Lecture 9

Now, $$\frac{A_{n-1} + A_{n+1}}{A_n} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

$$\Rightarrow \frac{A_{n-1} + A_{n+1}}{2A_n} = \frac{2\omega_0^2 - \omega^2}{2\omega_0^2}$$

At this point, we have nowhere to go. Let us write the solution of A_n as—

$$A_n = B \cos n\theta + C \sin n\theta$$

As we have discussed earlier, three parameters (three A's or two A's and ω) determines the whole set A_n, we have three undetermined constants B, C and θ that will determine A_n. Its a claim that it should work, but we have to prove this

Let us define, $$\cos\theta = \frac{A_{n-1} + A_{n+1}}{2A_n} = \frac{2\omega_0^2 - \omega^2}{2\omega_0^2}$$

Since $|\cos\theta| \leq 1$, we have a constraint that $\omega \leq 2\omega_0$. If $\omega = 0$, $\cos\theta = 1$ and $\omega = 2\omega_0$, $\cos\theta = -1$.

We already know, two A's and ω will determine all A's. Say, we know A_0 and A_1, along with ω. If there is wall, we obviously know A_0, which is simply 0. But let's be general here.

$$A_0 = B\cos(0\cdot\theta) + C\sin(0\cdot\theta)$$

$$\Rightarrow \quad B = A_0$$

and $$A_1 = B\cos(1\cdot\theta) + C\sin(1\cdot\theta)$$

$$\therefore \; C\sin\theta = A_1 - B\cos\theta$$

Since we found B, we then know C. So, the A_0, A_1 and ω uniquely determines B, C and θ. (θ is determined by ω using $\cos\theta = \frac{2\omega_0^2 - \omega^2}{2\omega_0^2}$).

So, this construction tells you that $A_n = B\cos n\theta + C\sin n\theta$ works for $n=0$ and $n=1$. We can now inductively show that it works for any n.

Say, $A_n = B\cos n\theta + C\sin n\theta$ works for $n-1$ and n. Then —

$$2\cos\theta = \frac{A_{n-1} + A_{n+1}}{2A_n} \Rightarrow A_{n+1} = 2\cos\theta\, A_n - A_n$$

$$\therefore A_{n+1} = 2\cos\theta\left[B\cos n\theta + C\sin n\theta\right] - \left[B\cos(n-1)\theta + C\sin(n-1)$$

$$= B\left[2\cos\theta\cos n\theta - \cos(n\theta-\theta)\right] + C\left[2\cos\theta\sin n\theta + \sin(n\theta-\theta)\right]$$

$$= B\left[2\cos\theta\cos n\theta - \left(\cos n\theta\cos\theta + \sin n\theta\sin\theta\right)\right] +$$

$$C\left[2\cos\theta\sin n\theta + (\sin n\theta\cos\theta - \cos n\theta\sin\theta)\right]$$

$$= B\left[\cos n\theta\cos\theta - \sin n\theta\sin\theta\right] + C\left[\cos n\theta\sin\theta + \sin n\theta\cos\theta\right]$$

$$= B\cos(n\theta+\theta) + C\sin(n\theta+\theta)$$

$$= B\cos(n+1)\theta + C\sin(n+1)\theta$$

which is the expected result for A_{n+1}. Since the inductive step is valid, the result is true for any n.

$$\therefore A_n = B\cos(n\theta) + C\sin(n\theta)$$

So, our solution is then, $x_n = A_n e^{i\omega t}$

$$= (B\cos n\theta + C\sin n\theta)e^{i\omega t}$$

and $x_n = A_n e^{-i\omega t} = (D\cos n\theta + E\sin n\theta)e^{-i\omega t}$

The general solution is then -

$$x_n(t) = (B\cos n\theta + C\sin n\theta)e^{i\omega t} + (D\cos n\theta + E\sin n\theta)e^{-i\omega t}$$

$$\cancel{= (B\cos n\theta + D\cos n\theta)}$$

$$= \cancel{(B+D)} \cancel{= B\cos n\theta + D\cos}$$

$$= \cos n\theta\,(Be^{i\omega t} + De^{-i\omega t}) + \sin n\theta\,(Ce^{i\omega t} + Ee^{-i\omega t})$$

And using our argument of x_n being real as before-

$$x_n(t) = \cos n\theta\, F(\omega t+\phi) + \sin n\theta\, G\cos(\omega t+\psi)$$

$\therefore x_n(t) = F\cos n\theta \cos(\omega t + \phi) + G \sin n\theta \cos(\ldots$

We can expand ~~tri~~ trigonometric terms as –

$$x_n(t) = F\cos n\theta\left[\cos\omega t\cos\phi - \sin\omega t\sin\phi\right] + G\sin n\theta\left[\begin{matrix}\cos\omega t\cos\psi\\ -\sin\omega t\sin\psi\end{matrix}\right]$$

$$= F\cos n\theta\cos\omega t\cos\phi - F\cos n\theta\sin\omega t\sin\phi + G\sin n\theta\cos\omega t\cos\psi - G\sin n\theta\sin\omega t\sin\psi$$

$$\boxed{\therefore x_n(t) = C_1\cos n\theta\cos\omega t + C_2\cos n\theta\sin\omega t + C_3\sin n\theta\cos\omega t + C_4\sin n\theta\sin\omega t} \quad (*)$$

with $C_1 = F\cos\phi$, $C_2 = -F\sin\phi$, $C_3 = G\cos\phi$ and $C_4 = -G\sin\psi$

These four undetermined constants can be found by four initial conditions — for example, $x_0(0)$, $\dot{x}_0(0)$, $x_1(0)$ and $\dot{x}_1(0)$. θ is determined from,

$$\theta = \cos^{-1}\left(\frac{2\omega_0^2 - \omega^2}{2\omega_0^2}\right)$$

and we have our solution given by (*) for N-masses.

A few remarks

1. Equation (*) suggests that x_n varies sinusoidally with position n as well as time t. But, there is an important difference to remember. Time t takes a continuous set of values, whereas the position

can only take discrete values $(n = 0, 1, 2, \ldots, N+1)$. If we define the equilibrium positions by $z = na$, where a is the separation between the masses, then,

$x_z(t)$

$$x_n(t) = C_1 \cos\left(\frac{z}{a}\theta\right)\cos\omega t + C_2 \cos\left(\frac{z}{a}\theta\right)\sin\omega t + \cdots$$

For a given value of a and θ, this is a sinusoidal function of z. But remember, z can only take discrete values given by $z = na$.

2. z represents the equilibrium position of the masses. For a particular mass, z is fixed. $x_z(t)$ denotes the position of a mass relative to its equilibrium position z. We could just write $x_z(t)$ as a two variable function $x(z,t)$. But for now, since z is discrete, we will just use $x_z(t)$.

3. The (*) equation gives the general solution for a given value of ω, that is for a given mode. The most general solution is not determined by just $x_0(0)$, $x_1(0)$, $\dot{x}_0(0)$ and $\dot{x}_1(0)$. To see why, lets plug the initial conditions in (*):

$$\therefore \quad x_0(0) = C_1 \quad \text{and} \quad x_1(0) = C_1\cos\theta + C_3\sin\theta$$

Given that we know ω and hence θ, these two equations determine C_1 and C_3. But, then again,

$$x_n(0) = C_1 \cos n\theta + C_3 \sin n\theta$$

Since we know C_1 and C_3, all the initial position is uniquely determined for the given ω. Similarly,

$$\cancel{\dot{x}_n(0) = \omega C_2 \qquad \dot{x}_n(t) =}$$

$$\dot{x}_n(0) = \omega C_2 \cos n\theta + \omega C_4 \sin n\theta \qquad \boxed{\text{Check}}$$

$$\therefore \dot{x}_0(0) = \omega C_2 \quad \text{and} \quad \dot{x}_1(0) = \omega C_2 \cos\theta + \omega C_4 \sin\theta$$

These two equation determines C_2 and C_4 and C_2 and C_4 determine all $\dot{x}_n(0)$. So, like before, the four initial condition $x_0(0)$, $x_1(0)$, $\dot{x}_0(0)$ and $\dot{x}_1(0)$ determines all other initial positions and velocities for a particular ω – that is particular normal mode.

Back to the problem

Now, consider N masses connected by springs, with wall on two sides. So, our boundary conditions are –

$$x_0(t) = x_{N+1}(t) = 0, \text{ for all } t.$$

If $x_0(t) = 0$, then from ⊛,

$$C_1 \cos \omega t + C_2 \cos \omega t = 0$$

$$\therefore C_1 = -C_2$$

C_1 and C_2 terms get cancelled out

$$\boxed{\therefore x_n(t) = C_3 \sin n\theta \cos \omega t + C_4 \sin n\theta \sin \omega t}$$ ⊛⊛

$$x_{N+1}(t) = C\left[\cos(N+1)\theta \cos \omega t - \cos(N+1)\theta \cos \omega t\right] + C_3 \sin(N+1)\theta \cos \omega t + C_4 \sin(N+1)\theta \sin \omega t$$

$$\Rightarrow 0 = \sin(N+1)\theta \left[C_3 \cos \omega t + C_4 \sin \omega t\right]$$

Now, the parantheses term can be zero for all t only if $C_3 = C_4 = 0$, which corresponds to no oscillation at all as per ⊛⊛, since $x_n(t) = 0$ for all n.

So,
$$\sin(N+1)\theta = 0$$

$$\Rightarrow \sin(N+1)\theta = \sin p\pi$$

$$\therefore \theta = \frac{p}{N+1}\pi \; ; \text{ with } p = 1, 2, 3, \ldots$$

The solution is then,

$$x_n(t) = \left(C_3 \cos \omega t + C_4 \sin \omega t\right) \sin\left(\frac{np\pi}{N+1}\right)$$

$$\therefore x_n(t) = C \cos(\omega t \mp \phi) \sin\left(\frac{np\pi}{N+1}\right)$$

$$\boxed{\therefore x_n(t) = A_n \cos(\omega t \mp \phi)}$$

with $A_n = C \sin\left(n \frac{p\pi}{N+1}\right)$

and, that's our amplitudes of the n^{th} mass.

We can now use the equation,

$$\cos\theta = \frac{2\omega_0^2 - \omega^2}{2\omega_0^2}$$

$$\Rightarrow \omega^2 = 2\omega_0^2 (1 - \cos\theta)$$

$$\Rightarrow \omega^2 = 2\omega_0^2 \cdot 2\sin^2\frac{\theta}{2}$$

$$\therefore \omega = 2\omega_0 \sin\left(\frac{p\pi}{2(N+1)}\right)$$

As we have found, different values of p will give different values of ω. These different values of ω corresponds to different normal modes. So, we have, the normal modes ~~charge~~ characterized by,

$$\omega_p = 2\omega_0 \sin\left(\frac{p\pi}{2(N+1)}\right)$$

and the amplitudes are given by,

$$A_{np} = C_p \sin\left(n \frac{p\pi}{N+1}\right)$$

with the final solution given by for the p^{th} mode,

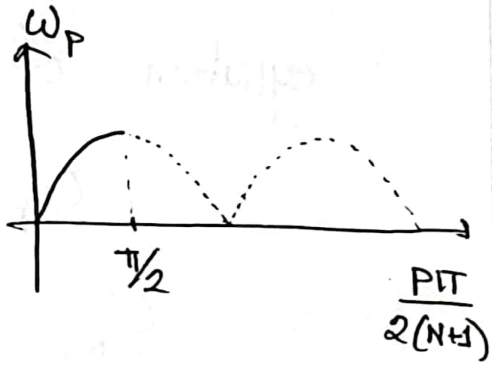
$$\boxed{x_{np} = A_{np} \cos(\omega_p t \mp \phi_p)}$$

Now

c

Now, for a N-particle system one would expect N number of normal modes. But then, there is a problem. We defined p to be any integer. So, there could be an infinite number of normal modes! That's absurd, right? Since, for sure we have seen for two and three masses there are two and three normal modes respectively. What's happening then?

Think about the frequency ω_p with mode number p. If we plot ω_p as a function of $\frac{p\pi}{2(N+1)}$, then this is exactly a sine curve, but we have modified this to have positive frequencies only.



As we go from $p=1$ to $p=N$, the ω_p has different different characteristic frequencies. For $p=N+1$, the argument is $\frac{\pi}{2}$ and we reach a maximum frequency of $\omega_{max} = 2\omega_0$. After that, its just the repetition of the same ω's on and on. So, $p>N$, does not really describe new modes!

Wait, we said ~~p>N~~ $p \geq N$ does not describe new motion. What about $p = N+1$? This is allowed and describes a unique motion. The thing is if you plug $p = N+1$ in the amplitude, then –

$$A_{n,\,N+1} = C_{N+1} \sin\left(\frac{n(N+1)\pi}{N+1}\right) = C_{N+1} \sin(n\pi) = 0$$

So, all the amplitudes are zero, and it does not describe anything interesting. So, upto N modes is what matters.

To show this mathematically, let's set $p = N+2$ in the equation of ω_p.

$$\omega_p = 2\omega_0 \sin\left(\frac{p\pi}{2(N+1)}\right)$$

$$\therefore\ \omega_{N+2} = 2\omega_0 \sin\left[\frac{(N+2)\pi}{2(N+1)}\right] \cancel{= 2\omega_0 \sin\Big[} = 2\omega_0 \sin\left[\frac{N\pi + 2\pi}{2(N+1)}\right]$$

$$\cancel{= 2\omega_0 \sin\left[\frac{(N+1)\pi + \pi}{2(N+1)}\right]}$$

$$= 2\omega_0 \sin\left[\pi - \frac{N\pi}{2(N+1)}\right]$$

$$= 2\omega_0 \sin\left(\frac{N\pi}{2(N+1)}\right)$$

$$= \omega_N$$

You can also show that, $\omega_{N+3} = \omega_{N-1}$, $\omega_{N+4} = \omega_{N-2}$ and so on.

You can show in a similar manner that, the amplitudes also repeat themselves for $p > N+1$, which ~~we~~ I will leave as an exercise.

Let's now see how the various modes look like. Well, before that, let's first verify our solution for N-mass for the previously developed $N=2$ and $N=3$ masses.

N=2 case

We will run $P=1$ to 2 for N mass, as per our discussion above.

$P=1$:
$$A_n = C_1 \sin\left(\frac{n\pi}{3}\right) \quad \text{with} \quad \omega = 2\omega_0 \sin\left(\frac{\pi}{6}\right)$$
$$\Rightarrow \omega = \omega_0$$

$A_1 = C_1 \sin(\pi/3)$
$A_2 = C_1 \sin\left(\frac{2\pi}{3}\right)$
$= C_1 \sin\left(\pi - \frac{\pi}{3}\right)$
$= C_1 \sin(\pi/3)$

$$\therefore A \propto \begin{pmatrix} \sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) \end{pmatrix} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$P=2$:
$$A \propto \begin{pmatrix} \sin\frac{2\pi}{3} \\ \sin\frac{4\pi}{3} \end{pmatrix} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{with} \quad \omega = 2\omega_0 \sin\frac{\pi}{3}$$
$$\Rightarrow \omega = \sqrt{3}\,\omega_0$$

And these result exactly matches with our result from previous sections! You can now go on checking for more N values, and you will find everything in order. For a homework, check for $N=3$ case whether it matches with our previous result.

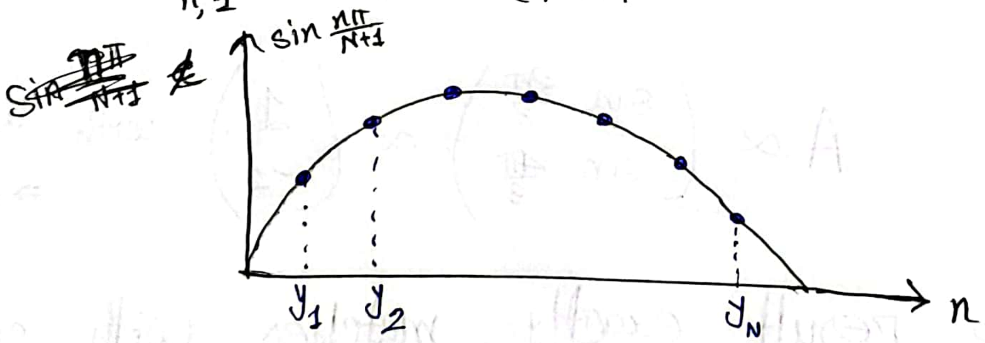
Let's now see how individual modes look like. We have,

$$x_{n,p} = A_{n,p} \cos(\omega_p t - \phi) \quad \text{with} \quad A_{n,p} = C_p \sin\left(n\frac{p\pi}{N+1}\right)$$

$$= C_p \sin\left(n\frac{p\pi}{N+1}\right) \cos(\omega_p t - \phi)$$

At a particular instant of time, $\cos(\omega_p t - \phi)$ is a constant for the p^{th} mode. It is only the $\sin\left(n\frac{p\pi}{N+1}\right)$ term that distinguishes the displacement of the particles. We draw the plot of $\sin\left(\frac{n\pi}{N+1}\right)$ as a function of n continuously from 0 to N+1, for the first mode $P=1$. The displacements are given by

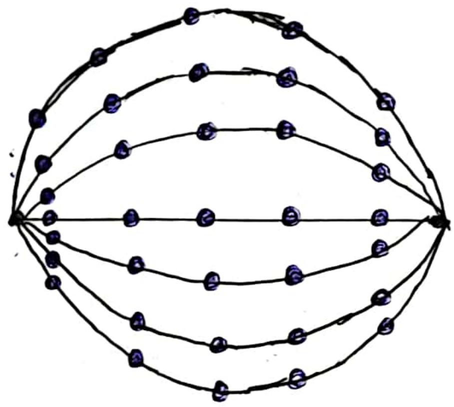
$$x_{n,1} = C_1 \sin\left(\frac{n\pi}{N+1}\right) \cos(\omega_1 t - \phi)$$



The position of the actual particles are given by the blue dots for the discrete values of $n = 1, 2, \ldots, N$.

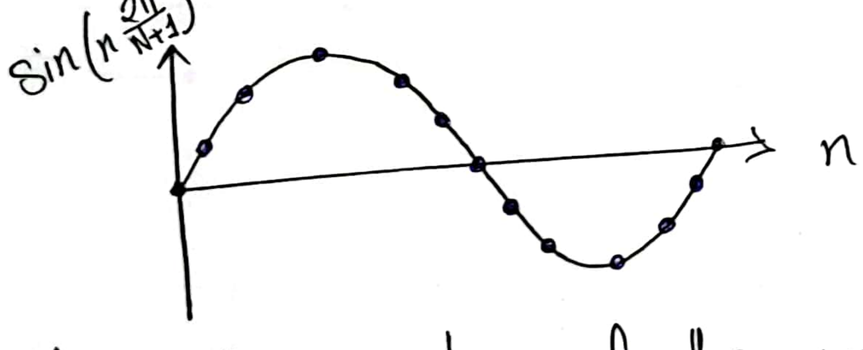
Basically you divide the horizontal axis into N+1 parts and put your N number of particles at the interval points on the curve to demonstrate the position of the particles.

Now as time goes on, the individual particles will oscillate sinusoidally, and it looks something like the graph shown below.



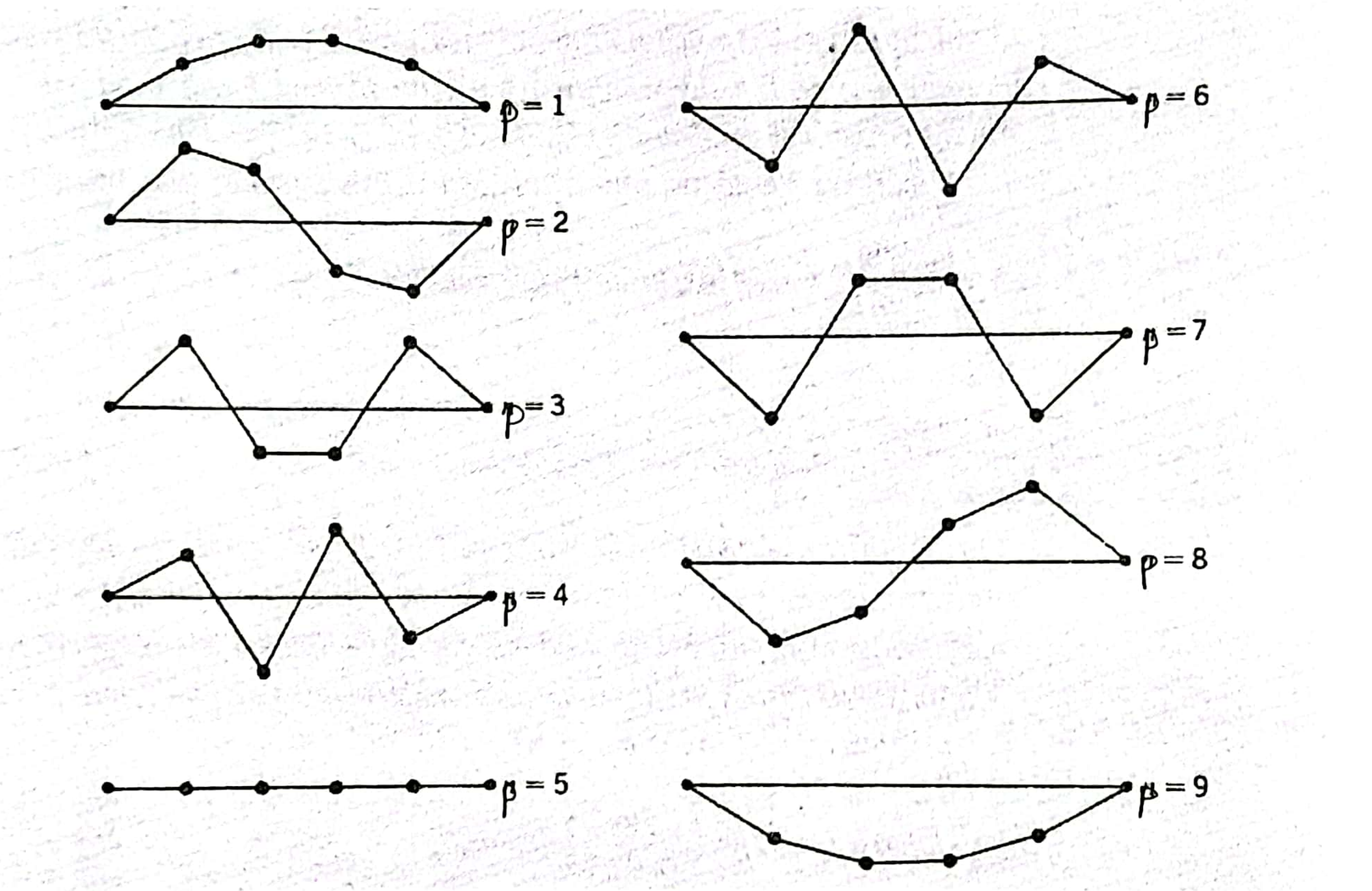
For the second mode, $p = 2$,

$$x_{n,2} = C_2 \sin\left(n \frac{2\pi}{N+1}\right) \cos(\omega_2 t - \phi)$$



It shows the position of the particles for the second mode for a particle number of 11. In this case, the middle mass never moves. Aparently, it happens for all odd numbered particle coupled oscillator. You shoul

You should be able to imagine how the individual particles move with time. The following figure shows the normal modes ranging from $p=1$ to $p=9$ for a four particle coupled oscillator. Its nice to see how $n=6$ exactly resembles $n=4$, $n=7$ resemble $n=3$ and so on. The connecting straight



lines was used in the ~~book~~ book of A. P. French ~~for~~ for a different purpose. Ignore this.