Physics 110

Chapter 1: Vectors

Mathematics & Physics

It is an interesting and mysterious fact about the universe that mathematics is the language of nature. Thus it is very important for us to know about the relevant mathematical structure in order to be able do physics. In fact, much of modern mathematics was discovered due to its need in expressing new physical laws.

For example, one of the co-discoverers of calculus was Sir Isaac Newton who needed calculus to express the laws of mechanics, gravitation and the simple pendulum.

Vectors and the calculus of vectors was formalized by the Scottish mathematical physicist Oliver Heaviside and the American physicist Josiah Willard Gibbs when they reformulated Maxwell's equations for electromagnetism in its modern form.

Vectors & Scalars in Mechanics:

Many physical quantities in physics are represented by a single number. For example,

i) The temperature of an object in thermal equilibrium

ii) The mass of an object

iii) The average speed of a moving object

iv) The number density of atoms in a given fluid

Such quantities in physics are called scalars [The mathematical definition of a scalar is slightly different.]

Similarly, in physics there are quantities which require both a magnitude (a positive) number and a direction for their description. Such quantities, with an important exception, is described by vectors.

Example of physical quantities described by vectors:

1. Displacement: Distance through which a object moves + the direction in which it moves.

2. Average velocity: The Displacement per unit time. Incidentally this introduces the notion of taking the time derivative of a vector (displacement).

3. Instantaneous velocity, average acceleration, instantaneous acceleration, force, electric field, magnetic field etc.

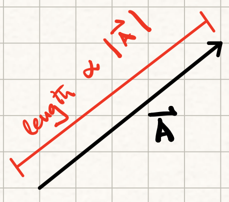
WARNING: In 3D space rotation has associated to it a direction given by the axis of rotation and a magnitudue given by the angle of rotation. But as we shall see rotations cannot be represented by vectors.

Vectors: A vector is a quantity which is an element of a set which satisfies a set of rules to be given below. Informally we may say a vector is a quantity that has both a magnitude (which is a positive number, i.e. ≥ 0) and a direction in 3D Euclidean space.

Vector Notation:

We denote a vector quantity by $\vec{A}$ or $\underline{A}$. In printed or online books a vector is often denoted by bold letter: $\mathbf{A}$. In this course we shall follow the convention $\vec{A}$.

Pictorially, we represent a vector by a straight line with an arrowhead. The length of the line is proportional to the magnitude of the vector:



The magnitude of a vector is represented by $\|\vec{A}\|$ or $|\vec{A}|$. The director of a vector $\vec{A}$ is denoted by $\hat{A}$. Thus we can write the vector $\vec{A}$ as:

$$\vec{A} = |\vec{A}|\,\hat{A}$$
$$= \hat{A}\,|\vec{A}|$$

This notation implies that

$$|\hat{A}| = 1.$$

$\hat{A}$ is also known as an unit vector.

Comments:

1. Although vectors are defined in reference to a fixed laboratory or some other 'fixed' physical system, vectors exist independent of any coordinate system.
2.

Equality of vectors:

When are two vectors $\vec{A}$ & $\vec{B}$ are considered to be equal to each other? Naturally the answer should be when:

$$\left.\begin{array}{ll} \text{i)} & |\vec{A}| = |\vec{B}| \text{ and} \\ \text{ii)} & \hat{A} = \hat{B} \end{array}\right\} \Leftrightarrow \vec{A} = \vec{B}$$

Note that in comparing the two vectors $\vec{A}$ and $\vec{B}$ we do not require them to be in the same place!

This implies that we can transport $\vec{A}$ (or $\vec{B}$) to $\vec{B}$ (or $\vec{A}$) but that does not change magnitude or direction the vector that is being transported. This is called parallel transportation because when we transport the vector $\vec{B}$ to the

Figure: In Euclidean space a vector that is parallely transported is independent of the path taken.

place where $\vec{A}$ is we make sure the at each 'moment' the transported $\vec{B}$ is parallel to the original $\vec{B}$.

This assumes that space through which we are transporting the vector is has Euclidean geometry. It is only in Euclidean space that unique parallel lines that extend to infinity exist. On curved spaces such as spheres or Lobachevski geometries no such notion of a unique parallel lines exist.

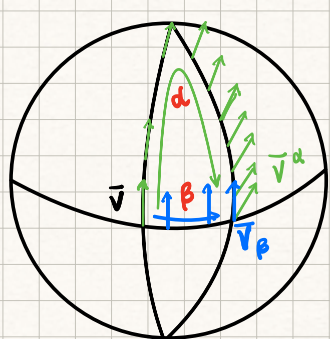
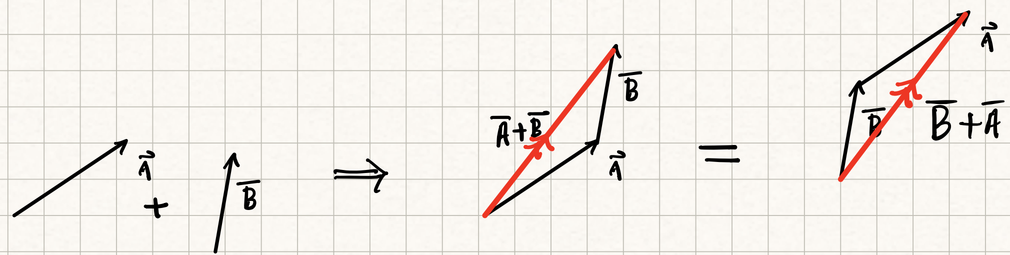


Figure: A sphere is an example of a non-Euclidean geometry. Locally (i.e. if you zoom in) the sphere looks like a flat Euclidean plane. If we use that intuition to parallely transport a vector $\bar{V}$ then we see that transported vector depends on the path taken! Thus the path α yields the vector $\bar{V}_\alpha$, while the path β yields the vector $\bar{V}_\beta$. It is patently clear that $\bar{V}_\alpha \neq \bar{V}_\beta$.

Vector Addition:

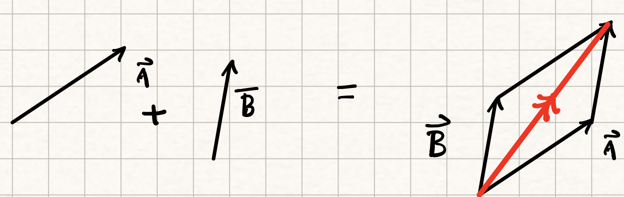
When can we add two vectors? From physical considerations both vectors need to be the same 'type'. For example we can add two displacement vectors but we can't add a displacement to a velocity vector.

To add two vectors $\vec{A}$ & $\vec{B}$ we parallely transport one of the vectors so that its base touches the tip of the other vector and then the vector from the base of the second vector to the tip of the first vector is the result of the addition. This vector is known as the resultant vector.



Note that resultant vector is the same regardless of which vector we parallely transport. Thus we can write $\vec{A}+\vec{B} = \vec{B}+\vec{A}$. Vector addition is commutative.

The symmetric nature of vector addition is revealed in the so-called parallelogram rule in which the vectors $\vec{A}$ and $\vec{B}$ form the adjacent arms of a parallelogram while the resultant is represented by the diagonal that goes through the common bases of the two vectors:



Comment:

Note that this way of adding two vectors $\vec{A}$ & $\vec{B}$ yields a quantity which, in general, has both magnitude and direction. Thus we say $\vec{A}+\vec{B}=\vec{C}$ where $\vec{C}$ is also a vector.

This property of vector addition called closure. Vector addition is a binary operation between two vectors which yields another vector.

Below we shall come across another binary operation where the result is not another vector but a scalar.

Negative of a Vector

The definition of the unit vector means that when we multiply a vector $\vec{A}$ by a real positive number α. This vector $\alpha\vec{A}$ has the same direction as $\vec{A}$ but its length is rescaled by a factor of α. This is a special case of scalar multiplication.

The idea of scalar multiplication can be easily extended to all $\alpha \in \mathbb{R}$, both positive & negative. A special case of interest is when $\alpha = -1$. In that case $\alpha\vec{A} = -\vec{A}$.

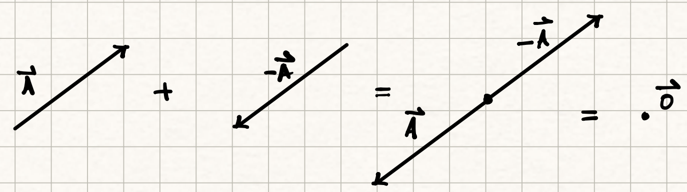
$-\vec{A}$ is a vector that has the same length as $\vec{A}$ but its direction is opposite. Graphically

$-\vec{A}$ is called negative vector of the $\vec{A}$ vector.

The Zero Vector/ The Null Vector

If we consider the addition $\vec{A}$ & its negative vector $-\vec{A}$ we see, using the rule of addition defined above that the resultant vector has zero magnitude and no direction. This is a special vector known as the zero vector or the null vector. The null vector is denoted by $\vec{0}$.

$$\vec{A} + (-\vec{A}) = \vec{0}$$



Comments:

1. The null vector has zero magnitude and no direction associated to it.
   $|\vec{0}| = 0.$
2. For any vector $\vec{A}$ we have $\vec{A} + \vec{0} = \vec{A}.$

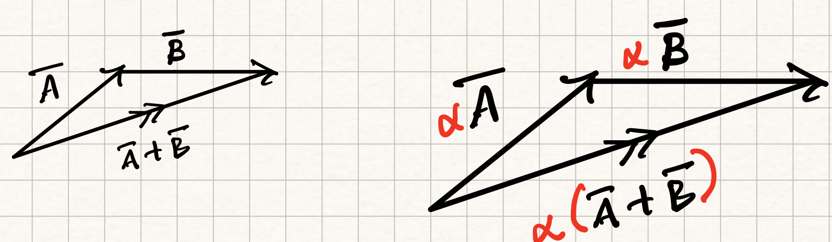
Properties of Scalar Multiplication:

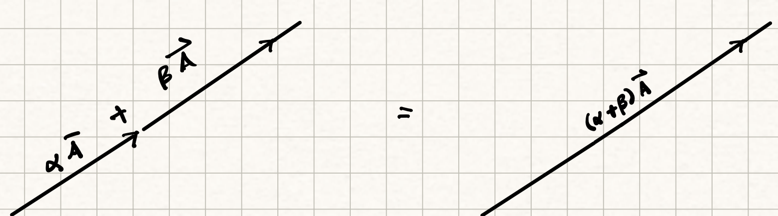
The following properties of scalar multiplication can be easily demonstrated:

$$\alpha(\vec{A} + \vec{B}) = \alpha\vec{A} + \alpha\vec{B} \quad \text{(Distributive law)}$$

$$(\alpha + \beta)\vec{A} = \alpha\vec{A} + \beta\vec{A}$$

Graphically:

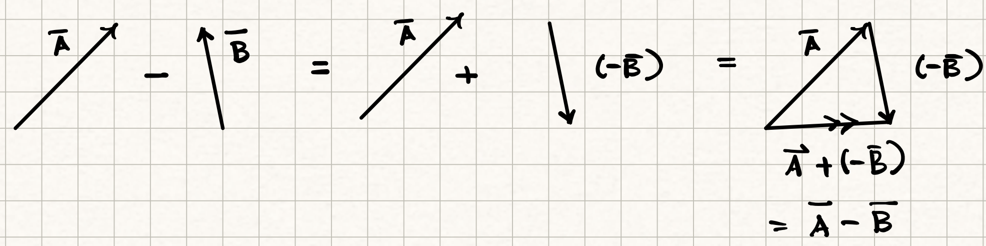


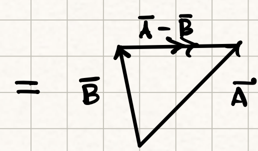


Vector Subtraction

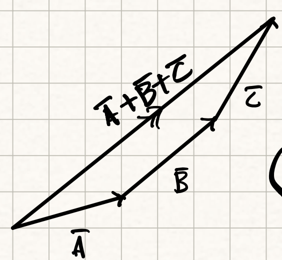
We can define subtraction in the following way:

$$\vec{A} + (-\vec{B}) := \vec{A} - \vec{B}$$





Associativity of Vector Addition:



It is straight-forward to show that

$$(\bar{A} + \bar{B}) + \bar{C} = \vec{A} + (\bar{B} + \bar{C}) \quad [\text{Associativity}]$$

Thus we can write $(\vec{A} + \vec{B}) + \vec{C} = (\vec{A} + \vec{B}) + \vec{C} := \vec{A} + \vec{B} + \vec{C}$

Mathematical Definition of a Vector:

A vector is an element of a set $\mathbb{V}$, known as a vector space over the reals $\mathbb{R}$, equipped with two binary operations

i) Vector Addition: $\quad + : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$

ii) Scalar Multiplication: $\quad \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$

st the following properties are satisfied:

1) If $\vec{A}, \vec{B} \in \mathbb{V}$ then $\vec{A} + \vec{B} \in \mathbb{V}$ (closure)

2) $\exists$ a unique vector, known as the zero vector or the null vector, denoted by $\vec{0}$, st $\vec{0} + \vec{A} = \vec{A}$ $\forall \vec{A} \in \mathbb{V}$.

3) For each vector non-zero vector $\vec{A} \in \mathbb{V}$ $\exists$ in $\mathbb{V}$ another unique vector $(-\vec{A})$ such that $\vec{A} + (-\vec{A}) = \vec{0}$.

4) Addition is commutative : $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

5) Addition is associative $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} = \vec{A} + \vec{B} + \vec{C}$

6) closure under scalar multiplication : $\forall \alpha \in \mathbb{R}$ & $\forall \vec{A} \in \mathbb{V}$

$\alpha \vec{A} \in \mathbb{V}$.

7) If $\alpha, \beta \in \mathbb{R}$ and $\vec{A}, \vec{B} \in \mathbb{V}$ then $\alpha \vec{A} + \beta \vec{B} \in \mathbb{V}$

8) $\alpha (\vec{A} + \vec{B}) = \alpha \vec{A} + \beta \vec{B}$

9) $(\alpha + \beta) \vec{A} = \alpha \vec{A} + \alpha \vec{B}$

Vector Products:

Several products between vectors can be defined. They are:

i) The scalar product or the dot product.

2) The cross-product.

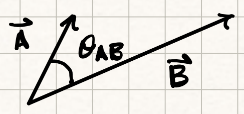
We discuss each in turn:

The Dot/Scalar product:

The scalar product or the dot product between two vectors $\vec{A}$ & $\vec{B}$ is defined to be

$$\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos \theta_{AB}$$

where θ_{AB} is the angle between the two vectors.



Comments:

1. The result of the dot product is a real number and <u>not</u> a vector.
2. The dot product is commutative:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$