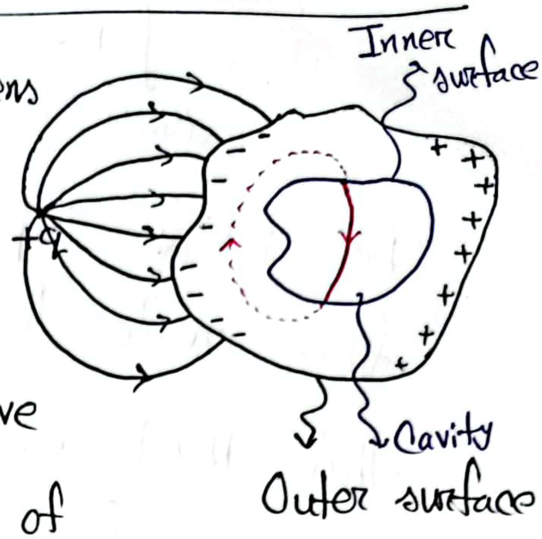
~~Induced~~

Lecture 7

Electric field inside a cavity of conductor without any charge

We already talked about what happens when we place a positive charge outside a conductor that is solid. But if there is a cavity inside, things don't change that much. Negative charges still pile up on the surface of the conductor that is close to the charge. Positive charges will pile up on the other side. Now, They will assemble themselve in such a manner, that there is no electric field inside the material of the conductor, such that we reach electrostatic equilibrium. Now, nothing is stoping the presence of electric field in the cavity of the conductor, since it won't break the electrostatic equilibrium.



Let's see whether there exists any electric field or not. Any electric field present inside the cavity, must start at one side of the inner surface and end at other. Let's choose a closed path (red colored), where some part of the loop is inside the cavity, and the other part is in the material. Now,

$$\oint_{\Gamma} \vec{E} \cdot d\vec{s} = 0$$

$$\therefore \int_{\text{Cavity}} \vec{E}\cdot d\vec{s} + \overbrace{\int_{\text{Material}} \vec{E}\cdot d\vec{s}}^{0} = 0$$

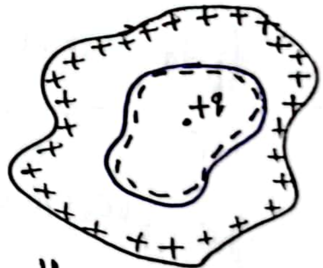
Since the E-field inside the material is zero.

$$\boxed{\therefore \int_{\text{Cavity}} \vec{E}\cdot d\vec{s} = 0}$$

Now if we look at the path traversed by the electric field inside the cavity, then $\int_{\text{Cavity}} \vec{E}\cdot d\vec{s}$ looks something that must be non-zero, unless the electric field itself is zero. So, we conclude that the electric field must be zero inside the cavity.

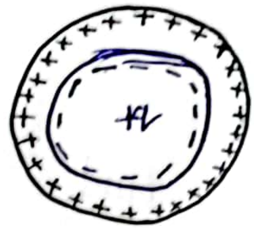
Induced charges in the conductor

Now, let's say, rather than the charge being outside the conductor, now we have a positive charge inside the cavity of the conductor. The positive charge will attract negative charges from the material on the inside surface. Correspondingly, there will be positive charges on the outer surface of the conductor. The question is, how will they assemble themselves? Will the distribution be uniform or not?

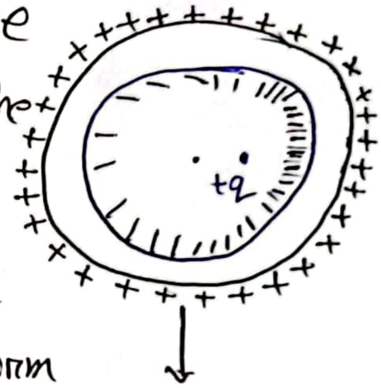


Now, if the conductor and the cavity both are in random shape, then the induced charge distribution will mostly be likely non-uniform. It depends on many factors. For example, consider first a spherical shell with a spherical cavity. The charge is placed at the center of the cavity. Each and every place on the material is equidistant from the charge $+q$. So, it will attract the charges uniformly.

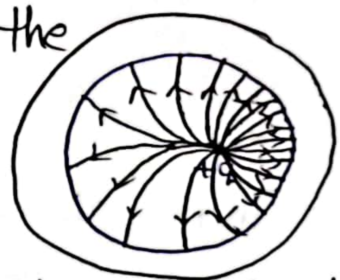
Hence, the negative charges will pile up on the inner surface, and consequently the negative charges will pile up on the outer surface, uniformly, like in the figure.



Now say, the charge is a bit offset from the center. Its now closer to a surface than the other one. So, we expect higher density of charges on the surface closer to the charge and lower on the other side. The field lines emitting from $+q$ will be denser on that side and lighter on the other.



But what happens to the charges on the outer surface? Are they non-uniform too? The answer is no. Because, on the outer surface, the charges do not have an electric field to tell them how to orient. So, if the surface is uniform, which is in this particular case, the charge will also distribute them uniformly. They will be now spread out on the outer surface of the sphere evenly.

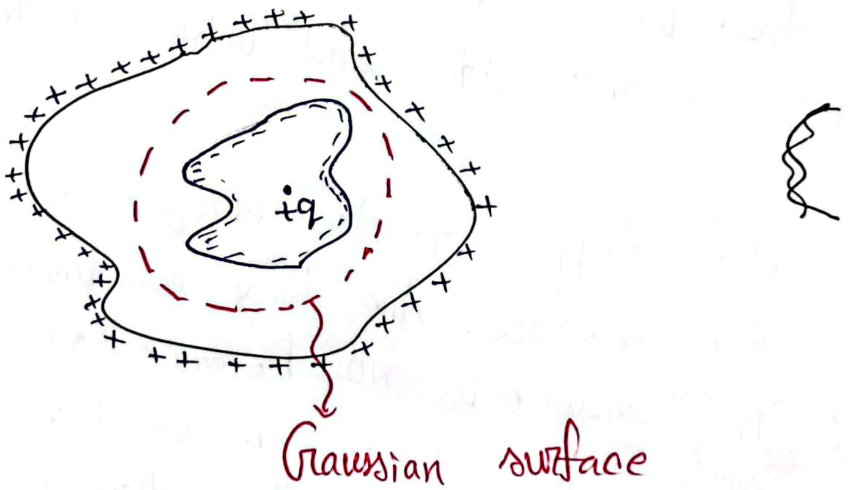


The induced charge density is very complicated he at least on the inner surface. The induced charges can be found by the following,

$$q_{ind,-} = \iint_{\substack{\text{Inner}\\\text{surface}}} \sigma_{ind,-}\, dA. \quad \Big| \quad q_{ind,+} = \iint_{\substack{\text{Outer}\\\text{surface}}} \sigma_{ind,+}\, dA$$

And it gets uglier when we try to find induced charges on non-uniform conductor with non-uniform cavity. By non-uniform, I mean without a symmetrical shape.

However, we have the mighty Gauss's law. Let's take a Gaussian surface, which is on the material of the conductor.



There is no electric field passing through the Gaussian surface, since $\vec{E} = \vec{0}$ in the material of the conductor. So,

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = 0$$

So, $\Phi_E = \dfrac{q_{enc}}{\epsilon_0} = 0 \quad \Rightarrow \quad q_{enc} = 0$

$$\therefore \quad q + q_{ind,-} = 0$$

$$\boxed{\therefore \; q_{ind,-} = -q} \quad !!!$$

Also, the total charge on the conductor before and after the induced charges were formed, must be conserved.

$$\therefore \; +q + q_{ind,-} + q_{ind,+} = +q$$

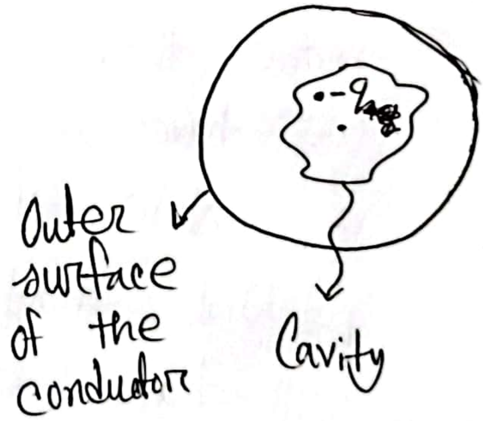
$$\therefore \; q_{ind,+} = -q_{ind,-}$$

$$\boxed{\therefore \; q_{ind,+} = +q}$$

This result is true for any shape of the conductor.

Problem 1

Find the electric field outside the spherical conductor with a cavity, at a radial distance r from the center of the sphere.



Uniqueness theorem

Poisson's equation was given by, $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$.

In the space where there are no charges, Poisson's equations reduces to Laplace's equation, given by,

$$\nabla^2\phi = 0$$

In Cartesian coordinates, Laplace's equation is written as-

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

We know, the charges are on the surface on the conductor. So, outside the surface of the conductor, there are no charges. So, the potential function ϕ should follow Laplace's equation. Now, to have a solution of this equation, we will need some boundary condition. Now, given some boundary condition, one might ask, is there no, one or more then one solution to the Laplace's equation. If we have a particular set-up, then there must be a state of the set-up, a stable state. So, the potential should follow some function, and we will have at least a solution for the Laplace's equation given some boundary conditions. Now, let's see the argument of the uniqueness theorem.

"The solution to Laplace's equation in some volume V is uniquely determined if ϕ is specified on the boundary surface.

So, if we have a set of boundary conditions for a set up of conductors, there exists one, and only one solution to the Laplace's equation.

<u>Proof</u>: We consider a volume where we want to find the potential ϕ, where the value of ϕ is known on the surface of that volume enclosing it. Say, there were two solutions, ϕ_1 and ϕ_2, each meeting the boundary condition, meaning $\phi_1 = \phi_2$ on the surface of the volume.

$$\therefore \quad \nabla^2 \phi_1 = 0 \quad \text{and} \quad \nabla^2 \phi_2 = 0$$

Consider another function $\phi_3 = \phi_1 - \phi_2$, which should also be the solution of the Laplace's equation. The reason is, Laplace's equation is linear. So, any linear combination of ϕ_1 and ϕ_2 will also be the solution of the Laplace's equation.

$$\nabla^2 (a\phi_1 + b\phi_2) = a\nabla^2\phi_1 + b\nabla^2\phi_2 = 0 + 0 = 0$$

Since, the boundary conditions are same for ϕ_1 and ϕ_2, so ϕ_3 will have the value of zero at the boundaries since, $\phi_3 = \phi_1 - \phi_2 = 0$ ($\phi_1 = \phi_2$ here).

Now, the solution to Laplace's equation doesn't have any local minimum or maximum except at the boundaries. This is due to the fact that, the solution of the Laplace's equation has the property of averaging value,

In 1D, the solution has the property that,

$$V(x) = \frac{1}{2}\left[V(x+a) - V(x-a)\right]$$

Same properties hold in 2D and 3D also, in a different manner.

So, if Φ_3 is 0 on the surface and at infinity, it must be zero everywhere.

$$\therefore \cancel{V_1} \qquad \therefore \Phi_1 = \Phi_2, \text{ everywhere.}$$

So, we have a unique solution.

Uniqueness theorem is also true for Poissons equation.

Now we would have,

$$\nabla^2 \Phi_1 = -\frac{\rho}{\epsilon_0} \qquad \nabla^2 \Phi_2 = -\frac{\rho}{\epsilon_0}$$

$$\therefore \nabla^2 \Phi_3 = \nabla^2 \Phi_1 - \nabla^2 \Phi_2 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$$

So, again we would have $\Phi_1 = \Phi_2$, since Φ_3 satisfies Laplace's equation.

We can use uniqueness theorem to prove that E-field is zero inside a conductor, as long as there is no charge inside (there may well be a cavity).

The conductor outer surface is an equipotential. So, we ~~ka~~ know the potential at the boundary. Since there are no charges inside, the potential function must satisfy the ~~Laan~~ Laplace's equation,

$$\nabla^2\phi = 0$$

Now, $\phi = \phi_0$ at the boundary — surface of the conductor. One solution of the Laplace's equation that also statisfies the boundary condition is obviously that $\phi = \phi_0$ everywhere inside the ~~cavity~~ conductor. By virtue of uniqueness theorem, this should be the only solution.

$\therefore$ ϕ = constant inside the conductor.

$\therefore$ $E = 0$, " " " .

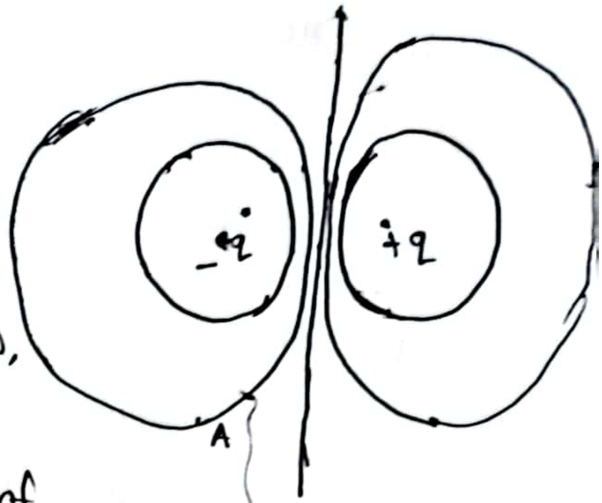
Now, this wouldn't have been true if there were charge inside the cavity. For one reason,

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

So, our previous guess solution that ϕ = constant wouldn't work here.

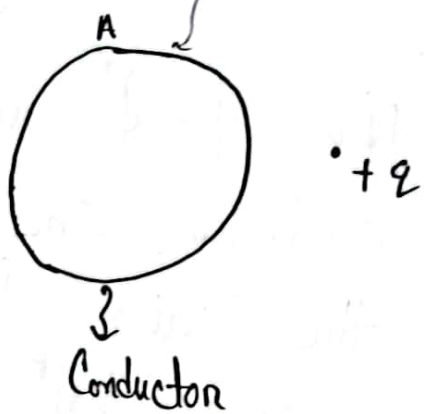
The classic image problem

Motivation



The figure shows the intersection of the equipotential surfaces with the page, for two charges, one positive and one negative.

Now consider, we replace one of the equipotential surfaces with a conductor, and set the potential on the surface of the conductor same as the equipotential from the charge assembly.

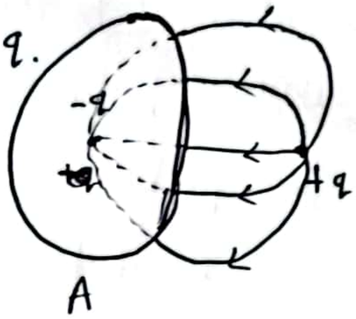


Now, our setup has no difference with the two point charge set up. The potential is same on the surface of the conductor, as on the equipotential surface. The E-fields will be perpendicular to the conducting surface, as it is in the equipotential surface.

Now, the potential is specified on the boundary. The potential statisfies Poisson's equation outside the conductor. So, the potential has a unique solution.

Hence, the potential outside the conductor will be same as the two-charge system. So, the electric field outside will look like as if there was an imaginary charge inside the conductor, $-q$.

This imaginary charge is called the image charge. So, we solved a new problem with the virtue of uniqueness theorem.



The classic image problem

Consider an infinite grounded conducting plane (meaning it could be slab also), so the plane is the surface of a conductor) on the XY plane. A point charge q is held at a distance d above the plane. You have to find the potential (and hece E-field) in the region above the plane. Now, the charge q will induce some charges on the surface. So the potential will be due to $+q$ and induced charge and the superposition of them.

The potential will satisfy Poisson's equation above the plane, with the boundary conditions —

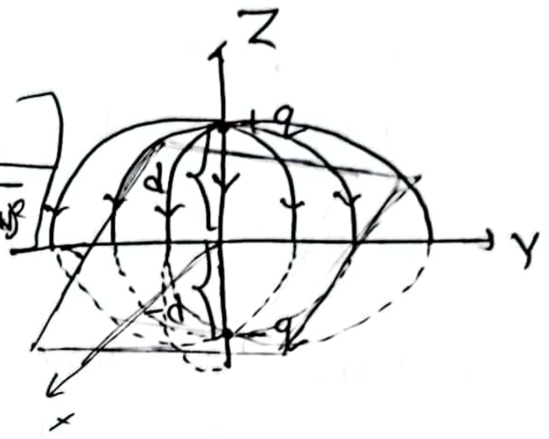
(i) $\phi = 0$, when $z = 0$

(ii) $\phi \to 0$, far from charges $(r^2 \gg d^2)$.

Now, consider a different problem, again a positive and negative charge. Our conductor ~~is the~~ can be represented by the flat equipotential surface at the midway between the charges. We might sense that the problem is solved. All we need an image charge, at a distance d inside the conductor.

Due to the charges,

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0} \cdot \left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}} \right]$$



This potential satisfies the boundary conditions —

(i) ~~$\phi(0,0,$~~ $\phi(x,y,0) = 0$

(ii) $\phi \to 0$, for $x^2+y^2+z^2 \gg d^2$

$r^2 \gg d^2$

Since ϕ meets the boundary conditions and ~~satisfies~~ satisfies Poisson's equation, this must be the solution to our problem! (Since uniqueness theorem says, this is the only solution.)

Induced surface charge

The electric field just outside the conductor will be normal to the surface of the conductor and equal to,

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$$\therefore \quad \vec{E} = \frac{\sigma}{\epsilon_0}\hat{k}$$

In terms of magnitude, $\sigma = \epsilon_0 E = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$

$$= -\epsilon_0 \cdot \frac{q}{4\pi\epsilon_0}\left[\frac{-\frac{1}{2}2(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} - \frac{-\frac{1}{2}\times 2(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}}\right]\Bigg|_{z=0}$$

$$= \cancel{-\frac{qd}{4\pi}\frac{}{\left[x^2+y^2+\right.}}$$

$$= -\frac{q}{4\pi}\frac{2d}{(x^2+y^2+d^2)^{3/2}} = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

$$\therefore \sigma(x,y) = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

So, the induced charge is negative if q is positive and maximum at $x=y=0$. The total induced charge will be given by,

$$q_{ind} = \iint \sigma\, dA = \iint \sigma\, dx\, dy$$

$$= \iint -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}\, dx\, dy$$

$$= \int_0^\infty\int_0^{2\pi} -\frac{qd}{2\pi(r^2+d^2)^{3/2}}\, r\, dr\, d\theta$$

$$= -\frac{qd}{2\pi}\times 2\pi \int_0^\infty \frac{r}{(r^2+d^2)^{3/2}}\, dr$$

$$= -qd \int_d^{\infty} \frac{P}{P^3}\, dP$$

$$= -qd \int_d^{\infty} \frac{1}{P^2}\, dP = -qd \cdot \left. \frac{P^{-2+1}}{-2+1} \right|_d^{\infty}$$

$$= +qd \left[\frac{1}{\infty} - \frac{1}{d} \right]$$

$$\boxed{\therefore\ q_{ind} = -q}$$

$r^2 + d^2 = P^2$

$\Rightarrow 2r \frac{dr}{dP} = 2P$

$\therefore\ r\,dr = P\,dP$

$r = 0,\quad P = d$