Classical Mechanics

Lecture # 3

Review:

In the last lecture we saw that Newton's law for a single particle

$$\vec{F} = \dot{\vec{p}},$$

where $\vec{p} = m\dot{\vec{x}}$, can be generalized for a collection of N interacting particles:

$$(\dagger) \qquad \boxed{\vec{F}^{(ext)} = M\ddot{\vec{R}}}$$

where $M = \sum_{i=1}^{N} m_i$ and $\vec{R} = \dfrac{\sum_{i=1}^{N} m_i \vec{r}_i}{M}$ is the centre of mass.

In deriving this result we also assumed that the forces among the particles that makes up the system satisfied Newton's third law of motion:

$$\vec{F}_{ij} = -\vec{F}_{ji}.$$

To compute $\vec{F}^{(ext)}$, the net external force on the system, we first computed the net force on each particle:

$$\vec{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij} + \vec{F}_i^{(ext)}$$

where $\vec{F}_i^{(ext)}$ is the net force on the i-th particle.
To find the total force we sum over $i = 1, \dots, N$ and we get

$$(*) \quad \text{———} \quad \sum_i \vec{F}_i = \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ij} + \sum_i F_i^{(ext)}$$

The first sum on the right-hand-side is over both i & j, exclud-ing $i = j$ cases (no self-force assumed). This sum can be written as:

$$\sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ij} = \sum_{\substack{i,j \\ i > j}} \vec{F}_{ij} + \sum_{\substack{i,j \\ i < j}} \vec{F}_{ij}$$

Newton's third law of motion tells us that $\vec{F}_{ij} = -\vec{F}_{ji}$.
In the second sum on the right-hand-side we can relabel i and j as j and i, and thus we get

$$\sum_{\substack{i,j \\ i > j}} \vec{F}_{ij} + \sum_{\substack{i,j \\ i < j}} \vec{F}_{ij} = \sum_{\substack{i,j \\ i > j}} \vec{F}_{ij} + \sum_{\substack{j,i \\ j < i}} \vec{F}_{ji}$$

$$= \sum_{\substack{i,j \\ i > j}} (\vec{F}_{ij} + \vec{F}_{ji}) = \sum_{\substack{i,j \\ i > j}} (\vec{F}_{ij} - \vec{F}_{ij}) = 0 .$$

Thus we see that Newton's third law implies that internal forces do not contribute to the net force on a system of particles.

Thus (*) becomes

$$\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(ext)}$$

The left-hand-side becomes, upon using Newton's second law for particles,

$$\sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i^{(ext)}$$

Using the definition of the centre of mass the LHS becomes

$$\frac{d}{dt} \sum_i m_i \dot{\vec{r}}_i = \frac{d}{dt} M \dot{\vec{R}} = M \ddot{\vec{R}} .$$

The RHS is the net external force. Thus we get:

$$\boxed{M\ddot{\vec{R}} = \vec{F}^{(ext)}}$$

The motion of extended bodies:

In analogy with the rotational equation for a single particle

$$\vec{\tau} = \dot{\vec{\ell}}$$

we can derive: $\boxed{\vec{\tau}^{(ext)} = \dot{\vec{L}}}$ (show)

where $\vec{L} = \sum_{i=1}^{N} \vec{\ell}_i$ is the total angular momentum and

$\vec{\tau}^{\,ext} = \sum_i \vec{r}_i \times \vec{F}^{(ext)}$ is net external torque (what happens to the torques due to the internal forces?).

Conservation Laws:

The most important conservation laws that arise from Newton's laws for multiparticle systems are:

1. Conservation of linear momentum: $\vec{F}^{(ext)} = 0 \Rightarrow \dot{\vec{P}} = 0$ $\Rightarrow \vec{P}$ is constant.

   When $M =$ constant $\vec{P} = M\dot{\vec{R}}$ and $\vec{P}$ constant means that the velocity $\dot{\vec{R}}$ of the centre of mass is a constant. This can be seen as a consequence of boost invariance.

2. Conservation of total angular momentum:

$$\vec{\tau}^{\,(ext)} = 0 \quad \Rightarrow \quad \vec{L} = \sum_i \vec{l}_i = \text{constant}.$$

   This has many important applications including deriving Kepler's second law.

Locality and Third law:

From the modern perspective there is something odd about the third law. Suppose we take the two body system where the distance between the particles is great. If the positions of the particles are $\vec{r}_1$ & $\vec{r}_2$, Newton's third law states that

$$\vec{F}_{12}(\bar{r}_1 - \bar{r}_2, \dots) = -\vec{F}_{21}(\bar{r}_2 - \bar{r}_1, \dots).$$

Where we expect the forces to depend on $\vec{r}_1 - \vec{r}_2$ due translation invariance. The $\cdots$ in the argument denotes any other dependence. If $|\bar{F}_{12}| = |\vec{F}_{21}| \neq 0$, then the force between them is some sort of non-contact force since we have assumed that the particles are far from each other.

Now suppose we perturb the position of particle 2: $\bar{r}_2 \to \bar{r}_2 + \delta\bar{r}_2$. We also assume that the source of the pertubation on particle 2 is local, meaning that $|\bar{r}_s - \bar{r}_2| << |\bar{r}_2 - \bar{r}_1|$ where $\bar{r}_s$ is roughly the location of the source of perturbation.

Since the force on particle 2 is due to some non-contact force due to particle 1, particle 2 experiences the force due to some field whose source is particle 1. Thus $\bar{r}_2 \to \bar{r}_2 + \delta\bar{r}_2$ induces a change $\bar{F}_{21} \to \bar{F}_{21} + \delta\bar{F}_{21}$.

Does this change in $\bar{F}_{21}$ lead to an immediate and instant change in $\bar{F}_{12}$? Since Newton's second law doesn't mention any time, we must assume that $\vec{F}_{12}$ changes instantaneously so as to preserve the third law.

But we know from electromagnetism that the fastest speed at which 'news' can travel is the speed of light (Maxwell) and so for far away objects the news of the perturbation of object 2 will reach object 1 after a time $t \sim \frac{|\vec{r}_1 - \vec{r}_2|}{c}$ where c is the speed of light.

This gedanken experiment is presented as an argument to think of the field as something as dynamic as the particles that interact with them. Maxwell's Theory of the electromagnetic field makes this manifest. In fact, Maxwell's equations put field theory as a natural extension of classical mechanics.

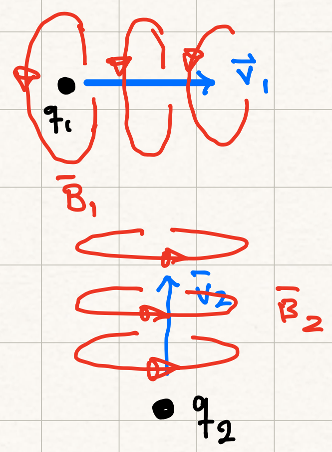
An example of an apparent violation of the Third law:

In fact the force on a charged particle due to the motion of another charged particle does not respect the most simple form of Newton's Third law. Suppose we have a charged particle with charge q_1 moving with a velocity $\vec{v}_1$. The magnetic field due to this charged particle is given by

$$\vec{B}_1 = \frac{\mu_0 q_1}{4\pi} \frac{\vec{v}_1 \times \hat{r}_1'}{|\vec{r}_1'|^2}$$

where $\vec{r}_1'$ is the location of the magnetic field in relation to the charged particle.

This is the Biot-Savart law for point particles.
The direction of the magnetic field is given by the right-hand rule of cross product:



Now suppose a second particle of charge q_2 is moving with velocity $\bar{v}_2$ near the 1st particle. The force on particle 2 due to the magnetic field of charge 1 is given by the Lorentz force law:

$$\vec{F}_{21} = q_2 \, \bar{v}_2 \times \bar{B}_1 .$$

If $\bar{v}_1$ and $\bar{v}_2$ are as shown in the diagram we can see that: $\bar{v}_2 \perp \bar{B}_1$ and $\bar{F}_{21}$ point to the right. But when we calculate $\vec{F}_{12} = q_1 \bar{v}_1 \times \bar{B}_2$, we find $\bar{v}_1 \parallel \bar{B}_2$ and so $\vec{F}_{21} = 0$. Thus it appears that Newton's third law is being violated.

This happens even when the particles are close to each other and so we can't use the speed of light excuse.

This would lead to the violation of conservation laws! But it doesn't because according to the theory of electromagnetism the electromagnetic field itself carries momentum and angular momentum. When those are taken into account third law is restored!

When is the third law valid:

In Newtonian physics we can assume the third law to be valid in cases whenever the 'field' is not dynamical. Thus when the force is electrostatic force or gravitational force we can assume Newton's third law to be valid. Magnetostatics is also included. Frictional forces have their origin in (quantum) electrostatic attraction between atoms and molecules and so these forces are also assumed to satisfy the third law.

Total Mechanical Energy

Consider the motion of an extended body — Its motion can be separated into the motion of the centre of mass and the motion of the particles relative to the centre of mass.

How does this division translate into a statement about mechanical energy?

Total Kinetic Energy:

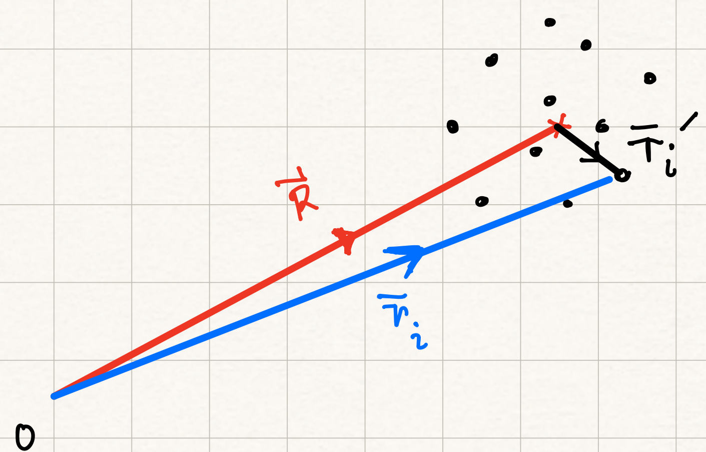
Let us consider a system of N iteracting particles. Then the total kinetic energy of the system is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$$

If we now introduce the centre of mass $\vec{R} = \dfrac{\sum_i m_i \vec{r}_i}{M}$

If the masses are constant we can write $\dot{\vec{R}} = \dfrac{\sum_i m_i \dot{\vec{r}}_i}{M}$

Let us also introduce $\vec{r}_i{}' = \vec{r}_i - \vec{R}$



$$\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \sum_i \frac{m_i}{2} \left[\dot{\vec{R}} + \dot{\vec{r}}_i{}'\right]^2$$

$$= \sum_i \frac{m_i}{2} \dot{\vec{R}}^2 + \sum_i \frac{m_i}{2} \dot{\vec{r}}_i'^2$$

$$+ \sum_i m_i \, \dot{\vec{R}} \cdot \dot{\vec{r}}_i'$$

The last term: $\sum_{i=1}^{N} m_i \dot{\vec{R}} \cdot \dot{\vec{r}}_i' = \dot{\vec{R}} \cdot \sum_{i=1}^{N} m_i \dot{\vec{r}}_i'$

Note that $\sum_{i=1}^{N} m_i \dot{\vec{r}}_i' = \sum_{i=1}^{N} m_i \left(\dot{\vec{r}}_i - \dot{\vec{R}} \right) = M\dot{\vec{R}} - M\dot{\vec{R}} = 0$.

And so we get:

$$T_{total} = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{\text{KE of the COM}} + \underbrace{\sum_i \frac{1}{2} m_i \vec{v}_i^2}_{\text{Internal KE}}$$

Recall the Work-Kinetic energy theorem:

$$T(t_2) - T(t_1) = \int_{\vec{r}_1(t_1)}^{\vec{r}_2(t_2)} \vec{F} \cdot d\vec{r}$$

Generalizing to a system of N particles:

$$\vec{F}_i = \vec{F}_i^{(ext)} + \sum_{j \neq i} \vec{F}_{ij}$$

Net external force on ith particle

Internal force on ith particle due to the j.th particle.

Work done on i.th particle

$$T_i(t_2) - T_i(t_1) = \int_{\vec{r}_i(t_1)}^{\vec{r}_i(t_2)} \vec{F}_i \cdot d\vec{r}_i$$

The change in total kinetic energy:

$$T(t_2) - T(t_1) = \sum_{i=1}^{N} \int_{\vec{r}_i(t_1)}^{\vec{r}_i(t_2)} \Big(\vec{F}_i^{ext} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{F}_{ij} \Big) \cdot d\vec{r}_i$$

$$= \sum_{i=1}^{N} \int_{\vec{r}_i(t_1)}^{\vec{r}_i(t_2)} \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \int_{\vec{r}_i(t_1)}^{\vec{r}_i(t_2)} \vec{F}_{ij} \cdot d\vec{r}_i$$

If the external forces are conservative as well as the internal forces then

$$\vec{F}_i^{(ext)} = -\vec{\nabla}_i V_i(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$$

$$\vec{F}_{ij} = -\vec{\nabla}_i V_{ij}(\vec{r}_1, \ldots, \vec{r}_N)$$

To obtain further simplification we assume $V_i = V_i(\bar{r}_i)$. For example, if we have system of particles in a gravitational field the gravitational potential energy of the i-th particle is independent of the other particles' positions.

For the internal forces, if we want $\vec{F}_{ij} = -\bar{F}_{ji}$ we should require $V_{ij} = V_{ji}$ and $V_{ij} = V_{ij}(|\bar{r}_1 - \bar{r}_2|)$. Such forces are known as central forces.

Under these circumstances we get:

$$V = \sum_i V_i(\bar{r}_i) + \sum_{i>j} V_{ij}(|\bar{r}_i - \bar{r}_j|)$$

And the total energy $H = T + V$ is conserved.