Chapter 3: Infinite Series

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PHY405 – Mathematical Physics
Fall 2024
Reference Texts: Mary L. Boas; Arfken, Weber and Harris.

1 Sequences and Series: What's the difference?

1.1 Sequences

A *sequence*, $\{a_n\}_{n=1}^{\infty}$, is a countably infinite set of numbers,

$$a_1, a_2, a_3, ..., a_n, ... \quad (1)$$

For instance the set of all natural numbers ($\mathbb{N}$) is such a sequence.

We say a sequence possesses a (finite) limit l,

$$\lim_{n\to\infty} a_n = l \ , \quad (2)$$

or, simply put, $a_n \to l$ as $n \to \infty$. More formally, it can be stated as, for every $\varepsilon > 0$, no matter how small, there exists a number N for which

$$|a_n - l| < \varepsilon \quad \forall \ n > N \ . \quad (3)$$

Note that, Eq.(2) (and Eq.(3)) is a **necessary** (but **not sufficient**) condition for convergence of the sequence. However, it is not always easy to find the limit and so it is helpful to view this from a different perspective.

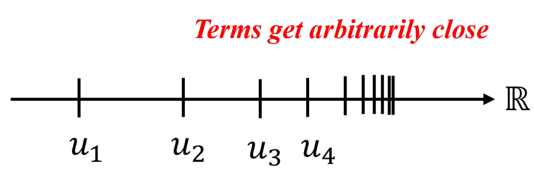

Figure 1: Sequence on $\mathbb{R}$ approaching a limit - terms cluster together around the value.

Consider Figure 1 which shows a sequence on the real number line and clearly we notice that the terms 'cluster' together as you go towards the right and so clearly the sequence approaches a limiting value. Without calculating the limit however, we can still say that for all $\varepsilon > 0$, $\exists \ N \in \mathbb{N}$, $\forall \ n \geq m \geq N$ such that,

$$\boxed{|a_n - a_m| < \varepsilon} \quad (\textbf{Cauchy Criterion} \text{ for convergence}) \ . \quad (4)$$

Any sequence with the property defined by Eq.(4) is known as a **Cauchy sequence** and by definition, for a sequence of real numbers, Cauchy sequence $\Leftrightarrow$ convergent sequence (*Completeness Axiom*).

1.2 Series

A *series* is a sequence of partial sums, i.e.

$$s_n = \sum_{k=1}^{n} a_k \ , \tag{5}$$

given that, $\{a_k\}_{k=1}^{\infty}$ is a sequence. The set of all these sums, $\{s_n\}_{n=1}^{\infty}$, itself forms a sequence. If this latter sequence has a limit S, i.e.,

$$s_n \to S \text{ as } i \to \infty \ , \tag{6}$$

then we say that the **infinite series**,

$$\lim_{n\to\infty} \sum_{k=1}^{i} a_k = \sum_{k=1}^{\infty} a_k \ , \tag{7}$$

possesses the limit S (or, converges to S), i.e.

$$\sum_{k=1}^{\infty} a_k = S \ . \tag{8}$$

By the Cauchy criterion, this will be true *if and only if*

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \ , \tag{9}$$

for any fixed $\varepsilon > 0$, whenever $n \geq m \geq N$. By using the definition of partial sum given in Eq.(5), one can show that an equivalent statement for Cauchy criterion is,

$$\boxed{\left| \sum_{k=m}^{n} a_k \right| < \varepsilon \quad \text{is equivalent to} \quad |s_n - s_m| < \varepsilon} \ , \tag{10}$$

for any fixed $\varepsilon > 0$, whenever $n \geq m \geq N$. Obviously, a **necessary** (but **not sufficient**) condition for a series

$$\sum_{k=1}^{n} a_k$$

to converge is that the sequence must obey,

$$\lim_{k\to\infty} a_k \to 0 \ , \tag{11}$$

otherwise the series is said to be **divergent**.

Example 1. Determine if the infinite series

$$\sum_{k=1}^{\infty} (-1)^k$$

converges or diverges.

Cauchy criterion: $\left|\sum_{k=m}^{n}(-1)^k\right|$ and let $m = N$ and $n = N+2$ then there are exactly three terms and we can calculate each of these for two cases namely, when N is odd and when N is even. Thus,

$$\left|\sum_{k=N}^{N+2}(-1)^k\right| = \left\{ \begin{array}{l} \text{even N: } |1+(-1)+1| \\ \text{odd N: } |-1+1+(-1)| \end{array} \right\} = 1$$

which $\Rightarrow \left|\sum_{k=m}^{n}(-1)^k\right| = 1$ which does not satisfy Cauchy criterion for all $\varepsilon > 0$, if for example we set $\varepsilon = 1$. Hence, the series is **not** convergent or alternatively we can say that the series diverges.

Example 2. A familiar series is the **geometric series**. Consider the series to be

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ...$$

where a is a constant and $r \geq 0$. Determine if the series converges.

Let's look at the $nth-$partial sum s_n (sum of first n terms):

$$s_n = \sum_{m=0}^{n-1} ar^m = a + ar + ar^2 + ar^3 + ... + ar^{n-1}, \tag{12}$$

and multiply both sides by r such that,

$$\begin{aligned} &\Rightarrow rs_n = r(a + ar + ar^2 + ar^3 + ... + ar^{n-1}) \\ &\Rightarrow rs_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^n \ . \end{aligned} \tag{13}$$

Now, perform (12) - (13) which yields,

$$\begin{aligned} &s_n(1-r) = a - ar^n \\ &\Rightarrow \boxed{s_n = \frac{a(1-r^n)}{1-r}} , \end{aligned} \tag{14}$$

which represents the sum of first n terms of the geometric series. From observation, clearly the series diverges when $|r| > 1$. As such, we restrict our attention to $|r| < 1$ so that for large n, r^n approaches zero and s_n has the limiting value of,

$$\begin{aligned} &\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \\ &\text{clearly, } r^n \to 0 \\ &\Rightarrow \boxed{s_\infty = \frac{a}{1-r}} , \end{aligned} \tag{15}$$

where s_∞ is a finite limit which the series approaches as $n \to \infty$ for the case when $|r| < 1$. Thus, in such cases, geometric series is convergent.

Example 3. Verify that the infinite geometric series: $\frac{2}{3}+\frac{4}{9}+...+\left(\frac{2}{3}\right)^n$, has a sum (or, in other words, converges).

Identify: $a=\frac{2}{3}$, $r=\frac{4/9}{2/3}=\frac{2}{3}$.

Now, using Eq.(13), the sum of the first n terms of the series is,

$$s_n=\frac{\left(\frac{2}{3}\right)\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}=2\left(1-\left(\frac{2}{3}\right)^n\right)$$

which clearly shows that as $n\to\infty$ then $s_n\to 2$, meaning that the series sums to 2. We can confirm this result by directly using Eq.(14) for $n\to\infty$ as,

$$s_\infty=\frac{2/3}{1-(2/3)}=2$$

.

2 Review of Calculating Limits

For the remainder of the class, let us brush up on the calculating limits.

Example 4. Find the limit as $n\to\infty$ of the sequence

$$\frac{(2n-1)^4+\sqrt{1+9n^8}}{1-n^3-7n^4}\ .$$

$$=\lim_{n\to\infty}\frac{(2n-1)^4+\sqrt{1+9n^8}}{1-n^3-7n^4}$$

$$=\lim_{n\to\infty}\frac{\frac{(2n-1)^4+\sqrt{1+9n^8}}{n^4}}{\frac{1-n^3-7n^4}{n^4}}=-\frac{19}{7}\ .$$

Example 5. Calculate, by applying L'Hôpital's rule, the limit,

$$\lim_{n\to\infty}\frac{\ln n}{n}$$

$$\Rightarrow\lim_{n\to\infty}\frac{1/n}{1}=0\ .$$

Example 6. Calculate the limit,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}}\ .$$

Let: $y = \frac{1}{n}$. Then evaluate,

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \ln y = -\lim_{n\to\infty} \frac{1}{n} \ln n = 0 \ .$$

Then,

$$\lim_{n\to\infty} y = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = e^0 = 1 \ .$$

2.1 The Harmonic Series

An important series to consider is the *harmonic series*,

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ... = \sum_{n=1}^{\infty} \frac{1}{n} \ . \tag{16}$$

At first glance, one notices that the series satisfies the preliminary test for convergence (Eq.(11)), i.e., the terms in the sequence approaches zero with increasing n. Let us apply the Cauchy criterion to verify this intuition. Recall,

$$\text{Cauchy criterion:} \qquad |s_{\tilde{n}} - s_{\tilde{m}}| < \varepsilon \text{ for any fixed } \varepsilon > 0 \ \forall \ \tilde{n} \geq \tilde{m} \geq N \in \mathbb{N} \ .$$

Let $\tilde{m} = n, \tilde{n} = 2n$ such that we have:

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

and for the adjacent series,

$$s_{2n} = ... + \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} \ .$$

Then, no matter how large n is, it must be true that,

$$\begin{aligned} |s_{2n} - s_n| = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} &\geq \frac{1}{n+n} + \frac{1}{n+n} + ... + \frac{1}{n+n} \\ &\geq n\left(\frac{1}{2n}\right) = \frac{1}{2} \ . \end{aligned} \tag{17}$$

Of course this violates the Cauchy criterion (at least for $\varepsilon = \frac{1}{2}$) and we can conclude that the sequence $s_{\tilde{n}}$ is not Cauchy or in other words, the harmonic series *diverges*.

3 Absolute and Conditional Convergence

Suppose we have a convergent series $\sum_{n=1}^{\infty} a_n$. If it is also true that the series $\sum_{n=1}^{\infty} |a_n|$ converges, we conclude that the original series **converges absolutely**. Otherwise, we say that the original series is **conditionally convergent**, i.e. it converges due to sign alterations. A sufficient condition for (at least) conditional convergence is provided by the following theorem due to Leibnitz:

If the terms of a series are of alternating sign and additionally, their absolute values tend to zero, i.e. $|a_n| \to 0$, monotonically, i.e. $|a_n| > |a_{n+1}|$ for sufficiently large n, then we can say

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

In absolutely convergent series, one can rearrange the terms without affecting the value of the sum. With conditionally convergent series, such rearrangements of terms may not only alter the value of the sum to any desired value, but can also make the conditionally convergent series to diverge!

Example 7. We can make a conditionally convergent series from the divergent harmonic series (Eq.(16)) by alternating every other sign:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = \ln 2 \ . \tag{18}$$

Clearly, each of $\{|a_n|\}_{n=1}^{\infty} \to 0$ and according to Leibnitz, this series is at least conditionally convergent. Now, let's multiple (18) by $\frac{1}{2}$:

$$\begin{aligned} &\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + ... = \frac{1}{2}\ln 2 \\ \Rightarrow\ &2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + ... = \frac{1}{2}\ln 2 \\ \Rightarrow\ &1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = \frac{1}{2}\ln 2 \ , \end{aligned} \tag{19}$$

upon rearranging and adding terms with the same denominators. But notice, (19) is exactly the same series as (18), giving different sums! This is obviously a contradiction and so it makes no mathematical sense. This is justified by Riemann's series theorem which states that making such random rearrangements for certain types of series (e.g. the alternating harmonic series) leads to contradictions such as this one.

Exercise 1. Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = \ln 2$.

3.1 A Theorem about Absolutely Convergent Series

For an absolutely convergent series then, one can rearrange the terms without changing the value of the sum. Moreover, if the two series,

$$S = \sum_{i=1}^{\infty} u_i \ , \quad T = \sum_{j=1}^{\infty} v_j$$

are absolutely convergent, then the series,

$$P = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_i \; v_j \ ,$$

formed from the product of their terms written in *any* order, is absolutely convergent; the value of the sum of the series is equal to the product of the individual series, i.e.,

$$P = ST \ .$$

3.2 Convergence Tests

The following tests can determine whether a given series is absolutely convergent or not.

3.2.1 Comparison Test

If $b_n > 0 \; \forall \; n$ and you know that the series,

$$\sum_{n=1}^{\infty} b_n$$

is convergent, then the series you are testing namely,

$$\sum_{n=1}^{\infty} a_n$$

is **absolutely convergent** if $|a_n| \leq b_n \; \forall \; n$, from some $nth-$term onwards. In words, this means that the absolute value of each term of the $a-$series is no larger than the corresponding term of the $b-$series.

If, on the other hand, $|a_n| \geq b_n > 0 \; \forall \; n$ and it is known that $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ is **not** absolutely convergent and $\sum_{n=1}^{\infty} a_n$ may converge conditionally or diverge.

Example 8. Test if

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + ...$$

is absolutely convergent.

From all the series we have discussed so far, we can use a geometric series with $r < 1$ to be the comparison series. So,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$$

Note that $\forall \; n > 3$, we have $\left| \frac{1}{n!} \right| < \frac{1}{2^n}$. Thus, we can conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n!} \text{ is absolutely convergent.}$$

3.2.2 Root Test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if, from a certain term onward,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \leq q < 1 \ , \tag{20}$$

where $q \geq 0$ is independent of n.

3.2.3 Ratio Test

The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if, from a certain term onward,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq q < 1 \ , \tag{21}$$

where $q \geq 0$ is independent of n.

Example 9. When does $\sum_{n=1}^{\infty} nq^n$ converge? By the root test:

$$\lim_{n\to\infty} \sqrt[n]{|nq^n|} = |q| \underbrace{\lim_{n\to\infty} \sqrt[n]{|n|}}_{=1} = |q| \ .$$

By the ratio test:

$$\lim_{n\to\infty} \left|\frac{(n+1)q^{n+1}}{nq^n}\right| = |q| \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = |q| \ .$$

We see that in both cases, the series is absolutely convergent if $|q| < 1$, and divergent otherwise.

Aside: If, for instance, from the ratio test you determine the limiting $q = 1$, then we have an indeterminate result, i.e. the given series neither converges nor does it surely diverge. We then implement other tests at our disposal.

3.2.4 Integral Test

If $f(x)$ is a continuous, monotonically decreasing real function of $x \in \mathbb{R}$ such that,

$$f(n) = |a_n| \ , \tag{22}$$

then

$$\sum_{n=1}^{\infty} |a_n| \text{ converges if } \int_1^{\infty} dx \ f(x) < \infty \ . \tag{23}$$

This test follows from the geometric interpretation of the integral as being the are under the graph can be made obvious by examining the graphs in Figure 2. The graphs shown are for testing the harmonic series, so

$$f(x) \equiv y = \frac{1}{x} \ \forall \ x \ .$$

The area of the rectangles are just the terms in the series. The curve on the left have the top edge of each rectangle above the curve (overestimating the area under the curve) while the curve on the

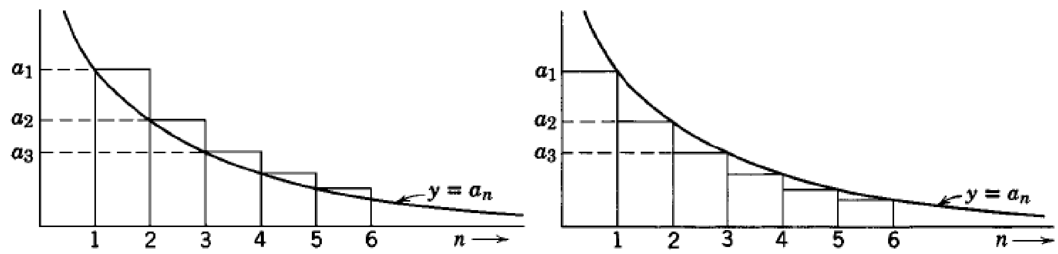

Figure 2: The integral tests for monotonically decreasing series.

right has the top edge of each rectangle below the curve (underestimating). Notice that the area under the curve is given by

$$\int_1^\infty a_x \, dx \ .$$

For the left curve, we can see

$$\sum_{n=1}^\infty a_n > \int_1^\infty a_n \, dn$$

while the

$$\sum_{n=1}^\infty a_n < \int_1^\infty a_n \, dn \ ;$$

if both of these integrals are finite (infinite) then we can say the series is convergent (divergent). The terms at the beginning of the series have nothing to do with convergence; the only requirement for convergence is that the integral is finite from a certain term onwards.

Example 10. Test the Riemann zeta function

$$\zeta(p) = \sum_{n=1}^\infty n^{-p} \ , \tag{24}$$

for convergence. Let $f(x) = x^{-p}$ and use the integral test, Now, we know for $p = 1$ we have,

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

which we have shown to be divergent. When $p \neq 1$, our integral becomes:

$$\begin{aligned}
&\int_1^\infty x^{-p} \, dx = \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^t \quad \text{for } x \geq 1 \\
&\Rightarrow \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] \\
&\text{if } 0 \leq p < 1 \text{ then: } \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] \to \infty \text{ and so } \sum_{n=1}^\infty n^{-p} \text{ diverges;} \\
&\text{if } p > 1 \text{ then: } \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right] = \lim_{t\to\infty} \left[\frac{1}{t^{p-1}(-p+1)} - \frac{1}{-p+1}\right] \\
&\qquad\qquad = -\frac{1}{1-p}(\text{finite}) \Rightarrow \sum_{n=1}^\infty n^{-p} \text{ converges.}
\end{aligned} \tag{25}$$

3.2.5 A Special Comparison Test

This test has two parts:

1. If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and $\dfrac{a_n}{b_n} \to$ finite limit, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} d_n$ is a **divergent series** of positive terms and $a_n \geq 0$ and $\dfrac{a_n}{b_n} \to l$ with $l > 0$ (up to ∞), then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 11. Test for convergence, the series:

$$s_n = \sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2} \ .$$

As $n \gg 3,\ s_n \simeq \dfrac{\sqrt{2n^2}}{4n^3} = \dfrac{\sqrt{2}}{4n^2}$ and as a comparison series we can thus consider,

$$b_n = \sum_{3}^{\infty} \frac{1}{n^2} \ .$$

Is $\{b_n\}_{n=3}^{\infty}$ convergent? Notice that this is just the zeta function series with $p = 2$ and thus we are sure it converges! Convince yourself of this by applying your favourite convergence test. Then, by the special comparison test we have,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}}{1/n^2} \right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n^2\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2} \right)$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n^3\sqrt{2 - 5/n + 1/n^2}}{n^3(4 - 7/n + 2/n^2)} \right) = \frac{\sqrt{2}}{4} \ ,$$

which shows that the series given converges.

4 Power Series

The concept of infinite series that we have built up thus far can be extended to include a sum of a sequence of functions, instead of a sequence of constants. The partial sums then become,

$$s_n(x) = a_1(x) + a_2(x) + a_3(x) + \ldots + a_n(x) \ , \tag{26}$$

and similarly the series sum becomes,

$$\sum_{n=1}^{\infty} a_n(x) = \lim_{n\to\infty} s_n(x) = S(x) \ . \tag{27}$$

These types of series are called *power series* because the terms are multiples of powers of x. Some examples of power series are:

$$\begin{aligned} &1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+...+\frac{(-x)^n}{2^n}+... \\ &x-\frac{(x+2)}{\sqrt{2}}+\frac{(x+2)^2}{\sqrt{3}}+...+\frac{(x+2)^n}{\sqrt{n+1}}+... \end{aligned} \tag{28}$$

4.1 Uniform Convergence

If for any arbitrarily small $\varepsilon > 0$, there exists a number N, **independent of** x in the interval $[a, b]$ such that,

$$|S(x) - s_n(x)| < \varepsilon, \qquad \forall\, n \geq N\ , \tag{29}$$

then the series is said to be **uniformly convergent** over that interval. Stated simply, this means that, for a uniformly convergent series of functions, one can always find a finite N such that the absolute value of the *tail* of the infinite series, $|\sum_{n=N+1}^{\infty} a_n(x)|$ is less than any arbitrarily small value of ε for all x in the given interval (including endpoints). This is the Cauchy criterion for a power series to be uniformly convergent. For a sequence of functions, the condition of uniform convergence becomes,

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N\ . \tag{30}$$

Let us evaluate some examples.

Example 12. Test the uniform convergence of the series,

$$S(x) = \sum_{n=0}^{\infty}(1-x)x^n$$

on the interval $[0, 1]$.

$$\begin{aligned} &\text{Realize that, } S(x) = (1-x)\ \sum_{n=0}^{\infty} x^n \\ &\Rightarrow S(x) = (1-x).\frac{1}{1-x} = 1 \text{ for the interval } [0,1)\ . \\ &\text{But at } x = 1,\ S(x=1) = 0\ . \end{aligned}$$

$$\text{Meaning: } \sum_{n=0}^{\infty}(1-x)x^n = \begin{cases} 1 \text{ for } 0 \leq x < 1 \\ 0 \text{ for } x = 1\ . \end{cases}$$

Thus, $S(x)$ is convergent for the entire interval $[0, 1]$, and since each term is non-negative, it is also **absolutely** convergent. The next thing we check for is the uniform convergence of the series in the given interval. If $x \neq 0$, the partial sum of the series is,

$$\begin{aligned} s_N &= (1-x)\ \sum_{n=0}^{N} x^n \\ \Rightarrow s_N &= (1-x)\ .\ \frac{1-x^N}{1-x} = 1 - x^N\ . \end{aligned}$$

Then the uniform convergence criterion becomes,

$$|S(x) - s_N(x)| = |1 - (1 - x^N)| = x^N < \varepsilon,$$

is not true for all values of N because there will always exist some value of x (close to 1) for which one can always find a arbitrarily smaller $\varepsilon > 0$. Thus, the series is **not** uniformly convergent. Note that, a series can thus be absolutely convergent but may not necessarily be uniformly convergent − there are independent concepts.

Example 13. Test the uniform convergence of the series,

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n+x} \ , \text{ in the interval } [0, b] \text{ with } b > 0.$$

Let, $f_n(x) = \dfrac{1}{x+n}$

and $f(x) = \lim_{n\to\infty} f_n(x) = 0 \qquad \forall\ x \in [0, b]$.

Then the uniform convergence test leads to

$\left|f_n(x) - f(x)\right| = \left|\dfrac{1}{x+n} - 0\right| = \left|\dfrac{1}{x+n}\right| < \varepsilon$

$\Rightarrow 1 < \varepsilon(x+n)$

$\Rightarrow n > \dfrac{1}{\varepsilon} - x$ decreasing with x such that the maximum value of n is $\dfrac{1}{\varepsilon}$.

A good thing check is if there exists any x for which $\dfrac{1}{\varepsilon} \to \infty$?

Just by looking at it, we can say the above condition is only satisfied for $x \to -\infty$

which is not part of the interval of convergence here.

Thus, there exists $n \geq N > \dfrac{1}{\varepsilon}$ for which

$|f_n(x) - f(x)| < \varepsilon\ \forall\ n \geq N$ and the series is uniformly convergent in the interval $[0, b]$.

4.2 Properties of Uniform Convergence

Consider the series of functions of a real variable,

$$f(x) = \sum_{n=1}^{\infty} g_n(x) \ .$$

For this, the following properties are satisfied:

1. If the g_n are continuous, we can integrate term by term if $\sum_n g_n$ is uniformly convergent over the domain of integration:

$$\int_a^b f(x)\ dx = \sum_{n=1}^{\infty} \int_a^b g(x)\ dx \ . \tag{31}$$

2. If the g_n and $g_n^\alpha = \frac{d}{dx} g_n$ are continuous, and $\sum_n g_n'$ is uniformly convergent, then we can differentiate term by term:
$$f'(x) = \sum_{n=1}^{\infty} g_n'(x) \ . \tag{32}$$

5 Theorems about Power Series and Some Techniques

Theorem 1. A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series with the same interval of convergence as the original series.

Theorem 2. Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence.

Theorem 3. One series may be substituted in another, provided that the values of the substituted series are in the interval of convergence of the other series.

Theorem 4. The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

Theorem 5. If a power series converges for one point, $x = x_0$, it converges *uniformly* and *absolutely* for all x satisfying
$$|x| \leq \eta, \ \ \eta > 0 \text{ but } \eta < |x_0| \ .$$

Example 14. Obtaining a power series for $\sin x$.

Assume: $\sin x = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$

Plug in $x = 0$: $\Rightarrow a_0 = 0$.

by theorem 1, differentiate term by term on both sides:

$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + ...$

plug in $x = 0$: $\Rightarrow a_1 = 1$.
continue this process and you will arrive at ..

$$-\cos x = 3.2a_3 + 4.3.2a_4 x + ... \Rightarrow a_3 = -\frac{1}{3!}$$

Ultimately leading to: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + ...$.

A series obtained in this manner is known as the **Taylor series** or the special variety of the series about the origin $x = 0$ known as the **Maclaurin series**. Formally we define the Taylor series as a power series of $(x - a)$ where a is a constant:
$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + .. + \frac{1}{n!}(x-a)^n f^{(n)}(a) + ... \tag{33}$$

while the Maclaurin series can be obtained by setting $a = 0$ above.

Example 15. Application of Taylor series. Evaluate

$$\lim_{n\to 0}\left(\frac{1-\cos x}{x^2}\right) .$$

$$\frac{1-\cos x}{x^2} = \frac{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-...}{x^2}$$
$$\text{Then : } \lim_{x\to 0}\frac{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-...}{x^2} = \frac{1}{2} .$$

5.1 Binomial Series

Another important application of the Maclaurin series is the derivation of the Binomial theorem, which we are not going to go into but the reader is directed to Arfken and Weber for the details. From direct application of Eq.(33) with $a = 0$, we arrive at the **binomial expansion**,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ... \tag{34}$$

which is convergent for $-1 < x < 1$. Note that Eq.(34) is valid for any m, whether or not it is an integer, and for both positive and negative values of m. For non-negative m, $(1+x)^m$ has a finite sum. The coefficients appearing in the expansion are denoted by,

$$\binom{m}{n} = \frac{m(m-1)(m-2)..(m-n+1)}{n!} . \tag{35}$$

In the special case that $m > 0,\ \&m \in \mathbb{Z}$, we may write the binomial coefficients as,

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} , \tag{36}$$

so that the binomial expansion terminates when $n = m$.

Example 16. The total relativistic energy of a particle of mass m and velocity v is,

$$E = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} ,$$

where c is the speed of light.

Realize this looks like we can apply the binomial expansion:

$$E = mc^2\left[1-\frac{1}{2}\left(-\frac{v^2}{c^2}\right)+\frac{3}{8}\left(-\frac{v^2}{c^2}\right)^2+...\right]$$
$$= mc^2+\frac{1}{2}mv^2+\frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right)+\frac{5}{16}\left(-\frac{v^2}{c^2}\right)^2+...$$

where the first term, mc^2, is the famed *rest-mass* energy. Thus,

$$E_{\text{Kinetic}} = \frac{1}{2}mv^2\left[1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\left(-\frac{v^2}{c^2}\right)^2 + ...\right] ,$$

and clearly, when $v \ll c$, the expression in the square brackets reduces to 1 and we see that the kinetic energy mathces the classical result.