Group Theory
Lecture #6

From Lie Groups to Lie Algebras:

Lie groups are groups whose elements are labelled by a set of continuous parameters α_i, $i = 1, \dots, N$. The number of independent parameters N is the dimension of the group.

Lie groups are manifolds with group structure:

If p & q are point on the manifold M then it is possible to associate two group elements g_p & g_q s.t.

$$g_p \circ g_q = g_r \quad \text{wher } g_i \in G \cong M$$

where r is also a point on M. Not all manifold can be given a group structure.

Thinking of a Lie group as a manifold allows us to think of the parameter space in a geometric, coordinate invariant way.

A Lie algebra is a description of a Lie group in the neighbourhood of the identity element e. Let us show a Lie algebra arises from a Lie group for a matrix group.

Let G be a matrix group of dim N. Then a generic element of G can be written as

$$g(\underline{\alpha}) = e^{i\,\underline{\alpha}\cdot\underline{X}}$$

where $\underline{\alpha}\cdot\underline{X} = \sum_{i=1}^{N} \alpha_i X_i$ with X_i as the generators. Then closure of the group implies

$$g(\underline{\alpha})\,g(\underline{\beta}) = g(\underline{\gamma})$$

or $$e^{i\underline{\alpha}\cdot\underline{X}}\, e^{i\underline{\beta}\cdot\underline{X}} = e^{i\underline{\gamma}\cdot\underline{X}}$$

If G is an Abelian group then

$$\underline{\gamma} = \underline{\alpha} + \underline{\beta}$$

so that we have $e^{i\underline{\alpha}\cdot\underline{x}} e^{i\underline{\beta}\cdot\underline{x}} = e^{i\underline{\beta}\cdot\underline{x}} e^{i\underline{\alpha}\cdot\underline{x}}$

For non-Abelian groups we expect there to be corrections

$$\underline{\gamma} = \underline{\alpha} + \underline{\beta} + \cdots$$

To find these corrections we write

$$i\underline{\gamma}\cdot\underline{x} = \log\left[e^{i\underline{\alpha}\cdot\underline{x}} e^{i\underline{\beta}\cdot\underline{x}} \right]$$

Let us now α_i & β_i to be small. We then use the formula $\log(1+x) \simeq x - \frac{x^2}{2} + \dots$ for small x.

Identifying $x = e^{i\underline{\alpha}\cdot\underline{x}} e^{i\underline{\beta}\cdot\underline{x}} - 1$ we get:

$$x \simeq \left(1 + i\underline{\alpha}\cdot\underline{X} - \frac{(\underline{\alpha}\cdot\underline{X})^2}{2}\right)\left(1 + i\underline{\beta}\cdot\underline{X} - \frac{(\underline{\beta}\cdot\underline{X})^2}{2}\right) - 1$$

$$= i(\underline{\alpha}+\underline{\beta})\cdot\underline{X} - \alpha_i\beta_j X_i X_j - \tfrac{1}{2}\alpha_i\cancel{\alpha_j X_i X_j}$$

$$- \tfrac{1}{2}\cancel{\beta_i\beta_j} X_i X_j$$

$$x^2 = [i(\underline{\alpha}+\underline{\beta})\cdot\underline{X}][i(\underline{\alpha}+\underline{\beta})\cdot\underline{X}]$$

$$= -[\alpha_i\alpha_j \cancel{X_i} X_j + \beta_i\cancel{\beta_j X_i} X_j$$

$$+ \alpha_i\beta_j X_i X_j + \beta_j\alpha_i X_j X_i]$$

Thus $$x - \frac{x^2}{2} = i(\underline{\alpha}+\underline{\beta})\cdot\underline{X} - \alpha_i\beta_j (X_iX_j) + \alpha_i\beta_j$$

$$\tfrac{1}{2}(X_iX_j + X_jX_i)$$

$$= i(\underline{\alpha}+\underline{\beta}) - \tfrac{1}{2}\alpha_i\cancel{\beta_j X_i} X_j - \tfrac{1}{2}\alpha_j\beta_i X_j X_i$$

$$+ \tfrac{1}{2}\alpha_i\cancel{\beta_j} X_i X_j + \tfrac{1}{2}\beta_i\alpha_j X_i X_j$$

$$= i(\underline{\alpha}+\underline{\beta})\cdot\underline{X} - \tfrac{1}{2}\alpha_j\beta_i [X_j, X_i]$$

For closure : $[X_j, X_i] = i f_{jik} X_k$

↳ coefficient of linear combination

with $\underline{\gamma} = \underline{\gamma}(\underline{\alpha}, \underline{\beta})$

$$\boxed{[X_i, X_j] = i f_{ijk} X_k}$$

This is known as the Lie algebra.

Comments:

1. The Lie algebra captures the noncommutative nature of the group in the nbhd of the identity element.

2. Lie algebras do not completely specify the global properties of the group.
   Two different Lie groups can have the same Lie algebra: $SO(3)$ & $SU(2)$ have the same Lie algebra $[J_i, J_j] = i\epsilon_{ijk} J_k$

2. One can introduce the notion of a Lie algebra where $L = \{X_i, 0\}$ form a vector space over $\mathbb{R}$ and one introduces a product $[\ ,\ ] : L \times L \to L$ called the Lie bracket.

Properties of the Lie bracket & The Lie algebra

Let us denote by $\tilde{g}$ the Lie algebra associated with The Lie gp G.

Then $\tilde{g}$ is The collection of linearly independent generators $\{X_i\}$. If we add to it the null element 0 then The set forms a vector space. The field chosen for the vector space is usually $\mathbb{R}$ but $\mathbb{C}$ is also possible. [Choosing $\mathbb{C}$ instead of $\mathbb{R}$ for the same set of generators will lead to a different vectors space.]

We define the following skew-symmetric product on $\tilde{g}$:

$$[X, Y] = -[Y, X] \quad \forall\, X, Y \in \tilde{g}$$

This product is known as the Lie bracket and it's the map

$$[\cdot\,,\cdot] : \tilde{g} \times \tilde{g} \to \tilde{g}$$

This is a non-associative product:

$$[[x, y], z] \neq [x, [y, z]]$$

and so we have to specify what is the rule for associativity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

is called the Jacobi identity.

Comment:

1. One can choose the basis of $\tilde{g}$ is a such that f_{ijk}, known as structure constants, can be chosen to be real.

2. Note that from definition

$$f_{ijk} = -f_{jik}. \quad \text{— (A)}$$

But for compact Lie groups the structure constants can be chosen to be completely antisymmetric:

$$f_{ijk} = - f_{ikj} \quad \text{— Ⓑ}$$

in addition to Ⓐ.