Quantum Mechanics II

Lecture Notes 3

Course Instructor:
Tilsna Ali

The Dirac Notation:

In the often used Dirac notation vectors in a Hilbert space are denoted by a 'Ket': $|\psi\rangle$. The addition of two vectors is written as:

$$|\varphi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \quad , \quad \text{where } c_i \in \mathbb{C}.$$

Note that for a given vector $|\varphi\rangle \in \mathcal{H}$, the inner product defines the map:

$$\langle\varphi|\cdot\rangle \equiv (\varphi, \cdot) : \mathcal{H} \longrightarrow \mathbb{C}.$$

This map is antilinear

$$((c_1\varphi_1 + c_2\varphi_2), \psi) = c_1^* (\varphi_1, \psi) + c_2^* (\varphi_2, \psi)$$

$$= c_1^* \langle\varphi_1|\psi\rangle + c_2^* \langle\varphi_2|\psi\rangle$$

$$\langle c_1\varphi_1 + c_2\varphi_2|\psi\rangle = (c_1^* \langle\varphi_1| + c_2^* \langle\varphi_2|)|\psi\rangle$$

The number of linearly independent maps is equal to the dimension of the Hilbert space. Thus the space of antilinear maps from $\mathbb{H} \to \mathbb{C}$ itself forms a vector space. This space is called a dual vector space $\mathbb{H}^*$

The inner product gives an isomorphism between $\mathbb{H}$ and $\mathbb{H}^*$. We can express this by choosing a set of orthonormal basis $\{|\alpha\rangle\}$:

$$\langle \alpha | \alpha' \rangle = \delta_{\alpha\alpha'}$$

Operators in the bracket notation:

We write $(\psi, \hat{A}\varphi) = \langle \psi | \hat{A} | \varphi \rangle$ where $\hat{A}$ is implicitly thought to act on $|\varphi\rangle$.

$\langle \psi | \hat{A}$, on the other hand, is dual to $A^\dagger |\psi\rangle$ since

$$\begin{aligned} (\psi, \hat{A}\varphi) &= (\hat{A}^\dagger \psi, \varphi) \\ &= (\varphi, \hat{A}^\dagger \psi)^* \\ &= \langle \varphi | A^\dagger | \psi \rangle^* . \end{aligned}$$

When $\hat{A}$ is Hermitian:

$$(\psi, A^\dagger \varphi) = (\psi, A\varphi) \quad \forall \ \psi \ \& \ \varphi.$$

$$\Rightarrow \quad \langle \psi | A^\dagger | \varphi \rangle = \langle \psi | A | \varphi \rangle$$

Projection Operators:

Suppose $\hat{A}$ is a Hermitian operator. Let α_i be its eigenvalues and $|a_i\rangle$ the corresponding eigenvector. Then we can express any vector $|\psi\rangle$ as a linear combination of the eigenvectors of $\hat{A}$:

$$|\psi\rangle = \sum_i c_i |a_i\rangle$$

Now suppose we want to know what is the j-th component of $|\psi\rangle$ in the basis $\{|\alpha_i\rangle\}$.

We Then define an operator $\hat{P}_j$ st.

$$\hat{P}_j |\alpha_j\rangle = |\alpha_j\rangle$$

$$\hat{P}_j |\alpha_i\rangle = 0 \quad \text{if } i \neq j.$$

These two equations can be expressed as

$$\hat{P}_i |\alpha_j\rangle = \delta_{ij} |\alpha_i\rangle \text{ (no sum)}$$

$\hat{P}_j$ is called a projection operator. Note that

$$\hat{P}_j |\psi\rangle = \hat{P}_j \sum_i c_i |\alpha_i\rangle$$

$$= \sum_i c_i \hat{P}_j |\alpha_i\rangle$$

$$= \sum_i c_i \delta_{ij} |\alpha_j\rangle$$

$$= c_j |\alpha_j\rangle$$

Comments:

1. $\hat{P}_j$ peels off the j-th component of $|\psi\rangle$

when expressed in the $\{|\alpha_i\rangle\}$ basis.

2. Note that the projection operators are basis dependent.

It is conventional to express $\hat{P}_j$ in the Dirac notation as

$$\hat{P}_j = |\alpha_j\rangle\langle\alpha_j|$$

Then $\langle\alpha_i|\alpha_j\rangle = \delta_{ij} \Rightarrow$

$$\hat{P}_j|\psi\rangle = |\alpha_j\rangle\langle\alpha_j| \sum_i c_i|\alpha_i\rangle$$

$$= |\alpha_j\rangle \sum_i c_i\langle\alpha_j|\alpha_i\rangle$$

$$= |\alpha_j\rangle \sum_i c_i\,\delta_{ij}$$

$$= c_j|\alpha_j\rangle.$$

Note that

$$\hat{P}_i^{\,2} = \hat{P}_i \quad (\text{Show})$$

and $$\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_j \quad (\text{no sum})$$

Finally, when we sum over <u>all</u> the projectors associated with a basis, we should get $\hat{\mathbb{1}}$:

$$\sum_i P_i = \sum_i |\alpha_i\rangle\langle\alpha_i| = \hat{\mathbb{1}} \quad (\text{show}).$$

When we express $|\psi\rangle$ in the basis $\{|\alpha_i\rangle\}$:

$$|\psi\rangle = \sum_i c_i |\alpha_i\rangle$$

$$\Rightarrow \quad c_i = \langle\alpha_i|\psi\rangle$$

Previously we wrote this as

$$c_i = (\alpha_i, \psi)$$

One of the most useful formulas:

$$\sum_i P_i = \hat{\mathbb{1}}$$

can expressed as:

$$\sum_i |\alpha_i\rangle\langle\alpha_i| = \hat{\mathbb{1}}$$

The Hermitian matrix $\hat{A}$ whose eigenvectors are $\{|\alpha_i\rangle\}$ can be expressed as

$$\hat{A} = \sum_i \alpha_i |\alpha_i\rangle\langle\alpha_i|$$

If $\hat{B}$ is another operator such that it commutes with $\hat{A}$ then $\hat{B}$ can be expressed as:

$$\hat{B} = \sum_i \beta_i |\alpha_i\rangle\langle\alpha_i| .$$

Since $\hat{A}$ & $\hat{B}$ commute, $\{|\alpha_i\rangle\}$ form simultaneous eigenstates.

$$[\hat{A}, \hat{B}] = \sum_{i,j} \alpha_i \beta_j |\alpha_i\rangle\langle\alpha_i|\alpha_j\rangle\langle\alpha_j|$$

$$- \sum_{i,j} \beta_i \alpha_j \, |\alpha_j\rangle\langle\alpha_j|\alpha_i\rangle\langle\alpha_i|$$

$$= \sum_{i,j} \alpha_i \beta_j \quad |\alpha_i\rangle \, \delta_{ij} \, \langle\alpha_j|$$

$$- \sum_{i,j} \beta_i \alpha_j \quad |\alpha_j\rangle \, \delta_{ji} \, \langle\alpha_i|$$

$$= \sum_{i} \alpha_i \beta_i \, |\alpha_i\rangle\langle\alpha_i|$$

$$- \sum_{i} \beta_i \alpha_i \, |\alpha_i\rangle\langle\alpha_i| = 0$$

The state $|\psi\rangle$ of a system in the position representation is defined to be the wave function:

$$\psi(x) = \langle x|\psi\rangle$$

where $\quad \hat{x}|x\rangle = x|x\rangle$

and $\quad \langle x|x'\rangle = \delta(x-x')$

Then a generic equation:

$$\hat{O}|\psi\rangle = |\varphi\rangle$$

becomes in the position representation:

$$\langle x|\hat{O}|\psi\rangle = \langle x|\varphi\rangle$$

$$\Rightarrow \int dx' \langle x|\hat{O}|x'\rangle\langle x'|\psi\rangle = \varphi(x)$$

[where we have used

$$\hat{\mathbb{1}} = \int dx\, |x\rangle\langle x|$$
]

$$\Rightarrow \quad \int dx \; \hat{O}_{xx'} \, \psi(x') = \varphi(x)$$

If $\quad \hat{O} = \hat{P}$

$$\hat{P}_{xx'} = i\hbar \frac{\partial}{\partial x'} \delta(x - x')$$

and we get :

$$-i\hbar \frac{\partial}{\partial x} \psi(x) = \varphi(x).$$