Lecture 10

Introduction to dimensional analysis

Any investigation in physics ultimately comes down to determining certain quantity which may depend on various other quantities that characterizes the phenomena under consideration. The problem therefore reduces to establishing relationship which can always be represented in the form –

$$f = f(q_1, q_2, \ldots, q_n)$$

The quantities $q_1, q_2, \ldots, q_n$ are called governing parameters which combines into a single term that bears the same dimension as f.

Dimensional and dimensionless quantities

If the numerical value of any quantity changes upon change of one unit of measurement to another unit of measurement, then the quantity is called a dimensional quantity.

Example: The angular frequency of a simple pendulum is given by, $\omega = \sqrt{g/L}$.

The numerical values of ω depends on g and L. If we change the unit of measurement from MKS to CGS, the numerical values of g, L and hence ω changes. These are dimensional quantities.

However, we can construct a quantity,

$$\Pi = \frac{\omega}{\sqrt{g/L}}$$

Upon changing the ~~quantity~~ unit of measurement from MKS to CGS, the numerical value of Π remains the same. So, Π is a dimensionless quantity.

Power-law monomial nature of dimensions

A function $f = f(a_1, a_2, \ldots, a_n)$ is called power law monomial is f can be expressed in terms of,

$$f = a_1^{\alpha} a_2^{\beta} \ldots a_n^{\gamma}.$$

In fact, all physical laws follow power law monomial. There is no such instances where dimension function can be expressed as. $a_1 e^{a_2}$ or $\sin a_3$ or $\log a_4$. or $a_1^2 + a_2^{-3}$ etc.

This power law monomial nature actually helps us to deduce the exact nature of some function or quantity even without detailed information. Let's see an example.

The speed of travelling wave through a string depends on the elastic property of the medium and intertia. For a string, the elastic property is the tension on the string T and the inertia term is the mass density μ.

$$\therefore\ v = v(T, \mu)$$

Power law monomial says,

$$[v] = [T]^{\alpha} [\mu]^{\beta}$$

$$\Rightarrow LT^{-1} = (MLT^{-2})^{\alpha} (ML^{-1})^{\beta} = M^{\alpha+\beta} L^{\alpha-\beta} T^{-2\alpha}$$

Equating both sides we get,

$$\alpha + \beta = 0 \qquad \text{and} \qquad -2\alpha = -1$$
$$\alpha - \beta = 1 \qquad\qquad \therefore \alpha = 1/2$$

$$\therefore \beta = -\frac{1}{2}$$

$$\therefore [v] = [T]^{1/2} [\mu]^{-1/2}$$

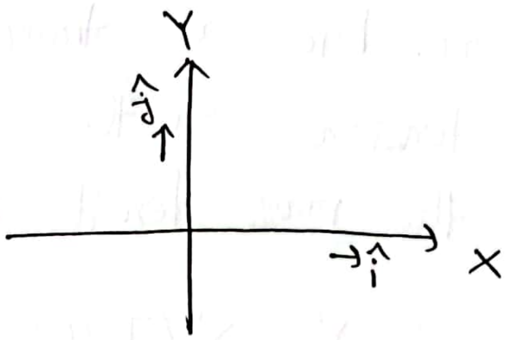
$$\therefore v \sim \sqrt{T/\mu}$$

Upto some dimensionless constant, the expression of the velocity is exact.

Reviews of coordinate systems and vectors

Cartesian coordinates and polar coordinates

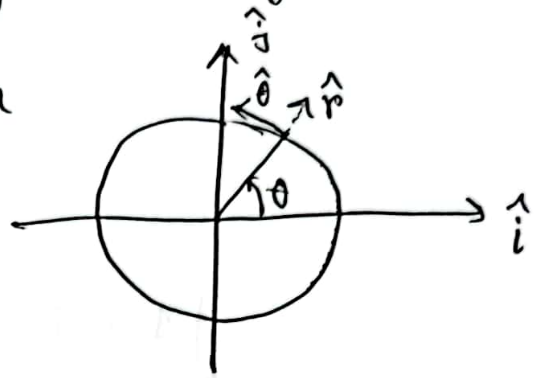
2D



In 2D cartesian coordinates, we define two unit vectors $\hat{i}$ and $\hat{j}$ along the X and Y axis. The unit vectors are typically in the directions of increasing X and Y.

If we move to polar coordinates, then the unit vectors are $\hat{r}$ and $\hat{\theta}$. $\hat{r}$ is typically away from the origin in outward direction and ~~changes~~ $\hat{\theta}$ is perpendicular to it and is in the direction of increasing θ. The major difference between the Cartesian and polar coordinate is that- $\hat{i}$ and $\hat{j}$ is fixed in terms of directions. ~~Howe-~~ However, $\hat{r}$ and $\hat{\theta}$ continuously can change directions

At any time, $\hat{r}$ and $\hat{\theta}$ can be expressed as—



$$\hat{r}(t) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta}(t) = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

if the particle is moving in a circle. The position of any particle at some time is exactly given by,

$$\vec{r}(t) = r\,\hat{r}(t).$$

<u>Vector dot and cross product</u>

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\,(A_yB_z - A_zB_y) + \hat{j}\,(A_zB_x - A_xB_z) + \hat{k}\,(A_xB_y - B_xA_y)$$

$\cancel{\hat{i}\cdot\hat{i} =}$

$$\left.\begin{aligned} &\hat{i}\cdot\hat{j} = \delta_{ij} \\ &\cancel{\hat{i}\times\hat{k} =} \\ &\hat{i}\times\hat{j} = \epsilon_{ijk}\,\hat{k} \end{aligned}\right\} \quad \begin{aligned} &\delta_{ij} = \text{Kronecker delta} \\ &\epsilon_{ijk} = \text{Levi-civita symbol} \end{aligned}$$

$$\delta_{ij} = \begin{cases} 1\,; & i=j \\ 0\,; & i\neq j \end{cases}$$

In general, we can express dot product as,

$$\vec{A}\cdot\vec{B} = \sum A_iB_i = A_iB_i \rightsquigarrow$$

Einstein sum convention → repeated indices are summed over

$$(\vec{A}\times\vec{B})_i = \epsilon_{ijk}A_jB_k$$

$$= \sum_{k=1}^{3}\epsilon_{11k}(A_1B_k) + \sum_{k=1}^{3}\epsilon_{12k}(A_2B_k) + \sum_{k=1}^{3}\epsilon_{13k}(A_3B_k)$$

$$= 0 + [0+0+A_2B_3] + [0+(-1)A_3B_2+0]$$

$$= A_2B_3 - A_3B_2 = A_yB_z - A_zB_y \text{ and so on.}$$