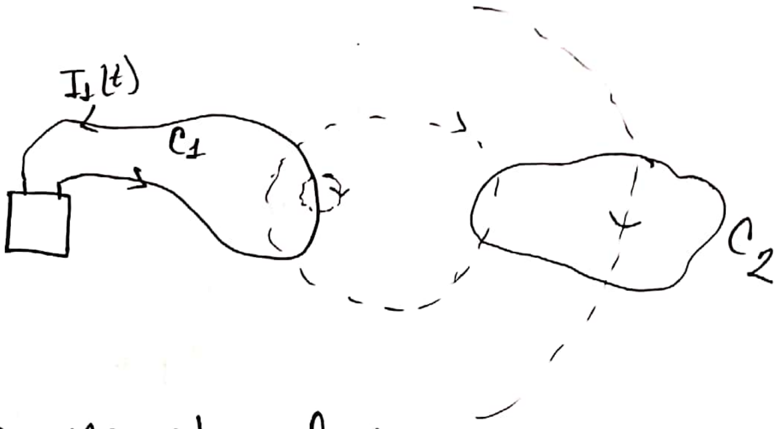
Lecture 17

Mutual inductance

Consider two conductor loops, C_1 and C_2 are fixed in position relative to one another. C_1 is connected to a battery and a current I is passing through it, a variable time-dependent current.



Say, the magnetic field due to current I_1 in the neighbourhood of C_1 is given by $\vec{B_1}$. The flux through C_2 due to this field is given by,

$$\phi_{2,1} = \int_{S_2} \vec{B_1} \cdot d\vec{A}$$

with S_2 being the surface whose boundary is C_2.

If the shape and relative position of the loops constant, the flux should be proportional to I_1.

$$\therefore \phi_{21} \propto I_1 \Rightarrow \phi_{21} = M_{21} I_1$$

Since the magnetic field itself is proportional to the current.

The constant of proportionality is called the mutual inductance. Now, say, the current is changing, and so, the flux through C_2 should also change, and an emf will be induced at C_2.

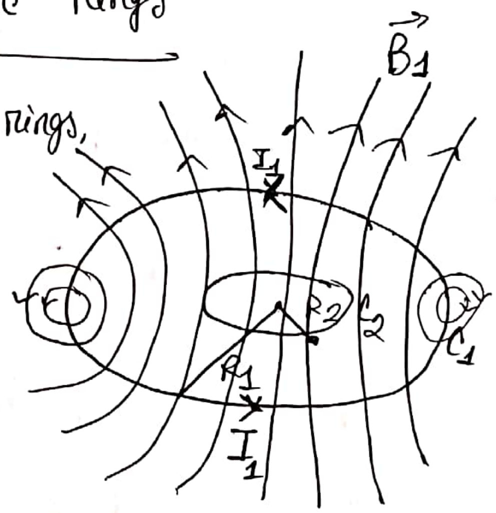
$$\mathcal{E}_{21} = -\frac{d\Phi_{21}}{dt} = -M_{21}\frac{dI_1}{dt}$$

Mutual inductance M_{21} depends on the geometry of the loops. The unit is Henry.

$$1 \text{ henry} = 1 \frac{\text{Volt. Second}}{\text{Ampere}} = 1 \text{ ohm-second}.$$

Mutual inductance for concentric rings

Consider two coplanar concentric rings, a small ring C_2 and a much larger ring C_1, assuming $R_2 \ll R_1$.



The mutual inductance,

$$M_{21} = \frac{\Phi_{21}}{I_1}$$

Now, $\Phi_{21} = \iint\limits_S \vec{B}\cdot d\vec{A}$. Since $R_2 \ll R_1$, the magnetic field is nearly uniform on S_2 and can be approximated by the magnetic field at the center.

At the center, $\vec{B} = \frac{\mu_0 I_1}{2R_1} \hat{k}$

$$\therefore \Phi_{21} = B \iint_{S_2} dA = \frac{\mu_0 I_1}{2R_1} \times \pi R_2^2 = \frac{\mu_0 \pi R_2^2}{2R_1} I_1$$

$$\therefore M_{21} = \frac{\mu_0 \pi R_2^2 I_1}{2R_1 I_1} \qquad \boxed{\therefore M_{21} = \frac{\mu_0 \pi R_2^2}{2R_1}}$$

$$\therefore \mathcal{E}_{21} = -M_{21} \frac{dI_1}{dt} = -\frac{\mu_0 \pi R_2^2}{2R_1} \frac{dI_1}{dt}$$

If current I_2 was flowing through C_2, then,

$$\Phi_{12} \propto I_2 \Rightarrow \Phi_{12} = M_{12} I_2$$

$$\boxed{\therefore \mathcal{E}_{12} = -M_{12} \frac{dI_2}{dt}}$$

Now, for a pair of circuits, M_{21} and M_{12} are equal.

<u>Reciprocity theorem:</u> For any two circuits, $M_{21} = M_{12}$.

<u>Proof:</u>

$$M_{21} = M_{12}$$

$$\Rightarrow \frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2} \Rightarrow \frac{\iint_{S_2} \vec{B}_1 . d\vec{A}}{I_1} = \frac{\iint_{S_1} \vec{B}_2 . d\vec{A}}{I_2}$$

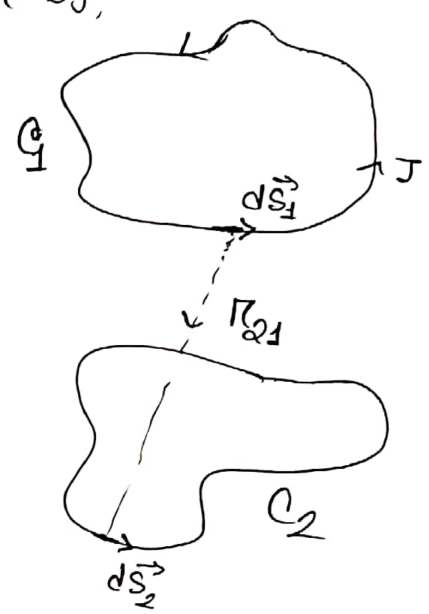
Let's use Stoke's theorem using vector potential.

$$\oint_C \vec{A}\cdot d\vec{s} = \iint_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a}$$

$$= \iint_S \vec{B}\cdot d\vec{a}$$

So, the line integral of vector potential ~~is~~ around a closed loop is equal to the surface integral of magnetic field on the surface enclosed by that loop.

Now, the vector potential is given by,



$$\vec{A}_{21} = \frac{\mu_0}{4\pi}\oint_{C_1} \frac{I_1\, d\vec{s}_1}{r_{21}}$$

Now, flux $\Phi_{21} = \iint_{S_2} \vec{B}_1\cdot d\vec{a}$

$$= \oint_{C_2} \vec{A}_{21}\cdot d\vec{s}_2$$

$$= \oint_{C_2} d\vec{s}_2 \cdot \frac{\mu_0}{4\pi}\oint \frac{I_1\, d\vec{s}_1}{r_{21}} = \frac{\mu_0 I_1}{4\pi}\oint_{C_2}\oint_{C_1} \frac{d\vec{s}_2\cdot d\vec{s}_1}{r_{12}}$$

Similarly, $\Phi_{12} = \frac{\mu_0 I_2}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\vec{s}_1\cdot d\vec{s}_2}{r_{21}}$

Now, $r_{12} = r_{21}$, since its just the distance (scalar). The double integral is just then to take ~~two~~ a pair of line elements, scalar product between divide them by the distance and sum over all such pairs. Since its a scalar product, the order should not affect and so,

$$M_{21} = M_{12} .$$

Self inductance

When the current through ~~E_1~~ C_1 is changing, the flux through it changes too. So, an electromotive force is induced on the loop C_1, itself.

$$\mathcal{E}_{11} = - \frac{d\Phi_{11}}{dt}$$

where Φ_{11} is the flux through C_1 of the field $\vec{B_1}$ due to the current I_1.

$$\text{Now,} \quad \Phi_{11} \propto I_1$$

$$\therefore \Phi_{11} = L_1 I_1 \qquad \therefore \cancel{\mathcal{E}_{11} = -L_1 \, d\Phi}$$

$$\therefore \cancel{\mathcal{E}_{11} = -L_1 \, d\Phi}$$

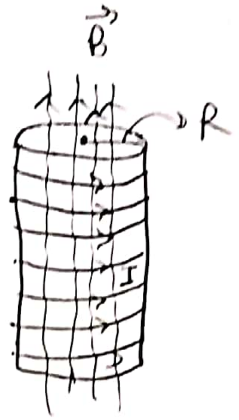
$$\therefore \mathcal{E}_{11} = -L_1 \frac{dI_1}{dt}$$

The constant L_1 is called the self inductance of the circuit. Lets drop the 1's and we have,

$$\mathcal{E} = -L \frac{dI}{dt}$$

$$\text{and} \quad L = \frac{\Phi}{I}$$

Inductance of a solenoid



$$\vec{B} = \mu_0 n I \hat{k}$$

with $n =$ number of turns per unit length.

$$\therefore \vec{B} = \frac{\mu_0 N I}{l} \hat{k}$$

Now,
$$L = \frac{\Phi}{I} = \frac{N \iint_S \vec{B} \cdot d\vec{A}}{I}$$

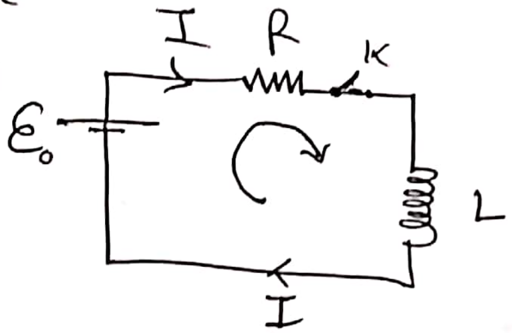
where $\iint_S \vec{B} \cdot d\vec{A}$ is the flux through ~~one~~ the surface of one loop

$$= \frac{N \iint_S \frac{\mu_0 N I}{l} \, dA}{I}$$

$$\therefore L = \frac{\mu_0 N^2 \pi R^2}{l}$$

Circuits containing self inductance

A circuit with just resistance and an emf source will drive a current $I = \frac{\mathcal{E}_0}{R}$ immediately after the switch is plugged.



But if we have an inductor in the circuit, then —

$$\mathcal{E}_0 - IR - L\frac{dI}{dt} = 0$$

If current is increasing, then to resist its increase, the upper end will be positive and lower end will be negative.

$$\Rightarrow L\frac{dI}{dt} = \mathcal{E}_0 - IR$$

$$\Rightarrow \int_0^{I(t)} \frac{dI}{\mathcal{E}_0 - IR} = \int_0^t \frac{1}{L} dt$$

$$\Rightarrow \int_{\mathcal{E}_0}^{\mathcal{E}_0 - I(t)R} \frac{-\frac{1}{R}}{u}\,du = \frac{1}{L}\int_0^t dt$$

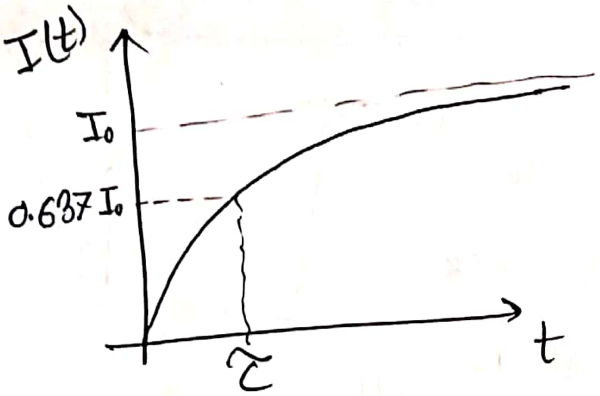
Let,
$u = \mathcal{E}_0 - IR$
$\Rightarrow \frac{du}{dI} = -R$
$I = 0, \quad u = \mathcal{E}_0$
$I = I(t), \quad u = \mathcal{E}_0 - I(t)R$

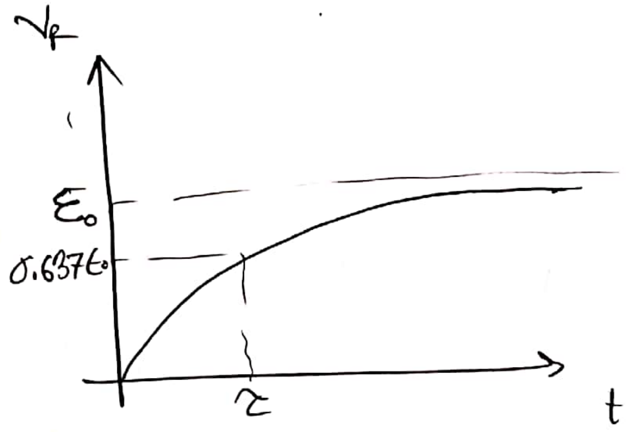
$$\Rightarrow \ln|u| \Big|_{\mathcal{E}_0}^{\mathcal{E}_0 - I(t)R} = -\frac{R}{L}t$$

$$\Rightarrow \ln \frac{\mathcal{E}_0 - I(t)R}{\mathcal{E}_0} = e^{-\frac{R}{L}t}$$

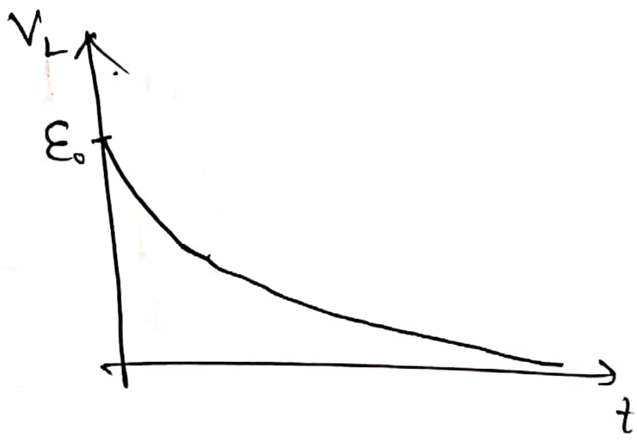
$$\therefore I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\boxed{\therefore I(t) = I_0\left(1 - e^{-t/\tau}\right)}$$ with $\tau = \frac{L}{R}$





$$V_R = I(t)R = \mathcal{E}_0\left(1 - e^{-t/\tau}\right)$$



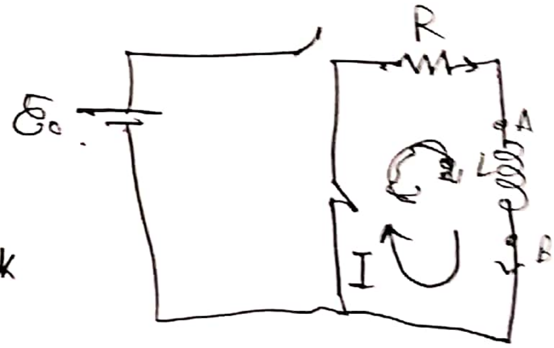
$$V_L = -L\frac{dI}{dt}$$

$$= -L\,\frac{d}{dt} I_0\left[0 + \frac{1}{\tau}e^{-t/\tau}\right]$$

$$= \cancel{-L\cdot IR} = -L\, I_0\, \frac{R}{L}\, e^{-t/\tau}$$

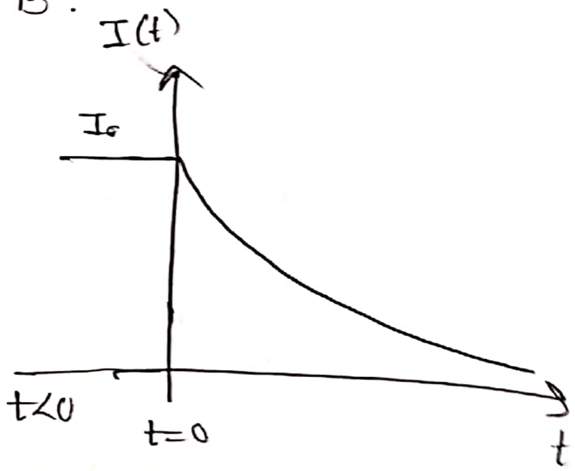
$$\boxed{\therefore V_L = \mathcal{E}_0 e^{-t/\tau}}$$

Now, let's try to discharge the circuit. If after full charging, we open the switch to drop the current from I_0 to 0, the back emf will be $L\frac{dI}{dt} \to \infty$. This could be catastrophic, not only from mathematical point of view. The high back emf will cause a spark between the open ends to keep the current from dropping, and people have been killed by this. Rather, we create an alternative path disconnecting the battery like shown in the figure



Now, since current tries to decrease, the positive of back emf is in the lower end B.

$$\therefore -IR - L\frac{dI}{dt} = 0$$



$$\Rightarrow \int_{I_0}^{I(t)} \frac{dI}{I} = -\frac{L}{R}\int_0^t dt$$

$$\Rightarrow \ln\frac{I(t)}{I_0} = -\frac{L}{R}t$$

$$\boxed{\therefore I(t) = I_0 e^{-t/\tau}}$$

Energy stored in magnetic field

$$dU = \oint I^2 R\, dt$$ (Energy dissipated through R)

$$\therefore U = \int_0^t I^2 R dt = \int_0^t I_0^2 e^{-\frac{2R}{L}t} R dt$$

$$= I_0^2 R \left. \frac{e^{-\frac{2R}{L}t}}{-2R/L} \right|_0^t$$

$$= -\frac{1}{2} I_0^2 L \left[\cancel{1} - e^{-\frac{2R}{L}t} - 1 \right]$$

$$\therefore U = \frac{1}{2} L I_0^2 \left[1 - e^{-\frac{2R}{L}t} \right]$$

After infinite time, $t \to \infty$, $U = \frac{1}{2} L I_0^2$

This is the energy stored in the magnetic field of the inductor. This is the same amount of work done by the battery to develop the current against the back emf

Now, for an inductor, $B = \frac{\mu_0 N I_0}{l} \Rightarrow I_0 = \frac{Bl}{\mu_0 N}$

and $L = \frac{\mu_0 N^2 \pi R^2}{l}$

$$\therefore U = \frac{1}{2} \frac{\mu_0 N^2 \pi R^2}{l} \times \frac{B^2 l^2}{\mu_0^2 N^2}$$

$\pi R^2 l$ = volume

$$\boxed{\therefore U = \frac{1}{2\mu_0} B^2 \times \text{volume}}$$

$\therefore$ Energy stored per unit volume is,

$$U' = \frac{1}{2\mu_0} B^2$$

To find total energy,

$$\boxed{U' = \frac{1}{2\mu_0} \iiint_{\text{All space}} B^2 \, d\tau}$$

<u>LC oscillating circuit</u>