Chapter 6: Ordinary Differential Equations

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Reference Texts: *Mathematical Methods in the Physical Sciences* by Mary Boas; *Mathematical Methods for Physicists* by Arfken, Weber and Harris.

1 Ordinary Differential Equations in Physics

Consider a mass-spring system, similar to the one shown below in Figure 1. The equation of motion

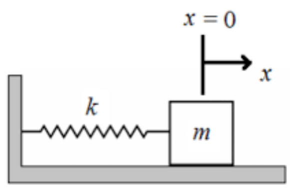


Figure 1: The Harmonic Oscillator

for such a system is a pretty straightforward result from general physics, i.e.,

$$\boxed{-kx = m\frac{d^2x}{dt^2}} \tag{1}$$

where $F_s = kx$ is the *restoring* (spring) force with the negative sign indicating that the force is in the opposite direction to the displacement, and m is the mass of the block. Once displaced from the equilibrium length, the system will oscillate indefinitely, considering the friction to be non-existent. Eq.(1) is an example of a **differential equation** since it contains derivatives; more specifically, this is an **ordinary differential equation** or ODE (only ordinary derivatives appear) which is of **second order** (the highest power of the derivative term is 2). Other examples of ODE's appearing in physics are

$$\begin{aligned}
&y' + xy^2 = 1, \qquad \text{first order } \textit{non-linear} \text{ ODE}\\
&xy' + y = e^x \qquad \text{first order } \textit{linear} \text{ ODE}\\
&\frac{dv}{dt} = -g \qquad \text{first order } \textit{linear} \text{ ODE}\\
&L\frac{dI}{dt} + IR = V \qquad \text{first order } \textit{linear} \text{ ODE}\ .
\end{aligned} \tag{2}$$

Notice that the non-linearity of the first equation of (2) is due to the y^2 term. All other equations are linear. Linear equations have the general form

$$a_0 y + a_1 y' + a_2 y'' + a_3 y''' + ... = b\ , \tag{3}$$

where a_i and b may be constant or functions of x. In physics, linear $n-$th order ODEs are plentiful and thus is a primary focus of this chapter.

Example 1. Consider the ODE $y' = \cos x$. We can guess the solution to this very easily since the **anti-derivative** of y is known. Thus, we have

$$y = \sin x + c$$

where c is the constant of integration. If we had a second order ODE, we would have two constants of integration.

In general, the solution for an $n-$th order ODE will contain n independent constants. These solutions, valid up to the addition of an arbitrary constant, are known as **general solutions** of the ODE in question. Often though, interesting solutions emerge when some **boundary conditions**, for constrained endpoints, are given; these may also be constraints set at $t = 0$, in which case they are called **initial conditions**.

Example 2. Find the distance covered by an object in free fall, in t seconds, if it starts from rest.

Equation of motion for the object is

$$\frac{d^2x}{dt^2} = g$$

and we want to determine $x(t)$. Thus,

$$\int \frac{d^2x}{dt^2} dt = g \int dt \Rightarrow \int \frac{dx}{dt} dt = \int (gt + v_0)\ dt \Rightarrow \boxed{x(t) = \frac{1}{2} gt^2 + v_0 t + x_0}$$

is the *general solution* to the free fall equation of motion. Given that the object starts (at $t = 0$) at rest (i.e. $v_0 = 0$) which yields

$$x(t = 0) = 0 = x_0$$

which gives the *particular solution*

$$\boxed{x(t) = \frac{1}{2} gt^2}\ .$$

2 Different Types of First-Order Equations and Methods to Solve Them

2.1 Separable Equations

Consider the integral,

$$\int f(x)\ dx\ ; -$$

notice that this is equivalent to the ODE

$$\frac{dy}{dx} = f(x)\ .$$

This is quite easy to solve by simply making some rearrangements as

$$dy = f(x)\ dx \tag{4}$$

where the variable y and x can be separately integrated. Equations of the form (4) are thus called **separable equations** which can be solved by integrating both sides independently.

Example 3. Solve the ODE: $xy' = y + 1$.

$$\begin{aligned}
& x\frac{dy}{dx}dx = (y+1)dx \\
& \Rightarrow x\ dy = (y+1)\ dx \\
& \Rightarrow \int \frac{dy}{y+1} = \int \frac{dx}{x} \\
& \Rightarrow \ln(y+1) = \ln x + \ln c \\
& \Rightarrow y + 1 = xc
\end{aligned} \tag{5}$$

which is the general solution of the given ODE. This can be graphed on the $xy-$plane to represent a family of curves, which in this context, will be straight lines through $(0, -1)$. Physically these lines could be electric field lines emanating radially from a positive charge at that point. Then, we know from elementary electrostatics, that one can draw equipotential surfaces around the charge which will look like circles (on a 2D plane) centered on $(0, -1)$. This is displayed in Figure 2.

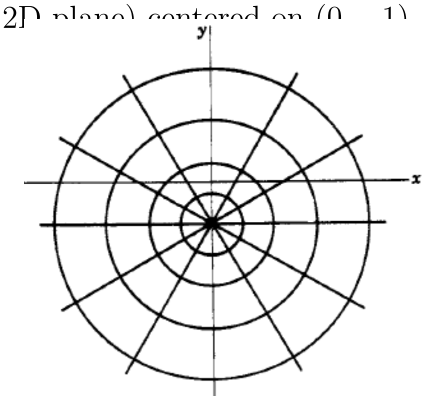


Figure 2: Family of curves displaying general solutions of the ODE given. Physically these may represent lines of electric field for a positive charge at $(0, -1)$. The circles are equipotential surfaces.

But why do they have to be circles? To answer this, we realize that the slope of any straight line in the graph is (using (5))

$$y' = c$$

and a rearrangement of (5) yields

$$c = \frac{y+1}{x} = y' \ .$$

At this point, recall that equipotential lines are perpendicular to the electric field lines. As such, electric field lines and equipotential surfaces are said to be **orthogonal trajectories**. To get further along in the discussion, we quote a general conclusion from analytic geometry that **the slope of two perpendicular lines are negative reciprocals of each other.** Using this, we determine that the slope of any equipotential surface to be

$$\begin{aligned}
& y' = -\frac{x}{y+1} \\
& \Rightarrow \frac{dy}{dx} = -\frac{x}{y+1} \\
& \Rightarrow \int (y+1)\ dy = -\int x\ dx \\
& \Rightarrow x^2 + (y+1)^2 = c_2 \ ,
\end{aligned} \tag{6}$$

that one recognizes to be the equation of a circle, with origin at $(0,-1)$ and radius c_2! Thus, we have successfully connected mathematics to physical phenomena, as a physicist should always do.

2.2 Linear First Order ODEs

Linear first order ODEs have the general form

$$y' + Py = Q \ , \tag{7}$$

where $P \equiv P(x),\ Q \equiv Q(x)$. For simplicity, let us assume that $Q(x) = 0$, in which case (7) becomes

$$\boxed{y' + Py = 0} \ , \tag{8}$$

which is a first order, linear, homogeneous, ODE. From (8) then we can get,

$$\begin{aligned}
&\frac{dy}{dx} = -Py \\
&\Rightarrow \ln y = -\int P(x)\ dx + c \\
&\Rightarrow y = e^{-\int P(x)\ dx + c} \\
&\Rightarrow y = e^c\ e^{-\int P(x)\ dx}; \qquad \text{define: } I \equiv \int P(x)\ dx\ ,\ A \equiv e^c \\
&\Rightarrow ye^{I(x)} = A \qquad \text{(differentiate w.r.t. } x) \\
&\Rightarrow \frac{dy}{dx}e^I + y\frac{d}{dx}(e^{I(x)}) = 0 \\
&\Rightarrow y'e^I + y\Big\{e^I\ \underbrace{\frac{dI}{dx}}_{=P}\Big\} = 0 \\
&\Rightarrow e^I(y' + Py) = 0\ ,
\end{aligned} \tag{9}$$

where e^I is an integrating factor and the term in parenthesis is recognized as the left-hand side of (8). This can be generalized to include **inhomogeneous** form of (7) such that (9) gets modified to

$$\begin{aligned}
&e^I(y' + Py) = Qe^I \\
\Rightarrow\ &\frac{d}{dx}(ye^I) = Qe^I \\
\Rightarrow\ &ye^I = \int Qe^I\ dx + c \\
\Rightarrow\ &\boxed{y = ce^{-I} + e^{-I}\int Qe^I\ dx}
\end{aligned} \tag{10}$$

which is the general solution of the first order, linear, inhomogeneous ODE of the form given by (7). The term ce^{-I} is then the solution to the homogeneous equation (9). The second term of (10) is one particular solution to (8).

Example 4. Solve the ODE: $(1 + x^2)y' + 6xy = 2x$.

Owing to the maximum power of y being 1, the ODE in question is linear, and to the first order. We want to employ the recipe above to solve this ODE. As such, let us recast the given equation in the form of (7). Thus, we have

$$y' + \underbrace{\frac{6x}{1+x^2}}_{=P} y = \underbrace{\frac{2x}{1+x^2}}_{=Q}$$

from which we identify

$$I \equiv \int P\ dx = \int \frac{6x}{1+x^2}\ dx = 3\ln(1+x^2)$$

where the last equality was a result of using a simple substitution procedure to solve the integral (left as an exercise to the reader). Then we have

$$e^I = e^{3\ln(1+x^2)} = e^{\ln(1+x^2)^3} \Rightarrow e^I = (1+x^2)^3\ .$$

Finally, using the general form of the solution given by (10) yields

$$ye^I = \int \frac{2x}{1+x^2}(1+x^2)^3\ dx = 2\int x(1+x^2)^2\ dx$$

$$\Rightarrow ye^I = \frac{1}{3}(1+x^2)^3 + c \Rightarrow \boxed{y = \frac{1}{3} + \frac{c}{(1+x^2)^3}}\ .$$

One encounters such equations and their solutions in many physics applications. A particular example is found in radioactive decay of nuclei, worked out in the book by Boas (Chapter 8, Section 3, Example 2).

2.3 Exact Equations

The differential equation

$$P(x,y)dx + Q(x,y)dy = 0 \tag{11}$$

is said to be an **exact** equation if there exists a function $\varphi(x,y)$ such that

$$P = \frac{\partial\varphi}{\partial x},\ Q = \frac{\partial\varphi}{\partial y}\ \Rightarrow P\ dx + Q\ dy = d\varphi\ = 0, \tag{12}$$

the solution for which is $\varphi(x,y) =$ constant. To determine if the function $\varphi(x,y)$ exists, we differentiate the expressions for P and Q in (12) as,

$$\frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial P(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

and these are consistent if and only if

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}\ . \tag{13}$$

One can always multiply a **non-exact** equation with an appropriate factor to make it an exact one. Also of note here is the fact that, *separability* and *exactness* are independent characteristics such that: **all separable ODEs are automatically exact, but not all exact ODE's are separable.**

Example 5. A Non-separable Exact ODE.

Consider the ODE: $$y' + \left(1 + \frac{y}{x}\right) = 0.$$

We want to recast it into the form for an exact equation. This is done by,

$$xdx\left[y' + \left(1 + \frac{y}{x}\right)\right] = 0$$

which leads to

$$x\ dx\ \frac{dy}{dx} + x\ dx + y\ dx = 0 \Rightarrow (x+y)\ dx + x\ dy = 0$$

where we can identify from (11) that

$$P(x,y) = x + y, \text{ and } Q(x,y) = x\ .$$

This equation is **not** separable but is it exact? To check this, we need to show that (13) is satisfied. Thus,

$$\frac{\partial P}{\partial y} = \frac{\partial(x+y)}{\partial y} = 1\ , \qquad \frac{\partial Q}{\partial x} = \frac{\partial(x)}{\partial x} = 1$$

which shows that the ODE is an exact equation! The solution, $\varphi(x,y)$ will be a constant and can be found as,

$$\begin{aligned}
\varphi &= \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy \\
&= \int_{x_0}^{x} (x+y)\ dx + \int_{y_0}^{y} x_0\ dy \\
&= \left[\frac{x^2}{2} + xy - \frac{x_0^2}{2} - x_0 y\right] + (x_0 y - x_0 y_0) \\
&= \frac{x^2}{2} + xy + \text{ constant terms} \\
&= \frac{x^2}{2} + xy + = C \quad \text{is the solution, as expected.}
\end{aligned}$$

2.4 Homogeneous Equations

An ODE is termed as a *homogeneous equation* (of order n) in x and y if the combined powers of x and y add to n in all the terms of $P(x,y)$ and $Q(x,y)$ when the equation is written in the form of (11). Another way we interpreted the term *homogeneous* was in the context of linear ODEs with the right-hand side set to 0, which is a bit different from the current context. For instance, the ODE $x^3\ dx + xy^2\ dy = 0$ is a homogeneous equation of order 3 (i.e. $n = 3$) since the combined powers of x and y in the ODE add up to 3 in all the terms. Rewriting (11) we have,

$$Q(x,y)dy = -P(x,y)dx \Rightarrow Q(x,y)\frac{dy}{dx} = -P(x,y)\frac{dx}{dx} \Rightarrow \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)} \equiv f\left(\frac{y}{x}\right)\ ,$$

from which we can conclude that a differential equation is homogeneous if it can be written as $y' =$ a function of y/x. Thus if we make a change of variables such as

$$v = \frac{y}{x} \Rightarrow y = xv\ ,$$

it yields a separable equation which can then be easily solved.

2.5 Isobaric Equations

An isobaric equation is a generalization of a homogeneous equation; before, equal *weights* were assigned to the order of x and y. In this case one modifies the definition of homogeneity by assigning different weights to x and y (and their corresponding differentials). If assigning unit weight to each x, dx and a weight m to each y, dy makes the ODE homogeneous, then the substitution $y = x^m v$ will lead to a separable equation. Let's examine this by looking at an example.

Example 6. An Isobaric ODE

Consider an isobaric ODE

$$(x^2 - y)\ dx + x\ dy = 0\ .$$

Let us assign a weight of 1 to each x and dx while to y and dy we assign a weight m each. Then,

$$\underbrace{x^2\ dx}_{w=3} - \underbrace{y\ dx}_{w=m+1} + \underbrace{x\ dy}_{w=1+m} = 0$$

and since each term must have the same overall weight we have,

$$3 = 1 + m \Rightarrow m = 2\ .$$

This means, to solve the given ODE we need to use the substitution $y = x^2 v$. This leads to

$$(1 + v)\ dx + x\ dv = 0$$

which can be separated to yield

$$\frac{dx}{x} = -\frac{dv}{1+v} \Rightarrow \ln x + \ln(1+v) = \ln c \Rightarrow x(1+v) = c\ .$$

From this we get,

$$v = \frac{c}{x} - 1 \rightarrow y = x^2 v = x^2\left(\frac{c}{x} - 1\right) \Rightarrow \boxed{y = cx - x^2}$$

is the solution to the ODE.

2.6 ODEs with Constant Coefficients

Another frequently occurring class of ODEs that aren't constrained to be of specific order, are linear and whose homogeneous terms have constant coefficients. The generic form of this type of ODEs is,

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = F(x)\ . \tag{14}$$

The homogeneous equation corresponding to (14) (i.e. with $F(x) = 0$) has solutions of the form

$$\begin{aligned} &y = e^{mx} \\ &\text{where, } m \text{ is a solution of the algebraic equation,} \\ &m^n + a_{n-1}m^{n-1} + ... + a_1 m + a_0 = 0\ . \end{aligned} \tag{15}$$

Exercise 1. Apply the method above to find the general solution to the ODE

$$M\frac{d^2y}{dt^2} = -ky\ .$$

3 Second Order Linear ODEs

3.1 Homogeneous Second Order Linear ODEs

A homogeneous, second order linear equation has the form

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0 \ , \tag{16}$$

where a_0, a_1, a_2 are constants. Let's study an example of such an equation.

Example 7. Solve the equation $y'' + 5y' + 4y = 0$.

We conveniently define $D \equiv d/dx$ which helps rewrite the given ODE as

$$D^2y + 5Dy + 4y = 0$$
$$\Rightarrow (D^2 + 5D + 4)y = 0$$
$$\Rightarrow (D+1)(D+4)y = 0$$

which leads to two equations

$$(D+1)y = 0 \quad \& \quad (D+4)y = 0$$

which gives,

$$Dy = -y \Rightarrow \frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -dx \Rightarrow \ln y + \ln c_1 = -x \Rightarrow y = c_1 e^{-x}$$

or similarly for the other equation we get,

$$y = c_2 e^{-4x} \ .$$

Thus, the general solution fir the given ODE is

$$y = c_1 e^{-x} + c_2 e^{-4x}$$

for the two linearly independent solutions. As such, we can consider e^{-x} and e^{-4x} to be basis vectors for 2D linear vector space and the general solution gives all the vectors of that space!

From line 2 of reorganizing of the ODE with the definition $D \equiv \frac{d}{dx}$, we have the equation of algebraic form

$$\boxed{D^2 + 5D = 4 = 0} \Rightarrow D = -1 \text{ and } -4 \text{ are the roots.}$$

The boxed equation is known as the **auxiliary equation** of the given ODE.

Thus, in the case when the roots, a & b, of the general auxiliary equation $(D-a)(D-b)y = 0$, for $a \neq b$, the general solution of the differential equation in question is a linear combination of e^{ax} and e^{bx}, i.e.

$$\boxed{y = c_1 e^{ax} + c_2 e^{bx} \text{ is the general solution of } (D-a)(D-b)y = 0,\ a \neq b \ .} \tag{17}$$

3.1.1 Complex Conjugate Roots of the Auxiliary Equation

If the roots of the auxiliary equation are distinct (unequal) and complex, i.e. $\alpha \pm i\beta$, the general solution of the differential equation under scrutiny is

$$\boxed{y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})\ .} \tag{18}$$

Using Euler's formula for complex numbers, i.e. $e^{\pm i\beta x} = \cos\beta x \pm i \sin\beta x$, one can also express (18) as

$$\boxed{y = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)\ ,} \tag{19}$$

where c_1, c_2 are arbitrary constants. Using trigonometric identities, (19) can also be written as

$$\boxed{y = ce^{\alpha x}\sin(\beta x + \gamma).} \tag{20}$$

3.1.2 Equal Roots of Auxiliary Equation

For the case of equal roots of the auxiliary equation, i.e. $a = b$, we rewrite the differential equation as

$$(D - a)(D - a)y = 0\ , \tag{21}$$

for which one solution is as before $y = c_1 e^{ax}$ but the second solution is not simply $y = c_2 e^{bx}$ since $a = b$. To determine the second solution, we use a substitution such that

$$u = (D - a)y \tag{22}$$

and plugging this in (21) yields

$$(D - a)u = 0 \tag{23}$$

from which we have the solution for u that

$$u = Ae^{ax}\ . \tag{24}$$

To yield the solution for y we can substitute (24) into (22) to get,

$$(D - a)y = Ae^{ax} \Rightarrow Dy - ay = Ae^{ax} \Rightarrow y' - ay = Ae^{ax} \tag{25}$$

which one identifies as a linear, first order ODE with the general solution of the form given by (10). As such, for unequal or repeated roots of the auxiliary equation, the general solution for (21) is

$$\boxed{y = (Ax + B)e^{ax}\ .} \tag{26}$$

Exercise 2. Verify Eq.(26) by solving the ODE in (25).

3.2 Inhomogeneous Second Order Linear ODEs

Equations of the form (16) (with zero on the right-hand side) describe **free oscillations** of mechanical or electrical systems, for instance, a spring-mass system allowed to oscillate on a table with no friction. Physically realistic systems however, often experience external forces which effect their motion, for instance friction in the previous example will lead to a *damping* of the oscillations.

Other external agents may be responsible for continued oscillation to counteract the damping and keep the amplitude of oscillation constant. These vibrations are termed as **forced oscillations** and the ODE describing them is of the form

$$
\begin{aligned}
a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = f(x) \\
\Rightarrow \boxed{\frac{d^2y}{dx^2} + \frac{a_1}{a_2}\frac{dy}{dx} + \frac{a_0}{a_2}y = F(x) \ ,}
\end{aligned} \tag{27}
$$

where $f(x)$ is known as the **forcing function** or sometimes also as the *source*. Clearly, the difference between (16) and (27) is the source term. As such, it is not far fetched to guess that the solution to (27) will be related to that of (16). In fact, the general solution to (27) is of the form

$$\boxed{y = y_c + y_p} \tag{28}$$

where y_c is the **complementary function** and the general solution to (16) while y_p is a **particular solution** of (27).

Example 8. Forced Oscillations: Consider a *real* $1D$ harmonic oscillator described by the ODE

$$-kx^2 + F_f = m\frac{d^2x}{dt^2}$$

where F_f is the frictional (or drag) force. Generally, for our purpose, we can safely consider the friction to be related to the velocity of the system as

$$F_f = -b\frac{dx}{dt}; \ b \text{ is a positive proportionality constant} \tag{29}$$

which leads to the ODE describing a **damped** harmonic oscillator,

$$\boxed{\frac{d^2x}{dt^2} + \gamma^2\frac{dx}{dt} + \omega^2 x = 0 \ ,} \tag{30}$$

where we have defined the variables $\gamma = \frac{b}{m}$ and $\omega = \frac{k}{m}$. The effect of damping is such that the amplitude will go down over time, ultimately leading to the vibration dying out. To counteract this, we can introduce a **periodic force** to force the oscillation to continue such that (31) assumes the slightly altered form

$$\boxed{\frac{d^2x}{dt^2} + \gamma^2\frac{dx}{dt} + \omega^2 x = F_0\cos\omega t} \ , \tag{31}$$

where the nonzero right hand side indicates an inhomogeneous ODE with a source term, for instance in this example we used, $F_0\cos\omega t$. (31) can then be solved by the method of exponentials (or other suitable method of choice).

3.2.1 Ways to Determine Particular Solutions

1. **By Inspection**: for simple looking ODE's, this works brilliantly. For instance, consider the ODE

$$y'' - 6y' + 9y = 8e^x$$

, it is easy to guess that the solution will be some multiple of e^x since that is what we have on the right hand side. Clearly, $y = 2e^x$ leads to the left hand side giving

$$2e^x - 6(2e^x) + 9(2e^x) = 8e^x \ ,$$

which is what is required to satisfy the equation. However, this method will not work for more complicated ODEs.

2. **Successive Integration of Two First-Order Equations**: let's look at how this works using an example.

   Example 9. We want to solve the ODE

$$y'' + y' - 2y = e^x \ .$$

   As a first step let's rewrite the equation as

$$(D^2 + D - 2)y = e^x \Rightarrow (D-1)(D+2)y = e^x \ .$$

   Then, set

$$u = (D+2)y$$

   which leads to

$$\begin{aligned} &(D-1)u = e^x \\ \Rightarrow\ &Du - u = e^x \\ \Rightarrow\ &u' - u = e^x \ , \end{aligned}$$

   which you should recognize is in the form of (7) and can be solved accordingly to yield the general solution

$$u = xe^x + c_1 e^x$$

   and replacing the substitution in y gives the ODE

$$(D+2)y = xe^x + c_1 e^x \Rightarrow y' + 2y = xe^x + c_1 e^x$$

   which is again a first-order linear ODE in the form of (7) and can be solved analogously as before. Finally, this yields the general solution for the original ODE in the question:

$$y = \frac{1}{3}xe^{3x} + c_1' + c_2 e^{-2x}$$

   where c_1', c_2 are constants of integration $-$ two constants due to the second-order nature of the ODE in question.

   In fact, for equations of the form where there is an exponential on the RHS, the particular solution can be assumed to have the following forms:

$$\boxed{y_p = \begin{cases} Ce^{cx}; c \text{ not equal to either } a \text{ or } b; \\ Cxe^{cx};\ \text{if } c \text{ equals } a \text{ or } b,\ a \neq b; \\ Cx^2e^{cx}; \text{if } c = a = b \ . \end{cases}} \tag{32}$$

3. **Use of Complex Exponentials**: Again, we take the help of an example and learn from it.

Example 10. Solve: $y'' + y' - 2y = 4\sin 2x$.

The trick lies in realizing that $e^{2ix} = \cos 2x + i\sin 2x$ and solving for a different variable Y such that

$$Y'' + Y' - 2Y = 4e^{2ix}$$

then if $Y = Y_R + iY_I$, the above equation is equivalent to the **two** equations:

$$\begin{aligned} Y_R'' + Y_R' - 2Y_R &= \text{Re }(4e^{2ix}) = 4\cos 2x \ , \\ Y_I'' + Y_I' - 2Y_I &= \text{Im }(4e^{2ix}) = 4\sin 2x \ . \end{aligned}$$

Immediately we notice that the second equation above is exactly the equation we were asked to solve. Thus, we realize that the solution of the the given ODE is the imaginary part of Y. Thus, $y_p = \text{Im }(Y_p)$. From the discussion in point 2, it is easy to assume a solution of the form

$$Y_p = Ce^{2ix}$$

and plugging this into the auxiliary equation for Y gives

$$\begin{aligned} &(-4 + 2i - 2)Ce^{2ix} = 4e^{2ix} \ , \\ &\Rightarrow C = \frac{4}{2i-6} = -\frac{1}{5}(i+3) \ , \\ &\Rightarrow Y_p = -\frac{1}{5}(i+3)e^{2ix} \ . \end{aligned}$$

Thus,

$$y_p = \text{Im }(Y_p) = \text{Im }\left\{ -\frac{1}{5}(i+3)e^{2ix} \right\} = \text{Im }\left\{ -\frac{1}{5}(i+3)(\cos 2x + i\sin 2x) \right\}$$

$$\Rightarrow \boxed{y_p = -\frac{1}{5}\cos 2x - \frac{3}{5}\sin 2x}$$

is the general solution to the given ODE.

4. **Method of Undetermined Coefficients**: A more general version of the discussion in points 2 and 3 concerns the inclusion of a polynomial multiplied to the exponential on the RHS. Specifically, for an auxiliary equation of the form

$$(D-a)(D-b)y = e^{cx}P_n(x) \tag{33}$$

where $P_n(x)$ is a polynomial of degree n, the particular solution is of the form

$$\boxed{y_p = \begin{cases} e^{cx}Q_n(x); c \text{ not equal to either } a \text{ or } b; \\ xe^{cx}Q_n(x); \text{ if } c \text{ equals } a \text{ or } b, \ a \neq b; \\ x^2e^{cx}Q_n(x); \text{if } c = a = b \ , \end{cases}} \tag{34}$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$, with *undetermined coefficients* to be found to satisfy the given ODE.

Example 11. Find the particular solution of

$$(D-1)(D+2)y = y'' + y' - 2y = 18xe^x \ .$$

In the notation of (33), we identify $a = 1, \ b = -2, \ c = 1, \ P_{n=1}(x) = 18x$. This means $Q_n(x) \equiv Q_1(x) = Ax + B$. Since, $c = a \neq b$, then by (34) we find,

$$y_p = xe^x(Ax + B) = e^x(Ax^2 + Bx)$$

from where we get

$$y_p' = e^x(Ax^2 + Bx + 2Ax + B) \ ,$$

$$y_p'' = e^x(Ax^2 + Bx + 4Ax + 2B + 2A) \ ,$$

and

$$y_p'' + y_p' - 2y_p = e^x(6Ax + 3B + 2A) = 18xe^x \ .$$

Comparison of coefficients very simply leads to $A = 3, B = -2$ and thus we have

$$y_p = (3x^2 - 2x)e^x \ ,$$

is the particular solution to the equation we started with.

5. **Superposition Principle**: Suppose we have a linear second order ODE

$$y'' + y' - 2y = e^x + 4\sin 2x + (x^2 - x)$$

which can be simplified by introducing operator notation such that,

$$(D-1)(D+2)y = e^x + 4\sin 2x + (x^2 - x)$$

and one immediately notices several terms on the right hand side. In the case of a **linear ODE** we can simply solve the equation for each individual term on the right hand side and add up the individual particular solutions to yield the full solution to the given ODE. In the present case we identify,

$$(D-1)(D+2)y = e^x \quad \text{has particular solution} \quad y_{p_1} = \frac{1}{3}xe^x \ ,$$

$$(D-1)(D+2)y = 4\sin 2x \quad \text{has particular solution} \quad y_{p_2} = -\frac{1}{5}(\cos 2x - 3\sin 2x)$$

and

$$(D-1)(D+2)y = (x^2 - x) \quad \text{has particular solution} \quad y_{p_3} = -\frac{1}{2}(x^2 + 1)$$

which leads to the complete solution by the *superposition principle*

$$y_p = \frac{1}{3}xe^x - \frac{1}{5}(\cos 2x - 3\sin 2x) - \frac{1}{2}(x^2 + 1)$$

4 Laplace Transform and its Applications to Solving ODEs

This section concerns a type of *integral transform* known as the *Laplace Transform* (L.T) which transforms a function $f(t)$ into another function $F(p)$ defined by

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-pt} \equiv F(p)} \ . \tag{35}$$

Our aim here is to ultimately use L.T's to solve linear ODEs. Let us do some examples to see how this works.

Example 12. For $f(t) = e^{at}$ determine $\mathcal{L}\{f(t)\}$.

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-pt}dt$$

$$\Rightarrow F(p) = \int_0^\infty e^{(a-p)t}dt = \left.\frac{e^{(a-p)t}}{a-p}\right|_0^\infty$$

now, this integral diverges in the case when $a \geq p$ and so we only care about the case $a < p$:

$$\Rightarrow \boxed{F(p) = -\frac{1}{a-p}} \ .$$

Example 13. The Heaviside step function is defined as

$$\Theta(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0; \end{cases} \tag{36}$$

We want to determine the

$$\mathcal{L}\{(\Theta(t-a))\}$$

which is a shifted step function, translated by an amount a along the $t-$axis.

$$F(p) = \int_0^\infty \Theta(t-a)e^{-pt}dt$$

$$\Rightarrow \int_a^\infty 1 \cdot e^{-pt}dt$$

$$\Rightarrow \boxed{F(p) = \frac{e^{-ap}}{p}} \ ,$$

where one notices the lower limit of the integral to be a; this is due to the fact that everywhere below $t = a$, $\Theta(t) = 0$, excluding the need to integrate over that region.

4.1 Properties of Laplace Transforms

1. Laplace transformation is a **linear operator** i.e.

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \text{ (linearity under addition)}$$

and

$$\mathcal{L}[cf(t)] = c\mathcal{L}[f(t)] \ .$$

2. **Existence Theorem**: if $f(t)$ is continuous and of *exponential order* with constant c, then

$$F(p) = \mathcal{L}\{\{(\sqcup)\}$$

exists for all $p > c$.

Aside: $f(t)$ is said to be of exponential order if

$$|f(t)| \leq ke^{ct}$$

for large values of t, for some constant k and c.

3. The inverse Laplace transform of a function $F(p)$

$$f(t) = \mathcal{L}^{-1}[f(t)]$$

yields a unique function $f(t)$.

Exercise 3. Show that $\mathcal{L}(e^{iat}) = \mathcal{L}(\cos at + i \sin at) = \mathcal{L}(\cos at) + i\mathcal{L}(\sin at)$.

4.2 Transformation of Derivatives under Laplace Transforms

As mentioned before, we would like to implement Laplace Transforms in solving second order linear ODEs. As such, we would need to how derivatives of order n, i.e. $y' = \frac{dy}{dt}, y'' = \frac{d^2y}{dx^2}, ...$ change under L.T.

$$\begin{aligned}
&\mathcal{L}(y'(t)) = \int_0^\infty y'(t)e^{-pt}dt \\
&\text{implement integration by parts with,} \\
&u = e^{-pt} \Rightarrow \frac{du}{dt} = -pe^{-pt}, \frac{dv}{dt} = y' \Rightarrow v = y(t) \\
&\Rightarrow \mathcal{L}(y'(t)) = -\underbrace{y(0)}_{y_0} + p\underbrace{\int_0^\infty y(t)e^{-pt}dt}_{\mathcal{L}(y(t))\equiv Y} \\
&\Rightarrow \boxed{\mathcal{L}(y'(t)) = pY - y_0} .
\end{aligned} \tag{37}$$

Similarly for the second derivative, realize that

$$\begin{aligned}
&y'' = (y')' \\
&\Rightarrow \mathcal{L}(y'') = p \underbrace{\mathcal{L}(y')}_{\text{plug in (37)}} - \underbrace{y'(0)}_{=y_0'} \\
&\Rightarrow \boxed{\mathcal{L}(y'') = p^2Y - y_0' - py_0} .
\end{aligned} \tag{38}$$

Example 14. Solve the ODE

$$y'' + 4y' + 4y = t^2e^{-2t}$$

with given initial conditions $y_0 = 0,\ y' = 0$ using Laplace Transform method.

Start by taking the L.T. of each of the term in the equation such that we have,

$\mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(t^2 e^{-2t})$

use (37) and (38) and implement initial conditions to yield :

$$\Rightarrow p^2 Y + 4pY + 4Y = \int_0^\infty t^2 e^{-(2+p)t}$$

the RHS is easy to work out by doing integration by parts twice. Thus we have,

$$\Rightarrow (p^2 + 4p + 4)Y = \frac{2}{(2+p)^3} \tag{39}$$

$$\Rightarrow (p+2)^2 Y = \frac{2}{(2+p)^3}$$

$$\Rightarrow \boxed{Y = \frac{2}{(p+2)^5}} .$$

But we have only solved for the transformed variable, Y, and still have to solve for the variable of interest y in the ODE. So,

$$\begin{aligned} Y \equiv \mathcal{L}(y) &= \frac{2}{(p+2)^5} \\ \Rightarrow y &= \mathcal{L}^{-1}\left[\underbrace{\frac{2}{(p+2)^5}}_{\tilde{F}(p)}\right] . \end{aligned} \tag{40}$$

Carrying out such inverse transformations is not so easy, but thankfully we have access to computers and L.T. tables. Such a table is listed in Figure 3. From the table one notes the function $f(t)$ in the first column that most closely resembles the source term in the given ODE. As such, we identify $L6$, where $f(t) = t^k e^{-at}$ for which the transformed function Y is

$$\begin{aligned} F(p) &= \frac{k!}{(p+a)^{k+1}} \text{ with } k = 4,\ a = 2 \\ \Rightarrow F(p) &= \frac{4!}{(p+2)^5} = 12\underbrace{\frac{2}{(p+2)^5}}_{\tilde{F}(p)} \\ \Rightarrow \tilde{F}(p) &= \frac{F(p)}{12} . \end{aligned} \tag{41}$$

This leads to,

$$\begin{aligned} y &= \mathcal{L}^{-1}(\tilde{F}(p)) \\ \Rightarrow y &= \mathcal{L}^{-1}(\frac{F(p)}{12}) \\ \Rightarrow y &= \frac{1}{12}\underbrace{\mathcal{L}^{-1}(F(p))}_{f(t)=t^4e^{-2t}} \\ \Rightarrow & \boxed{y = \frac{t^4 e^{-2t}}{12}} , \end{aligned} \tag{42}$$

which is the particular solution we need! An incredible feature of using L.T's to solve ODEs is that we do not need to find integration constants and general solutions but rather we are directly led to the particular solution which satisfies the given boundary (or initial) conditions.

Table of Laplace Transforms

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
L1	1	$\frac{1}{p}$	$\text{Re } p > 0$
L2	e^{-at}	$\frac{1}{p+a}$	$\text{Re } (p+a) > 0$
L3	$\sin at$	$\frac{a}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L4	$\cos at$	$\frac{p}{p^2+a^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L5	$t^k,\ k > -1$	$\frac{k!}{p^{k+1}}$ or $\frac{\Gamma(k+1)}{p^{k+1}}$	$\text{Re } p > 0$
L6	$t^k e^{-at},\ k > -1$	$\frac{k!}{(p+a)^{k+1}}$ or $\frac{\Gamma(k+1)}{(p+a)^{k+1}}$	$\text{Re } (p+a) > 0$
L7	$\frac{e^{-at} - e^{-bt}}{b-a}$	$\frac{1}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
L8	$\frac{ae^{-at} - be^{-bt}}{a-b}$	$\frac{p}{(p+a)(p+b)}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
L9	$\sinh at$	$\frac{a}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
L10	$\cosh at$	$\frac{p}{p^2-a^2}$	$\text{Re } p > \lvert \text{Re } a \rvert$
L11	$t \sin at$	$\frac{2ap}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L12	$t \cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L13	$e^{-at} \sin bt$	$\frac{b}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
L14	$e^{-at} \cos bt$	$\frac{p+a}{(p+a)^2+b^2}$	$\text{Re } (p+a) > \lvert \text{Im } b \rvert$
L15	$1 - \cos at$	$\frac{a^2}{p(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L16	$at - \sin at$	$\frac{a^3}{p^2(p^2+a^2)}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L17	$\sin at - at \cos at$	$\frac{2a^3}{(p^2+a^2)^2}$	$\text{Re } p > \lvert \text{Im } a \rvert$

Table of Laplace Transforms (continued)

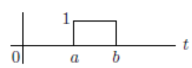
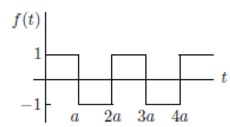
	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
L18	$e^{-at}(1 - at)$	$\frac{p}{(p+a)^2}$	$\text{Re } (p+a) > 0$
L19	$\frac{\sin at}{t}$	$\arctan \frac{a}{p}$	$\text{Re } p > \lvert \text{Im } a \rvert$
L20	$\frac{1}{t} \sin at \cos bt,$ $a > 0,\ b > 0$	$\frac{1}{2}\left(\arctan \frac{a+b}{p} + \arctan \frac{a-b}{p}\right)$	$\text{Re } p > 0$
L21	$\frac{e^{-at} - e^{-bt}}{t}$	$\ln \frac{p+b}{p+a}$	$\text{Re } (p+a) > 0$ $\text{Re } (p+b) > 0$
L22	$1 - \text{erf}\left(\frac{a}{2\sqrt{t}}\right),\quad a > 0$ (See Chapter 11, Section 9)	$\frac{1}{p} e^{-a\sqrt{p}}$	$\text{Re } p > 0$
L23	$J_0(at)$ (See Chapter 12, Section 12)	$(p^2+a^2)^{-1/2}$	$\text{Re } p > \lvert \text{Im } a \rvert$; or $\text{Re } p \geq 0$ for real $a \neq 0$
L24	$u(t-a) = \begin{cases} 1, & t > a > 0 \\ 0, & t < a \end{cases}$ (unit step, or Heaviside function)	$\frac{1}{p} e^{-pa}$	$\text{Re } p > 0$
L25	$f(t) = u(t-a) - u(t-b)$ 	$\frac{e^{-ap} - e^{-bp}}{p}$	All p
L26		$\frac{1}{p} \tanh\left(\frac{1}{2}ap\right)$	$\text{Re } p > 0$
L27	$\delta(t-a),\ a \geq 0$ (See Section 11)	e^{-pa}	
L28	$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$ $= g(t-a)u(t-a)$	$e^{-pa}G(p)$ [$G(p)$ means $L(g)$.]	
L29	$e^{-at}g(t)$	$G(p+a)$	

Table of Laplace Transforms (continued)

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$
L30	$g(at),\ a > 0$	$\frac{1}{a} G\left(\frac{p}{a}\right)$
L31	$\frac{g(t)}{t}$ (if integrable)	$\int_p^\infty G(u)\,du$
L32	$t^n g(t)$	$(-1)^n \frac{d^n G(p)}{dp^n}$
L33	$\int_0^t g(\tau)\,d\tau$	$\frac{1}{p} G(p)$
L34	$\int_0^t g(t-\tau)h(\tau)\,d\tau = \int_0^t g(\tau)h(t-\tau)\,d\tau$ (convolution of g and h, often written as $g * h$; see Section 10)	$G(p)H(p)$
L35	Transforms of derivatives of y (see Section 9): $L(y') = pY - y_0$ $L(y'') = p^2Y - py_0 - y_0'$ $L(y''') = p^3Y - p^2y_0 - py_0' - y_0''$, etc. $L(y^{(n)}) = p^nY - p^{n-1}y_0 - p^{n-2}y_0' - \cdots - y_0^{(n-1)}$	

Figure 3: Table of Laplace Transforms and their Inverse Transformations (from Boas)

5 Green's Function

Consider the linear, homogeneous equation,

$$Lu(x) = f(x) \tag{43}$$

where $u,\ f$ are functions over some domain Ω and L is a linear differential operator. It is often the case that differential operators have associated inverses, such that one might expect a solution to (43) to be of the form

$$u(x) = \int_\Omega G(x, x')f(x')dx' \tag{44}$$

where the *kernel* of the integral operator is called the **Green's function,** $G(x, x')$, after the British mathematician George Green, who pioneered this method of solving ODEs. Physically, (43) describes the response $u(x)$ at some location x, to some external impact force (the source) $f(x)$. For instance, in an elastic system, the extending force is the source while the change in length of the string is the response to the force. But for the response at a single location implies the existence of a **point source** at each x' which leads to a roadblock - mathematically, there is no continuous function which is nonzero at only a single location.

5.1 Dirac Delta Function

The conundrum is resolved by defining objects like the **Dirac delta function** which only behave like functions under integrations. These are known as **generalized functions**. The Dirac delta function is defined as

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \tag{45}$$

which satisfies the property

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{46}$$

that can graphically be represented by a single spike at $x = 0$.

5.1.1 Properties of the $\delta-$function

1. For any continuous function $\phi(x)$, we get:

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} \phi(x)\delta_\epsilon(x - x') = \phi(x'), \tag{47}$$

   which is interpreted as the delta function integral "picks of" the functional value at $x = x'$ value.

2. If we now take the integral over the derivative of the $\delta-$function, i.e. $\delta'(x - x')$ we have,

$$\int_{-\infty}^{\infty} \phi(x)\delta'(x - x')dx = -\phi(x') \tag{48}$$

   and in fact for $n-$th order derivative we have,

$$\int_{-\infty}^{\infty} \phi(x)\delta^n(x - x')dx = -(-1)^n\phi^{(n)}(x')\ . \tag{49}$$

We can generalize the above relations to $3-D$ as,

$$\begin{aligned}
&\delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\\
&\text{where,}\quad \vec{\mathbf{x}}_0=\{x_0,y_0,z_0\}.\\
&\text{Then, we have}\quad \int\int\int_V \delta(\vec{\mathbf{x}}-\vec{\mathbf{x}}_0)f(\vec{\mathbf{x}})dV=f(\vec{\mathbf{x}}_0)\ .
\end{aligned} \tag{50}$$

Refer to your textbook (Boas) to study other similar properties of the $\delta-$function.

5.2 Motivation for Using Green's Function

Consider the well-known Poisson's equation,

$$\nabla^2\phi=-f(r) \tag{51}$$

where $f(r)$ is the source term. In electrostatics, this is the charge density ρ i.e.

$$f(r)=\frac{\rho}{\varepsilon_0}\ .$$

Poisson's equation, being an inhomogeneous one, is harder to solve compared to Laplace's equation,

$$\nabla^2\phi=0 \tag{52}$$

where the source is absent. Moreover, chances are that if we could solve (51) for a particular $f(r)$, the recipe can seldom be retained for a different $f(r)$. George Green, the great British mathematician, had a brilliant realization that if one replaced $f(r)$ with the delta function, i.e.

$$\delta(\mathbf{r}-\mathbf{r}')$$

where $\mathbf{r}'$ is an arbitrary position in three dimensional space. Then (51) becomes,

$$\boxed{\nabla^2 G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}')} \tag{53}$$

which is known as **Green's equation** and $G(\mathbf{r},\mathbf{r}')$ is the Green's function. Green's equation makes life a lot easier since the delta function forces the equation to be a homogeneous one for all points where $\mathbf{r}\neq\mathbf{r}'$. The utility of Green's method to solving Green's equation lies in the fact that once we know what $G(\mathbf{r},\mathbf{r}')$ is, we can obtain the solution to the Poisson's equation, $\phi(\mathbf{r})$, for **any** source term $f(r)$ such that we get,

$$\boxed{\phi(\mathbf{r})=\int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d^3r'} \tag{54}$$

where $d^3r'=dx'\ dy'\ dz'$ is the volume element over the domain of the integral. By substituting (54) into (51), it is easy to see that the form of $\phi(\mathbf{r}')$ satisfies Poisson's equation.

5.3 1D Poisson Equation Solved Using Green's Method

We aim to solve

$$\frac{d^2\phi}{dx^2}=-f(x) \tag{55}$$

which satisfy Dirichlet boundary conditions

$$\phi(x=0) = \phi(x=1) = 0 \ . \tag{56}$$

Then we implement Green's recipe to get,

$$\frac{\partial^2}{\partial x^2} G(x,x') = -\delta(x-x') \tag{57}$$

which satisfy the same boundary conditions as the original equation such that we have

$$G(0,x') = G(1,x') = 0 \ , \tag{58}$$

with x' lying in between 0 and 1. This means that the solution to (55) can be written as,

$$\phi(x) = \int_0^1 G(x,x') f(x') dx' \ . \tag{59}$$

Convince yourself that (59) indeed satisfies the 1D Poisson equation. Now, notice that whenever $x \neq x'$, we have

$$\frac{\partial^2 G}{\partial x^2} = 0 \tag{60}$$

which is the 1D Laplace equation and this holds in the limit $x \to x'$, evaluated from the left as well as from the right. Thus the Green's function is a piecewise function given by,

$$G(x,x') = \begin{cases} G_<(x,x'), & x < x'; \\ G_>(x,x'), & x > x'; \end{cases} \tag{61}$$

As such, we must solve four second order ODEs which requires four boundary conditions and thus far we are given two. It is quite easy to see that

$$\begin{aligned} \frac{\partial^2 G_<}{\partial x^2} = 0 \Rightarrow G_<(x,x')|_{x=0} = A(x')x + B(x')|_{x=0} \Rightarrow B(x') = 0 \\ \Rightarrow \boxed{G_<(x,x') = Ax} \end{aligned} \tag{62}$$

while,

$$\begin{aligned} \frac{\partial^2 G_>}{\partial x^2} = 0 \Rightarrow G_>(x,x')|_{x=1} = C(x')x + D(x')|_{x=1} \Rightarrow C(x') = -D \\ \Rightarrow \boxed{G_>(x,x') = C(x-1)} \ . \end{aligned} \tag{63}$$

By definition of Green's functions, the piecewise functions must agree at the boundaries (**continuity**) such that we demand,

$$\begin{aligned} &G_<(x,x')|_{x=x'} = G_>(x,x')|_{x=x'} \\ &(62),(63) \Rightarrow Ax' = C(x'-1) \\ &\text{which is true when } \boxed{G(x,x') = \begin{cases} C(x'-1)x, & x < x'; \\ C(x-1)x', & x > x'; \end{cases}} \end{aligned} \tag{64}$$

Another part of the definition of Green's function is that the change in its derivatives suffers from a discontinuity such that,

$$\begin{aligned}
&\left.\frac{\partial G}{\partial x}\right|_{x'-\varepsilon}^{x'+\varepsilon} = -1 \\
&\Rightarrow \lim_{\varepsilon\to 0^+}\frac{\partial G}{\partial x} - \lim_{\varepsilon\to 0^-}\frac{\partial G}{\partial x} = -1 \\
&\Rightarrow \frac{\partial G_>}{\partial x} - \frac{\partial G_<}{\partial x} = -1 \\
&\Rightarrow C = -1 \text{ upon using (62), (63) .}
\end{aligned} \tag{65}$$

Thus, we have evaluated the Green's function for the problem to be

$$\boxed{G(x,x') = \begin{cases} x(1-x'), & x < x'; \\ x'(1-x), & x > x' . \end{cases}} \tag{66}$$

When substituted for G in (59) and accounting for the piecewise nature of the function, we get,

$$\boxed{\phi(x) = (1-x)\int_0^x x'(x-1)f(x')dx' + \int_x^1 x(x'-1)f(x')dx'}\ , \tag{67}$$

which can then lead to the solution to Poisson's equation for arbitrary source $f(x')$. With one stone, you have successfully killed a wide array of birds!

5.4 A General ODE and Implementation of Green's Method

If we are given a general second order ODE of the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x) \tag{68}$$

then the Green's function satisfies the following general properties:

1. Green's equation takes the form

$$a(x)\frac{\partial^2 G}{\partial x^2} + b(x)\frac{\partial G}{\partial x} + c(x)G(x,x') = \delta(x-x')$$

   with the solution to the original ODE given in terms of G by

$$f(x) = \int_a^b G(x,x')f(x')dx'$$

   with $a, b \in [x, x']$.

2. $G(x, x')$ must satisfy the same boundary conditions as the original ODE.

3. $G(x, x')$ is continuous at $x = x'$ such that

$$\boxed{G(x,x')|_{x^-\to x'} = G(x,x')|_{x^+\to x'}}\ .$$

4. The first derivative suffers from a discontinuity such that

$$\boxed{\left.\frac{\partial G}{\partial x}\right|_{x^+\to x'} - \left.\frac{\partial G}{\partial x}\right|_{x^-\to x'} = \frac{1}{a(x')}}\ .$$