Group Theory
Lecture # 4

Ref: Jones.

Group Homomorphism:

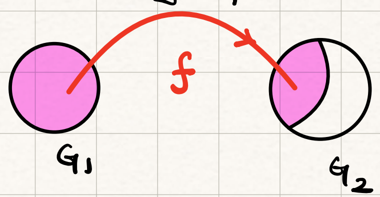
Let G_1 & G_2 be two groups. If $f : G_1 \to G_2$ s.t. for $g_1, g_1' \in G_1$
$f(g_1)$ & $f(g_1') \in G_2$ is such that $f(g_1) \circ_{G_2} f(g_1') = f(g_1 \circ_{G_1} g_1')$
then f is called a group homomorphism.

Comments:

1. Group hom is a map that preserves the group structure.
2. Group hom doesn't have to be one-to-one.
3. Group hom maps e_1 to e_2 [show]

Defn: Im f $= \{ g_2 \in G_2 \mid g_2 = f(g_1) \ \forall \ g_1 \in G_1 \}$.

Show: The image of f is a subgroup of G_2

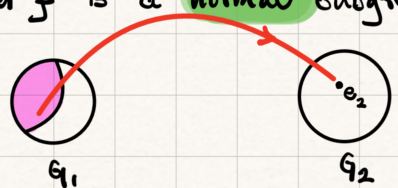


[Shaded area is Im f]

Defn. Ker f $= \{ g_1 \in G_1 \mid f(g_1) = e_2 \}$

Show:

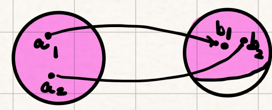
The Kernel f is a normal subgroup of G_1.



[Shaded area is Ker f]

Comments:

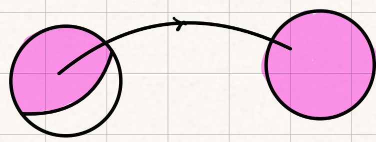
1. A hom is 1-to-1 if for each $a \in A$, f maps to a unique element in B. i.e. If If $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$. 1-to-1 map are also known as injective maps (or injections).



2. A hom is onto if the image of f is the full set B.

   $\forall\, b \in B\ \exists\ a \in A$ st $f(a) = b$.

   An onto map is also known as a surjective map or a surjection.



3. If a map is one-to-one and onto then the map is invertible. Such maps are called bijective maps or bijections. Thus if $f: A \to B$ is 1-to-1 & onto then $\exists\ f^{-1}: B \to A$.

   A hom that is bijective is called an isomorphism.

An Example

Let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z} = \{\underline{0}, \underline{1}, \underline{2}, \underline{3}\}$. This set forms a group under addition mod 4.

Now let us consider the group $(\mathbb{Z}/5\mathbb{Z})^{\times}$. This is the multiplication group mod 5.

The elements of this group consists of the four equivalence classes:

$\bar{1} = \cdots, -4, 1, 6, 11, 16, \ldots$

$\bar{2} = \cdots, -3, 2, 7, 12, 17, \ldots$

$\bar{3} = \cdots, -2, 3, 8, 13, 18, \ldots$

$\bar{4} = \cdots, -1, 4, 9, 14, 19, \ldots$

Group multiplication is usual multiplication between representives of the equivalence classes:

$$\bar{2} \cdot \bar{3} = 7 \times 8 = 56 = \bar{1}$$

Multiplication table for $\mathbb{Z}_4$:

+	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{0}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{0}$	$\underline{1}$
$\underline{3}$	$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$

$\Rightarrow$

·	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

with $e = \underline{0}$, $a = \underline{1}$, $b = \underline{2}$, $c = \underline{3}$

Now consider the multiplication table for $(\mathbb{Z}/5\mathbb{Z})^{\times}$

$\times$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

=

$\times$	$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{3}$
$\bar{2}$	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{1}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$	$\bar{2}$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{2}$	$\bar{4}$

=

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

with $e = \bar{1}$, $a = \bar{2}$, $b = \bar{4}$, $c = \bar{3}$

Thus we see that the map $f : \mathbb{Z}_4 \to (\mathbb{Z}/5\mathbb{Z})^{\times}$, $f: \underline{0} \mapsto \bar{1}$, $\underline{1} \mapsto \bar{2}$, $\underline{2} \mapsto \bar{4}$, $\underline{3} \mapsto \bar{3}$

forms an isomorphism.

The First Isomorphism Theorem:

Recall the two exercises:

If $f: A \to B$ is a group hom then:

i) The $\operatorname{Im} f$ is a subgroup of B and

ii) The $\operatorname{Ker} f$ is a normal subgroup of A.

These two results allow us to prove the **first isomorphism theorem**:

Let $f: G \to G'$ be a group hom between two groups G & G'. Let $\operatorname{Ker} f \subset G$ be denoted as K. Then the first isomorphism theorem states that:

$$G/K = \operatorname{Im} f$$

Proof: We know that K is a normal subgroup. Thus G/K is a quotient group. On the other hand we know that $\operatorname{Im} f \subset G'$ is also a group.

Elements of G/K are cosets of the form gK. On the other hand, elements of $\operatorname{Im} K$ are of the form $f(g)$. Thus the proof of the theorem boils down to proving the following facts:

i) Isomorphism $f(g) \longleftrightarrow gK$

ii) Checking that the map preserves the group structure

i) 1. The map $\mathrm{Im} f \to G/K$ is well defined.

If $f(g_1) = f(g_2) \Rightarrow g_1 K = g_2 K$ i.e. $g_1 \sim g_2$.

Proof of i) 1.: $f(g_1) = f(g_2)$

$$[f(g_1)]^{-1} f(g_1) = [f(g_1)]^{-1} f(g_2)$$

$$e' = [f(g_1)]^{-1} f(g_2)$$

$$e' = f(g_1^{-1} g_2) \quad \{\text{since } f \text{ is a group hom.}\}$$

This means that $g_1^{-1} g_2$ is in the kernel of f. $g_1^{-1} g_2 \in K \Rightarrow g_1^{-1} g_2 = k$ for some $k \in K$. $\Rightarrow g_2 = g_1 k \Rightarrow g_2 \sim g_1$.

2. The map $G/K \to \mathrm{Im} f$ is well defined.

Proof: If $g_1 \sim g_2$ then $g_1 = g_2 k$ for some $k \in K$. Then

$$f(g_1) = f(g_2 k)$$

$$= f(g_2) f(k)$$

$$= f(g_2) e'$$

$$= f(g_2).$$

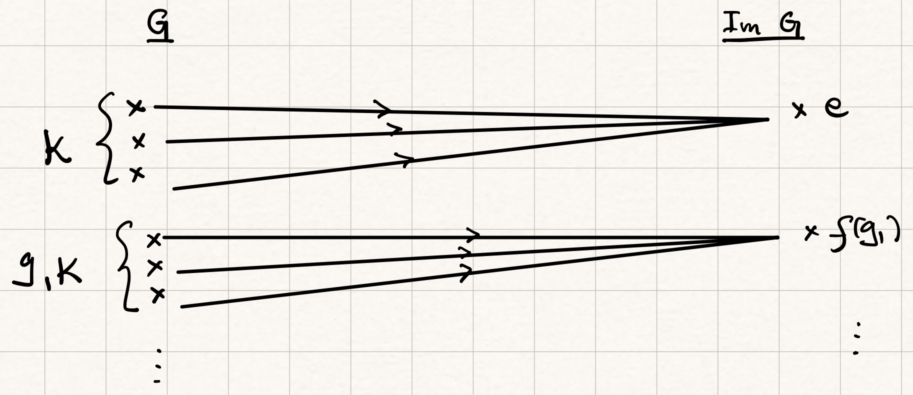
ii) Group structure:

$$(g_1 K) \cdot (g_2 K) = g_1 g_2 K$$

$$f(g_1) f(g_2) = f(g_1 g_2)$$

Comments:

1. The Geometric interpretation of the Theorem is as follows:



2. In the proof of the 1st isomorphism Theorem there are <u>two</u> notions of gp homs that are being used. A given gp hom f which is being used to build an induced isomorphism between $G/\ker f$ and $\operatorname{Im} f$.

An Application:

Let $G_1 = SU(2)$ and $G_2 = SO(3)$. Then the homomorphism:

$$f : SU(2) \to SO(3)$$

$$f : u \mapsto R(u) \text{ given by } u(\vec{\sigma}\cdot\vec{x})u^{\dagger} = \vec{\sigma}\cdot(R(u)\,\vec{x}) \quad \forall\, \vec{x} \in \mathbb{R}^3$$

has as its Kernel $K = \mathbb{Z}_2 < SU(2)$. Thus by the first isomorphism theorem we have $SU(2)/\mathbb{Z}_2 \cong SO(3)$.