Classical Mechanics
Lecture # 11

Poisson Brackets & Canonical Transformations:

We now explore some formal aspects of classical mechanics in terms of a mathematical object called the Poisson bracket. Poisson brackets reveal the algebraic structure of conserved quantities. They also put the equations of classical mechanics in a form that closely reflects the equations of quantum mechanics as was formulated by Werner Heisenberg.

The Poisson Bracket:

Let $f(q,p)$ & $g(q,p)$ be functions of phase space coordinates q_i and p_i. Then the Poisson bracket between f & g is defined by:

$$\{f,g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Following are some of the properties of the Poisson bracket:

1. Skew-symmetry: $\{f,g\} = -\{g,f\}$

2. Linearity: $\{\alpha f+\beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$ for $\alpha, \beta \in \mathbb{R}$.

3. Leibniz: $\{fg, h\} = f\{g,h\} + \{f,h\}g$

4. Jacobi Identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

<u>Comments:</u>

1. All these properties are also satisfied by the generators of a Lie group. The collection of these generators are known as a Lie algebra.

For example let us take the rotation group in 3D: SO(3). If we consider three independent rotation $R_{\hat{x}}(\theta_1)$, $R_{\hat{y}}(\theta_2)$, $R_{\hat{z}}(\theta_3)$ around the $\hat{x}$, $\hat{y}$, & $\hat{z}$ axes. The generators J_a of these transformations can be defined as:

$$R_{\hat{a}}(\theta_a) = \exp\left[i\,\theta_a J_a \right] \quad (\text{no sum})$$

Then it is easy to show that J_a have forms such that

$$[J_1, J_2] = i J_3\,, \quad [J_2, J_3] = i J_1\,, \quad [J_3, J_1] = i J_2$$

or $$\boxed{[J_a, J_b] = i\,\epsilon_{abc} J_c}$$

This is the Lie algebra of SO(3), also known as the angular momentum algebra.

Although the Lie algebra of the SO(3) group have been defined in terms of 3x3 matrices J_a, this algebra has representations in different dimensions. For example for spin $1/2\hbar$ particles in quantum mechanics the rotations of the two-component wave function is affected by the 2x2 unitary (not orthogonal) matrices:

$$U_a(\theta_a) = \exp\left[i \frac{\theta_a}{\hbar} \tau_a \right] \text{ (no sum)}$$

where $\tau_a = \frac{\hbar}{2}\sigma_a$ with σ_a as the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

τ_a are the generators and they satisfy

$$[\tau_a, \tau_b] = i\hbar\, \epsilon_{abc} \tau_c$$

which is the angular momentum algebra if we set $\hbar = 1$.

It is easy to see that the commutator $[A,B]$ satify all four properties of the Poisson bracket.

2. The poisson bracket between the phase space coordinates become:

$$\{q_i, q_j\} = 0$$

$$\{p_i, p_j\} = 0$$

and $$\{q_i, p_j\} = \delta_{ij}.$$

The time evolution of function $f(q,p,t)$ on phase space can be written as:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Proof: $$\frac{df}{dt} = \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}$$

$$= \{f, H\} + \frac{\partial f}{\partial t}.$$

3. Suppose $Q(q,p)$ is a time independent conserved quantity. Then

$$\frac{dQ}{dt} = 0$$

$$\Rightarrow \quad \{Q, H\} = 0$$

If $P(q,p)$ is another quantity such that

$$\{P, H\} = 0$$

then by the Jacobi identity:

$$\{P,\{Q,H\}\} + \{Q,\{H,P\}\} + \{H,\{P,Q\}\} = 0$$

$$\Rightarrow \qquad \{H,\{P,Q\}\} = 0$$

$$T \equiv \{P,Q\}$$

is also conserved:

$$\frac{dT}{dt} = 0$$

Thus we see that conserved quantities form a closed (Lie) algebra under the Poisson bracket.

Examples:

1. Angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

In components:

$$L_a = \epsilon_{abc}\, r_b\, p_c$$

$$L_1 = r_2 p_3 - r_3 p_2, \quad L_2 = r_3 p_1 - r_1 p_3$$

$$L_3 = r_1 p_2 - r_2 p_1$$

Then $\{L_1, L_2\} = \dfrac{\partial L_1}{\partial r_i}\dfrac{\partial L_2}{\partial p_i} - \dfrac{\partial L_1}{\partial p_i}\dfrac{\partial L_2}{\partial r_i}$

$$= (\delta_{i2} p_3 - \delta_{i3} p_2)(r_3 \delta_{i1} - r_1 \delta_{i3})$$

$$- (r_2 \delta_{i3} - r_3 \delta_{i2})(\delta_{i3} p_1 - \delta_{i1} p_3)$$

$$= \delta_{i2}\underbrace{\cancel{\delta_{i1}}}_{=0} r_3 p_3 - \delta_{i2}\underbrace{\cancel{\delta_{i3}}}_{=0} r_1 p_3 - \delta_{i3}\underbrace{\cancel{\delta_{i1}}}_{=0} p_2 r_3 + \delta_{i3}\delta_{i3} r_1 p_2$$

$$- \delta_{i3}\delta_{i3} r_2 p_1 + \delta_{i3}\underbrace{\cancel{\delta_{i1}}}_{=0} r_2 p_3 + \delta_{i2}\underbrace{\cancel{\delta_{i3}}}_{=0} r_3 p_1 - \delta_{i2}\underbrace{\cancel{\delta_{i1}}}_{=0} r_3 p_3$$

$$= r_1 p_2 - r_2 p_1 = L_3$$

Thus we have $\{L_1, L_2\} = L_3$ and similarly we have $\{L_2, L_3\} = L_1$ and $\{L_3, L_1\} = L_2$.

Thus if L_1 and L_2 are conserved then so must L_3. The total (angular-momentum)^2:

$$L^2 = L_j L_j$$

then

$$\{L^2, L_i\} = L_j \{L_j, L_i\} + \{L_j, L_i\} L_j = 0$$

So L^2 is also conserved.

Canonical Transformations:

When discussing the Euler-Lagrange equations we saw that a general coordinate transformation

$$q_i \to \tilde{q}_i = \tilde{q}_i(q_i, t)$$

leave the form of the Euler-Lagrange equation invariant. We can similarly ask what general transformation of the phase space coordinates can we make that will leave the form of Hamilton's equations invariant?

To do this let us define a wedge product $\wedge$ such that

$$d\xi_i \wedge d\xi_j = -d\xi_j \wedge d\xi_i$$

where $\xi_i \in \{q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n\}$

Note that the antisymmetry of the wedge product implies

$$d\xi_i \wedge d\xi_i = 0 \quad \text{(no sum)}$$

We then introduce a 2-form

$$\Omega_2 = dq_i \wedge dp_i$$

If we define a $2n \times 2n$ matrix J by

$$\Omega = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$$

Then we can express Ω_2 as:

$$\Omega_2 = \frac{1}{2} \Omega_{ij} \, d\xi_i \wedge d\xi_j$$

with $\xi_i = (q_i, p_i)$

Now if we make a general linear transformation:

$$\xi_i \rightarrow \zeta_i = \zeta_i(\xi_i)$$

Then
$$dQ_i = \frac{\partial Q_i}{\partial q_j} dq_j + \frac{\partial Q_i}{\partial p_j} dp_j$$

$$dP_i = \frac{\partial P_i}{\partial q_j} dq_j + \frac{\partial P_i}{\partial p_j} \partial p_j$$

Then $$dQ_i \wedge dP_i = \frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial q_\ell} dq_j \wedge dq_\ell + \frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial p_\ell} dq_j \wedge dp_\ell + \frac{\partial Q_i}{\partial p_j}\frac{\partial P_i}{\partial q_\ell} dp_j \wedge dq_\ell + \frac{\partial Q_i}{\partial p_j}\frac{\partial P_i}{\partial p_\ell} dp_j \wedge dp_\ell$$

$$= \frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial q_\ell} dq_j \wedge dq_\ell + \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial P_i}{\partial p_\ell} - \frac{\partial Q_i}{\partial p_\ell}\frac{\partial P_i}{\partial q_j}\right) dq_j \wedge dp_\ell + \frac{\partial Q_i}{\partial p_i}\frac{\partial P_i}{\partial p_\ell} dp_i \wedge dp_\ell$$

We see that $\frac{\partial Q_i}{\partial q_j}, \frac{\partial P_i}{\partial q_\ell}, \dots$ are all elements of the Jacobian matrix:

$$J = \begin{pmatrix} \frac{\partial \vec{Q}}{\partial \vec{q}} & \frac{\partial \vec{Q}}{\partial \vec{p}} \\ \frac{\partial \vec{P}}{\partial \vec{q}} & \frac{\partial \vec{P}}{\partial \vec{p}} \end{pmatrix}$$

$$\mathcal{J}_{ij} = \frac{\partial \zeta_i}{\partial \xi_j}$$

And so we can write

$$\tilde{\Omega}_2 = dQ_i \wedge dP_i$$

$$= \frac{1}{2} J_{ij} \, d\zeta_i \wedge d\zeta_j$$

$$= \frac{1}{2} J_{ij} \, \mathcal{J}_{ik} \, d\xi_k \wedge \mathcal{J}_{jl} \, d\xi_l$$

So if we require $\tilde{\Omega}_2 = \tilde{\Omega}$ then

$$\mathcal{J}_{ik} \, J_{ij} \, \mathcal{J}_{jl} = J_{kl}$$

$$\Rightarrow \boxed{\mathcal{J}^T \Omega \mathcal{J} = \Omega}$$

Thus the transformations need to be of the form that satisfy the above condition. Matrices $\mathcal{J}$ that satisfy this condition form the group known as $Sp(2n, \mathbb{R})$ — the symplectic group.

The invariant form Ω_2 is called the symplectic form. The transformations that leave J_2 invariant are known as ==Canonical transformations==

Exercises: 1. Express Hamilton's equation using the form

$$\Omega = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

Then show that the equations have the same form under a canonical transformation.

2. Show that the Poisson bracket $\{f, g\}$ of two functions $f(q,p)$ and $g(q,p)$ are is invariant under a canonical transformation.

Comment:

1. In the form language the element is

$$dV = \underbrace{\Omega_2 \wedge \Omega_2 \wedge \Omega_2 \dots \wedge \Omega_n}$$

n copies

Since Ω_2 is invariant under a canonical transformation, the volume element is as well.