Lecture 2

Using time translation symmetry to find the solution

Since our system has time translation symmetry, if $x(t)$ is a solution, then so should $x(t+\alpha)$. Now, let's think about the simplest relation that can exists between $x(t+\alpha)$ and $x(t)$. $x(t)$ can just be multiplied by an overall constant factor that is dependent on α. And its pretty much reasonable. The reason is obviously the other property — linearity. If $x(t)$ is a solution, so is some constant multiplied with $x(t)$. (The equation has to be homogenous also, Like ours at hand)

So,
$$x(t+\alpha) = h(\alpha)\, x(t)$$

Now, as we have seen earlier, the solution might pop out to be complex, so we will leave floor for this possibility. Let's set $t=0$ in the equation.

$$x(\alpha) = h(\alpha)\, x(0)$$

$$\therefore h(\alpha) = \frac{x(\alpha)}{x(0)}$$

We can always multipy our solution $x(t)$ with a constant so that $x(0)=1$. This simplifies our current calculation

$$\therefore h(\alpha) = x(\alpha)$$

$$\therefore x(t+\alpha) = x(t)\, x(\alpha)$$

Now, if $\alpha = t$, then, $x(2t) = x(t)^2$

If $\alpha = 2t$, then, $$x(3t) = x(t)\, x(2t) = [x(t)]^3$$

and so on. In general,

$$x(Nt) = [x(t)]^N \quad \text{——} \; (1)$$

Now consider a very small time $t = \epsilon \ll 1$. We can do a Taylor (Maclaurin) expansion of $x(t)$ to approximate $x(\epsilon)$

$$x(t) = x(0) + \left.\frac{dx}{dt}\right|_{x=0} t + \left.\frac{d^2x}{dt^2}\right|_{x=0} \frac{t^2}{2!} + \cdots$$

This is the expansion of $x(t)$ ~~are~~ about $t=0$. Now, for small variation $t = 0+\epsilon$,

$$x(\epsilon) = x(0) + x'(0)\,\epsilon + O(\epsilon^2)$$

$$\therefore \; x(\epsilon) = 1 + x'(0)\,\epsilon = 1 + H\epsilon \quad \text{with } H = x'(0)$$

Now, from (1), $$x(N\epsilon) = [x(\epsilon)]^N$$

For any time $t = N\epsilon$, we can write,

$$\underset{x(N\epsilon)}{x(t)} = \lim_{N\to\infty} \left[x\left(\tfrac{t}{N}\right)\right]^N = \lim_{N\to\infty}\left[1 + \frac{Ht}{N}\right]^N$$

$$= e^{Ht}$$

So, as long as we have an ~~solution~~ equation that obeys time translation invariance and linearity (and is homogenous), the solution should be of the form,

$$x(t) = e^{Ht}$$

** It doesn't matter whether the solution is coming from SHM or DE or other place, as long as linearity and time translation symmetry is obeyed, the solution will be of this form.** Now, let's concentrate on our equation.

$$\frac{d^2}{dt^2} x(t) + \omega^2 x(t) = 0$$

$$\Rightarrow H^2 x(t) = -\omega^2 x(t)$$

$$\therefore H = \pm i\omega$$

So, $x(t) = e^{\pm i\omega t}$ and following the previous procedure,

$$\boxed{x(t) = A\cos(\omega t + \phi) = Re[Ae^{-i(\omega t + \phi)}]}$$

We will use the (**) result later with great importance.

Energy in SHM

When the block is in SHM, we have the kinetic energy of the block and potential energy of the spring.

$$K.E. = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \qquad P.E. = \frac{1}{2} kx^2$$

Total mechanical energy, $E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} kx^2$

If there are no external forces acting on the system, the total energy must be conserved. You can verify this using $x(t) = A\cos(\omega t + \phi)$ as the solution.

$$E = \frac{1}{2}kA^2$$

You can also find the equation of motion by differentiating the energy equation with respect to time—

$$\frac{1}{2} m\, 2\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + \frac{1}{2}k \times 2x\frac{dx}{dt} = \frac{d}{dt}(E)$$

$$\therefore \quad m\frac{d^2x}{dt^2} + kx = 0$$

This is obviously the equation of motion found by the force method. The time period of oscillation is simply, $T = 2\pi/\omega = 2\pi\sqrt{m/k}$

Small oscillations and SHM

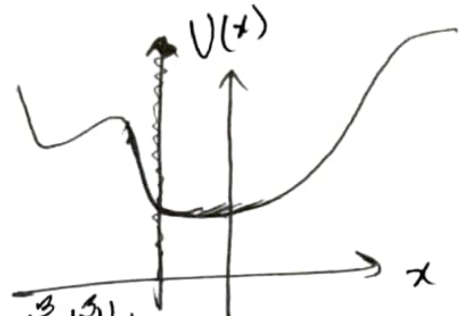
If we have a well behaved potential energy function, having at least a minimum, then for small perturbations about that minimum, the motion will be a SHM.

The conservative force can be derived from a potential given by,

$$F(x) = -\frac{dU}{dx}$$

Now, consider a random potential energy function with a local minimum at $x=0$.

Let's perform a Taylor expansion of $U(x)$ around $x=0$.



$$\therefore U(x) = U(0) + \frac{x}{1!}\left.\frac{dU}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{d^3U}{dx^3}\right|_{x=0} + \cdots$$

Since we are at a minimum, $\left.\frac{dU}{dx}\right|_{x=0} = 0$. For small oscillations, ignoring the higher order terms $(O(x^3))$,

$$U(x) = U(0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x=0} x^2$$

Now, we can set the zero of the potential energy wherever we want, at $x=0$ also. If $U(0)=0$, then,

$$U(x) = \frac{1}{2}kx^2$$

where we defined a constant $k = \left.\frac{d^2U}{dx^2}\right|_{x=0}$.

The force will then be given by,

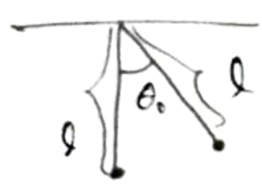
$$F = -\frac{dU}{dx} = -kx$$

which is nothing but the Hooke's law force. So, for a sufficiently small oscillation about a minimum of a potential energy, the oscillation will be SHM.

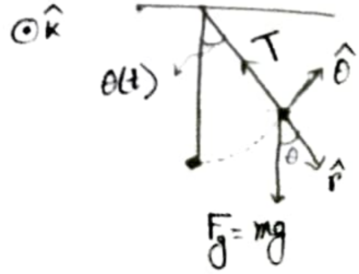
Although, the argument wouldn't have been true if the potential was not a smooth function so that the first or second derivative of potential is not defined at the minimum (stable equilibrium) point. In that case, we wouldn't be able to perform a Taylor expansion. On the other hand, it could be that $\frac{d^2V}{dx^2}\big|_{x=0}$ vanishes. Other than two exceptional cases, any small oscillation about a stable equilibrium point will be a simple harmonic motion.

~~Siple~~ Simple pendulum

Force approach



Say, there is a small mass attached at one end of a massless (that is very light) rod. For an initial displacement of θ_0, say, at some time t, the pendulum makes an angle $\theta(t)$ with the vertical axis.



Now, $\vec{T} = -T\hat{r}$ and $\vec{F_g} = mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta}$

Here, $\vec{F_\theta} = -mg\sin\theta\,\hat{\theta}$ is the restoring force that tries to bring the pendulum towards the equilibrium.

If $\theta > 0$, $F_\theta < 0$ and $\theta < 0$, $F_\theta \to 0$, for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Now, $a_\theta = \alpha l$

Newton's second law along $\hat{\theta}$ direction,

$$F_\theta = m a_\theta$$

$$\Rightarrow -mg\sin\theta = m\alpha l$$

$$\Rightarrow l\frac{d^2\theta}{dt^2} = -g\sin\theta \qquad \therefore \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

The equation is not SHM equation, but it describe some periodic motion. Anyways, we can take the small angle approximation. In the limit $\theta \approx 0$, $\sin\theta \approx \theta$,

and so,
$$\frac{d^2\theta}{dt^2} = -\omega^2\theta \qquad \text{with} \quad \omega = \sqrt{g/l}$$

We already know the solution.

$$\theta(t) = \theta_0 \cos(\omega t + \phi)$$

for sufficiently small θ_0.

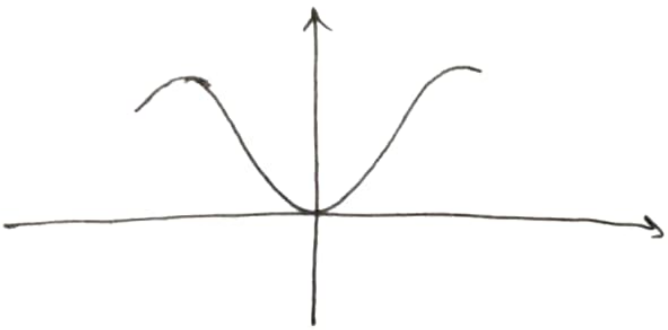
The potential energy function is given by,

$$U = mgl(1-\cos\theta)$$

$$\rightarrow U(\theta) = mgl\left[1-\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)\right]$$

$$\therefore U(\theta) = \frac{1}{2}(mgl)\theta^2 \quad \text{for } \theta \approx 0$$

So, the potential energy as a function of θ looks like a SHM potential energy, under small oscillations



$(1-\cos\theta)$ potential energy

Energy approach

$$U = mgl\,(1-\cos\theta) \qquad K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\,\frac{d\theta}{dt}\right)^2$$

$v = \omega_0 r$

$$M.E. = mgl\,(1-\cos\theta) + \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2$$

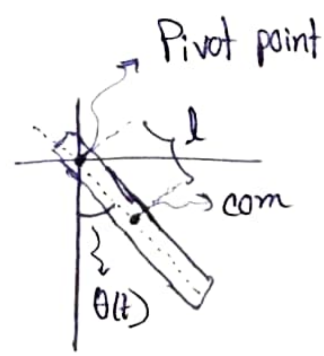
$$\Rightarrow \frac{d}{dt}(M.E.) = -\,mgl\,(-\sin\theta)\,\frac{d\theta}{dt} + \frac{1}{2}m\,2l^2\,\frac{d\theta}{dt}\,\frac{d^2\theta}{dt^2}$$

$$\boxed{\therefore \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta}$$

And we get to the same equation.

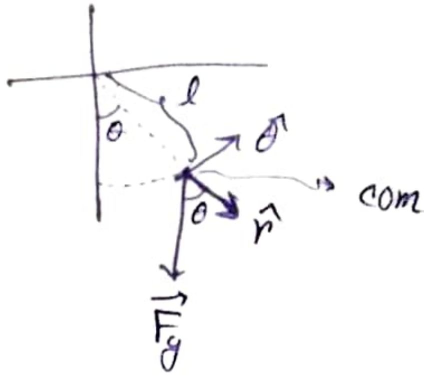
A massive rod pivoted on a point (thin)

Say, we have a rod like object with its center of mass (com) and pivot point as shown in the diagram. Let's draw a force diagram and find the



equation of motion.

The torque with respect to the pivot point is given by,



$$\vec{\tau} = \vec{l} \times \vec{F_g}$$

$$= l\,\hat{r} \times (mg\cos\theta\,\hat{r} - mg\sin\theta\,\hat{\theta})$$

$$= -mgl\sin\theta\,\hat{k}$$

$\alpha = \odot$

Now, $\quad \tau = I\alpha$

$$\Rightarrow -mgl\sin\theta = \left(\frac{1}{12}ML^2 + ml^2\right)\alpha$$

Say, $L = \beta l$

$$\Rightarrow -mgl\sin\theta = \left(\frac{1}{12}m\beta^2 l^2 + ml^2\right)\alpha$$

$$\Rightarrow -g\sin\theta = \left(\frac{\beta^2}{12} + 1\right) l \,\frac{d^2\theta}{dt^2}$$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{\left(\frac{\beta^2}{12}+1\right)l}\sin\theta$$

For small angle limit, $\quad \dfrac{d^2\theta}{dt^2} = -\omega_0^2\,\theta$

with $\quad \omega_0 = \sqrt{\dfrac{g}{\left(\frac{\beta^2}{12}+1\right)l}}$

You can calculate the time period using this equation with $T = \frac{2\pi}{\omega}$. This type of pendulums are used to calculate the magnitude of ~~gravitay~~ gravitational acceleration g.

A nice way to calculate the time period

Let's consider the total energy in general with no non-conservative forces.

$$\frac{1}{2} m\dot{x}^2 + V(x) = E$$

$$\Rightarrow \quad \frac{1}{2} m\dot{x}^2 = E - V(x) \quad \Rightarrow \quad \dot{x}^2 = \frac{2}{m}\left(E - V(x)\right)$$

$$\therefore \quad \frac{dx}{dt} = \sqrt{\frac{2}{m}}\sqrt{E - V(x)}$$

We are taking the positive square root, implying $\frac{dx}{dt} > 0$. It won't be a problem since we will restrict only to first quarter interval.

$$\therefore \quad dt = \frac{dx}{\sqrt{\frac{2}{m}}\sqrt{E - V(x)}}$$

$$\boxed{\therefore \quad \int dt = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}}}$$

Consider the first quarter period. The ~~pendula~~ block/pendulum moves a distance from $x = 0$ to $x = A$. Then,

$$\int_0^{T/4} dt = \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{E - V(x)}}$$

For mass-spring system, $V(x) = \frac{1}{2}kx^2$.

$$\therefore \quad t\Big|_0^{T/4} = \sqrt{\frac{m}{2}} \int_0^A \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}}$$

$$\therefore \quad \frac{T}{4} = \sqrt{\frac{m}{2}}\sqrt{2} \int_0^A \frac{dx}{\sqrt{2E - kx^2}}$$

$$\therefore\ T = 4\sqrt{m}\,\frac{1}{\sqrt{2E}} \int_0^A \frac{dx}{\sqrt{1-\frac{K}{2E}x^2}}$$

$$= \frac{2\sqrt{2m}}{\sqrt{E}} \int_0^A \frac{dx}{\sqrt{1-Px^2}} \qquad \text{with} \quad P = \frac{K}{2E}$$

Here, $P > 0$.

Let, $\sqrt{P}\,x = \sin u$ $\qquad x = 0,\ u = \sin^{-1}(\sqrt{P}.0) = 0$

$\Rightarrow \sqrt{P}\,dx = \cos u\, du$ $\qquad x = A,\ u = \sin^{-1}(\sqrt{P}A)$

$$\therefore\ T = \frac{2\sqrt{2m}}{\sqrt{E}} \int_0^{\sin^{-1}(\sqrt{P}A)} \frac{\cos u\, du/\sqrt{P}}{\sqrt{1-\sin^2 u}}$$

$$= \frac{2\sqrt{\frac{2m}{P}}}{\sqrt{E}} \int_0^{\sin^{-1}(\sqrt{P}A)} du$$

$$\therefore\ T = \frac{2\sqrt{\frac{2m}{P}}}{\sqrt{E}} \sin^{-1}(\sqrt{P}A)$$

$$= \frac{2\sqrt{\frac{2m}{K/2E}}}{\sqrt{E}} \sin^{-1}\left(\sqrt{\frac{K}{2E}}\,A\right)$$

$$= 4\sqrt{\frac{m}{K}}\ \sin^{-1}\left(\sqrt{\frac{K}{2\times\frac{1}{2}KA^2}}\,A\right)$$

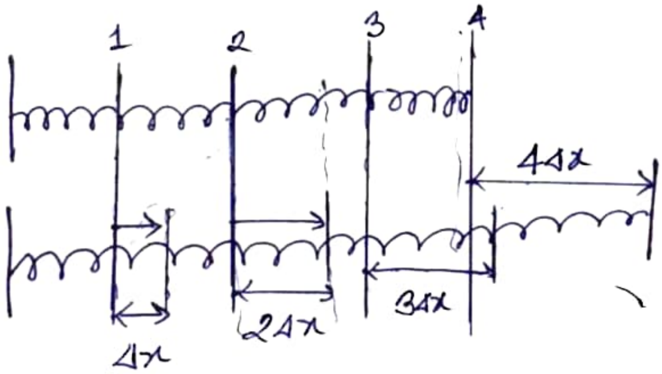
$$= 4\sqrt{\frac{m}{K}}\ \sin^{-1}(1)$$

$$= 4\sqrt{\frac{m}{K}} \times \frac{\pi}{2}$$

$$\boxed{\therefore\ T = 2\pi\sqrt{\frac{m}{K}}}$$

Calculations with massive spring

If the spring has mass, say M_S, we can't simply ignore the kinetic energy of the spring. It will contribute to the total energy and the time period will change. But its not simply $\frac{1}{2}M_S v^2$, since all part of the spring will not move with same velocity. Think about the extension of the spring when it ~~stre~~ is being stretched. Let's first divide the spring into four parts. If all parts of the spring extends uniformly, then the first part extends by Δx, the second part will extend by Δx + the Δx ~~ex~~ due to the extension of the first part.



Since all the displacement happens at same time, then—

$$\Delta t = \frac{\Delta x}{v_1} = \frac{2\Delta x}{v_2} = \frac{3\Delta x}{v_3} = \frac{4\Delta x}{v_4}$$

$$\therefore v_2 = 2v_1, \quad v_3 = 3v_1, \quad v_4 = 4v_1$$

Assuming the block is attached with the 4th spring, if the blocks velocity is v, then, $v = 4v_1$

$$\therefore v_1 = v/4, \quad v_2 = \frac{v}{2}, \quad v_3 = \frac{v}{3}, \ldots$$

The kinetic energy will then be given by,

$$K_s = \sum_{i=1}^{4} \frac{1}{2}\frac{M_s}{4}(v_i)^2 = \frac{1}{2}\cdot\frac{M_s}{4}\sum_i v_i^2$$

$$= \frac{1}{2}\frac{M_s}{4}\sum_i \left(\frac{i}{4}v\right)^2$$

Say, the spring has a length l, and let the distance measured from a fixed end be s $(0 \leq s \leq l)$. Consider an element of spring lying between s and $s+ds$. Its mass is given by,

$$dM = \frac{M}{l}ds$$

The velocity of this segment will be given by,

$$v_s = \frac{s}{l}v \quad \text{(Compare with the discrete case)}$$

$$\therefore dK_s = \frac{1}{2}\left(\frac{M}{l}ds\right)\left(\frac{s}{l}v\right)^2$$

$$= \frac{M}{2l^3}v^2 s^2 ds$$

Then, total kinetic energy of the spring, $K_s = \int dK$

$$= \frac{M}{2l^3}v^2\int_0^l s^2 ds$$

$$= \frac{1}{6}Mv^2$$

Now, total energy, $\frac{1}{2}mv^2 + \frac{1}{6}Mv^2 + \frac{1}{2}kx^2 = E$

$$\Rightarrow \frac{1}{2}\left(m+\frac{M}{3}\right)v^2 + \frac{1}{2}kx^2 = E$$

which will give an expression for ω as, $\omega = \sqrt{\frac{k}{m+M/3}}$