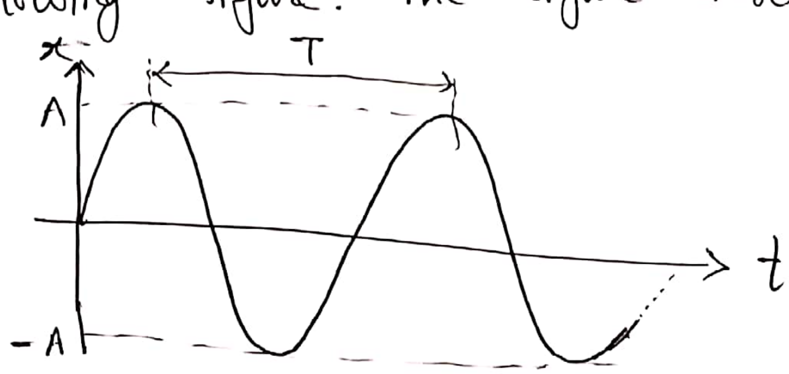
Lecture 1

"After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We can't even say "vibration" properly without the tip of the tongue oscillating..... Even the atoms of which we are constituted vibrate".

— R.E.D. Bishop

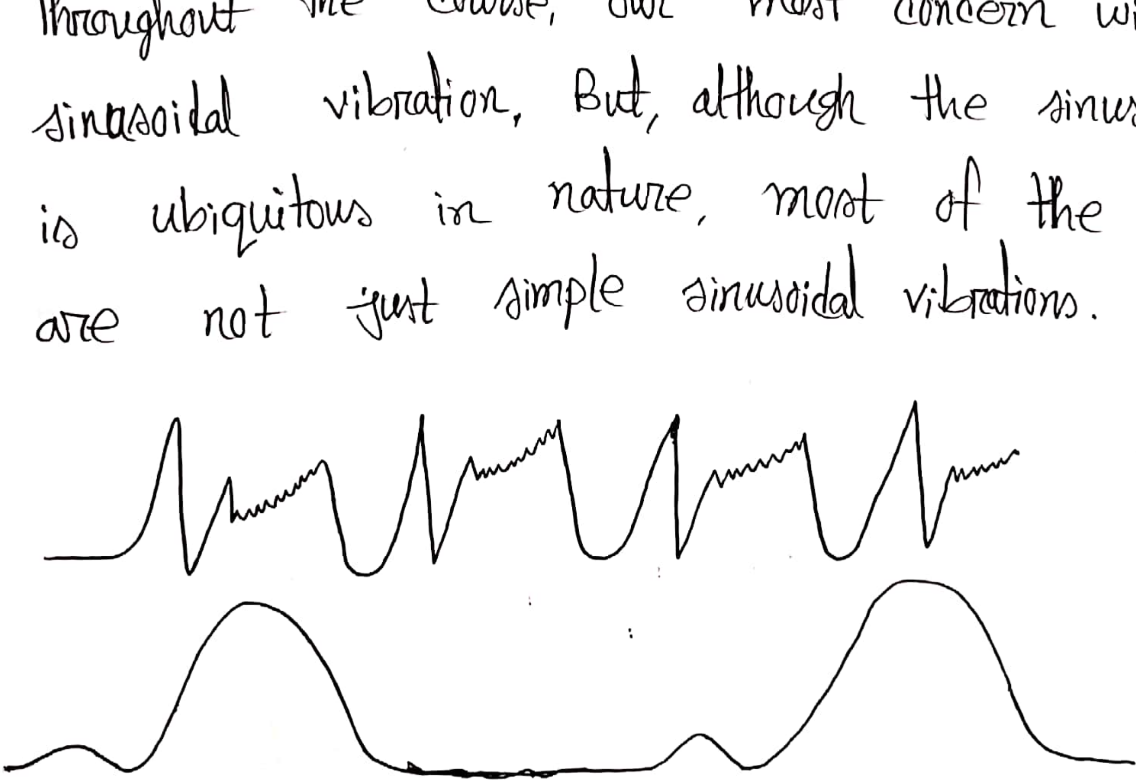
Vibrations and oscillations are so much ubiquitous in nature. They are everywhere. From the vibration of the mosquito wings to the heartbeat, from the smallest atomic vibration to vibration of earth in an earthquake, we experience vibrations/oscillations in many different length and time scales. Oscillations are everywhere in the physics realm. The study of oscillations are thus as important as it can be for studying different ~~br~~ branches of physics.

All these different oscillations have one thing in common— periodicity. The pattern of movement/displacement repeats itself over and over again. The simplest periodic f one may think of is a sinusoidal wave, as shown in the following figure. The figure describes the displace-



ment of ~~the~~ some arbitrary object from a fixed origin as a function of time. One can clearly see that the oscillation is periodic — with a time period T, after which the motion repeats itself. This type of oscillation might be found from the vibrations of a tuning fork.

Throughout the course, our most concern will be on the sinusoidal vibration, But, although the sinusoidal vibration is ubiquitous in nature, most of the vibrations are not just simple sinusoidal vibrations.



Pressure variation in a cat's heart

human heart

These complicated ~~a~~ oscillations are for sure not sinusoidal. They can be as complicated as it can be, but they are periodic. It might seem that, we are leaving an ocean of things by just focusing on the simple sinusoidal oscillations. Obviously, these simple sinusoidal vibrations are not rare. We encounter them often now and then. Many system behaves to have simple sinusoidal oscillations under small displacements.

But, there is a deeper mathematical reason. The profound importance of purely sinusoidal vibrations can be found in a famous theorem proposed by French ~~Mat~~ mathematician J. B. Fourier. Any periodic vibrations, no matter how complicated they are, can be constructed from a set of purely sinusoidal oscillations of frequencies $(\omega, 2\omega, 3\omega, \dots)$ with appropriately choosen amplitudes. Its an infinite ~~sum~~ series made up of a fundamental frequency and its harmonics. The series is called Fourier series. So,

$$f(x) = \sum_{n=0}^{\infty} \left(a_n \sin(n\omega_0 x) + b_n \cos(n\omega_0 x)\right)$$

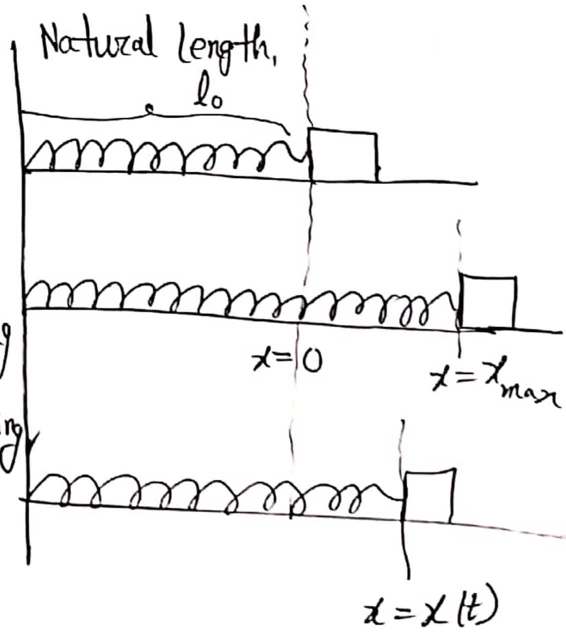
$$= b_0 + \sum_{n=1}^{\infty} \left[a_n \sin(n\omega x) + b_n \cos(n\omega x)\right]$$

Anyways, the Fourier series is a strong tool to divide any periodic oscillation into a series of sinusoidal ones. So, it again comes down to the study of simple sinusoidal oscillations. That's why we will be, for now mostly focused on sinusoidal oscillations. Later, we may bring Fourier series in action.

Simple harmonic motion – the harmonic oscillator

You may already have learnt about harmonic oscillator in your mechanics course. We will start with this.

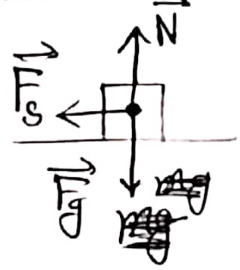
Consider a block with mass m, free to slide on a frictionless track, attached to a nearly massless spring with its other end attached to a fixed wall.



Say, at some time, the block is stretched to the right by some distance, say, x_{max}. We are assuming an ideal spring. So, while stretching the spring should be uniformly stretched from the fixed wall.

So, the only coordinate we will think about is the position of the block from the equilibrium. So, there is only one degree of freedom in the system. Degrees of freedom is the number of coordinates that must be specified in order to determine the configuration completely.

Let's do a free body diagram for the block. Gravitational force is exactly nullified by the normal force provided by the surface to the block. The only relevent force comes from the stretching or ~~comp~~ compressing the spring — the restoring spring force. This force is a function of position from the equilibrium. In simplest and generalized terms, the spring force can then be written



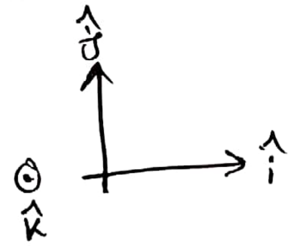
as, $$F_s(x) = -\left(kx + k_1 x^2 + k_2 x^3 + \cdots\cdots\right)$$

where $k, k_1, k_2, \ldots$ are constants preserving the dimension of force. You can think of this as a Maclaurin expansion of $F_s(x)$. The negative sign is here to remind you that the force is a restoring force.

However, if the spring is ideal, the constants $k_1, k_2, \ldots$ becomes irrelevant and,

$$F(x) = -kx \text{ ------- } (1)$$

Any spring obeying equation (1) is called a Hookean spring. Real life springs are not exactly Hookean. But, if your ranges of x are small enough, that is, for small displacements, the higher order terms contributes little and the spring approximately behaves ideal. In vector notation, we write,



$$\vec{F_s} = -K\vec{x}$$

with our choice of coordinates as shown. If $\vec{x} = +ve\ \hat{i}$, $\vec{F_s} = -ve\ \hat{i}$ and if $\vec{x}\ \cancel{\vec{F_s}} = -ve\ \hat{i}$, then $\vec{F_s} = +ve\ \hat{i}$, and everything makes sense perfectly. We can now use Newton's second law to write,

$$ma = -Kx \quad \text{(we dropped the vector notations)}$$

$$\Rightarrow m\frac{d^2}{dt^2}x(t) = -Kx(t)$$

$$\Rightarrow \frac{d^2}{dt^2}x(t) = -\frac{K}{m}x(t)$$

$$\therefore \frac{d^2}{dt^2}x(t) = -\omega^2 x \quad \text{with } \omega = \sqrt{K/m}, \text{ a constant}$$

for now, consider ω to be just a constant. Later we will see ω is called angular frequency having a dimension of T^{-1}.

This is the equation of motion for the system. Since we have only one degree of freedom, there is only one equation of motion. The equation involving $x(t)$ and its derivatives is called a differential equation. The equation is actually a second order, homogenous, linear differential equation. To understand what it means, let's consider a general form of the differential equation—

$$\alpha \frac{d^2}{dt^2} x(t) + \beta \frac{d}{dt} x(t) + \gamma x(t) = f(t)$$

The order of the equation is two— meaning the highest order of derivatives that the equation contains is two. The equation is linear since $x(t)$ appears at most to the power one in the terms. If all the terms ~~are~~ involve exactly one power of x, then the equation would have been called homogenous. Obviously this equation is not homogenous as the right hand side contains no term with $x(t)$. However, if the right hand side is zero, like exactly our equation of motion, the the equation becomes homogenous.

Now, our current equation is linear.

$$\frac{d^2}{dt^2} x(t) = - \omega^2 x(t)$$

Linearity of the equation plays a very important role here. It ensures that if $x_1(t)$ and $x_2(t)$ are solutions of the differential equation, then any linear combination of $x_1(t)$ and $x_2(t)$ is also a solution to the differential equation. So, the most general solution will be given by,

$$x_{12}(t) = A x_1(t) + B x_2(t)$$

You can do a quick check: $\frac{d^2}{dt^2} x(t) + \omega^2 x(t) = 0$ — (2)

Now, $$\frac{d^2}{dt^2}\left[A x_1(t) + B x_2(t)\right] + \omega^2 \left[A x_1(t) + B x_2(t)\right]$$

$$= A \frac{d^2}{dt^2} x_1(t) + A\omega^2 x_1(t) + B \frac{d^2}{dt^2} x_2(t) + B\omega^2 x_2(t)$$

$$= A\left[\frac{d^2}{dt^2} x_1(t) + \omega^2 x_1(t)\right] + B\left[\frac{d^2}{dt^2} x_2(t) + \omega^2 x_2(t)\right]$$

But since $x_1(t)$ and $x_2(t)$ are solutions to equation (2), then the coefficients of A and B are both 0.

$$\therefore \frac{d^2}{dt^2} x_{12}(t) + \omega^2 x_{12}(t) = 0$$

$\therefore x_{12}(t)$ is also a solution to the differential equation as given. But if we have an ~~inhomogen~~ inhomogenous liner DE, the linearity plays a different role. We will come back to this later.

Solution to the differential equation:

$$\frac{d^2}{dt^2}x \cancel{+} = -\omega^2 x$$

We will make an ansatz. That is, we will guess a solution. Our solution is clearly something, upon differentiating twice, it returns the same functions. We all know that exponential functions has this property. So, our trial solution is—

$$x(t) = e^{rt}$$

$$\therefore \frac{d^2x}{dt^2} = r^2 e^{rt}$$

$$\therefore r^2 e^{rt} = -\omega^2 e^{rt}$$

In the limit when t does not go to infinity we can write,

$$r^2 = -\omega^2$$

$$\therefore r = \pm i\omega$$

So, we have two possible solutions — $e^{i\omega t}$ and $e^{-i\omega t}$. Any linear combination will be the general solution to the equation.

$$\therefore x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

where C_1 and C_2 are not necessarily real co-efficients. In general, they are complex.

But, if you are not familiar with i and complex numbers, let's have a brief overview.

The square root of -1, called i, and all multiples of this are called imaginary numbers. A complex number is a sum of real and imaginary number. We can denote a complex number as—

$$z = x + iy$$

with x and y being real.

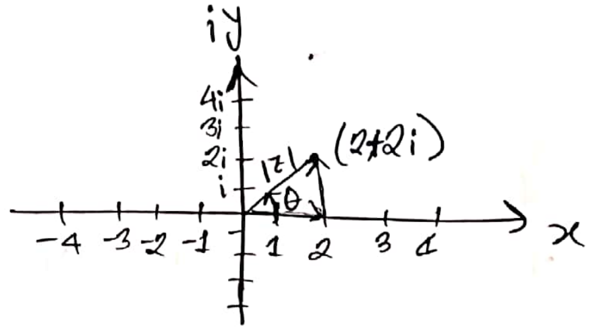
The real part is written as: $Re(z) = x$

imaginary part is written as: $Im(z) = y$

The complex conjugate is defined as, $z^* = x - iy$, that is by changing the sign of i.

$$\therefore \; Re(z) = \frac{z + z^*}{2} \quad \text{and} \quad Im(z) = \frac{z - z^*}{2i}$$

Since complex number is represented by two real numbers, it can be thought of as a two dimensional vector, with component (x, y) as along the x- and y- axis. We can then make the complex plane with axes x and iy.



i can now be considered as an instruction to perform a counter-clockwise rotation of $90°$ upon whatever it precedes. To find ia, we traverse a distance a along the x-axis and rotate by $90°$ to end up a displacement a along the Y-axis. To form i^2a, we again make a $90°$ rotation which lands it to $-a$, ensuring that i^2 is -1 and so on. The absolute value of z is given by,

$$|z| = \sqrt{a^2+b^2} = \sqrt{zz^*}$$

The argument is given by,

$$\underset{\theta}{\arg(z)} = \begin{cases} \tan^{-1}(b/a) ; a \geq 0 \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi ; a < 0 \end{cases}$$

Its the counterclockwise angle that z makes with the positive X-axis.

The complex exponentials

The Maclaurin expansion of $\sin\theta$ and $\cos\theta$ is given by,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

Now, the complex number can be written as,

$$z = |z|\cos\theta + i|z|\sin\theta$$

But,
$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \cdots$$

$$= \cancel{1+i\theta}$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= e^{i\theta}$$

$$\therefore e^{i\theta} = \cos\theta + i\sin\theta$$

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots}$$

$$\boxed{\therefore z = |z| e^{i\theta}}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The complex exponential ~~is~~ very helpful for ~~to~~ performing algebras with complex numbers, as we will see. Now lets get back to our solution.

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t} = (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$

Since our $x(t)$ is real, we require –

$$\text{Re}\,(C_1 - C_2) = 0 \quad \text{—(i)}$$

$$\text{and} \quad \text{Im}\,(C_1 + C_2) = 0 \quad \text{—(ii)}$$

such that the coefficients of C_1 and C_2 are real.

equation (i) and (ii) can only be satisfied if,

$$C_1 = \bar{C}_2 \quad \text{and} \quad C_2 = \bar{C}_1$$

$$\therefore x(t) = (C_1 + \bar{C}_1)\cos\omega t + i(C_1 - \bar{C}_1)\sin\omega t$$

$$\therefore x(t) = 2\,Re(C_1)\cos\omega t + i \cdot 2i\,Im(C_1)\sin\omega t$$

$$\boxed{\therefore x(t) = A\cos\omega t + B\sin\omega t}$$

with A and B as specified.

We can also express $x(t)$ as a function of only one trigonometric function.

$$x(t) = A\cos\omega t + B\sin\omega t = A\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) + B\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$$

$$= \cancel{A+} \quad = A\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) - iB\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2}\right)$$

$\frac{c + c^*}{2}$

$$= \frac{1}{2}\left[(A + iB)e^{-i\omega t} + (A - iB)e^{i\omega t}\right]$$

$$= Re\left[(A+iB)e^{-i\omega t}\right]$$

$$= Re\left[c\,e^{i\phi}e^{-i\omega t}\right]$$

$$= Re\left[c\,e^{-i(\omega t - \phi)}\right]$$

$$= C\cos(\omega t - \phi)$$

with $C = \sqrt{A^2 + B^2}$ and $\phi = \arg(A + iB)$.

In general, we can write,

A and ϕ are ~~HHEH~~ unknown constants that has to be found using initial conditions

$$x(t) = A\cos(\omega t \pm \phi)$$
$$= \text{Re}\left[A e^{i(\omega t+\phi)}\right]$$

We have written the dummy variable C as A

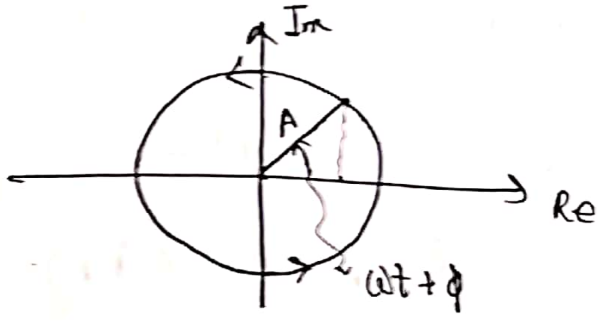
So, the solution to the SHM DE is really the rotation in the complex plane. We can always work with the complex solution, perform all the algebra, and finally just take out the real part as we will only need it. Complex exponentials makes calculations extremely simple rather than trigonometric ones. So, we will use the complex solution often now and then.

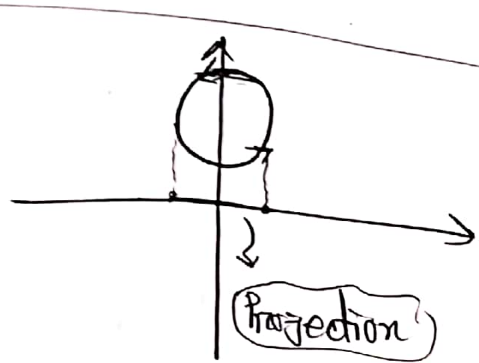
$$z_1 = \cos\theta_1 + i\sin\theta_1$$
$$z_2 = \cos\theta_2 + i\sin\theta_2$$

$$z_1 z_2 = i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$= i\sin(\theta_1+\theta_2) + \cos(\theta_1+\theta_2)$$

In terms of complex exponential:

$$z_1 z_2 = e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$





Properties of the SHM solution:

1. **Linearity:** If $x_1(t)$ and $x_2(t)$ are solutions of SHM then any linear combination is also a solution.

$$\therefore x(t) = Ax_1(t) + Bx_2(t)$$

2. **Time translation invariance/symmetry:** Time translation symmetry tells, if $x(t)$ is a solution, so is $x(t+\alpha)$. So, if the time is translated by an amount α, then the physics looks the same. So, it doesn't matter when you start the clock, you will find same periodic motion, just with different initial condition. Mathematically speaking, our equation was

$$\frac{d^2}{dt} x(t) + \omega^2 x(t) = 0$$

Now,

$$\frac{d^2}{dt} x(t+\alpha) + \omega^2 x(t+\alpha)$$

Here,

$$\underbrace{\frac{d}{dt} x(t+\alpha)}_{\frac{d}{dt}x(t')} = \cancel{\frac{d}{d(t+\alpha)}}\; \underbrace{\frac{d}{dt}(t+\alpha)\,\frac{d}{d(t+\alpha)}\,x(t+\alpha)}_{\frac{dt'}{dt}\,\frac{dx(t')}{dt'}}$$

$$= (1+0)\,\frac{dx(t')}{dt'} \qquad \boxed{\text{Via chain rule}}$$

$$\therefore \frac{d}{dt}x(t') = \frac{dx(t')}{dt'}$$

$$\therefore \quad \frac{d^2 x(t')}{dt'^2} + \omega^2 x(t') = 0$$

We can also check this by using the exponential solution.

$$x(t) = Ae^{i(\omega t+\phi)}$$

$$x(t+\alpha) = Ae^{i(\omega t+\omega\alpha+\phi)} = e^{i\omega\alpha} \cdot Ae^{i(\omega t+\phi)}$$

$$= Ae^{i[\omega t+(\phi+\omega\alpha)]}$$

So, time translation is nothing but a rotation in the complex plane. All the Physics should be the same.

<u>Finding the undetermined constants</u>

$$x(t) = A\cos(\omega t+\phi) \quad \text{and} \quad x'(t) = \frac{dx(t)}{dt} = -\omega A\sin(\omega t+\phi)$$

$$x(0) = A\cos\phi \qquad x'(0) = -\omega A\sin\phi$$

$$\therefore \quad A = \sqrt{x(0)^2 + \left(\frac{x'(0)}{A\omega}\right)^2}$$

$$\phi = \tan^{-1}\left(-\frac{x'(0)}{x(0)\,\omega}\right)$$

If $x(0) = 0$, then, $A = \dfrac{x'(0)}{\omega} \Rightarrow V(0) = \omega A$

$\phi = 90^\circ$

Anyways, using proper initial conditions, you can always find the undetermined constants.