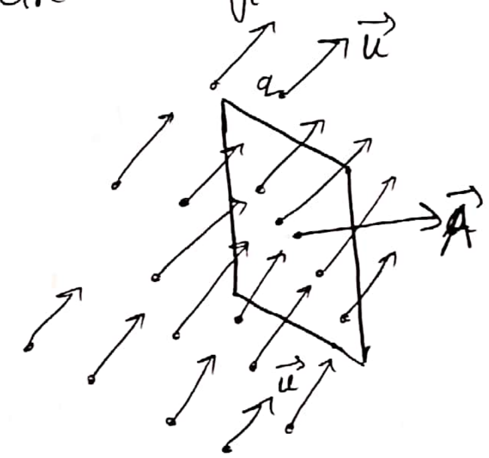
Electric Current and Current Density

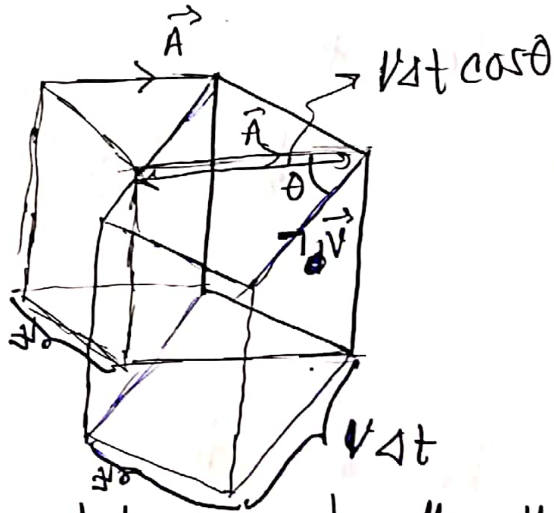
Electric current is basically the amount of charge passing through a particular area in a given instant of time.

$$\therefore I_{avg} = \frac{\Delta Q}{\Delta t}$$

The unit of current is ampere which is equivalent to Coulomb/second.

Consider that there are n charged particle per unit cubic meter, on average, all moving with the same vector velocity $\vec{v}$, and carrying the same charge q. Now, consider a frame of area A fixed in some orientation and the area vector $\vec{A}$ makes an angle θ with $\vec{u}$.





The question is, how many particles are travelling through the frame within a time interval Δt. The particles that will be able to pass through the frame A are those contained in the blue volume. The area of the blue volume coincides with the frame area for ~~one~~ side. We have drawn another black cuboid, which is exactly in alligned with the frame A. So, the volume will be given by $= A\cos\theta \cdot V\Delta t$

$$= \vec{A} \cdot \vec{V} \, \Delta t$$

So, the total charge passing through the frame A will be $q \times (n \vec{A} \cdot \vec{V} \Delta t)$. So, current through the frame, I_A, is,

$$I_A = \frac{q \times (n \vec{A} \cdot \vec{V} \Delta t)}{\Delta t} = nq \vec{A} \cdot \vec{V}$$

Now, the particles in general can have different velocities. Say, particles with velocity $\vec{V_1}$ has a volume charge density n_1, $\vec{V_2}$ with n_2 and so on. The current will then be given by,

$$I_A = n_1 q_1 \vec{A} \cdot \vec{V_1} + n_2 q_2 \vec{A} \cdot \vec{V_2} + \cdots$$

$$= \sum_{K=1}^{N} n_K q_K \vec{A} \cdot \vec{V_K} = \vec{A} \cdot \sum_{K=1}^{N} n_K q_K \vec{V_K}$$

We then define a new quantity, current density as.

$$\vec{J} = \sum_K n_K q_K \vec{V_K}$$

which is independent of the frame area.

$$\therefore I_A = \vec{J}.\vec{A}$$

In general, the surface might not be as simple as the rectangular surface, and we generalize the current through any surface as,

$$\boxed{I_A = \iint_S \vec{J}.d\vec{A}}$$

In the case of current conduction, the carriers are electrons with an amount of charge $q=-e$. The average velocity of the electrons can be found by averaging over all the electrons. If n_K number of electrons are found to have $\vec{V}_k$ velocity per unit volume, and N_e is the total number of electrons per unit volume, then the average velocity will be given by,

$$\vec{\bar{V}} = \frac{1}{N_e}\sum_{K=1}^{N} n_K \vec{V}_K$$

Previously we had, $\vec{J} = \sum_K n_K q_K \vec{V}_K = -e\sum_K n_K \vec{V}_K$

$$\boxed{\therefore \vec{J} = -eN_e\vec{\bar{v}}}$$

$$\boxed{\therefore \vec{J} = -N_e e\langle\vec{v}\rangle}$$

If we write the volume charge density as, $\rho_e = -eN_e$

then,

$$\boxed{\vec{J} = \rho_e\langle\vec{v}\rangle}$$

We

Microscopic view of Ohm's law

Drude model of conductivity

Paul Drude gave his theory of conductivity in 1900, based on the discovery of electron by Sir J. J. Thomson. At that time, the idea of atom wasn't even that solid. Anyways, ~~Drude~~ Drude had two very important assumptions.

1. The velocity of electron just after the scattering from an atom or electron is completely random.

2. There is, on ~~aver~~ average a characteristic time ~~const~~ interval τ, between each subsequent collision.

We will try to be more refined in our calculations than done by Paul Drude, and try to point where these approximations break.

We will consider that, in a material, there are some equal number of positive ions and negative charges, each carrying charge of e in magnitude. Say, the ~~charge~~ number density of them are N. The mass of positive ion is M_+ and negative ion is M_-. If there is no electric field, then say, the positive ions and ~~elect~~ negative charges are moving in random velocities. The mean free path of them are much much larger than the molecular diameters. So, effectively most of the times they are free, except when they encounter a collision with the molecules.

The ~~molecules~~ ions will move in straight line until they come close to an atom/molecule, when they will be scattered to a new velocity. Momentum and kinetic energy will be conserved in the ~~collis~~ collision process. After that, the ion will again start moving freely, until catching another collision. After some such collisions, ~~th~~ a particular ion will have a velocity that doesn't have any correlation with its initial velocity. Statistically speaking if we have N number of ions initially moving in the same direction (with $N \to \infty$), after a few collision, there ~~velocity~~ velocities will be totally random w.r.t. on another, producing an average of zero.

Now, say there is some constant electric field in some direction $\vec{E}$. Consider that at $t=0$, ~~the~~ an ion is subjected to a collision and has a velocity immediately after the collision $\vec{v}^c$. The electric field will impart a force $q\vec{E}$ on the ion, which will cause a change in momentum of $q\vec{E}t$. So, the momentum will now be, before another collision $= M\vec{v}^c + q\vec{E}t$, where t is the time between subsequent collision. Now, the average momentum at any time of all the ions will be given by,

$$M\langle\vec{v}\rangle = \frac{\sum_i M\vec{v}_i^{\,c} + e\vec{E}t_i}{N}$$

with $\vec{v}_i^{\,c}$ = immediate velocity after last collision of ith ion

t_i = time passed after last collision.

$$\therefore \quad M\langle\vec{v}\rangle = \frac{1}{N}M\sum_i \vec{v}_i^{\,c} + e\vec{E}\,\frac{\sum_i t_i}{N}$$

Now, the first term on the right hand side will just be zero, if we consider the ~~veloc~~ velocities after the collision to be completely random.

$$\therefore \quad M\langle\vec{v}\rangle = e\vec{E}\langle t\rangle$$

where $\langle t\rangle$ is the average time between two collisions (!). So, for a positive ion, the average velocity is,

$$\langle \vec{V}_+ \rangle = \frac{e\vec{E}\langle t_+ \rangle}{M_+}$$

$$\therefore \vec{J} = Ne\langle \vec{v}_+ \rangle = \frac{Ne^2 \vec{E}\langle t_+ \rangle}{M_+}$$

For negative ion,

$$\vec{J} = N(-e) \cdot \frac{-e\vec{E}\langle t_- \rangle}{M_-} = Ne^2 \vec{E} \frac{\langle t_- \rangle}{M_-}$$

If we have both positive and negative ions, then,

$$\vec{J} = Ne^2 \left[\frac{\langle t_+ \rangle}{M_+} + \frac{\langle t_- \rangle}{M_-} \right] \vec{E}$$

For metals, we know only electrons move inside the metal, and so,

$$\vec{J} = \frac{Ne^2\langle t_- \rangle}{M_-} \vec{E} \Rightarrow \boxed{\vec{J} = \sigma \vec{E}} \quad \text{—(i)}$$

where we define the conductivity as,

$$\sigma = \frac{Ne^2 \langle t_- \rangle}{M_-}$$

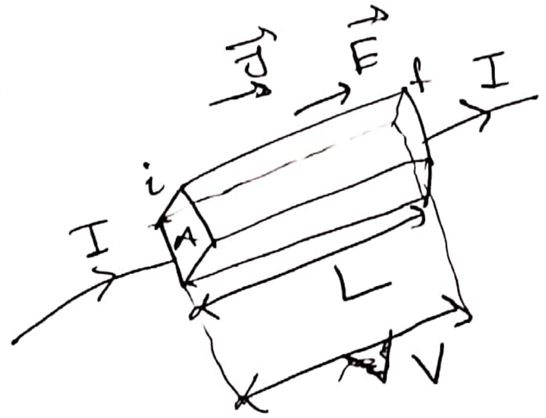
The average time between collisions was taken by Drude as the characteristic time τ, and so,

$$\boxed{\sigma = \frac{Ne^2 \tau}{M_e}}$$

Equation (i) is the microscopic Ohm's law.

Now, I is the surface integral of $\vec{J}$ over a cross section of the conductor, which implies that I is proportional to J. On the other hand the potential difference between two points $\phi_2 - \phi_1 = V$ is the line integral of the electric field $\vec{E}$ on a path through the conductor from one point to another point. This a hint that I should be proportional to V.

Consider the metal solid rod of cross sectional area of A and length L. A steady current flows from one end to the othere.



$$J = \frac{I}{A}$$

$$\phi_f - \phi_i = V = -\int_i^f \vec{E} \cdot d\vec{s} = -\int_i^f E\, dl = -E(f - i) = -E \times L$$

$$\therefore E = \frac{V}{L} \text{ in term of magnitude}$$

Now, $J = \sigma E$

$$\Rightarrow \frac{I}{A} = \sigma \frac{V}{L}$$

$$\boxed{\therefore I = \frac{\sigma A}{L} V}$$

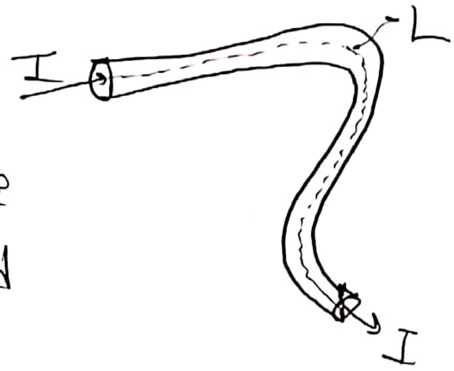
We define $R = \frac{L}{\sigma A} = \frac{\rho L}{A}$, called the resistance of the wire, with $\rho = \frac{1}{\sigma}$ to be the resistivity of the rod.

$$\boxed{\therefore I = \frac{1}{R} V}$$

This is macroscopic view of Ohm's law.

The equation for resistance also works perfectly for bent rods, as long as the rod is surrounded by insulators (also true for straight rod).

Since no current can leak, the bent rod acts the same as the straight rod. We can measure the length L along the wire and



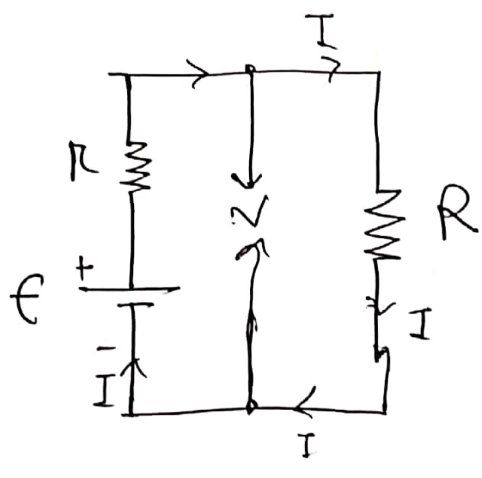
$R = \frac{\rho L}{A}$ will perfectly apply.

Electromotive force

A potential difference between two points will generate a current, but only for a while until the potentials of the points becomes the same.

For a continuous steady current, we need some source that will constantly maintain the potential difference. The source are found in the form of batteries, made out of cells, where the source of energy is chemical.

When a cell is fully charged, there is a potential difference between the two terminals, which we will call the electromotive force ϵ. If the cell terminals are connected with an external resistance R, then the potential difference between the cell terminals drops a little below ϵ, to V, ~~in~~ and a steady current $I = \frac{V}{R}$ flows. The difference in ϵ and V is produced by the resistance of the electrolyte ~~itself~~ of the cell, through which the current passes. If the internal resistance is r, then we can draw a circuit as the following.



The current I can be found by writing,

$$Ir + IR = \epsilon$$

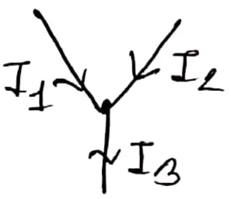
$$\boxed{\therefore I = \frac{\epsilon}{R+r}}$$

$\boxed{V = \epsilon - Ir}$, $V = IR$

Kirchoff's law

(1) At a node point of a circuit, the algebraic sum of the currents into the node must be zero,

This is basically the conservation of charge in circuit language.



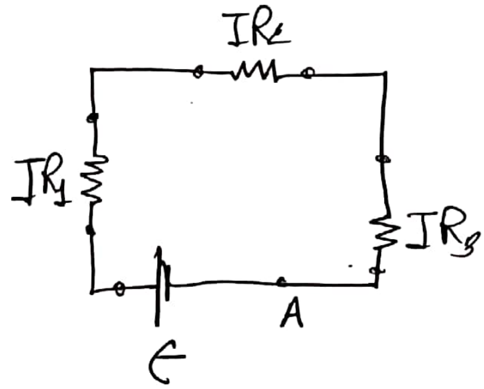
$\therefore\ I_1 + I_2 - I_3 = 0$

(ii) The sum of the potential differences taken in order around a loop of the circuit, a path starting and ending at the same node, is zero.

This is basically due to the conservative property of electrostatic field.

$$\therefore \oint \vec{E}.d\vec{s} = 0$$

$\therefore$ All the potential differences must add up to zero.

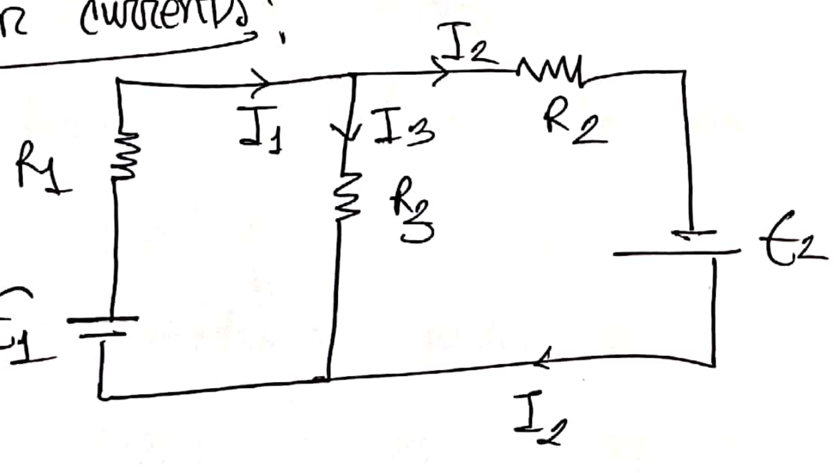


Starting from A, the line integral of $\vec{E}$ field will give,

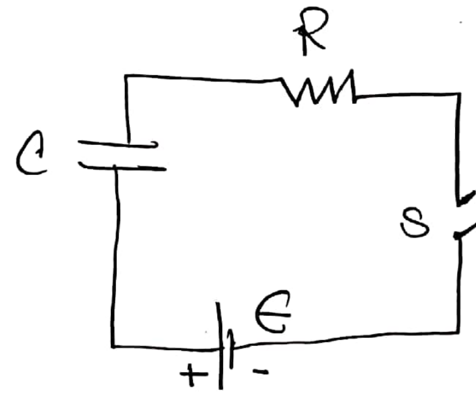
$$+\epsilon - IR_1 - IR_2 - IR_3 = 0$$

$$\therefore\ \epsilon = I(R_1 + R_2 + R_3)$$

Solve for currents:



RC circuit



Consider the circuit shown with a capacitor and a resistor. The DC voltage source has an emf $\mathcal{E}$. ~~At time $t=0$, the switch~~ The capacitor is initially uncharged since the switch S is open. At time $t = 0$, the switch is closed. Before closing the switch, the

the capacitor plates are at the same poten[tial] and hence acts like a short circuit. Immediately after $t=0$, the voltage across the resistance becomes the same as the emf and current starts to flow. The current is given by,

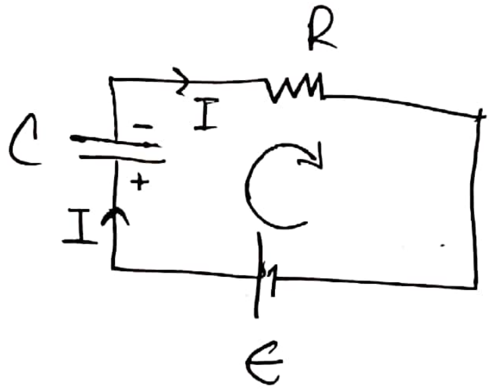
$$I_0 = \frac{\epsilon}{R}$$

As the current flows through the circuit, charges begin to accumulate on the capacitor plates. Due to the electric field, the electron moves in one plate and leave from another in the same amount to the ~~left~~ and hence, create the accumulation of charges on the capacitor. As charges build up, there builds up a potential difference across the capacitor plates, V_c.

$$V_c(t) = \frac{q(t)}{C}$$

As more and more charges accumulate, the potential difference increases more and more, and after a particular time, it equals the emf ϵ, and the current stops flowing. The reason can be also seen from electrostatics. As more and more charges accumulate on the plates, it ~~is~~ becomes harder and harder to deposit more charges due to the repulsion of already present charges on the plate.

Now, using KVL we get,



$$-\epsilon + V_c + IR = 0$$

$$\Rightarrow \frac{q(t)}{C} + \frac{dq(t)}{dt} R - \epsilon = 0$$

$$\Rightarrow \frac{dq(t)}{dt} R = \epsilon - \frac{q(t)}{C}$$

$$\Rightarrow \frac{dq(t)}{\epsilon - \frac{q(t)}{C}} = \frac{1}{R} dt \Rightarrow \int_{q=0}^{q=q} \frac{dq(t)}{\epsilon - \frac{q(t)}{C}} = \int_{t=0}^{t=t} \frac{1}{R} dt$$

$$\Rightarrow -C \ln\left|\epsilon - \frac{q(t)}{C}\right| \Big|_0^q = \frac{1}{R} t$$

$$\Rightarrow \ln\left|\epsilon - \frac{q(t)}{C}\right| - \ln \epsilon = -\frac{1}{RC} t$$

$$\Rightarrow \ln \frac{\epsilon - \frac{q(t)}{C}}{\epsilon} = -\frac{1}{RC} t$$

$$\Rightarrow \frac{\epsilon - q(t)/C}{\epsilon} = e^{-\frac{t}{RC}}$$

$$\Rightarrow \epsilon - \frac{q(t)}{C} = \epsilon e^{-t/RC}$$

$$\boxed{\therefore\ q(t) = C\epsilon\left[1 - e^{-t/RC}\right]}$$

$$\therefore\ q(t) = CE\left[1 - e^{-t/RC}\right]$$

The maximum charge stored in each plate is denoted by, $\underbrace{Q = CE}_{\text{at } t=\infty}$; as the maximum voltage across the capacitors is $V_{c,max} = E$.

$$\therefore\ q(t) = Q\left(1 - e^{-t/\tau}\right)$$

where, $\tau = RC$ is called the time constant of an RC circuit.

At $t = \tau$, $\quad q(t) = Q\left(1 - \frac{1}{e}\right) = 0.63\,Q$

So, after the first time constant, the capacitor charges to 63% of its maximum value.

Now, $\quad I(t) = \dfrac{dq(t)}{dt} = Q\left[0 + \frac{1}{RC}\, e^{-t/RC}\right]$

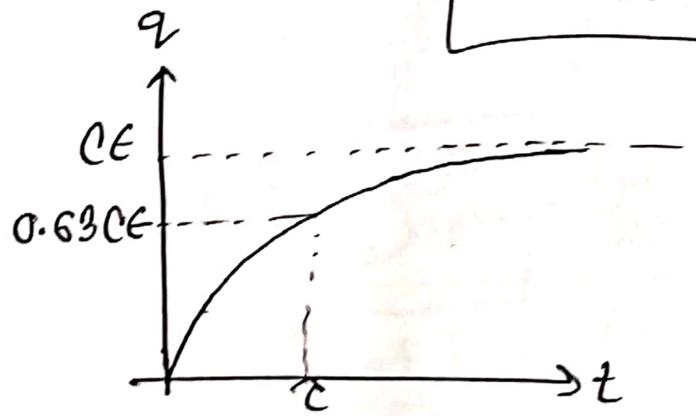
$$\therefore\ I(t) = \left(\frac{E}{R}\right) e^{-t/\tau}$$

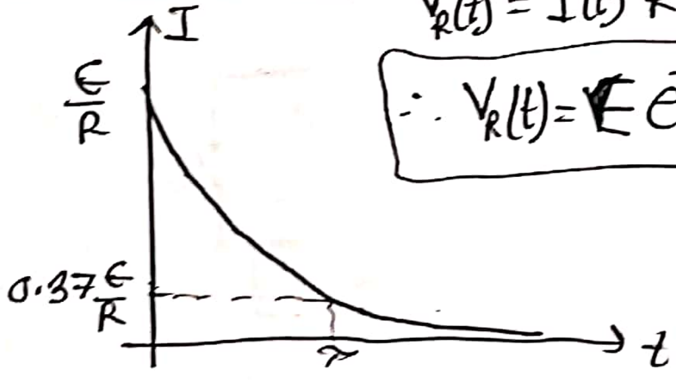
$$\boxed{\therefore\ I(t) = I_0\, e^{-t/\tau}}$$

$\boxed{t = \tau,\ I(t) = 0.37 I_0}$

$V_R(t) = I(t)\,R$

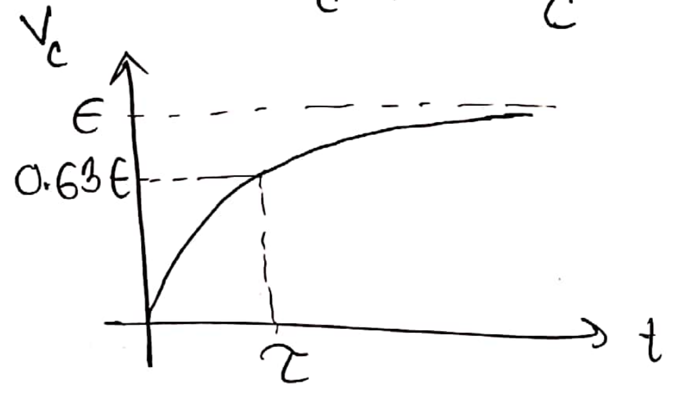
$\boxed{\therefore\ V_R(t) = E\, e^{-t/\tau}}$





Voltage across capacitor,

$$V_c(t) = \frac{q(t)}{C} = \epsilon\left(1-e^{-t/\tau}\right)$$



$$\therefore\ \boxed{q(t) = Q\left[1-e^{-t/\tau}\right]} \qquad \boxed{I(t) = I_0 e^{-t/\tau}} \qquad \boxed{V_c(t) = \epsilon\left[1-e^{-t/\tau}\right]}$$

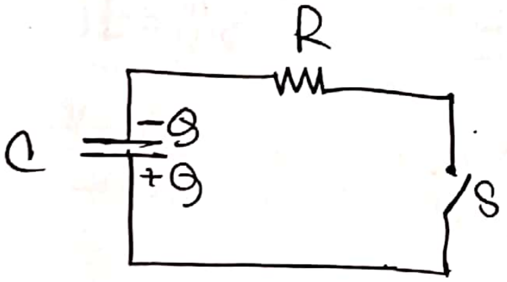
As $t \to \infty$, $q(t) \to Q$ with $Q = C\epsilon$

As $t \to \infty$, $I(t) \to I_0$ with $I_0 = \frac{\epsilon}{R}$

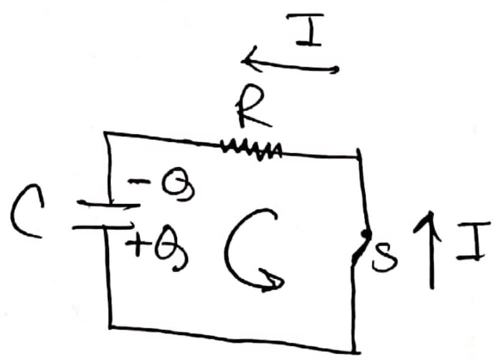
As $t \to \infty$, $V_c(t) \to \epsilon$ with $V_{c\,max} = \epsilon$

Discharging of a capacitor

Now, suppose initially the capacitor has been charged to some value Q. Initially the switch S is open and $V_c = \frac{Q}{C}$. However, the potential difference ~~acr~~ across the resistor is zero since there is no current flow, so $I=0$.



$t<0$



$t=0$

At time $t=0$, the switch is closed and the capacitor starts discharging which acts like a voltage source now to drive the current across the circuit. The current develops as the charges from the capacitor plates starts moving to other plate due to the voltage difference. Applying KVL again,

$$V_C - IR = 0$$

$$\Rightarrow \frac{q(t)}{C} - IR = 0 \quad \Rightarrow \quad \frac{q(t)}{C} = -\frac{dq(t)}{dt} R$$

$$\Rightarrow -\frac{dq(t)}{q(t)} = \frac{1}{RC} dt \quad \Rightarrow -\int_{q(t)=Q}^{q(t)=q(t)} \frac{dq(t)}{q(t)} = \int_0^t \frac{1}{RC} dt$$

$$\Rightarrow -\ln|q(t)|\Big|_Q^{q(t)} = \frac{1}{RC} t$$

$$\Rightarrow \cancel{\frac{\ln q(t)}{\ln}} \rightarrow \ln \frac{q(t)}{Q} = -\frac{1}{RC} t$$

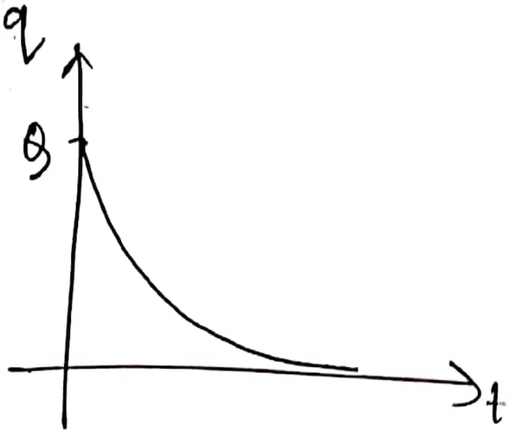
$$\Rightarrow q(t) = Q e^{-t/\tau}$$

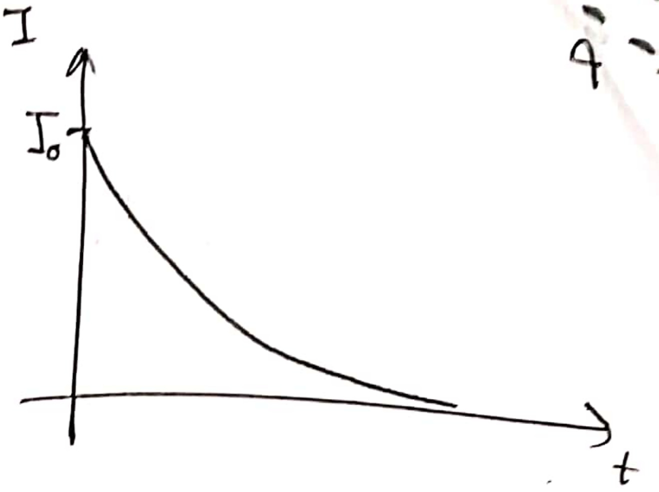
$$\boxed{\therefore q(t) = Q e^{-t/\tau}}$$

$$V_C(t) = \frac{q(t)}{C} = \frac{Q}{C} e^{-t/\tau} \qquad \boxed{\therefore V_C(t) = V_0 e^{-t/\tau}}$$

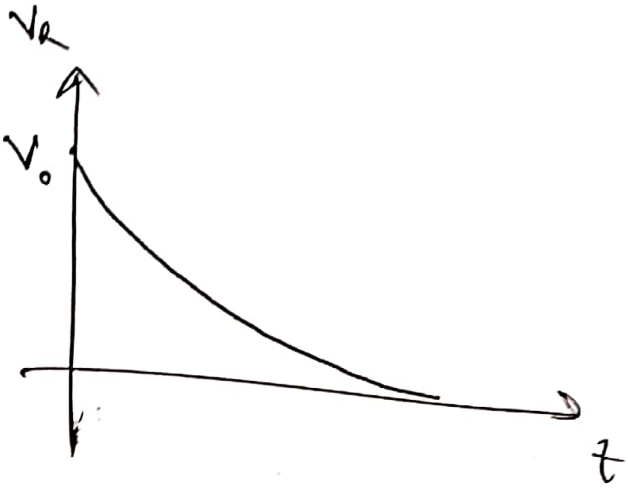
$$I(t) = -\frac{dq(t)}{dt} = \frac{Q}{RC} e^{-t/\tau} \qquad V_R(t) = IR$$

$$\boxed{\therefore I(t) = I_0 e^{-t/\tau}} \qquad \boxed{\therefore V_R(t) = V_0 e^{-t/\tau}}$$


q
Q
t


I
I0
t


Vc
V0
t


VR
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t