Lecture 11

Fourier Series

Motivation to study Fourier series

Let's look at the famous Taylor and Maclaurin series

Taylor series

Taylor series is a way of expressing a function that is continuous and which is differentiable infinitely many times at some point. We expand a function about a point in terms of the algebraic power series.

Before going to Taylor series, let's first talk about the Maclaurin series.

Let's take a function $f(x)$ and say it is continuous and infinitely differentiable at $x=0$. Now, let's write,

$$f(x) = a + bx + cx^2 + dx^3 + \cdots$$

We have to find the coefficients $a, b, c, \ldots$

$$f(0) = a$$

$$f'(x) = b + 2cx + 3dx^2 + \cdots$$

$$f'(x)\big|_{x=0} = b$$

$$f''(x) = 2c + 6dx + \cdots \qquad f'''(x) = 6d + \cdots$$

$$f''(x)\big|_{x=0} = 2c \qquad f'''(x)\big|_{x=0} = 6d$$

$$\Rightarrow c = \frac{f''(x)\big|_{x=0}}{2} \qquad d = \frac{f'''(x)\big|_{x=0}}{6}$$

Finally, $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \cdots$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

This is the Maclaurin series of $f(x)$, where we have extended the function about the point $x=0$.

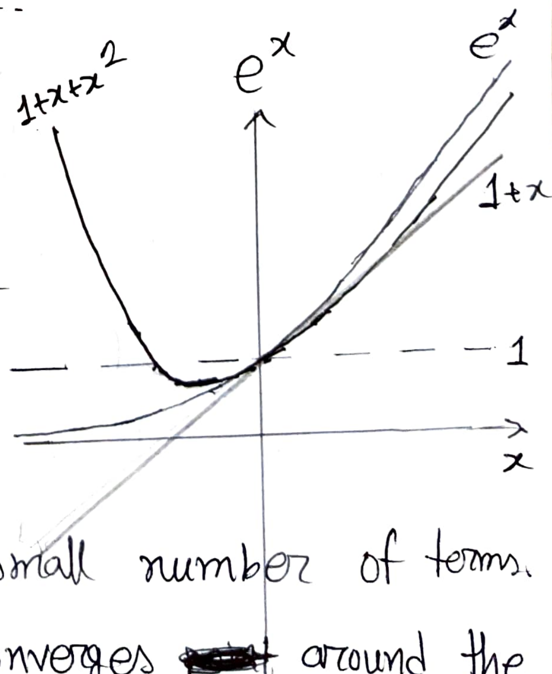
Let's choose to expand e^x.

$$e^x = a + bx + cx^2 + dx^3 + fx^4 + \cdots$$

$$e^x = e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 + \cdots$$

$$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As we add terms, the series on the right hand side approaches to match the function e^x, but it mostly matches around the point $x=0$. The function away from $x=0$ doesn't match that much for small number of terms.



So, Taylor/Maclaurin series converges ~~at~~ around the point as you increase more and more terms.

Taylor series is a generalization of the Maclaurin series (Maclaurin series is ~~[illegible]~~ a special case of Taylor series). In Taylor series, we expand the function around any arbitrary point, say $x=a$.

Then, $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$

So, what are the shortcomings of the Taylor expansion?

→ You have to pick a particular point to expand around. If a system is translationally invariant, then you don't really have a specific point to expand around, you might have to choose an arbitrary point.

→ The point around which you are expanding must have to be infinitely differentiable there.

→ The convergence is slow, as it is limited to some neighbourhood around the point where we are expanding. Its not good if you are interested in large domain, since you will have to incorporate many many terms.

→ For discontinuous functions, you can only use Taylor series for some function only in a limited domain. ~~For examp~~

So, we need something else to overcome these difficulties. That's where Fourier series come.

Fourier Series

Joseph Fourier was trying to solve the heat equation, which was basically a PDE. No prior solution was achieved before Fourier. He tried to model a complicated heat source as a linear combination of sine and cosine waves — which we now know as Fourier series.

In Fourier series, we expand a function as a linear combination of sines and cosines. Since these functions are periodic, we need our function to be also periodic (we will later see that its also possible for non-periodic functions). Along with the periodicness, there are few other conditions. This conditions are known as Dirichlet conditions. The conditions are—

(i) The function has to be periodic.

(ii) It has to be single valued and continuous. However, a finite number of discontinuities over a period can be allowed.

(iii) It must have finite number of maxima and minima over one period.

(iv) The integral over one period of $|f(x)|$ must converge.

Now, why can't we just use only sine or only cosine function?

$\sin x$ is an odd function, meaning,

$$f(-x) = -f(x) \quad ; \quad \sin(-x) = -\sin x$$

where, $\cos x$ is an even function, meaning,

$$f(-x) = f(x) \quad ; \quad \cos(-x) = \cos x$$

We can't express an odd function with a series of cosines; similarly, any even function can't be expressed in terms of sine functions. However, the odd functions can be written as a linear combination of sines only and the even functions can be expressed ~~in ter~~ as a linear combination of cosines only.

Now, take any function, $f(x)$.

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$

$$= f_{even}(x) + f_{odd}(x)$$

So, any function can be written ~~as~~ in terms of an odd and an even part. Hence, we can write any function in terms of a linear combination of sine and cosine series.

Example: $f(x) = x^2 \Rightarrow x^2 = \frac{1}{2}\left[(x)^2 + (-x)^2\right] + \frac{1}{2}\left[(x)^2 - (-x)^2\right]$

$f(x) = e^x \Rightarrow e^x = \frac{1}{2}\left[e^x + e^{-x}\right] + \frac{1}{2}\left[e^x - e^{-x}\right]$

$$= \underset{\text{even}}{\cosh x} + \underset{\text{odd}}{\sinh x}$$

In general, we write the Fourier series for any function $f(x)$ as —

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(n \frac{2\pi}{L} x\right) + \sum_{n=0}^{\infty} b_n \sin\left(n \frac{2\pi}{L} x\right) \quad (1)$$

where, L is the length of one period. If $L = 2\pi$, then

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx)$$

With the 2π introduced in trigonometric function the $n=1$ term has a period of L, $n=2$ term has a period of $\frac{L}{2}$ and so on. For any integer n, an integral number of complete oscillation will then for sure fit into L. So, the expression (1) has a period of at most L.

It is a standard convention (but not necessary) to isolate the $n=0$ term and write the Fourier series as–

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(n \frac{2\pi}{L} x\right) + b_n \sin\left(n \frac{2\pi}{L} x\right) \right]$$

Now, this way of representing $f(x)$ is the same thing as writing the function in orthogonal basis of sine and cosine. If two vectors are orthogonal, then the dot product / inner product of this two vectors are zero. For the case of Cartesian unit vectors,

$$\hat{i} \cdot \hat{i} = 1 \;,\quad \hat{i} \cdot \hat{j} = 0 \quad \text{and so on.}$$

In compact notation, we write the orthogonality as–

$$\hat{i} \cdot \hat{j} = \delta_{ij}$$

where δ_{ij} is the Kronecker Delta function, defined as–

$$\delta_{ij} = \begin{cases} 0 \,; & i \neq j \\ 1 \,; & i = j \end{cases}$$

But sines and cosines are functions. How can define orthogonality here? We first consider the integral,

$$\int_0^L \cos\left(n \frac{2\pi}{L} x\right) \cos\left(k \frac{2\pi}{L} x\right) dx$$

$$= \frac{1}{2} \int_0^L \left[\cos\left\{(n+k)\frac{2\pi}{L} x\right\} + \cos\left\{(n-k)\frac{2\pi}{L} x\right\} \right] dx$$

Both the terms are ~~identically~~ zero, upon integrating, if $n \neq k$. Because both the terms in the integrand undergo an integral number of complete oscillations over the interval 0 to L, so the area under the curve must be zero. But, if $n = k$, then the second term is just 1 and the integral gives L.

$$\therefore \int_0^L \cos\left(n \frac{2\pi}{L} x\right) \cos\left(k \frac{2\pi}{L} x\right) dx = \begin{cases} 0 ; & n \neq k \\ \frac{L}{2} ; & n = k \end{cases}$$

You can similarly show that,

$$\int_0^L \sin\left(n \frac{2\pi}{L} x\right) \sin\left(k \frac{2\pi}{L} x\right) dx = \begin{cases} 0 ; & n \neq k \\ \frac{L}{2} ; & n = k \end{cases}$$

However, the integral, $\int_0^L \cos\left(n \frac{2\pi}{L} x\right) \sin\left(k \frac{2\pi}{L} x\right) dx = 0$

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for all n and k.

$$\int_0^L \cos\left(n\frac{2\pi}{L}x\right)\sin\left(k\frac{2\pi}{L}x\right)dx \cancel{= \frac{1}{2}\int_0^L\left[\sin\left\{(n+m)\frac{2\pi}{L}x\right\}+\sin\left\{(n-m)\frac{2\pi}{L}x\right.\right.}$$

$$= \frac{1}{2}\int_0^L \left[\sin\left\{(n+k)\frac{2\pi}{L}x\right\}+\sin\left\{(n-k)\frac{2\pi}{L}x\right\}\right]dx$$

$= 0$ for all n and k (verify).

We then have,

$$\int_0^L \cos\left(n\frac{2\pi}{L}x\right)\cos\left(k\frac{2\pi}{L}x\right)dx = \frac{L}{2}\delta_{nk}$$

$$\int_0^L \sin\left(n\frac{2\pi}{L}x\right)\cos\left(k\frac{2\pi}{L}x\right)dx = \frac{L}{2}\delta_{nk}$$

$$\int_0^L \cos\left(n\frac{2\pi}{L}x\right)\cos\left(k\frac{2\pi}{L}x\right)dx = 0$$

So, the integral of the product of any of these two functions is zero unless they are the same function. We then call these function orthogonal, and define the inner product of two functions as -

$$\int f(x)\,g(x)\,dx$$

So, we can say, Fourier series of a function $f(x)$ expands it in the basis of sines and cosines. The linear independence is pretty much instructive here. If, $\quad c_1 \sin\left(1 \cdot \frac{2\pi}{L}x\right) + c_2 \cos\left(1 \cdot \frac{2\pi}{L}x\right) + c_3 \sin\left(2 \frac{2\pi}{L}x\right)$

$$= 0,$$

then $\quad c_1 = c_2 = c_3 = \cdots = 0$ for any non-zero basis.

Finding Fourier co-efficients

Let's take the average of $f(x)$ over one period.

$$\frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}a_0\int_0^L dx + \frac{1}{L}\sum_{n=1}^{\infty}\int_0^L a_n \cos\left(n\frac{2\pi}{L}x\right)dx$$

$$+ \frac{1}{L}\sum_{n=1}^{\infty}\int_0^L b_n \cos\left(n\frac{2\pi}{L}x\right)dx$$

$$= \frac{1}{L}a_0 \cdot L + \frac{1}{L}\sum_{n=1}^{\infty} a_n \cancelto{0}{\int_0^L \cos\left(n\frac{2\pi}{L}x\right)dx} + \frac{1}{L}\sum_{n=1}^{\infty} b_n \cancelto{0}{\int_0^L \sin\left(n\frac{2\pi}{L}x\right)dx}$$

$$\therefore \frac{1}{L}\int_0^L f(x)\,dx = a_0 \qquad \boxed{\therefore a_0 = \frac{1}{L}\int_0^L f(x)\,dx}$$

For finding a_n, we multiply $f(x)$ with $\cos\left(k\frac{2\pi}{L}x\right)$ and integrate over one period.

$$\therefore \frac{1}{L}\int_0^L f(x)\cos\left(k\frac{2\pi}{L}x\right)dx = \frac{1}{L}a_0 \cancelto{0}{\int_0^L \cos\left(k\frac{2\pi}{L}x\right)dx}$$

$$+ \frac{1}{L}\sum_{n=1}^{\infty} a_n \int \cos\left(n\frac{2\pi}{L}x\right)\cos\left(k\frac{2\pi}{L}x\right)dx$$

$$+\sum_{n=1}^{\infty} b_n \int_0^L \sin\left(n\frac{2\pi}{L}x\right)\cos\left(k\frac{2\pi}{L}x\right)dx \nearrow 0$$

$$\Rightarrow \frac{1}{L}\int_0^L f(x)\cos\left(k\frac{2\pi}{L}x\right)dx = \frac{1}{L}\sum_{n=1}^{\infty} a_n\, \delta_{nk}\frac{L}{2} = \frac{1}{L}a_k\frac{L}{2}$$

Since we will ~~d~~ only have non-zero term for $n=k$.

$$\therefore\ a_k = \frac{2}{L}\int_0^L f(x)\cos\left(k\frac{2\pi}{L}x\right)dx$$

Similarly, multiplying $f(x)$ by $\sin\left(k\frac{2\pi}{L}x\right)$ and integrating,

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(k\frac{2\pi}{L}x\right)dx$$

Since k is just a dummy index, we have -

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(n\frac{2\pi}{L}x\right)dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi}{L}x\right)dx$$

and hence, the coefficients can be calculated. So, we can expand the function in Fourier series. The limit of integration doesn't have to be 0 to L strictly. Any interval over one period is okay and will give the same co-efficients.

Symmetry about points other than $x=0$

We have already discussed – if any function has an even symmetry about $x=0$, then $b_n=0$ for all n and it can be expressed in terms of cosines only. Similarly, if it has an odd symmetry about $x=0$ then $a_n=0$ for all n. But what about symmetry about any other point?

Say, $f(x)$ has an even or odd symmetry about, say $x=\frac{L}{4}$. Then, $f\left(\frac{L}{4}-x\right)=\pm f\left(x-\frac{L}{4}\right)$. Let's define, $x-\frac{L}{4}=s$. So,

$$f(-s)=\pm f(s)$$

Now, $$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(n\frac{2\pi}{L}x\right)dx$$

$$=\frac{2}{L}\int_0^L f(s)\sin\left[n\frac{2\pi}{L}s+n\frac{2\pi}{L}\cdot\frac{L}{4}\right]ds$$

$$=\frac{2}{L}\int_0^L f(x)\sin\left[n\frac{2\pi}{L}s+n\frac{\pi}{2}\right]ds$$

Now, $\sin\left(n\frac{2\pi}{L}s+n\frac{\pi}{2}\right)=\sin\left(n\frac{2\pi}{L}s\right)\cos\left(n\frac{\pi}{2}\right)+\cos\left(n\frac{2\pi}{L}s\right)\sin\left(n\frac{\pi}{2}\right)$

If $n =$ even, then $\sin\left(n\frac{\pi}{2}\right) = 0$, $\cos\left(n\frac{\pi}{2}\right) = \pm 1$.
So, left hand side of equation (1) is a function of only $\sin\left(\frac{2\pi}{L} n s\right)$, and so it's an odd function of s.

If $n =$ odd, then $\cos\left(n\frac{\pi}{2}\right) = 0$ and $\sin\left(n\frac{\pi}{2}\right) = \pm 1$, and hence L.H.S. of (1) is an even function of s.

Hence, if $f(s)$ is even and n is even, then the integral is zero [since $f(s)$ is even and $\sin\left(n\frac{2\pi}{L}s + n\frac{\pi}{2}\right)$ is odd, then the integrand is odd]. If $f(s)$ is odd and n is odd, then the integral is also zero, since the integrand is again odd.

Integration of an odd function over a symmetric interval is zero:

$$\int_{-x}^{+x} f(x)\,dx = \int_{-x}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{0}^{x} f(-x)\,dx + \int_{0}^{x} f(x)\,dx = -\int_{0}^{x} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0$$

$$\int_{-x}^{0} f(x)\,dx = \int_{y=x}^{y=0} f(-y)(-dy)$$

$$= -\int_{x}^{0} f(-y)\,dy = \int_{0}^{x} f(-y)\,dy$$

$x = -y$
$\Rightarrow dx = -dy$
If $x = -x$, $y = x$
$x = 0$, $y = 0$

Conclusions: (i) If $f(x)$ is even about $\frac{L}{4}$, then $b_{2n} = 0$. Similarly, you can prove, $a_{2n+1} = 0$.

even coefficients ... odd coefficients.

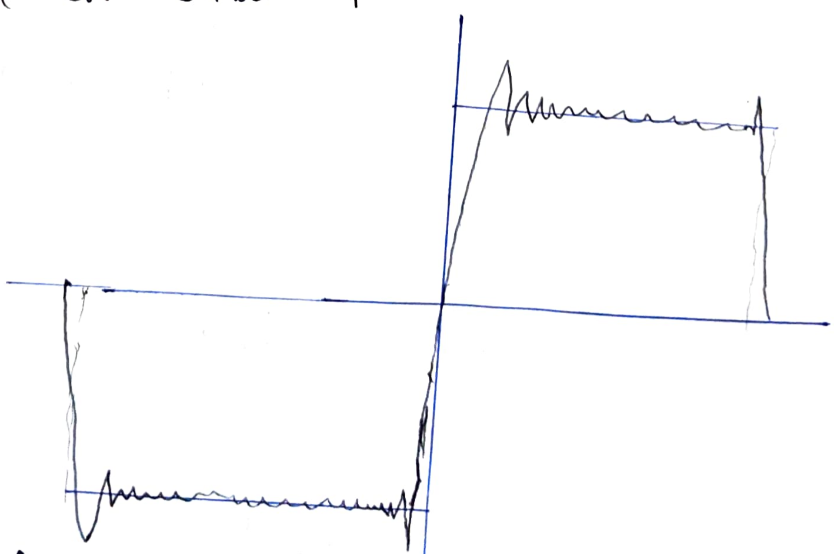
(ii) If $f(x)$ is odd about $\frac{L}{4}$, then $a_{2n} = 0$ and $b_{2n+1} = 0$.

Discontinuous function

At a point of finite discontinuity x_d, the Fourier series converges to,

$$\frac{1}{2} \lim_{\epsilon \to 0} \left[f(x_d+\epsilon) + f(x_d-\epsilon) \right]$$

So, it assumes an average value at the point of finite discontinuity. At a discontinuity, the Fourier series sum of the function overshoots its value, which doesn't fade away even adding numerous terms. This is known as Gibb's phenomenon.



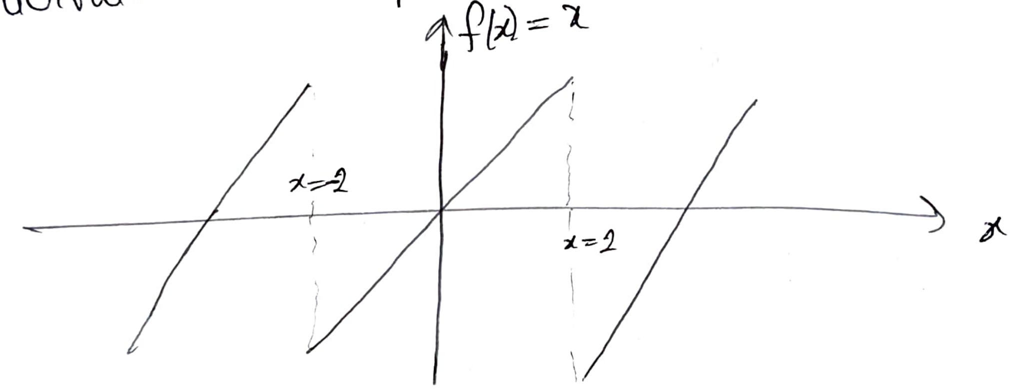
Fourier series of a square wave, which has a discontinuity at $x=0$. See how the series assumes a value of 0 at $x=0$.

Series at $x=0$ is $= \frac{1}{2}\left[f(0+\epsilon) + f(0-\epsilon)\right]$

$= \frac{1}{2}\left[1+(-1)\right] = 0$

Non-periodic function

Fourier series only works for periodic functions. But most of the functions are non-periodic. What do we do then? There is a way. We can make a function periodic if we repeat the function in a certain interval. For example –



$f(x) = x$ is now periodic over $[-2,2]$.

Complex Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{2\pi}{L}x\right) + b_n \sin\left(n\frac{2\pi}{L}x\right)\right]$$

Now, $\cos\left(n\frac{2\pi}{L}x\right) = \dfrac{e^{in\frac{2\pi}{L}x} + e^{-in\frac{2\pi}{L}x}}{2}$

and $\sin\left(n\frac{2\pi}{L}x\right) = \dfrac{e^{in\frac{2\pi}{L}x} - e^{-in\frac{2\pi}{L}x}}{2i}$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{in\frac{2\pi}{L}x} + e^{-in\frac{2\pi}{L}x}}{2} + b_n \frac{e^{in\frac{2\pi}{L}x} - e^{-in\frac{2\pi}{L}x}}{2i} \right]$$

$$= a_0 + \left[\sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\frac{2\pi}{L}x} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-in\frac{2\pi}{L}x} \right]$$

$$\cancel{= a_0 + \left[\sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\frac{2\pi}{L}x} + \sum_{n=-\infty}^{-1} \frac{a_{-n} + ib_{-n}}{2} e^{in\frac{2\pi}{L}x} \right]}$$

$$\cancel{= a_0 + \left[\sum_{n=1}^{\infty} \bar{C}_n e^{in\frac{2\pi}{L}x} + \sum_{n=-\infty}^{-1} C_{-n} e^{in\frac{2\pi}{L}x} \right]}$$

$$= a_0 + \left[\sum_{n=1}^{\infty} \bar{C}_n e^{in\frac{2\pi}{L}x} + \sum_{n=1}^{\infty} C_n e^{-in\frac{2\pi}{L}x} \right]$$

Introducing $k = -n$, we get in the second sum we get

$$f(x) = a_0 + \left[\sum_{n=1}^{\infty} \bar{C}_n e^{in\frac{2\pi}{L}x} + \sum_{k=-\infty}^{-1} \bar{C}_{-k} e^{ik\frac{2\pi}{L}x} \right]$$

Since k is just a dummy index, we write,

$$f(x) = a_0 + \left[\sum_{n=1}^{\infty} \bar{C}_n e^{in\frac{2\pi}{L}x} + \sum_{n=-\infty}^{-1} \bar{C}_{-n} e^{in\frac{2\pi}{L}x} \right]$$

~~We define C_n for negative n's as: $C_n = \bar{C}_{-n}$ for $n = -1, -2, \ldots$~~

Along with it, defining $C_0 = a_0$, we can write –

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{2\pi}{L}x}}$$

with, $C_n = \frac{1}{2}(a_n - ib_n)$, for $n = 1, 2, \dots$

$C_{-n} = \frac{1}{2}(a_{-n} + ib_{-n})$, for $n = -1, -2, \dots$

$C_0 = a_0$

For finding the coefficients C_n, we multiply both sides of $f(x)$ by $e^{-ikx\frac{2\pi}{L}}$ and integrate over x.

So, exponentials are forming the basis now

$$\int_0^L f(x)\, e^{-ikx\frac{2\pi}{L}} dx = \sum_{n=-\infty}^{\infty} \int_0^L C_n e^{i(n-k)\frac{2\pi}{L}x} dx$$

If $n \neq k$, then, $C_n \int_0^L e^{i(n-k)\frac{2\pi}{L}x} = 0$ (prove).

So, the only surviving term is $n = k$.

$$\therefore \int_0^L f(x)\, e^{-inx\frac{2\pi}{L}} dx = C_n \int_0^L e^{i(n-n)\frac{2\pi}{L}x} dx = C_n \times L$$

$$\boxed{\therefore C_n = \frac{1}{L} \int_0^L f(x)\, e^{-i\frac{2\pi}{L}nx} dx}$$

Parseval's theorem

Let's consider two functions $f(x)$ and $g(x)$.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2\pi}{L}nx}$$

$$g(x) = \sum_{n=-\infty}^{\infty} d_n e^{i\frac{2\pi}{L}nx}$$

Same period

Now, $f(x) \cdot g^*(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{2\pi}{L}nx} g^*(x)$

$$\Rightarrow \frac{1}{L}\int_0^L f(x) g^*(x)\,dx = \sum_{n=-\infty}^{\infty} C_n \frac{1}{L}\int_0^L g^*(x) e^{i\frac{2\pi}{L}nx}\,dx$$

$$= \sum_{n=-\infty}^{\infty} C_n \frac{1}{L}\int_0^L \left[g(x) e^{-i\frac{2\pi}{L}nx}\right]^* dx$$

$$= \sum_{n=-\infty}^{\infty} C_n d_n^*$$

If $f(x) = g(x)$, then,

$$\frac{1}{L}\int_0^L |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |C_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This is Parseval's theorem.

~~Fourier transform~~ Few remarks

(i) If $f(x)$ is real, then, $C_{-n} = C_n^*$. This must be so since the exponentials are already complex conjugates of each other. That's what we also see from our workings assuming $f(x)$ to be real.

(ii) If $f(x)$ is even, then $C_n = C_{-n}$ (so that only the cosines survives). If additionally $f(x)$ is real, then $C_n = C_n^*$ implies that C_n must be real.

(iii) If $f(x)$ is odd, the only the sine terms survives. So, $C_n = -C_{-n}$. If along with it $f(x)$ is real, then $C_{-n} = C_n^*$ implies that all C_n's are purely imaginary.

Fourier transform

In Fourier series, we had, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\frac{2\pi}{L}x}$

But this only worked for periodic functions. But how can we write this ~~in terms~~ for a non-periodic function over the whole domain?

The key here is to consider the function to be periodic on an interval $-\frac{L}{2}$ to $\frac{L}{2}$, with $L\to\infty$. So, we say that the function has an infinite period.

We can write, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{iK_n x}$ with $K_n = n\frac{2\pi}{L}$. The difference between successive K_n is, $dK_n = \frac{2\pi}{L}\, dn = \frac{2\pi}{L}$. Now, since we are interested in $L\to\infty$ limit, dK_n is very small and K_n is essentially a continuous function. We can then write,

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$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{iK_n x} (dn)$$

We can mu[illegible] by dn since [illegible] just 1.

$$= \sum_{n=-\infty}^{\infty} C_n e^{iK_n x} \left(\frac{L}{2\pi} dK_n \right)$$

Since, dK_n is very small, this sum is essentially an integral, and we write –

$$f(x) = \int_{-\infty}^{\infty} \left(C_n \frac{L}{2\pi} \right) e^{iK_n x} \, dK_n = \int_{-\infty}^{\infty} C(K_n) e^{iK_n x} dK_n$$

We had, $C_n = \frac{1}{L} \int_{-\infty}^{\infty} f(x) e^{-in\frac{2\pi}{L}x} dx = \frac{1}{L} \int_{-\infty}^{\infty} f(x) e^{-iK_n x} dx$

$$\boxed{\therefore \; C(K_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iK_n x} dx}$$

Since we are considering K_n to be a continuous variable essentially, then, we drop the ~~susbcr~~ subscript n, write $C(K_n)$ as $C(K)$ or to be more general, we write it as $\tilde{f}(K)$.

$$\therefore \; f(x) = \int_{-\infty}^{\infty} \tilde{f}(K) e^{iKx} \, dK$$

$$\tilde{f}(K) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iKx} \, dx$$

$\tilde{f}(K)$ is called the Fourier transform of $f(x)$ and

$f(x)$ is called the inverse Fourier transform of $\hat{f}(k)$. One could make things symmetric by defining $\tilde{f}(k) = \sqrt{2\pi}\, C(k)$. In that definition,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

If your function is a function of time, then -

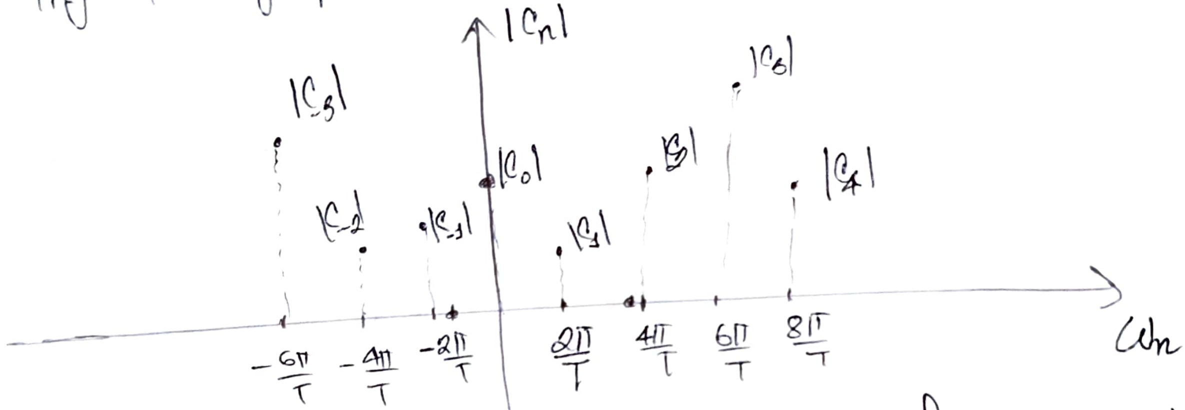
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

with T being the period and $\omega_n = \frac{2\pi n}{T}$ giving the n^{th} frequency.

$\tilde{f}(k)$ tells you how much $f(x)$ is made up of e^{ikx} and $f(x)$ tells you how much $\tilde{f}(k)$ is made up of e^{-ikx}, just like the Fourier series. To be more precise, $f(x)$ ~~$\hat{f}(k)\,dk$~~ is made up of e^{ikx} terms by an amount $\hat{f}(k)\,dk$ in the range k and $k+dk$.

Now, think about what Fourier transform is giving you. It basically transform your function from time space to frequency space, from position space to k space (we will define k later). Let's try to graph the frequency spectrum:



This would be the frequency spectrum for a series. So, $\tilde{f}(\omega_n)$ shows you the relative amplitudes (coefficients) of the basis corresponding to the frequencies. In Fourier transform, this discrete graph becomes an integral curve and we get $\hat{f}(\omega)$. This ~~e~~ graphs contains as much information as $f(t)$ does.

FT has applications in many brances of Physics. We will encounter in our course, later in courses like QM, Optics etc. Often you will find a problem difficult to solve in say, original space. Then what you do is do a FT, move to the conjugate space, solve the problem (may be easier), and do a reverse FT to get back your solution in original space.