Lecture 10

Capacitance and Capacitors

Consider an isolated conductor carrying a charge Q and the potential on the conductor is given by, say ϕ_0. Now, the charge on the conductor Q is proportional to the potential ϕ_0. But the exact proportionality depends on the size and shape of the conductor. We can write,

$$Q \propto \phi_0 \implies Q = C\phi_0$$

C is called the capacitance of the conductor. So, if the charge on the conductor was doubled, so would have been the capacitance. However, exactly how much potential will be attributed to a conductor depends on the capacitance for a given amount of charge.

The unit of capacitance is, $\frac{\text{Coulomb}}{\text{Volt}}$. This is generally denoted as farad.

$$\therefore 1 \text{ farad} = 1 \text{ C/V}$$

For an isolated spherical conductor with charge Q, we know that the potential is given by,

$$\phi_0 = \frac{1}{4\pi\epsilon_0} \frac{Q}{r},$$

where r is the radius of the conductor.

Now, $C = \frac{Q}{\Phi_0} = 4\pi\epsilon_0 r$

Now, to get a sense of how gigantic unit a Farad is, consider a spherical conductor with a radius of earth. The capacitance is,

$$C = 4\pi\epsilon_0 \times 6\times10^{6} \text{ Farad} = 7\times10^{-4} \text{ F}$$

In our regular use, we generally deal with micro-farad (μF) $= 10^{-6}$ F and picofarad (pF) $= 10^{-12}$ F etc.

There is an interesting fact about the unit Farad

$$\epsilon_0 = \frac{C}{4\pi r}$$

So, the unit of ϵ_0 can be expressed as Farad/meter. Due to this dimensional relation, the capacitance will always involve one factor of ϵ_0 and one net power of length. So, the ~~capacit~~ capacitance of any given shaped conductor will scale as the linear dimension of the conductor.

The concept of capacitance is more useful when we talk about a number of conductors. The most important of them is the one where there are two conductors with opposite charges, $+q$ and $-q$ on them.

The capacitance here is defined as the ratio of the magnitude of charge on each conductor and the potential difference between the conductors.

$$\therefore \cancel{Q} \qquad \therefore \cancel{C = \frac{Q}{|\Delta\phi|}}$$

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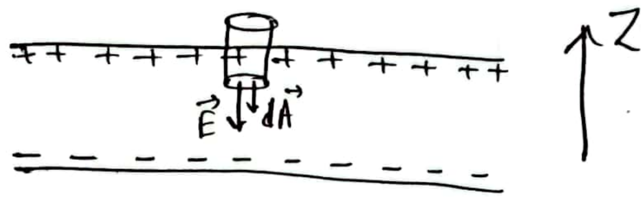
The object comprising of the two conductors, may be with electrical connections and insulating materials between the conductors is called a capacitor.

Parallel plate capacitor

The most common and simplest capacitor is the parallel plate capacitor. Two similar flat parallel plates are arranged, separated by a distance d. Let the area of the parallel plates are both A. Now, if the area of the conductor A is much greater than the distance d (might seem absurd), then the electric field between the plates will be fairly uniform.

We know that, the electric field due to an infinite sheet/slab of charges are uniform everywhere. The approximation also holds if the observation point is very close to a finite sheet. Let's first calculate the electric field in the space between two infinite parallel conducting slab, with excess charges Q and

$-Q$ on them. Now, as soon as the excess ch[arges] were given in each conductor, they will move out to the outer surface. But, there is an important point. When we arrange the parallel slabs close to each other, the excess charges attract each other on the plates, and technically all we have is all the excess charges residing on the surface which is facing each other. So, technically all we have is two infinite plane of charges.



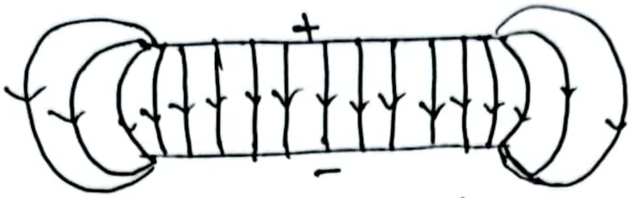
We know the electric field now will be from the positive to negative charges, and will be uniform. So, we take a Gaussian pillbox and use Gauss's law.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\Rightarrow \quad EA = \frac{\sigma A}{\epsilon_0} \qquad \therefore E = \frac{\sigma}{\epsilon_0}$$

$$\boxed{\therefore \vec{E} = -\frac{\sigma}{\epsilon_0}\hat{k}}$$

Now, this calculation is valid for infinite parallel plate capacitor, where the field is uniform. However, in real life, we do not have the luxury to have such infinite plates. There are edge effects in real life capacitors. The field lines are approximately as shown below



So, the field is nearly uniform, except at the edges. But for the sake of calculations, consider the electric field to be uniform totally. The potential difference between the plates will be given by,

$$\phi_+ - \phi_- \quad \cancel{\phi_+ - \phi_-} = -\int_{z_-}^{z_+} \vec{E}\cdot d\vec{s} = -\int_{z_-}^{z_+} -\frac{\sigma}{\epsilon_0}\hat{k}\cdot dz\,\hat{k}$$

$$= \frac{\sigma}{\epsilon_0}(z_+ - z_-) = \frac{\sigma}{\epsilon_0} d$$

$$\therefore \sigma = \frac{\epsilon_0(\phi_+ - \phi_-)}{d}$$

Now, $Q = \sigma A = A\,\dfrac{\epsilon_0(\phi_+ - \phi_-)}{d}$

where Q is the charge in one plate. So, we see that the charge on the plates is proportional to the potential difference of the plates. The constant of proportionality is what we call the capacitance.

$$\therefore C = \frac{Q}{\Delta\phi} \qquad \boxed{\therefore C = \frac{\epsilon_0 A}{d}}$$

You can very clearly see that the capacitance only depends on the geometry and size of the capacitors, and nothing else. For a given conductor arrangements, $\frac{Q}{\Delta\phi}$ is always a constant.

Now, you might say, what about the edge effects? Well, ~~we~~ we need computers and all. We can write the charge on the capacitor correctly with a correction factor f by,

$$Q = \frac{\epsilon_0 A (\phi_+ - \phi_-)}{d} f$$

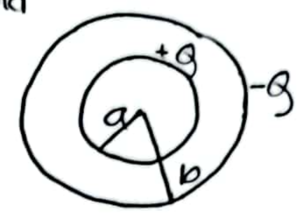
Now, consider two parallely placed conducting disc of radius R. Here is a list of correction factor as a function of d/R. You can see, our approximated value is astonishingly correct for low d/R values.

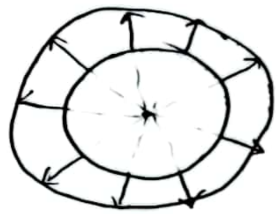
d/R	f
0.2	1.286
0.1	1.167
0.05	1.094
0.02	1.042
0.01	1.023

So, as long as our plate sizes are larger compared to the distance between the plates, we are good to use our approximated ~~f~~ formula.

Spherical capacitor

Consider two conducting spherical shells that are concentric. Let, there are charges $+Q$ and $-Q$ on the inner and outer shells respectively. Due to the properties of the conductors, all the charges will reside on the outer and inner surface of the inner and outer shell respectively. Now, the field due to the outer shell is basically zero, as we already know that there is no electric field inside a spherical shell with uniform charge density. So, the field is entirely due to the inner shell and the field lines will be spherically symmetric.





$$\phi_+ - \phi_- = -\int_{r=b}^{r=a} E\,\hat{r} \cdot dr\,\hat{r} = -\int_b^a E\,dr$$

$$= -\int_b^a K\frac{Q}{r^2}\,dr = -KQ\left[-\frac{1}{r}\right]_b^a$$

$$\therefore \Delta\phi = KQ\left[\frac{1}{a} - \frac{1}{b}\right]$$

So, the capacitance, $C = \dfrac{Q}{\Delta\phi} = \dfrac{Q}{\frac{1}{4\pi\epsilon_0} Q\left[\frac{1}{a} - \frac{1}{b}\right]}$

$$\boxed{\therefore \quad C = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}} = \frac{4\pi\epsilon_0 ab}{b-a}$$

If the outer shell extend to infinity, meaning $b \to \infty$, then $C = 4\pi\epsilon_0 a$, and we regain the expression for the capacitance of a spherical conductor.

Also, if the distance between the two shells is much smaller than b, then essentially $d = b - a$ is much smaller than b, then essentially $r \approx a \approx b$ and the area to ~~be~~ if we define be $4\pi r^2$,

$$C = \frac{4\pi r^2 \epsilon_0}{d} = \frac{\epsilon_0 A}{d}.$$