Last Time:

Hooke's Law for a linear spring

$$\vec{F} = -k\,\Delta\vec{x}$$

In one dimension, we can write $F = -kx$

According to Newton's law:

$$ma = -kx$$

$$\Rightarrow \ddot{x} = -\frac{k}{m}x \qquad \text{where} \qquad \ddot{x} \equiv \frac{d^2x}{dt^2}$$

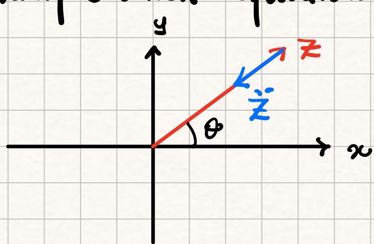
To solve let us look at a more general equation. Let $z(t) = x(t) + iy(t)$, where $i = \sqrt{-1}$. We propose to solve:

$$① \text{ ——— } \boxed{\ddot{z}(t) = -\frac{k}{m}z(t)} \qquad \begin{cases} \ddot{x}(t) = -\frac{k}{m}x(t) & \text{— Real Part} \\ \ddot{y}(t) = -\frac{k}{m}y(t) & \text{— imaginary part} \end{cases}$$

then take either the real or imaginary part of the solution.

Since Newton's law's are 2nd order we expect two integration constants.

The geometric interpretation equation ① is:



This is suggestive of uniform circular motion and we adopt the ansatz

$$② \text{ ——— } \boxed{z(t) = A\,e^{i(\omega t + \varphi)}}$$

where A & φ are arbitrary constants to be determined by initial conditions.

Putting ② ⇒ ①

$$-\omega^2 A e^{i(\omega t+\varphi)} = -\frac{k}{m} A e^{i(\omega t+\varphi)}$$

$$\Rightarrow \quad \omega = \pm\sqrt{\frac{k}{m}}$$

Solution

$$z(t) = A e^{i(\omega t+\varphi)}$$

For $\omega > 0$ we have rotation in counterclockwise sense.

For $\omega < 0$ we have " " clockwise sense.

Using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we get:

$$x(t) = A\cos(\omega t+\varphi)$$

and $y(t) = A\sin(\omega t+\varphi)$

Before we analyze the solution $x(t) = A\cos(\omega t+\varphi)$ we make hugely important observation:

$$\boxed{\text{Simple harmonic motion in 1D can be thought of as the real or imaginary part of uniform circular motion on the 2d complex plane.}}$$

Simple Harmonic Motion:

Let us look at the solution:

$$x(t) = A\cos(\omega t+\varphi)$$

Note $[x] = [A] = \text{length}$

$[\omega] = \text{time}^{-1} = [\text{frequecy}]$

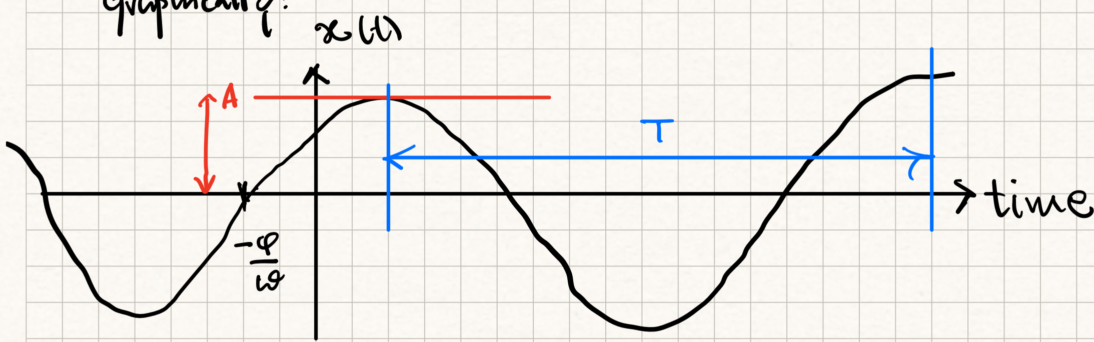
$[\varphi] = 0$

Cosine takes maximum / minimum value $+1/-1$.

A → Amplitude

φ → "phase" determines displacement at $t=0$.

Graphically:



To find the interpretation of φ we set the argument of cosine to zero:

$$\omega t + \varphi = 0$$

$$\Rightarrow \quad \varphi = -\omega t$$

Time period is the time between two maxima or minima:

$$\omega T = 2\pi \quad \Rightarrow \quad T = \frac{2\pi}{\omega}$$

Frequency: $\quad f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \Rightarrow \quad \omega = 2\pi f.$

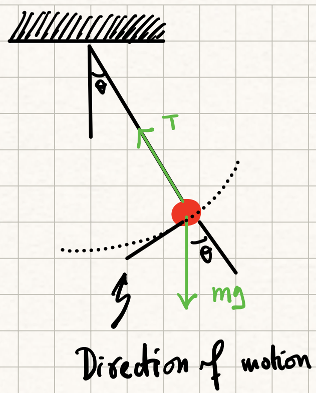
<u>Another Example of Simple Harmonic Motion</u>:

Simple Pendulum:

$T = mg\cos\theta$

$$\boxed{mg\sin\theta = -ml\ddot{\theta}}$$

Newton's Law



For small θ:

$\sin\theta \approx \theta$

$$\Rightarrow \quad g\theta = -l\ddot{\theta}$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g}{l}\theta$$

c.f. $\quad \ddot{x} = -\frac{k}{m}x = -\omega^2 x$

$$\Rightarrow \quad \theta(t) = \theta_M \cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{g}{l}}$$

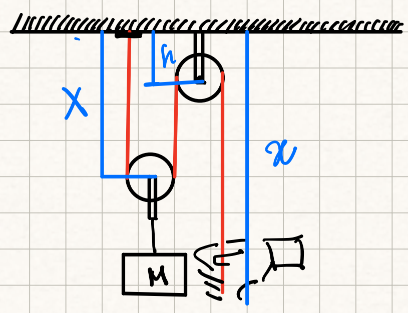
$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Comments:

1. The time period of the simple pendulum is independent of its mass due to the equality of inertial and gravitational masses. Newton had entertained the possibility that the gravitational masses of different material might be different. He performed experiments with a hollow bob filled with different material to test his hypothesis. He could not find any noticeable difference in the time period.

Simple Mechanical Machines:

A simple system of pulleys can allow us to multiply the force or change the acceleration of bodies. So it's worth studying their properties. Consider the following system of pullies. We want to find out how the acceleration of the mass M is related to the acceleration through which the hand pulls down the rope.



Let the length of the rope be l and the radius of the pullies be R. Then

$$l = X + \pi R + (X-h) + \pi R + (x-h)$$

Taking the second time derivative

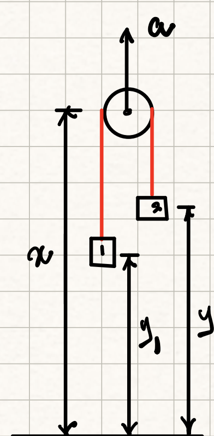
$$0 = \ddot{X} + 2\ddot{x} \Rightarrow \ddot{x} = \frac{\ddot{X}}{2}$$

$$2T - (M+\mu)g = \ddot{X}$$

$$\text{If } \ddot{x} = 0 \Rightarrow T = \frac{(M+\mu)g}{2}$$

$$\text{For } M \gg \mu \quad T = \frac{Mg}{2}$$

Atwood's Machine:



A pully of radius R is being pulled up at acceleration a. Two objects of masses M_1 & M_2 are connected by a rope of length l. what are the accelerations of the two objects?

Equation for object 1 : $T - m_1 g = m \ddot{y}_1$ ——— ①

Equation for object 2: $T - m_2 g = m_2 \ddot{y}_2$ ——— ②

We have three unknowns y_1, y_2 & T but we have only 2 dynamical equations.

To deal with this situation we use the constraint that the length of the rope l is equal to:

$$l = (x - y_1) + \pi R + (x - y_2)$$

Differentiating twice with respect to time and using $\ddot{x} = a$, we get

$$2a = \ddot{y}_1 + \ddot{y}_2 \qquad \text{———} (3)$$

From (1) & (2) we get:

$$m_1(g + \ddot{y}_1) = m_2(g + \ddot{y}_2)$$

$$g + \ddot{y}_1 = \frac{m_2}{m_1}(g + 2a - \ddot{y}_1)$$

$$\ddot{y}_1 + \frac{m_2}{m_1}\ddot{y}_1 = g\left(\frac{m_2}{m_1} - 1\right) + \frac{2am_2}{m_1}$$

$$\ddot{y}_1 \frac{m_1 + m_2}{\cancel{m_1}} = g\frac{m_2 - m_1}{\cancel{m_1}} + \frac{2am_2}{\cancel{m_1}}$$

$$\ddot{y}_1 = \frac{g(m_2 - m_1) + 2am_2}{m_1 + m_2} = \frac{(2a+g)m_2 - m_1 g}{m_1 + m_2}$$

Similarly $\ddot{y}_2 = \dfrac{(2a+g)m_1 - m_2 g}{m_1 + m_2}$

$$T = \frac{m_1 (2a+g)m_2 - m_1^2 g}{m_1 + m_2} + m_1 g$$

$$= \frac{2m_1 m_2 a + m_1 m_2 g - \cancel{m_1^2 g} + \cancel{m_1^2 g} + m_1 m_2 g}{(m_1 + m_2)}$$

$$= \frac{2(a+g)m_1 m_2}{(m_1 + m_2)}$$