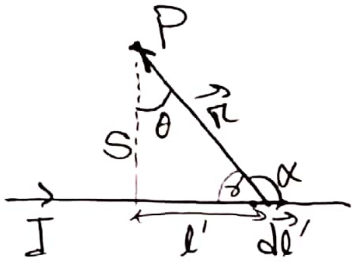
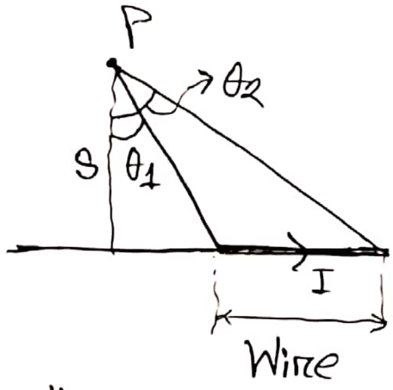
Lecture 14

Magnetic field produced by a long straight wire carrying current I

We want to calculate the magnetic field at a distance s from a current carrying wire.





In terms of direction, $d\vec{l}' \times \hat{r}$ will point out of the page ($\odot$). We can find the magnitude by converting every ~~coordinate~~ variables in terms of the angle θ. As shown in the second picture, the boundaries of the wire can just be characterized by θ_1 and θ_2 given a point P where we want to calculate the magnetic field.

Now, $$|d\vec{l}' \times \hat{r}| = dl' \sin\alpha = dl' \sin(180^\circ - \gamma)$$
$$= dl' \sin(180^\circ - (90^\circ - \theta))$$
$$= dl' \sin(90^\circ + \theta) = dl' \cos\theta$$

Now, $$l' = s\tan\theta \Rightarrow \frac{dl'}{d\theta} = s\sec^2\theta$$
$$\therefore dl' = \frac{s}{\cos^2\theta}\, d\theta$$

Also, $$s = r\cos\theta \Rightarrow \frac{1}{r^2} = \frac{\cos^2\theta}{s^2}$$

$$\therefore \; B = \frac{\mu_0 I}{4\pi} \int_{\theta=\theta_1}^{\theta=\theta_2} \overset{\frac{s}{\cos^2\theta}d\theta}{\cancel{dl'}} \cos\theta \times \frac{\cos^2\theta}{s^2}$$

$$= \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{s}{\cos^2\theta} \times \frac{\cos^3\theta}{s^2}\, d\theta = \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta$$

$$\boxed{\therefore \; B = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1)}$$

The equation above gives the magnetic field for a finite wire with steady current flowing. But how can finite wire has steady current (current must go somewhere). Perhaps, it can give us the contribution to the magnetic field of a section of wire of some closed circuit.

For an infinite wire, $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. So, we get,

$$B = \frac{\mu_0 I}{4\pi s} (1-(-1)) = \frac{\mu_0 I}{2\pi s}$$

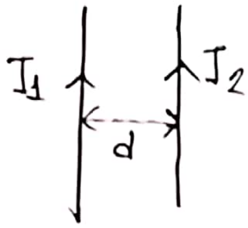
One interesting fact is that, the magnetic field also falls off as: $\frac{1}{s}$, just like the electric field due to an infinite ~~wero~~ wire.

In general, the magnetic field circles around the wire, in accordance with the right hand rule.

$$\boxed{\therefore \; \vec{B} = \frac{\mu_0 I}{2\pi s} \, \hat{\phi}}$$

We can calculate the force between two parallel, long current carrying wires at a distance d, by using the result obtained in the last page.

The magnetic field at (2) due to (1) is,



$$B = \frac{\mu_0 I_1}{2\pi d}, \text{ into the page.}$$

The Lorentz force on the wire will then be given by,

$$\vec{F} = I_2 \int d\vec{l} \times \vec{B} = I_2 \int dl \, \frac{\mu_0 I_1}{2\pi d} \, \hat{j} \times (-\hat{k})$$

$$= \frac{\mu_0 I_1 I_2}{2\pi d} \int dl \, (-\hat{i})$$

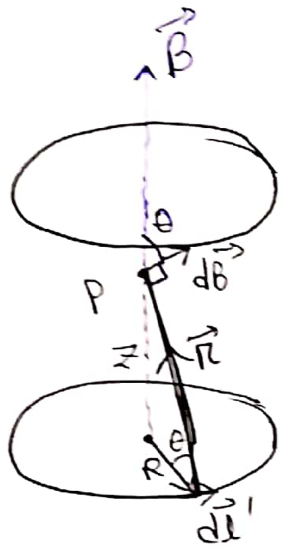
The force is obviously infinite if the wires are infinite. However, the force per unit length is,

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

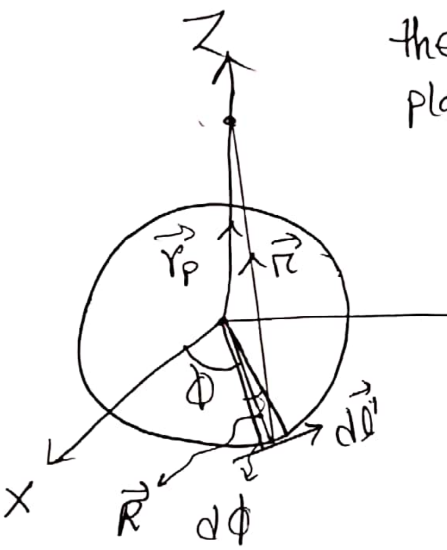
and the force is attractive. You can also check that wire (1) also experiences the same force. If the currents were in opposite directions, then the wires would have repelled each other.

Magnetic field a distance z above from the center of a circular loop of radius R carrying a steady current I

Consider the infinitesimal magnetic fields created by a length element $d\vec{l}'$. As $d\vec{l}'$ is integrated over the circular loop, $d\vec{B}$ also sweeps out a cone.



Loop is in the XY plane



The horizontal components will be cancelled out due to symmetry. So, magnetic field will point in $\hat{k}$ direction

$$\therefore B = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{r^2} \cos\theta$$

while $\cos\theta$ comes out as the ~~ra~~ component along the vertical direction. Now, r and $\cos\theta$ both are constant.

$$\therefore B = \frac{\mu_0 I}{4\pi} \frac{\cos\theta}{r^2} \int dl'$$

$$= \frac{\mu_0 I}{4\pi} \cdot \frac{R}{r \cdot r^2} \times 2\pi R$$

$$= \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{3/2}}$$

Now, $\vec{r} = \vec{r}_P - \vec{R}$

$$= z\hat{k} - [R\cos\phi\, \hat{i} + R\sin\phi\, \hat{j}]$$

$$d\vec{l}' = R\, d\phi\, \hat{\phi}$$

$$= R\, d\phi\, [-\sin\phi\, \hat{i} + \cos\phi\, \hat{j}]$$

Now, $$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \hat{r}}{r^2}$$

$\cancel{= \frac{\mu_0 I}{4\pi}}$ Now, $d\vec{l}' \times \vec{r}$

$$= R\, d\phi \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi & \cos\phi & 0 \\ -R\cos\phi & -R\sin\phi & z \end{vmatrix}$$

$$= R\, d\phi\, [z\cos\phi\, \hat{i} + \hat{j}\, z\sin\phi + R\, \hat{k}]$$

$$r = \sqrt{R^2 + z^2}$$

$$\therefore B = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$$

$$\therefore \vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{R\,d\phi\left[z\cos\phi\,\hat{i} + z\sin\phi\,\hat{j} + R\hat{k}\right]}{(R^2+z^2)^{3/2}}$$

$$\therefore B_x = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} zR\cos\phi\, d\phi = 0$$

$$B_y = 0$$

$$B_z = \frac{\mu_0 I}{4\pi} \frac{R^2}{(R^2+z^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$\boxed{\therefore B_z = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}}$$

If we want to calculate at the center of the circle, then,

$z=0$ $\quad \boxed{\therefore B_z = \frac{\mu_0 I}{2R}}$

The divergence and curl of $\vec{B}$

For straight line currents

For an infinite straight wire, $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$

Now, the line integral of $\vec{B}$ along ~~around~~ a circular path around the wire is given by,

$$\oint \vec{B}\cdot d\vec{s} = \oint \frac{\mu_0 I}{2\pi s}\hat{\phi}\cdot s\,d\phi\,\hat{\phi} = \frac{\mu_0 I}{2\pi}\oint d\phi$$

$$= \frac{\mu_0 I}{2\pi} \times 2\pi$$

$$\therefore \oint \vec{B}\cdot d\vec{s} = \mu_0 I$$

Now, it doesn't have to be a circle around the wire

Any closed loop, will give the same result.

Say, the current is flowing along the z-axis

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi} .$$

In cyllindrical coordinate, $d\vec{s} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$

$$\therefore \oint \vec{B} \cdot d\vec{s} = \int \frac{\mu_0 I}{2\pi s}\, s\, d\phi = \mu_0 I$$

But, if the loop doesn't enclose the wire, then the $\oint d\phi = 0$, since you are just coming back to the same angle.

Now, if we had a bundle of wires carrying different/same currents, enclosed by our loop, then, since magnetic field also obeys superposition,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

We know, $I_{enc} = \iint \vec{J} \cdot d\vec{A}$

where the integral is taken over any surface bounded by the loop.

$$\therefore \oint_{\partial S} \vec{B} \cdot d\vec{s} = \mu_0 \iint_{S} \vec{J} \cdot d\vec{A}$$

$$\Rightarrow \iint_S (\vec{\nabla}\times\vec{B})\cdot d\vec{A} = \mu_0 \iint_S \vec{J}\cdot d\vec{A} \quad \text{(via Stoke's theorem)}$$

Since this is true for the same surface on both sides where the integration is running over,

$$\boxed{\therefore\ \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}} \qquad \boxed{\oint \vec{B}\cdot d\vec{s} = \mu_0 J_{enc}}$$

So, we have found curl of $\vec{B}$ for infinite straight wire, and this gives Ampere's law.

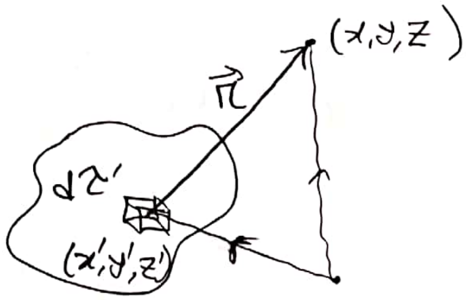
Divergence and curl of $\vec{B}$ for any current

For a volume current, Biot - Savart law is given by.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

where we are calculating the magnetic field at a point $\vec{r} = (x,y,z)$ in terms of integral over the current distribution $\vec{J}(x',y',z')$. So, the integration is over the primed coordinate and we are calculating magnetic field in the unprimed coordinate.

$$\therefore \vec{\mathscr{r}} = (x-x')\,\hat{i} + (y-y')\,\hat{j} + (z-z')\,\hat{k}$$



$$d\tau' = dx'\,dy'\,dz'$$

$$\vec{B} = \vec{B}(x,y,z)$$

$$\vec{J} = \vec{J}(x',y',z')$$

Now, $\vec{\nabla}.\vec{B} = \vec{\nabla}. \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$

$$= \frac{\mu_0}{4\pi}\int \vec{\nabla}.\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

Now, $\vec{\nabla}.(\vec{P}\times\vec{Q}) = \vec{Q}.(\vec{\nabla}\times\vec{P}) - \vec{P}.(\vec{\nabla}\times\vec{Q})$

$\cancel{\therefore \vec{\nabla}.\vec{B} = \frac{\mu_0}{4\pi}\int\left[\vec{J}.\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) + \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\vec{\nabla}\times\vec{J})\right] d\tau'}$

$\therefore \vec{\nabla}.\vec{B} = \frac{\mu_0}{4\pi}\int\left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.(\vec{\nabla}\times\vec{J}) - \vec{J}.\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right] d\tau$

Now, $\vec{\nabla}\times\vec{J} = \vec{0}$, since the curl is w.r.t. unprimed coordinate and $\vec{J}$ doesn't depend on it. Also,

$$\vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{0}$$

as you can check.

$$\boxed{\therefore \vec{\nabla}.\vec{B} = 0}$$

$(\vec{P}\times\vec{Q})_i = \epsilon_{ijk} P_j Q_k$

$\vec{\nabla}\times\vec{P} = \epsilon_{ijk}\frac{\partial P_k}{\partial x_j}$

$\therefore \vec{\nabla}(\vec{P}\times\vec{Q}) = \frac{\partial}{\partial x_i}\left[\epsilon_{ijk} P_j Q_k\right]$

$= \epsilon_{ijk}\frac{\partial P_j}{\partial x_i} Q_k + \epsilon_{ijk}\frac{\partial Q_k}{\partial x_i} P_j$

$\cancel{= Q_k\left(\epsilon_{kij}\frac{\partial P_j}{\partial x_i}\right) - P_j\epsilon_{jik}\frac{\partial Q_k}{\partial x_i}}$

$= Q_k\left(\epsilon_{kij}\frac{\partial P_j}{\partial x_i}\right) - P_j\epsilon_{jik}\frac{\partial Q_k}{\partial x_i}$

$= \vec{Q}.(\vec{\nabla}\times\vec{P}) - \vec{P}.(\vec{\nabla}\times\vec{Q})$

For curl of $\vec{B}$, $\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau$

~~Now,~~ $\cancel{\vec{\nabla}\times(\vec{P}\times\vec{Q}) = \vec{P}(\vec{\nabla}.\vec{Q}) - \vec{Q}}\ (\vec{Q}.\vec{\nabla})\vec{P}$

Now, $\vec{\nabla} \times (\vec{P} \times \vec{Q}) = \vec{P}(\vec{\nabla} \cdot \vec{Q}) - (\vec{P} \cdot \vec{\nabla})\vec{Q} +$
$- \vec{Q}(\vec{\nabla} \cdot \vec{P}) + (\vec{Q} \cdot \vec{\nabla})\vec{P}$

$\therefore \vec{\nabla} \times \left(\vec{J} \times \frac{\hat{r}}{r^2}\right) = \vec{J}\left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right) - (\vec{J} \cdot \vec{\nabla})\frac{\hat{r}}{r^2} - \frac{\hat{r}}{r^2} \cdot (\vec{\nabla} \cdot \vec{J})$
$+ \left(\frac{\hat{r}}{r^2} \cdot \vec{\nabla}\right) \vec{J}$

Now, since $\vec{J}$ doesn't depend on the unprimed coordinates, so, we can drop the last two terms, involving the derivatives of $\vec{J}$ with respect to the unprimed coordinate.

$\therefore \vec{\nabla} \times \left(\vec{J} \times \frac{\hat{r}}{r^2}\right) = \vec{J}\left(\vec{\nabla} \cdot \frac{\hat{r}}{r^2}\right) - (\vec{J} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2}$

Now, $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi \delta^3(\vec{r})$, as calculated previously.

Also, $(\vec{J} \cdot \vec{\nabla})\frac{\hat{r}}{r^2} = -(\vec{J} \cdot \vec{\nabla}')\frac{\hat{r}}{r^2}$

Because, $\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

We know, $\frac{\partial}{\partial x} f(x-x') = -\frac{\partial}{\partial x'} f(x-x')$

~~Use just the chain rule:~~ $\cancel{\frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} [f(x-x')]}$

$$\left\{ \begin{aligned} & \frac{\partial(x-x')}{\partial x} \cdot \frac{\partial}{\partial(x-x')} f(x-x') \\ & \equiv -\frac{\partial(x-x')}{\partial x'} \cdot \frac{\partial}{\partial(x-x')} f(x-x') \end{aligned} \right.$$

loop

The x component is then,

$$(\vec{J}\cdot\vec{\nabla}')\,\frac{x-x'}{r^3} = \vec{\nabla}'\cdot\left[\frac{x-x'}{r^3}\,\vec{J}\right] - \left(\frac{x-x'}{r^3}\right)(\vec{\nabla}'\cdot\vec{J})$$

For steady current, $\vec{\nabla}'\cdot\vec{J} = 0$

$$\therefore (\vec{J}\cdot\vec{\nabla}')\,\frac{x-x'}{r^3} = \vec{\nabla}'\cdot\left[\frac{x-x'}{r^3}\,\vec{J}\right]$$

So, in contribution to the integral is,

$$\iiint_V \vec{\nabla}'\cdot\left(\frac{x-x'}{r^3}\vec{J}\right)d\tau' = \oint_S \frac{x-x'}{r^3}\,\vec{J}\cdot d\vec{A}'$$

Now, the integration was over a volume where the current is enclosed. On the boundary surface, the current is zero (all the currents are inside), so the surface integral vanishes.

$$\therefore \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,4\pi\delta^3(\vec{r}-\vec{r}')\,d^3r'$$

$$\boxed{\therefore \vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})}$$

This is obviously the Ampere's law in differential form. We can find the integral form by the following:

$$\iint(\vec{\nabla}\times\vec{B})\cdot d\vec{A} = \oint \vec{B}\cdot d\vec{s} = \mu_0\iint \vec{J}\cdot d\vec{A}$$

$$\boxed{\therefore \oint \vec{B}\cdot d\vec{A} = \mu_0 I_{enc}}$$

where I_{enc} is the current enclosed by an Amperian loop that encloses the current and is the boundary of the surface where current is passing.

The fact that divergence of $\vec{B}$ is zero, ensures that there is no magnetic monopole. If there was a point like magnetic charge g, it would have give rise to a magnetic field like,

$$\vec{B} = \frac{1}{4\pi} \cdot \frac{g\,\hat{r}}{r^2}$$

But $\vec{\nabla}\cdot\vec{B} = 0$ ensures that there is no magnetic monopole. And we haven't yet found any. But that doesn't mean magnetic monopoles doesn't exist. Actually, magnetic monopoles are ubiquitous in theories that goes beyond Standard model. May be detecting magnetic monopoles are not compatible with our current or near future technology, or they are very rare, but may be someday we will observe it and rewrite $\vec{\nabla}\cdot\vec{B} = \rho_m$.

The vector potential

In electrostatics, we saw that $\vec{\nabla}\times\vec{E} = \vec{0}$, and consequently we defined a scalar potential V such that $\vec{E} = -\vec{\nabla}V$. It was always possible to write since curl of a ~~divergence~~ gradient is always zero. Now, in magnetostatics we don't have the zero curl of $\vec{B}$, but the divergence of $\vec{B}$ is zero. We can then define a vector potential $\vec{A}$ given by,

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

since, the ~~gra~~ divergence of a curl is zero.

$$\therefore \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0, \text{ always.}$$

With this introduction, we can rewrite Ampere's law as,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

$$\rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad \text{—— (1)}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\nabla^2 \vec{A} = \nabla^2 A_x \hat{i} + \nabla^2 A_y \hat{j} + \nabla^2 A_z \hat{k}$$

We can solve for $\vec{A}$ from the equation above and hence calculate $\vec{B}$.

Gauge transformation

Now, in the case of electric potential, the function V was not unique. Remember, we could add any constant with V without altering the physical significance of $\vec{E}$. In the case of vector potential, this is also true. There are many vector potentials giving rise to the same magnetic field $\vec{B}$. The reason is, curl of a gradient is always zero. So, we can add gradient of any scalar function λ with $\vec{A}$, but $\vec{B}$ will remain the same.

$$\therefore \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \overset{0}{\vec{\nabla} \times (\vec{\nabla}\lambda)}$$

$$\therefore \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A}$$

Such a transformation of $\vec{A}$ is called a gauge transformation. Such gauge transformation plays a key role in theoretical physics. For our purposes now, we can use this freedom to make our problem simpler. We can always make a gauge transformation λ such that the vector potential $\vec{A}'$ satisfies,

$$\vec{\nabla} . \vec{A}' = 0$$

This choice is called the Coulomb gauge.

Suppose, we have some $\vec{A}$ that gives us the required magnetic field, and so, $\vec{B} = \vec{\nabla} \times \vec{A}$. But, the divergence of $\vec{A}$ is not zero, rather,

$$\vec{\nabla} . \vec{A} = \Psi(\vec{r})$$

We now make the transformation, $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$, which has a divergence –

$$\vec{\nabla} . \vec{A}' = \vec{\nabla} . \vec{A} + \nabla^2 \lambda = \Psi(\vec{r}) + \nabla^2 \lambda$$

If we now want $\vec{\nabla} . \vec{A}' = 0$, then we just need to find the appropriate gauge transformation λ such that,

$$\nabla^2 \lambda = - \Psi(\vec{r})$$

But this equation is nothing but Poisson's equation with a typical solution,

$$\lambda = \frac{1}{4\pi} \int \frac{\Psi(\vec{r}')}{r} d\tau'$$

if $\Psi(\vec{r}')$ goes to zero at infinity.

Compare with, $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

$$\therefore V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}') d\tau'}{r}$$

if $\rho(\vec{r}')$ goes to zero at infinity

With this condition, equation (1) can be written as

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \implies \left.\begin{array}{l} \nabla^2 A_x = -\mu_0 J_x \\ \nabla^2 A_y = -\mu_0 J_y \\ \nabla^2 A_z = -\mu_0 J_z \end{array}\right\} \text{In Cartesian coordinate system.}$$

There is only three Poisson's equations to solve to find $\vec{A}$. But we already can read the solution as—

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\mathscr{r}}\, d\tau' \quad \Big|\ \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

We can use this equation to calculate $\vec{A}$ and hece $\vec{B}$ by taking the curl of $\vec{A}$.

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{\mathscr{r}}\, d\tau'$$

$$= \frac{\mu_0}{4\pi}\int \vec{\nabla}\times \frac{\vec{J}(\vec{r}')}{\mathscr{r}}\, d\tau'$$

Now, $\vec{\nabla}\times(P\vec{Q}) = P(\vec{\nabla}\times\vec{Q}) + \vec{\nabla}P\times\vec{Q}$

$$\therefore\ \vec{\nabla}\times\frac{\vec{J}(\vec{r}')}{\mathscr{r}} = \frac{1}{\mathscr{r}}\left[\vec{\nabla}\times\vec{J}(\vec{r}')\right] + \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\times\vec{J}(\vec{r}')$$

Since the $\vec{\nabla}$ is on the unprimed coordinate, so $\vec{\nabla}\times\vec{J}(\vec{r}') = 0$

$$\therefore\ \vec{\nabla}\times\frac{\vec{J}(\vec{r}')}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\times\vec{J}(\vec{r}') = \vec{J}(\vec{r}')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\boxed{\therefore\ \vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'} \rightarrow \text{This is exactly the Biot-Savart law.}$$

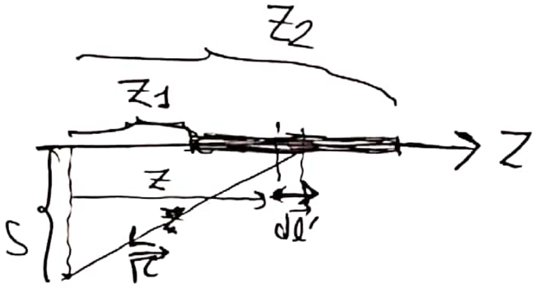
For line and surface currents, we can define the vector potential as,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\,dl'}{r} = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{r}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\,dA'}{r}$$

Calculation of vector potential

For a straight finite wire carrying current I



$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{dl'}{r}$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{dz\,\hat{z}}{\sqrt{s^2+z^2}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2} \frac{dz}{\sqrt{s^2+z^2}}$$

$\cancel{s^2+z^2=r^2}$

$z = s\tan\theta$

$\Rightarrow dz = s\sec^2\theta\, d\theta$

$\theta_1 = \tan^{-1}\frac{z_1}{s}$

$\theta_2 = \tan^{-1}\frac{z_2}{s}$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\int_{\tan^{-1}\frac{z_1}{s}}^{\tan^{-1}\frac{z_2}{s}} \frac{s\sec^2\theta\, d\theta}{s\sec\theta}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\int \sec\theta\, d\theta = \frac{\mu_0 I}{4\pi}\hat{z}\,\ln|\tan\theta+\sec\theta|\Big|_{\tan^{-1}\frac{z_1}{s}}^{\tan^{-1}\frac{z_2}{s}}$$

$$\cancel{= \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left(\frac{z}{s}+\right.\right.}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left(\frac{z_2}{s}+\frac{\sqrt{s^2+z_2^2}}{s}\right)-\ln\left(\frac{z_1}{s}+\frac{\sqrt{s^2+z_1^2}}{s}\right)\right]$$

$$\boxed{\therefore \vec{A} = \frac{\mu_0 I}{4\pi} \ln \frac{z_2+\sqrt{z_2^2+s^2}}{z_1+\sqrt{z_1^2+s^2}}\ \hat{z}}$$

Vector potential for infinite current carrying wire

If the current carrying wire is infinite, we can not use our general formula for finding vector potential, since the current doesn't go to zero at infinity. But, we can always use some trick to solve the problem.

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \qquad \text{and} \qquad \vec{B} = \vec{\nabla}\times\vec{A}$$

In cylindrical coordinate, $\vec{\nabla}\times\vec{A} = \frac{1}{s}\begin{vmatrix} \hat{s} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_s & sA_\phi & A_z \end{vmatrix}$

$$\therefore (\vec{\nabla}\times\vec{A})_{\hat{\phi}} = \hat{\phi}\left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)$$

Now, our wire was shooting to infinity in the z direction. The magnetic field didn't have any dependency on z, so should $\vec{A}$.

$$\therefore \frac{\partial A_s}{\partial z} = 0$$

$$\therefore (\vec{\nabla}\times\vec{A})_\phi = -\frac{\partial A_z}{\partial s}\hat{\phi} \Rightarrow \vec{B} = -\frac{\partial A_z}{\partial s}\hat{\phi}$$

$$\Rightarrow \frac{\mu_0 I}{2\pi s} = -\frac{\partial A_z}{\partial s} \qquad \therefore A_z = \frac{\mu_0 I}{2\cancel{s}\pi}\ln s$$

$$\boxed{\therefore \vec{A} = -\frac{\mu_0 I}{2\pi}\ln s\ \hat{z}}$$

You can now check whether $\vec{\nabla}\times\vec{A} = \vec{B}$ and $\vec{\nabla}\cdot\vec{A} = 0$ or not.