Chapter 2: Preliminary Calculus

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This chapter is concerned with reviewing the basics of calculus. Arguably, this subject has the most wide ranging applications in physics, compared to any other mathematical formalism. To have the most complete review, I have chosen to break this chapter down into two sections – the first section deals only with differential calculus while integral calculus is covered in the second section.

1 Differential Calculus

Say, we have a function $f(x)$ and we are interested in examining the dependence of the function on x (its *argument*). More specifically, if we want to see how quickly or slowly $f(x)$ changes as we change x, then we employ the techniques of *differentiation.* Many examples of physical variables are defined as differentials (or changes) of other physical variables such as, velocity (rate of change of position $x(t)$ with time t), acceleration (rate of change of velocity $v(t)$ with time t), etc.

1.1 Differentiation from first principles

Consider the function $f(x)$ as shown in Figure 1.

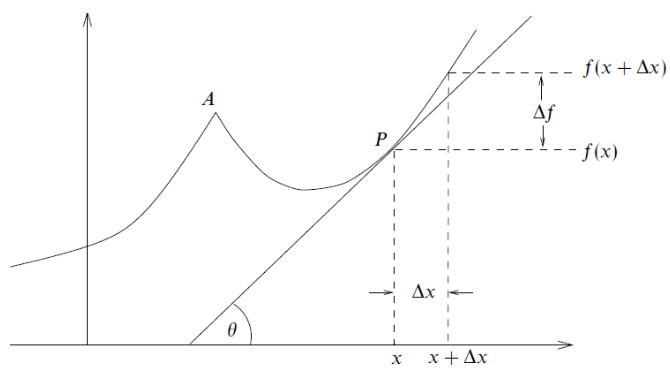

Figure 1: The graph of $f(x)$ under investigation.

Near any point P, the value of the function changes by an amount Δf as x changes by a small amount Δx. Then the slope of the tangent to the graph at P is approximately $\Delta f/\Delta x$. Mathematically, the true value of the gradient or the *first derivative* of $f(x)$ at P is defined as,

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}, \tag{1}$$

provided that the limit exists. If indeed the limit exists, say, at a point $x=a$ then $f(x)$ is said to be *differentiable* at a. In (1) $\Delta x \to 0$ can be obtained from either positive or negative $x-$axis but for the definition to be true, both limits must agree. Furthermore, once differentiability is confirmed at $x=a$, then it necessarily true that $f(x)$ is also *continuous* at $x=a$. In a non-technical sense, one can think of a continuous function at $x=a$ as having no sudden jumps in the value of $f(x)$

at that point. However, the converse statement is not necessarily true. For instance, in Figure 1, notice the 'kink' at A: if you calculate the two limits of the gradient as $\Delta x \to 0$ from positive and negative values. you would find the limits to be different, even though $f(x)$ is continuous at A.

From Eq.(1) it follows that an $n-th$ order derivative of the function $f(x)$ is defined as,

$$\boxed{f^{(n)}(x) \equiv \frac{df(x)}{dx} \equiv \lim_{\Delta x \to 0} \frac{f^{(n-1)}(x+\Delta x) - f^{(n-1)}(x)}{\Delta x}} \ , \tag{2}$$

where n denotes the order of the derivative such that $f'(x) \equiv f^{(1)}$, $f''(x) \equiv f^{(2)}$, etc and formally $f^{(0)}(x) \equiv f(x)$

Example 1. Calculate the first derivative of $f(x) = x^2$ with respect to x, from first principles.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} \end{aligned}$$

$$\Rightarrow f'(x) = 2x \ .$$

From this, it follows a general procedure to calculate derivatives in a shorthand manner which is defined for a function $f(x) = x^n$ as,

$$\boxed{f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}} \ , \tag{3}$$

which easily translates for higher order derivatives as well. Derivatives of some simple functions are

$$\begin{aligned} &\frac{d}{dx}(e^{ax}) = ae^{ax}, && \frac{d}{dx}(\ln ax) = \frac{1}{x}, \\ &\frac{d}{dx}(\sin ax) = a\cos ax, && \frac{d}{dx}(\cos ax) = -a\sin ax, \\ &\frac{d}{dx}(\tan ax) = a\sec^2 ax, && \frac{d}{dx}(\sec ax) = a\sec ax \tan ax \ . \end{aligned}$$

1.2 Common Rules of Differentiation

- **Differentiation of Products:** If we have a function which is itself defined as a product of two or more functions, i.e., say

  $$f(x) = u(x)v(x) \ ,$$

  then,

  $$f'(x) \equiv \frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \ ,$$

which is commonly referred to as the product rule. This formula can be generalized to products of more than two functions very easily.

- **Quotient Rule:** If instead we have a function of the form

$$f(x) = \frac{u(x)}{v(x)}$$

then

$$f'(x) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \ .$$

- **Chain Rule:** We also need to deal with functions of functions such as $f(x) = u(x)^n$. In that case, from the chain rule it follows that

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \ .$$

Example 2. Find the derivative of $f(t) = 2at$, with respect to x where $x = at^2$.

By the chain rule we have,

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx} = 2a\left(\frac{1}{2at}\right) = \frac{1}{t} \ .$$

2 Integral Calculus

There are a number of ways of interpreting what an *integral* is. In the following sections we review the basics of integral calculus using the two most commonly used interpretations.

2.1 Integration as an Inverse Differentiation

A differential of a sum of variables can be expanded as,

$$d(x + y + z + ...) = dx + dy + dz + ...,$$

which means that a reverse of such an operation will yield,

$$\int (dx + dy + dz + ...) = \int dx + \int dy + \int dz + ..., \tag{4}$$

and so for any expression comprised of the sum or differences of terms of any form, we must integrate term by term to get the integral of the total expression. Similarly, in differential calculus,

$$d(x + c) = dx + dc = dx,$$

since c is a constant and does not vary. In the reverse process then c must reappear as,

$$\int dx = \int dx + \int dc = x + c \tag{5}$$

and so a constant must always be added to the result of an indefinite integral since we do not know before we integrate if there was an added constant in the original function. Similarly, if we integrate the well-known formula for differentiation with respect to x we get,

$$\int dx \frac{d}{dx} x^n = \int n x^{n-1} dx \Rightarrow \int n x^{n-1} dx = x^n \Rightarrow n \int x^{n-1} dx = x^n \ ,$$

since n is a constant. So how do we carry out the integral without the coefficient without compromising the integrity of our result? Consider the differential of $d(x^{m+1})$,

$$\begin{aligned}
& d(x^{m+1}) = (m+1)\ x^{(m+1)-1}\ dx \\
\Rightarrow\ & d(x^{m+1}) = (m+1)\ x^m\ dx \\
\Rightarrow\ & \frac{1}{(m+1)} d(x^{m+1}) = x^m\ dx \\
\Rightarrow\ & \int \frac{1}{(m+1)} d(x^{m+1}) = \int x^m\ dx \\
\Rightarrow\ & \frac{1}{(m+1)} \int d(x^{m+1}) = \int x^m\ dx \\
\Rightarrow\ & \boxed{\int x^m\ dx = \frac{x^{(m+1)}}{(m+1)} + c}.
\end{aligned}$$

This is the general prescription for deriving all integral formulae that can be obtained by reversing the result of the corresponding differentiation. Try out the following exercises for your benefit.

Exercise 1. Show that $\int \cos\theta = -\sin\theta$.

Exercise 2. Show that $\int sec^2\theta = \tan\theta$.

Exercise 3. Show that $\int \frac{dx}{x} = \ln x$.

2.1.1 Integration of Products of Functions

Say we have two functions of x: $u(x)$, $v(x)$, and their product gives a new function $f(x) = u(x)v(x)$. Then, starting from differentiation of products,

$$\begin{aligned}
& \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{(product rule)} \\
\text{integrating both sides} \ \Rightarrow\ & \int \frac{d}{dx}(uv)\ dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \\
\Rightarrow\ & uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx \\
\Rightarrow\ & \boxed{\int u\ \frac{dv}{dx}\ dx = uv - \int v\ \frac{du}{dx}\ dx}\ .
\end{aligned} \tag{6}$$

Although it does not give us the exact value of the integral $\int u\frac{dv}{dx}dx$ on the left, Eq.(6) gives the integral partially by stating that it is equal to uv minus another integral. Hence, this process is named *partial integration* or *integration by parts*.

Example 3. Determine $\int \ln x \; dx$.

Let $I = \int \ln x \; dx$ and then assume,

$$u = \ln x, \;\; \frac{dv}{dx} = 1$$
$$\text{so, } \frac{du}{dx} = \frac{1}{x} \text{ and } v \equiv \int \frac{dv}{dx} = x \; .$$
$$\text{Thus, } I = uv - \int v \frac{du}{dx} dx,$$
$$= x \ln x - \int x \frac{1}{x} dx$$
$$\Rightarrow \boxed{I = x \ln x - x + c} \; .$$

2.2 Integration as a Continuous Sum

We have seen how integration is just the reverse of differentiation. However, we had given an independent definition of differentiation (in terms of limits) but we have not done so for integration. Specifically what is meant by the symbol $\int$? To put it simply, to integrate an expression as $\int dx$ simply means one is *summing* all infinitesimally small parts dx such that it yields

$$\int dx = x \; .$$

Recall that we can plot a general function $f(x)$ on the $xy-$plane by setting $y = f(x)$. Let Figure 2 represent such a setup. Say, we want to calculate the area under the graph.

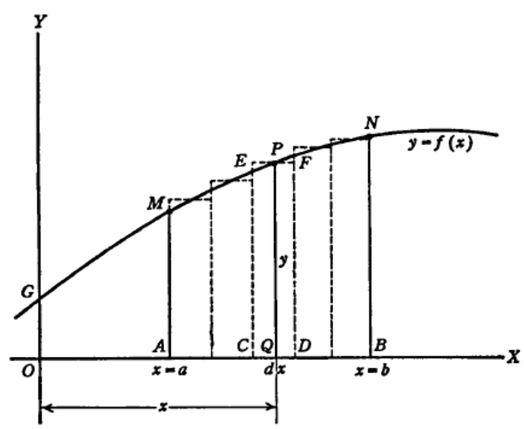

Figure 2: Integration as area under a graph

One can imagine slicing up the region under $f(x)$ into small rectangles between the regions bounced by $x = a$ and $x = b$. The slicing is done in a way such that the rectangles have infinitesimally small widths of dx each. If one calculates area of a single rectangle (say, EFCD) we get,

$$dA = (CD)(EC) = y \; dx$$

and the total area under the graph is determined by summing each rectangle as,

$$\boxed{\int dA = A = \int y \; dx} \; . \tag{7}$$

2.2.1 Application of Integration - Length of a Curve

The differential length of a curve, ds, at any point on the curve is measured instantaneously *along the tangent* at that point. As the tangent changes its direction continually to follow the curve, however, the sum of all the length differentials ds is the total length s along the curve. Consider the curve shown in Figure 3 below.

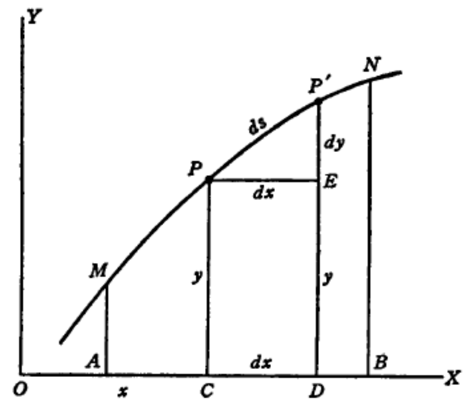

Figure 3: Length of a Curve

If we assume that the separation of P and P' is infinitesimally small such that we can approximate PEP' to be a small right-triangle such that,

$$\begin{aligned}
&(ds)^2 = (dx)^2 + (dy)^2 \\
\Rightarrow\ &\frac{(ds)^2}{(dx)^2} = 1 + \frac{(dy)^2}{(dx)^2} \\
\Rightarrow\ &\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\
\Rightarrow\ &\int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
\Rightarrow\ &\boxed{s = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}\,,
\end{aligned} \tag{8}$$

for any two points on a curve $x = a$ and $x = b$.