Classical Mechanics

Lecture # 5

Constrained Dynamics

More often than not the system we are considering will be constrained. This means that there will be relationships connecting the 3N coordinates of the N particles. So the number of degrees of freedom will be reduced.

For example, when we consider a simple harmonic oscillator the system is assumed to be in two dimensions. Thus the position of the bob will be described by two numbers in the Cartesian coordinate system: x & y. But because of the string x & y are not independent and are related by

$$x^2 + y^2 = \ell^2$$

where ℓ is the length (length of the string + radius of the bob). Thus for this system the number of degrees of freedom is only 1.

Holonomic Constraints:

The constraints discussed above ($z=0$, $x^2+y^2-\ell^2=0$) are examples of holonomic constraints. In general, if the coordinates of a system with N particles satisfy

$$f_\alpha(x_A, t) = 0 \qquad \alpha = 1, 2, \dots, 3N-n$$

We can find n independent generalized coordinates such that

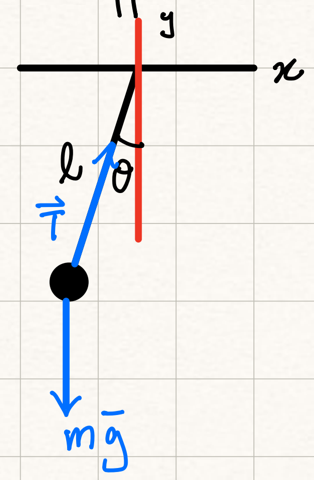
$$x_A = x_A(q_1, q_2, \dots, q_n).$$

It is also true that $\dfrac{\partial f_\alpha(x_A,t)}{\partial q_i} = 0 \quad \forall\ \alpha \ \&\ i.$

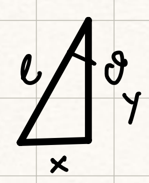
The presence of $3N-n$ holonomic constraints mean the number of degrees of freedom is reduced to n This is the dimension of the configuration space.

The SHO as an example:

Consider Newton's law applied to the SHO.



$$\bar{T} + m\bar{g} = m\ddot{\bar{r}}$$



In Cartesian coordinates:

$$m\ddot{x} = -Tx/l \quad \text{and} \quad m\ddot{y} = mg - Ty/l$$

In polar coordinates we can write this equation as:

$$\underbrace{\frac{m}{\ell}\left(\frac{d(\ell\theta)}{dt}\right)^2}_{\left(\frac{mv^2}{\ell}\right)\ \text{Centripetal Force}} = T - mg\cos\theta$$

$$\Rightarrow \boxed{T = m\ell\dot{\theta}^2 + mg\cos\theta}$$

and

$$\underbrace{m(\ell\ddot{\theta})}_{\text{Tangential acceleration}} = -mg\sin\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\left(\frac{g}{\ell}\right)\sin\theta}$$

Comments:

1. We see that to solve for the motion of the particle we also need to solve for the tension of the string.

Lagrangian Approaches:

What is the Lagrangian of the simple harmonic oscillator?

We might be temped to write

$$L \stackrel{?}{=} \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy$$

Due to origin being 'above'.

But this would be incorrect as this would imply that the SHO has two degrees of freedom (we have already incorporated the $z=0$ constraint). There are two ways to incorporate the constraint $x^2+y^2-\ell^2=0$:

1) Introduce for each holonomic constraint one new degree of freedom, known as a Lagrange multiplier, into the Lagrangian:

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy + \underbrace{\frac{\mu}{2}}_{\text{Lagrange Multiplier}} (\underbrace{x^2+y^2-\ell^2}_{\text{Constraint (L.H.S.)}})$$

We then work with the E-L equation for <u>all</u> coordinates: x, y & μ. So now in addition to the unconstraint equations of motion:

$$m\ddot{x} = \mu x \quad \text{and} \quad m\ddot{y} = mg + \mu y$$

we also get the μ e.o.m.: $\quad x^2+y^2-\ell^2=0.$

If we compare with Newton's equation we see that

$$\mu = -T/\ell.$$

If we want to properly incorporate the tension we can use the Lagrangian:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy - V(r) \qquad \left[V(r) = V(x^2+y^2) \right]$$

Where $r = \sqrt{x^2+y^2}$ and $V(r)$ is the potential that gives rise to the tension force: $\vec{T} = -\left(\hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial x} \right)$

In this picture the string is not an object of fixed length and its tension is described by the function $V(r)$.

To derive the equations of motion let us use polar coordinates:

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + mgr\cos\theta - V(r)$$

Equations of motion are:

$$\theta: \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$$

$$\Rightarrow \; -mgr\sin\theta - \frac{d}{dt}(mr^2 \dot{\theta}) = 0$$

$$\Rightarrow \; \boxed{mgr\sin\theta + 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0}$$

$$r: \quad \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}} \right) = 0$$

$$\boxed{mr\dot{\theta}^2 + mg\cos\theta - \frac{\partial V}{\partial r} = m\ddot{r}}$$

If we now impose the constraint $r = l$ and $\dot{r} = \ddot{r} = l$ and $T = \left.\frac{\partial V}{\partial r}\right|_{r=l}$, we get

$$\ddot{\theta} = -\left(\frac{g}{l}\right)\sin\theta$$

$$T = ml\dot{\theta}^2 + mg\cos\theta$$

Which are exactly the equation of motion we arrived at using Newton's law.

Comments:

1. In the Lagrangian with $V(r)$ we are allowing for the length of the string to change in accordance with the potential. Any restoring force (such as tension) ultimatey comes about from deforing the object (in this case the string) that is imposing the constraint.

2. In setting $\dot{r} = \ddot{r} = 0$ and evaluting $\left.\frac{\partial V}{\partial r}\right|_{r=l}$ we are saying that we are ignoring the stretching of the string which is too small to be measured. Here we were careful not to impose the constraint too soon.

2) The second approach is when we do not want to know anything about the constraining forces. In this case we use the constraint to express the Lagrangian in terms of the physical degrees of freedom:

$$L = L(q_i, \dot{q}_i, t)$$

In the case of the SHO we can write:

$$L = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell \cos\theta$$

Then we get only one equation for the physical d.o.f.

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow \quad -\cancel{m}g\cancel{\ell}\sin\theta - \cancel{m}\ell^{\cancel{2}}\ddot{\theta} = 0$$

$$\Rightarrow \quad \ddot{\theta} = -\left(\frac{g}{\ell}\right)\sin\theta \,.$$

Note that in this approach there is no mention of the tension force

Formal Discussion on holonomic constraints:

Let us generalize what we have seen for the SHO so far. For a system with N particle described by 3N generalized coordinates x^A, $A = 1, \dots, 3N$ and $(3N-n)$ holonomic constraints

$$f_\alpha(x^A, t) = 0 \qquad \alpha = 1, 2, \dots, 3N-n$$

One can describe the dynamics using the Lagrangian:

$$L' = L(x^A, \dot{x}^A, t) + \mu_\alpha f_\alpha(x^A, t)$$

where $L(x^A, \dot{x}^A, t)$ is the Lagrangian for the unconstraind system and μ_α are new degrees of freedom known as Lagrange multipliers. The equation of motion for x^A then become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) - \frac{\partial L}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A}$$

and the EL equations for the Lagrange multipliers become the constraint equation:

$$f_\alpha(x^A, t) = 0 .$$

Another approach is to find the independent generalized coordinates $q_1, q_2, \dots, q_n$ that solve the constraint equations

$$x_A = x_A(q_1, q_2, \dots, q_n)$$

And then express the unconstrained Lagrangian in terms of q_i and $\dot{q}_i$:

$$\tilde{L}(q_i, \dot{q}_i, t) = L(x_A(q_i), \dot{x}_A(q_i), t)$$

Then if work with the Lagrangian

$$L' = \tilde{L} + \mu_\alpha f_\alpha$$

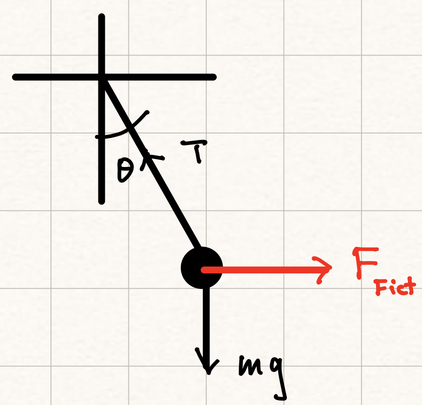
then the e.o.m. for q_i become:

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}_i}\right) - \frac{\partial \tilde{L}}{\partial q_i} = \mu_\alpha \frac{\partial f_\alpha}{\partial q_i}$$

But $\dfrac{\partial f_\alpha}{\partial q_i} = 0$ and so we see that we get the constrained equations of motion from $\tilde{L}(q_i, \dot{q}_i, t)$.

Examples:

1. An accelerometer: Consider a simple pendulum-like device which is in a car that is accelerating at a constant acceleration.

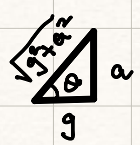


Newton's law in an inertial frame gives us:

$$T\cos\theta = mg \quad \sim \text{Equilibrium in the } z\text{-direction}$$

$$T\sin\theta = ma$$

$$\Rightarrow \quad a = g\tan\theta \Rightarrow \tan\theta = \frac{a}{g}$$



$$\cos\theta = \frac{g}{\sqrt{g^2+a^2}}$$

$$T = m\sqrt{g^2+a^2}$$

In a non-inertial frame attached to the car, the bob of the pendulum is stationary. And so in this frame the bob experiences an extra force $F_{fict.} = -ma$.

Let us see how this comes about from the Lagrangian formulation.

Let $\tilde{x} = x + \frac{1}{2}at^2 = l\sin\theta + \frac{1}{2}at^2$

$\tilde{y} = y = l\cos\theta$

$$\dot{\tilde{x}} = l\dot{\theta}\cos\theta + at$$
$$\dot{\tilde{y}} = -l\dot{\theta}\sin\theta$$

$\tilde{x}$ and $\tilde{y}$ are coordinates in the inertial frame.

So we start with the Lagrangian in an inertial frame and then change the coordinates to the non-inertial frame. Our theorem about the form of the EL equations mean that we can use the Lagrangian in the non-inertial frame to derive the E-L equations.

$$L = \frac{1}{2}m(\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2) + mg\tilde{y}$$

$$= \frac{1}{2}m((l\dot{\theta}\cos\theta + at)^2 + l^2\dot{\theta}^2\sin^2\theta) + mgl\cos\theta$$

$$= \frac{1}{2}m(l^2\dot{\theta}^2\cos^2\theta + a^2t^2 + 2lat\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta) + mgl\cos\theta$$

$$= \frac{1}{2}m(l^2\dot{\theta}^2 + a^2t^2 + 2lat\dot{\theta}\cos\theta) + mgl\cos\theta$$

EL:

$$\frac{\partial L}{\partial \theta} = -mla\cancel{t}\,\dot{\theta}\sin\theta - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + mlat\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mla\cos\theta - mlat\,\cancel{\dot{\theta}\sin\theta}$$

Then we get:

$$\cancel{m}l^2\ddot{\theta} + \cancel{m}la\cos\theta + \cancel{m}gl\sin\theta = 0$$

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{a}{l}\cos\theta}$$

Comments:

1. The second term on the RHS is proportional to a and is thus a 'pseudo' force that arises due to the acceleration.

2. The $\frac{1}{2}ma^2t^2$ term in the Lagrangian plays no role in the equation of motion. It is an example of a total time derivative term that we can add or subtract from the Lag-

rangian that has no effect on the equation of motion.

Solution:

Let $\theta = \theta'$ be the equilibrium solution when $\ddot{\theta} = 0$ and let η be small oscillations about θ':

$$\theta(t) = \theta' + \eta(t)$$

Then the equation of motion becomes:

$$\ddot{\eta} = -\frac{g}{\ell}(\sin\theta' \cos\eta + \cos\theta' \sin\eta)$$
$$- \frac{a}{\ell}(\cos\theta' \cos\eta - \sin\theta' \sin\eta)$$

Note that at equilibrium $\quad g \sin\theta' + a \cos\theta' = 0$

$$\Rightarrow \tan\theta' = -\frac{a}{g}.$$

$$\sin\theta' = \frac{-a}{\sqrt{a^2+g^2}} \qquad \cos\theta' = \frac{g}{\sqrt{a^2+g^2}}$$

In the small angle approximation $\cos\eta \simeq 1$ and $\sin\eta = \eta$:

$$\ddot{\eta} = -\frac{g}{\ell}(\sin\theta' + \cos\theta' \cdot \eta) - \frac{a}{\ell}(\cos\theta' - \sin\theta' \eta)$$

$$= -\frac{1}{l}\underbrace{\left(g\sin\theta' + \cos\theta'\right)}_{=0} - \frac{1}{l}\left(g\cos\theta' - a\sin\theta'\right)\eta$$

due to $\ddot{\theta}' = 0$

$$\boxed{\ddot{\eta} = -\frac{\sqrt{g^2+a^2}}{l}\,\eta}$$

If we define $\omega^2 = \frac{\sqrt{g^2+a^2}}{l}$, which has dimensions of $(\text{time})^{-2}$, the general solution is:

$$\theta(t) = \theta_0 \sin(\omega t + \varphi)$$

Thus we see that acceleration impacts the frequency and in the limit $a \to 0$ we get back the usual result: $\omega \to \sqrt{g/l}$.