutions for $f_2(x) = 0$ is given by,

$$x = \frac{-2 \pm \sqrt{4+24}}{6} \Rightarrow x_\pm = -\frac{1}{3}(1 \mp \sqrt{7})\ .$$

Thus, $f(x) = 0$ has four real, simple roots: $x = -1, 2, -\frac{1}{3}(1-\sqrt{7})$ and $-\frac{1}{3}(1+\sqrt{7})$.

2 Coordinate Geometry - A Quickfire Review

The standard form for the equation of a straight-line graph is,

$$y = mx + c \tag{11}$$

which shows that the *dependent variable* y varies linearly on the *independent variable* x. Here, m represents the tangent of the angle that the line makes with the $x-$axis (i.e. the *slope* or *gradient* of the graph) whilst c denotes the $y-$intercept. Alternatively, one can rewrite Eq.(11) in the form,

$$ax + by + k = 0 \tag{12}$$

wherein one can identify,

$$m = -\frac{a}{b} \text{ and } c = -\frac{k}{b}$$

by comparing the two equations. Notice, Eq.(12) treats x and y on a more symmetrical basis such that the intercepts on the two axes are $-k/a$ and $-k/b$ respectively.

Often in physics applications, we want to display results of an analysis on a simple plot. As such, power-law relations are usually cast into straight-line form and then plotted on a logarithmic scale. Consider the relationship $y = Ax^n$, where A is some constant. Using law of logarithms we get,

$$\ln y = \ln(Ax^n) = \ln A + \ln x^n$$

which in straight-line form becomes,

$$\ln y = n \ln x + \ln A \ , \tag{13}$$

and we can identify the slope (or m) of the $\ln y$ graph given by the power n and intercept on the $\ln y-$axis as being given by $\ln A$.

The other standard coordinate forms of two-dimensional forms that one should recall are those concerned with *conic sections* – named so because they can all be obtained by taking suitable sections across a double cone. The general form of the equation obtained in this manner is,

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \tag{14}$$

which can be rearranged to represent four generic forms: *ellipse*, *parabola*, *hyperbola* and a *degenerate pair of straight lines*. If the axes of symmetry are made to coincide with the coordinate axes, the first three take the forms

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1 \qquad \text{(ellipse)}, \tag{15}$$

$$(y-\beta)^2 = 4a(x-\alpha) \qquad \text{(parabola)}, \tag{16}$$

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1 \qquad \text{(hyperbola)}. \tag{17}$$

Here, a, b are coordinates and (α, β) is usually taken to be the origin and gives the 'center' of the curve. Additionally, one recognizes that the circle represented by,

$$(x-\alpha)^2 + (y-\beta)^2 = a^2 \qquad \text{(circle)} \tag{18}$$

as being the special case of an ellipse for which $b = a$.

3 Partial Fractions

We will need to know methods to simplify ratios of polynomials of the form $f(x) = \dfrac{g(x)}{h(x)}$. In such a setup, the behavior of $f(x)$ is crucially determined by the location of the zeros of $h(x)$ since $f(x)$ changes rapidly when x is close to those α_i which are the roots of $h(x) = 0$. To make such behavior explicit, one expresses $f(x)$ as a summation of terms of the form $A/(x-\alpha)^n$, where A is a constant, α is one of the roots of $h(x) = 0$ and n is a positive integer. Writing a function in this manner is known as expressing it in *partial fractions*, and this has wide-ranging applications. Let us review the steps through an example.

Example 4. Suppose we want to express the following function in partial fractions,

$$f(x) \equiv \frac{g(x)}{h(x)} = \frac{4x+2}{x^2+3x+2}.$$

Convince yourself that the factors of $h(x)$ are $(x+1)$ and $(x+2)$ which identify *distinct*, real roots of $h(x) = 0$. As such, we can write,

$$\frac{4x+2}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}.$$

The goal is to determine the values of A and B and while there are multiple ways one can do this, the straightforward way is to simplify the RHS of the equation and compare coefficients from both sides. Hence,

$$\text{RHS} \;\equiv\; \frac{A(x+2)+B(x+1)}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

which leads to,

$$\frac{4x+2}{(x+1)(x+2)} = \frac{(A+B)x+(2A+B)}{(x+1)(x+2)}$$

and comparing coefficients yields,

$$4 = A+B, \qquad\qquad 2 = 2A+B \Rightarrow B = 2-2A$$

thus the first expression becomes

$$4 = A+2-2A \Rightarrow 2 = -A \Rightarrow \boxed{A=-2} \Rightarrow \boxed{B=6}\,.$$

Thus our function simplifies to,

$$\boxed{f(x) = \frac{4x+2}{x^2+3x+2} \equiv -\frac{2}{x+1} + \frac{6}{x+2}}\,.$$

There are complicated and special cases where the above recipe needs to be modified but that is left to the reader to stress over. Consider the following exercises to brush up on partial fractions.

Exercise 1. Find the partial fraction decomposition of the function,

$$\frac{x^3+3x^2+2x+1}{x^2-x-6}\,.$$

Hint: try long-division or factorisation before applying the recipe from the example above.

Exercise 2. Find the partial fraction decomposition of the function,

$$\frac{x^5-2x^4-x^3+5x^2-46x+100}{(x^2+6)(x-2)^2}\,.$$

Hint: try long-division first and be a little careful before directly applying the recipe from the example above. There is a subtlety in the application here due to the squared polynomial in the denominator.