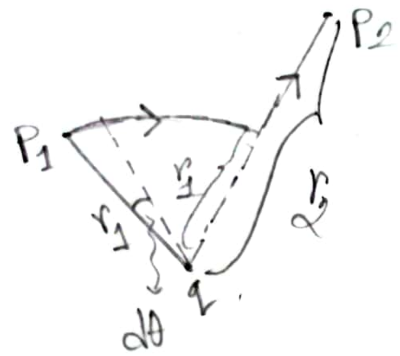
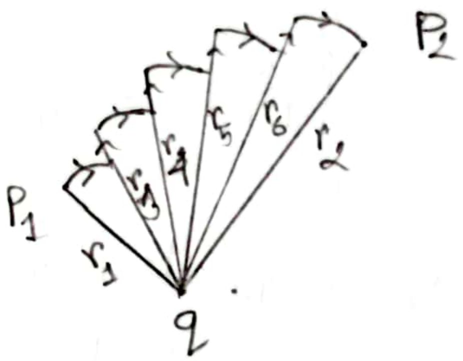
Lecture 5

Electric potential

Line integral of the electric field

Let us first calculate the line integral of the electric field produced by a point charge q from point P_1 to P_2, as shown in the figure 1. The line integral can easily be computed if we first move along an arc of radius r_1 upto the line r_2, and then move radially outward along r_2 to reach P_2.





$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\hat{r}$$

$$\therefore \int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{\theta=0}^{\theta=\theta} \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}\cdot r_1 d\theta\,\hat{\theta} + \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}\cdot dr\,\hat{r}$$

$$= 0 + \frac{q}{4\pi\epsilon_0}\cdot -\frac{1}{r}\Big|_{r_1}^{r_2}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Now, you could take a different path, as shown in the second picture, to reach from P_1 to P_2.

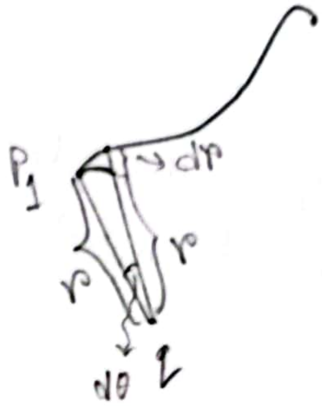
$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s} = \int_{\theta=0}^{\theta=\theta_1} \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}\cdot r_1 d\theta\,\hat{\theta} + \int_{r=r_1}^{r_3} \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\hat{r}\cdot dr\,\hat{r} + \cdots$$

$$= 0 + \frac{1}{4\pi\epsilon_0} q\left[\frac{1}{r_1} - \frac{1}{r_3}\right] + 0 + \frac{1}{4\pi\epsilon_0} q\left[\frac{1}{r_3} - \frac{1}{r_4}\right] + \cdots + \left[\frac{1}{4\pi\epsilon_0} q \left[\frac{1}{r_6} - \frac{1}{r_2}\right]\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Now you can take any random path, and the infinitesimal displacement in polar coordinate can always be written as —

$$d\vec{S} = r\,d\theta\,\hat{\theta} + dr\,\hat{r}$$



So, for any arbitrary path.

$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\int_{P_1}^{P_2} \frac{1}{r^2}\hat{r}\cdot\left[r\,d\theta\,\hat{\theta} + dr\,\hat{r}\right]$$

The contribution from the $\hat{\theta}$ part will always give you a zero and hence,

$$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{S} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \text{ for any path.}$$

Now, the electric field could come from many charges. or in general from a distribution. But the superposition principle gives,

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots + \vec{E}_n$$

So, $$\int_{P_1}^{P_2} \vec{E}\cdot d\vec{S} = \int_{P_1}^{P_2} \vec{E}_1\cdot d\vec{S} + \int_{P_1}^{P_2} \vec{E}_2\cdot d\vec{S} + \int_{P_1}^{P_2} \vec{E}_3\cdot d\vec{S} + \cdots + \int_{P_1}^{P_2} \vec{E}_n\cdot d\vec{S}$$

Since the individual line integrals does not depend on any particular path, so, for any electric field $\vec{E}$, the line integral of $\vec{E}$ is path independent.

$$\therefore \int_{P_1}^{P_2} \vec{E}.d\vec{s} \text{ is the same for any arbitrary path.}$$

Since the line integral doesn't depend on a particular path, rather only depends on the endpoints, so, we can write,

$$\int_{P_1}^{P_2} \vec{E}.d\vec{s} = \phi(P_1) - \phi(P_2)$$

where $\phi(P_1)$ and $\phi(P_2)$ are some scalar numbers at P_1 and P_2, ~~with r_1 and r_2 being the radial distance of P_1 and P_2~~ from the charge.

$$\therefore \phi(P_2) - \phi(P_1) = -\int_{P_1}^{P_2} \vec{E}.d\vec{s}$$

$$\therefore \phi_{21} = -\int_{P_1}^{P_2} \vec{E}.d\vec{s}$$

where, ϕ_{21} is a single valued scalar function of the two positions P_1 and P_2, called the potential difference between these two points.

Multiplying both sides with a q, we get,

$$q\Phi_{2,1} = -\int_{P_1}^{P_2} q\vec{E}\cdot d\vec{s} = -\int_{P_1}^{P_2} \vec{F}_e\cdot d\vec{s}$$

The right hand side is the work done ~~against~~ by an external agent ~~the electric force~~ to move a positive charge q from P_1 to P_2. So, here we have our physical meaning for the potential difference. Its the work done per unit charge ~~against~~ in the electric field by an external agent to move the charge from P_1 to P_2. The unit of potential difference is hence $\frac{\text{Joule}}{\text{Coulomb}}$ which is commonly written as volt.

Although we now have the potential difference between two points, we do not have an exact potential function at a particular point P. However, since we are only worried about the potential difference we can set a reference point $P_1 = P_{ref}$, and calculate the potential difference between any point P and P_{ref}. We can then define the potential difference as a function of P only, and we call that the potential function w.r.t. the reference point P_{ref}.

$$\therefore \quad \boxed{\phi(P) = -\int_{P_{ref}}^{P} \vec{E}\cdot d\vec{s}}$$

This is the same thing as setting the potential at the reference point to be zero, as can be seen from the above equation. Finally the electric potential is the work done by an external agent per unit charge for moving the charge from P_{ref} to P.

For a point charge, we often set the reference point at infinity.

$$\therefore \quad \phi(P) = -\int_{r_{ref}}^{r} \vec{E}\cdot d\vec{s} = \frac{-q}{4\pi\epsilon_0}\left[\frac{1}{r_{ref}} - \frac{1}{r}\right]$$

$$r_{ref} = \infty \implies \boxed{\phi(P) = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r}}$$

where r is the radial distance of the point P from that charge. For a collection of charges, the potential energy at a point P is given by,

$$\phi(P) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r_{P,i}}$$

with $r_{P,i}$ being the distance of P from the i^{th} charge.

For a continuous distribution of charges,

$$\phi(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{over all vol}} \frac{\rho\, dV}{r}$$

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Electric field as a gradient of potential

We have,
$$\int_{P_1}^{P_2} \vec{E}.d\vec{s} = \phi(P_1) - \phi(P_2) \quad \text{--- (1)}$$

But from vector calculus, the fundamental theorem of calculus for the gradient tells,

$$\int_{P_1}^{P_2} (\vec{\nabla}\phi).d\vec{s} = \phi(P_2) - \phi(P_1) \quad \text{--- (11)}$$

Comparing equation (1) and (11) we can write,

$$\vec{E} = -\vec{\nabla}\phi$$

So, the electric field can be written as a negative gradient of potential.

Now, since the line integral is not path dependent,

$$\therefore \oint \vec{E}.d\vec{s} = 0, \text{ for any closed path.}$$

Using Stoke's theorem, we get,

$$\oint_{\partial S} \vec{E}.d\vec{s} = \iint_S (\vec{\nabla}\times\vec{E}).d\vec{A}$$

$$\therefore \iint_S (\vec{\nabla}\times\vec{E}).d\vec{A} = 0 \text{ ; for any surface.}$$

Since the integral is zero for any arbitrary surface, so, the integrand itself must be zero.

$$\boxed{\vec{\nabla}\times\vec{E}=\vec{0}} \quad \text{———} (*)$$

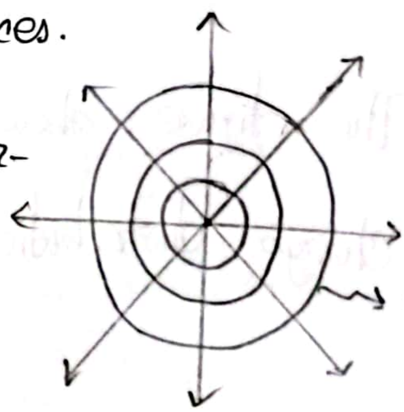
This result could also be found from the relation that

$$\vec{\nabla}\times(\vec{\nabla}\phi)=\vec{0}.$$

Equation (*) is Maxwell's second equation for electrostatics.

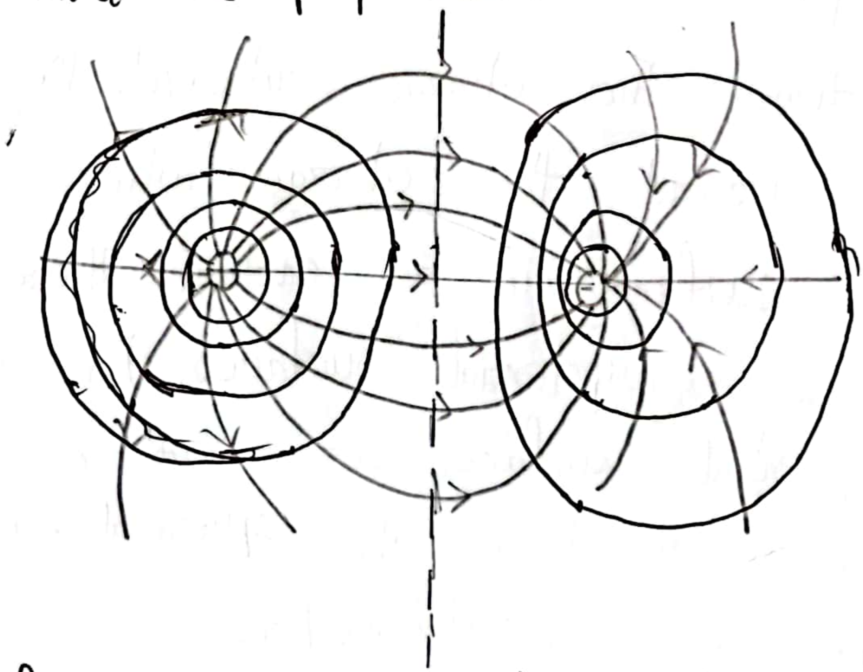
Field lines and equipotential surfaces

For a positive point charge, we know that the field lines diverges from the charge outward. We can draw surfaces around the charge where the potential on the surface is a constant. These surfaces are called equipotential surfaces. For a point charge, the equipotential surfaces are ~~point~~ spheres centered at the point charge. The spherical surfaces shown in the figure are equipotential surfaces.



$\phi =$ constant

Now, the electric field must be perpendicular to the equipotential surfaces. Because if the electric field is not perpendicular to the surface, then, there will be some component of the electric field along the surface. Any component of the electric field along the surface will change the potential on the surface, and the surface will no longer be an equipotential. So, field lines are always perpendicular to the equipotential surface. This is also evident from the fact that electric field is the gradient of potential. Gradient is in the direction where the function is changing most rapidly. Since the most rapid change from a point on the equipotential surface must be the perpendicular direction, so the electric field must be perpendicular to the equipotential surfaces.



The figure shows equipotential surfaces of a dipole charge distribution.

Potential from an electric field for an infinite line of charge

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

perpendicular

We choose an arbitrary reference point P_1 at a perpendicular distance r_1 from the wire. Then, to move a unit charge from P_1 to P_2, which is at a perpendicular distance of r_2 is given by,

$$\phi_{21} = -\int_{r_1}^{r_2} \vec{E}.d\vec{s} = -\int_{r_1}^{r_2} \frac{\lambda}{2\pi\epsilon_0 r}\cdot\hat{r}\cdot\left[dr\,\hat{r} + r\,d\theta\,\hat{\theta}\right]$$

$$\therefore \phi_{21} = -\frac{\lambda}{2\pi\epsilon_0}\ln r\Big|_{r_1}^{r_2} = -\frac{\lambda}{2\pi\epsilon_0}\ln r_2 + \frac{\lambda}{2\pi\epsilon_0}\cdot\ln r_1$$

If r_1 is my reference point, then for any point r, we can define the potential function as,

$$\boxed{\phi(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln r_2 + \text{constant}}$$

If we want to recover the electric field, then —

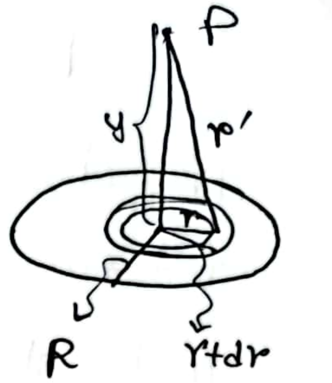
$$\vec{E} = -\vec{\nabla}\phi = -\hat{r}\frac{d\phi}{dr} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

where, there is no angular dependence.

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi}\hat{\psi}$$

Potential for a charged non-conducting disk

Considering the reference point at infinity,



$$\phi(P) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\, dA}{r'}$$

$$= \frac{1}{4\pi\epsilon_0}\int_{r=0}^{r=R} \frac{\sigma \cdot 2\pi r\, dr}{\sqrt{r^2+y^2}}$$

$$= \frac{\sigma}{2\epsilon_0}\int_0^R \frac{r}{\sqrt{r^2+y^2}}\, dr$$

$$u = r^2+y^2$$

$$\Rightarrow \frac{du}{dr} = 2r$$

$$\Rightarrow du = 2r\, dr$$

$r=0$, $u=y^2$

$r=R$, $u=R^2+y^2$

$$= \frac{\sigma}{2\epsilon_0}\int_{y^2}^{R^2+y^2} \frac{du}{2\sqrt{u}}$$

$$= \frac{\sigma}{4\epsilon_0}\cdot \frac{\sqrt{u}}{1/2}\Bigg|_{y^2}^{R^2+y^2}$$

$$\boxed{\therefore\ \phi(P) = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+y^2} - |y|\right]}$$

So, the potential is an even function of y. At y

$$\phi(P) = \frac{\sigma R}{2\epsilon_0}$$