Chapter 4: Complex Numbers and Functions

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Reference Texts: Mary L. Boas; Arfken, Weber and Harris.

1 Introduction

Complex number are defined in terms of *imaginary unit*, i, with the property

$$i^2 = -1 \ . \tag{1}$$

As such, a general *complex* number can be written in the form

$$z = x + iy \ , \tag{2}$$

where x, y are real numbers. Another common form of expressing a complex number is in coordinate form, as an *ordered* pair of two variables,

$$z \equiv (x, y) \ , \tag{3}$$

where the ordering is crucial – the first entry is the 'real part' of z while the second entry is the 'imaginary part' and thus, generally,

$$z = (x, y) \neq (y, x) \ .$$

Another form commonly used to express complex numbers is

$$z = \text{Re } z + i \text{ Im } z \ . \tag{4}$$

Complex numbers follow the usual rules of addition and multiplication as real numbers: If,

$$\begin{aligned}
&z_1 = x_1 + iy_1 \ , \\
&z_2 = x_2 + iy_2 \ , \\
&\text{then,} \\
&z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \ , \\
&\text{while,} \\
&z_1 z_2 = x_1 x_2 + iy_1 x_2 + ix_1 y_2 + i^2 y_1 y_2 \\
&\qquad = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \ .
\end{aligned} \tag{5}$$

We can define the **complex conjugate** of z, denoted by z^* (or $\bar{z}$), by reversing the sign of i in Eq.(2) such that,

$$z^* = x - iy \ . \tag{6}$$

Thus, some quick algebra leads to

$$\begin{aligned}
\text{Re } z &= \frac{z + z^*}{2} \\
\text{Im } z &= \frac{z - z^*}{2i}
\end{aligned} \tag{7}$$

and further note that,

$$zz^* = x^2 + y^2 \tag{8}$$

is purely real and non-negative, and thus we define the **modulus**, or *magnitude*, or *absolute value* of z as,

$$|z| = \sqrt{zz^*} = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2} \ , \tag{9}$$

where the positive square root is implied.

We can proceed to give a simple geometric interpretation of complex numbers, by thinking of them as two-dimensional vectors, as shown in Figure 1.

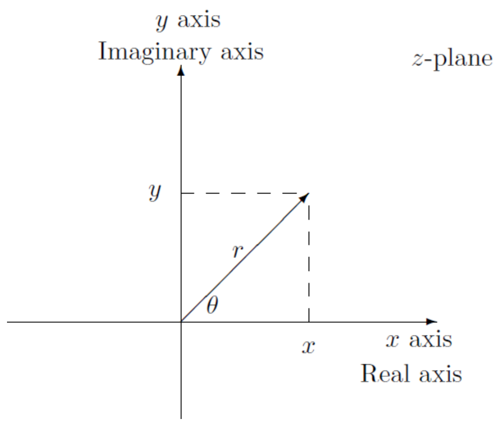

Figure 1: Geometrical Interpretation of $z = x + iy$. Sometimes called an **Argand Diagram**. Note that the reflection of the vector **r**, in the $x-$axis, would be the complex conjugate.

Here, the length of the vector is the magnitude of the complex number,

$$r = |z| \ , \tag{10}$$

and the angle the vector makes with the real axis is θ,

$$\tan\theta = \frac{y}{x}, \tag{11}$$

and where the quadrant that θ lies in is determined by the signs of x and y and θ is called the **argument** or *phase* of z denoted as

$$\theta = \arg\ z \ . \tag{12}$$

There is an arbitrariness in the choice of θ since one can always add an arbitrary multiple of 2π to θ without changing z,

$$\theta \to \theta + 2\pi n, \ \text{ where } n \text{ is an integer, then } z \to z \ . \tag{13}$$

Thus, it is often convenient to define a single-valued argument function of z. By convention, the **principal value** of arg z is the phase angle which satisfies the inequality

$$-\pi < \arg\ z \le \pi \ , \tag{14}$$

which is always measured in *radians*.

2 De Moivre's Theorem

A complex number expressed as in Eq.(2) is known as the *rectangular form* since $x,\ y$ are rectangular coordinates of the point representing the complex number z. We can extend this idea to the *polar representation* of a complex number from the Argand diagram in Figure 1 as,

$$z = r\cos\theta + i\ r\sin\theta = r(\cos\theta + i\sin\theta)\ , \tag{15}$$

such that if we have two complex numbers as defined in Eq.(5) but expressed in polar form,

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)\ ,$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)\ ,$$

then their product becomes,

$$\begin{aligned} z_1 z_2 &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i[\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1]\} \\ &= r_1 r_2[\cos(\theta_1+\theta_2) + i(\sin\theta_1+\theta_2)]\ , \end{aligned} \tag{16}$$

from which we notice that the moduli of complex numbers multiply,

$$|z_1 z_2| = |z_1||z_2|\ , \tag{17}$$

while their arguments add,

$$\arg(z_1 z_2) = \arg\ z_1 + \arg\ z_2\ . \tag{18}$$

Now, if we choose a unit vector (recall, this is a vector with unit magnitude, i.e. $|r| = 1$) then Eq.(15) reduces to,

$$z = \cos\theta + i\sin\theta\ ,$$

and successive powers follow a simple pattern:

$$\begin{aligned} z^2 &= \cos 2\theta + i\sin 2\theta\ , \\ z^3 &= \cos 3\theta + i\sin 3\theta\ , \\ &\ldots \\ z^n &= \cos n\theta + i\sin n\theta\ , \end{aligned} \tag{19}$$

where n is a positive integer. This is known as *De Moivre's Theorem.*

2.1 Application of De Moivre's Theorem - Calculating Roots

Suppose we are asked to find all the nth roots of unity, i.e., all the solutions to the equation,

$$z^n = 1\ , \tag{20}$$

where n is a positive integer. If we take the polar form, we have

$$z = \rho(\cos\phi + i\sin\phi)\ ,$$

which by De Moivre's theorem gives,

$$\rho^n(\cos n\phi + i\sin n\phi) = 1\ ,$$

which implies that

$$\rho = 1, \text{ and } n\phi = 2\pi k \ ,$$

where k is any integer. Thus, the nth root of unity has the form

$$z = \cos\left(\frac{2\pi k}{n}\right) + i\left(\sin\frac{2\pi k}{n}\right) \ . \tag{21}$$

These are distinct for $k = 0, 1, 2, ..., n-1$; apart from these values of k, the roots repeat. Thus, there are n distinct nth roots of unity. For instance, if $n = 8$, the roots can be plotted in the complex plane, as shown in Figure 2.

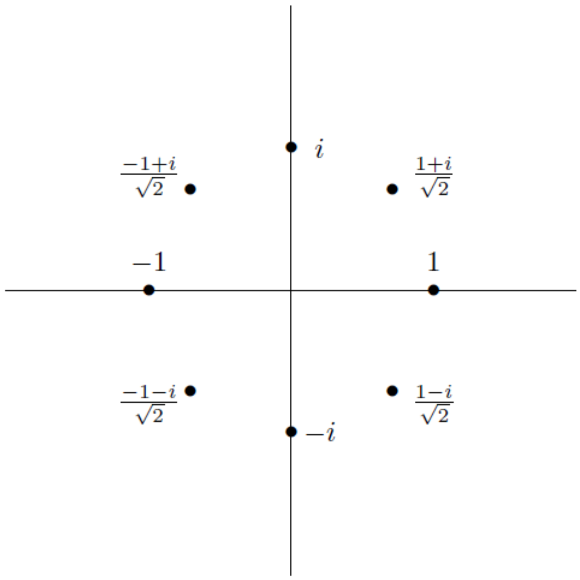

Figure 2: 8th roots of unity.

3 Functions in the Complex Domain

We have seen that the fundamental mathematical operations for complex numbers obey the same rules as those for real numbers. We can extend this to defining functions of complex variables; one implication of this is the notion that if a function (of real numbers) is represented by a power series, then within the interval of convergence of the power series, one can use such series with complex values of the expansion. This notion is called the **permanence of the algebraic form**.

3.1 The Exponential Function

We can define the exponential function for z as,

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + ... \ . \tag{22}$$

Replacing $z \to iz$, Eq.(22) assumes the form,

$$\begin{aligned} e^{iz} &= 1 + iz + \frac{1}{2!}(iz)^2 + \frac{1}{3!}(iz)^3 + \frac{1}{4!}(iz)^4 + ... \\ &= \left[1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ...\right] + i\left[z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ...\right] \ , \end{aligned} \tag{23}$$

where the rearrangement and grouping of the terms in Eq.(23) is allowed since the power series expansion of the exponential function is absolutely convergent for all z (left to the reader as practice to **verify** this statement). Notice that,

$$1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - ... \equiv \cos z$$

and

$$z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - ... \equiv \sin z$$

and thus we can express Eq.(23) in the form,

$$\boxed{e^{iz} = \cos z + i \sin z} \tag{24}$$

which is known as the *Euler formula* for expressing complex exponentials in terms of trigonometric functions. Defining the complex conjugate for Eq.(24)

$$e^{-iz} = \cos z - i \sin z \ ,$$

one can, via addition and subtraction of e^{iz} and e^{-iz}, yield simplified expressions for

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \ . \end{aligned} \tag{25}$$

Similar extensions to this also hold for hyperbolic functions,

$$\begin{aligned} \cosh z &= \frac{e^{z} + e^{-z}}{2} \\ \sinh z &= \frac{e^{z} - e^{-z}}{2} \ , \end{aligned} \tag{26}$$

and comparing the trigonometric functions with their corresponding hyperbolic functions lead to the conclusion that

$$\begin{aligned} \cosh iz &= \cos z \ , \\ \sinh iz &= i \sin z \ . \end{aligned} \tag{27}$$

Exercise 1. Prove equations (25) and (27).

Using De Moivre's theorem and Euler's formula, we can express a complex number z as,

$$z = re^{i\phi}, \qquad \text{and} \qquad z^n = r^n e^{in\phi} \ , \tag{28}$$

which is unique when $n \in \mathbb{Z}$ while for fractional powers (or roots) we have,

$$z = re^{i\phi}, \qquad \text{and} \qquad z^{1/n} = r^{1/n} e^{i\phi/n} \ , \tag{29}$$

but the result is not unique since the roots are always unique only up to a constant multiple of 2π. Figure 3 illustrates the multiple values of $z^{1/n} \equiv 1^{1/3},\ i^{1/3},\ (-1)^{1/3}$.

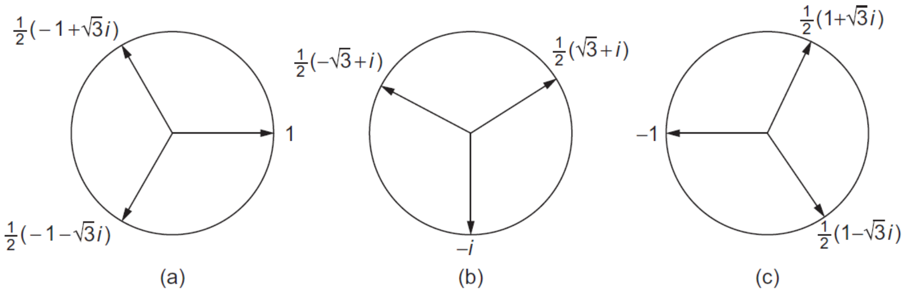

Figure 3: Cube roots of: (a) $1^{1/3}$, (b) $i^{1/3}$, (c) $(-1)^{1/3}$ are all multi-valued.

3.2 Logarithm

In polar representation, we can write out the multivalued complex logarithm function as,

$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta \ , \tag{30}$$

while one must remember that,

$$\ln z = \ln(re^{i(\theta+2n\pi)}) = \ln r + i(\theta + 2n\pi) \ , \tag{31}$$

holds true for **any** positive or negative integer n. Thus, $\ln z$ has an infinite number of values corresponding to all possible choices of n, for a given z.

4 Complex Power Series and Circle of Convergence

Following the discussions on the complex valued exponential and logarithm functions, we can generalize the notion of power series expanded in powers of z as,

$$\sum c_n z^n \ , \tag{32}$$

where $z = x + iy$ and the c_n are complex-valued coefficients. Some examples include,

$$\begin{aligned} &1 - z + \frac{z^2}{2} - \frac{z^3}{3} + \frac{z^4}{4} + \ldots \ , \\ &1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \ldots \end{aligned} \tag{33}$$

Let us investigate the convergence of

$$\sum_0^\infty c_n z^n$$

using the root test. The test states that if

$$\lim_{n\to\infty} |z| \sqrt[n]{|c_n|} < 1 \ ,$$

the series converges while if,

$$\lim_{n\to\infty} |z| \sqrt[n]{|c_n|} > 1 \ ,$$

the series diverges. Therefore, the power series converges within a **circle of convergence** of radius ρ (*the radius of convergence*), where

$$\rho = |z| = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}} \ ,$$

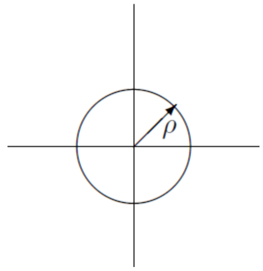

Figure 4: Circle of convergence of power series with radius of convergence denoted with ρ.

and diverges outside that circle, as shown in Figure 4. However, more detailed examination is required to test whether the series converges *on* the circle of convergence.

Example 1. Test the convergence of the series

$$\sum_{n=0}^{\infty} \frac{(z+1-i)^n}{3^n n^2} \,.$$

By the root test, we have

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{(z+1-i)^n}{3^n n^2}} = \lim_{n\to\infty} \frac{|z+1-i|}{|3n^{2/n}|} = \frac{|z+1-i|}{3}$$

and the given series converges when,

$$\frac{|z+1-i|}{3} < 1 \Rightarrow |z-(-1+i)| < 3$$

which is just the circle $|z| = 3$ with centre shifted as $(0,0) \to (-1,1)$ as shown in Figure 5. Thus, the series is convergent inside the red circle and divergent outside.

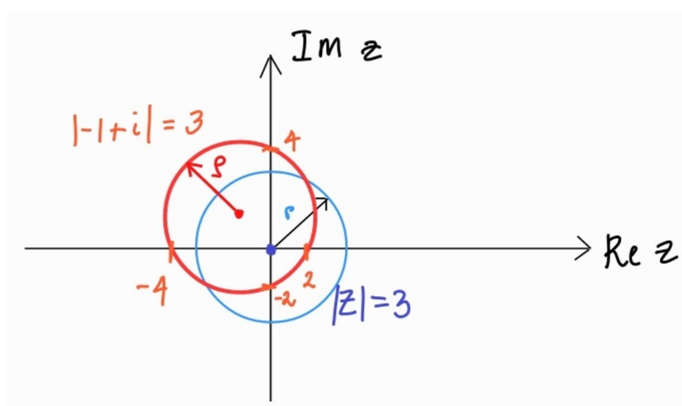

Figure 5: Circle of convergence of $\sum_{n=0}^{\infty} \frac{(z+1-i)^n}{3^n n^2}$.

Recall: the equation of a circle is: $(x-a)^2 + (y-b)^2 = r^2$ where (a,b) are the coordinates of the centre of the circle and r is the radius. In the complex domain this can be written as

$$|z - z_0| = \rho$$

where $z_0 \equiv x_0 + iy_0$ is the centre of the circle and ρ is the radius, for any arbitrary complex number z.