# Conservation of Momentum

The interactions of particles is central to mechanics. These interactions can happen in through both contact and non-contact forces. If external forces are absent then the total linear momentum (i.e. the vector sum of the momentum vectors of the individual particles) remains constant through the interaction process.

For two particles with initial momenta $\vec{p}_1$ & $\vec{p}_2$ and final momenta $\vec{p}_1'$ & $\vec{p}_2'$ we have, in the absence of external forces,

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$$

where $\vec{p} = M\vec{v}$ is the linear momentum.

In physics a brief and localized interaction between particles is called a collision. Thus conservation of momentum implies that the total momentum of two particles colliding will be constant as long as there are no external net force.

There are two types of collision:

1. Elastic collisions: Collisions in which the total kinetic energy of the initial particles is equal to the total kinetic energy of the final particles.
2. Non-elastic collisions: Collisions in which some of the initial kinetic energy is converted into internal energy (such as heat) of the constituent particles.

In both cases, the total momentum is conserved.

Derivation of Momentum conservation from Newton's third law:

$$\vec{F}_{12} = m\vec{a}_1 = m\frac{d\vec{v}_1}{dt} = \frac{d\vec{p}_1}{dt}$$

$$\vec{F}_{21} = m\vec{a}_2 = m\frac{d\vec{v}_2}{dt} = \frac{d\vec{p}_2}{dt}$$

But according to Newton's third law:

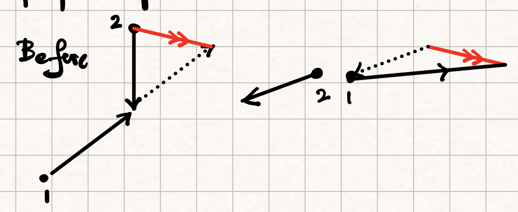
$$\vec{F}_{12} + \vec{F}_{21} = 0 = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt}$$

$$\Rightarrow \int_{t_i}^{t_f} \frac{d\vec{q}_1}{dt}\, dt + \int_{t_i}^{t_f} \frac{d\vec{q}_2}{dt}\, dt = 0$$

$$\Rightarrow \int_{\vec{p}_1}^{\vec{p}_1'} d\vec{q}_1 + \int_{\vec{p}_2}^{\vec{p}_2'} d\vec{q}_2 = 0$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$$

Graphically:




Comments:

1. The conservation of momentum is a vector equation. So the total momentum before and after point along the same line.
2. Conservation of momentum alone does not solve collision problems.

<u>Ex:</u>

1. Suppose we have two particles of identical mass. One is at rest while the other is in motion with velocity $\vec{v}$. Suppose the two particles stick together after collision. What is the velocity of the combined system after collision?

$$\vec{p}_1 = M\vec{v} \qquad \vec{p}_2 = 0$$

$$\vec{p}_1 + \vec{p}_2 = M\vec{v} = \vec{p}_1' + \vec{p}_2' = 2M\vec{v}'$$

$$\Rightarrow \vec{v}' = \frac{\vec{v}}{2}$$

2. If instead of sticking together the first particle is brought to rest. what is the vel-

ocity of the second particle after the collision?

$$\vec{p}_1 = M\vec{v} \qquad \vec{p}_2 = 0$$

$$\vec{p}_1{}' = 0 \qquad \vec{p}_2{}' = M\vec{v}\,'$$

$$\Rightarrow \quad \vec{v} = \vec{v}\,'$$

## Dynamics of a system of particles:

Suppose we we have a system of N particles with $\vec{f}_i$ being the force on the ith particle. Then according to Newton's second law:

$$\vec{f}_i = \frac{d\vec{p}_i}{dt}$$

The force on the i-th particle can be divided into two types:

1. Force on the particle from an external source
2. " " " " " other particles within the system

$$\vec{f}_i^{\,ext} + \vec{f}_i^{\,int} = \frac{d\vec{p}_i}{dt}$$

Now if we sum over all the particles then all the internal forces will cancel and we shall have:

$$\sum_{i=1}^{N} \vec{f}_i^{\,ext} = \vec{F}^{ext} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N) = \frac{d}{dt}\vec{P}^{tot}$$

$$\boxed{\vec{F}^{ext} = \frac{d\vec{P}^{tot}}{dt}}$$

This is Newton's 2nd law for a system of particles.

## Centre of Mass

Newton's 2nd law for a composite system

$$\vec{F} = \frac{d\vec{P}}{dt}$$

looks exactly like Newton's 2nd law for a point particle.

If $M = \sum_{i=1}^{N}$ is the total mass of the system we may ask if the whole system behaves like a particle? Since the system may be an extensive s stem we have to define the $\vec{R}$ at which the our fictitious particle is located.

We define $\vec{R}$ by:

$$\vec{F} = \frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2}$$

$$\text{since} \quad \vec{P} = \sum_{i=1}^{N}\vec{p}_i = \sum_{i=1}^{N} m_i\vec{v}_i \quad \text{we get}$$

$$\frac{d^2}{dt^2}\sum_{i=1}^{N} m_i\vec{r}_i = M\frac{d^2\vec{R}}{dt^2}$$

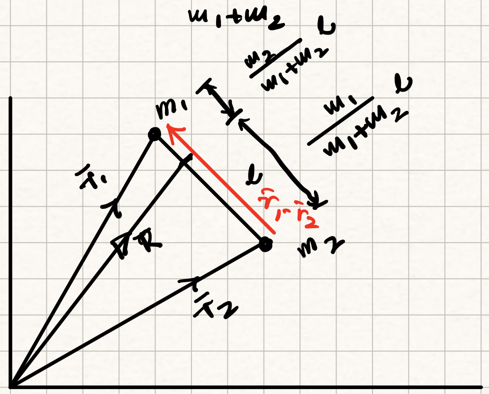
$$\Rightarrow \quad \vec{R} = \frac{\sum_{i=1} m_i\vec{r}_i}{M}$$

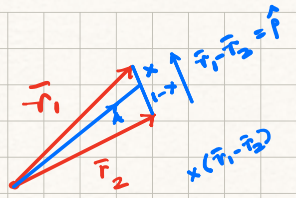
$\vec{R}$ is known as the centre of mass. It is the average position of the particles weighted by their masses.

Ex: What is the centre of mass of a system that consists of two particles of masses $m_1$ & $m_2$ seperated by a 'massless' rod of length $\ell$?

If $\vec{r}_1$ & $\vec{r}_2$ are the positions of the two masses then:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$






$$\vec{A}+x(\vec{r}_1-\vec{r}_2)=\vec{r}_1$$

$$\vec{A}=-x(\vec{r}_1-\vec{r}_2)+\vec{r}_1$$

$$=\vec{r}_1(1-x)+x\vec{r}_2$$

$$x=\frac{m_2}{m_1+m_2}$$

$$1-x=\frac{m_1}{m_1+m_2}$$

$$\vec{A}=\frac{\vec{r}_1 m_1+\vec{r}_2 m_2}{m_1+m_2}$$