Group Theory
Lecture # 1

Motivations for Studying Group Theory:

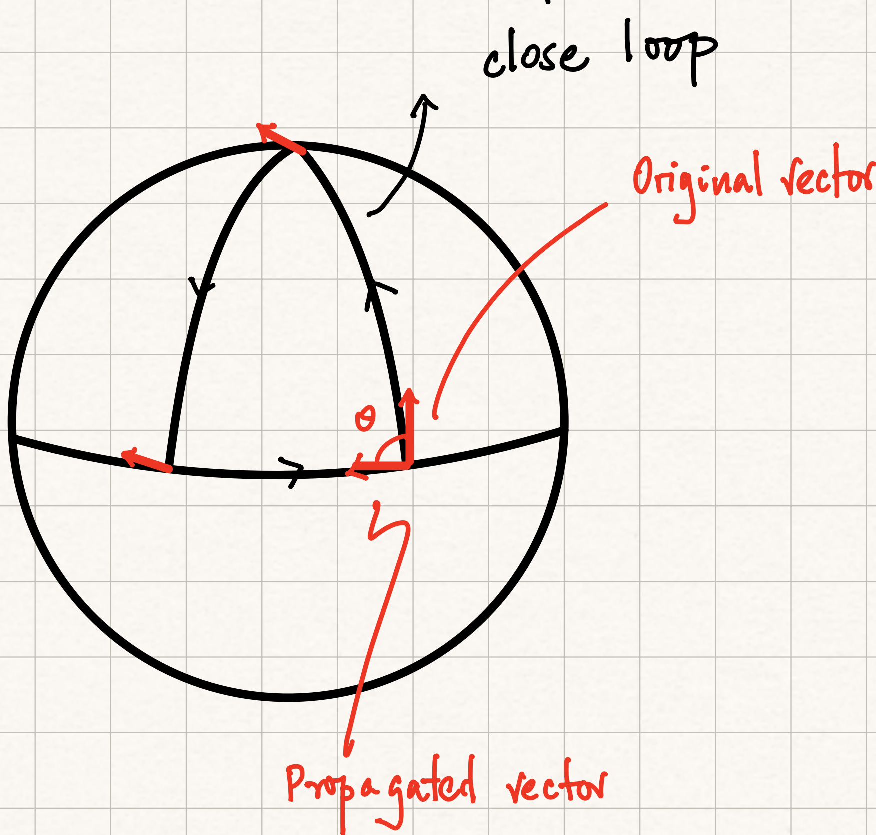
Groups are ubiquitous in both physics and Mathematics. The applications of group theory in mathematics are so numerous that any list of applications will necessarily be incomplete.

In mathematics, groups arose from the work of Évariste Galois. Galois found a way to solve polynomial equations of arbitrary order. A few other places where groups arise in mathematics are:

Geometry of homogeneous spaces. Spaces which look the same from every point are called homogeneous spaces. For example S^n (n-dimensional sphere), $\mathbb{E}^n$ (n-dimensional Euclidean space), H^n (n-dimensional hyperbolic space), M^n (n-dimensional Minkowski space), dS^n (n-dim de Sitter space), AdS^n (n-dim anti-de Sitter space) are all examples of homogeneous spaces. Homogeneous spaces can be thought of in terms of quotient of groups and their subgroups.

For example it is easy to see that the group $SO(3)$ is the symmetry group of the two-dimensional sphere S^2. However every point on the sphere is left invariant by the group $SO(2)$ which is a particular subgroup of $SO(3)$. S^2 is a homogeneous space and we can write it as $S^2 \cong SO(3)/SO(2)$.

Holonomy Groups: In Riemannian geometry curvature of manifolds are classified in terms of something called holonomy groups. For example, if we parallelly transport a small vector around a closed loop on a sphere then the vector will get rotated by an amount that depends on the path:



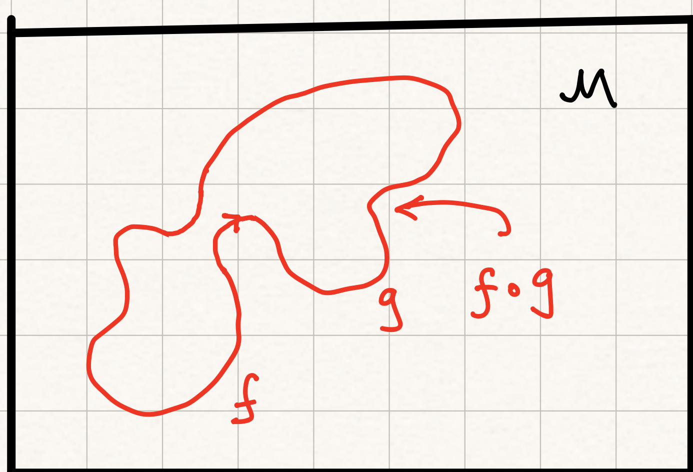
It should clear that although the rotation is path-dependent it must be an element of $SO(2)$. For a given element of $SO(2)$ one can always find a closed path that will rotate a small vector by that element.

S^2 is an example of a homogeneous space and its curvature is characterized by the holonomy group $SO(2)$.

On the hand the two dimensional Euclidean plane $\mathbb{E}^2$ is also a homogeneous space but parallel propagation of any vector around a closed loop does not rotate that vector. Therefore the holonomy group of $\mathbb{E}^2$ is the identity $\mathbb{1}$.

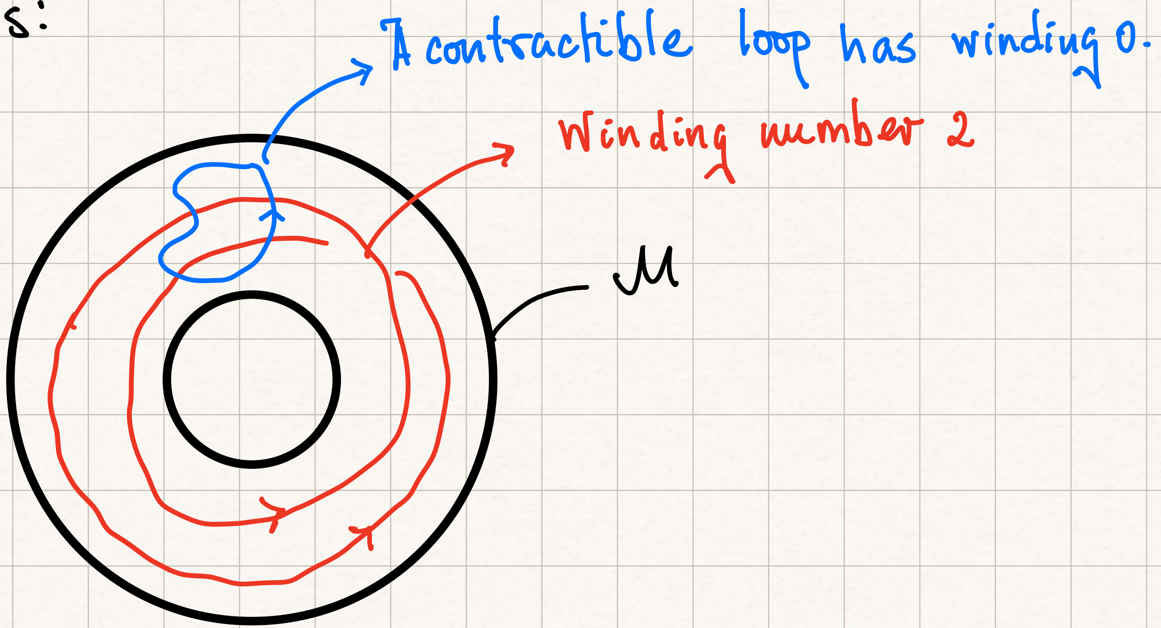
Thus we see that holonomy groups classify homogeneous spaces by their curvature.

Homotopy Groups: Let M be a manifold. [A manifold is a set for which the open neighbourhood of any member looks like an open neighbourhood of $\mathbb{R}^n$.] Then we consider the space of continuous maps from S^1 into M. Two maps f & g are considered homotopic if $\exists$ a continuous map $h(t)$ with $t \in [0,1]$ st $h(t)$ continuously interpolates from $f = h(0)$ to $g = h(1)$. Then the equivalence classes of homotopically equivalent maps can be given an Abelian group structure in the following way:



The homotopy class of constant maps from S^1 into $\mathcal{M}$ form the identity element e of this group. This group is called the first homotopy group of $\mathcal{M}$ or $\pi_1(\mathcal{M})$. If the topology of a manifold is such that $\pi_1(\mathcal{M}) = \mathbb{1}$ then we say that the manifold is simply connected. Intuivity this means that any loop on $\mathcal{M}$ (a loop is simply a map from S^1 into $\mathcal{M}$) is contractible to a point.

An example of a manifold for which $\pi_1(\mathcal{M}) \neq \mathbb{1}$ is the circle or an annulus:



The elements of the fundamental group of an annulus or a circle are loops with different winding numbers: $\pi_1(\mathcal{M}) = \mathbb{Z}$.
The fundamental groups can be generalized to higher homotopy groups $\pi_n(\mathcal{M})$ which classify homotopic maps from S^n into $\mathcal{M}$.

It is intuitively clear that $\pi_1(S^n) = \mathbb{1}$. [Rubber bands fall of a sphere.]

Some applications in physics:

Quantum Mechanics: A major application of group theory and its off-shoot representation theory is quantum mechanics. Eugene Wigner showed that the space of states of a quantal system furnishes a representation of the symmetry group of that system.

The rotation group SO(3) and its 'double cover' SU(2) play an important role in the energy levels of atoms and molecules. Wigner also showed that different kinds of fundamental particles are also classified according to different representations of the Poincaré group.

Particle Physics: Just as groups play an important part in describing the space-time symmetries of quantum mechanical systems, they also describe the variety of different kinds of quarks that govern much of strong interactions. In the 1950s Murray Gell-Mann discovered 'quarks' by conjecturing that behind the zoo of strongly interacting particles are three quarks which furnish an (approximate) representation of the group SU(3).

This ultimately lead to gauge theories and the standard model for particle physics whose particles form representations of the group $SU(3) \times SU(2) \times U(1)$ and leads to the unification of the weak nuclear force and electromagnetism.

Defn of a Group:

A group G is a set with a binary operation $\circ : G \times G \to G$, called the group multiplication, that satisfies the following properties:

[Note that the definition of the binary operation implies that the set is closed under this operation. I.e., if $g_1, g_2 \in G$ then $g_1 \circ g_2 \in G$.]

1. Associativity: Composition is associative: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

2. Existence of identity: $\exists$ a unique element e, known as the identity that satifies:

$$g \circ e = e \circ g = g \quad \forall g \in G.$$

3. Existence of an inverse: For each element $g \in G$ $\exists$ a unique element g^{-1} which is called the inverse element:

$$g \circ g^{-1} = g^{-1} \circ g = e$$

Comments:

1. Note that although closure requires $g_1 \circ g_2 \in G$ and $g_2 \circ g_1 \in G$, these two are not required to be the same elements. Groups for which $g_1 \circ g_2 = g_2 \circ g_1$ $\forall$ g_1 and $g_2 \in G$ are called Abelian groups.

2. The inverse element g^{-1} of g need not be different from g. For example, for the simplest non-trivial group $\mathbb{Z}_2 = \{1, -1\}$ with the group multiplication defined by ordinary arithmetic multiplication we have $e = 1$, $g = -1$, and $g^{-1} = -1$.

3. A group can be discrete or continuous. For discrete group one has a discrete label for each element. A discrete group can be finite or infinite. Groups which have one or more continuous real parameters necessarily have an infinite number of elements.

Some Examples of Groups:

The Translation Group: Consider three-dimensional Euclidean space $\mathbb{E}^3$. Any element of $\mathbb{E}^3$ can be written as a vector

$$\vec{x} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}.$$

We can add to it a constant vector $\vec{a}$ or $\vec{b} \in \mathbb{E}^3$ to each element $\vec{x} \in \mathbb{E}^3$.

$$\vec{x}\,' = \vec{x} + \vec{a}$$
$$\vec{x}\,'' = \vec{x} + \vec{b}$$

or we can add to it $\vec{a} + \vec{b}$:

$$\vec{x}\,''' = \vec{x} + (\vec{a} + \vec{b}).$$

Adding $\vec{a}$ or $\vec{b}$ from $\mathbb{E}^3$ takes $\vec{x} \in \mathbb{E}^3$ to another element in $\mathbb{E}^3$.

It is clear that this operation is a symmetry of $\mathbb{E}^3$. [It is only part of the whole symmetry group of $\mathbb{E}^3$.]

The $\mathbb{E}^3$ from which we are drawing $\vec{a}$ or $\vec{b}$ is an Abelian group where the group multiplication is vector addition and the $\mathbb{E}^3$ whose vectors we are translating is the representation on which the group is acting.

For an element $\vec{a} \in \mathbb{E}^3$, the inverse element is $-\vec{a}$. And for the identity element we have the zero vector $\vec{0}$.

The Rotation Group:

The rotation group are continuous linear transformations of $\vec{x} \in \mathbb{E}^3$, that leave the sphere centred at the origin invariant.

Thus if we represent the vector $\vec{x}$ by a column matrix:

$$\vec{x} \rightarrow \underline{x} = (x, y, z)^T \quad \text{and } R \text{ is a } 3\times3 \text{ matrix}$$

s.t. $\quad \underline{x}' = R\,\underline{x} \quad$ s.t.

$$\underline{x}^T \underline{x} = \underline{x}'^T \underline{x}' = c^2, \quad c \in \mathbb{R}$$

Then $R^T R = \mathbb{1}$ or $R^T = R^{-1}$.

It is not hard to verify that all 3x3 matrices that satisfy

$$R^T = R^{-1}$$

form a group with the group multiplication given by matrix multiplication. This group is called $O(3)$ or the orthogonal group of 3x3 matrices.

The rotations form a subset of $O(3)$, and it is called $SO(3)$ because for rotations

$$\text{Det } R = +1.$$

But in general for $R \in O(3)$ satisfy $\det R = \pm 1$.

Exercises:

Show that all 3×3 matrices R that satisfies

$$R^T = R^{-1}$$

form a group with the group multiplication given by matrix multiplication.

Show that all group elements R continuously connected to the identity satisfy $\det R = +1$.

Show that $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3)$ ~~cannot~~ is not continuously connected to the identity.