Group Theory
Lecture #7

The exponential map and one-parameter Abelian subgroups

Given a Lie algebra element X, one can use exponentiation to generate the Lie group element $e^{i\alpha M}$ for some $\alpha \in \mathbb{R}$. This map is called the exponential map and it is of the form:

$$\exp: I \times \tilde{g} \longrightarrow G$$

$$\exp: \alpha \times X \longmapsto \exp(i\alpha X)$$

where I is an interval of $\mathbb{R}$. $\tilde{g}$ is the Lie algebra and G is the corresponding Lie group.

The exponential map must satisfy the following properties:

1. $\exp(0) = e$ where e is the identity element.
2. $\exp(isX)\exp(itX)$

$$= \exp(i(s+t)X)$$

hence the set of all exponential maps for a given X form a one-parameter Abelian subgroup of G.

3. $\frac{1}{i}\frac{d}{dt}\exp(itX)\Big|_{t=0} = X$.

Example:

Let $G = SU(2)$. The Lie algebra of $SU(2)$ is

$$\widetilde{su(2)} = \{T_1, T_2, T_3\}$$

If we take $T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then elements of the 1-parameter Abelian subgroup has the form

$$\exp(itT_3) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}$$

Show that T_1 & T_2 also generate 1-parameter Abelian subgroups.

The Adjoint Representation

Recall the definition of a representation:

A representation is a map from a group G into a set of matrices D such that the group structure is preserved:

$$\text{If} \quad g_1 g_2 = g_3 \quad \text{for } \forall \; g_1, g_2, g_3 \in G$$

Then $\quad D(g_1) D(g_2) = D(g_3).$

In particular $\quad D(e) = \mathbb{1}.$

The dimension of the matrices D is the **dimension of the representation**.

Comment:

1. The definition of representation is valid for both discrete and continuous groups.

2. The dimension of a rep is in general

different from the dimension of a Lie group. A Lie group of a given dimension can have an infinite number of representations of different dimensions.

The Adjoint Representation of a Lie Group:

Given a Lie group G and an associated Lie algebra $\tilde{g}$ one can define an action of G on $\tilde{g}$. This action is called an adjoint action Adj:

For $g \in G$ and $X \in \tilde{g}$, the Adj action is defied by

$$\mathrm{Adj}_g(X) = g X g^{-1}.$$

Comments:

1. The Adj. action preserves the Lie algebra: If $\{T_a\} \cong \tilde{g}$ and $\{\tilde{T}_a = g T_a g^{-1}\}$ for some $g \in G$ then

$$[\tilde{T}_a, \tilde{T}_b] = i f_{ab}{}^{c} \tilde{T}_c$$

2. The Adj. action maps the Lie algebra $\tilde{g}$ onto itself.

The Adj action induces a representation C that acts on the Lie algebra $\tilde{g}$.

$$\mathrm{Adj}_{g_1 g_2}(X) = g_1 g_2 X g_2^{-1} g_1^{-1}$$

$$= g_1 \,\mathrm{Adj}_{g_2}(X)\, g_1^{-1}$$

$$= \mathrm{Adj}_{g_1}(\mathrm{Adj}_{g_2}(X))$$

This representation is called the adjoint representation of G. Let us denote the matrices of the adjoint rep. of G by $C_a{}^b$:

$$\mathrm{Adj}_g(T_a) = g T_a g^{-1} = C(g)_a{}^b T_b$$

$$\mathrm{Adj}_{g_1 g_2}(T_a) = \mathrm{Adj}_{g_1}(\mathrm{Adj}_{g_2}(T_a))$$

$$= \mathrm{Adj}_{g_1}(C_a{}^b(g_2) T_b)$$

$$= C_a{}^d(g_1)\, C_d{}^b(g_2)\, T_b$$

$$= \mathrm{Adj}_{g_3}(T_a) = C_a{}^b(g_3)\, T_b$$

Since T_b are arb. elements of $\tilde{g}$ $\Rightarrow$

$$C_a{}^d(g_1)\, C_d{}^b(g_2) = C_a{}^b(g_1 g_2)$$

Thus we can write

$$\tilde{T}_a = C_a{}^b(g)\, T_b$$

The Adjoint Representation of a Lie Algebra:

The Adj action of a Lie group G on a Lie algebra $\tilde{g}$ induces an action of the Lie algebra $\tilde{g}$ on itself. It is called the adjoint map:

$$\mathrm{adj}_X(Y) := [X, Y]$$

To see why this is natural let us expand Adj action for infinitesimal element $g \simeq \mathbb{1} + i\theta_a T_a$

where T_a are generators $\in \tilde{g}$.

Then $$\mathrm{Adj}_g(T_b) \simeq \mathbb{1} + i\theta_a [T_a, T_b]$$

$$= \mathbb{1} + i\theta_a \,\mathrm{adj}_{T_a}(T_b)$$

We can also expand $C(g)$ as a power series in θ_a:

$$C(\mathbb{1} + i\,\theta_a T_a)_b{}^c\, T_c \simeq \mathbb{1} + i\theta_a (F_b)_a{}^c\, T_c$$

Comparing these two we get:

$$(F_b)_a{}^c\, T_c = [T_a, T_b]$$

$$(F_b)_a{}^c\, T_c = i f_{ab}{}^c\, T_c$$

$$\boxed{(F_a)_b{}^c = -i f_{ab}{}^c}$$

$F_a \to \dim\tilde{g} \times \dim\tilde{g}$ matrices. F_a satisfies the Lie algebra $\tilde{g}$. This can be shown using the Jacobi identity.

Claim:

The matrices F_a form a rep of $\tilde{g}$. To prove this we start with the Jacobi identity:

$$[T_a,[T_b,T_c]]+[T_b,[T_c,T_a]]+[T_c,[T_a,T_b]]=0$$

using $[T_a,T_b]= i f_{ab}{}^{c}\, T_c$ we get:

$$-f_{ad}{}^{f}\, f_{bc}{}^{d} - f_{bd}{}^{f}\, f_{ca}{}^{d} - f_{cd}{}^{f}\, f_{ab}{}^{d} = 0$$

$$(F_b)_c{}^{d}(F_a)_d{}^{f} - (F_a)_c{}^{d}(F_b)_d{}^{f} = i f_{ba}{}^{d}(F_d)_c{}^{f}$$

$$\boxed{[F_a,F_b]= i f_{ab}{}^{c}\, F_c}$$

Example:

If $G=SU(2)$ and $\tilde{g}=\widetilde{su(2)}$. Then the generators of the adjoint reps are:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(J_i)_{jk} = -i\,\epsilon_{ijk}$$

show that $[J_i, J_j] = i\,\epsilon_{ijk} J_k$

use: Use explicit rep above or use:

$$\epsilon_{ilm}\,\epsilon_{jnm} = \delta_{ij}\,\delta_{ln} - \delta_{in}\,\delta_{lj}$$