# Classical Mechanics

## Lecture 6

### Noether's Theorem and Conserved Quantities:

One of the most profound realizations in physics is that conserved quantities such as energy, momentum, angular momentum, electric charge etc. are result of symmetries of physical systems.

This is a theorem in classical mechanics and classical field theory and it is known as Noether's theorem named after Emmy Noether. Noether's theorem has far-reaching consequences in all branches of theoretical physics.

### Constants of motion:

A constant of motion is a quantity that remains unchanged along the physical path of the system in configuration space. Thus $Q(q_i(t))$ is a constant of motion if

$$\frac{dQ}{dt} = 0$$

for $q_i(t)$ a solution to the Euler-Lagrange equation.

Noether's Theorem says that for each continuous symmetry of the Lagrangian there exist a conserved quantity.

## Canonical Momenta:

So far we have worked with generalized coordinates and generalized velocities. An important quantity that we can define in lieu of generalized velocity is the momentum canonically conjugate to coordinates. So for each generalized coordinate $q_i(t)$ the momentum canonically conjugate to it is defined as:

$$\boxed{p_i(t) = \frac{\partial L}{\partial \dot{q}_i(t)}}$$

In the Hamiltonian formulation of classical mechanics the conjugate momenta will play an important role.

## Noether's Theorem:

For each continuous symmetry of the Lagrangian, there exists a conserved quantity.

## Proof:

Let $L(q_i, \dot{q}_i, t)$ be the Lagrangian of a system. Suppose we make an infinitesimal change

$$q_i(t) \rightarrow q_i(t) + \delta q_i(t)$$

which results in the following infinitesimal change of the Lagrangian:

$$L \to L + \delta L \qquad \text{(0)}$$

We can express $\delta L$ as:

$$\delta L = \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \qquad \text{(1)}$$

Next consider the equations of motion which we can write as:

$$\frac{\delta S}{\delta q_i(t)} = 0 \qquad \text{(2)}$$

Since the action $S = \int L \, dt$ we can write the LHS as:

$$\frac{\delta S}{\delta q_i(t)} = \int \left\{ \frac{\partial \mathcal{L}}{\partial q_i(t')} \delta(t-t') + \frac{\partial \mathcal{L}}{\partial \dot{q}_i(t')} \frac{\partial}{\partial t'} \delta(t-t') \right\} dt'$$

We use the Dirac delta identity

$$\int f(x) \frac{d}{dx} \delta(x-x') \, dx = -\int \frac{df(x)}{dt} \cdot \delta(x-x') \, dx$$

and write the above as:

$$\frac{\delta S}{\delta q_i(t)} = \int \left\{ \frac{\partial L}{\partial q_i(t')} - \frac{d}{dt'}\left( \frac{\partial L}{\partial \dot{q}_i(t')} \right) \right\} \delta(t-t')\, dt$$

$$= \frac{\partial L}{\partial q_i(t)} - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i(t)} \right)$$

We can use this to rewrite equation ① as:

$$\delta L = \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\delta S}{\delta q_i} \cdot \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$\delta L = \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) + \frac{\delta S}{\delta q_i} \delta q_i$$

Now, we identify as the Noether charge

$$Q \equiv \frac{\partial L}{\partial \dot{q}_i} \delta q_i$$

Then we have:

$$\frac{dQ}{dt} = \delta L - \frac{\delta S}{\delta q_i}\delta q_i$$

Thus if the equations of motion are satisfied

$$\frac{\delta S}{\delta q_i} = 0$$

and the Lagrangian is invariant $\delta L = 0$ under $q_i \to q_i + \delta q_i$, then

$$\boxed{Q = \frac{\partial L}{\partial \dot{q}_i}\delta q_i}$$

is conserved.

Under certain circumstances the Lagrangian is not invariant but is a total derivative under the transformation (0):

$$L \longrightarrow L + \frac{dF}{dt} \quad \Rightarrow$$

in that case the conserved current is

$$\boxed{Q = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - F.}$$

## Comments:

1. Noether's Theorem is a constructive theorem. It gives us an explicit form of the conserved charge.

2. $Q$ is a constant of motion and the Theorem states that $Q$ is conserved for the solutions of EL equations.

## Some Applications:

1. Cyclic or ignorable coordinates:

Suppose one (or more) of the more generalized coordinates are missing from $L$. Let us denote this specific coordinate as $q_\alpha$. Then

$$\frac{\partial L}{\partial q_\alpha} = 0$$

i.e. $q_\alpha(t) \to q_\alpha(t) + \varepsilon$ is a symmetry of $L$ when $\varepsilon$ is a constant.

Such coordinates are called cyclic or ignorable coordinates. We then find that momenta canonically conjugate to the cyclic coordinates $q_\alpha$:

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$$

are constants of motion.

$$\mathcal{Q} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \overbrace{(\varepsilon \delta_{i\alpha})}^{\delta q_i}$$

$$= \varepsilon \frac{\partial L}{\partial \dot{q}_\alpha} = \varepsilon p_\alpha$$

$$\Rightarrow \quad \frac{d\mathcal{Q}}{dt} = 0 \Leftrightarrow \frac{dp_\alpha}{dt} = 0$$

**2. Time translation invariance:**

If the Lagrangian has no explicit time dependence, i.e.,

$$\frac{\partial L}{\partial t} = 0$$

then under a translation in the time direction

$$t \to t + \varepsilon$$

The generalized coordinates change as:

$$q_i(t) \longrightarrow q_i(t+\varepsilon)$$

$$= q_i(t) + \varepsilon\, \dot{q}_i(t)$$

and the Lagrangian $L(t)$ also change as:

$$L(t) \longrightarrow L(t+\varepsilon) \simeq L(t) + \varepsilon \frac{dL}{dt}$$

And so
$$\begin{cases} \delta q_i = \varepsilon\, \dot{q}_i \\ \delta L = \varepsilon \dfrac{dL}{dt} \end{cases}$$

Then the conserved charge is

$$Q = \sum_i p_i \dot{q}_i - L \equiv H$$

It is called the Hamiltonian and it has the dimension of energy. It is natural to interpret it as total energy.

For example, if we consider N particles interacting via a potential $V(\vec{x}_1, \vec{x}_2, \dots \vec{x}_N)$ then the Lagrangian is:

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x}_1, \dots, \vec{x}_N)$$

$$= \sum_i \frac{1}{2} m \dot{q}_i^2 - V(q_1, \dots, q_{3N})$$

Then $\mathcal{H} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - \frac{1}{2} \sum_i m \dot{q}_i^2 + V(q_1 \dots q_{3N})$

$$= \sum_i m \left( \dot{q}_i^2 - \frac{1}{2} \dot{q}_i^2 \right) + V(q_1 \dots q_{3N})$$

$$= \sum_i \frac{1}{2} m \dot{q}_i^2 + V(q_1 \dots q_{3N})$$

$$= \sum_i \frac{p_i^2}{2m} + V(q_1 \dots q_{3N})$$

$$= \quad KE + PE.$$

## Homogeneity of space

For this example and the next we shall deal with a particular form of interaction potential between the particles. Let $\bar{r}_i$ and $\bar{r}_j$ be the position vectors of two particles that we denote as the i-th and j-th particle. Then we assume that their interaction potential is of the form:

$$V_{ij} \equiv V(|\vec{r}_i - \vec{r}_j|)$$

Such a potential is obviously translation invariant: $\bar{r}_i \longrightarrow \bar{r}_i + \vec{a}$

and $\bar{r}_j \longrightarrow \bar{r}_j + \vec{a}$

for $\vec{a}$ a constant vector (i.e., time-independent).

The interaction force $\vec{F}_{ij}$ (or $\vec{F}_{ji}$) that arises from such an interaction potential is along the line joining the two particles.

Such potentials (forces) are called central potentials (forces). Newton's law of gravitation:

$$V_N = G \frac{M_1 M_2}{|\bar{r}_1 - \bar{r}_2|}$$

or Coulomb's interaction:

$$V_c = k \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

are both examples of central forces. On the other hand, forces due to the magnetic fields generated by moving charges are <u>not</u> central.

Now we write down the Lagrangian of N point particles interacting with each other via central potentials:

$$L = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2} - V$$

where $V = \sum_{i>j} V(|\vec{r}_i - \vec{r}_j|)$

If we translate the whole system by a constant displacement vector $\vec{a}$:

$$\vec{r}_i \longrightarrow \vec{r}_i + \vec{a}$$

we see that L remains invariant.

The dispacement vector can be witten as

$$\vec{a} = a\,\hat{a}$$

where $\hat{a}$ is the unit vector in the $\vec{a}$ direction and $a = \|\vec{a}\|$.

For an infintesimal dispacement we have

$$\vec{a} = \underbrace{\varepsilon}_{\text{small}}\,\hat{a}$$

Then $\delta \vec{r}_i = \varepsilon\,\hat{a}$ and so the conserved quantity is

$$\mathcal{Q}_{\hat{a}} = \sum_i \frac{\partial L}{\partial \dot{\vec{x}}_i} \cdot \hat{a}$$

$$= \sum_i \vec{p}_i \cdot \hat{a}$$

where $\vec{p}_i$ is the momentum of the ith particle. Now we can choose $\hat{a}$ to be $\hat{x}$, $\hat{y}$, and $\hat{y}$ and these will tell us that the x-, y-, and z-component of the total momentum

$$\vec{P} = \sum_i \vec{p}_i$$

is conserved.

Thus we see that the translation symmetry of the Lagrangian leads to the conservation of total momentum.

We know that the homogeneity of space is a symmetry of Euclidean space. The Lagrangian of N interacting particles with central potential respects that symmetry.

There are simple systems which break this symmetry. For example a single particle in a non-trivial potential has the Lagrangian:

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x})$$

is not invariant $\vec{x} \to \vec{x} + \vec{a}$ if $V(\vec{x})$ is not a constant potential. In this case the momentum $\vec{p}$ is not conserved.

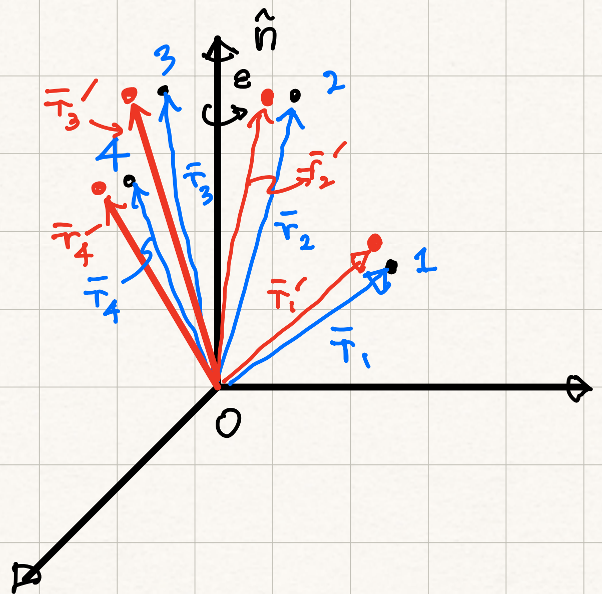
Q: This of a physical situation when this latter example is true.

# Isotropy of Space:

3D Euclidean space is isotropic. meaning that all direction are equivalent. Consider a system of N particles whose Lagrangian is given by:

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - V$$

where $V = \sum_{i>j} V(|\bar{r}_i - \bar{r}_j|)$




Now if we rotate the whole system by a small angle $\varepsilon$ around the $\hat{n}$ axis then it should be clear that each position vector $\vec{r}_i$ will change

<u>only</u> in their components which are orthogonal to $\hat{n}$. And so the new rotated vectors will be given by

$$\vec{r}_i \to \vec{r}_i^{\,\prime} = R_{\hat{n}}(\varepsilon)\,\vec{r}_i$$

$$\simeq \vec{r}_i + \varepsilon\,\hat{n} \times \vec{r}_i$$

This shows that $\delta \vec{r}_i = \varepsilon\,\hat{n} \times \vec{r}_i$

It should also be clear that under rotation:

a) Since $\vec{r} \cdot \vec{r}$ remains invariant so does the kinetic energy term in $L$.

b) Each $|\vec{r}_i - \vec{r}_j|$ remain invariant and so the potential energy term also remain invariant.

Thus $L$ remains invariant under rotation around <u>any</u> $\hat{n}$. Thus this system respects the isotropy of $\mathbb{R}^3$. The conserved quantity is then a vector quantity (why) given by:

$$\vec{Q} = \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \times \vec{r}_i$$

$$= \sum_i \vec{p}_i \times \vec{r}_i$$

$$= \sum_i \vec{l}_i$$

$$= \vec{L} \text{ (total angular momentum)}$$

## Comments

1. We see that corresponding to translation invariance in time and space we have energy (the Hamiltonian) and linear momentum conservation. We shall see when we study special relativity space and time will become one unified object and the corresponding conserved quantity will be a four-dimensional vector $p^\mu = (\frac{E}{c}, p_x, p_y, p_z)$ known as the energy-momentum 4-vector.

2. The momentum that is canonically conjugate to a cyclic coordinate is known as an integral of motion. The reason for this name should be clear from the following. If $q_\alpha$ is the cyclic coordinate then it follows from the Euler Lagrange equation:

$$\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

$$\Rightarrow \quad 0 - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0$$

$$\Rightarrow \quad \frac{dp_\alpha}{dt} = 0$$

$$\Rightarrow \quad \int \frac{dp_\alpha}{dt}\, dt = 0$$

$$\Rightarrow \quad p_\alpha = c$$