Group Theory
Lecture 10

The Problem:

We have seen the defining representations of the various groups. These are irreducible representations. The problem that we want to address is how to make larger irreducible representations from these more 'fundamental' ones.

Here we shall use what are known as tensor methods. In this lecture we discuss tensor methods for the orthogonal groups while in the next lecture we discuss SU(N) groups.

Vector Representations of the Orthogonal Groups

Let us consider the defining representation of the O(N) group:

If V_i is an N-dimensional vector then $R_{ij} \in O(N)$ if V_i transform as

$$V_i \longrightarrow V_i' = R_{ij} V_j \quad \text{st} \quad V_i V_i = V_i' V_i'.$$

$$\Rightarrow \quad R_{ij} = (R^{-1})_{ji} \quad [\ R^T = R^{-1}\ ]$$

This representation is known as the vector representation.

Examples of vector representations from physics:

1. In Newtonian mechanics displacement $\Delta\vec{x}$, velocity $\vec{v}$, momentum $\vec{p}$, acceleration $\vec{a}$ are all vector representation of O(3) group. Note that under parity $P: \vec{x} \to -\vec{x}$ $\Delta\vec{x} \longrightarrow -\Delta\vec{x}$, $\vec{p} \to -\vec{p}$, etc. but $\vec{L} = \vec{r} \times \vec{p} \longrightarrow \vec{L}$. Since $\vec{L}$ does not change its sign under parity like the other vectors $\vec{L}$ is known as a pseudo-vector.
2. The electric field $\vec{E}$ & and the magnetic field $\vec{B}$ are also in the vector rep of O(3).
3. SO(1,3) or O(1,3) vector are $V^\mu \longrightarrow V'^\mu = \Lambda^\mu{}_\nu V^\nu$. V^μ are contravariant vectors. We can show that covariant vectors $V_\mu \to V'_\mu = \Lambda_\mu{}^\nu V_\nu$ can be derived from this representation.

Tensor Methods:

While the electric and magnetic fields are $O(3)$ vectors under a Lorentz transformation $O(1,3)$ they transfer as a rank-2 antisymmetric tensor:

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}.$$

In general if we are given vector representations V_i, W_i, U_i etc we can make larger, newer representations by taking their tensor products:

$$T_{ij} = V_i W_j \longrightarrow T'_{ij} = R_{il} R_{jk} T_{lk}.$$

But in general such a tensor product is not irreducible. To express the tensor products as direct sums of irreps we need the invariant tensors.

O(N) invariant tensor:

The tensor δ_{ij} is an invariant tensor because:

$$\begin{aligned}
\delta_{ij} \to \delta'_{ij} &= R_{il} R_{jk} \delta_{lk} \\
&= R_{il} (R^T)_{kj} \delta_{lk} \\
&= (R \mathbb{1} R^T)_{ij} \\
&= (R R^{-1})_{ij} = (\mathbb{1})_{ij} = \delta_{ij}.
\end{aligned}$$

O(1,3) invariant tensor:

$\eta_{\mu\nu}$ is an invariant tensor since

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu} \quad \text{according } \Lambda^T \eta \Lambda = \eta.$$

SO(N) invariant tensors:

For $N=3$ ϵ_{ijk} is invariant under an $SO(3)$ transformation because:

$$\begin{aligned}
\epsilon_{ijk} \to \epsilon'_{ijk} &= R_{il} R_{jm} R_{kn} \epsilon_{lmn} \\
&= \det R \; \epsilon_{ijk}
\end{aligned}$$

Note that for $R \in SO(3)$ $\det R = +1$

and so ϵ_{ijk} is an invariant tensor under $SO(3)$. But since $R = P$

has det -1, under parity $\epsilon_{ijk} \longrightarrow \epsilon'_{ijk} = -\epsilon_{ijk}$.

Thus ϵ_{ijk} is not invariant under $O(3)$.

For general N, $\epsilon_{a_1 \dots a_N}$ is an $SO(N)$ invariant tensor.

Decomposition of tensor product of vector representation in terms of smaller irreps:

Let V_i & W_i be two vector rep of $O(N)$. Then we can write their tensor product as $T_{ij} = V_i W_j$. But this is not an irrep. To construct the irreps we can do the following:

$T_{ii} = \delta_{ij} V_i W_j \equiv T$

T is a scalar (trivial) rep of $O(N)$:

$$T \to T' = \delta_{ij} R_{il} R_{jk} V_l W_k = \delta_{lk} V_l W_k = T$$

$T^{ST}_{ij} = T_{(ij)} - \frac{\delta_{ij}}{N} T$ is symmetric and tracefree.

It has $\underbrace{\frac{N^2 - N}{2}}_{\text{Off diagonal}} + \underbrace{N}_{\text{Diagonal}} - \underbrace{1}_{\text{Trace}} = \frac{N^2 + N}{2} - 1$ components.

$T^{A}_{ij} = T_{[ij]}$ is antisymmetric (and automatically trace free)

It has $\frac{N^2 - N}{2}$ components

Although it is not completely unambiguous we can denote an n dim rep of $O(N)$ by n. Thus we see

$$T_{ij} = \begin{cases} T & \text{— } 1 \text{ components} \\ T^{ST}_{ij} & \text{— } \frac{1}{2}N(N+1) - 1 \text{ ''} \\ T^{A}_{ij} & \text{— } \frac{1}{2}N(N-1) \text{ ''} \end{cases}$$

can be written as:

$$N \otimes N = 1 \oplus \frac{N(N+1)}{2} - 1 \oplus \frac{N(N-1)}{2}.$$

Using the Levi-Civita Tensor:

If we have a rank $(N-P)$ tensor of $SO(N)$ we can construct a pseudo-tensor of rank p using ϵ. Thus

$$L_i = \epsilon_{ijk} \Delta x_j p_j$$

Under a rotation $L_i \to L_i' = \epsilon_{ijk} R_{jl} R_{km} \Delta x_l p_m$

But $\epsilon_{ijk} R_{il} R_{jm} R_{kn} = \epsilon_{lmn} \Rightarrow \epsilon_{ijk} \underbrace{R_{il} R_{jm} R_{kn} R_{sl}}_{\delta_{is}} = \epsilon_{lmn} R_{sl}$

$$\Rightarrow \quad \epsilon_{sjk} R_{jm} R_{kn} = \epsilon_{lmn} R_{sl}$$

$$\Rightarrow \quad \epsilon_{ijk} R_{jm} R_{kn} = R_{il} \epsilon_{lmn}$$

$$\Rightarrow L_i' = R_{il} \epsilon_{lmn} \Delta x_m p_n = R_{il} L_l$$

$\Rightarrow$ L_i is an $SO(3)$ tensor.