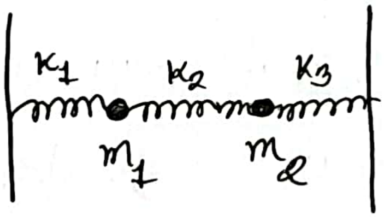
Lecture 7

Coupled oscillators

So far, we have been concerned with a single object, may be a single mass-spring system or a mass connected via two springs. We have encountered forced oscillation, that too including a single object. Now, let's extend our ideas to more than one oscillator. Obviously, if the objects/masses are let to oscillate independently, with no connection between them, they would have just oscillate with their natural frequency. But, if the masses are connected to each other, things gets interesting. We will later find that, coupled systems like this exhibits more than one natural (or normal) frequencies, and the general motion is a combination of oscillations at all different normal frequencies. We call such systems coupled oscillator. One example of such system is molecules. We can model molecules as a system of masses (atoms) coupled together by springs (actually there are no springs; however, the force between atoms takes a Hooke's law form around their equilibrium position). We will be then able to understand such system if we understand coupled oscillator correctly.

Two masses and three springs

We will first consider a coupled oscillator as shown in the figure. All the springs are



unstretched at equilibrium. Let, x_1 and x_2 denote the displacement of left and right mass from their equilibrium position (rightward positive and leftward negative). The middle spring will then be stretched or compressed by a distance of $x_2 - x_1$ (think about this).

Now, the equation of motion for mass m_1 can be written as—

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

For m_2, we can write,

$$m_2 \ddot{x}_2 = -k_3 x_2 - k_2 (x_2 - x_1)$$

Think about x_2 and x_1 both positive, with $x_2 > x_1$. In which direction do you expect m_1 and m_2 to feel forces?

We can rewrite these equations as—

$$\left.\begin{aligned} m_1 \ddot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 \ddot{x}_2 &= k_2 x_1 - (k_2 + k_3) x_2 \end{aligned}\right\} \quad \text{— (1)}$$

These equations are coupled differential equations. Note that, any motion in m_1 is not just restricted to it; it also affects the motion of m_2 and vice-versa. The equations are coupled, since both x_1 and x_2 appears in both the equations. How do we solve them then? We will see two procedures to solve them.

First method

The first method is easy to perform, but it is only limited to systems having possible symmetries and some guesses. If the situation is more complicated, this method will be hard to apply, along with the guessing part. But we will go through this method as it will provide some insight about the problem.

Let's add these equations, and then we get —

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = -K_1 x_1 - K_3 x_2 \quad \text{——— } (2)$$

Subtracting the two equations gives us —

$$m_1\ddot{x}_1 - m_2\ddot{x}_2 = -(K_1+2K_2)x_1 + (K_3+2K_2) \quad \text{——— } (3)$$

But, the equations (2) and (3) are still coupled. As we previously said, we need particular symmetry for this method to work out properly. Let's take, $m_1 = m_2 = m$ and $K_1 = K_3 = K$ and rewrite $K_2 = \kappa$. Now, we do have a symmetry. Left mass is now connected to springs with spring constant K and κ, and so does the right mass, and in a similar fashion.

Now, equations (1) and (3) becomes –

$$m(\ddot{x}_1+\ddot{x}_2) = -K(x_1+x_2) \quad \text{——} (4)$$

$$m(\ddot{x}_1-\ddot{x}_2) = -(K+2\kappa)(x_1-x_2) \quad \text{——} (5)$$

If we define two new coordinates given by $q_1 = x_1+x_2$ and $q_2 = x_1-x_2$, then we can write,

$$m\ddot{q}_1 = -Kq_1 \quad \text{——} (6) \qquad \text{and} \qquad m\ddot{q}_2 = -(K+2\kappa)q_2 \quad \text{——} (7)$$

The solutions of equation (6) and (7) are well known and are given by,

$$q_1(t) = A_s \cos(\omega_s t+\phi_s) \quad \text{with} \quad \omega_s = \sqrt{\frac{K}{m}}$$

and $$q_2(t) = A_f \cos(\omega_f t+\phi_f) \quad \text{with} \quad \omega_f = \sqrt{\frac{K+2\kappa}{m}}$$

Surely, $\omega_f < \omega_s$, and so $T_f > T_s$. So, 's' stands for slow oscillation and 'f' stands for fast oscillation.

We found something very interesting. The actual motion of left mass and right mass can and might get as complicated as it can be, but $q_1 = x_1+x_2$ and

$q_2 = x_1 - x_2$ will always oscillate under simple harmonic oscillation. We can now easily find x_1 and x_2, as—

$$x_1 = \frac{q_1+q_2}{2} \quad \text{and} \quad x_2 = \frac{q_1-q_2}{2}$$

$\therefore \cancel{x_1 = \frac{A_s+A_f}{2}}$

$$\therefore x_1 = \frac{A_s}{2}\cos(\omega_s t+\phi_s) + \frac{A_f}{2}\cos(\omega_f t+\phi_f) \quad \text{and}$$

$$x_2 = \frac{A_s}{2}\cos(\omega_s t+\phi_s) - \frac{A_f}{2}\cos(\omega_f t-\phi_f).$$

And, we found the solutions. But this method only worked for the symmetric case. As we have already seen, this method of adding and subtracting would not be that useful if the masses were unequal and the spring constants were different, like we have in the actual problem. Let's now introduce the second and systematic method, that will work out for complicated scenerios as well.

Second method

Let's concentrate on the equations that we have at hand.

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -(K_1+K_2)x_1 + K_2x_2 \\ m_2\ddot{x}_2 &= K_2x_1 - (K_2+K_3)x_2 \end{aligned}\right\} \text{—— (1)}$$

We can write these two equation in a compact matrix form as –

$$M\ddot{\mathbf{x}} = -K\mathbf{x} \quad \text{—— (11)}$$

with,

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}$$

We will start solving for the equations with the idea ~~assumption~~ that, we look for solutions where both masses move with the same frequency. Such kind of motion might not exist, but we can try to find. We will eventually find that there always exists such kind of motion. Let's then guess,

$$x_1 = A_1 e^{i\omega t} \quad \text{and} \quad x_2 = A_2 e^{i\omega t}$$

We will finally take the real part of the complex solution for our actual motion.

In matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t} \quad \text{or,} \quad X = Ae^{i\omega t}$$

You can directly plug $X = Ae^{i\omega t}$ in equation (11) or individual x_1 and x_2 in equation (1). Plugging in (11) gives u

$$M(i\omega)^2 A e^{i\omega t} = -KAe^{i\omega t}$$

$$\Rightarrow -\omega^2 MA e^{i\omega t} = -KA e^{i\omega t}$$

$$\cancel{\therefore (K-\omega^2)} \quad \therefore (K-\omega^2 M) A = 0$$

This is a matrix equation. Expanding this we get –

$$\begin{pmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{—— (*)}$$

$$\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix} - \omega^2 \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

The trivial solution to this equation is obviously $A_1 = A_2 = 0$, which corresponds to the masses where the do not move at all. This is a solution, but not the one we are particularly interested in. If the matrix $(K-\omega^2 M)$ has non-zero determinent, the the inverse matrix exists and we can multiply both sides by the inverse matrix, and we are left with the trivial solutions $A_1 = A_2 = 0$. But, if the inverse of the matrix does not exist, that is, the ~~determinen~~ determinent of the ~~mad~~ matrix is zero, it is only then non-zero solutions are possible.

So, for finding non-trivial solutions, it is enough to set the det $(K-\omega^2 M) = 0$.

$$\therefore \; (k_1+k_2-\omega^2 m_1)(k_2+k_3-\omega^2 m_2) - k_2^2 = 0$$

$$\Rightarrow k_1k_2 + k_1k_3 - \omega^2 k_1 m_2 + k_2^2 + k_2k_3 - \omega^2 k_2 m_2$$
$$- \omega^2 k_2 m_1 - \omega^2 k_3 m_1 + \omega^4 m_1 m_2 - k_2^2 = 0$$

$$\Rightarrow \omega^4 m_1 m_2 - \omega^2 [k_1 m_2 + k_2 m_2 + k_2 m_1 + k_3 m_1) + [k_1k_2 + k_1k_3 \cancel{+2k_2^2}] = 0 \quad \text{——} (*)$$

$$\Rightarrow a\omega^4 - b\omega^2 + c = 0 \quad (**)$$

$$\therefore \; \omega^2 = \frac{b \pm \sqrt{b^2-4ac}}{2a}$$

One can find values of ω^2 from here and plug in (*) to find the values of corresponding A_1 and A_2, and hence the solution.

Let's try to find the solution for our symmetric case. If we plug in $k_1 = k_3 = k$ and $k_2 = \kappa$ and $m_1 = m_2 = m$ in (*) and (**), then —

$$\begin{pmatrix} k+\kappa-\omega^2 m & -\kappa \\ -\kappa & k+\kappa-\omega^2 m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{——} (1)$$

and $\cancel{\omega^4 m^2 - \omega^2 m(2k+2\kappa) + (k\kappa + k^2 + 2\kappa^2) = 0}$

$$\cancel{\therefore \; \omega^2 = \frac{2m(k+\kappa) \pm \sqrt{}}{}}$$

$$\det[K-\omega^2 m] = (k+\kappa-\omega^2 m)^2 - \kappa^2 = 0$$

$$\Rightarrow (K+K' - \omega^2 m)^2 = K'^2$$

$$\Rightarrow K+K'-\omega^2 m = \pm K'$$

$$\therefore \omega^2 = \frac{K}{m} \quad \text{and} \quad \omega^2 = \frac{2K'+K}{m}$$

Plugging the results in (1) we get,

$$\omega^2 = \frac{K}{m}: \begin{pmatrix} K+K'-K & -K' \\ -K' & K+K'-K \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow K' \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A_1 - A_2 \\ -A_1 + A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore A_1 - A_2 = 0 \quad \text{and} \quad -A_1 + A_2 = 0$$

$$\therefore A_1 = A_2 \qquad \therefore A_1 = A_2$$

For $\omega^2 = \frac{2K'+K}{m}$: $\begin{pmatrix} K+K'-2K'-K & -K' \\ -K' & K+K'-2K'-K \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow K' \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -A_1 - A_2 \\ -A_1 - A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore A_1 = -A_2 \quad \text{and} \quad A_1 = -A_2$$

So, for $\omega_s = \frac{K}{m}$, $A_1 = A_2$ and $\omega_f = \frac{2K+K}{m}$,

$A_1 = -A_2$. This means, when the masses move with ω_s frequency, they are in phase. When they move with ω_f frequency they are exactly out of phase.

Now, the general solution must be a linear combination of these solutions.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_s t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_s t} + C_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_f t} + C_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_f t}$$

$$\therefore x_1 = C_1 e^{i\omega_s t} + C_2 e^{-i\omega_s t} + C_3 e^{i\omega_f t} + C_4 e^{-i\omega_f t}$$

$$= C_1 \left[\cos(\omega_s t) + i\sin(\omega_s t)\right] + C_2 \left[\cos(\omega_s t) - i\sin(\omega_s t)\right] + C_3 \left[\cos(\omega_f t) + i\sin(\omega_f t)\right] + C_4 \left[\cos(\omega_f t) - i\sin(\omega_f t)\right]$$

~~Taking the real part,~~

$$\cancel{x_1 = (C_1 + C_2)\cos(\omega_s t) + (C_3 + C_4)\cos(\omega_f t)}$$

We already know that, for x_1 to be real, $C_1 = \bar{C_2}$ and $C_3 = \bar{C_4}$ and finally,

$$x_1(t) = A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)$$

Similarly, $x_2(t) = A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi_f)$

And

And, the result exactly agrees with the result that we found using the first method.

Normal modes and normal coordinates

If, $A_f = 0$, then, $x_1 = x_2 = A_s \cos(\omega_s t + \phi_s)$

So, both masses move with the same frequency and phase. They move together to the right and together to the left. In this case, the middle spring is never stretched or compressed. It is technically not there. This makes sense as ω_s contain the natural frequency of one single spring-mass system.

This type of motion, where both the masses move with same frequency, is called a normal mode. In this case, both masses has the same amplitude. This could be found by stretching (or compressing) both the masses by same amount in the same direction and then releasing them.

If $A_s = 0$, then, $x_1 = A_f \cos(\omega_f t + \phi_f)$ and

$x_2 = -A_f \cos(\omega_f t + \phi_f)$

Here both the masses move with the same frequency ω_f, with same amplitude, but they move in opposite directions. This is another normal mode. It could have been found by stretching ~~to~~ (or compressing) both the mass by same amount, but in opposite directions.

It is suggestive that, any arbitrary motion of ~~the~~ the system is a linear combination of the normal modes. But it might be difficult to tell in complicated scenerios what these normal modes could be.

If we add x_1 and x_2, then, x_1+x_2 oscillates with frequency ω_s ; similarly x_1-x_2 oscillates with a frequency ω_f only. x_1+x_2 is called a normal coordinate corresponding to normal mode with frequency ω_s. x_1-x_2 is the second normal coordinate corresponding to the normal mode with frequency ω_f.

Weakly coupled oscillator : Beats (again)

Consider the same coupled oscillator, but the middle spring having a very small spring constant $K' \ll K$. Let's apply some suitable initial conditions—

At $t=0$: $\dot{x}_1=0$, $\dot{x}_2=0$, $x_1=0$ and $x_2=A$.

So, we are just displacing the right mass by A and letting it go from that position with zero initial velocity, without harming the left mass.

$$x_1(t) = A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f);$$

$$\therefore \quad 0 = A_s \cos\phi_s + A_f \cos\phi_f \quad \text{—— (I)}$$

For $x_2(t)$: $\quad A = A_s \cos\phi_s - A_f \cos\phi_f \quad$ —— (II)

$$\text{(I)}+\text{(II)} \Rightarrow \quad A = 2A_s \cos\phi_s \quad \text{—— } (*)$$

$$\text{(I)}-\text{(II)} \Rightarrow \quad -A = 2A_f \cos\phi_f \quad \text{—— } (**)$$

$$\dot{x}_1 = -\omega_s A_s \sin(\omega_s t + \phi_s) - \omega_f A_f \sin(\omega_f t + \phi_f)$$

$$\Rightarrow \quad 0 = -\omega_s A_s \sin\phi_s - \omega_f A_f \sin\phi_f \quad \text{—— (III)}$$

For $\dot{x}_2$: $\quad 0 = -\omega_s A_s \sin\phi_s + \omega_f A_f \sin\phi_f \quad$ —— (IV)

(III) + (IV) $\Rightarrow$

$$-2\omega_s A_s \sin\phi_s = 0$$

$$\therefore \phi_s = n\pi$$

(III) − (IV) $\Rightarrow \quad -2\omega_f A_f \sin\phi_f = 0$

$$\therefore \phi_f = n\pi$$

Now, from $(*)$ and $(**)$, (taking for $n=0$)

$$\left.\begin{aligned} A &= 2A_s \\ -A &= 2A_f \end{aligned}\right\} \quad \begin{aligned} A_s &= A/2 \\ A_f &= -A/2 \end{aligned}$$

$$\therefore x_1 = \frac{A}{2}\left(\cos\omega_s t - \cos\omega_f t\right)$$

$$x_2 = \frac{A}{2}\left(\cos\omega_s t + \cos\omega_f t\right)$$

Now, $\omega_s = \sqrt{\frac{K}{m}}$ and $\omega_f = \sqrt{\frac{2\kappa + K}{m}}$

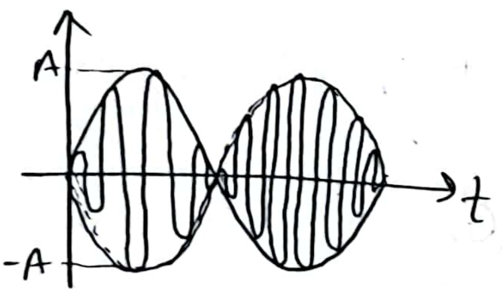
If $\kappa << K$, then, $\omega_f > \omega_f$, but the difference is very small. If we write –

$$\omega_s = \frac{\omega_s + \omega_f}{2} \cancel{+} - \frac{\omega_f - \omega_f}{2} = \Omega - \epsilon$$

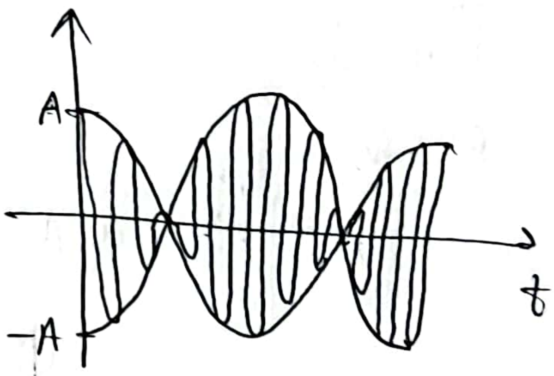
$$\omega_f = \frac{\omega_s + \omega_f}{2} + \frac{\omega_f - \omega_s}{2} = \Omega + \epsilon$$

$$\therefore x_1 = \frac{A}{2}\left[\cos(\Omega - \epsilon)t - \cos(\Omega + \epsilon)t\right] = A\sin\Omega t \sin\epsilon t$$

$$x_2 = \frac{A}{2}\left[\cos(\Omega - \epsilon)t + \cos(\Omega + \epsilon)t\right] = A\cos\Omega t\cos\epsilon t$$



x_1



x_2

So, at first, mass 2 will oscillate, while ~~m~~ mass 1 remain stationary. But as time goes on, the amplitude of 2 decreases with amplitude of 1 increasing to the maximum. This process goes on forever, with energy being transferred from one mass to another.