Group Theory
Lecture # 9

Finite Dimensional Unitary Representations of SU(2):

Here we construct the finite dimensional unitary representations of the Lie algebra of SU(2). The representations constructed here are irreducible. The construction given here form the backbone of the representation theory of all Lie algebras. The importance of of this construction cannot be overemphasized.

Let us start with the Lie algebra of SU(2):

$$[J_a, J_b] = i\,\epsilon_{abc} J_c$$

[Here we follow the physics convention of labelling the generators of SU(2) by J_a.]

In components the non-trivial components of this Lie algebra are:

$$[J_1, J_2] = iJ_3$$

$$[J_2, J_3] = iJ_1$$

$$[J_3, J_1] = iJ_2$$

In addition we have $J_a^\dagger = J_a$

Let us define a different basis for the Lie algebra:

$$J_\pm = J_1 \pm iJ_2$$

$$J_3 = J_3$$

We now take $\{J_+, J_-, J_3\}$ as the basis of the Lie algebra. Note that $J_\pm$ are not hermitian operators:

$$J_+^\dagger = (J_1 + iJ_2)^\dagger = J_1^\dagger - iJ_2^\dagger = J_1 - iJ_2 = J_-$$

Similarly $J_-^\dagger = J_+$.

In terms of the new basis the Lie algebra becomes:

Show:

$$[J_3, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_3$$

Recall that the rank of $SU(N)$ is $N-1$ (the number of diagonal, traceless, hermitian matrices). And so rank of $SU(2) = 1$.

We choose the cartan subalgebra to be J_3.

The Quadratic Casimir Invariant of SU(2)

Schur's Lemma tells us that any matrix that commutes with all the matrices of an irreducible rep must be proportional to the identity. At the Lie algebra level this translates to any operator / matrix that commutes with the whole of the Lie algebra must be proportional to the identity matrix.

It is easy to show that

$$\bar{J}^2 = J_1^2 + J_2^2 + J_3^2$$

satisfies

$$[\bar{J}^2, J_a] = 0 \quad \forall a$$

And thus $\bar{J}^2 = \lambda \mathbb{1}$.

$\bar{J}^2$ is known as the Casimir invariant of $SU(2)$. $\bar{J}^2$ has an important role in labelling the vector space of a representation. Note that the generators (elements of the Lie algebra) will map $V_n \to V_n$, where V_n is the vector space on which the irrep acts on. Since the Casimir is $\bar{J}^2 = \mu \mathbb{1}$ on that vector space, the eigenvalue μ will label the vector space on which that irrep acts on. Thus different irreps are <u>labelled</u> by different eigenvalues of $\bar{J}^2$. In physics, $\bar{J}^2$ has the interpretation of the square of the total angular momentum operator.

States of the irrep

From linear algebra we know that the simultaneous eigenvectors of the maximal set of commuting hermitian operators form a complete and orthonormal basis for a vector space. In our case this maximal set is $\{\bar{J}^2, J_3\}$. But we see that $\bar{J}^2$ acts as the identity operator on all the states. Thus we label the vectors by

$$\vec{J}^2 |\lambda, m\rangle = \lambda |\lambda, m\rangle$$

$$J_3 |\lambda, m\rangle = m |\lambda, m\rangle$$

where both λ & m are real as both $\vec{J}^2$ & J_3 are hermitian. Furthermore $\lambda \geq 0$ since $\vec{J}^2$ is positive definite.

The action of $J_\pm$ on $|\lambda, m\rangle$.

Given $|\lambda, m\rangle$, let us ask what is J_3 eigenvalue of $J_+ |\lambda, m\rangle$?

$$J_3 (J_+ |\lambda, m\rangle) = J_+ J_3 |\lambda, m\rangle + J_+ |\lambda, m\rangle \quad [\text{using } [J_3, J_+] = +J_+]$$

$$= J_+ m |\lambda, m\rangle + J_+ |\lambda, m\rangle \quad [\text{using } J_3 |\lambda, m\rangle = m |\lambda, m\rangle]$$

$$= (m+1)(J_+ |\lambda, m\rangle)$$

Thus we see that the result of applying J_+ on the vector $|\lambda, m\rangle$ is to increase J_3 eigenvalue. Thus we can write:

$$J_+ |\lambda, m\rangle = C_{m+1} |\lambda, m+1\rangle$$

where C_{m+1} is a constant of proportionality.

Similarly we can establish that the action of J_- lowers the J_3 eigenvalue by 1:

$$J_3 (J_- |m\rangle = (m-1)(J_- |\lambda, m\rangle)$$

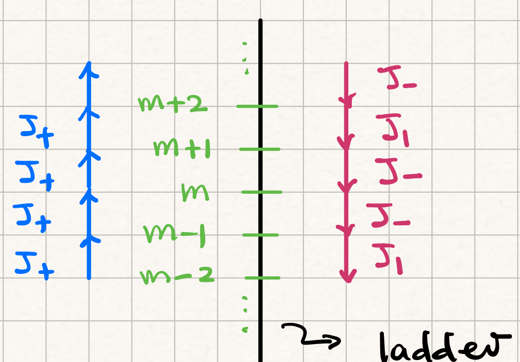
And so
$$J_- |\lambda, m\rangle = D_{m-1} |\lambda, m-1\rangle$$

For some constant D_{m-1}.

Since J_+ (J_-) increases (decreases, respectively) the J_3 eigenvalue we name them:

$$\left.\begin{array}{l} J_+ \longrightarrow \text{Raising Operator} \\ J_- \longrightarrow \text{lowering Operator.} \end{array}\right\} \text{Ladder operators}$$

Pictorially:



By assumption we have:

$$\langle \lambda, m | \lambda', m' \rangle = \delta_{\lambda\lambda'} \delta_{mm'}$$

But since we are just dealing with an irrep we set $\lambda = \lambda'$ and so the

the basis vectors we are dealing with satisfy:

$$\langle \lambda, m | \lambda, m' \rangle = \delta_{m,m'}.$$

We now use orthonormality to find c & D:

$$\langle \lambda, m+1 | J_+ | \lambda, m \rangle = \langle \lambda, m+1 | c_{m+1} | \lambda, m+1 \rangle$$

$$= c_{m+1} \langle \lambda, m+1 | \lambda, m+1 \rangle$$

$$\langle \lambda, m+1 | J_+ | \lambda, m \rangle = c_{m+1} \qquad (1)$$

On the other hand from $J_- | \lambda, m+1 \rangle = D_m | \lambda, m \rangle$

If we take hermitian conjugate of this equation we get:

$$\langle \lambda, m+1 | J_+ = D_m^* \langle \lambda, m |$$

And then we take the inner product with $|\lambda, m\rangle$:

$$\langle \lambda, m+1 | J_+ | \lambda, m \rangle = D_m^* \langle \lambda, m | \lambda, m \rangle$$

$$\langle \lambda, m+1 | J_+ | \lambda, m \rangle = D_m^* \qquad (2)$$

Thus we get:

$$c_{m+1} = D_m^*$$

Thus we can express both the raising & lowering actions using just cs:

$$J_- | m \rangle = c_m^* | m-1 \rangle$$

$$\langle m | J_+ = \langle m-1 | c_m$$