Chapter 3: Infinite Series

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PHY405 – Mathematical Physics
Fall 2024
Reference Texts: Mary L. Boas; Arfken, Weber and Harris.

1 Sequences and Series: What's the difference?

1.1 Sequences

A *sequence*, $\{a_n\}_{n=1}^{\infty}$, is a countably infinite set of numbers,

$$a_1, a_2, a_3, ..., a_n, ... \tag{1}$$

For instance the set of all natural numbers ($\mathbb{N}$) is such a sequence.

We say a sequence possesses a (finite) limit l,

$$\lim_{n\to\infty} a_n = l \ , \tag{2}$$

or, simply put, $a_n \to l$ as $n \to \infty$. More formally, it can be stated as, for every $\epsilon > 0$, no matter how small, there exists a number N for which

$$|a_n - l| < \epsilon \quad \forall \ \ n > N \ . \tag{3}$$

Note that, Eq.(2) (and Eq.(3)) is a **necessary** (but **not sufficient**) condition for convergence of the sequence. However, it is not always easy to find the limit and so it is helpful to view this from a different perspective.

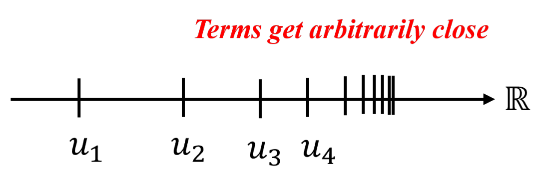

Figure 1: Sequence on $\mathbb{R}$ approaching a limit - terms cluster together around the value.

Consider Figure 1 which shows a sequence on the real number line and clearly we notice that the terms 'cluster' together as you go towards the right and so clearly the sequence approaches a limiting value. Without calculating the limit however, we can still say that for all $\epsilon > 0$, $\exists \ N \in \mathbb{N}$, $\forall \ n \geq m \geq N$ such that,

$$\boxed{|a_n - a_m| < \epsilon} \quad (\textbf{Cauchy Criterion} \text{ for convergence}) \ . \tag{4}$$

Any sequence with the property defined by Eq.(4) is known as a **Cauchy sequence** and by definition, for a sequence of real numbers, Cauchy sequence $\Leftrightarrow$ convergent sequence (*Completeness Axiom*).

1.2 Series

A *series* is a sequence of partial sums, i.e.

$$s_n = \sum_{k=1}^{n} a_k \ , \tag{5}$$

given that, $\{a_k\}_{k=1}^{\infty}$ is a sequence. The set of all these sums, $\{s_n\}_{n=1}^{\infty}$, itself forms a sequence. If this latter sequence has a limit S, i.e.,

$$s_n \to S \text{ as } i \to \infty \ , \tag{6}$$

then we say that the **infinite series**,

$$\lim_{n\to\infty} \sum_{k=1}^{i} a_k = \sum_{k=1}^{\infty} a_k \ , \tag{7}$$

possesses the limit S (or, converges to S), i.e.

$$\sum_{k=1}^{\infty} a_k = S \ . \tag{8}$$

By the Cauchy criterion, this will be true *if and only if*

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon \ , \tag{9}$$

for any fixed $\epsilon > 0$, whenever $n \geq m \geq N$. By using the definition of partial sum given in Eq.(5), one can show that an equivalent statement for Cauchy criterion is,

$$\boxed{\left| \sum_{k=m}^{n} a_k \right| < \epsilon \quad \text{is equivalent to} \quad |s_n - s_m| < \epsilon} \ , \tag{10}$$

for any fixed $\epsilon > 0$, whenever $n \geq m \geq N$. Obviously, a **necessary** (but **not sufficient**) condition for a series

$$\sum_{k=1}^{n} a_k$$

to converge is that the sequence must obey,

$$\lim_{k\to\infty} a_k \to 0 \ , \tag{11}$$

otherwise the series is said to be **divergent**.

Example 1. Determine if the infinite series

$$\sum_{k=1}^{\infty} (-1)^k$$

converges or diverges.

Cauchy criterion: $\left|\sum_{k=m}^{n}(-1)^k\right|$ and let $m = N$ and $n = N+2$ then there are exactly three terms and we can calculate each of these for two cases namely, when N is odd and when N is even. Thus,

$$\left|\sum_{k=N}^{N+2}(-1)^k\right| = \left\{ \begin{array}{ll} \text{even N:} & |1+(-1)+1| \\ \text{odd N:} & |-1+1+(-1)| \end{array} \right\} = 1$$

which $\Rightarrow \left|\sum_{k=m}^{n}(-1)^k\right| = 1$ which does not satisfy Cauchy criterion for all $\epsilon > 0$, if for example we set $\epsilon = 1$. Hence, the series is **not** convergent or alternatively we can say that the series diverges.

Example 2. A familiar series is the **geometric series**. Consider the series to be

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + ... + ar^{n-1} + ...$$

where a is a constant and $r \geq 0$. Determine if the series converges.

Let's look at the $nth-$partial sum s_n (sum of first n terms):

$$s_n = \sum_{m=0}^{n-1} ar^m = a + ar + ar^2 + ar^3 + ... + ar^{n-1}, \tag{12}$$

and multiply both sides by r such that,

$$\begin{aligned} &\Rightarrow rs_n = r(a + ar + ar^2 + ar^3 + ... + ar^{n-1}) \\ &\Rightarrow rs_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^n\ . \end{aligned} \tag{13}$$

Now, perform (12) - (13) which yields,

$$\begin{aligned} &s_n(1-r) = a - ar^n \\ &\Rightarrow \boxed{s_n = \frac{a(1-r^n)}{1-r}}\ , \end{aligned} \tag{14}$$

which represents the sum of first n terms of the geometric series. From observation, clearly the series diverges when $|r| > 1$. As such, we restrict our attention to $|r| < 1$ so that for large n, r^n approaches zero and s_n has the limiting value of,

$$\begin{aligned} &\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \\ &\text{clearly, } r^n \to 0 \\ &\Rightarrow \boxed{s_\infty = \frac{a}{1-r}}\ , \end{aligned} \tag{15}$$

where s_∞ is a finite limit which the series approaches as $n \to \infty$ for the case when $|r| < 1$. Thus, in such cases, geometric series is convergent.

Example 3. Verify that the infinite geometric series: $\dfrac{2}{3}+\dfrac{4}{9}+...+\left(\dfrac{2}{3}\right)^n$, has a sum (or, in other words, converges).

Identify: $a=\dfrac{2}{3}$, $r=\dfrac{4/9}{2/3}=\dfrac{2}{3}$.

Now, using Eq.(13), the sum of the first n terms of the series is,

$$s_n=\frac{\left(\frac{2}{3}\right)\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}=2\left(1-\left(\frac{2}{3}\right)^n\right)$$

which clearly shows that as $n\to\infty$ then $s_n\to 2$, meaning that the series sums to 2. We can confirm this result by directly using Eq.(14) for $n\to\infty$ as,

$$s_\infty=\frac{2/3}{1-(2/3)}=2$$

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2 Review of Calculating Limits

For the remainder of the class, let us brush up on the calculating limits.

Example 4. Find the limit as $n\to\infty$ of the sequence

$$\frac{(2n-1)^4+\sqrt{1+9n^8}}{1-n^3-7n^4}\ .$$

$$=\lim_{n\to\infty}\frac{(2n-1)^4+\sqrt{1+9n^8}}{1-n^3-7n^4}$$

$$=\lim_{n\to\infty}\frac{\frac{(2n-1)^4+\sqrt{1+9n^8}}{n^4}}{\frac{1-n^3-7n^4}{n^4}}=-\frac{19}{7}\ .$$

Example 5. Calculate, by applying L'Hôpital's rule, the limit,

$$\lim_{n\to\infty}\frac{\ln n}{n}$$

$$\Rightarrow\lim_{n\to\infty}\frac{1/n}{1}=0\ .$$

Example 6. Calculate the limit,

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}}\ .$$

Let: $y = \frac{1}{n}$. Then evaluate,

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \ln \left(\frac{1}{n}\right)^{\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty} \ln y = -\lim_{n\to\infty} \frac{1}{n} \ln n = 0 \ .$$

Then,

$$\lim_{n\to\infty} y = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = e^0 = 1 \ .$$