Classical Mechanics

Lecture #2

Generalization to many particles

Newtons laws would not be useful if they only applied to single particles. However, with the help of the third law, it turns out we can derive a simple form of Newton's 2nd law that describes extended objects. These extended objects can be thought of a collection of point particles of a continuum of small parts. For now we assume that our system consists of a large number of discrete small particles. Let the total number of particles be N (where N can be large). We also assume that the particles interact with each other, i.e., they exert forces on each other. Then if we take the ith particle from this collection, then the total force on that particle would be:

$$\vec{F}_i = \underbrace{\sum_{\substack{j=1 \\ (j\neq i)}}^{N} \vec{F}_{ij}}_{\text{Force on ith particle due to the j-th particle.}} + \overbrace{\vec{F}_i^{ext}}^{\text{Force on the ith particle from outside the system of N particles.}}$$

We assume that the particle does not apply any force on itself. Then the total force on the whole system will be the sum of the forces on all the particles:

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i$$

$$= \sum_{i=1}^{N} \left\{ \sum_{j \neq i} \vec{F}_{ij} + \vec{F}_i^{ext} \right\}$$

$$= \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ij} + \sum_i \vec{F}_i^{ext}$$

$$= \left\{ \left(\sum_{j<i} \vec{F}_{ij} + \sum_{j>i} \vec{F}_{ij} \right) + \vec{F}_i^{ext} \right\}$$

$$\begin{array}{cc} & \begin{matrix} i=1 & 2 & 3 & \cdots & N \end{matrix} \\ \begin{matrix} j=1 \\ 2 \\ 3 \\ \vdots \\ N \end{matrix} & \underbrace{\begin{pmatrix} 0 & \vec{F}_{21} & \vec{F}_{31} & \cdots & \vec{F}_{N1} \\ \vec{F}_{12} & 0 & \vec{F}_{32} & \cdots & \vec{F}_{N2} \\ \vec{F}_{13} & \vec{F}_{23} & 0 & \cdots & \\ \vdots & \cdots & & 0 & \cdots \\ \vec{F}_{1N} & \cdots & & & 0 \end{pmatrix}}_{\text{vector sum over all elements}} \end{array}$$

Let us write $\vec{F}^{ext} = \sum_{i=1}^{N} \vec{F}_i^{ext}$ as the net external force.

Now what about the first term: $\sum_{j<i} \vec{F}_{ij} + \sum_{j>i} \vec{F}_{ij}$?

This is Newton's third law comes to play which states:

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

Then we see that the first term becomes:

$$\sum_{j<i} \vec{F}_{ij} - \sum_{j>i} \vec{F}_{ji} = 0$$

And so we have $\vec{F}^{ext} = \sum_{i=1}^{N} \vec{F}_i$

But according to Newton's 2nd law applied to the point particles we have:

$$\vec{F}_i = \dot{\vec{p}}_i$$

Then we have

$$\vec{F}^{ext} = \sum_i \dot{\vec{p}}_i$$

Now define the total mass of the system to be

$$M = \sum_i m_i$$

↳ masses of the individual particles

and the centre of mass $\vec{R}$:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M} \Rightarrow M\dot{\vec{R}} = \sum_i \vec{p}_i$$

Then $M\ddot{\vec{R}} = \sum_i \dot{\vec{p}}_i$ and so we have

$$\boxed{\vec{F}^{(ext)} = M\ddot{\vec{R}}}$$

which has the same form as Newton's law for a single particle.

* This allows us to use Newton's second law to describe the motion of the centre of mass of large and extended objects such as planets, a stone, a star etc.