Chapter 7: Special Functions

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Reference Texts: *Mathematical Methods in the Physical Sciences* by Mary Boas

1 Introduction

While we must resort to being satisfied with approximate solutions to some *unsolvable problems*, physically it is often desirable to generate exact results. The route to the solution often require solving improper integrals and special functions provide us with a tool to extract exact values of complicated integrals. That is enough reason to study this topic in as much detail as the limited time in our hands will permit.

2 Factorial Function

Consider the improper integral

$$\int_0^\infty e^{-\alpha x}dx = \int_0^\infty -\frac{1}{\alpha}e^{-\alpha x}\bigg|_0^\infty = \frac{1}{\alpha}\ . \tag{1}$$

Convince yourself that repeatedly differentiating (1) with respect to α on both sides lead to another improper integral of the form,

$$\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}} \tag{2}$$

which, in the special case when $\alpha = 1$ becomes,

$$\boxed{\int_0^\infty x^n e^{-x} = n!}\ , \tag{3}$$

and this is the **factorial function**, for $n \in \mathbb{Z}^+$. It is easy, from this integral, to see why $0! = 1$.

3 Gamma Function

The generalization of (3) to include non-integer values of n, is known as the **gamma function**, denoted by $\Gamma(p)$, defined by

$$\boxed{\Gamma(p) \equiv \int_0^\infty x^{p-1}e^{-x}dx}\ ;\ p > 0\ \&\ p \in \mathbb{R}\ . \tag{4}$$

Comparing (3) and (4) leads to realizing that

$$p = n + 1$$

so that,

$$
\begin{aligned}
\Gamma(n+1) &= \int_0^\infty x^{n+1-1}e^{-x} = n! \text{ from (3) while} \\
\Gamma(n) = \Gamma((n+1)-1) &= \int_0^\infty x^{(n+1-1)-1}e^{-x} = (n-1)! \text{ from (3).}
\end{aligned} \tag{5}
$$

Thus, in summary we can define,

$$
\boxed{\begin{aligned}\Gamma(n+1) = n! \\ \Gamma(n) = (n-1)!\end{aligned}} \tag{6}
$$

for all $n \in \mathbb{Z}^+$. For general (non-integer) values $p,\ p > -1$, we usually write the definition as,

$$
\boxed{\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx = p!} \tag{7}
$$

which follows from (5). A useful identity involving the Γ function is the **recursion relation**,

$$
\boxed{\Gamma(p+1) = p\Gamma(p)} \tag{8}
$$

which can be derived from a straightforward application of integration by parts on (7).

Exercise 1. Derive the recursion relation for the Γ function ((8)).

Example 1. Let us now apply the recursion relation to simplify Γ function expressions. Consider $\Gamma(9/4)$.

$$\Gamma(9/4) = \Gamma(5/4+1) = (5/4)\Gamma(5/4)$$

$$\Rightarrow \Gamma(9/4) = (5/4)\Gamma(1/4+1) = (5/4)(1/4)\Gamma(1/4)$$

$$\Rightarrow \boxed{\Gamma(9/4) = \frac{5}{16}\Gamma(1/4).}$$

Example 2. Express the following integral as a Γ functions.

1. $\int_0^\infty e^{-x^4} dx$

   First we implement a substitution: $u = x^4 \Rightarrow x = u^{1/4} \Rightarrow dx = \frac{1}{4}u^{-3/4}du$ which leads to

   $$\int_0^\infty e^{-x^4} dx = \frac{1}{4}\int_0^\infty e^{-u}u^{-3/4}du = \frac{1}{4}\underbrace{\int_0^\infty u^{1/4-1}e^{-u}du}_{\Gamma(1/4)} = 1/4\Gamma(1/4) = \Gamma(1/4+1) = \Gamma(5/4)\ .$$

Exercise 2. Express each of the following integrals as Γ functions.

1. $\int_0^\infty x^{2/3}e^{-x}dx$,
2. $\int_0^1 (\ln x)^{1/3}dx$.

3.1 Gamma Function of Negative Numbers

From (8), it is straightforward to get,

$$\boxed{\Gamma(p) = \frac{1}{p}\Gamma(p+1)} \tag{9}$$

which defines $\Gamma(p)$ for $p < 0$. Let's see how this works in practice.

Example 3. Determine $\Gamma(-0.3)$.
With $p = -0.3$, by using (9) we get,

$$\Gamma(-0.3) = \frac{1}{-0.3}\Gamma(-0.3+1) \Rightarrow \Gamma(-0.3) = -\frac{10}{3}\Gamma(0.7) \ .$$

Example 4. Determine $\Gamma(-1.3)$.
With $p = -1.3$ plugged into (9) we get

$$\Gamma(-1.3) = \frac{1}{-1.3}\Gamma(-0.3) = \frac{10}{3}\frac{10}{3}\Gamma(0.7) = \frac{100}{9}\Gamma(0.7) \ .$$

Notice that as $p \to 0$, $\Gamma(p)$, as defined in (9), diverges. Additionally, we know $\Gamma(1) = 1$ from which it follows that $\Gamma(p)$ **diverges at all negative integers**. From the two examples above, we also note that in the intervals between the negative integers, $\Gamma(p)$ is alternatively positive and negative, for instance,

$$-1 < p < 0 \to \Gamma(-0.3) < 0$$

while

$$-2 < p < -1 \to \Gamma(-1.3) > 0$$

and so on. Besides (9) there is also another important functional that the Γ function satisfies, known as **Euler reflection formula** given by,

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p} \ ,$$

where p is not an integer.

3.2 $\Gamma(1/2)$

Using the definition of the gamma function we can write

$$\Gamma(1/2) = \int_0^\infty t^{1/2-1}e^{-t}dt = \int_0^\infty t^{-1/2}e^{-t}dt \ . \tag{10}$$

By implementing a substitution such that $u = t^{1/2}$, (10) changes to

$$\Gamma(1/2) = 2\int_0^\infty e^{-u^2}du = \int_{-\infty}^\infty e^{-u^2}du \tag{11}$$

where the last equality is achieved by exploiting the *even nature* of the integrand. Integrals of this form are known as **Gaussian integrals** and traditional methods of integral calculus fail to tackle this. As such, we use wisdom from Euler to solve this. First step is to realize that one can write (11) as,

$$\Gamma(1/2) = \sqrt{\int_{-\infty}^{\infty} e^{-u^2} du}$$

$$= \sqrt{\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy} = \sqrt{\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dA}$$

which in polar coordinates we have: $dA = dxdy = rdrd\theta,\ r^2 = x^2 + y^2$

$$\Rightarrow \Gamma(1/2) = \sqrt{\int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr}$$

where we can implement another substitution $w = r^2 \rightarrow dw = 2rdr$ leading to

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi} \underbrace{\sqrt{\int_0^{\infty} e^{-w} dw}}_{=1}$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}\ .$$

3.3 Beta Function

Another function of interest is the beta function, defined as

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \tag{12}$$

where $p, q > 0$. The $\beta-$function has some important properties and identities that are worth noting.

1. $\beta(p,q) = \beta(q,p)$, i.e. $\beta-$function is symmetric in the arguments.
2. A change of variables in (12) such that $x = y/a$ leads to an equivalent description for the $\beta-$function,
$$\boxed{\beta(p,q) = \frac{1}{a^{p+q-1}} \int_0^a y^{p-1}(a-y)^{q-1} dy}\ . \tag{13}$$
3. Another change of variables $x = sin^2\theta$ in (12) leads to the trigonometric form of the $\beta-$function i.e.
$$\boxed{\beta(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1} d\theta}\ . \tag{14}$$
4. A third change of variables such that $x = \dfrac{y}{1+y}$ in (12) yields an improper integral representation of the $\beta-$function:
$$\boxed{\beta(p,q) = \int_0^{\infty} \frac{y^{p-1}}{(1+y)^{p+q}}}\ . \tag{15}$$

3.4 $\beta-$function and its Relation to $\Gamma-$function

Starting with the integral definition of the Γ function i.e.

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

and making the substitution

$$t = y^2$$

leads to the expression,

$$\Gamma(p) = 2\int_0^\infty y^{2p-1} e^{-y^2} dy \ . \tag{16}$$

Following this, we can define another gamma function of a second variable q such that,

$$\Gamma(q) = 2\int_0^\infty x^{2q-1} e^{-x^2} dx \ , \tag{17}$$

and multiplying (16) and (17) yields,

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty dx\ dy\ x^{2q-1} y^{2p-1} e^{-(x^2+y^2)} \ . \tag{18}$$

If we make the switch to polar coordinates which sets

$$r^2 = x^2 + y^2,\ x = r\cos\theta,\ y = r\sin\theta,\ dxdy = rdrd\theta$$

then (18) becomes,

$$\begin{aligned} \Gamma(p)\Gamma(q) &= (4)\left(\frac{1}{2}\right)\Gamma(p+q)\left(\frac{1}{2}\right)\beta(p,q) \\ \Rightarrow & \boxed{\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}} \ . \end{aligned} \tag{19}$$

Exercise 3. Using the identity (19), evaluate the integral

$$\int_0^\infty \frac{x^3\ dx}{(1+x)^5} \ .$$

4 A Tribute to Richard Feynman

We've met the Gaussian integral in this chapter, specifically in (11), i.e. the integral of the form

$$\int_{-\infty}^\infty e^{-x^2} dx = 2\underbrace{\int_0^\infty e^{-x^2} dx}_{\equiv I} \tag{20}$$

and we have employed Euler's trick to evaluate it. Richard Feynman gives us a better (arguably) trick which is quite unique. He demands that we first *parameterize* the integral by defining an independent function inside the integral, of some other variable, which would lead to a simpler integral. In that spirit, he asks us to instead consider the seemingly unrelated integral,

$$
\begin{aligned}
& f(t) = \int_0^\infty \frac{e^{-t^2(1+x^2)}}{1+x^2}\, dx \\
& \text{take the derivative w.r.t. } t \Rightarrow f'(t) = \int_0^\infty \frac{-2t(1+x^2)e^{-t^2(1+x^2)}}{1+x^2}\, dx \\
& \Rightarrow f'(t) = 2te^{-t^2} \int_0^\infty e^{-t^2x^2} dx \\
& \text{now let } u = tx \Rightarrow du = tdx \text{ so that} \\
& \Rightarrow f'(t) = \frac{-2te^{-t^2}}{t} \underbrace{\int_0^\infty e^{-u^2} du}_{=I} \\
& \Rightarrow f'(t) = -2e^{-t^2} I \qquad (21) \\
& \text{and integrating both sides w.r.t. } t \\
& \Rightarrow f(\infty) - f(0) = -2I \int_0^\infty e^{-t^2} dt \\
& \Rightarrow 0 - \underbrace{\int_0^\infty \frac{1}{1+x^2} dx}_{=\arctan(\infty)-\arctan(0)} = 2I^2 \\
& I^2 = \left(\frac{1}{2}\right)\frac{\pi}{2} \\
& \Rightarrow I = \frac{\sqrt{\pi}}{2} \ .
\end{aligned}
$$

Going back to the original (Gaussian) and using what we evaluated I to be, we get

$$
\begin{aligned}
& \int_{-\infty}^\infty e^{-x^2} dx = 2I \\
\Rightarrow & \int_{-\infty}^\infty e^{-x^2} dx = 2\frac{\sqrt{\pi}}{2} \qquad (22) \\
\Rightarrow & \boxed{\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}} \ .
\end{aligned}
$$

I hope you realized how powerful Feynman's trick is - you started by solving a seemingly unrelated integral to the one you were originally given and got the answer to the given integral anyway! The technique is quite dense for beginners, however a good amount of practice lets one solve hefty (and even complex valued) integrals in a straightforward manner. For fun, try the following exercises.

Exercise 4. Implement Feynman's trick to evaluate the following integrals:

1. $\int_0^1 \frac{x^7-1}{\log x} dx$, **hint:** use $f(t) = \int_0^1 \frac{(x^t-1)}{\log x} dx$ as your parameterized function and see how it works out.

2. $\int_0^\infty \frac{\sin x}{x} dx$;

Hint: we aim to differentiate under the integral sign. To that end, parameterize the integral by placing the parameter (say, t) so that something from the integral, unrelated to t, gets simplified.

5 Relation of Gaussian Integrals and Gamma Functions

A general Gaussian integral can be written as

$$I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \ , \tag{23}$$

which can be extended to include,

$$I(a,b) = \int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}}\ e^{b^2/4a} \ . \tag{24}$$

Recall the relations,

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \ ,$$

$$\Gamma(n+1) = \int_0^{\infty} x^{n} e^{-x} dx \ .$$

If we define the integral,

$$I_n(a) = \int_{-\infty}^{\infty} x^n e^{-ax^2} dx \ , \tag{25}$$

we realize that whenever n is odd, the integral vanishes since the integrand is an odd function. While for even n, by using a substitution of the form $y = ax^2$, one is led to the integral

$$I_n(a) = \frac{1}{a^{\frac{n+1}{2}}} \underbrace{\int_0^{\infty} y^{\frac{n-1}{2}} e^{-y} dy}_{\Gamma(\frac{n+1}{2})} \ , \tag{26}$$

and thus we have the relation,

$$\boxed{\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = \frac{\Gamma(\frac{n+1}{2})}{a^{\frac{n+1}{2}}}} \ , \tag{27}$$

which often pop up in quantum mechanics calculations, especially in topics of perturbation theory.