Classical Mechanics
Lecture #7

The Two-Body Central Force Problem:

The two body central force problem is an extremely important problem. For inverse square law forces, i.e.,

$$V(r) \sim \frac{1}{r}$$

This leads to the motion of a planet around the and the motion of two charged particles ignoring electromagnetic radiation.

For many potentials the two body problem is integrable (i.e., has a closed-form solution). Early in the 20th century Henri Poincaré tried to solve the three body problem. It was later realized that there is no solution to the general three body problem. It is also a problem that is chaotic meaning that a small change in the initial condition leads to a rapid breakdown in predictability:

$$\delta \vec{x}(t) \sim \delta \vec{x}(0)\, e^{\lambda_L t}$$

λ_L is known as the Lyapunov exponent and is one of the characteristics of classical chaos.

We shall study two aspects of the two-body problem:

1) Its bound orbits ($E \leq 0$).

2) Scattering of particles from a central potential.

<u>The Lagrangian:</u>

$$L = \frac{1}{2} m_1 \dot{\bar{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\bar{r}}_2^{\,2} - V(|\bar{r}_1 - \bar{r}_2|) \quad (1)$$

This system has six degrees of freedom: $\bar{r}_1$ & $\bar{r}_2$ which are the coordinates of the two particles.

However, if we introduce the centre-of mass coordinates:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

and $\vec{r} = \vec{r}_1 - \vec{r}_2$

Then L can be written as: (Show)

$$L = \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \qquad (2)$$

where $M = m_1+m_2 \rightarrow$ total mass

and $\mu = \frac{m_1 m_2}{m_1+m_2} \rightarrow$ reduced mass

In the form of L given in equation 2 we see that the centre of mass $\vec{R}$ is an ignorable coordinate and so the velocity of $\vec{R}$ is constant:

$$M\dot{\vec{R}} = \text{Constant}$$

Thus the relative dynamics is captured by the remaining 3 degrees of freedom $\vec{r}$ which describes the motion of a 'fictitious' mass $\mu = \frac{m_1 m_2}{m_1+m_2}$. When one of the masses is much greater than the other, say $m_1 \gg m_2$, $\mu \approx m_2$.

Thus for the sun-earth system μ accurately describes the mass of the earth.
$(M_\odot \approx 333{,}000\ M_E)$.

However the Lagrangian ② is rotationally invariant and so the angular momentum

$$\vec{l} = \vec{r} \times \vec{p}$$

is conserved. Since $\vec{l}$ and $\vec{p}$ are orthogonal to each other the motion of the body takes place in the plane orthogonal. Since $\vec{L}$ is conserved the plane in which the motion takes place is does not change.

Thus we have reduced the problem to

a problem in 2-D. Setting $r_z =$ Constant we then get:

$$L = \frac{1}{2}\mu(\dot{r}_x^2 + \dot{r}_y^2) - V(r)$$

Introducing polar coordinates

$$r_x = r\cos\theta$$
$$r_y = r\sin\theta$$

$$\dot{r}_x = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$\dot{r}_y = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\boxed{L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)}$$

Here we see that θ is an ignorable coordinate and so

$$\boxed{\ell = \mu r^2\dot{\theta}}$$

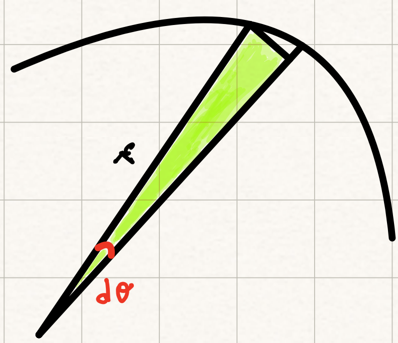
is a conserved quantity.

In fact l is nothing but the magnitude of the angular momentum $\vec{l}$.

When we fixed the plane of motion to be orthogonal to $\vec{l}$ we used two of its components.

The constancy of l immediately leads to Kepler's second law:

The areal velocity of the particle is constant:



$dA = \frac{1}{2} r^2 d\theta$ ~ the area of the triangle which subtends a small angle $d\theta$ at the origin.

Then $$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{m}$$

The radial equation is:

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

$$\Rightarrow \quad \boxed{m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r} = f(r)}$$

where $f(r) = -\frac{\partial V}{\partial r}$ is the force towards the centre.

<u>Solving the equations of motion:</u>

We can rewrite the equations of motion:

$$m r^2 \dot{\theta} = \ell$$

and

$$m\ddot{r} - \frac{m^2 r^4 \dot{\theta}^2}{m r^3} = -\frac{dV}{dr}$$

$$m\ddot{r} - \frac{\ell^2}{m r^3} = -\frac{dV}{dr}$$

We can write the latter as:

$$m\ddot{r} = -\frac{d}{dr}\left(\frac{1}{2}\frac{\ell^2}{mr^2} + V(r)\right)$$

But using $\dfrac{dg(r)}{dt} = \dfrac{dg(r)}{dr}\dfrac{dr}{dt} = \dot{r}\dfrac{dg}{dr}$

we get:

$$m\ddot{r}\,\dot{r} = \frac{d}{dt}\left(\frac{1}{2}\frac{\ell^2}{mr^2} + V(r)\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left(\frac{1}{2}\frac{\ell^2}{mr^2} + V(r)\right)$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)\right\} = 0$$

Integrating which we get:

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\dot{r}^2\dot{\theta}^2 + V(r)}$$

as a constant of motion. Since our Lagrangian is time-translation invariant we could have used Noether's theorem to write down E.

This is a 1st order differential equation. If we eliminate $\dot{\theta}^2$ in favour of the constant ℓ we get:

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}$$

$$dt = \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr'^2}\right)}} \quad \text{③}$$

This is the formal solution in terms of a quadrature. Whether one can actually do the integral will depend on $V(r')$!

Once the integral has been done and we have

$$t = t(r)$$

one can invert it and get $r = r(t)$.

Then solution to the θ equation follows:

$$\theta = l \int_0^t \frac{dt'}{mr^2(t)} + \theta_0 \qquad \text{— (4)}$$

Equivalent One Dimensional Problem:

Equations 3 & 4 give formal solutions to the problem at hand but it's hard to extract any useful physical information from them.

To obtain physical insight we use the radial equal of motion

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{dV}{dr}$$

and rewrite it as:

$$m\ddot{r}^2 = -\frac{\partial}{\partial r} V_{eff}(r)$$

with $V_{eff} = V + \dfrac{l^2}{2mr^2}$

Thus we have reduced the radial dynamics of the system to a one dimensional system whose potential is given by the effective potential

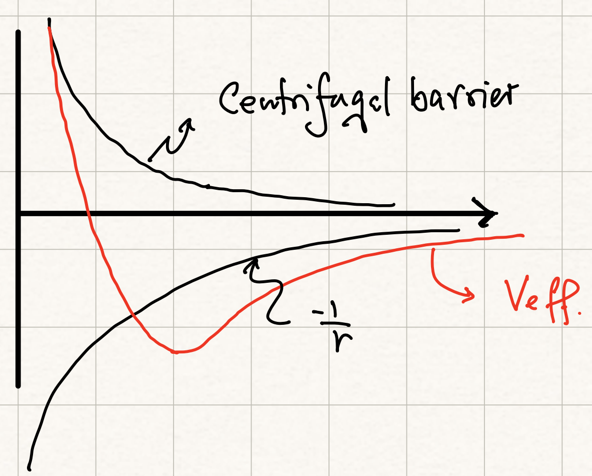
$$V_{eff}(r) = V(r) + \frac{l}{2mr^2}$$

The second term is an angular momentum dependent term known as the centrifugal barrier or the angular momentum barrier.

Example of $V_{eff}(r)$:

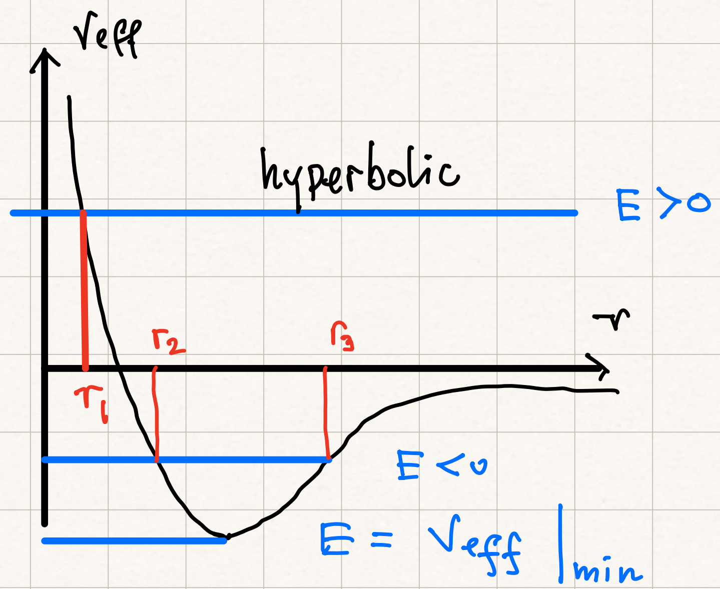
Suppose $V(r) = -\frac{k}{r}$

Then V_{eff} looks like:



Given $V_{eff}(r)$ for the inverse square law we can classify the orbits according to the total energy

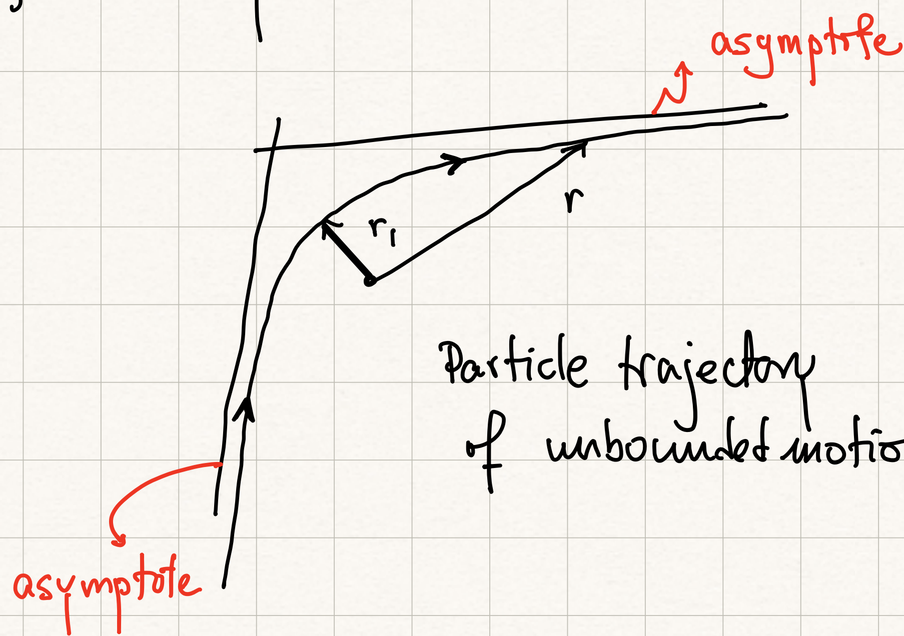
$$E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$$



When $E > 0$, we see that our effective particle comes in from $r = \infty$ and reaches $r = r_1$ at which $E = V_{eff}$. The particle cannot get any closer since otherwise the potential energy would be greater than E and the kinetic energy would be negative.

r_1 is the turning point. From a 2-D perspective, Since we know that the angular momentum $l = mr^2\dot{\theta}$ must be constant shows that the angle will rapidly change as the particle nears the centre of the potential.

Qualitatively the orbit will look like



Particle trajectory of unbounded motion.

As we lower E and E = 0 the qualitative feature of the trajectory will remain the same. But for $E < 0$ but $E > V_{eff}\big|_{min}$ the trajectory will have two 'turning points' r_2 & r_3.

While it is clear that for this case the orbit will be bounded (i.e. it cannot escape to infinity) it is not obvious that the orbits will form closed curves. The fact that for inverse square law (and Hooke's law) the orbits form closed curves is known as Bertrand's Theorem.

Explicit Solutions:

For the inverse square law we can find form of the orbit, i.e., how r is related to θ.

From

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

and

$$dt = \frac{mr^2}{\ell}\, d\theta$$

we can eliminate dt and obtain:

$$d\theta = \frac{\ell}{m r^2} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V(r) - \frac{\ell^2}{2mr^2}\right)}}$$

$$\theta = \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2mE}{\ell^2} - \frac{2mV(r)}{\ell^2} - \frac{1}{r^2}}} + \theta_0$$

We can set $V = -\frac{K}{r}$ and change the integration variable to

$$u = \frac{1}{r}$$

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mK}{\ell^2}u - u^2}}$$

The indefinite integral is of the form:

$$I = \int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}}$$

Note that in our case $\gamma = -1$ and so we want to solve

$$-\left(x^2 + \frac{\beta}{\gamma}x + \frac{\alpha}{\gamma}\right) = -(x - x_+)(x - x_-)$$

where $$x_\pm = \frac{-\frac{\beta}{\gamma} \pm \sqrt{\frac{\beta^2}{\gamma^2} - 4\frac{\alpha}{\gamma}}}{2}$$

$$= \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\gamma}$$

Then $$I = \frac{1}{\sqrt{-\gamma}} \int \frac{dx}{\sqrt{-(x-x_+)(x-x_-)}}$$

let $y = -(x - x_+)$ then $x = -y + x_+$

$$I = -\frac{1}{\sqrt{-\gamma}} \int \frac{dy}{\sqrt{y(-y + x_+ - x_-)}}$$

$$= -\frac{1}{\sqrt{-\gamma}} \frac{1}{\sqrt{x_+ - x_-}} \int \frac{dy}{\sqrt{y\left(1 - \frac{y}{x_+ - x_-}\right)}}$$

$$z = \frac{y}{x_+ - x_-}$$

$$I = -\frac{1}{\sqrt{-\gamma}} \frac{(x_+ - x_-)}{\left(\sqrt{x_+ - x_-}\right)^2} \int \frac{dz}{\sqrt{z\,(1-z)}}$$

$$= \frac{-1}{\sqrt{-\gamma}} \int \frac{dz}{\sqrt{z(1-z)}}$$

$$z = \cos^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$dz = -2\sin\theta\cos\theta\, d\theta \qquad = 2z - 1$$

$$I = +\frac{1}{\sqrt{-\gamma}} \int \frac{2\,\cancel{\sin\theta}\,\cancel{\cos\theta}\quad d\theta}{\cancel{\cos\theta}\;\cancel{\sin\theta}}$$

$$= \frac{2\theta}{\sqrt{-\gamma}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}(2z-1)$$

$$= \frac{1}{\sqrt{-\gamma}} \cos^{-1}\left(\frac{2y}{x_+ - x_-} - 1\right)$$

$$= \frac{1}{\sqrt{-\gamma}} \cos^{-1} \frac{-2x+2x_+ - x_+ + x_-}{x_+ - x_-}$$

$$= \frac{1}{\sqrt{-\gamma}} \cos^{-1} \frac{x_+ + x_- - 2x}{x_+ - x_-}$$

Now $x_+ - x_- = \sqrt{\frac{\beta^2 - 4\alpha\gamma}{\gamma}} = \frac{\sqrt{q}}{\gamma}$

So $I = \frac{1}{\sqrt{-\gamma}} \cos^{-1} \frac{\gamma(-\beta/\gamma - 2x)}{\sqrt{q}}$

$$I = \frac{1}{\sqrt{-\gamma}} \cos^{-1} - \left(\frac{\beta + 2\gamma x}{\sqrt{q}} \right)$$

We now put

$$\alpha = \frac{2mE}{l^2}, \beta = \frac{2mk}{l^2}, \gamma = -1.$$

And obtain:

$$\theta = \theta' - \cos^{-1} \frac{\frac{\ell^2 u}{mK} - 1}{\sqrt{1 + \frac{2E\ell^2}{mK^2}}}$$

$$\Rightarrow \boxed{\frac{1}{r} = \frac{mK}{\ell^2}\left(1 + \sqrt{1 + \frac{2E\ell^2}{mK^2}} \cos(\theta - \theta')\right)}$$

ℓ, E, θ' are integration constant. The fourth constant will determine the initial position of the mass m.

The solution above can be compared to

$$\frac{1}{r} = c\left[1 + e\cos(\theta - \theta')\right]$$

which is the equation of a conic with one of the foci at the centre and θ' as as a turning angle.

We can idenify the eccentricity

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$e > 1$, $\quad E > 0 \quad$ hyperbola

$e = 1$, $\quad E = 0 \quad$ parabola

$e < 1$, $\quad E < 0 \quad$ ellipse

$e = 0 \quad E = -\frac{mk^2}{2l^2} \quad$ circle.

For the elliptical orbit we can find the radii at the turning points by noticing that at those points the radial velocity is zero and so if we set $\dot{r} = 0$ in

$$E = \frac{1}{2} m \cancel{\dot{r}}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$$

the solution to the resulting quadratic equations give the turning points.

The equation is

$$r^2 + \frac{k}{E} r - \frac{l^2}{2mE} = 0$$

The semi-major axis is

$$a = \frac{r_1 + r_2}{2} = -\frac{k}{2E}$$

And the eccentricity in terms of a:

$$e = \sqrt{1 - \frac{l^2}{mka}}$$

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta'}$$