Group Theory
Lecture #3

SU(2) & SO(3)

SU(2):

The set of 2x2 complex-valued unitary matrices with unit determinants form a group under matrix multiplication. This group is known as SU(2).

Let $U \in SU(2)$. Let us represent U as:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$$

$$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

On the other hand

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Unitarity implies

$$U^\dagger = U^{-1}$$

or $$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \frac{1}{\operatorname{Det} U}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \quad a^* = \frac{1}{\operatorname{Det} U} d$$

and $$b^* = \frac{-1}{\operatorname{Det} U} \cdot c$$

These two equations cut down 8 real parameters of a 2×2 complex matrix to 4 real parameters.

Let us put $a = x+iy$ and $b = v+iw$ with $x, y, v, w \in \mathbb{R}$. Then the $\operatorname{Det} U = 1$ condition gives us another condition:

$$|a|^2 + |b|^2 = 1$$

or $$\boxed{x^2 + y^2 + v^2 + w^2 = 1}$$

This is the equation of an S^3. Thus we see that $SU(2) \cong S^3$.

Also note that $\operatorname{Dim} SU(2) = 3$.

The Generators of SU(2):

The Three-sphere is a connected manifold and so it means that all group elements of SU(2) are continuously connected.

Let U_θ be an element of SU(2) that depends on a real parameter θ such that $U_\theta \to \mathbb{1}$, when $\theta \to 0$.

This means that $\exists$ a 2x2 Hermitian matrix T st

$$U_\theta = \exp[i\theta T]$$

where $\exp[i\theta T] = 1 + i\theta T + \frac{(i\theta)^2}{2!} T^2 + \cdots$

To note that T has to be Hermitian we note that

$$U_\theta^\dagger = 1 - i\theta T^\dagger + \frac{(-i\theta)^2}{2!}(T^\dagger)^2 + \cdots$$

$$= \exp[-i\theta T^\dagger]$$

But $U_\theta^{-1} = \exp[-i\theta T]$ and so from

$$U_\theta^{-1} = U_\theta^\dagger$$

one gets $T^\dagger = T$.

On the other hand using matrix identity:

$$\log \mathrm{Det}\, X = \mathrm{Tr} \log X$$

one gets $\quad \mathrm{Det}\, U_\theta = 1$

$$\Rightarrow \quad \mathrm{Tr}\, T = 0.$$

For infinitesimal $\delta\theta$:

$$U_{\delta\theta} \simeq \mathbb{1} + i\delta\theta\, T$$

Since SU(2) is three dimensional we can choose three real parameters $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ to specify an element of SU(2). Thus there must exist three linearly-independent and traceless matrices T_1, T_2, T_3 so that a general element of SU(2) can be written as

$$U = \exp\left[i\, \vec{\theta} \cdot \vec{T} \right]$$

where $\vec{T} = (T_1, T_2, T_3)$.

These matrices are called the generators of SU(2).

The Pauli matrices σ_i:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are a good basis for 2x2 Hermitian, traceless matrices and so we set:

$$\tau_i = \frac{\sigma_i}{2}.$$

The generators of SU(2) satisfy the Lie algebra:

$$\boxed{[\tau_i, \tau_j] = i\,\epsilon_{ijk}\,\tau_k.}$$

This equation is actually three equations:

$$[\tau_1, \tau_2] = i\,\tau_3, \quad [\tau_2, \tau_3] = i\,\tau_2$$

and $[\tau_3, \tau_2] = i\,\tau_1$.

Representations of Groups:

The fundamental representation of $SU(2)$ is given by 2×2 dimensional unitary matrices with unit determinant acting on $\mathbb{C}^2$.

Thus if $\psi \in \mathbb{C}^2$ with $\psi = (\psi_1, \psi_2)^T$, then an action of $U \in SU(2)$ on ψ:

$$\psi' = U\psi$$

s.t. $$\psi'^\dagger \psi' = \psi^\dagger U^\dagger U \psi = \psi^\dagger \psi .$$

Another important representation of $SU(2)$ is the action of on the Hermitian matrix of the form:

$$X = \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix}$$

$$X \longrightarrow X' = UXU^\dagger .$$

This representation is called the adjoint representation. Note that:

$$X'^\dagger = (UXU^\dagger)^\dagger = U^\dagger X^\dagger U = U^\dagger X U = X' .$$

And the Hilbert-Scmidt inner product between two matrices A & B given by $\mathrm{Tr}(A^\dagger B)$ yields:

$$\frac{1}{2}\mathrm{Tr}(X^\dagger X) = \frac{1}{2}\mathrm{Tr}\, X^2 = \frac{1}{2}\mathrm{Tr}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}\begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix}$$

$$= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} x_3^2 + x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 + x_3^2 \end{pmatrix}$$

$$= x_1^2 + x_2^2 + x_3^2 = \vec{x}\cdot\vec{x}.$$

Thus we see that under an adjoint action of $SU(2)$ the Hermitian matrix X transforms in such a way that the Euclidean norm of the three dimensional real vector $\bar{x} \in \mathbb{R}^3$ remains invariant:

$$\vec{x}\cdot\vec{x} \to \vec{x}'\cdot\vec{x}' = \vec{x}\cdot\vec{x}$$

This smells very much like rotation and it will be if we can show:

$$\vec{x}' = R(U)\vec{x}$$

where $R(U)$ is an orthogonal unit-determinant 3×3 matrix. To show this we consider:

$$\begin{aligned}
\mathrm{Tr}\, X\sigma_i &= \mathrm{Tr}\, \vec{x}\cdot\vec{\sigma}\,\sigma_i \\
&= \sum_j x_j\, \mathrm{Tr}\, \sigma_j\sigma_i \\
&= \frac{1}{2}\sum_j x_j\, \mathrm{Tr}\{\sigma_i, \sigma_j\}
\end{aligned}$$

$$= \frac{1}{\cancel{2}} \sum_j x_j \,\mathrm{Tr}\, \cancel{2}\, \delta_{ij} \mathbb{1}_2$$

$$= \quad 2 x_i$$

And so $x_i = \frac{1}{2} \mathrm{Tr}\, X \sigma_i$

For an infinitesimal unitary transformation:

$$U \simeq \mathbb{1} + i \vec{\theta} \cdot \vec{I}$$

Thus $X' = X + i \sum_j \theta_j [I_j, X]$

$$= X + i \sum_{j,k} \theta_j \frac{1}{2} x_k [\sigma_j, \sigma_k]$$

$$= X + i \sum_{j,k} \frac{1}{\cancel{2}} \theta_j x_k \cancel{2} i \epsilon_{jkl} \sigma_l$$

$$= X - \sum_{j,k,l} \epsilon_{ljk} \theta_j x_k \sigma_l$$

Multiplying both sides by σ_i and taking the trace we get:

$$2 x_i' = 2 x_i - \sum_{j,k,l} 2 \epsilon_{ljk} \theta_j x_k \delta_{il}$$

$$x_i' = x_i - \sum_{j,k} \epsilon_{ijk}\, \theta_j\, x_k$$

$$[\,\vec{x} \simeq \vec{x} - \theta\, \hat{n} \times \vec{x}\,]$$

$$\Rightarrow \qquad \vec{x}\,' = R(\vec{\theta})\, \vec{x}$$

for infinitesimal $\vec{\theta}$.

Comments:

1. The proof is completed by showing that $[F_i]_{jk} = i\,\epsilon_{ijk}$ are the elements of the matrices F_i that generate $SO(3)$.

2. We see that for U and $-U$, we get the same element $R \in SO(3)$. U and $-U$ correspond to the antipodal points on S^3. The subgroup $\mathbb{Z}_2 = (\mathbb{1}, -\mathbb{1}) < SU(2)$ form a normal subgroup. Thus we get:

$$SO(3) \cong \frac{SU(2)}{\mathbb{Z}_2} \cong S^3/\mathbb{Z}_2$$

3. It can be shown that SO(3) is not simply connected and that

$$\pi_1(SO(3)) = \mathbb{Z}_2$$

where π_1 is the fundamental group or the first homotopy group of SO(3).