# Lecture 16

## Maxwell's correction to Ampere's law

There is a particular inconsistency in our old Ampere's law. It stated,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{or} \quad \oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

We will address this inconsistency separately through differential and integral form.

Starting with the differential form:

We write two of the Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—(I)} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{—(II)}$$

The divergence of a curl is always zero. Taking the divergence on both sides of (I) we get,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot \left(\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{B})$$

We can commute $\vec{\nabla} \cdot$ and $\frac{\partial}{\partial t}$ since they are independent of each other.

Now, $\vec{\nabla} \cdot \vec{B} = 0$, always.

$\therefore$ $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = 0$ and everything is fine.

Lets now take divergence ~~of~~ on both sides of equation (11).

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0(\vec{\nabla}\cdot\vec{J})$$

Now, $\vec{\nabla}\cdot\vec{J} = 0$ only in the static case, where steady current is flowing. But if we leave the regime of steady current, Ampere's law can't be right (or complete). Because, the left hand side is mandatorily zero, where the right hand side is non-zero if the current is not steady.

To account for this inconsistency, we must have some additional term on the right hand side. Tracking back to continuity equation, we get-

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0\,\vec{\nabla}\cdot\vec{E}) = -\vec{\nabla}\cdot\left(\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

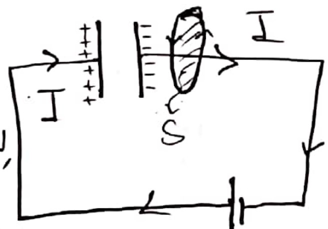
So, if we add the extra term $\epsilon_0\frac{\partial\vec{E}}{\partial t}$ in Ampere's law along with $\vec{J}$, the divergence of $\vec{J}$ will exactly cancel out ~~the~~ by the extra term with divergence. So, if we write,

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

the divergence of $\vec{\nabla}\times\vec{B}$ becomes zero, and we are fine again.

Although it might seem only suggestive by Maxwell, but it serves well to eradicate the inconsistency. The confirmation of the theory came, in 1888 with Hertz's experiment of electromagnetic wave.

Let's look at this from a different perspective. Consider a capacitor charging/discharging. Since there is a current, there should be an induced magnetic field. Let's take an Amperian loop as shown in the figure. Now, the surface that ~~is~~ has a boundary of our Amperian loop can be taken in many different manner. The simplest surface is the disc, whose boundary is the circle that we are considering as our Amperian loop.

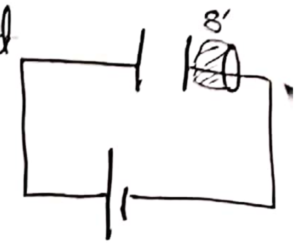



But we could as well take the bag sized surface. It also has the boundary of the Amperian loop.

First, let's take the disc surface. Current passing through the disk surface is obviously I. So, we write,

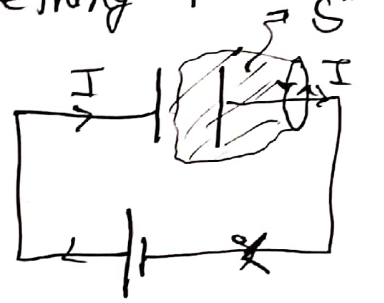
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Everything is nice and fine so far. Now take the bag sized surface. The current passing through the surface is still I, and Ampere's law stands still with dignity.




$$\therefore \oint \vec{B}\cdot d\vec{S} = \mu_0 I$$

Now, take the end of the bag sized surface in between the capacitor. Now, something peculiar happened. No current is passing through the surface.




$$\therefore \oint \vec{B}\cdot d\vec{S} = 0$$

That can't be true. Its the same system, as before. The line integral of $\vec{B}$ should not vary with our choice of surface. Something must be missing here.

In the region between the capacitor plates, there is no current for sure. But there is ~~obviously~~ a changing electric field, which develops due to the concentration of charges on the plates.

$$\text{Now,}\quad \frac{dQ}{dt} = \epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) = \epsilon_0 A \frac{d}{dt}\left(\frac{QA}{\epsilon_0}\right)$$

$$= \epsilon_0 A \frac{d\vec{E}}{dt}$$

Because, $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0}$

$$\therefore \frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt} = \epsilon_0 \cancel{A} \frac{d}{dt}(EA)$$

In general, $\frac{dQ}{dt} = \epsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{A} = \epsilon_0 \frac{d\Phi_E}{dt}$

This term is called the displacement current. But its not actually a current. It just is something that has unit of current. For historical reason, since Maxwell called it so, it is still used. We can add this displacement current to I, and Ampere's law now stands as,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \left( I + \epsilon_0 \frac{d\Phi_E}{dt} \right)$$

$$\therefore \boxed{\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}}$$

This equation now tells you that, along with a current that can generate a magnetic field, a changing electric field can also generate a magnetic field.

And, with that, we have our complete Maxwell equations.

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

$$\iint_S \vec{E}.d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

$$\iint_S \vec{B}.d\vec{A} = 0$$

$$\oint \vec{E}.d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B}.d\vec{s} = \mu_0 I + \mu_0\epsilon_0\frac{d\Phi_E}{dt}$$

Now, let's consider the source free Maxwell's equations. So, there will be no charge and currents. We have,

$$\vec{\nabla}.\vec{E} = 0 \qquad \vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (I)} \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \quad \text{--- (II)}$$

Taking the curl on both sides of equation (I) and (II) we get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\left(\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\cancel{\vec{\nabla}.\vec{E}})^{0} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow -\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\therefore \nabla^2\vec{E} = +\mu_0\epsilon_0\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\epsilon_0\,\vec{\nabla}\times\left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}(\cancel{\vec{\nabla}.\vec{B}})^{0} - \nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\Rightarrow -\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\Rightarrow \nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2 \vec{B}}{\partial t^2}$$

Now, these has the exact forms of three dimensional wave equations,

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

with velocity,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

And thus, the electromagnetic wave travels at the speed of ~~light~~ c, which now can be calculated to be 299, 792, 458 $ms^{-1}$. Later, the experiment by Fizeau, ~~pr~~ calculated the speed of light to be exactly this amount. Maxwell then wrote, " Light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena". The speed of light is exactly c, with which em wave travels.

With the fact that $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$, we can write,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial (c\vec{B})}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial (c\vec{B})}{\partial t} \text{ --- (I)} \qquad \vec{\nabla} \times (c\vec{B}) = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \text{ --- (II)}$$

You can see an apparent symmetry in equation (I) and (II) with $\vec{E}$ and $c\vec{B}$! Indeed, in the case of em wave, at each instance of time,

$$E = cB.$$