Lecture 6

Superposition of periodic motions

Many physical situations involve simultaneous application of two or more harmonic vibrations to the system at the same time. If the system is linear, then the resultant of two or more harmonic vibrations will be taken to be simply the sum of the individual vibrations. We will be interested in such systems.

The superposed vibrations of equal frequency

Let's say, two SHM are described by the following ~~equation~~ equations (of same frequency) -

$$x_1(t) = A_1 \cos(\omega t + \phi_1)$$

$$x_2(t) = A_2 \cos(\omega t + \phi_2)$$

We have already seen, the combination of them will again be a sinusoidal motion. Let's prove this.

$$\therefore \quad x(t) = x_1(t) + x_2(t)$$

$$= A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$$

$$= A_1 \cos\omega t \cos\phi_1 - A_1 \sin\omega t \sin\phi_1 + A_2 \cos\omega t \cos\phi_2 - A_2 \sin\omega t \sin\phi_2$$

$$= (A_1 \cos\phi_1 + A_2 \cos\phi_2)\cos\omega t - (A_1 \sin\phi_1 + A_2 \sin\phi_2)\sin\omega t$$

$$= C\cos\omega t - D\sin\omega t$$

which can be written as,

$$x(t) = A\cos(\omega t + \phi)$$

with $A = \sqrt{C^2 + D^2}$ and $\phi = \tan^{-1}\frac{D}{C}$

With a little algebra, you will be able to show that,

$$\left.\begin{aligned} A &= \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 + \phi_2)} \quad \text{and} \\ \phi &= \tan^{-1}\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}. \end{aligned}\right\} \text{---} ①$$

Superposed vibrations with different frequency: beats

Let's now consider superposition of two SHM with ~~ap~~ amplitudes A_1 and A_2 and with different frequency ω_1 and ω_2. To make calculations clean, we are assuming the phase constants are zero.

$$\therefore x_1 = A_1\cos\omega_1 t \qquad x_2 = A_2\cos\omega_2 t$$

Now, $x = x_1 + x_2$

$$= A_1\cos\omega_1 t + A_2\cos\omega_2 t$$

$$= A_1\cos\left(\frac{\omega_1+\omega_2}{2}t + \frac{\omega_1-\omega_2}{2}t\right) + A_2\cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{\omega_1-\omega_2}{2}t\right)$$

$$= A_1 \cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) - A_1 \sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$+ A_2 \cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) + A_2 \sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$= (A_1+A_2)\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cos\left(\frac{\omega_1-\omega_2}{2}t\right) + (A_2-A_1)\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$= C\cos\left(\frac{\omega_1+\omega_2}{2}t\right) + D\sin\left(\frac{\omega_1+\omega_2}{2}t\right)$$

with $C = (A_1+A_2)\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$ and $D = (A_2-A_1)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$

$$\therefore\ x(t) = A\cos\left(\frac{\omega_1+\omega_2}{2}t \mp \phi\right)$$

with ~~A and~~ $A = \sqrt{C^2+D^2}$ and $\phi = \tan^{-1}\frac{D}{C}$.

This won't be simple cosine wave, as the amplitude A and ϕ will now be dependent on t (check this). However, if $\omega_1 = \omega_2 = \omega$, then,

$$C = A_1 + A_2 \quad \text{and} \quad D = 0$$

Then, $x(t) = A\cos(\omega t)$

with $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = A_1 + A_2$

You can compare this result with equations (1), with $\phi_1 = \phi_2 = 0$, and you will see everything checks out perfectly. Then it just becomes a cosine ~~w~~ function with amplitude $A_1 + A_2$.

Beats

Interesting phenomenon occurs when $A_1 = A_2 = A$ and the frequencies are different.

$$\therefore x_1(t) = A\cos\omega_1 t \quad \text{and} \quad x_2(t) = A\cos\omega_2 t$$

$$\therefore x(t) = A\left[\cos\omega_1 t + \cos\omega_2 t\right]$$

$$= 2A\cos\frac{\omega_1+\omega_2}{2}t \cos\frac{\omega_1-\omega_2}{2}t$$

$\omega = 2\pi f$

$T = \frac{2\pi}{\omega} = \frac{1}{f}$

Let's think about two very close frequencis— say $f_1 = 100\,Hz$ and $f_2 = 102\,Hz$. So, the angular frequency of the first cosine term is 202π and the angular frequency of the second cosine term is 2π. So, the time period of oscillation is $\frac{1}{101}$ s ~~and~~ of the first cosine term and 1 s for the ~~second~~ second cosine term.

The effective motion then is, a rapidly oscillating function within ± 1, scaled by a very slowly oscillating function within $\pm 2A$. In other language, it is said that the amplitude is being modulated.

$$\therefore x(t) = A_m \cos\frac{\omega_1+\omega_2}{2}t \quad \text{with } A_m = 2A\cos\frac{\omega_1-\omega_2}{2}t$$

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The zeros of the modulating amplitudes occur for–

$$A_m = 0$$

$$\Rightarrow 2A \cos \frac{\omega_1 - \omega_2}{2} t = 0$$

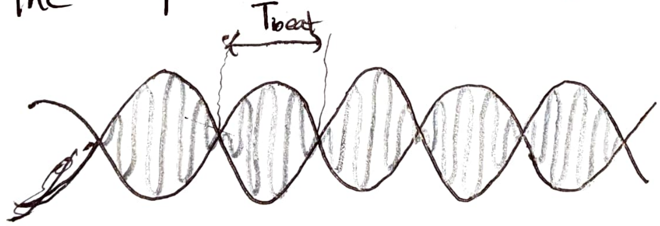
$$\rightarrow \cos \frac{\omega_1 - \omega_2}{2} t = \cos (2n+1) \frac{\pi}{2} \quad ; \; n = 0, 1, \ldots$$

$$\therefore \; t = (2n+1) \frac{\pi}{\omega_1 - \omega_2}$$

We define the time period of beats by the time between subsequent zeros.

$$\therefore \; T_{beat} = (2\times 1 + 1) \frac{\pi}{\omega_1 - \omega_2} - (2\times 0 + 1) \frac{\pi}{\omega_1 - \omega_2} = \frac{2\pi}{\omega_1 - \omega_2}$$

The actual time period of the modulation is just the $2T_{beat}$. The position of the particle looks like –



The beat phenomenon is common in many systems. If two tuning forks are vibrated with same amplitude but with a different frequency, there is a periodic large sound, and then zero sound with a periodic manner.

Superposition of two mutually perpendicular oscillation

So far, we have only discussed harmonic oscillation along one dimension. What if there are two harmonic oscillation in two mutually perpendicular directions? This is exactly the scenerio that was in assignment 1, question 4(d). Let's analyze the motion of a particle subjected to these two mutually perpendicular harmonic vibrations.

Let's consider two SHM along perpendicular directions

$$x_1(t) = A_1 \cos(\omega t - \phi_1) \quad \text{--- (I)}$$

$$y_1(t) = A_2 \cos(\omega t \mp \phi_2) \quad \text{--- (II)}$$

We are considering motions with same angular frequency ω and phase difference $\phi = \phi_1 - \phi_2$.

Let us try to find the trajectory equation. We have,

$$\cancel{y(t) = A_2 \cos(\omega t - \alpha + \alpha - \beta)}$$

$$\cancel{= A_2 \cos[(\omega t - \alpha) + (\alpha -}$$

$$y(t) = A_2 \cos(\omega t - \phi_2) = A_2 \cos[(\omega t - \phi_1) + (\phi_1 - \phi_2)]$$

$$= A_2 [\cos(\omega t - \phi_1) \cos(\phi_1 - \phi_2) - \sin(\omega t - \phi_1) \sin(\phi_1 - \phi_2)]$$

We want to express y in terms of x and the constants.

From (i), $\cos(\omega t-\phi_1) = \frac{x}{A_1}$

$$\therefore \sin(\omega t-\phi_1) = \sqrt{1-\cos^2(\omega t-\phi_1)} = \sqrt{1-\frac{x^2}{A_1^2}}$$

$$\therefore y = A_2 \frac{x}{A_1}\cos\phi - A_2\sqrt{1-\frac{x^2}{A_1^2}}\sin\phi$$

$$\Rightarrow y = \frac{A_2 x\cos\phi}{A_1} - \frac{A_2\sqrt{A_1^2-x^2}}{A_1}\sin\phi$$

$$\Rightarrow A_1 y = A_2 x\cos\phi - A_2\sqrt{A_1^2-x^2}\sin\phi$$

$$\Rightarrow A_1 y - A_2 x\cos\phi = -A_2\sqrt{A_1^2-x^2}\sin\phi$$

$$\Rightarrow A_1^2 y^2 - 2A_1A_2xy\cos\phi + A_2^2x^2\cos^2\phi = A_2^2(A_1^2-x^2)\sin^2\phi$$

$$\Rightarrow A_1^2 y^2 - 2A_1A_2xy\cos\phi + A_2^2x^2 = A_2^2A_1^2\sin^2\phi \quad \text{——— (1)}$$

Special cases

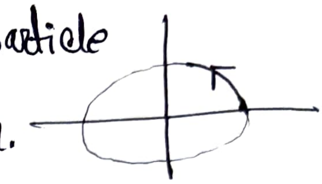
(i) If $\phi = \pm m\frac{\pi}{2}$ where m is an odd number.

From (1) $\Rightarrow$ $A_1^2y^2 + A_2^2x^2 = A_2^2A_1^2$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

(assuming $A_1 \neq A_2$)

which is an equation of an ellipse whose principle axes lie along the X and Y axes. If we take $\phi_1 = 0$ and $\phi_2 = \frac{\pi}{2}$, then, after $t=0$, the x begins to decrease from its initial positive value and ~~at~~ y begins increase ($x = A_1 \cos \omega t$ and $y = A_2 \sin \omega t$). So, the first location at $t=0$ is $(A_1, 0)$ and x decreases as y increases. So, the particle moves in counterclockwise elliptical path.



(ii) If $A_1 = A_2 = A$ and $\phi = \pm m \frac{\pi}{2}$, then,

$$x^2 + y^2 = A^2$$

which is an equation of ~~ellipse~~. circle.

(iii) If $\delta = 0$, then,

$$\cancel{A_1^2 A_2^2} \quad A_1^2 y^2 - 2A_1 A_2 xy + A_2^2 x^2 = 0$$

$$\Rightarrow \quad (A_1 y - A_2 x)^2 = 0$$

$$\therefore \quad y = \frac{A_2}{A_1} x$$

which is an equation of a straight line. So, the particle moves along a straight line.

(iv) If $\delta = \pm \pi$, ~~$m = 1, 2, 3, \ldots$~~ $(m \in Z)$

$$A_1^2 y^2 + 2A_1 A_2 xy + A_2^2 x^2 = 0$$

$$\therefore \quad y = -\frac{A_2}{A_1} x$$

which again is a straight line.

But what happens if the frequencies are not same?

Let's first take consider,

$$\left.\begin{aligned} x(t) &= A_1 \cos(\omega_1 t) \\ \text{and} \quad y(t) &= A_2 \cos(\omega_2 t + \delta) \end{aligned}\right\} \text{So, the phase difference is } \delta.$$

Let's take, $\omega_2 = 2\omega_1 \quad | \omega_1 = \omega$

$$\therefore \quad y = A_2 \cos(2\omega t + \delta)$$

$$= A_2 \left[\cos(2\omega t)\cos\delta - \sin(2\omega t)\sin\delta\right]$$

$$= A_2 \left[(2\cos^2\omega t - 1)\cos\delta - \overset{2\sin\omega t\cos\omega t}{\cancel{\sin 2\omega t}}\sin\delta\right]$$

and $\quad x = A_1 \cos(\omega t) \quad \Rightarrow \quad \cos\omega t = \dfrac{x}{A_1}$

$$\therefore \quad y = A_2 \left[\left(\frac{2x^2}{A_1^2} - 1\right)\cos\delta - 2\sqrt{1 - \frac{x^2}{A_1^2}}\,\frac{x}{A_1}\sin\delta\right]$$

$$\Rightarrow \frac{y}{A_2} = 2\left(\frac{x}{A_1}\right)^2\cos\delta - \cos\delta - 2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{A_1^2}}\sin\delta$$

$$\Rightarrow \frac{y}{A_2} + \cos\delta - 2\left(\frac{x}{A_1}\right)^2\cos\delta = -2\frac{x}{A_1}\sqrt{1 - \frac{x^2}{A_1^2}}\sin\delta$$

$$\Rightarrow \left[\left(\frac{y}{A_2} + \cos\delta\right) - 2\left(\frac{x}{A_1}\right)^2\cos\delta\right]^2 = 4\frac{x^2}{A_1^2}\left(1 - \frac{x^2}{A_1^2}\right)\sin^2\delta$$

$$\Rightarrow \left(\frac{y}{A_2} + \cos\delta\right)^2 - 4\left(\frac{x}{A_1}\right)^2\cos\delta\left(\frac{y}{A_2} + \cos\delta\right) + 4\left(\frac{x}{A_1}\right)^4\cos^2\delta$$
$$= 4\frac{x^2}{A_1^2}\sin^2\delta - 4\left(\frac{x}{A_1}\right)^4\sin^2\delta$$

$$\cancel{\Rightarrow \left(\frac{y}{A_2} + \cos\delta\right)^2 + 4\frac{x^2}{A_1^2}\left[\frac{x^2}{A_1^2}\cos^2\delta - \frac{y}{A_2}\cos\delta\right.}$$

$$\therefore \left(\frac{y}{A_2}+\cos\delta\right)^2 + 4\frac{x^2}{A_1^2}\left[\cancel{\frac{x^2}{A_1^2}\cos\delta}\ \frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\cos\delta\right] = 0$$

Now, if, $\delta = 0$, then –

$$\left(\frac{y}{A_2}+1\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\right) = 0$$

$$\Rightarrow \left(\frac{y}{A_2}+1\right)^2 - 2\cdot\frac{2x^2}{A_1^2}\left(\frac{y}{A_2}+1\right) + \left(\frac{2x^2}{A_1^2}\right)^2 = 0$$

$$\therefore \left[\left(\frac{y}{A_2}+1\right) - \frac{2x^2}{A_1^2}\right]^2 = 0$$

The equation above represents two coincident parabola.

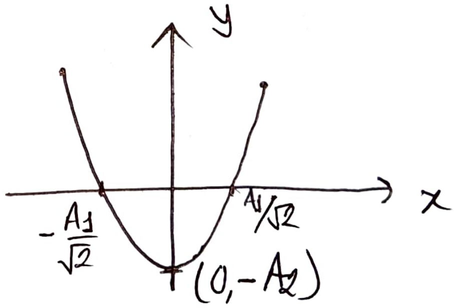
$$\frac{y}{A_2}+1 - \frac{2x^2}{A_1^2} = 0$$

$$\Rightarrow \frac{y+A_2}{A_2} = \frac{2x^2}{A_1^2} \Rightarrow (y+A_2) = \frac{2A_2}{A_1^2}x^2$$

If, $x = 0$, $y = -A_2$

$y = 0$, $A_2\left[\frac{2}{A_1^2}x^2 - 1\right] = 0$

$$\therefore x = \pm\frac{A_1}{\sqrt{2}}$$



Now, if $\frac{\omega_1}{\omega_2}$ is rational, then the Lissajous curves will be closed, meaning the particle will repeat the same path. But if $\frac{\omega_1}{\omega_2}$ is not rational, the path is open, that is the particle will move in different paths throughout the time.

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Proof: First, say $\frac{\omega_1}{\omega_2}$ is rational. We can write,

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}$$

The time periods are, $T_1 = \frac{2\pi}{\omega_1}$ and $T_2 = \frac{2\pi}{\omega_2}$

$$\therefore \frac{T_1}{T_2} = \frac{\omega_2}{\omega_1} = \frac{q}{p}$$

$$\therefore pT_1 = qT_2$$

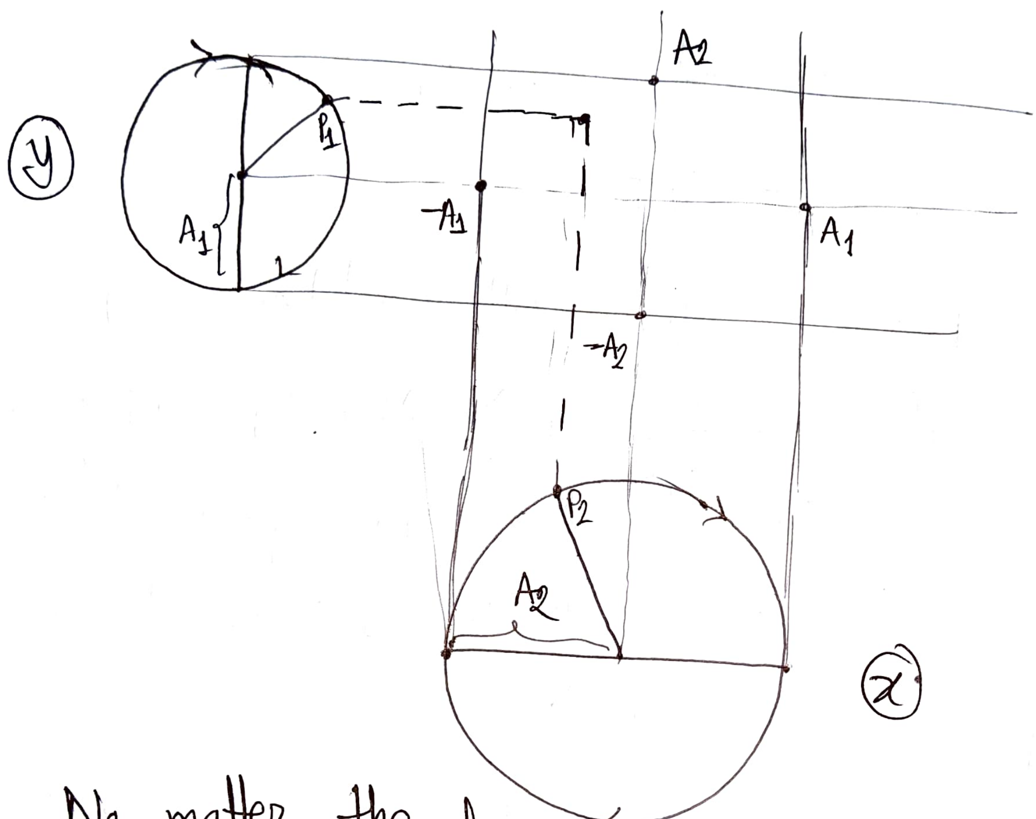
So, p periods of T_1 exactly matches q periods of T_2. So, after a time $\tau = \text{Max}\{pT_1, qT_2\}$, the particle will return to the same point of the curve, and start the motion again. So, τ can be considered time period of this path.

If, $\frac{\omega_1}{\omega_2}$ is irrational, then, there is no such p and q for which $pT_1 = qT_2$. So, the curve will never return to the starting point and repeat the motion. So, the curve will be open.

Graphical representation that the motion of the particle is restricted to a rectangle of width $2A_1$ and $2A_2$.

$$x = A\cos(\omega t + \phi_1)$$

$$y = A\cos(\omega t + \phi_2)$$



No matter the frequency or phase difference between the two SHMs, the trajectory will always be confined in the rectangle of width $2A_1$ and length $2A_2$.