Group Theory
Lecture #2

Defn: Subgroup

Let G be a group with a group multiplication $\circ : G \times G \to G$. Then a subset $H \subset G$ that also forms a group with the same multiplication is called a subgroup.

Comments:

1. The identity element $e \in G$, necessarily also belongs to H.

2. The whole group G and just the identity element $\{e\}$ also are also subgroups but they are improper subgroups. Any subgroup $H \subset G$ that are different from these two are called proper subgroups

Examples:

1. $(\mathbb{R}, +)$ is a group. $(\mathbb{Z}, +)$ is a subgroup.

2. Rotations in a given plane is a subgroup of $SO(3)$. This subgroup is called $SO(2)$. There are infinitely different $SO(2)$ subgroups of $SO(3)$. $SO(3)$ is non-Abelian but $SO(2)$ is Abelian.

Def^{n}: Order of a finite group

If a group is finite then the number of elements of the group is called the order of the group.

Ex: The order of $\mathbb{Z}_2$ is 2.

Def^{n}: Coset

Suppose G is a group and H is a subgroup thereof. Then for each $g \in G$ the left coset of H in G is the set:

$$gH := \{gh \mid h \in H\}.$$

Comments:

1. The left coset of any element $h \in H$ is H itself. Note that $e \in H$.

2. One can similarly define the right coset $Hg = \{hg \mid h \in H\}$.

3. A left coset or a right coset is not necessarily a group.

4. Suppose that $g_1 \neq g_2$ but $g_1 \in g_2H$. Then $g_2 \in g_1H$ and

   $g_1H = g_2H$ as sets. Given $G = \{e, g_1, g_2, \dots, g_n\}$ and

$H = \{e, h_1, \ldots, h_m\}$, a subgroup, the cosets:

$$H,\ g_1H,\ g_2H, \ldots, g_{K-1}H$$ [where $g_1, g_2, \ldots, g_{K-1}$ are all distinct.]

divide the set G into K disjoint subsets so that

$$G = H \cup g_1H \cup g_2H \cup \cdots \cup g_{K-1}H.$$

4. Cosets divide the group G into equivalence classes. Two elements g_1 & g_2 are equivalent $g_1 \sim g_2$ if $g_1 = g_2 h$ for some $h \in H$.

Exercise: Let G be a finite group of order n and H is a sub roup of order m. Then show that $m|n$, wher m/n is a positive integer.

Cosets and Equivalence Relationship

Belonging to a coset is an equivalence relationship. Here we prove this fact. For a relationship to be an equivalence relationship it has to satisfy three conditions:

1. Reflexivity: $g \sim g$
2. Symmetry: If $g_1 \sim g_2$ then $g_2 \sim g_1$.
3. Transitivity: If $g_1 \sim g_2$ and $g_2 \sim g_3 \Rightarrow g_1 \sim g_3$.

Comments:

1. It is a general property of an equivalence relationship ($\sim$) that it divides a set H into disjoint non-empty subsets whose union give back the set itself. This fact is important enough to warrent that you attempt to prove this. Each subset is called an equivalence class. The set of these subsets is denoted by $G/\sim$.

Coset Space and Normal Subgroups:

The collection of cosets $H, g_1H, g_2H, \ldots, g_kH$ form a set $G/H = G/\sim$ called the coset space. In general, the coset space cannot be given a group structure. E.g., let $g_1 \sim g_1'$ and $g_2 \sim g_2'$. Then it is not true that $g_1 g_2$ is coset equivalent to $g_1' g_2'$.

Normal Subgroup or Invariant Subgroup:

A subgroup $H \subset G$ is called a normal subgroup or an invariant subgroup if $\forall\ g \in G$ one has $g h_1 = h_2 g$. Or equivalently $h_1 = g^{-1} h_2 g$.

This implies that the left coset gH and the right coset Hg are the same.

Comments:

1. We write $gHg^{-1} = H$ $\forall g \in G$ for H normal.

2. When H is normal the coset space can be given a group structure. If $g_1 \sim g_1' \Rightarrow g_1 = g_1' h_1$ and $g_2 \sim g_2' \Rightarrow g_2 = g_2' h_2$. Now consider $g_1 g_2 = g_1' h_1 g_2' h_2$. But since H is normal, $h_1 g_2' = g_2' h_1'$ and so we can write

$$g_1 g_2 = g_1' g_2' h_1' h_2 = g_1' g_2' (h_1' h_2) = g_1' g_2' h_3 .$$

Thus we see that $g_1 g_2 \sim g_1' g_2'$ (closure).
The identity element $e \sim h, \forall h \in H$.

3. The coset space when its a group is called the quotient group. G is called an extension of G/H by H.

4. If G is an Abelian group then all subgroups are automatically normal: $gh = hg \Rightarrow h = g^{-1} h g \; \forall g$. Then G/H is also an Abelian group: $\mathbb{R}^1/\mathbb{Z} \cong S^1$.

Exercise:

If $g_1 H$ and $g_2 H$ are two cosets and H is a normal subgroup of G then one can introduce the binary operation between two cosets by:

$$(g_1 H) \circ (g_2 H) := (g_1 g_2) H .$$

Show that this operation provides the coset space G/H with structure of a group.

An Example from Continuous Groups:

SU(2): The set of 2x2 complex valued unitary matrices form a group called $U(2)$. The elements $M \in U(2)$ that further satisfy $\det M = 1$ form a subgroup known as $SU(2)$.

$$SU(2) = \{ M \in GL(2,\mathbb{C}) \mid M^\dagger = M^{-1} \text{ and } \det M = 1 \}.$$

$$\text{Let } \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$M^\dagger = M^{-1} \Rightarrow \quad \begin{array}{l} a^* = d \\ -b = c^* \end{array}$$

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \Rightarrow \det M = 1 \Rightarrow |a|^2 + |b|^2 = 1$$

$$\left. \begin{array}{l} a = x + iy \; , \; x, y \in \mathbb{R} \\ b = w + iv \; , \; w, v \in \mathbb{R} \end{array} \right\} \to x^2 + y^2 + w^2 + v^2 = 1$$

Thus we see that the parameter space of $SU(2)$ is S^3. We can write $SU(2) \cong S^3$.

Representations of a Group:

For a group we must distinguish it from its realizations in terms of matrices. A group is an abstract object but its realizations in terms of matrices is called a representation.

So a representation D of a group G is a map from G into the set of $n \times n$ dimensional matrices D s.t. the group structure is preserved:

$$\boxed{\begin{gathered} \text{if} \qquad g_1 g_2 = g_3 \\ D(g_1) D(g_2) = D(g_1 g_2) := D(g_3). \end{gathered}}$$

where the product between the matrices $D(g)$ are matrix multiplications.

For every representation there is a natural vector space the matrices $D(g)$ act upon. Often this vector space is also refered to as the representation of the group. It should be clear from the context which object we are referring to.

Example: From its definition we have seen that the $SU(2)$ group consists of 2×2 unitary matrices $U^\dagger = U^{-1}$ with unit determinant $\det U = 1$. This is also a representation and it is called the fundamental or the defining representation.

The vector space that the fundamental representation of $SU(2)$ acts on is $\mathbb{C}^2$. If the context is clear this representation is known as the $\underline{2}$ of $SU(2)$.

As we shall soon see there are many (an infinite number) of higher dimensional matrix representations of $SU(2)$. These are extremely important for applications of quantum mechanics.