Group Theory
Lecture #5

In considering the isomorphism $SU(2)/\mathbb{Z}_2 \cong SO(3)$, we have claimed that $\mathbb{Z}_2$ is a normal subgroup. $\mathbb{Z}_2 \subset SU(2)$ is an example of being in the centre of $SU(2)$: $\mathbb{Z}(SU(2))$.

Def^n. The Centre of a Group

The centre $Z(G)$ of a group G are elements which commute will all elements of G.

Comments:

1. $Z(G)$ itself is an abelian normal subgroup.

   $a \in Z(G)$ if $ag = ga \;\; \forall g \in G$.

   $\Rightarrow \; a^{-1}ag = a^{-1}ga \Rightarrow g = a^{-1}ga \Rightarrow ga^{-1} = a^{-1}g$

   Thus $a^{-1} \in Z(G)$. e is obviously in $Z(G)$. $\Rightarrow Z(G)$ is a group.

   It is also abelian since $ag = ga$ is valid for all $g \in Z(G)$.

   Since $g\,Z(G)\,g^{-1} = Z(G) \Rightarrow Z(G)$ is a normal subgroup.

2. Since $Z(G)$ is a normal subgroup it means that $G/Z(G)$ is a quotient group. $G/Z(G)$ is called the inner automorphism group of G.

Def^n. Direct Product of Groups

If G_1 & G_2 are two groups then we may define the direct product group $G_1 \times G_2$ by defining the multiplication between ordered pairs $(g_1, g_2) \in G_1 \times G_2$ as $(g_1, g_2) \circ (g_1', g_2') := (g_1 g_1', g_2 g_2')$.

Comments:

1. $G_1 \simeq \{(g_1, e_2)\}$ & $G_2 \simeq \{(e_1, g_2)\}$ form normal subgroups of $G_1 \times G_2$.
2. If the product group is $G \times G$ then there is another subgroup, known as the

diagonal subgroup G which consists of elements (g,g) which is <u>not</u> a normal subgroup.

<u>Def^n. Conjugate Elements:</u>

Two elements a & $b \in G$ are said to be conjugate to each other if $\exists$ an element $g \in G$ st. $a = gbg^{-1}$. g is known as the conjugating element & it need not be unique. i.e. if a and b are conjugate elements there may exist g & g' with $g \neq g'$ st $a = gbg^{-1}$ <u>and</u> $a = g'bg'^{-1}$.

<u>Comments:</u>

1. Conjugacy is an equivalence relationship.
2. All equivalence relationships <u>partition</u> a set into disjoint subsets known as equivalence classes.

<u>Def^n. Conjugacy classes</u>

The equivalence classes under conjugacy are called conjugacy classes.

1. Conjugacy classes are equivalence classes.

2. Conjugacy classes consist of transformations which in a geometric sense are equivalent. For example, if we consider rotations in $SO(3)$, one can show that all rotations by a given angle fall into the same conjugacy class. This reflects the fact that such rotations are all equivalent under rotation of the axes.

Examples of Important Discrete Groups

The Cyclic Group C_n

The cyclic group C_n for $n \geq 2$ an integer consists of the set:

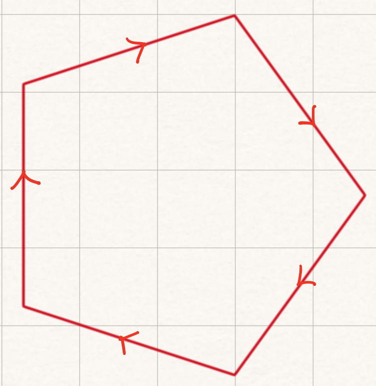
$$C_n = \{ \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}, \alpha^n = e \}.$$

C_n is always abelian and isomorphic to $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ with the group multiplication being addition modulo n.

Comments:

1. C_n is the symmetry group of an n-sided regular polygon with directed sides.

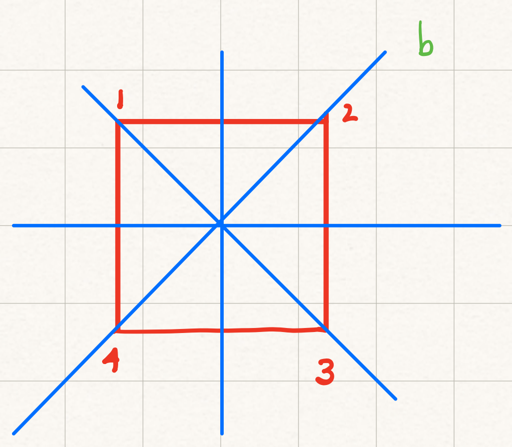
E.g. C_5 is the symmetry group of the shape below:



The Dihedral Group D_n

The dihedral group D_n is the symmetry group of n-sided regular polygons <u>without</u> directed sides.

Example D_4:

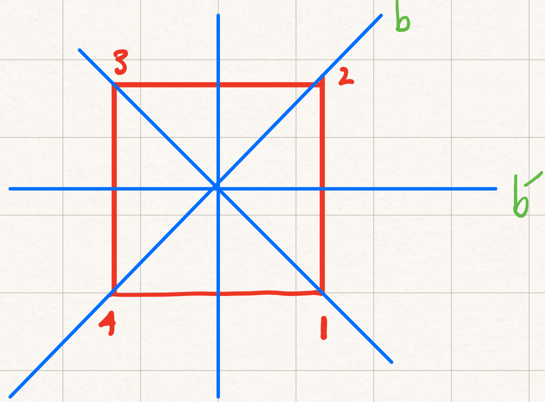


The dihedral group D_4 consists of the cyclic group C_n and reflections about the medians (shown by blue lines).

$C_4 = \{ e^{2\pi i n/4} \mid n = 0, 1, 2, 3 \}$

Under $e^{2\pi i/4}$ vertex $1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1$

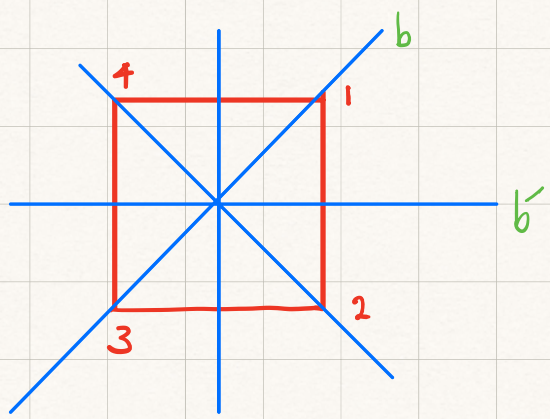
Under b: $2 \to 2$, $4 \to 4$, $1 \to 3$, $3 \to 1$.



Note that the ordering of the vertices have changed. Any further reflection about any median will change the ordering back to the original one. E.g. if w pick b'.

e

b': $1 \longrightarrow 2$, $2 \longrightarrow 1$, $3 \longrightarrow 4$, $4 \longrightarrow 3$



This should convince you that there are $2n$ group elements. In general if a is an element that rotates the shape and b is a reflection then $a^{-1} = b\,a\,b$.

Comments:

1. In abstract algebra D_n is called D_{2n}.
2. The centre of D_n depends on whether n is odd or even:

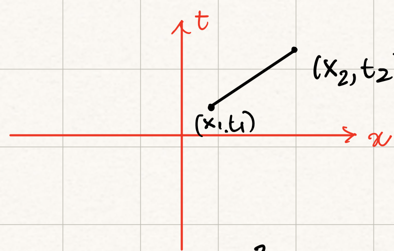
$$n = 2k+1 : \; Z(D_{2k+1}) = \{e\}$$

$$n = 2k : \; Z(D_{2k}) = \{e, a^k\} \simeq \mathbb{Z}_2 .$$

The Lorentz Group in 2D:

The group of symmetries that define special relativity is called the Lorentz group. Later we shall consider the full Lorentz group but now let us consider the Lorentz group in a 1+1 dimensional spacetime.

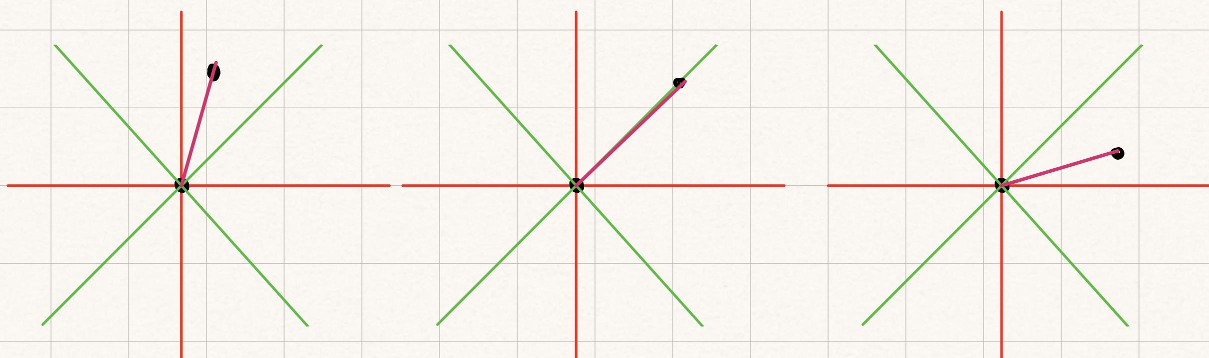
In special relativity one considers a space-time. According to Einstein's special



Theory of relativity the $(\text{interval})^2$ $\Delta s^2 = -c^2(t_2-t_1)^2 + (x_2-x_1)^2$ remains invariant under a transformation of inertial reference frame.

These transformations are known as Lorentz transformations.

Δs^2 can be positive, zero or negative.



Timelike separation $\Delta s^2 < 0$ | Lightlike or null separation $\Delta s^2 = 0$ | Spacelike separation $\Delta s^2 > 0$

Let $x_i^\mu = (t_i, x_i)$ & choose the speed of light $c=1$. WLOG we can choose $x_1^\mu = (t, x)$ & $x_2^\mu = (0,0)$. Then

$$\Delta s^2 = x^\mu x^\nu \eta_{\mu\nu} \quad \text{[Einstein convention employed]}$$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ~ Metric tensor of $\mathbb{R}^2$. $\mathbb{R}^{1,1} \cong \mathbb{R}^2$ with $\eta_{\mu\nu}$ as the metric. Known as Minkowski space.

In matrix language $\Delta s^2 = x^T \eta x$.

Lorentz transformations are defined to be the set of linear transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ st.

$$\Delta s^2 = x^T \eta x = x'^T \eta x'.$$

Comments:

1. For continuous groups we differentiate between $U(1)$ (or $SO(2)$) & $\mathbb{R}^1$ (or $SO(1,1)$) i.e. topology of the parameter manifold plays an important role.

This motivates the definition of a Lie group.

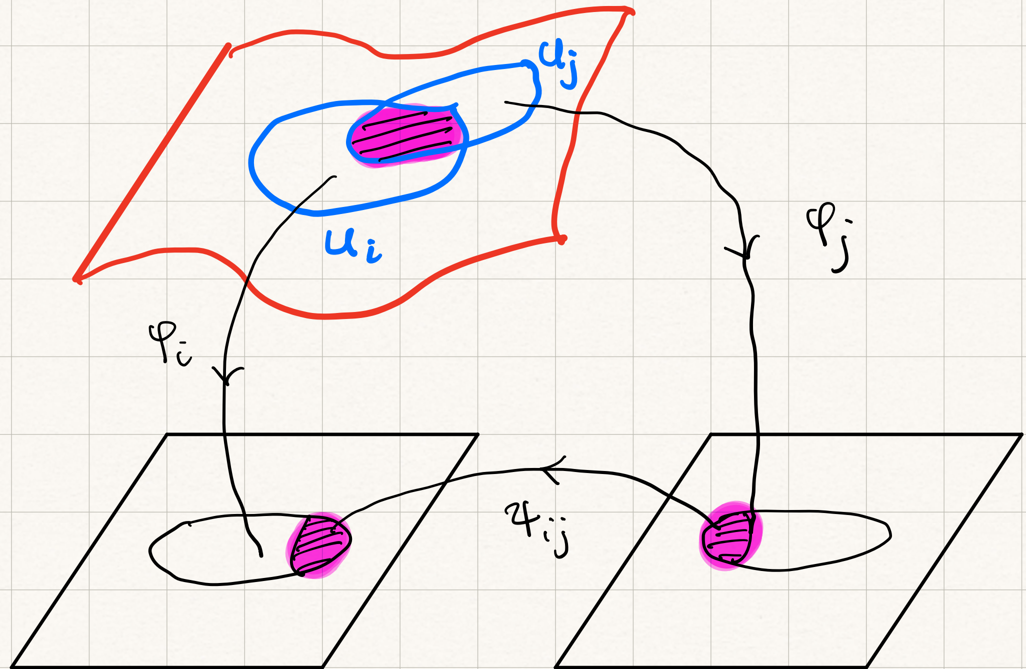
Defn : Differentiable Manifold

M is called an n-dimensional differentiable manifold if:

1. M is a topological space
2. M is equipped with a family of pairs $\{(U_i, \varphi_i)\}$.
3. Where $\{U_i\}$ is a family of open sets which covers M in the sense $\bigcup_i U_i = M$. φ_i is a homeomorphism from U_i onto an open subset U_i'

of $\mathbb{R}^n$.

4. For two U_i & U_j st $U_i \cap U_j \neq \emptyset$ the transition functions $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable (C^∞).

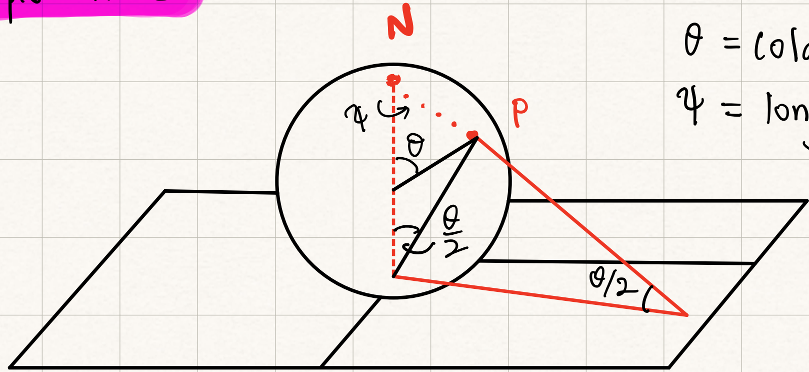


Topological Space:

Let X be a set and let $T = \{U_i \mid i \in I\}$ be a collection of subsets of X. The pair (X, T) is a topological space if:

1. $\emptyset, X \in T$.
2. For any subcollection of I, the family $\{U_j \mid j \in J \subset I\}$ satisfies $\bigcup_{j \in J} U_j \in T$.
3. If K is any finite subcollection of I, the family $\{U_k \mid k \in K\}$ satisfies $\bigcap_{k \in K} U_k \in T$.

An Example: $M = S^2$



θ = colatitude
φ = longitude

$0 < \theta < \pi$
$0 \leq \varphi < 2\pi$

One chart is $U_1 = M \setminus \{N, S\}$ $\quad\quad \varphi_1(p) = (\theta, \varphi)$

A second chart is $\varphi_2(p) = (x, y) = \left(2 \cot\frac{\theta}{2} \cos\varphi,\ 2\cot\frac{\theta}{2} \sin\varphi\right)$

$-\infty < x, y < \infty \quad\quad U_2 = M \setminus \{N\}$

Comments:

1. Note that the transition functions are continuous.
2. Charts are arbitrary way of describing a manifold. The geometry should not depend on the chart. Condition 3 of the definition of a manifold ensures that.

Def^n. Lie Group

A Lie group is a group whose parameter space is also a manifold.

Ex: 1. $SO(1,1)$ is a group with the parameter space $\mathbb{R}^1$. Such Lie groups are called non-compact Lie groups since the parameter space is a non-compact manifold.

2. $U(1) \cong SO(2)$ is an example of a compact group manifold.

3. What is the manifold that corresponds to $SO(3)$? In order to answer this we introduce another group: $SU(2)$.

Lie Groups & Lie Algebras of Matrix Groups

So far we have met several Lie groups such as $U(1)$, $SO(2)$, $O(2)$, $SO(3)$, $O(3)$, $SU(2)$ (compact groups) and $\mathbb{R}^1$, $SO(1,1)$ (non-compact groups).

We have defined a Lie group:

A Lie group G is a group which is also a manifold. In other words, in addition to the group multiplication $G \times G \longrightarrow G$, a Lie group also inherits all the additional structure of a manifold such as continuity, differentiability of charts and topology.

Comment:

1. A Lie group can be compact or non-compact. Example of a compact Lie group is $U(1)$. Example of a non-compact Lie group is $\mathbb{R}^1$.
2. Groups are abstract objects but most of the Lie groups are defined in terms of matrices. A given matrix group can have many (even infinite) number of representations. These representations are isomorphism from the Lie group in question into a subgroup of the appropriate $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ matrix group. Representation theory is an important part of group theory and we shall delve into representation theory in a few lectures.

Def^n. Representation of a Group

Consider an n-dimensional complex vector space V_n. Let T be a linear map

$$T: V_n \longrightarrow V_n$$

$$\text{st } T: \vec{x} \longrightarrow \vec{x}' = T\vec{x}$$

T must have the property $T(\alpha\vec{x} + \beta\vec{y}) = \alpha T\vec{x} + \beta T\vec{y} \;\; \forall\; \vec{x},\vec{y} \in V_n$ and $\alpha, \beta \in \mathbb{C}$.

If the map T is one-to-one $\exists\; T^{-1}$ st: $T^{-1}T\vec{x} = TT^{-1}\vec{x} = \mathbb{I}\vec{x} = \vec{x}$

Let G be a group. If for each element $g \in G$ $\exists$ a map st

$$T(g_1 \circ g_2) = T(g_1)\,T(g_2)$$

then we say that the operators $T(g)$ for an n-dimensional liner representation of the group G.

Matrix Representation

Consider the vector space V_n. If we now introduce a basis in V_n then we can represent $T(g)$ as an $n \times n$ matrix denoted by $D(g)$. The set of matrices $D(g)$ $\forall$ $g \in G$ is called an n-dimensional matrix representation of the group G. Defining an orthonormal basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ in V_n the action of $T(g)$ on $\bar{e}_k$ is given by:

$$T(g)\,\bar{e}_k = \sum_i D_{ik}(g)\,\vec{e}_i$$

Thus we can write $\vec{x}' = T\vec{x}$ as:

$$x_i' = \sum_j D_{ij}(g)\,x_j$$

Matrix Lie Groups

Many Lie groups are <u>defined</u> in terms of matrices. This is known as the defining representation of the group. Let us see how the Lie algebra of matrix groups arise naturally from the matrix representation

The Different Kinds of Matrix Groups

We have already met $GL(n,\mathbb{R})$ & $GL(n,\mathbb{C})$ as matrix groups. Here are some other useful matrix roups that arise as subgroups of $GL(n,\mathbb{R})$ & $GL(n,\mathbb{C})$.

$SL(n,\mathbb{R})$ & $SL(n,\mathbb{C})$:

$SL(n,\mathbb{R})$ is the group of $n\times n$ matrices with real entries such that the determinant of each matrix is 1. $SL(n,\mathbb{R})$ can be seen as the group that is generated by all elements of $GL(n,\mathbb{R})$ which are generated by $[g,h] = g^{-1}h^{-1}gh$. $[g,h]$ is called the commutator of g & h and it should be clear that

$$\det[g,h] = 1 \quad \text{for } GL(n,\mathbb{R}).$$

Note that $SL(n,\mathbb{R})$ is generated by $[GL(n,\mathbb{R}), GL(n,\mathbb{R})]$. This does not mean that products of two commutators is also a commutator.

$SL(n,\mathbb{C})$ is similarly all $n\times n$ complex valued matrices that have determinant 1.

Comment:

1. The dimension of $GL(n,\mathbb{R})$ is n^2, while the dimension of $SL(n,\mathbb{R})$ is n^2-1.

Orthogonal Groups $O(n)$:

The orthogonal group $O(n)$ consists of $n\times n$ real matrices O that satisfy:

$$O^T O = \mathbb{1}$$

In components: $O_{ij}O_{ik} = \delta_{jk} \quad \sim \quad n + \frac{n^2-n}{2} = \frac{2n+n^2-n}{2} = \frac{n^2+n}{2}$ conditions

$\Rightarrow$ Dimension of $O(n)$ is as follows:

$O(n)$ has n^2 real parameters. The $O^TO=\mathbb{1}$ are $n+\frac{n^2-n}{2}$ conditions. Thus we get $n^2-n-\frac{n^2-n}{2} = \frac{2n^2-2n-n^2+n}{2} = \frac{n^2-n}{2} = \frac{n}{2}(n-1)$

Ex: $\text{Dim } SO(2) = \frac{2}{2}(2-1) = 1$

$\text{Dim } SO(3) = \frac{3}{2}(3-1) = 3$

$\text{Dim } SO(4) = \frac{4}{2}(4-1) = 6$

Generators:

$O \simeq \mathbb{1} + i\theta_a T_a$, $a = 1, \dots, \frac{n}{2}(n-1)$

$O^T O = \mathbb{1} \Rightarrow T_a^T = -T_a$ antisymmetric. $\Rightarrow T_a^\dagger = T_a$ since Ts are pure im.

We introduce a clever labelling scheme for the generators $T^{ab} = -T^{ba}$, $a,b = 1,2,\dots n$

Note that $T^{aa} = 0$. There are precisely $\frac{n^2-n}{2}$ linearly independent generators

$T^{ab} \to$ generates rotations in the ab plane of an n-dimensional Euclidean space.

In details: $(T^{ab})_{cd} = i(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c)$

The Lie algebra:

Ex: $[T^{ab}, T^{cd}] = i(\delta^{ac} T^{bd} - \delta^{bc} T^{ad} - \delta^{ad} T^{bc} + \delta^{bd} T^{ac})$

Defn The Cartan Subalgebra

The maximal set of commuting generators of a Lie algebra for the Cartan subalgebra (CSA). The number of elements in CSA is called the rank of the group. The dim of the CSA is called the rank of the group.

Comment:

1. For the SO(n) group we can easily find the rank by considering rotations in independent planes. Thus if $n = 2K$ then the rank is K. If $n = 2K+1$ the rank is still K as the extra direction does not have another direction to pair up with to create a plane of rotation.

We can choose the generator of the CSA as $T^{12}, T^{34}, \dots T^{2K-1,2K}$

O(n,m)

We saw that the Lorentz group in 2D was $O(1,1)$. Similarly Lorentz group in 4D Minkowski space is $O(1,3)$. $O(1,3)$ is defined by

$$O^T \eta O = \eta$$

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Similarly $O(n,m)$ is defined in terms of

$$\eta_{\mu\nu} = \begin{pmatrix} \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{n} & \\ & m\left\{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}\right. \end{pmatrix}$$

Comment:

1. $SO(1,3) \subset O(1,3)$ is the group of continuous Lorentz transformations. These are the transformations continuously connected to identity. Infact if we consider the rotations and boosts of relativity then there is further restriction. For such cases we require $\Lambda^0{}_0$ element of $\Lambda^\mu{}_\nu \in SO(1,3)$ to be positive, i.e., $\Lambda^0{}_0 > 0$. Such transformations ensure that future pointing vectors cannot be transformed into a past pointing vector and vice-versa. This restricted subgroup of $SO(1,3)$ is called the proper, orthochronous Lorentz group and is sometimes denoted as $SO(1,3)^+$.

2. $SO(2,n)$ is the conformal group in n-dim Minkowski space. Conformal transformations do not preserve lengths of vectors but preserves angles between them. The conformal group has important applications to string theory, quantum field theory and quantum gravity. Conformal symmetry describes many condensed matter systems at 2nd order phase transitions.

U(n)/SU(n):

Consists of $n\times n$ complex matrices U that satisfy:

$$U^\dagger U = 1 \Rightarrow U^\dagger = U^{-1} \quad (U(n) \text{ group})$$

and $\det U = 1$.

U^2 has $2n^2$ real parameters. $U^\dagger U = 1$ gives $n + n(n-1)$ real conditions. The det $=1$ gives another real condition. $\dim SU(n) = 2n^2 - \cancel{n} - n^2 + \cancel{n} - 1 = n^2 - 1$.

Generators:

Let $U = e^{i\delta_a T_a}$, $\delta_a = $ real

$$\Rightarrow U^\dagger = U^{-1} \Rightarrow T_a^\dagger = T_a. \quad \text{Det } U = 1 \Rightarrow \text{Tr } T_a = 0.$$

Relationship Between $U(n)$ & $SU(n)$:

$$\frac{SU(n)\times U(1)}{\mathbb{Z}_n} \cong U(n)$$

Proof: Construct the hom $f: SU(n)\times U(1) \to U(n)$ by

$$(U, e^{i\varphi}) \longmapsto e^{i\phi} U$$

$$\text{Ker} f: \quad e^{i\phi} U = \mathbb{1} \quad \Rightarrow \quad \det U = \det e^{-i\phi}$$

$$\Rightarrow U = e^{-i\phi} \qquad 1 = e^{-i\phi n}$$

$$\Rightarrow e^{-i\phi} = e^{2\pi i k/n}, \quad k = 0, 1, \dots n-1$$

And so $\text{Ker} f \cong \mathbb{Z}_n$

By 1st Isomorphism Theorem: $\dfrac{SU(n)\times U(1)}{\mathbb{Z}_n} \cong U(n)$

Generators of $SU(n)$:

The set of generators consists of the following matrices:

a) $H_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$ $\cdots$ $H_{n-1} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$

These matrices form the CSA: $[H_i, H_j] = 0$.

b) $\frac{n(n-1)}{2}$ Real symmetric matrices with $a > b$.

$$(T^{ab})_{cd} = \frac{1}{2}(\delta^a_c \delta^b_d + \delta^a_d \delta^b_c) \qquad \text{Tr } T^{ab} = T^{ab}_{cc} = \frac{1}{2} \quad \delta^a_c \delta^b_c = \delta^a_b = 0$$

since $a \neq b$.

c) $\frac{n(n-1)}{2}$ imaginary, antisymmetric matrices

$$(\tilde{T}^{ab})_{cd} = \frac{i}{2}\left(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c\right)$$

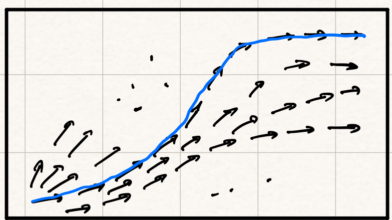
Total: $n-1 + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = \frac{2\cancel{n}-2+n^2-\cancel{n}+n^2-\cancel{n}}{2} = \frac{2n^2-2}{2} = n^2-1.$

<u>Comments:</u>

1. The SU(2) group is very important in quantum mechanics as it is the group of rotations of quantum systems.
2. SU(3), SU(2), U(1), are also the group of gauge transformation of the standard model of particle physics. SU(5) is a proposed GUT group.

<u>Sp(2n, ℝ)</u>

In classical mechanics, the state of an n-particle system is expressed by specifying the point on a 6n-dimensional space known as the phase space. The evolution of a system is specified by a Hamiltonian which via Hamilton's equations specify a vector field on the phase space.



If we organize the generalized coordinates and the canonically conjugate momenta we find that the form of Hamilton's equation is unchanged as long as the linear transformations on v leave a certain skew-symmetric bilinear form invariant

$$\Omega(gv_1, gv_2) = \Omega(v_1, v_2)$$

Explicity, for $d=1$, $v = \begin{pmatrix} q \\ p \end{pmatrix}$ the invariant form is

$$\Omega(v, v') = v^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v' = qp' - q'p$$

And the linear transformation is

$$v \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} v \quad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

Then $\Omega(v, v') = \Omega(gv, gv')$

$$\Rightarrow \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \alpha\delta - \beta\gamma \\ -\alpha\delta + \beta\gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$$

This is known as the $Sp(2, \mathbb{R})$ group. Incidentally $Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$.

Generators of $Sp(2, \mathbb{R})$:

$$g = e^{\lambda L}$$

$$\frac{d}{d\lambda} \left[g^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right]$$

Set $\lambda = 0$

$$L^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L = 0$$

$$\Rightarrow L = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lie algebra: $[E, F] = G$, $[G, E] = 2E$, $[G, F] = -2F$

Some subgroups of $Sp(2, \mathbb{R})$:

1. $e^{tG} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in \mathbb{R}_+$.

$\Rightarrow$ $Sp(2, \mathbb{R})$ is non-compact.

2. $e^{\theta(E-F)} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2).$

The symplectic Group in d dimensions:

$$g^T \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} g = \begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix}$$

$$\Rightarrow L = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

where A, B, C are $d \times d$ real matrices, with $B = B^T$ & $C = C^T$.

Dim: $A \to d^2$ real parameters. $B \to \frac{d^2-d}{2} + d = \frac{d^2+d}{2} = \frac{d(d+1)}{2}$

$C \to \frac{d(d+1)}{2}$

$\text{Dim } Sp(2d, \mathbb{R}) = d^2 + 2\frac{d(d+1)}{2} = 2d^2 + d.$

For $d=1$ $\dim(Sp(2,\mathbb{R}) = 3.$

Heisenberg Group

The Heisenberg algebra is the famous commutation relationship of quantum mechanics: $[x, p] = i\hbar$

We can rewrite it as $[X, Y] = Z$, $[X, Z] = 0$, $[Y, Z] = 0$

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So a typical ement of the Heisenberg group is $A(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, with $a, b, c \in \mathbb{R}$

Multiplication Rule is: $A(a,b,c)\,A(a',b',c') = A(a+a', b+b', c+c'+ab')$

$A(a,b,c)^{-1} = A(-a, -b, -c+ab).$

Non-abelian group with $A(0,0,c) \simeq \mathbb{R}$ the centre.