Classical Mechanics
Lecture 12

Maxwell's Equations and Galilean Symmetry

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{— Gauss's Law}$$

$\varepsilon_0 \rightarrow$ Permittivity of vacuum

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{— No magnetic charges}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{— Faraday's law}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \quad \text{— Ampère's Law with Maxwell's Correction.}$$

$\mu_0 \rightarrow$ Permeability of vacuum

Maxwell's equations unify electricity and magnetism. But they also contain within them the demise of Newtonian space-time. The symmetry of Newtonian space-time is given by the Galilean group which consists of space-time translations rotations and boosts:

i) $x^\mu \rightarrow x^\mu + a^\mu$, $x^\mu = (t, x, y, z)$

ii) $r_i \rightarrow R_{ij}\, r_j$, $r_j = (x, y, z)$

iii) $r_i \rightarrow r_i + v_i t$, where $\vec{v}$ is constant velocity vector.

So according to Galilean relativity the physics should be invariant under boosts.

But Maxwell's equations are not invariant under boosts. A clear way to see this is to note that Maxwell's equations predict wave equations:

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} - \vec{\nabla}^2 \vec{E} = 0 \qquad \text{(show)}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \vec{\nabla}^2 \vec{B} = 0 \qquad \text{(show)}$$

Dimensional analysis shows that electromagnetic waves (oscillations of $\vec{E}$ & $\vec{B}$) propagate at the speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

It was a remarkable discovery that the electrostatic constant $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m and the magnetic constant $\mu_0 = 1.257 \times 10^{-6}$ N/A^2 combine to give $c = 3 \times 10^8$ m/s, the speed of light. This proved that light is an electromagnetic wave.

But this opened up a new conundrum: Waves are understood to be disturbances in a medium. For example sound waves are oscillations in the fluid medium of air. Water waves are disturbances in water etc. But an electromagnetic wave can apparently propagate through vacuum.

This implies two possibilities as far as Newton's laws are concerned:

i) There is a preferred frame in which the speed of light is $c = 3 \times 10^8$ m/s. Therefore, Maxwell's equations are only valid in that reference frame.

It was conjectured that there exists a medium surrounding the earth which set the preferred reference frame. This medium was dubbed the 'luminoferous aether' (= light bearing aether) which was only sensitive to light.

ii) The symmetry of Maxwell's equations are more fundamental than the symmetry of Newton's equations.

If the aether exists then it should be able to detect earth's motion through it. Michaelson and Morley, and many others tried to detect the aether using very clever electromagnetic experimental setups but none of them were able to detect it.

Albert Einstein famously rejected the symmetry of Newton's equations and adopted the symmetry Maxwell's equations as more fundamental. This means that according to Einstein inertial reference frames are those in which Maxwell's equations have the same form. This implies that the speed of light must be the same in all inertial frames.

Einstein was able to derive Newtonian mechanics from his theory in the limit where the velocity the particles were small compared to the speed of light.

The Postulates:

The postulates adopted by Einstein were as follows:

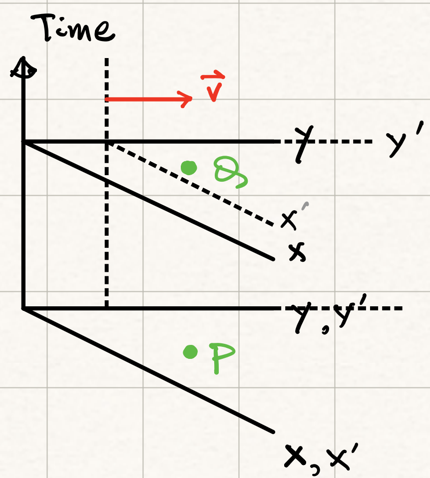
1. The physical laws take the same form in all inertial

frames.

2. The maximum speed at which anything can travel is the same in all inertial frames. In vacuum the speed of light is assumed to be the maximum speed. Therefore, the speed of light is the same in all inertial frames.

Galilean Relativity vs. Einsteinian Relativity:

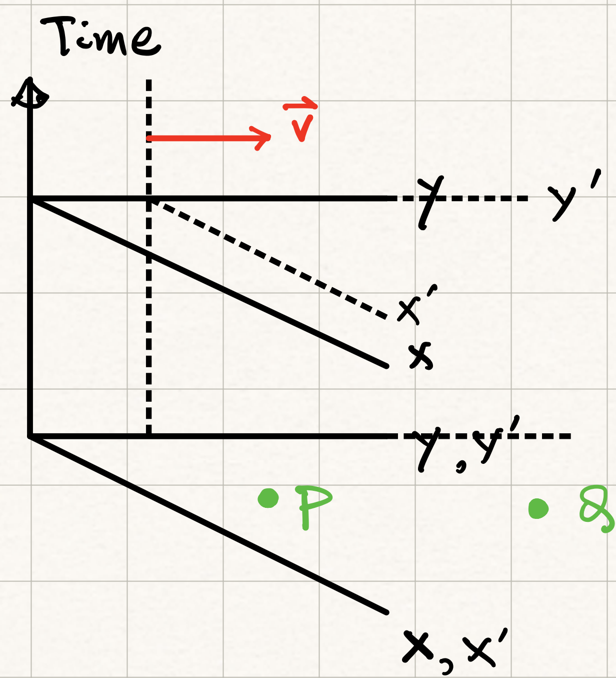
In Galilean Relativity whether something happens at the same spatial point depends on our choice of coordinate systems:



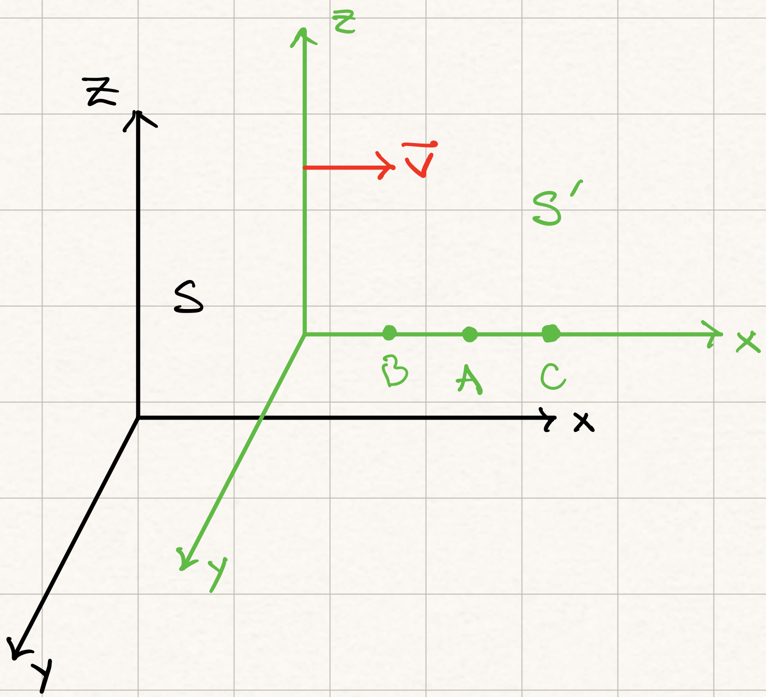
For example, in the figure above the points P & Q are shown in two inertial frames S & S'. S' is moving along the y axis of S with a relative velocity and they coincide at time $t = t' = 0$. It is easy to see that P & Q (which are known as events) have the same spatial coordinates in frame S but they have different spatial coordinates in frame S'. This is Galilean relativity. But the time elapsed between P & Q is the same in both frames. Galilean relativity is said to have absolute time.

Def[n]: An event is a point in space-time. It is characterized by four numbers (t, x, y, z) in a given reference frame.

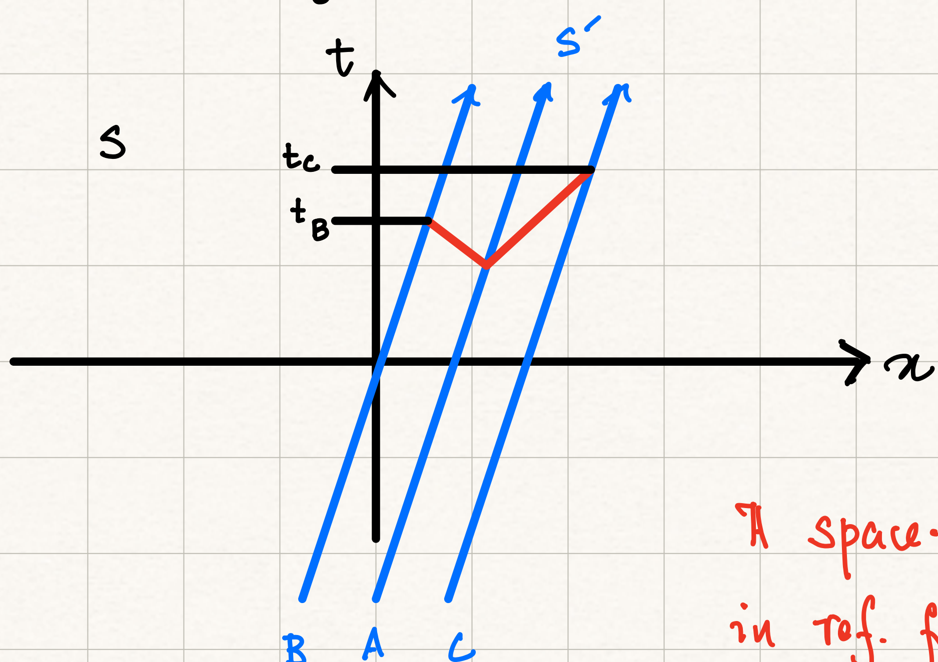
Note that in Galilean relativity, if two events are simultaneous (i.e. if they share the same time coordinate), they are simultaneous in all inertial reference frames.



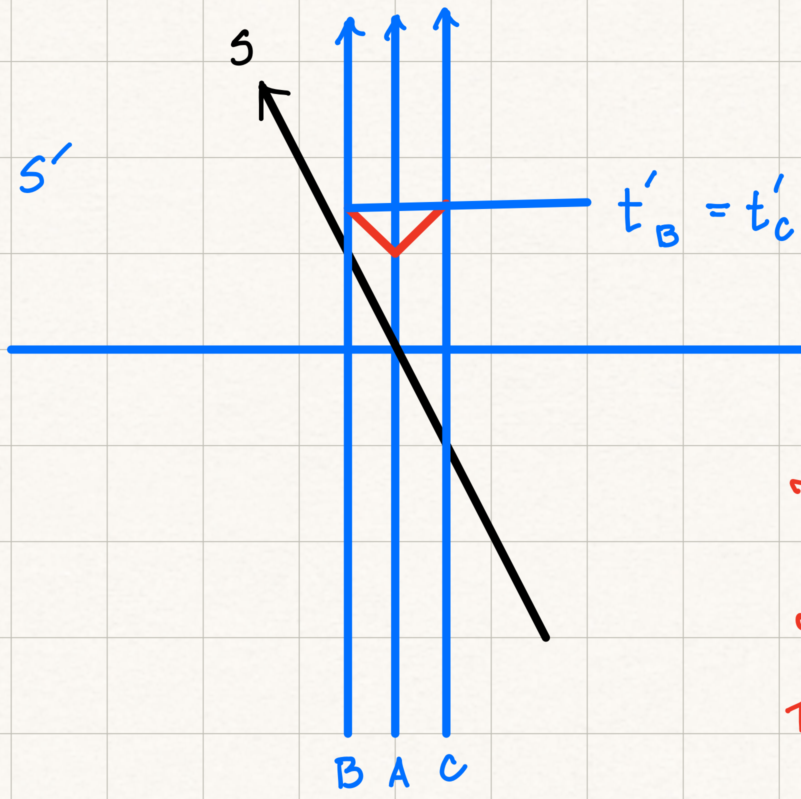
But in Einstein's framework if two events are simultaneous in one inertial frame they are not necessarily simultaneous in other reference frames.



Consider the scenario in which a source of light A and two detectors B & C are stationary in a reference frame S' which is moving at a constant velocity $\vec{v}$ w.r.t. the ref. frame S. The distance between A & the two detectors are assumed to be equal. Then a flash of light emanating from A reaches the detectors B and C at the same time in ref. frame S'.



A space-time diagram in ref. frame S.



A space time diagram of the same scenario in ref. frame S'.

But on the other hand in the reference frame S the detector at the light emanating from A moves has to travel a shorter distance to reach B compared to C. As a result the light is detected at B at a t value smaller than the t value when the light reaches the detector C. Thus we see that the detection of light at B and C are not simultaneous events in ref. frame S.

Formal Developments:

One expects that the physics should be invariant under a change of reference frames involving boost by a constant velocity. But we have seen that Maxwell's equations and the wave equations that result from them seem to be valid in only one Galilean reference frame.

But below we shall see that if we adopt Einstein's principle of relativity then Maxwell's equations are valid under different boosted reference frames but now the formula for boost transformations change radically.

Events

A point in space-time is called an event. Suppose P is an event then in the S coordinate system it will be represented by four numbers:

$$S: \qquad x^{\mu}_{P} = (ct_P, x_P, y_P, z_P)$$

Now suppose a flash of light emanates from P and it propagates to Q. Then it must be true that

$$c^2(t_Q - t_P)^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2$$

where $x^{\mu}_{Q} = (t_Q, x_Q, y_Q, z_Q)$ are the coordinates of the event Q in S.

Now suppose we boost to a different reference frame S' where the coordinates of the same events P & Q are given by:

$$x'^{\mu}_{P} = (t'_P, x'_P, y'_P, z'_P)$$

$$x'^{\mu}_{Q} = (t'_Q, x'_P, y'_P, z'_P)$$

Since, according to the principle of relativity, the speed of light is the same in all reference frames it must be that

$$c^2 (t'_Q - t'_P)^2 = (x'_Q - x'_P)^2 + (y'_Q - y'_P)^2 + (z'_Q - z'_P)^2$$

We can express this result in terms of an interval S_{12} defied as:

$$S: \quad S^2_{QP} = c^2 (t_Q - t_P)^2 - (x_Q - x_P)^2 - (y_Q - y_P)^2 - (z_Q - z_P)^2$$

Then the interval in S' frame is:

$$S': \quad S'^2_{QP} = c^2 (t'_Q - t'_P)^2 - (x'_Q - x'_P)^2 - (y'_Q - y'_P)^2 - (z'_Q - z'_P)^2 .$$

Then the principle of relativity tells us that for P & Q connected by a flash of light we must have:

$$S^2_{PQ} = S'^2_{PQ} = 0. \qquad [\text{Note that } S^2_{PQ} = S^2_{QP}]$$

When two events can be connected by a flash of light we say that the interval between them is light like or null:

$$S^2_{PQ} = 0.$$

We have seen that null intervals are null in all refrence frames related by boost because of the constancy of the speed of light

$$S^2_{PQ} = S'^2_{PQ} = 0.$$

Invariance of arbitrary interval:

Suppose P & Q are not necessrily light-like separated. Then how are S^2_{PQ} & S'^2_{PQ} related?

Exercise: Show that $S_{PQ}^2 = S'^2_{PQ}$ for S & S' related by rotations and space-time translations.

In order to derive the transformations that connect S and S' that result from boosts, let us consider the case when P & Q are infinitesimally close to one another:

$$S: \qquad ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

In S' it is given by

$$ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

We can work with ds & ds' and write:

$$ds = \frac{\partial s}{\partial s'} ds'$$

$$ds = a\, ds'$$

Note that this is consistent with $ds = ds' = 0$.

Now we make certain assumptions about space-time:

1. Space-time is homogeneous
2. Space-time is isotropic

This means that ds or ds' cannot depend on the direction of the boost velocity $\vec{v}$! Thus we write:

$$ds = a(v)\, ds'.$$

Now, let S_1 & S_2 be two reference frames that are related to S by boosts involving velocities $\vec{v}_1$ and $\vec{v}_2$:

$$ds = a(v_1)\, ds_1, \quad ds = a(v_2)\, ds_2$$

$$\text{—— ①} \qquad\qquad \text{—— ②}$$

But S_1 is related to S_2 by the relative velocity $\bar{v}_1 - \bar{v}_2 \equiv \bar{v}_{12}$:

$$ds_1 = a(v_{12})\, ds_2 \qquad \text{—— ③}$$

From equation ①, ②, & ③ we get

$$ds_1 = \frac{a(v_2)}{a(v_1)}\, ds_2 = a(v_{12})\, ds_2$$

Thus we get $\dfrac{a(v_2)}{a(v_1)} = a(v_{12})$

The left-hand-side of this equation is independent of the directions of $\vec{v}_1$ & $\vec{v}_2$ but the right-hand-side depends on the relative direction between $\vec{v}_1$ and $\vec{v}_2$ since:

$$v_{12}^2 = v_1^2 + v_2^2 - 2v_1 v_2 \cos\theta_{12}$$

Thus we have a contradiction unless we assume that a is independent of the boost velocity and we then have

$$a = 1$$

$$\Rightarrow \qquad ds = ds'$$

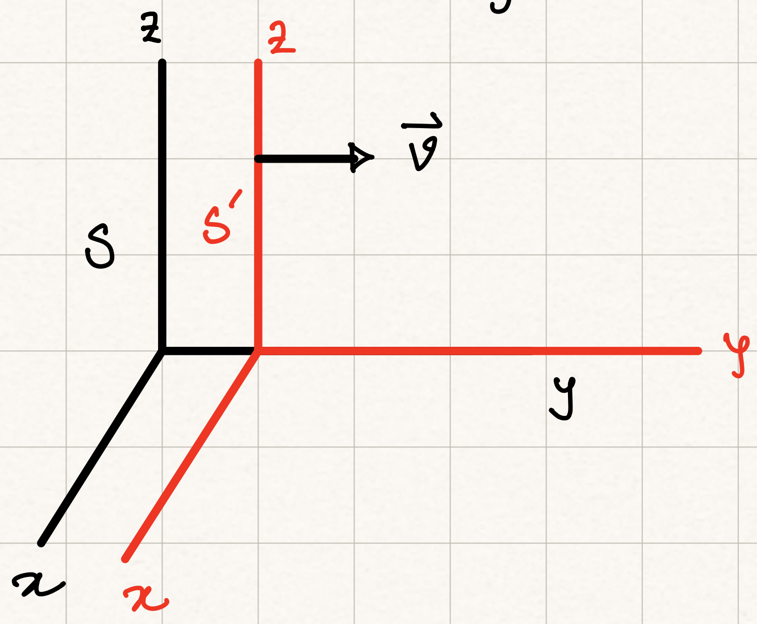
Thus we see that the interval

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is invariant under boosts, rotations and space-time translations. Thus ds^2 is called the invariant interval.

Lorentz Transformations:

We have seen that both rotations and translations are consistent with Einsteinian relativity. So now we figure out how to transform between two inertial reference frames which are related by a boost.



Let S and S' be two reference frames where S' moves with a constant velocity $\vec{v}$ w.r.t. S. We can choose our coordinate system in such a way so that at $t = t' = 0$ the axes of the two reference frame coincide.

Now, without loss of generality (WLOG), one can assume that the coordinates of the event P in S are given by $x_P^\mu = (0, 0, 0, 0)$. Then it follows from the definition of S' that $x'^{\mu}_{P} = (0,0,0,0)$.

So now for any P & Q we can write

$$S^2_{PQ} = c^2 t^2_Q - x^2_Q - y^2_Q - z^2_Q$$

and
$$S'^2_{PQ} = c^2 t'^2_Q - x'^2_Q - y'^2_Q - z'^2_Q$$

To reduce notational clutter we drop the P & Q labels. We have shown that

$$S^2 = S'^2.$$

We assume that under a boost

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

wher $\Lambda^{\mu}{}_{\nu}$ is a 4x4 constant matrix that transforms the Cartesian coordinates x^{μ} in S to x'^{μ} in S'.

We now introduce a rank 2 tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is called a metric tensor. With the help of $\eta_{\mu\nu}$ we can express s^2 as:

$$s^2 = x^\mu x^\nu \eta_{\mu\nu}$$

$$= c^2t^2 - x^2 - y^2 - z^2 .$$

But we also have:

$$s'^2 = x'^\mu x'^\nu \eta_{\mu\nu}$$
$$= (\Lambda^\mu{}_\alpha x^\alpha)(\Lambda^\nu{}_\beta x^\beta)\eta_{\mu\nu}$$
$$= x^\alpha x^\beta (\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu})$$

But since $s'^2 = s^2 = x^\alpha x^\beta \eta_{\alpha\beta}$

It follows that $\boxed{\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta} .}$

In matrix form this can be written as:

$$\Lambda^T \eta_{\mu\nu} \Lambda = \eta_{\alpha\beta}$$

where $(\Lambda^T)_\alpha{}^\mu = \Lambda^\mu{}_\alpha$

In analogy to the rotation matrices $R \in SO(3)$ which satisfy

$$R^T \mathbb{1} R = \mathbb{1}$$

Λ matrices belong to the $SO(1,3)$ group.

Explicit form of Λ:

Let S' be obtained from S by boosting in the x-direction. Then we have

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

with

$$\Lambda = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The condition $\Lambda^T \eta \Lambda = \eta$ implies

$$\det \Lambda \cdot \det \eta \cdot \det \Lambda = \det \eta$$

$$\Rightarrow \quad (\det \Lambda)^2 = 1.$$

Since we are looking continuous transformations we choose

$$\det \Lambda = +1.$$

Then we have $\alpha\delta - \gamma\beta = 1$

The condition $\Lambda^T \eta \Lambda = \eta$ also gives:

$$\alpha\beta = \gamma\delta$$
$$\alpha^2 - \gamma^2 = 1$$

and $$\beta^2 - \delta^2 = -1$$

The solutions of these equations can be guessed:

$$\alpha = \cosh \xi$$
$$\gamma = \sinh \xi$$

and $\alpha = \delta$ and $\gamma = \beta$. [check.]

Thus we see that for boost in the x-axis one gets:

$$\Lambda = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[The signs have been chosen for future convenience.

These transformations $\Lambda \in SO(1,3)$ are called the Lorentz transformations. These include boosts in the y- and z-directions as well as the three rotations:

$$\Lambda(R) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

where R is a 3×3 rotation matrix.

Ex: Write down the Lorentz transformation matrices for boosts in the y- and z-directions.

Boosts in terms of velocity

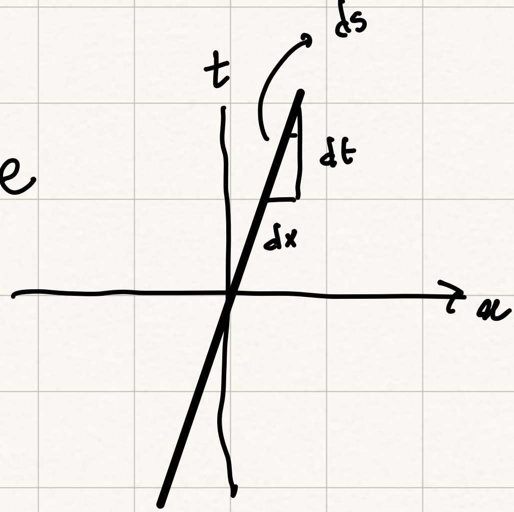
The parameter ζ in the boost Λ is called rapidity. Rapidity in related to the boost velocity $\vec{v}$.

Let S' be boosted in the x-direction with some velocity $\vec{v} = (v_x, 0, 0)$.

Then $$\vec{v} = \frac{d\vec{x}}{dt} = \left(\frac{dx}{dt}, 0, 0\right)$$

But $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\varsigma & -\sinh\varsigma \\ -\sinh\varsigma & \cosh\varsigma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$

implies: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\varsigma & +\sinh\varsigma \\ +\sinh\varsigma & \cosh\varsigma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$



The coordinate of the world line of the origin of S' in its own reference frame is

$$x' = 0, \quad t' = t'$$

And so $\quad x = ct \sinh\varsigma$

and $\quad t = t \cosh\varsigma$

Thus we see $\frac{1}{c}\frac{x}{t} = \frac{v_x}{c} = \tanh\varsigma$

Since $\cosh^2\varsigma - \sinh^2\varsigma = 1$

we get $1 - \tanh^2\varsigma = \frac{1}{\cosh^2\varsigma}$

$$1 - \frac{v_x^2}{c^2} = \frac{1}{\cosh^2\varsigma}$$

$$\Rightarrow \quad \cosh\varsigma = \frac{1}{\sqrt{1 - \frac{v_x^2}{c^2}}}$$

$$\sinh\varsigma = \sqrt{\cosh^2\varsigma - 1} = \sqrt{\frac{1}{1-\frac{v^2}{c^2}} - 1} = \frac{v_x/c}{\sqrt{1 - v_x^2/c^2}}$$

Thus we get

$$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}} & \frac{v_x/c}{\sqrt{1-\frac{v_x^2}{c^2}}} & 0 & 0 \\ \frac{v_x/c}{\sqrt{1-\frac{v_x^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The sign of v_x determines the direction of the boost.

Time Dilation, Proper Time & the Twin Paradox:

In Newtonian physics the Galilean group of transformations leave

$$c^2 dt^2 + dx^2 + dy^2 + dz^2$$

invariant. But in Einsteins new geometry the Lorentz transformations leave

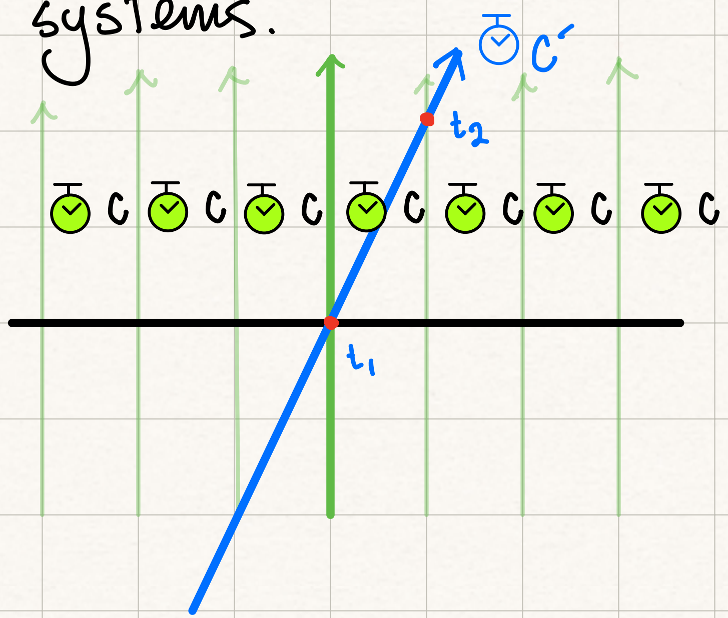
$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad \text{invariant.}$$

⌈ One can also define the invariant interval as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

It is the relative sign that is important. ⌋

This new exotic structure of space-time means that time is not absolute as can be seen from the Lorentz transformations. Thus we can ask how clocks measure time in two different coordinate systems.



Let C be set of stationary clocks which are at regular spatial intervals in the reference frame S. On the other hand C' is a clock that is moving at a constant velocity as shown in the figure.

All the clocks in S are synchronized. Let the clock C' be identical in construction as the clocks

c. We then compare time measured by c' to the clock c that the moving clock is passing at the moment.

Let t_1 & t_2 be two times shown in c then

$$t_1 = t_1' \cosh\varsigma + x_1' \sinh\varsigma$$

and $$t_2 = t_2' \cosh\varsigma + x_2' \sinh\varsigma$$

But in S', the clock c' is stationary and so $x_1' = x_2'$. Thus

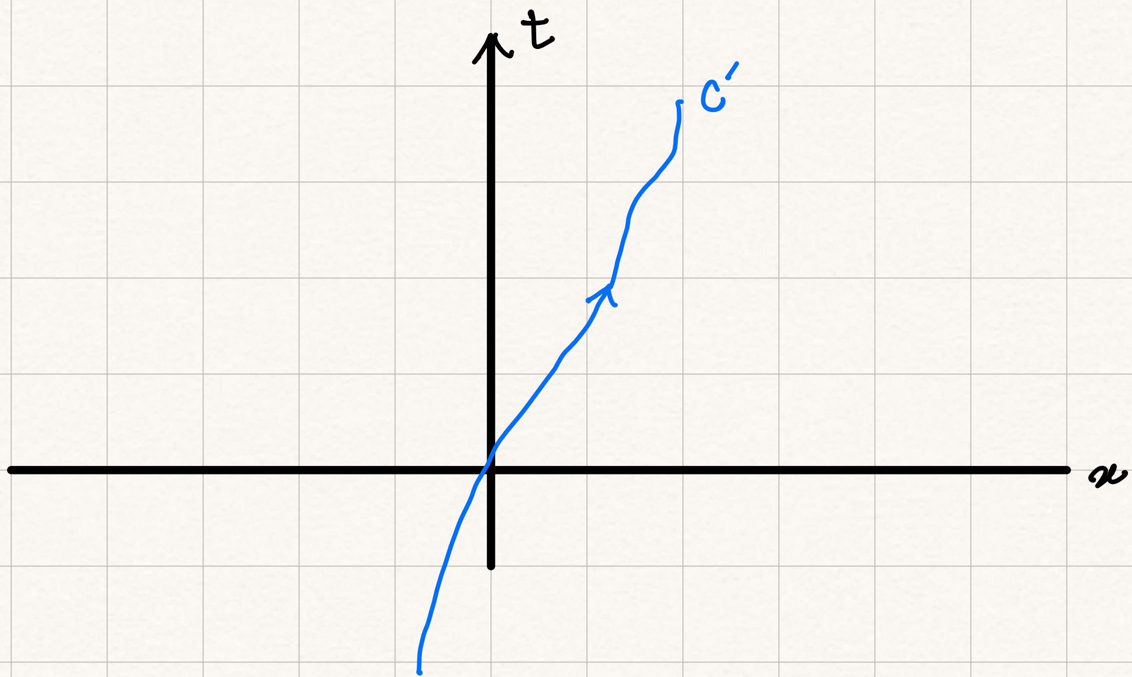
$$\Delta t = \Delta t' \cosh\varsigma = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

Since $\cosh\varsigma \geq 1$ we see that $\Delta t \geq \Delta t'$. |

Thus a moving clock always runs slower compared to a stationary clock. This is known as time dilation

Proper Time

The trajectory of c' is an example of a world line. Let us now consider the world line of a clock c' that is moving in an arbitrary manner.



What is the time measured by the clock c'? At each instant of time in S we can consider the clock c' to be at rest in a boosted frame S' which is moving with a velocity which coincides with the velocity of c'. Then

$$d\sigma^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$= c^2 dt'^2 + 0$$

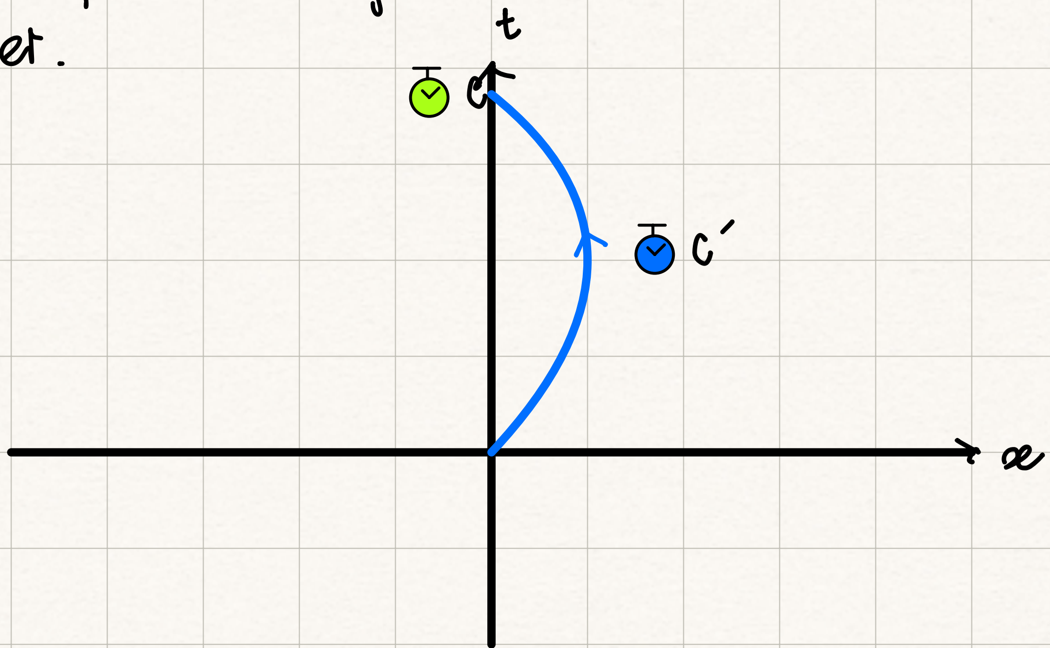
Thus $$dt' = dt\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}}$$

$$\boxed{t_2' - t_1' = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} \cdot dt}$$

This elapsed time as measured by the moving clock is called its proper time. Since the integrand $\sqrt{1-\frac{v^2}{c^2}}$ is either 1 or less we see that the proper time is shorter than time measured in any other frame.

The Twin Paradox

When the clock C' travels in a straight world line (i.e., the velocity of C' is constant) then from the point of view of S' each clock in S also runs slower.



We have also seen that the time measured by an arbitrarily moving clock C' is always less than stationary clock. Now if C' follows a world line that allows it to meet up with a clock C that is stationary in S then $t_2' - t_1' < t_2 - t_1$.

But since from the point of view of C' the stationary clock is always running slower would that not imply that $t_2 - t_1$ is smaller than $t_2' - t_1'$ and thereby raise a paradox?

The resolution of this paradox lies in the observation that the situations between C and C' are not symmetric. When we are measuring time C' we are always measuring against a clock that is stationary in S. But since the frame of C' is not inertial, the measurement of time in S is not measured against a set of clocks that are stationary in S'. In fact there is no S', since the motion of C' involves acceleration.

Thus one can use the Lorentz transformation in S frame one cannot do the same in the (non-inertial frame) of clock C'.

Length Contraction

Let us now consider a rigid rod of length l that is stationary in S.

If the at given time t the end points are at at spatial coordinates $\bar{x}_1$ & $\bar{x}_2$ then

$$l = |\bar{x}_1 - \bar{x}_2|$$

To measure its length in S we must ensure that $t_1' = t_2' = t'$

$$x_1 = -t \sinh\zeta + x_1' \cosh\zeta$$
$$x_2 = -t \sinh\zeta + x_2' \cosh\zeta$$

and so $l = |\vec{x}_1' - \vec{x}_2'| \cosh\zeta = l' \cosh\zeta$

Thus we see that $l > l'$ and so the a moving rod is shorter than a stationary one.

Velocity Addition Formula

Suppose two observers are travelling in opposite directions with velocities v_1 & v_2 in the x-direction. Then according to Galilean relativity the velocity of the second observer in the reference frame of the first observers would be $v_2 - v_1$.

If the two observers have velocities in the opposite directions then they would be travelling faster each other's frame with respect to the original frame. Then if the velocity addition formula were to be correct then if $v_1 = \frac{c}{2}$ and $v_2 = -\frac{c}{2}$ then the relative velocity would be the same as the speed of light. Thus the simple velocity formula $\vec{v}_{12} = \vec{v}_1 \pm \vec{v}_2$ cannot be correct in relativity. To find the correct formula we note that two successive boosts $\Lambda_x(\zeta_1)$ and $\Lambda_x(\zeta_2)$ along the x-axis give us:

$$\Lambda_x(\zeta) \equiv \Lambda_x(\zeta_1 + \zeta_2) = \Lambda_x(\zeta_2)\,\Lambda_x(\zeta_1) \quad [\text{Show}]$$

where ζ_i are rapidities. This implies:

$$\tanh(\zeta_1 + \zeta_2) \equiv \tanh\zeta = \frac{v}{c}$$

But $$\tanh(\zeta_1 + \zeta_2) = \frac{\tanh\zeta_1 + \tanh\zeta_2}{1 + \tanh\zeta_1 \tanh\zeta_2} = \frac{(v_1 + v_2)/c}{1 + v_1 v_2/c^2}$$

Thus $$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}.$$ The Newtonian limit is $\frac{v_1 v_2}{c^2} \ll 1$.

In that limit we get $v = v_1 + v_2$.

Causal Structure:

The form of ds^2 or S^2_{PQ} show that there are three kinds of intervals:

1. Time-like : when $S^2_{PQ} > 0$.

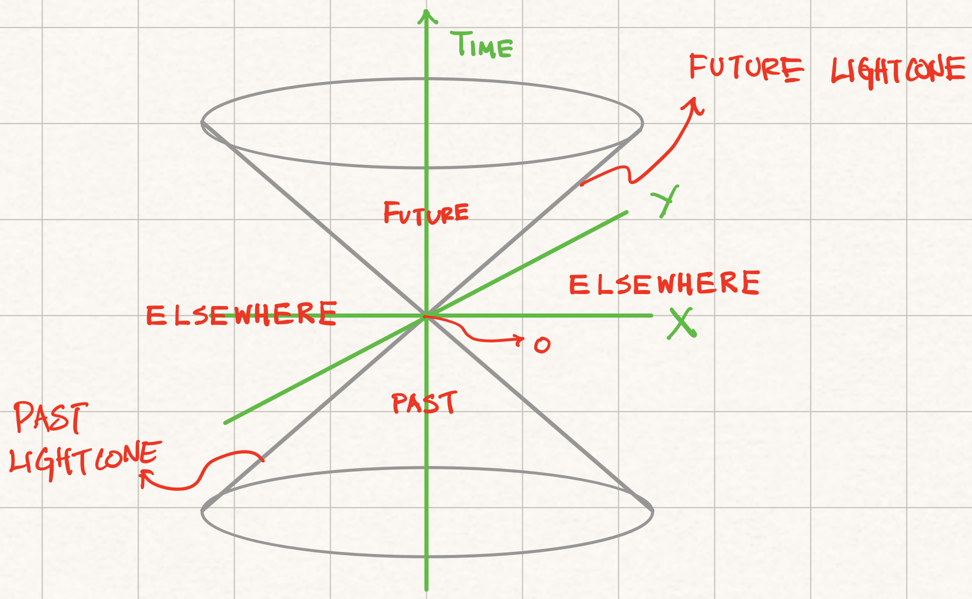
2. Space-like : when $S^2_{PQ} < 0$.

3. Light-like or Null: when $S^2_{PQ} = 0$.

The major difference between Galilean and Einsteinian relativity is that in the former two parties can interact arbitrarily fast even if they are separated by an arbitrary distance. [Think of Newton's law of gravity and/or the third law.]

In Einstein's relativity two events can be causally connect only if their separation is time-like or light-like.

Thus for each event space-time can be divided into three different regions:



All the points on and inside the future light-cone of O can be causally affected by signals and particles emanating from O.

Likewise all points on and the past light cone of O are points that can influence O.

All other events are causally disconnected from O. This region is called elsewhere.

Relativistic Tensors:

The fundamental laws of nature must be the same in all inertial frames. This is only possible if physical observables transform in a well-defined manner under Lorentz transformations.

In technical terms this means that all physical quantities must be tensorial.

The prototypical tensorial object in relativity is

$$dx^\mu = (c\,dt, dx, dy, dz)$$

with $\mu = 0, 1, 2, 3$. This is because under a Lorentz transformation:

$$dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\rho} dx^\rho$$
$$= \Lambda^\mu{}_\rho \, dx^\rho$$

Any quantity that transforms in this way is called a contravariant vector:

$$V^\mu \longrightarrow V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

An example of a contravariant vector is the 4-velocity vector defined by:

$$v^\mu = \frac{dx^\mu(\tau)}{d\tau}$$

where $x^\mu(\tau)$ is the worldline of a particle and τ is its proper time.

Comment:

1. The 4-velocity of a particle measures is velocity through space and time. The rate of change of its 4-position is measured with respect to its proper time τ.

Then if we change reference frames then its world line in the new frame is:

$$x'^{\mu}(\tau) = \Lambda^{\mu}{}_{\nu}\, x^{\nu}(\tau)$$

So $$v'^{\mu} = \frac{dx'^{\mu}}{d\tau} = \Lambda^{\mu}{}_{\nu} \frac{dx^{\nu}(\tau)}{d\tau}$$

$\{$Since $\Lambda^{\mu}{}_{\nu}$ are independent of τ.$\}$

$$\Rightarrow \quad \underline{v'^{\mu} = \Lambda^{\mu}{}_{\nu}\, v^{\nu}}$$

We know that ds^2 is invariant. Such an object is called a scalar.

Given $\quad ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$

This should equal

$$ds'^2 = \eta'_{\mu\nu}\, dx'^\mu dx'^\nu$$

where $\eta'_{\mu\nu}$ is the $\eta_{\mu\nu}$ in frame S'. Since $dx'^\mu = \Lambda^\mu{}_\nu\, dx^\nu$ we get

$$\eta'_{\mu\nu}\, \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$$

$$\boxed{\Rightarrow \quad \eta'_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu\, \eta_{\alpha\beta}}$$

This is an example of a rank 2 tensor with covariant indices.

Λ^{-1} are also Lorentz transformations. Therefore,

$$(\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu\, \eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\Rightarrow \quad \boxed{\eta'_{\mu\nu} = \eta_{\mu\nu}}$$

$\eta_{\mu\nu}$ is a special rank-2 tensor with two covariant indices which is invariant under Lorentz transformation.

Given a contravariant vector V^μ we can define a covariant vector by:

$$V_\mu := \eta_{\mu\nu} V^\nu$$

Then $V_0 = V^0$, $V_1 = -V^1$, $V_2 = -V^2$, $V_3 = -V^3$.

Then V_μ transforms as:

$$V_\mu \to V'_\mu = (\Lambda^{-1})^\sigma{}_\mu V_\sigma .$$

One can define $\Lambda_\mu{}^\sigma := (\Lambda^{-1})^\sigma{}_\mu$ Then we can write this as:

$$V_\mu \to V'_\mu = \Lambda_\mu{}^\sigma V_\sigma .$$

Inverse metric:

We can define $\eta^{\mu\nu}$ as:

$$\eta^{\mu\alpha}\eta_{\alpha\beta} = \delta^{\mu}_{\beta}$$

And so $\eta^{\mu\nu} \rightarrow \eta'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\eta^{\alpha\beta}$

From requiring $V_\mu V^\mu$ to be scalar it follows that

$$\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\eta^{\alpha\beta} = \eta^{\mu\nu}$$

Just we can use the metric tensor to lower an index $V_\mu = \eta_{\mu\nu}V^\nu$, we can use the inverse metric $\eta^{\mu\nu}$ to raise indices:

$$V^\mu = \eta^{\mu\nu}V_\nu .$$

In particular

$$\eta^{\mu\alpha}\eta^{\nu\beta}\eta_{\alpha\beta}$$

$$= \eta^{\mu\alpha}\delta^{\nu}_{\alpha} = \eta^{\mu\nu}$$

$$\Rightarrow \quad \eta^{\mu\alpha}\eta^{\nu\beta}\eta_{\alpha\beta} = \eta^{\mu\nu}$$

Comments:

1. The metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ can be used to convert a covariant index $(\ldots_\alpha\ldots)$ into contravariant index $(\ldots^\alpha\ldots)$ and vice versa.

The definition:

$$\Lambda_\mu{}^\alpha = (\Lambda^{-1})^\alpha{}_\mu$$

implies $\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\alpha\beta}$ — $*$

Thus we see if we set:

$$\boxed{\Lambda_\mu{}^\alpha = \eta_{\mu\nu}\eta^{\alpha\beta}\Lambda^\nu{}_\beta} \longrightarrow (\dagger)$$

Then LHS of $(*)$ becomes:

$$(\eta_{\mu\theta}\eta^{\alpha\delta}\Lambda^\theta{}_\delta)(\eta_{\nu\psi}\eta^{\beta\varphi}\Lambda^\psi{}_\varphi)\eta_{\alpha\beta}$$

$$= \Lambda^\theta{}_\delta \Lambda^\psi{}_\varphi \eta_{\mu\theta}\eta_{\nu\psi}\eta^{\delta\varphi}$$

$$= (\Lambda^\theta{}_\delta \Lambda^\psi{}_\varphi \eta^{\delta\varphi})\eta_{\mu\theta}\eta_{\nu\psi}$$

$$= \eta^{\sigma\varphi} \eta_{\mu\sigma} \eta_{\nu\varphi} = \eta_{\mu\nu} = \text{RHS}.$$

Thus we see that our definition

$$\Lambda_\mu{}^\alpha := (\Lambda^{-1})^\alpha{}_\mu$$

is consistent with $\Lambda_\mu{}^\alpha = \eta_{\mu\nu} \eta^{\alpha\beta} \Lambda^\nu{}_\beta$

<u>Ex</u>: Show that the completely antisymmetric 4-dim Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ with $\epsilon_{0123} = +1$ is invariant under a Lorentz transformation.
Use a boost in a specific direction to show this.

A general tensor $W_{p_1 p_2 \cdots p_n}{}^{q_1 q_2 \cdots q_m}$

is said to have $\binom{m}{n}$ rank [m contra-variant indices and n covariant indices] if they transform as:

$$W_{p_1 \cdots p_n}{}^{q_1 \cdots q_m}$$

$$\rightarrow \Lambda^{r_1}{}_{p_1} \cdots \Lambda^{r_n}{}_{p_n} \Lambda^{q_1}{}_{l_1} \cdots \Lambda^{q_m}{}_{l_m} \times W_{r_1 \cdots r_n}{}^{l_1 \cdots l_m}$$

Relativistic Energy-Momentum:

In non-relativistic physics we saw that energy and momentum were related time and space. The energy of a system was conserved if the system had a Lagrangian that had no explicit time dependence, and the momentum was conserved if the Lagrangian was invariant under spatial translation.

In relativistic selting Lorentz transformations mix up time and space. And so we may legitimely ask whether energy and momentum have separate existence.

We have already seen that the 4-velocity $v^\mu = \frac{dx^\mu}{d\tau}$ is a rank-1 contravariant tensor. Let us introduce analogously the 4-momentum of a particle by:

$$p^\mu = m \frac{dx^\mu}{d\tau}$$

where m is the inertial mass of the particle.

Recall that $d\tau = \sqrt{1-\frac{v^2}{c^2}}\, dt$. Then we can see that

$$p^0 = mc\frac{dt}{d\tau} = \frac{mc}{\sqrt{1-v^2/c^2}}$$

We see that $p^0 c$ has the dimension of energy. Thus it is tempting to identify it as the energy of a particle. But it is not its pure kinetic energy because in the Newtonian limit:

$$p^0 c \approx mc^2 + \frac{1}{2}mv^2 + \cdots$$

Thus we see we can associate an energy mc^2 with the particle even when it is at rest. Thus defining

$E = p^0 c$ we get

$E = mc^2$ when $\vec{v} = 0$. mc^2 is known as the rest of the particle. With the above definition of E we can write the 4-momentum as

$$p^\mu = \left(\frac{E}{c}, \bar{p}\right).$$

The quantity $p_\mu p^\mu$ is a Lorentz scalar. On dimensional grounds

$$p_\mu p^\mu = m^2c^2$$

Since $p_\mu = \left(\frac{E}{c}, -\vec{p}\right)$ we get

$$E^2 - \bar{p}^2 c^2 = m^2 c^4$$

or $$E = \sqrt{\bar{p}^2 c^2 + m^2 c^4}$$

Thus we see that in the $\vec{p} \to 0$ limit we get $E = mc^2$. On the other hand, for photons $m \to 0$ we get $E = |\bar{p}|c$ as is familiar for electromagnetism.

Comment:

1. We see that energy is now the 'time' component of a 4-dimensional contravariant vector. And it mixes with the spatial momentum components under a Lorentz boost:

$$p^\mu \to p'^\mu = \Lambda^\mu{}_\nu \, p^\nu$$

$$\begin{pmatrix} p'^0 \\ p'^1 \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \end{pmatrix}$$

The Maxwell Tensor:

The special theory of relativity has its origins in electromagnetism. Then Maxwell's equations better be invariant under Lorentz transformations. But the electric and magnetic fields $\vec{E}$ and $\vec{B}$ are 3-vectors and it is not obvious what their 4-dimensional counter-parts should be.

It turns out that the correct 4-dimensional point of view is to look at them as the components of rank-2 anti-symmetric contravariant tensor:

$$F^{0i} = -\frac{E_i}{c} , \quad F^{i0} = -F^{0i} = \frac{E_i}{c} , \quad i = 1,2,3$$

$$F^{ij} = -\epsilon_{ijk} B_k , \quad F^{ji} = -F^{ij} , \quad i,j = 1,2,3$$

We may write it in a matrix form:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & B_x \\ E_z/c & -B_y & -B_x & 0 \end{bmatrix}$$

Then $F_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} F^{\alpha\beta}$

$$= \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & B_x \\ -E_z/c & -B_y & -B_x & 0 \end{bmatrix}$$

Since it is a rank 2 tensor, under a Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

we get $\quad F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$

If the Lorentz transformation is

$$\Lambda = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And so

$$E_x' = E_x \,, \quad E_y' = \gamma(E_y - v B_z) \,, \quad E_z' = \gamma(E_z + v B_y)$$

$$B_x' = B_x \,, \quad B_y' = \gamma\left(B_y + \frac{v}{c^2} E_z\right), \quad B_z' = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

where $\quad \gamma := \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

Maxwell's equations then become:

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu} = J^\mu \, , \quad \text{with } J^\mu = (c\rho, \bar{J})$$

and

$$\frac{\partial}{\partial x^\mu} F_{\nu\rho} + \frac{\partial}{\partial x^\nu} F_{\rho\mu} + \frac{\partial}{\partial x^\rho} F_{\mu\nu} = 0$$

Comment:

1. The first covariant equation transforms as a rank 1 tensor equation.

2. The second covariant equation transforms as a rank-3 tensor equation.

3. 'Covariant' also means something that transforms as a tensor.