Critical damping

Now, let's consider the damping scenerio which is at the border of underdamping and overdamping. This is called critical damping, which occurs for $\frac{\gamma}{2} = \omega_0$. The general solution then becomes,

$$x(t) = (c_1 + c_2)\, e^{-\frac{\gamma}{2}t} = c e^{-\frac{\gamma}{2}t}$$

In the first glance, one might think the solution is complete. However, its a second order ODE. So, we should find two independent solutions with two undetermined constants. To emphasize the reasoning, think about initial position and velocity.

$$x(t) = c e^{-\frac{\gamma}{2}t} \Rightarrow \cancel{x(0)} \Rightarrow x(0) = c$$

$$v(t) = -\frac{\gamma}{2} c\, e^{-\frac{\gamma}{2}t} \Rightarrow v(0) = -\frac{\gamma}{2} x(0)$$

So, $v(0)$ is determined by $x(0)$. But we can set any velocity we want at $x(0)$. So, we must be missing another solution. But how?

Going back to the characteristic solution, we see–

$$\lambda_{\pm} = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

So, for $\frac{\gamma}{2} = \omega_0$, we are left with only one value of λ and that's where the problem pops out.

Now, there are several methods for finding another hidden solution. We will use the method, where we start from the underdamped or overdamped case and take the $\frac{\gamma}{2} \to \omega_0$ limit to constitute the solution.

The solution for the underdamped case is –

$$x(t) = e^{-\frac{\gamma}{2}t}\left[A\cos\omega t + B\sin\omega t\right]$$

~~Dividing the whole solutions by an overall constant is still a solution. Let's~~ take,

$$\cancel{x(t) = \frac{1}{\omega} e^{-\frac{\gamma}{2}t}\left[A\cos\omega t + B\sin\omega t\right]}$$

$$\cancel{\therefore x(t) = e^{-\frac{\gamma}{2}t}\left[\frac{A}{\omega}\cos\omega t + B\frac{\sin\omega t}{\omega}\right]}$$

$$\cancel{\lim_{\omega\to 0} x(t) = e^{-\frac{\gamma}{2}t}[}$$

The first solution : $x_1(t) = e^{-\frac{\gamma}{2}t} A\cos\omega t$

$$x_2(t) = e^{-\frac{\gamma}{2}t} B\frac{\sin\omega t}{\omega}$$

Dividing $x_2(t)$ by ω is still a solution. So, we set our solution to,

$$x(t) = e^{-\frac{\gamma}{2}t}\left[A\cos\omega t + B\frac{\sin\omega t}{\omega}\right]$$

$$\lim_{\omega\to 0} x(t) = e^{-\frac{\gamma}{2}t}\left[A + Bt\right]$$

$$\lim_{\omega\to 0}\frac{\sin\omega t}{\omega} = \lim_{\omega\to 0}\frac{t\sin\omega t}{\omega t} = t\lim_{\omega\to 0}\frac{\sin\omega t}{\omega t} = t$$

$$\cancel{\lim_{\omega\to 0}\frac{\sin\omega t}{\omega} = \lim_{\omega\to 0}\frac{t\sin\omega t}{\omega t} = \lim}$$

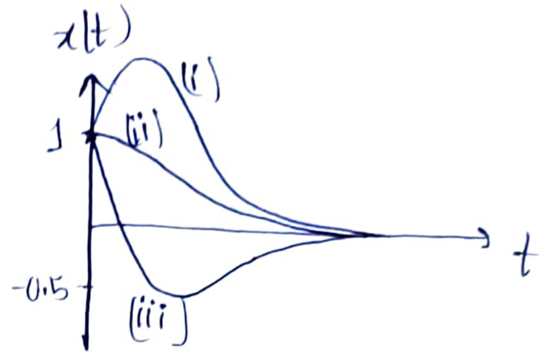
$$\cancel{\lim_{\omega\to 0}\frac{\sin\omega t}{\omega}}$$

t

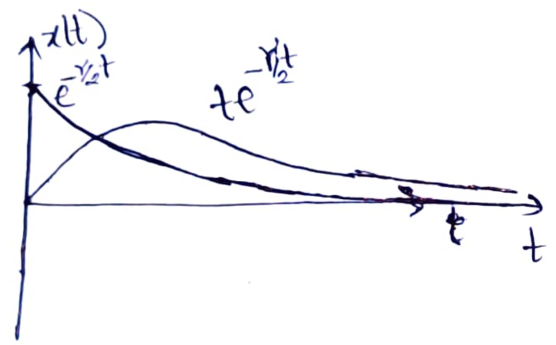
$\therefore$ The general solution for critically damped case is —

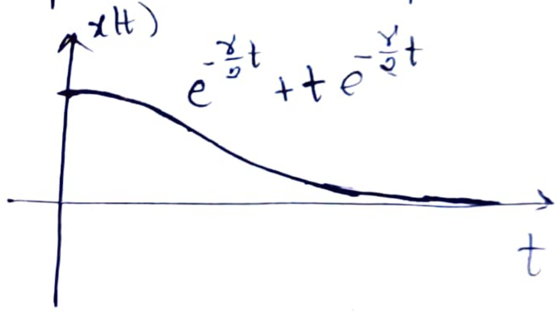
$$x(t) = (A+Bt)\,e^{-\frac{\gamma}{2}t}$$

The plots looks like the plots of overdamped case, but one important difference. The critically damped motion converges to the origin in quickest possible manner.



Its quicker than both underdamped and overdamped case.





(i) For critical damping, the motion goes to zero like $e^{-\frac{\gamma}{2}t}$ (since the Bt term is inconsequential compared to the exponential term). The overdamped motion goes to zero like $e^{-\Gamma_- t}$ $(\Gamma_+ > \Gamma_-)$. Since $\Gamma_- = \frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$, so $\Gamma_- < \frac{\gamma}{2}$.

$\therefore$ $x_{\text{critical}}(t) < x_{\text{overdamped}}(t)$ for large t.

(ii) The underdamped motion reaches the origin first, but it doesn't stay there. It oscillates back and forth around the origin for a large amount of time. The envelope decreases as $e^{-\frac{\gamma}{2}t}$, with $\frac{\gamma}{2} < \omega_0$. For critical damping, it decreases as $e^{-\frac{\gamma}{2}t} = e^{-\omega_0 t}$. So, its the critical damping, for which the motion reaches the origin as quickly as possible without bouncing around.

A familiar system that is close to critical damping is the combination of springs and shock absorbers in an automobile. The damping must be large enough to prevent bouncing (underdamping) but as well as not that large so that it takes long time ~~to~~ for the springs to settle down.

[You can look for variation of parameters for another way of finding the critically damped solution].

Forced oscillation

Driven and damped oscillation

Let's now think about the case of driven and damped oscillator. Let's say, we impose a periodic driving force with driving frequency ω_d. So, the equation of motion looks like —

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0 \cos\omega_d t$$

$$\therefore \frac{d^2x}{dt^2} + \omega_0^2 x + \gamma\frac{dx}{dt} = F_0 \cos\omega_d t \quad \text{—— (I)}$$

We have to keep track of the frequencies. ω_0 is the natural frequency, $\sqrt{\frac{k}{m}}$ of the simple undamped oscillator. $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ is the frequency of the under-damped oscillator. ω_d is the frequency of the driving force, which can have arbitrary values.

This is, for the first time we are having an inhomogenous equation. The general form is given by,

$$\alpha\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \eta x = F(t) \quad \text{—— (II)}$$

If $x_1(t)$ is a solution to the equation with a driving force $F_1(t)$ and $x_2(t)$ is a solution to the driving force $F_2(t)$, then,

$$\alpha\frac{d^2x_1(t)}{dt^2} + \beta\frac{dx_1(t)}{dt} + \eta x_1(t) = F_1(t)$$

$$\alpha\frac{d^2x_2(t)}{dt^2} + \beta\frac{dx_2(t)}{dt} + \eta x_2(t) = F_2(t)$$

$$\alpha\frac{d^2}{dt^2}\left[x_1(t)+x_2(t)\right] + \beta\frac{d}{dt}\left[x_1(t)+x_2(t)\right] + \eta\left[x_1(t)+x_2(t)\right] = F_1(t)+F_2(t)$$

So, $x_1(t) + x_2(t)$ is a solution for driving force $F_1(t) + F_2$
In general, $x(t) = A\,x_1(t) + B\,x_2(t)$ will be a solution to the driving force $AF_1(t) + BF_2(t)$. This is true for any number of forces.
The reason why we are emphasizing on this superposition idea so much is that, we are trying to solve the motion for a periodic driving force (like $\cos\omega_d t$ or $\sin\omega_d t$). The previous paragraph tells us that we can always find the solution to a force that is linear combination of sinusoidal forces with different multiplying constant. Now, ~~that~~ any general driving force can be written as the sum of $\cos\omega t$ and $\sin\omega t$ terms (and their harmonics) with appropriately choosen amplitudes. So, if we can figure out the solution for the driving force $F_0\cos\omega_d t$, we will eventually be able to obtain the solution for an arbitrary force.

Now, we can exploit the linearity to find the general solution. Remember that, the solution to equation (11) is not linear. Any solution multiplied with a constant is no longer a solution. You can check by yourself. Try $A x_1(t)$ assuming $x_1(t)$ is a solution, in equation (11).

$$\therefore A\left[\alpha\frac{d^2x_1(t)}{dt^2} + \beta\frac{dx_1(t)}{dt} + \eta\, x_1(t)\right] = F(t)$$

$\therefore AF = F$

Similarly, even if $x_1(t)$ and $x_2(t)$ are solutions of equation (ii), $Ax_1(t) + Bx_2(t)$ is not the general solution.

To find the most general solution, lets invoke the idea that $Ax_1(t)$ and $Bx_2(t)$ will be a solution for the total driving force $AF_1(t) + BF_2(t)$. —— (*)

Say, we have a particular solution $x_p(t)$. to the ODE (ii), that corresponds to force $F(t)$. Another solution $x_h(t)$, which corresponds to the no driving force situation, that is homogenous counterpart of (ii).

As per our discussion (*), $x(t) = x_p(t) + x_h(t)$ must also be a solution to the force $F(t) + 0 = F(t)$. And this will be the most general solution to ODE (ii). We already know the solution $x_h(t)$, we will just have to find a particular solution $x_p(t)$, and then, we are done. The reason why $x_p(t)$ can't be the general solution is that, it lacks ~~underminind~~ ~~c~~ undeterminind constants. For a second order ODE, we must have two undetermined constants that would be found by initial conditions. The general solution gives the floor to that, and we will see this shortly.

Saying all the pros and cons, lets now actually solve the equation for a particular solution. Our equation at hand is —

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = F_0/m \cos \omega_d t \quad \text{—(I)}$$

Let's rewrite the equation of motion in terms of the complex driving force $F(t) = F_0 e^{i\omega_d t}$. We can write,

$$\frac{d^2 z}{dt^2} + \gamma \frac{dz}{dt} + \omega_0^2 z = \frac{F(t)}{m} = \frac{F_0}{m} e^{i\omega_d t} \quad \text{—(II)}$$

Why can we use the complex driving force, which is not physical along with the complex positions? Well, we can always ~~tav~~ find the complex solution, with complex driving force, and take the real counterpart of z to find the actual physical solution corresponding to actual real force. To prove our point, we first think about how differentiation works with complex numbers. Differentiation commutes with the act of taking the real part of a complex number

$$\therefore \mathrm{Re}\left[\frac{d}{dt}(a+ib)\right] = \frac{da}{dt} = \frac{d}{dt}\left[\mathrm{Re}(a+ib)\right]$$

~~Taking~~ We can write our original equation as –

$$\mathrm{Re}\left[\frac{d^2 z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z\right] = \mathrm{Re}\left[\frac{F_0}{m}e^{i\omega_d t}\right]$$

$$\Rightarrow \frac{d^2}{dt^2}\left[\mathrm{Re}(z)\right] + \gamma\frac{d}{dt}\left[\mathrm{Re}(z)\right] + \omega_0^2\,\mathrm{Re}(z) = \frac{F_0}{m}\cos\omega_d t$$

So, if $z(t)$ is a solution to equation (ii), then the real part of $z(t)$ is a solution to the original equation.

To find the solution of (ii), we will guess a solution. Since the driving force is periodic, we expect to get a periodic solution. Now, because the R.H.S. has a term involving $e^{i\omega_d t}$ the solution should be of the same form as $e^{i\omega_d t}$. So, we look for solution,

$$z(t) = A e^{i\omega_d t}$$

The reason is, the frequency must be ω_d, is that, on the left hand side, you have derivatives of $z(t)$, and the function $z(t)$ itself. But since derivatives can't change the frequency, any other frequency other than ω_d has no chance in satisfying the equation. So, the frequency must be ω_d.