Classical Mechanics

Lecture #4

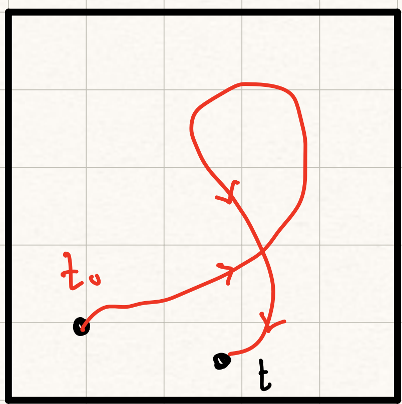
Degrees of freedom and generalized coordinates

Consider N particles in 3 dimensions. The instantaneous positions of these particles are described by 3N real numbers. We then say that our system consists of 3N degrees of freedom. This statement is true independent of the coordinate system. We denote by q^A with $A=1,2,\dots,3N$ the 3N numbers that describe the degrees of freedom and call them generalized coordinates. The generalized coordinates at any given time give the full configuration of the system and the space of generalized coordinates is called the configuration space.

A point on the configuration space specifies the positions of all the degrees of freedom at a given instant of time. However the generalized coordinates at a given instant are not enough to predict the evolution of the system. To do that one also needs the time derivatives of the generalized coordinates, known as the generalized velocities: $\dot{q}^A = \frac{dq^A}{dt}$, $A = 1,2,3,\dots,3N$.

Alternatively one also needs the generalized coordinates at a <u>different</u> instant of time. This is because Newton's law is a second order ODE and one needs <u>twice</u> the number of boundary conditions as the number of degrees of freedom.

The equations of motion are relationships which give us $q^A(t)$ given $q^A(t_0)$ and $\dot{q}^A(t_0)$ for a specific time.



Time evolution of a system in configuration space

The Lagrangian $L(q^A, \dot{q}^A, t)$ is a function of the generalized coordinates, the generalized velocities and time in general. The action S is defined to be:

$$S[q] = \int_{t_1}^{t_2} L(q^A, \dot{q}^A, t)\, dt$$

Each path in the configuration space is a different function

of t. Thus a change in q^A will change the numerical value of the action S.
Thus the action $S: \mathcal{F}_t \to \mathbb{R}$, i.e., it is a map from the space of functions (of t) to the real number. Such an object is called a functional. The action is a functional.

If $q_1^A(t)$ and $q_2^A(t)$ are two different functions then in general $S[q_1] \neq S[q_2]$. If we change a path by a tiny amount $q^A(t) \to q^A(t) + \delta q^A(t)$, $\delta q^A(t)$ is small, we would expect
$S[q] \longrightarrow S[q'] = S[q] + \delta S[q, \delta q]$.

Then $\lim_{\delta q \to 0} \dfrac{S[q'] - S[q]}{\delta q} \equiv \dfrac{\delta S}{\delta q(t)}$ is called the variational derivative. As you shall see in the tutorial, the variational derivative is simply the partial derivative in disguise.

The Principle of Least Action

[Also known as: The principle of stationary action or Hamilton's principle.]

We are now ready the state an alternative to Newton's law. It is known as the principle of least action (which is actually a mishomer):

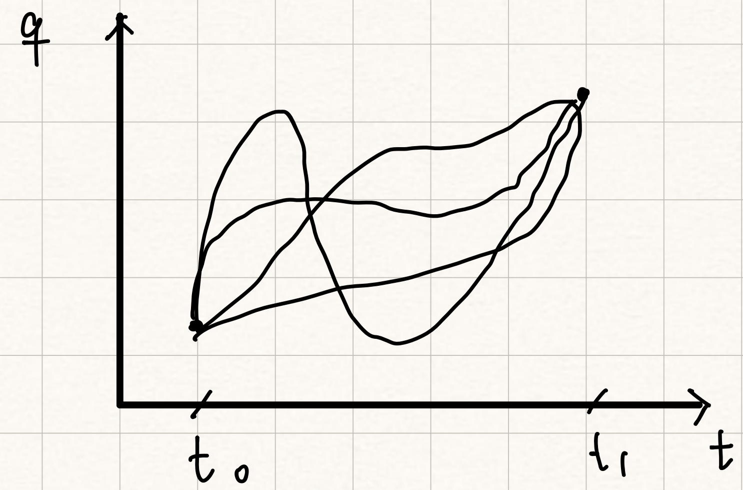
Every physical system is described by a Lagrangian function $L(q^A, \dot{q}^A, t)$. The physical path of the system is such that a variation of that path leads to a stationary action:

$$\delta S \Big|_{q_{\text{physical}}} = 0 .$$

This principle seems far from the equations of motion (Newton's law) which are ordinary 2nd order differential equations. We shall now show that the consequence of the principle of least action are second order differential equations.

Euler-Lagrange Equations

Suppose we have a physical system whose generalized coordinates at time t_0 and t_1 are given by $q^A(t_0)$ and $q^A(t_1)$. There are infinitely many functions that connect $q^A(t_0)$ and $q^A(t_1)$:



But given the boundary conditions $q^A(t_0)$ and $q^A(t_1)$ there is only one physical path $q^A_{Phy}(t)$ that connects them.

If we take an arbitrary path $q_1^A(t)$ subject to the boundary condition $q_1^A(t_0) = q_0^A$ and $q_1^A(t_1) = q_1^A$ and vary it $q_1^A \to q_1^A + \delta q_1^A$ such that $\delta q_1^A(t_0) = \delta q_1(t_1) = 0$, then δS will not vanish in general. But for the physical path $q^A_{Phy}(t)$, its variation will lead to $\delta S = 0$.

$q^A(t) \to q^A(t) + \delta q^A(t)$ then implies:

$$S[q] = \int_{t_0}^{t_1} L(q_A, \dot{q}_A, t)\, dt \to S[q + \delta q] = \int_{t_0}^{t_1} L(q_A + \delta q_A, \dot{q}_A + \delta \dot{q}_A, t)\, dt$$

$$S[q+\delta q] = \int_{t_0}^{t_1} \left(L + \frac{\partial L}{\partial q^A}\delta q^A + \frac{\partial L}{\partial \dot{q}^A}\delta \dot{q}_A \right) dt$$ [Implicit sum over repeated index.]

And so $\delta S = S[q+\delta q] - S[q]$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^A}\delta q^A + \frac{\partial L}{\partial \dot{q}^A}\delta \dot{q}^A \right) dt$$

Now in the second term use the fact $\delta \dot{q}^A = (\delta q^A)^{\cdot}$ to do integration by parts:

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^A}\delta q^A + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\delta q^A \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A} \right)\delta q^A \right) dt$$

The second term is a total derivative and so using the fundamental theorem of algebra we get:

$$\int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\delta q^A \right) dt = \frac{\partial L}{\partial \dot{q}^A}\delta q^A \Big|_{t_0}^{t_1} = 0$$

since $\delta q^A(t_0) = \delta q^A(t_1) = 0$.

Thus we get: $\delta S = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A} \right) \right] \delta q^A \, dt$. For the physical path this vanishes:

$$\int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right)\right] \delta q^A \, dt = 0$$

The variations δq^A are small but arbitrary. Thus the above condition is satisfied if the quantity in the square brackets is satisfied:

$$\boxed{\frac{\partial L}{\partial q^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_A}\right) = 0}$$

These are the Euler-Lagrange equations. The solutions to these equations give us the physical path in the configuration space.

<u>An Example:</u> A single particle. The Lagrangian for a single particle is given by:

important

$$L = T - V$$

where $T = \frac{1}{2} m \dot{\vec{r}}^{\,2}$ and $V = V(\vec{r})$ is the potential energy. Then the different terms in the Euler-Lagrange equtions are:

$$\frac{\partial L}{\partial q^A}: \quad -\frac{\partial V}{\partial r_i} = F_i \quad \Bigg| \quad -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_A}\right): \quad -\frac{d}{dt}\left(\frac{1}{2}\cdot 2 m \dot{r}_i\right) = -m\ddot{r}_i$$

And so the equation of motion becomes:

$$F_i = m\ddot{r}_i = \dot{p}_i$$

$$\bar{F} = \dot{\vec{p}}$$

Comments:

1. The variantial priciple (yet another name for Hamilton's principle) only deals with paths and makes no reference to reference frames. Thus it is valid for all frames, inertial or non-inertial.

2. The form of the Euler-Lagrange equations remain invariant under coordinate transformations.

3. The Lagrangian is not unique. Two Lagrangians, L & L', if they are related by a total time derivate of form given by:

$$L'(q^A, \dot{q}^A, t) = L(q^A, \dot{q}^A, t) + \frac{d}{dt} f(q^A, t)$$

Then we get the same equation of motion since $\delta S' = \delta S$.

4. Symmetrics of the Lagrangian leads to conserved quantities.

Coordinate Invariance of the E-L equations:

The Lagrangian is a scalar quantity. This means that the equations of motion, the Euler-Lagrange equations, holds in any coordinate system.

Let $x_1(t), x_2(t), \dots, x_{3N}(t)$ be a set of (generalized) coords for our system. Let the new coordinate system be:

$$q_a = q_a(x_1, x_2, \dots, x_{3N}, t) \quad , \quad a = 1, 2, \dots, 3N$$

This means that $\dot{q}_a = \frac{\partial q_a}{\partial x_A} \dot{x}_A + \frac{\partial q_a}{\partial t}$ {Summation over repeated index}

The relationship between x_A & q_a can be inverted as long as $\det\left(\frac{\partial q_a}{\partial x_A}\right)$ is non-zero.

$$\dot{x}_A = \frac{\partial x_A}{\partial q_a} \dot{q}_a + \frac{\partial x_A}{\partial t}$$

We also have $\frac{\partial \dot{x}_A}{\partial \dot{q}_a} = \frac{\partial}{\partial \dot{q}_a}\left(\frac{\partial x_A}{\partial q_b} \dot{q}_b + \frac{\partial x_A}{\partial t}\right)$

$$= \frac{\partial x_A}{\partial q_b} \delta_{ab} = \frac{\partial x_A}{\partial q_a}$$

Thus $\frac{\partial L}{\partial q_a} = \frac{\partial L}{\partial x_A} \frac{\partial x_A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_A} \frac{\partial \dot{x}_A}{\partial q_a}$

$$= \frac{\partial L}{\partial x_A} \frac{\partial x_A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_A}\left(\frac{\partial^2 x_A}{\partial q_a \partial q_b} \dot{q}_b + \frac{\partial^2 x_A}{\partial q_a \partial t}\right)$$

$$\frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}_A}\frac{\partial \dot{x}_A}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}_A}\frac{\partial x_A}{\partial q_a}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_A}\right)\frac{\partial x_A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}_A}\left\{\frac{\partial^2 x_A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x_A}{\partial q_a \partial t}\right\}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = \left\{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_A} - \frac{\partial L}{\partial x_A}\right\}\frac{\partial x_A}{\partial q_a}$$

Since $\det \frac{\partial x_A}{\partial q_a} \neq 0 \Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_A}\right) - \frac{\partial L}{\partial x_A} = 0$$

This shows that the form of the Euler-Lagrange equations are unchanged regardless of the coordinate system used.

<u>Comment:</u>

The only thing that we assumed about the coordinate transformation is that it is invertible. Nothing was assumed about whether either coordinate system was intertial.