a complex number.

$$\therefore \quad Re\left[\frac{d}{dt}(a+ib)\right] = \frac{da}{dt} = \frac{d}{dt}\left[Re\,(a+ib)\right]$$

~~Taking~~ We can write our original equation as –

$$Re\left[\frac{d^2 z}{dt^2} + \gamma\frac{dz}{dt} + \omega_0^2 z\right] = Re\left[\frac{F_0}{m}e^{i\omega_d t}\right]$$

$$\Rightarrow \frac{d^2}{dt^2}\left[Re(z)\right] + \gamma\frac{d}{dt}\left[Re(z)\right] + \omega_0^2\, Re(z) = \frac{F_0}{m}\cos\omega_d t$$

So, if $z(t)$ is a solution to equation (ii), then the real part of $z(t)$ is a solution to the original equation.

To find the solution of (ii), we will guess a solution. Since the driving force is periodic, we expect to get a periodic solution. Now, because the R.H.S. has a term involving $e^{i\omega_d t}$ the solution should be of the same form as $e^{i\omega_d t}$. So, we look for solution,

$$z(t) = A\,e^{i\omega_d t}$$

The reason is, the frequency must be ω_d, is that, on the left hand side, you have derivatives of $z(t)$, and the function $z(t)$ itself. But since derivatives can't change the frequency, any other frequency other than ω_d has no chance in satisfying the equation. So, the frequency must be ω_d.

Let's see whether we can find a solution for any corresponding value of A. Let's plug $z(t) = Ae^{i\omega_d t}$ in equation (ii). We get –

$$i^2\omega_d^2 Ae^{i\omega_d t} + i\omega_d \gamma A e^{i\omega_d t} + \omega_0^2 A e^{i\omega_d t} = \frac{F_0}{m} e^{i\omega_d t}$$

$$\Rightarrow A\left(-\omega_d^2 + i\omega_d\gamma + \omega_0^2\right) = \frac{F_0}{m}$$

$$\therefore A = \frac{F_0/m}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}$$

So, $z(t) = \dfrac{F_0/m}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d} e^{i\omega_d t}$ is a solution to (ii).

Taking the real part –

$$x(t) = \mathrm{Re}[z(t)] = \mathrm{Re}\left[\frac{F_0/m}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d} e^{i\omega_d t}\right]$$

$$= \mathrm{Re}\left[\frac{F_0/m}{\omega_0^2 - \omega_d^2 + i\omega_d\gamma}(\cos\omega_d t + i\sin\omega_d t)\right]$$

$$= \mathrm{Re}\left[\frac{F_0/m}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}\cos\omega_d t + i\frac{F_0/m}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d}\sin\omega_d t\right]$$

$$\cancel{= \mathrm{Re}\left[\frac{F_0/m}{\quad}\right.}$$

Now, $$\frac{1}{\omega_0^2 - \omega_d^2 + i\gamma\omega_d} = \frac{\omega_0^2 - \omega_d^2 - i\gamma\omega_d}{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}$$

$$\therefore x(t) = \frac{F_0}{m} Re \left[\frac{\omega_0^2 - \omega_d^2}{(\omega_0^2-\omega_d^2)^2 + \gamma^2\omega_d^2} \cos\omega_d t - \frac{i\gamma\omega_d}{(\omega_0^2-\omega_d^2)+\gamma^2\omega_d^2} \cos\omega_d t \right.$$

$$\left. + \frac{\gamma\omega_d}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2} \sin\omega_d t + i \frac{\omega_0^2-\omega_d^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2} \cos\omega_d t \right]$$

$$\boxed{\therefore x(t) = \frac{F_0/m\,(\omega_0^2-\omega_d^2)}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2} \cos\omega_d t + \frac{F_0/m\,\gamma\omega_d}{(\omega_0^2-\omega_d^2)+\gamma^2\omega_d^2} \sin\omega_d t}$$

$$\boxed{\tan\delta = \frac{\gamma\omega_d}{\omega_0^2-\omega_d^2}}$$

$$A = \sqrt{\frac{(F_0/m)^2(\omega_0^2-\omega_d^2)^2 + (F_0/m)^2 + \gamma^2\omega_d^2}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]^2}}$$

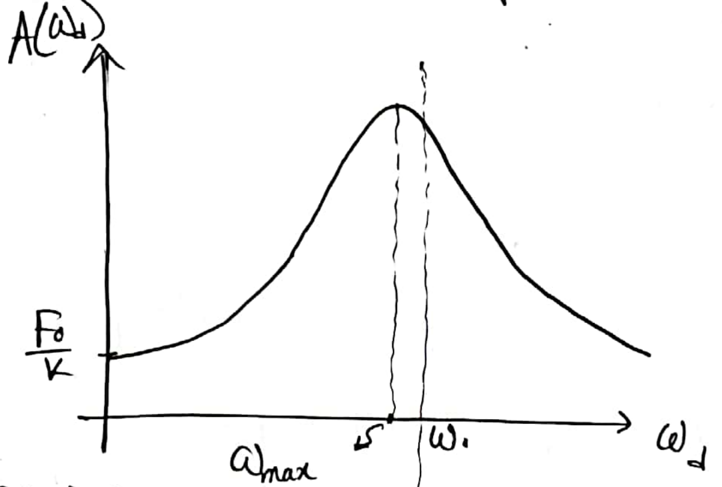
$$= \frac{F_0}{m}\sqrt{\frac{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}{\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]^2}}$$

$$\boxed{\therefore A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}}}$$

$$\boxed{\therefore x(t) = A\cos(\omega_d t - \delta)}$$

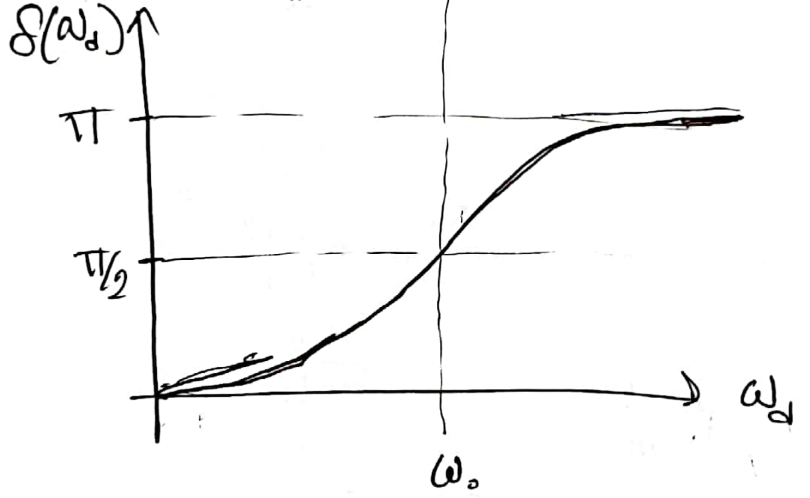
So, the particular solution is just like simple harmonic motion solution. But, there is a catch, a very important one. A and δ in the equation of $x(t)$ are NOT dependent on initial conditions. They are fixed.

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_d^2)^2 + \gamma^2\omega_d^2}} \quad \text{and} \quad \delta = \tan^{-1}\frac{\gamma\omega_d}{\omega_0^2-\omega_d^2}$$



If $\omega_d \to 0$, $A \to \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{K}$

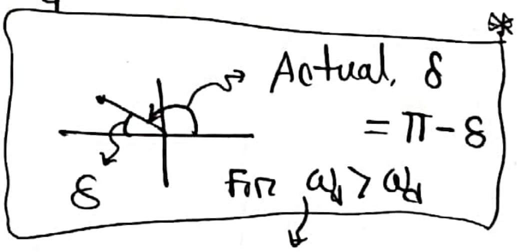
$\omega_d \to \infty$, $A \to 0$



$\omega_d \to 0, \quad \delta \to 0$

$\omega_d = \omega_0, \quad \delta = \pi/2$

$\omega_d \to \infty, \quad \delta \to \pi$



In the second quadrant

(i) Let's find the driving frequency for which the amplitude A is maximum.

~~A-0~~

$\Rightarrow$ A will the maximum, when the denominator is minimum. Setting the derivative of the under root of the denominator to zero gives us –

$$\frac{d}{d\omega_d}\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]=0$$

$$\Rightarrow 2(\omega_0^2-\omega_d^2)(-2\omega_d)+2\gamma^2\omega_d=0$$

$$\Rightarrow 2\omega_d\left[2\omega_0^2-2\omega_d^2+\gamma^2\right]=0$$

$$\therefore 2\omega_0^2-2\omega_d^2+\gamma^2=0$$

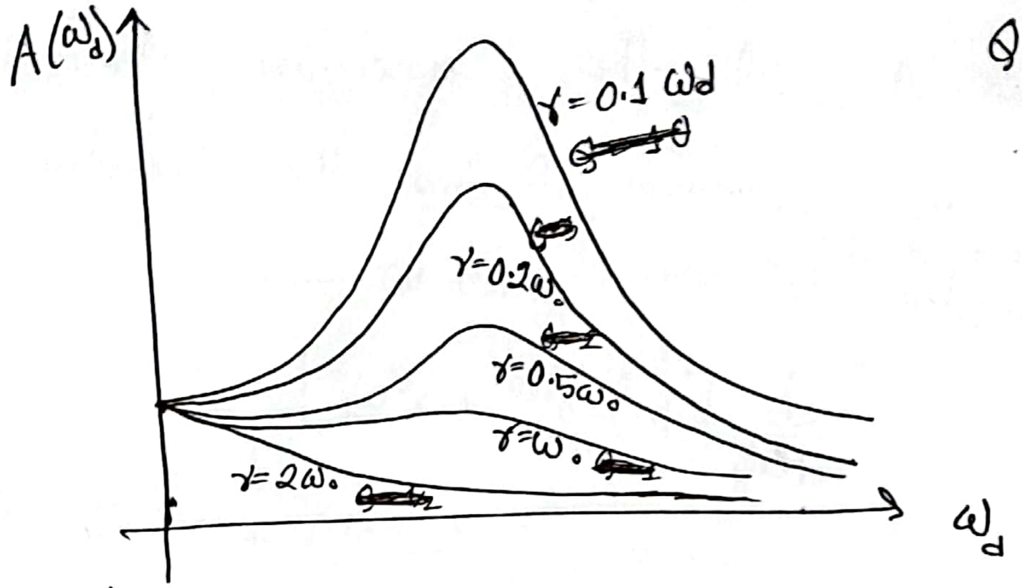
$$\therefore \omega_{d,max}=\sqrt{\omega_0^2-\frac{\gamma^2}{2}}$$

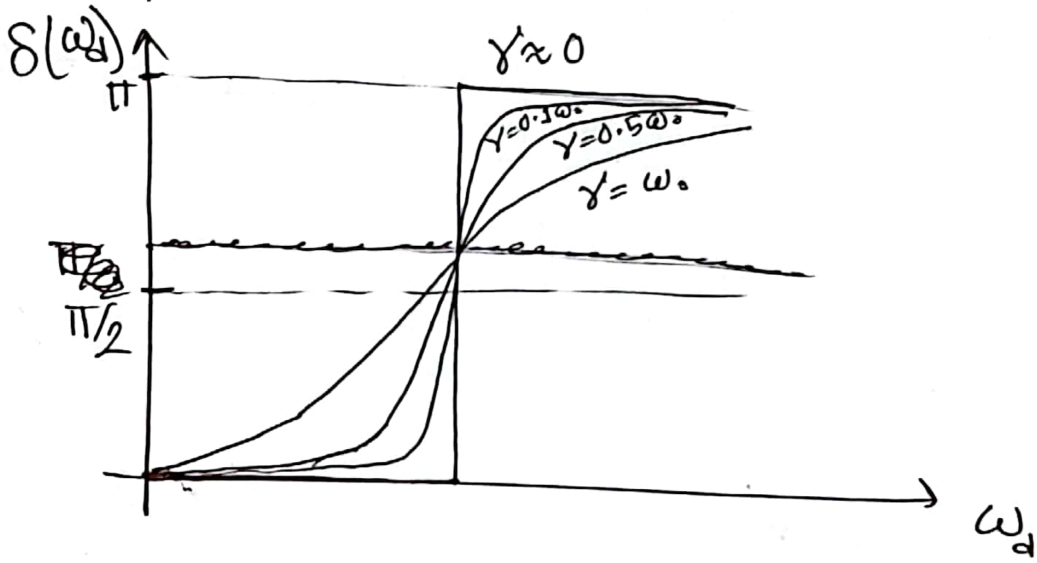
So, $A(\omega_d)=max$, when $\omega_{d_{max}}=\sqrt{\omega_0^2-\frac{\gamma^2}{2}}$. For small damping, such that $\gamma \ll \omega_0$, $\omega_{d,max}\approx\omega_0$. If $\gamma=\sqrt{2}\,\omega_0$, then maximum occurs at $\omega_d=0$. If $\gamma>\sqrt{2}\,\omega_0$, then there is no maximum for $\omega_d>0$. The graph just decreases as ω_d increases.

We are interested in the small damping region. So, the maximum amplitude occurs at $\omega_{d_{max}}=\omega_0$ and the phase lag is $\pi/2$: Then,

$$A_{max}=A(\omega_0)=\frac{F_0}{m\gamma\omega_0}$$

$Q = \frac{\omega_0}{\gamma}$





$$A(\omega_d) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}}$$

$$= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_d^2)^2+\omega_0^2\omega_d^2/Q^2}}$$

$$= \frac{F_0/m}{\sqrt{(\omega_0\omega_d)^2\left(\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}\right)^2+\left(\frac{\omega_0\omega_d}{Q}\right)^2}}$$

$$= \frac{1}{\omega_0\omega_d}\frac{F_0}{m}\frac{1}{\sqrt{\left(\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}\right)^2+\frac{1}{Q^2}}}$$

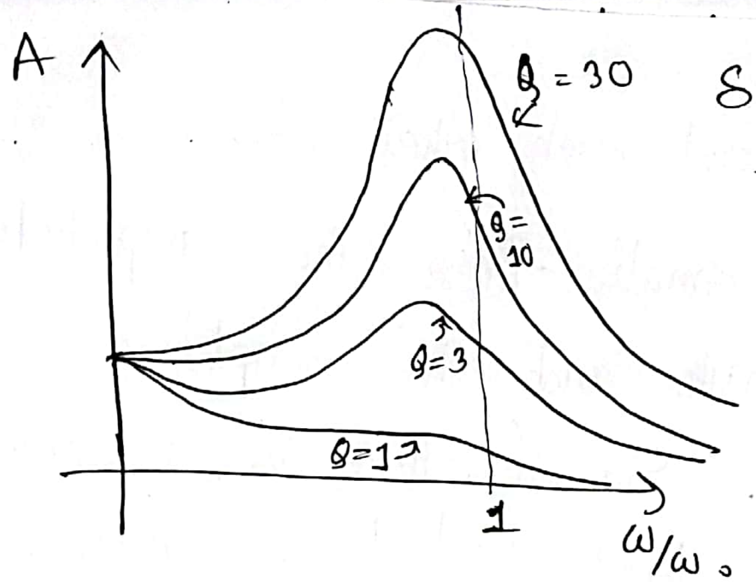
$$\boxed{\therefore A(\omega_d) = \frac{F_0}{K}\frac{\omega_0/\omega_d}{\sqrt{\left(\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}\right)^2+1/Q^2}}}$$

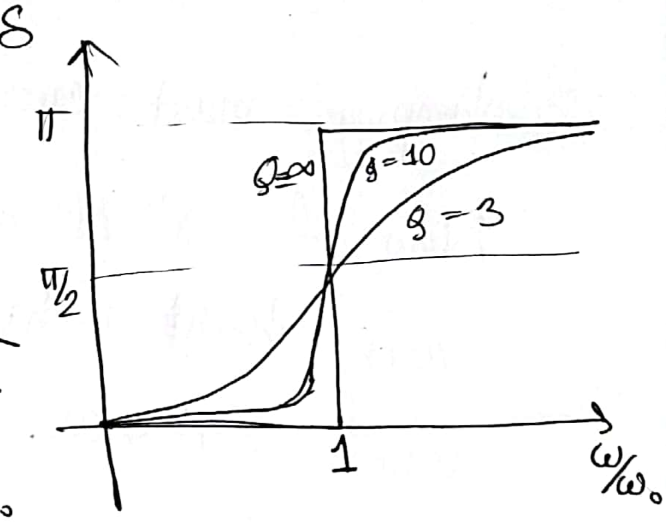
$$\tan\delta = \frac{\gamma\omega_d}{\omega_0^2-\omega_d^2}$$

$$= \frac{\frac{\omega_0}{Q}\omega_d}{\omega_0^2-\omega_d^2}$$

$$= \frac{\omega_0\omega_d/Q}{\omega_0\omega_d\left(\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}\right)}$$

$$\boxed{\therefore \delta = \tan^{-1}\left[\frac{1/Q}{\frac{\omega_0}{\omega_d}-\frac{\omega_d}{\omega_0}}\right]}$$





Resonance

When $\omega_d = \omega_0$, $\quad \phi = \frac{\pi}{2}$, $\quad A = \frac{F_0}{m\gamma\omega_0}$

$$\therefore x(t) = \frac{F_0}{m\gamma\omega_0} \cos\left(\omega_d t - \frac{\pi}{2}\right)$$

$$\therefore x(t) = \frac{F_0}{m\gamma\omega_0} \sin(\omega_d t)$$

Now, the damping force is, $F_{damping} = -bv$

$$= -(\gamma m)\dot{x}$$

$$= -\gamma m \frac{F_0}{m\gamma\omega_0}\omega_d \cos(\omega_d t)$$

Since $\omega_d = \omega_0$,

$$F_{damping} = -F_0 \cos(\omega_d t) = -F_0 \cos(\omega_0 t)$$

So, if $\omega_d = \omega_0$, then the damping force is exactly equal to the driving force, and they cancel each other out. This makes sense, because the system is oscillating with frequency ω_0, which is the frequency of normal motion. So, the effect of driving and damping must cancel each other out.

Now, if γ is small, then the amplitude is very ~~hight~~ high, and the amplitude is maximum when $\omega_d \approx \omega_0$. So, if there is a driving force with frequency almost equal to the natural frequency of the system, the system oscillates with maximum ~~frequen~~ amplitude. This phenomenon is called resonance. The phase lag here in resonance is $\pi/2$.

Some comments on phase lag

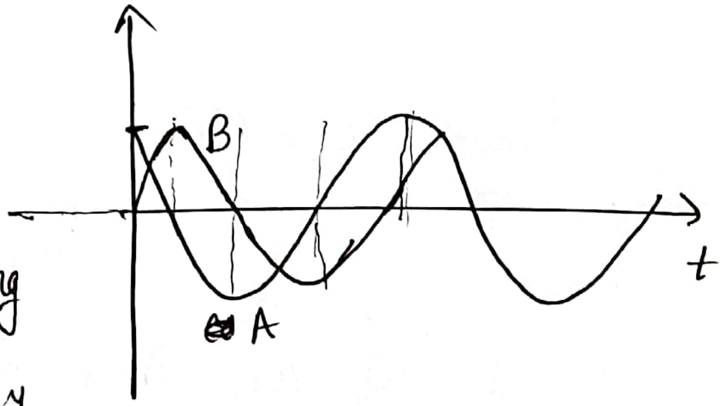
What do we mean by phase lag? Consider two functions,

$$A = \cos \omega t \quad \text{and} \quad B = \sin \omega t$$

Consider the plot of them.

We can write,

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$



We then say, A is leading B by $\frac{\pi}{2}$ and conversely B has a phase lag of $\frac{\pi}{2}$. So, B lags A by an amount of $\frac{\pi}{2}$.

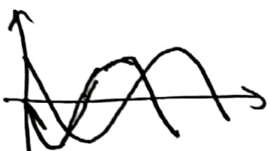
Consider that the position of a particle is given by,

$$x(t) = A\sin(\omega t+\phi)$$

$$\therefore v(t) = \omega A\cos(\omega t+\phi)$$

For $\phi = 0$, (or any value) $v(t)$ always lead $x(t)$ by $\frac{\pi}{2}$. It would be same if $x(t) = A\cos(\omega t+\phi)$. Then $\vec{v}(t) = -\omega A\sin(\omega t+\phi)$

$$\cos\left(\omega t+\phi+\frac{\pi}{2}\right) = -\sin(\omega t+\phi)$$



$-\sin\left(\omega t - \frac{\pi}{2}\right) = \cos\omega t$

So, $x(t)$ always lags by $\frac{\pi}{2}$ of $\vec{v}(t)$.

Similarly in our driven oscillation case,

$$F(t) = F_0\cos(\omega_d t)$$

$$x(t) = A\cos(\omega_d t - \phi)$$

So, we say, $x(t)$ lags $F(t)$ by a phase ϕ.

Now, $\phi = \tan^{-1}\dfrac{\gamma\omega_d}{\omega_0^2-\omega_d^2}$

If $\omega_d \ll \omega_0$, $\phi \approx 0°$, and the position $x(t)$ is almost in phase with the driving force. For $\omega_d = \omega_0$, the oscillator is $90°$ lagging behind the driving force. As ω_d increases and in the limit $\omega_d \gg \omega_0$, the oscillator is completely out of phase (~~180°~~) and lagging $180°$ behind the driving force.