Group Theory
Lecture # 8

Reducibility of Representation

Let D be a matrix representation of the group G. If $\forall g \in G$ $\exists$ a matrix S, independent of g, st that

$$\tilde{D}(g) \equiv S^{-1} D(g) S = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}$$

Then we say that D is a reducible representation.

Comment:

1. For a reducible representation

$$\tilde{D}(g)\tilde{D}(g') = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix} \begin{pmatrix} A(g') & C(g') \\ 0 & B(g') \end{pmatrix}$$

$$= \begin{pmatrix} A(g)A(g') & A(g)C(g') + C(g)B(g') \\ 0 & B(g)B(g') \end{pmatrix}$$

$$= \begin{pmatrix} A(gg') & A(g)C(g') + C(g)B(g') \\ 0 & B(gg') \end{pmatrix}$$

For finite & compact groups it can shown that $C(g)=0$. If that is the case then we say that the representation is completely reducible or decomposable.

2. For a reducible rep. the action of A & B on vector spaces V_A & V_B don't mix V_A & V_B. The dim of V_A & V_B are dim of A & B and the dim of the representation.

Unitarity Theorem

For applications to physics it is important that the matrices repres-

representing the groups be unitary. This is because in quantum mechanics transformations on a quantum system results in unitary transformations on the Hilbert space of states. So if a transformation represented by the elements of some group constitute a symmetry of the system then we expect amplitudes

$\langle\psi|\phi\rangle$ to remain unchanged. This will happen only if:

$$|\psi\rangle \to U(g)|\psi\rangle$$

$$|\phi\rangle \to U(g)|\phi\rangle$$

$$\langle\psi|\phi\rangle \longrightarrow \langle\psi|U^\dagger(g)\,U(g)|\phi\rangle = \langle\psi|\phi\rangle$$

$$\text{if} \quad U^\dagger(g)U(g) = 1.$$

Motivation for the Unitarity Theorem:

The physical motivation for the unitarity theorem comes mainly from quantum mechanics. In quantum mechanics, the state of a physical system is represented by a vector in a Hilbert space. A Hilbert space is a complex vector space (finite or infinite in dimensions) with a positive definite norm.

We represent the states by $|\psi\rangle$, $|\phi\rangle$ etc. and their duals by $\langle\psi|$, $\langle\phi|$, etc. In general, the inner product $\langle\psi|\phi\rangle \in \mathbb{C}$ but $\langle\phi|\phi\rangle \geq 0$ with the equality holding iff $|\phi\rangle = 0$.

When we make a transformation on a physical system represented by some group element $g \in G$ (G could be rotation group, the translation group etc.) this induces a linear transformation of the state of the system in Hilbert space:

$$|\psi\rangle \longrightarrow L|\psi\rangle$$

It was proved by Wigner that L must be either a unitary or anti-unitary operator in order for inner products $\langle \psi | \phi \rangle$ to remain invariant.

The Unitarity Theorem

To prove the unitarity theorem we use the rearrangement lemma. This lemma was set as an exercise for finite groups and can be extended easily to compact Lie groups. Let us remind ourself of the rearrangement lemma:

If G is a finite group and $f(g)$ is a function then the rearrangement lemma states that:

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(g'g) = \sum_{g \in G} f(gg')$$

for any arbitrary gp element g'. The sum in the rearrangement lemma must be over all the gp elements.

For Lie groups there is a natural measure known as the Haar measure. Compact Lie groups have finite volume with respect to this measure. So for compact Lie groups we can extend this theorem by replacing the sum by an integration over the whole group using the Haar measure:

$$\sum_{g\in G} \longrightarrow \int_G d\mu(g) \quad \text{[Haar measure]}$$

Then we get $\displaystyle\int_G d\mu(g) f(g) = \int_G d\mu(g) f(g'g) = \int_G d\mu(g) f(gg')$.

The unitarity theorem

All finite groups, and by extension, all compact Lie groups, have (finite dimensional) representations in terms of unitary matrices.

Proof:

We use the rearrangement lemma (with the Haar measure for compact Lie groups).

We proceed by assuming there exists a representation but it is not unitary and show how to construct a unitary representation from it.

Let $\tilde{D}(g)$ be the matrices of a non-unitary representation of a group G (G is assumed to be finite or a compact Lie group).

Now, consider the matrix H defined by

$$H = \sum_{g\in G} \tilde{D}^{\dagger}(g)\,\tilde{D}(g)$$

{For G, a compact Lie group we define $H = \displaystyle\int_G d\mu(g)\, \tilde{D}^{\dagger}(g)\,\tilde{D}(g)$ }

It is easy to see that H is hermitian:

$$H^\dagger = \sum_{g\in G} (\tilde{D}^\dagger(g)\tilde{D}(g))^\dagger$$
$$= \sum_{g\in G} \tilde{D}^\dagger(g)(\tilde{D}^\dagger(g))^\dagger$$
$$= \sum_{g\in G} \tilde{D}^\dagger(g)\tilde{D}(g)$$
$$= H$$

Next we show that H is invariant under conjugation by $\tilde{D}^\dagger(g')$ & $\tilde{D}(g')$

$$\tilde{D}^\dagger(g') H \tilde{D}(g') = \tilde{D}^\dagger(g')(\sum_{g\in G} \tilde{D}^\dagger(g)\tilde{D}(g))\tilde{D}(g')$$
$$= \sum_{g\in G} \tilde{D}^\dagger(g')\tilde{D}^\dagger(g)\tilde{D}(g)\tilde{D}(g')$$
$$= \sum_{g\in G} (\tilde{D}(g)\tilde{D}(g'))^\dagger \tilde{D}(g)\tilde{D}(g')$$
$$= \sum_{g\in G} \tilde{D}(gg')^\dagger \tilde{D}(gg')$$
$$= \sum_{g\in G} \tilde{D}(g)^\dagger \tilde{D}(g)$$
$$= H.$$

Since H is hermitian it can be diagonalized by a unitary transformation U:

$$\rho^2 \equiv U^\dagger H U$$
$$= U^\dagger \sum_{g\in G} \tilde{D}(g)^\dagger \tilde{D}(g) U$$

where $U^\dagger U = 1$.

The eigenvalues of H are real since it is a hermitian matrix, but we can show that they are also positive, and hence the notation ρ^2 for the diagonalized version.

Next we want to show that H has positive eigenvalues. But first let us look at a simpler problem. We show that any matrix of the form $M^\dagger M$ has positive-definite eigenvalues.

Let $M\psi = \psi'$ for ψ & ψ' vectors.

and $M^\dagger \psi' = c\,\psi$

Then $M^\dagger M \psi = c\,\psi$

& ψ is an eigenvector of $M^\dagger M$. Since $M^\dagger M$ is a hermitian matrix c is a real number.

If we left multiply $M^\dagger M\psi = c\psi$ by $\psi^\dagger$:

$$\psi^\dagger M^\dagger M \psi = \psi'^\dagger \psi' \geq 0$$

$$c\,\psi^\dagger\psi \geq 0$$

$$\Rightarrow c \geq 0.$$

Let $(e_j)_k = \delta_{jk}$

Since ρ^2 is diagonal let us consider

(no sum) $$(\rho^2)_{ii} = e_i^\dagger \rho^2 e_i$$
$$= \sum_{g\in G} e_i^\dagger u^\dagger \tilde{D}(g)^\dagger \tilde{D}(g) u\, e_i$$

Let us define $\phi_i(g) \equiv \tilde{D}(g) u\, e_i$. $\phi(g)$ is a vector.

Thus $(\rho^2)_{ii} = \sum_{g\in G} \phi_i^\dagger \phi_i$ [no sum over i]

Since $\phi_i^\dagger(g)\,\phi_i(g) \geq 0$ we get $(\rho^2)_{ii} \geq 0$

Let us define $\rho \equiv \sqrt{\rho^2}$.

And define $\boxed{D(g) \equiv \rho\, u^\dagger \tilde{D}(g)\, u\, \rho^{-1}}$

We can define p^{-1} exist because the only way $\det p = 0$ is if $\det \tilde{D}(g) = 0$ which is not allowed since otherwise $\tilde{D}^{-1}(g) = \tilde{D}(g^{-1})$ does not exist.

We now show that $D(g)$ is unitary:

$$D^{\dagger}(g) = (p^{-1})^{\dagger} u^{\dagger} \tilde{D}^{\dagger}(g) u p^{\dagger}$$

$$= p^{-1} u^{\dagger} \tilde{D}^{\dagger}(g) u p$$

$$D^{\dagger}(g) D(g) = p^{-1} u^{\dagger} \tilde{D}^{\dagger}(g) u p \, p u^{\dagger} \tilde{D}(g) u p^{-1}$$

$$= p^{-1} u^{\dagger} \tilde{D}^{\dagger}(g) u p^{2} u^{\dagger} \tilde{D}(g) u p^{-1}$$

$$= p^{-1} u^{\dagger} \tilde{D}^{\dagger}(g) u u^{\dagger} H u u^{\dagger} \tilde{D}(g) u p^{-1}$$

$$= p^{-1} u^{\dagger} \tilde{D}^{\dagger}(g) H \tilde{D}(g) u p^{-1}$$

$$= p^{-1} u^{\dagger} H u p^{-1}$$

$$= p^{-1} p^{2} p^{-1}$$

$$= \mathbb{1}.$$

It is easy to show that $D(g)$ is a representation since $\tilde{D}(g)$ is. QED.

Schur's Lemma :

If D is an irreducible representation of some group G & B is a matrix s.t. $[B, D(g)] = 0 \ \forall g \in G$ then $B = \lambda \mathbb{1}$ where $\mathbb{1}$ is the identity matrix.

Proof: We may assume WLOG that $B^{\dagger} = B$. Because if B isn't we can replace B by $B + B^{\dagger}$ which is hermitian and commutes with $D(g) \ \forall g \in G$.

Since B is hermitian we can diagonalize it using a unitary transformation:

$$B \to U^{\dagger} B U$$

We also apply the same unitary transformation to all the $D(g)$s. Then in the basis in which B is diagonal we can write:

$$(BD(g))^{i}{}_{j} = (D(g)B)^{i}{}_{j}$$

$$\Rightarrow B_{ii}\, D(g)^{i}{}_{j} = D(g)^{i}{}_{j}\, B_{jj} \quad \text{(no sum over any index)}$$

Since this true for all g & i and j. For the non-zero elements $D(g)^{i}{}_{j}$ we get

$$B_{ii} = B_{jj}$$

And so $B = \lambda \mathbb{1}$.

If $D(g)^{i}{}_{j} = 0 \ \forall g$, $i \neq j$ then $D(g)$ is not an irrep. E.g.

These elements

$$D(g) = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}$$

In which case

$$B = \begin{pmatrix} \lambda \mathbb{1} & 0 \\ 0 & \mu \mathbb{1} \end{pmatrix}.$$

An aside on structure constants:

Using the adjoint representation we can define a metric on the Lie algebra:

$$g_{ab} = \mathrm{Tr}(F_a F_b)$$

Using this metric we can lower the index of the structure constants:

$$f_{abc} \equiv g_{cd} f_{ab}{}^{d}$$

Furthermore, it can be shown for compact Lie algebras that

$$f_{abc} = f_{[abc]}$$

i.e., the structure constants are completely antisymmetric.