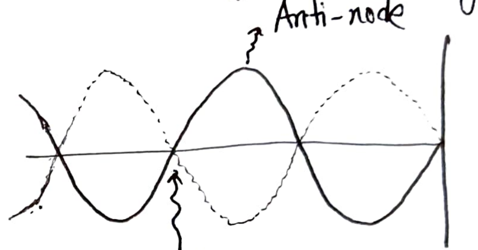
Lecture 14

Reflection and standing wave from semi-infinite String — fixed end

Consider a rightward moving sinusoidal travelling wave. The string is semi-infinite, meaning its one end is at $x=0$, which is a fixed end and goes of to infinity $x<0$. The general form of a sinusoidal rightward moving travelling wave is given by,



$$\Psi_i(x,t) = A\sin(\omega t - kx + \phi)$$

Since it will hit the rigid wall, the reflected wave is given by,

$$\Psi_r(x,t) = -\Psi_i(-x,t) = -A\sin(\omega t + kx + \phi)$$

So, the total wave is,

$$\begin{aligned}
\Psi(x,t) &= \Psi_i(x,t) + \Psi_r(x,t) \\
&= A\sin(\omega t - kx + \phi) - A\sin(\omega t + kx + \phi) \\
&= 2A\cos\left[\frac{2\omega t + 2\phi}{2}\right]\sin\left[\frac{-2kx}{2}\right] \\
&= -2A\sin(kx)\cos(\omega t + \phi)
\end{aligned}$$

The resultant wave is a standing wave, that just oscillates up and down with the same phase. The

boundary condition is satisfied since at $x=0$,
$\psi(x,t)=0$ for all t.

Free end

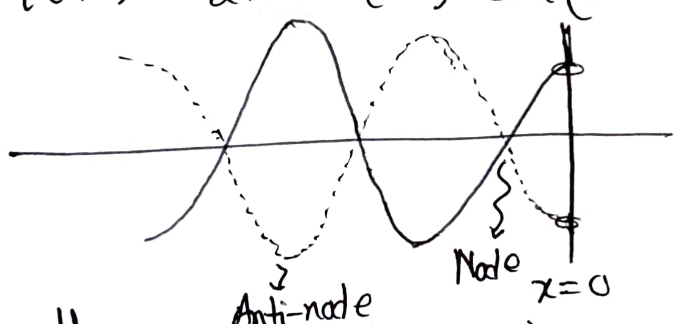
We connect one end of a semi-infinite string with a massless ring that can slide across a frictionless rod. The rod is necessary for creating the tension in the string. The total wave now is,

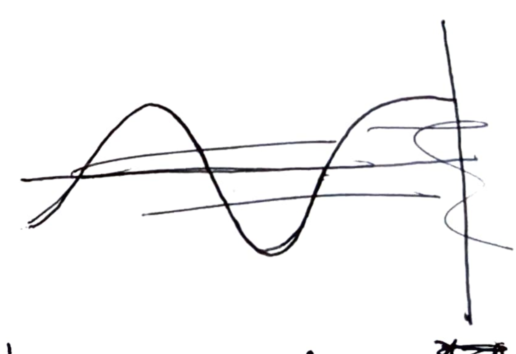
$$\psi(x,t) = \psi_i(x,t) + \psi_R(x,t)$$

$$= A\sin(\omega t - kx + \phi) + A\sin(\omega t + kx + \phi)$$

$$= 2A\sin\left(\frac{2\omega t + 2\phi}{2}\right)\cos\left(\frac{-2kx}{2}\right)$$

$$\therefore \psi(x,t) = 2A\cos(kx)\sin(\omega t + \phi)$$

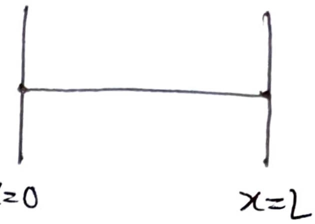




The resulting wave is again a standing wave of the form shown in the figure. You can see that the free massless ring slides across the rod as a function of $\cos(kx)$.

Normal modes in a string connected to two fixed end

Consider the string connected to two fixed ends at $x=0$ and $x=L$. So, the boundary conditions are,



$$\psi(0,t) = \psi(L,t) = 0$$

Now, we found in Lecture 12 that, the most general solution to the wave equation for an elastic string is given by,

$$\psi(x,t) = D_1 \cos kx \cos\omega t + D_2 \cos kx \sin\omega t + D_3 \sin kx \sin\omega t + D_4 \sin kx \cos\omega t$$

Now, $\psi(0,t) = D_1 \cos\omega t + D_2 \sin\omega t$

$$\Rightarrow \quad 0 = D_1 \cos\omega t + D_2 \sin\omega t$$

Since the equality is valid for all t, the co-efficients D_1 and D_2 must be zero. You can also argue from the orthogonality of $\sin\omega t$ and $\cos\omega t$. If $\cos\omega t$ and $\sin\omega t$ forms the basis, then if $D_1 \cos\omega t + D_2 \sin\omega t = 0$, it does mean $D_1 = D_2 = 0$.

Think about orthogonal bases in 3D Cartesian coordinate $\hat{i}$, $\hat{j}$ and $\hat{k}$. If $\cancel{a\hat{i}+5\hat{j}\ 6\hat{k}}$ $a\hat{i}+b\hat{j}+c\hat{k}=\vec{0}$, then $a=b=c=0$.

Now, $\Psi(x,t) = D_3 \sin kx \sin \omega t + D_4 \sin kx \cos \omega t$

Again, $\Psi(L,t) = 0$

$$\Rightarrow (D_3 \sin \omega t + D_4 \cos \omega t) \sin kL = 0$$

If $D_3 \sin \omega t + D_4 \cos \omega t = 0$, then $D_3 = D_4 = 0$ and we do not have any oscillation. So,

$$\sin kL = 0 \Rightarrow \sin kL = \sin n\pi$$

$$\therefore K = n\frac{\pi}{L} \quad \text{for} \quad n = 0, 1, 2, \cdots$$

For each value of ~~k~~ n, we get a different k. Let's label them as, $K_n = \frac{n\pi}{L}$

Now, $V = \frac{\omega}{k} \Rightarrow \omega = VK$

$$\boxed{\therefore \omega_n = \frac{n\pi V}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}}$$

Each of the ω_n represents a normal mode of the string. Clearly,

$$\omega_n = n\omega_1 \quad \text{if} \quad \omega_1 = \frac{\pi}{L}\sqrt{\frac{T}{\mu}}$$

So, there is a fundamental angular frequency corresponding to the lowest normal mode. All other frequencies are just integer multiples of ω_1.

Now, for the n^{th} normal mode,

$$\psi_n(x,t) = (D_{3,n} \sin\omega_n t + D_{4,n} \cos\omega_n t) \sin k_n x$$

$$\boxed{\therefore \psi_n(x,t) = D_n \sin(\omega_n t + \phi_n) \sin\left(\frac{n\pi}{L} x\right)} \quad (1)$$

The most general wave on the string is obviously found by linear combination of all the normal modes.

$$\therefore \psi(x,t) = \sum_{n=0}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) \sin(\omega_n t + \phi_n)$$

The $n=0$ mode contributes nothing here though.

Now, for a particular mode, (1) describes standing waves. The points reaches their maximum at the same time and also crosses zero displacement at the same time. So, the modes describes standing waves. Each of the normal modes represents different standing waves that are possible to accomodate in the string.

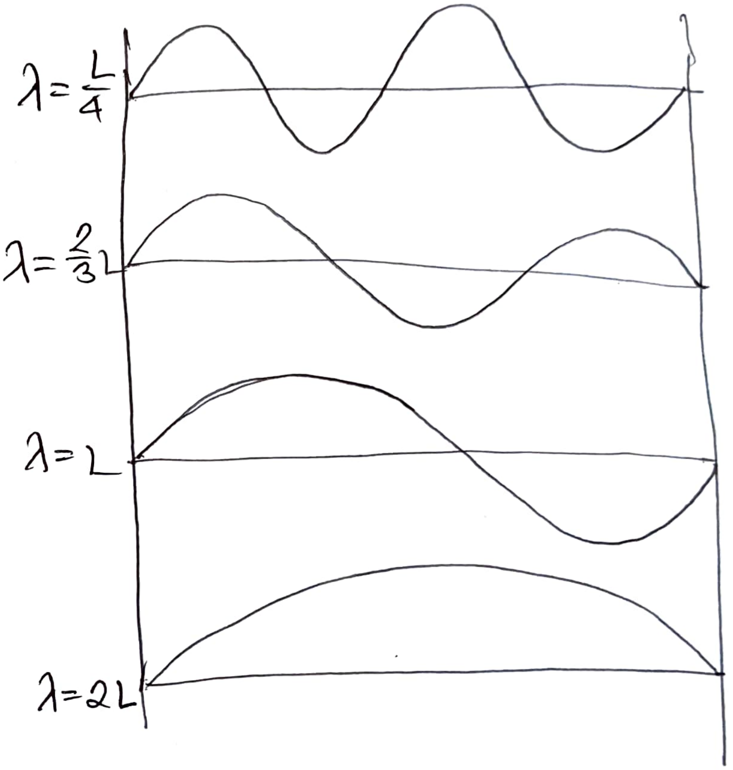
To get a view of how the normal modes looks like, consider the wavelengths.

$$K = \frac{2\pi}{\lambda}$$

$$\Rightarrow \frac{n\pi}{L} = \frac{2\pi}{\lambda_n} \qquad \boxed{\therefore \lambda_n = \frac{2L}{n\cancel{L}}}$$

~~For n=1, $\lambda = \frac{2}{L}$~~

$\therefore$ ~~$L = \frac{2}{\lambda}$~~



For $n=1$, $\lambda = 2L$

$\therefore L = \frac{\lambda}{2}$

So, only a half of the wavelength can be possible.

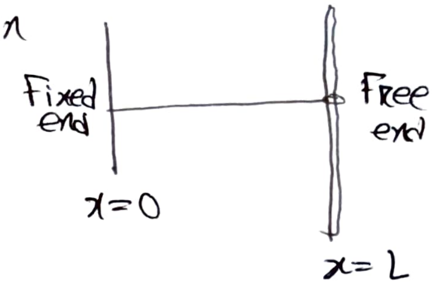
For $n=2$, $\lambda = L$.

$\therefore L = \lambda = 2 \cdot \frac{\lambda}{2}$

So, two half wavelengths, as in one complete wavelength is possible and so on. The figure shown the $n=1,2,3$ and 4 normal modes. All of them describes standing waves, where the particles just oscillate up and down in phase with ~~the same~~ different amplitudes given by $A\sin(\frac{n\pi}{L}x)$.

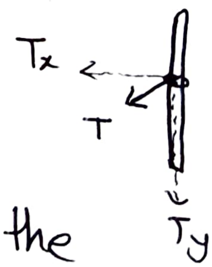
One fixed and another free end

In this case, one boundary condition is obvious.



$$\psi(0,t) = 0 \quad \text{—(i)}$$

Think about the boundary condition at $x=L$. The ring is massless. So, there will be a zero net force on it in both transverse and horizontal direction. There is no transverse force on the ring since the rod is frictionless. For there to be zero horizontal force, the end of the ring must be horizontal always. If the end of the string is not horizontal, that is make some angle with the horizontal axis, then there must be some transverse force, which can't be nullified. The horizontal force is be nullified by the force from the rod. So, the end of the string must be horizontal.



$$\therefore \left.\frac{\partial \psi}{\partial x}\right|_{x=L} = 0 \quad \text{—(ii)}$$

Using first boundary condition like before we get,

$$\psi(x,t) = (D_3 \sin \omega t + D_4 \cos \omega t) \sin kx$$

Now, $$\frac{\partial \Psi}{\partial x} = \left(D_3 \sin \omega t + D_4 \cos \omega t\right) K \cos Kx$$

$$\left.\frac{\partial \Psi}{\partial x}\right|_{x=L} = 0$$

$$\Rightarrow \left(D_3 \sin \omega t + D_4 \cos \omega t\right) K \cancel{\sin k} \cos KL = 0$$

$$\Rightarrow \cos KL = 0$$

$$\Rightarrow \cos KL = \cos(2n+1)\frac{\pi}{2} \qquad \boxed{\therefore K_n = (2n+1)\frac{\pi}{2L}}$$

with $n = 0, 1, 2, \ldots$

$$\boxed{\therefore \omega_n = (2n+1)\frac{\pi v}{2L}} \rightarrow \text{Normal mode frequencies}$$

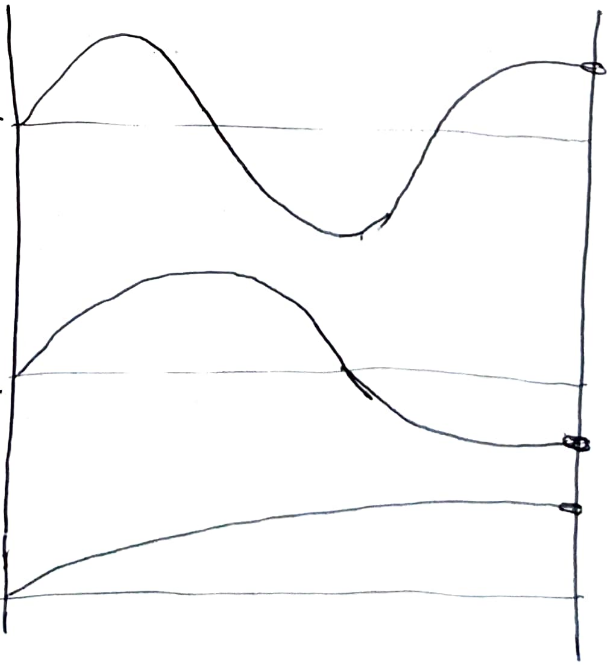
Again, $$\lambda_n = \frac{2\pi}{K_n} \qquad \boxed{\therefore \lambda_n = \frac{4L}{2n+1}}$$

The general solution is: $$\Psi(x,t) = \sum_{n=0}^{\infty} D_n \sin(\omega_n t + \phi) \sin(k_n x)$$

$\rightarrow L = \frac{5}{4}\lambda_2,\ n = 2,\ \lambda_2 = \frac{4}{5}L$

$\rightarrow L = \frac{3}{4}\lambda_1,\ n = 1,\ \lambda_1 = \frac{4}{3}L$

$L = \frac{\lambda_0}{4} \leftarrow n = 0,\ \lambda_0 = 4L$



H.W. : Two free ends.

Forced harmonic vibrations of stretched string

We have seen that the vibrations of string with end/s fixed are limited to fundamental frequencies and their integral multiples. But just like in SHM we introduced forced vibrations, with the help of which we could oscillate a system with a driven frequency, we are going to do so here.

Consider that the string is fixed at $x = L$ and at $x = 0$ someone is wiggling the string sinusoidally with an amplitude B. The boundary conditions now are -

$$\psi(0,t) = B\cos\omega t \quad \text{and} \quad \psi(L,t) = 0$$

Since the wave equation still holds, in the steady state we expect the solution of the form,

$$\psi(x,t) = f(x)\cos\omega t$$

Since $f(x)$ is not zero at both ends, we simply can't take $f(x) = A\sin kx$. Rather, we take,

$$f(x) = A\sin(kx+\alpha)$$

$$\therefore \psi(x,t) = A\sin(kx+\alpha)\cos\omega t$$

Now, $\psi(L,t) = 0$

$$\Rightarrow \quad A \sin(KL+\alpha) \cos\omega t = 0$$

This must be true for all times.

$$\therefore \sin(KL+\alpha) = 0 \quad \text{is a must.}$$

$$\Rightarrow \sin(KL+\alpha) = \sin n\pi$$

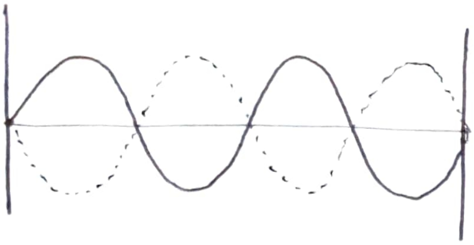
$$\Rightarrow KL+\alpha = n\pi \qquad \therefore \alpha = n\pi - KL = n\pi - L\frac{\omega}{V}$$

Again, $\psi(0,t) = B\cos\omega t$

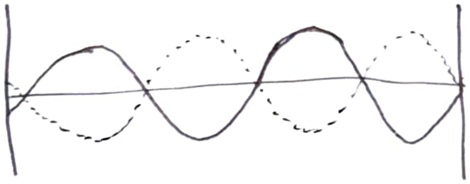
$$\Rightarrow A \sin\alpha \cos\omega t = B\cos\omega t$$

$$\Rightarrow A = \frac{B}{\sin\alpha} = \frac{B}{\sin\left(n\pi - \frac{\omega L}{V}\right)}$$

Now, whenever $n\pi - \frac{\omega L}{V} \to 0$ or $\omega \to \frac{n\pi V}{L}$, the amplitude is maximum. But $\frac{n\pi V}{L}$ is exactly the normal mode frequencies of a string connected to two fixed ends. So, the effect of the periodic driving force is then to create the standing waves, which are the normal modes. Near the resonance frequency $\omega = \frac{n\pi V}{L}$, a small driving amplitude B will be enough to excite a normal mode with substantial amplitude. Very small B means $x=0$ acts almost like a node (fixed end) and we can create as many normal modes as we want.



Fixed end



Forced vibration

Using initial conditions to predict the wave motion: Fourier analysis in action

We have solved for the wave in a string that is connected between two fixed ends. The general solution is,

$$\Psi(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t + \phi_n)$$

But, you require A_n and phase constant ϕ_n to be determined. It may at first seem tough, but you know Fourier Series now.

$$\Psi(x,0) = \sum_{n=0}^{\infty} A_n \sin\phi_n \sin\frac{n\pi x}{L}$$

$$\therefore \Psi(x,0) = \sum_{n=0}^{\infty} C_n \sin\frac{n\pi x}{L}$$

Using the idea of Fourier series, you know that the co-efficients C_n are simply,

$$C_n = \frac{2}{L}\int_0^L \Psi(x,0)\sin\left(n\pi\frac{x}{L}\right)dx$$

$$\therefore A_n \sin\phi_n = \frac{2}{L}\int_0^L \Psi(x,0)\sin\left(n\pi\frac{x}{L}\right)dx \longrightarrow (\mathrm{I})$$

And,
$$\dot{\Psi}(x,0) = \sum_{n=0}^{\infty} \omega_n A_n \sin\left(n\pi\frac{x}{L}\right)\cos(\cancel{\omega_n t}^{\,0}+\phi_n)$$

$$\Rightarrow \dot{\Psi}(x,0) = \sum_{n=0}^{\infty} D_n \sin\left(n\pi\frac{x}{L}\right)$$

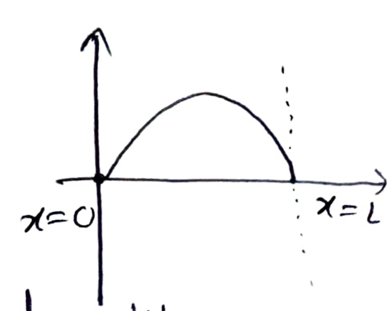
$$\therefore D_n = \frac{2}{L}\int_0^L \dot{\Psi}(x,0)\sin\left(n\pi\frac{x}{L}\right)dx$$

$$\therefore \omega_n A_n \cos\phi_n = \frac{2}{L}\int_0^L \dot{\Psi}(x,0)\sin\left(n\pi\frac{x}{L}\right)dx \longrightarrow (\mathrm{II})$$

Solving (I) and (II) gives you A_n and ϕ_n for all normal modes, and the whole pattern of the wave.

Example 1

We talked about the normal modes in the string connected to fixed ends. But how do we achieve one using initial conditions? Say, you want to find the first normal mode. For this, we make the string like a half sine curve and release it from rest.



$$\therefore \Psi(x,0) = \Psi_0 \sin\left(\pi\frac{x}{L}\right) \rightarrow \text{Initial condition}$$

and $\dot{\psi}(x,0) = 0 \longrightarrow$ Second initial condition.

From (II) we get, $\cos\phi_n = 0$

$$\therefore \phi_n = (2n+1)\frac{\pi}{2}$$

From (I) $\Rightarrow$ $A_n \sin\left[(2n+1)\frac{\pi}{2}\right] = \frac{2}{L}\psi_0\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

For $n=1$, $A_1 = \frac{2}{L}\psi_0 \int_0^L \sin\frac{\pi x}{L}\sin\frac{\pi x}{L}\,dx$

Now, $\int_0^L \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{L}{2}\delta_{1,n}$

$$\therefore A_1 = \frac{2}{L}\psi_0\cdot\frac{L}{2}\delta_{1,1} = \psi_0$$

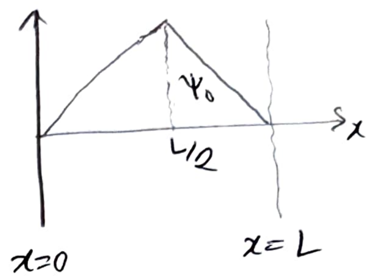
$$A_{n\neq 1} = \frac{2}{L}\psi_0\cdot\frac{L}{2}\delta_{1,n} = 0$$

$$\boxed{\therefore \psi(x,t) = \psi_0 \sin\left(\frac{\pi x}{L}\right)\cos\omega_1 t}$$

So, the wave function is the same as the first normal mode, the standing wave pattern. So, to oscillate the string in a normal mode, all you have to do is make the initial shape of the string like a sinusoidal one.

Example 2

Let's say, you pluck the string in the middle by an amount Ψ_0 and release from rest. What will be the corresponding wave.



Since we are releasing from rest,

$$\phi_n = (2n+1)\frac{\pi}{2}$$

$$\therefore \Psi(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left[\omega_n t + (2n+1)\frac{\pi}{2}\right]$$

Now, $$\Psi(x,0) = \begin{cases} 2\Psi_0 \frac{x}{L} \; ; & 0 \leq x < L/2 \\ 2\Psi_0 - 2\Psi_0 \frac{x}{L} \; ; & \frac{L}{2} < x \leq L \end{cases}$$ Slope of the string $= \frac{2\Psi_0}{L}$

Now, $$\Psi(x,0) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L}$$ $\boxed{\cos\left((2n+1)\frac{\pi}{2}\right) = 0}$

$$\therefore A_n = \frac{2}{L}\int_0^L \Psi(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} 2\Psi_0 \frac{x}{L} \sin\left(\frac{n\pi x}{L}\right) dx + \int_{L/2}^{L} \left(2\Psi_0 - 2\Psi_0 \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx\right]$$

$$= \frac{2}{L}\left[\frac{2\Psi_0}{L} \cdot \frac{L^2\left(2\sin\frac{\pi n}{2} - \pi n \cos\left(\pi\frac{n}{2}\right)\right)}{2\pi^2 n^2} - \frac{\Psi_0 L\left(2\sin\{n\pi\} - 2\sin\frac{n\pi}{2} - \pi n \cos\frac{n\pi}{2}\right)}{\pi^2 n^2}\right]$$

If $n =$ even, then, $A_n = 0$ (check).

If $n = \text{odd}$, then, $A_n = \frac{2\Psi_0}{\pi^2 n^2} \cdot 2(-1)^{n+1} + \frac{2\Psi_0}{\pi^2 n^2} \cdot 2(-1)^{n+1}$

$$\therefore A_n = \frac{8\Psi_0}{\pi^2 n^2}(-1)^{n+1}$$

$$\therefore A_1 = \frac{8\Psi_0}{\pi^2}, \quad A_3 = -\frac{8\Psi_0}{9\pi^2}, \quad A_5 = \frac{8\Psi_0}{25\pi^2}, \cdots$$

$$\boxed{\therefore \Psi(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8\Psi_0}{\pi^2 n^2}(-1)^{n+1} \sin\left(\frac{n\pi x}{L}\right) \cos(n\omega t)}$$