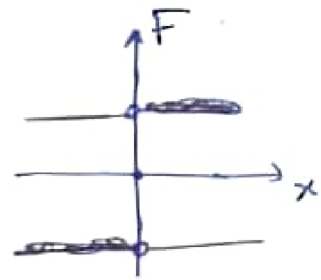
# Lecture 3




## A non-linear oscillator

Consider the equation of motion,

$$m \frac{d^2}{dt^2} x(t) = \begin{cases} -F_0 & ; x > 0 \\ F_0 & ; x < 0 \\ 0 & ; x = 0 \end{cases}$$

The equation describes a particle with mass $m$, subjected to a force to the left, $-F_0$, when the particle is to the right of the origin $(x>0)$ and vice versa. The force is zero at $x=0$. Due to the restoring nature of the force, the motion will be periodic. But it won't be SHM, as we will see. Even if you define a potential energy, it will grow linearly on both sides of $x$. But the force has a discontinuity at $x=0$. So, you wouldn't possibly be able to do a Taylor series expansion of potential energy around $x=0$ and use the idea of small angle approximation. However, we can find the ~~eq~~ solution by a different manner.

Say, at $t=0$, the particle is at the origin, but it will have a velocity, say $+v$ and is moving to the right of the origin. You can think this as the particle started with zero initial velocity at some distance $-A$ from the origin. Due to the force, it accelerates and gains some velocity $v$ when it reaches the origin.

The particle then moves to the right and decelerates with a constant acceleration,

$$a = -\frac{F_0}{m}$$

$$\therefore x(t) = vt - \frac{1}{2}\frac{F_0}{m}t^2 \quad \text{for } t \leq \tau$$

where $\tau = \frac{2mv}{F_0}$ is the time to reach $+A$ and then get back to the origin.

$\Delta P = F_0 \tau$

$\Rightarrow mv - (-mv) = F_0 \tau$

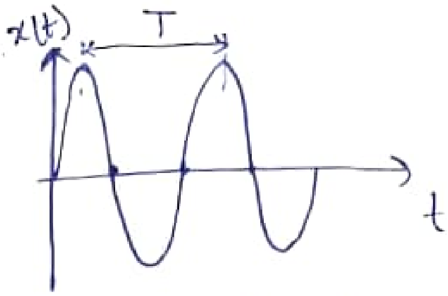
$\therefore \tau = \frac{2mv}{F_0}$

Now, after time $t = \tau$, the particle has reached the origin, with a velocity $v$, but now in opposite direction. After it crosses the origin, there is the same opposing force acting on it. Now,

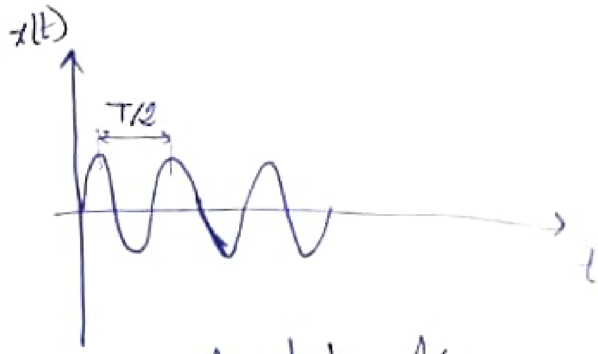
$$x(t) = -v(t-\tau) + \frac{F_0}{2m}(t-\tau)^2 \quad ; \quad \tau \leq t \leq 2\tau$$

and the process repeats itself.

The time period is strictly dependent on velocity $v$, and hence on the amplitude. The time period is proportional to the amplitude.




Amplitude = A




Ampitude = A/2

To express $x(t)$ in terms of amplitude, we have,

$$A = x\left(\frac{\tau}{2}\right) = v\frac{\tau}{2} - \frac{F_0}{2m}\frac{\tau^2}{4}$$

$$\therefore A = v \times \frac{2mv}{2F_0} - \frac{F_0}{2m} \times \frac{4m^2v^2}{4F_0^2}$$

$$= \frac{\cancel{2}mv^2}{F_0} - \frac{\cancel{2}mv^2}{2F}$$

$$= \frac{mv^2}{2F_0}$$

$$\therefore v = \sqrt{\frac{2F_0A}{m}}$$

$$\therefore \quad x(t) = \sqrt{\frac{2F_0A}{m}}\, t - \frac{F_0}{2m} t^2 \; ; \; 0 \leq t \leq \tau$$

$$x(t) = -\sqrt{\frac{2F_0A}{m}}\,(t-\tau) + \frac{F_0}{2m}(t-\tau)^2 \; ; \; \tau \leq t \leq 2\tau$$

## Damped oscillation

The SHM described before are found, almost nowhere in reality. The oscillations never continue forever, rather dies out. The reason is that, there are resistive forces. Consider first the case of the spring-mass system, but now immersed in a viscous liquid/fluid. The resistive force is velocity dependent and is given by,

$$f(v) = b_1 v + b_2 v^2 \quad \Big| \quad \vec{f}(v) = -(b_1 v + b_2 v^2)\,\hat{v}$$

where $v$ is the magnitude of velocity. So, the resistive force has two parts — one proportional to $v$ and one to $v^2$. We can find a critical velocity by equating the two terms.

$$b_1 v_c = b_2 v_c^2$$

→ where both of the terms have equal strength.

$$\therefore v_c = \frac{b_1}{b_2}$$

If the velocity is small compared to the critical velocity, that is, $v \ll v_c = b_1/b_2$, then the first term is the dominant term. Then, the equation can be written as,

$$m\frac{d^2x}{dt^2} = -kx - bv$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{—(1)}$$

with $\gamma = b/m$ and $\omega_0 = \sqrt{k/m}$

The equation is linear, has time translation symmetry and homogenous. We have previously shown that, the solution must be an exponential.

$$\therefore x(t) = e^{\lambda t}$$

Plugging this in equation (1) we get,

$$\lambda^2 e^{\lambda t} + \gamma\lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$$

$$\Rightarrow \quad (\lambda^2 + \gamma\lambda + \omega_0^2)e^{\lambda t} = 0$$

$$\boxed{\therefore \lambda^2 + \gamma\lambda + \omega_0^2 = 0} \quad (t \not\to \infty)$$

This is called the characteristic equation. The solutions of $\lambda$ are obviously,

$$\lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

$$\therefore \lambda_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

The general solution then can be written as—

$$x(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}$$

Underdamped case/ Light damping

For $\frac{\gamma}{2} < \omega_0$, that is $\frac{b}{2m} < \sqrt{\frac{k}{m}}$ or $b^2 < 4mk$,

$$\lambda_\pm = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = -\frac{\gamma}{2} \pm i\omega$$

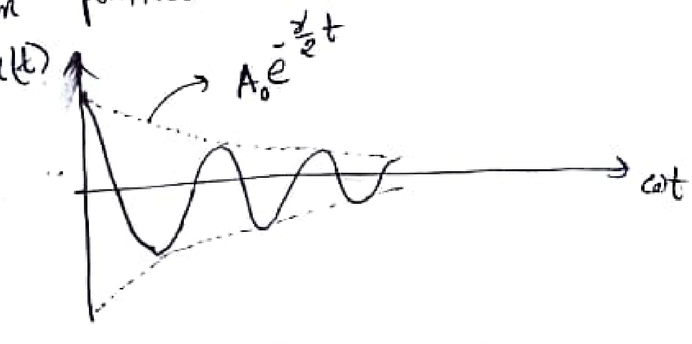
with $\omega = \sqrt{\omega_0^2 - \gamma^2/4}$

$$\therefore \; x(t) = C_1 e^{-\frac{\gamma}{2}t} e^{i\omega t} + C_2 e^{-\frac{\gamma}{2}t} e^{-i\omega t}$$

$$\Rightarrow x(t) = e^{-\frac{\gamma}{2}t}\left[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right]$$

$$\boxed{\therefore \; x(t) = e^{-\frac{\gamma}{2}t} A_0 \cos(\omega t + \phi)} = A(t)\cos(\omega t + \phi)$$

So, the motion is like a sinusoidal one, with exponentially decreasing amplitude. The oscillation frequency $\omega$ is less than the frequency of free motion $\omega_0$.
As time goes on, the factor $e^{-\frac{\gamma}{2}t}$ decreases the overall sinusoidal amplitudes, where the particle oscillates around the equilibrium position infinite times.



We can introduce a quality factor $Q = \frac{\omega_0}{\gamma}$.

Then,

$$\omega^2 = \omega_0^2 - \frac{\omega_0^2}{4Q^2} = \omega_0^2 \left(1 - \frac{1}{4Q^2}\right)$$

Since $\omega_0$ and $\gamma$ have same dimension, $Q$ is dimensionless. If $Q \gg 1$,

$$\omega \approx \omega_0$$

So, high value of $Q$ indicates less damping and vice-versa. One can also define a time constant, for which, the amplitude,

$$A(t) = A_0 e^{-t/\tau} \quad \text{with } \tau = \frac{2}{\gamma} = \frac{2m}{b}$$

$\tau$ is basically a characteristic time, after which the amplitude off oscillation falls down by a factor of $\frac{1}{e}$.

Now,

$$v = \frac{dx}{dt} = \left[-\frac{\gamma}{2} e^{-\frac{\gamma}{2}t} A_0 \cos(\omega t + \phi) - A_0 e^{-\frac{\gamma}{2}t} \omega \sin(\omega t + \phi)\right]$$

$$\therefore v = A_0 e^{-\frac{\gamma}{2}t} \left[-\frac{\gamma}{2}\cos(\omega t+\phi) - \omega \sin(\omega t + \phi)\right]$$

The total energy as a function of time is then -

$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$= \frac{1}{2}kA_0^2 e^{-\frac{\gamma}{2}t \times 2} \cos^2(\omega t+\phi) + \frac{1}{2} m A_0^2 e^{-\frac{\gamma}{2}t\times 2} \left[-\frac{\gamma}{2}\cos(\omega t + \phi) - \omega \sin(\omega t+\phi)\right]^2$$

Let's take $\phi = 0$ to make the calculations clean.

$$E = \frac{1}{2} KA_0^2 e^{-\gamma t} \cos^2 \omega t + \frac{1}{2} m A_0^2 e^{-\gamma t} \left( -\frac{\gamma}{2} \cos\omega t - \omega \sin\omega t \right)^2$$

$$\boxed{\frac{K}{m} = \omega_0^2}$$

$$\therefore K = m\omega_0^2$$

$$= \frac{1}{2} m A_0^2 e^{-\gamma t} \Big[ \omega_0^2 \cos^2 \omega t + \frac{\gamma^2}{4} \cos^2 \omega t + 2 \times \frac{\gamma}{2} \omega \cos\omega t \sin\omega t + \omega^2 \sin^2 \omega t \Big]$$

$$= \frac{1}{2} m A_0^2 e^{-\gamma t} \Big[ \omega_0^2 \cos^2 \omega t + \frac{\gamma^2}{4} \cos^2 \omega t + \gamma\omega \sin\omega t \cos\omega t + \left( \omega_0^2 - \frac{\gamma^2}{4} \right) \sin^2 \omega t \Big]$$

$$= \frac{1}{2} m A_0^2 e^{-\gamma t} \Big[ \omega_0^2 (\cos^2 \omega t + \sin^2 \omega t) + \frac{\gamma^2}{4} (\cos^2 \omega t - \sin^2 \omega t) + \frac{\gamma\omega}{2} 2 \sin\omega t \cos\omega t \Big]$$

$$\boxed{\therefore E(t) = \frac{1}{2} m A_0^2 e^{-\gamma t} \left[ \omega_0^2 + \frac{\gamma^2}{4} \cos 2\omega t + \frac{\gamma\omega}{2} \sin 2\omega t \right]}$$

This is the exact form of energy. If $\gamma = 0$, it becomes simply, $E = \frac{1}{2} m A_0^2 \omega_0^2 = \frac{1}{2} K A_0^2$.

Let's work with an approximation. If $\gamma$ is very small, the amplitude is roughly constant over one or a few oscillations. If we take the average of energy for a few time periods, then –

$$\langle E(t) \rangle = \frac{1}{2} m A_0^2 e^{-\gamma t} \omega_0^2$$

$$= \frac{1}{2} K A_0^2 e^{-\gamma t}$$

The average of the oscillatory terms ($\cos 2\omega t$ and $\sin 2\omega t$) is simply zero over one or more complete oscillations.

So, ~~ener~~ average energy of this damped oscillator decreases exponentially in the small $\gamma$ limit, that is, for light damping.

<u>Overdamped case</u>

If now, $\frac{\gamma}{2} > \omega_0$, that is, $\frac{b^2}{4m^2} > \frac{k}{m}$ or, $b^2 > 4km$,

then –

$$x(t) = C_1 e^{-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\, t} + C_2 e^{-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\, t}$$

~~Let us defi~~

$$\cancel{\therefore x(t) = C_1 e^{-\lambda_+ t} + C_2 e^{-\lambda_- t}}$$

~~In this case, $\lambda_+$ and $\lambda_-$ are both positive.~~

Let us define, $\Gamma_\pm = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

$$\therefore x(t) = C_1 e^{-\Gamma_- t} + C_2 e^{-\Gamma_+ t}$$

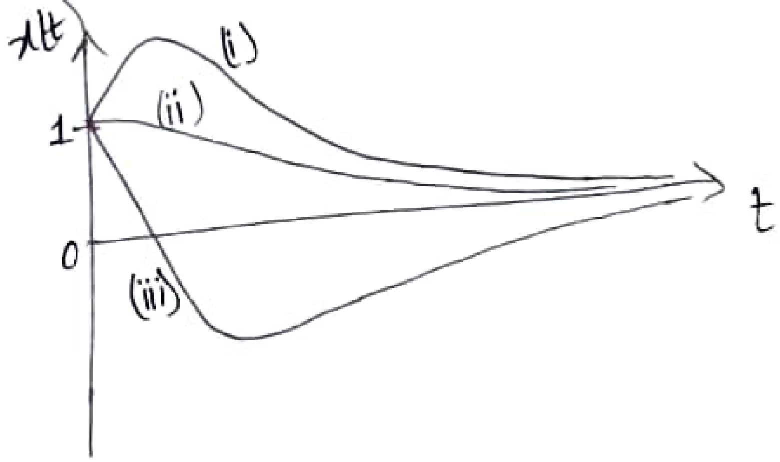
with both $\Gamma_\pm$ being positive.

So, $x(t)$ is the sum of two decreasing exponentials. So, $x(t)$ will decrease rapidly with time. Since, $\Gamma_+ > \Gamma_-$, so $e^{-\Gamma_+ t}$ decays more quickly than the $e^{-\Gamma_- t}$ term. So, for large value of $t$, we are essentially left with $e^{-\Gamma_- t}$. The plots of $x(t)$ will look like any three of the following figure –

(i) If the mass is thrown away from the origin

(ii) If it is released from rest (from some position far from the origin)

(iii) If it is thrown back ~~away~~ towards the origin.




In any case, the curve can cross the origin at most once. To see this, let's set,

$$x(t) = 0$$

$$\Rightarrow C_1 e^{-\Gamma_- t} + C_2 e^{-\Gamma_+ t} = 0$$

$$\Rightarrow -\frac{C_1}{C_2} = e^{(\Gamma_- - \Gamma_+)t}$$

$$\Rightarrow \ln\left(-\frac{C_1}{C_2}\right) = (\Gamma_- - \Gamma_+)t$$

$$\boxed{\therefore\ t = \frac{1}{\Gamma_- - \Gamma_+} \ln\left(-\frac{C_1}{C_2}\right)}$$

So, we have only one positive solution for the values of $C_1$ and $C_2$ such that, $-\frac{C_1}{C_2} < 1$ and $-\frac{C_1}{C_2} > 0$. The other possibilities of possible combination of $C_1$ and $C_2$ don't provide any meaningful solution for $t > 0$.

So, in the case of overdamping, the mass just slowly goes to the origin.