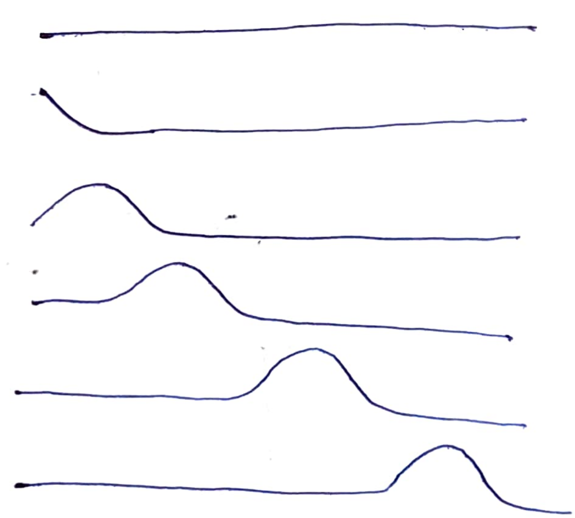
Lecture 13

Wave pulses

From our discussion so far, you may think a wave as something that involves a whole succession of crests and troughs, ~~b~~ like a sinusoidal function, that goes on forever. But, this is by no means necessary. It occurs that, there are many cases where a single, isolated ~~pul~~ pulse of disturbance, travels from one place to another through a medium. Consider an elastic rope and twitch one end and then hold still. That short twitch will now travel throughout the rope, preserving the same shape and travelling with a constant velocity.

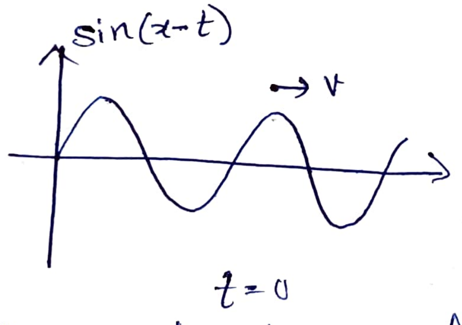


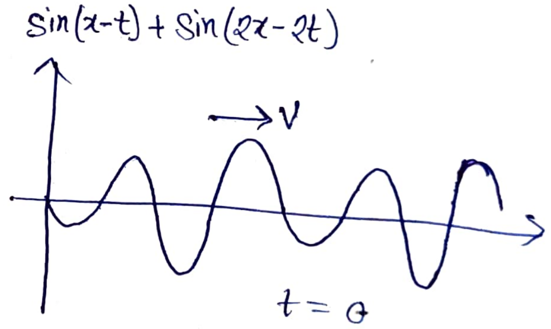
The shape of the pulse can be any shape, up to the condition that the velocity of the propagation of the pulse is determined by $v=\sqrt{\frac{T}{\mu}}$.

But, how is this a wave? And will it satisfy the wave equation?

Actually, you can construct a pulse of any shape by superimposing travelling waves of different frequencies. We have shown that the solution to the wave equation for a particular mode can be written as a linear combination of travelling waves. To get the exact mode, you have to plug in necessary boundary conditions.

Consider a travelling wave given by $A\sin(k_1 x - \omega_1 t)$, which is travelling to the right with a velocity $v = \frac{\omega_1}{k_1}$. Now the superposition of this t.w. with another t.w. given by $B\sin(k_2 x - \omega_2 t)$, will be a new travelling wave with velocity v if $v = \frac{\omega_2}{k_2}$, that is they have the same velocity.





So, the combination of travelling (sinusoidal) waves with same velocity gives you a travelling wave of a different shape. By virtue of Fourier series, you can now construct any periodic shape by superposition of the sinusoidal travelling waves. You can think the superposition of travelling waves at $t=0$. For example, the superposition shown in the second graph above at $t=0$ is $(\sin x + \sin 2x)$. Once you get the desired shape, all you have to do is add the appropriate time parameters [here it is $\sin(x-t) + \sin(2x-2t)$] to see the time evolution of the travelling wave of your desired shape. So, you can create any shape, and that shape is guranted to be a linear combination of the normal modes by virtue of Fourier series. And since the wave equation is linear, the superposition of all the normal modes is also a solution to the wave equation. So, the random shaped travelling disturbance is also a wave.

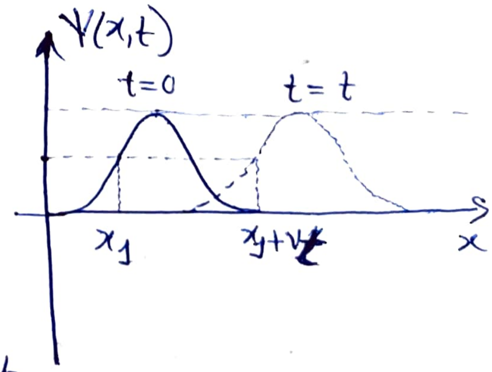
Now, for a pulse, the pattern/shape might be limited to a certain region in space (then that shape travels as time goes on). So, this isn't periodic. But of course we can think this random shaped disturbance has a very high (infinite) spatial period and write it as sum (integral) of the normal modes which then travels with time just like our previous periodic shape of finite period. This idea was captured in Fourier tansformed and we talked about how the integral of the normal modes gives you travelling wave, as a function of the form $f(x-vt)$ in the last part of lecture 12. So, pulses just fit perfectly in our description of ~~t~~ travelling wave of the form $f(x-vt)$ which preserves its shape and is travelling to the right. $f(x+vt)$ will the just ~~repo~~ represent a pulse travelling to the left.

Motion of wave pulse in constant shape

Consider a pulse moving from left to right, and at time $t=0$, it is described by the equation:

$$\psi(x,0) = f(x)$$

where $\psi(x,t)$ is the wave function that gives the displacements of particles from their equilibrium positions.



If the pulse is travelling as a whole with a velocity v, then a particular displacement that existed at x_1 will now exist at $x_2 = x_1 + vt$.

Mathematically,

$$\psi(x,t) = f(x-vt)$$

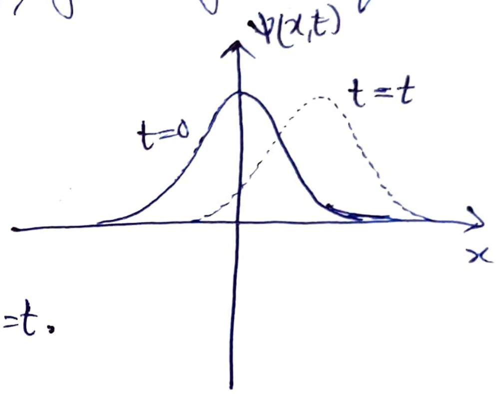
If the pulse was moving from right to left, then,

$$\psi(x,t) = g(x+vt)$$

We have already discussed in lecture 12 that any function of the form $f(x \pm vt)$ satisfies the wave equation. So, these travelling pulses describes the wave in a rope.

Consider a pulse, moving left to right, given by the equation,

$$\psi(x,t) = \frac{b^3}{b^2+(x-vt)^2} \quad ; \; b>0$$



The figure shows a sketch of the pulse at $t=0$ and at a later time $t=t$.

You can check the maximum of $\psi(x,t)$ occurs at $x=0$ at $t=0$ and has a value $=b$. It is very important to understand how the motion of a wave in the direction of propagation (here, x) is a direct consequence of particle displacement that are purely in transverse (ψ) direction to understand transverse waves. The transverse displacement of every point to the left of the peak is decreasing and to the right of the peak is increasing with time. It is an automatic consequence of these motions of the points that the peak of the pulse occurs at larger and larger value of x as time goes on.

and hence, the pulse travels to the right.

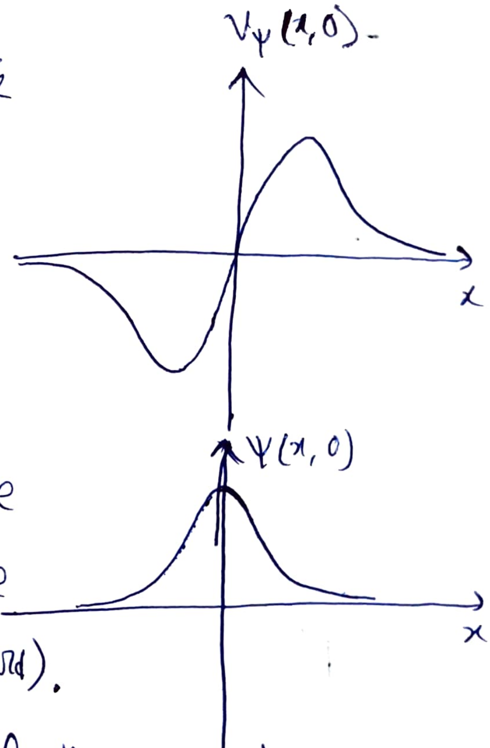
To illustrate this, let's calculate the transverse velocities of the points of the given pulse. The velocity of any point at a position x will be given by,

$$V_\psi = \frac{\partial \Psi(x,t)}{\partial t}$$

The partial derivative implies that we are considering the velocity of a particle at a fixed position x as a function of time. We get,

$$V_\psi(x,t) = \frac{2b^3(x-vt)\,v}{\left[b^2 + (x-vt)^2\right]^2}$$

At $t=0$, $$V_\psi(x,t) = \frac{2b^3 v x}{\left[b^2+x^2\right]^2}$$

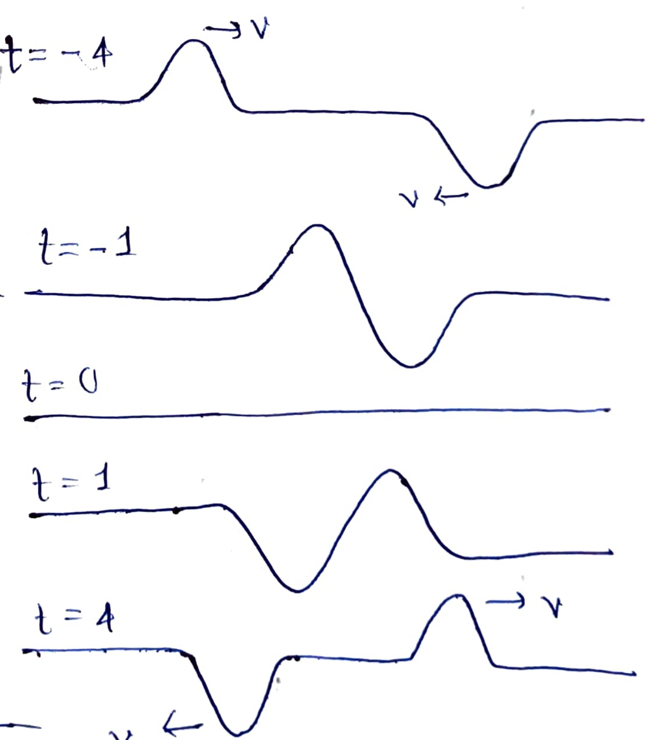


You can clearly see, the points to the left of the peak has a negative velocity (downward) and right of the peak has a positive velocity (upward).

Thus the transverse movements of the particles shifts the pulse towards right as a function of time. Of course, the velocity distribution itself is moving to right, since its a function of $(x-vt)$ too. It ensures that the lefward points to the peak always goes down and rightward points move up, to make the pulse moving to the right. You can plot $\Psi(x,t)$ in Desmos and see this visually by yourself.

Superposition of two wave pulses

What happens if we superimpose two wave pulses travelling in opposite directions? It occurs that, two travelling ~~pult~~ pulse in opposite directions can pass right through each other. Consider two symmetric pulses travelling opposite to each other. They are exactly alike, except for the fact that one is positive and another is negative. As they pass through each other, there comes a time when the displacement of the string is zero. The superposition is given by,



$$\Psi(x,t) = \frac{b^3}{b^2+(x-vt)^2} - \frac{b^3}{b^2+(x+vt)^2}$$

$\Psi(x,t)$ determines the displacements of the points at any time. At $t=0$, $\Psi(x,t)=0$. The oppositely travelling pulses were travelling, say some time $t<0$

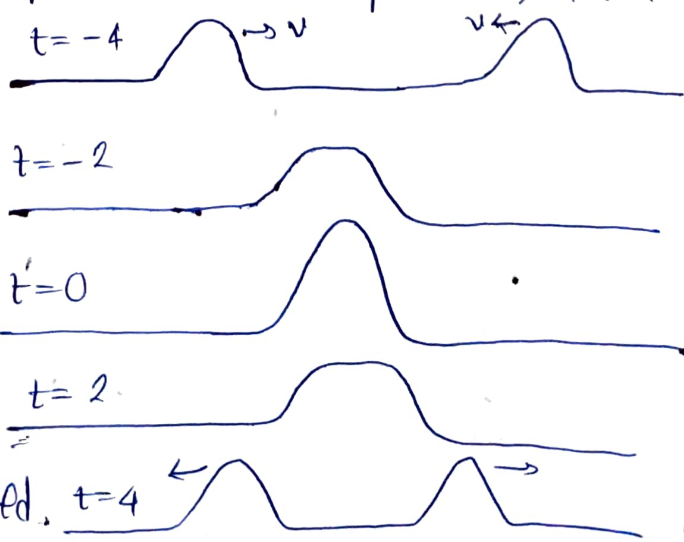
and at $t=0$, the exactly annihilate each other to have $\Psi(x,0)=0$. But each of them carry positive energies, which can't simply be washed away. You expect the pulses to reappear, and they do reappear. But what preserves the memory when the string is at zero displacement? Its the velocity. The velocity of the points is given by,

$$V_\Psi(x,t) = \frac{2b^3(x-vt)v}{[b^2+(x-vt)^2]^2} + \frac{2b^3(x+vt)v}{[b^2+(x+vt)^2]^2}$$

So, at $t=0$, although the displacements adds up to give $\Psi(x,0)=0$, the velocities really add up to give a non-zero velocity distribution. So, when $\Psi=0$, the velocities aren't, and that's why the pulses reappear.

~~Reflection of~~

However, if both the pulses were positive, then the pattern will look something shown here.

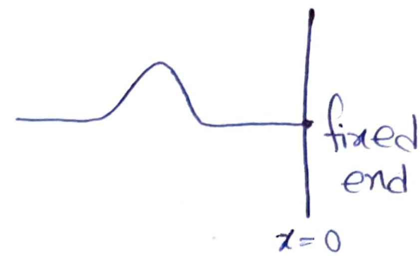


At $t=0$, now,

$$\Psi(x,0) = \frac{b^3}{b^2+x^2} + \frac{b^3}{b^2+x^2}.$$

So, your displacement gets doubled.

Reflection of wave pulses from free and fixed ends — a pictorial description with virtual pulses

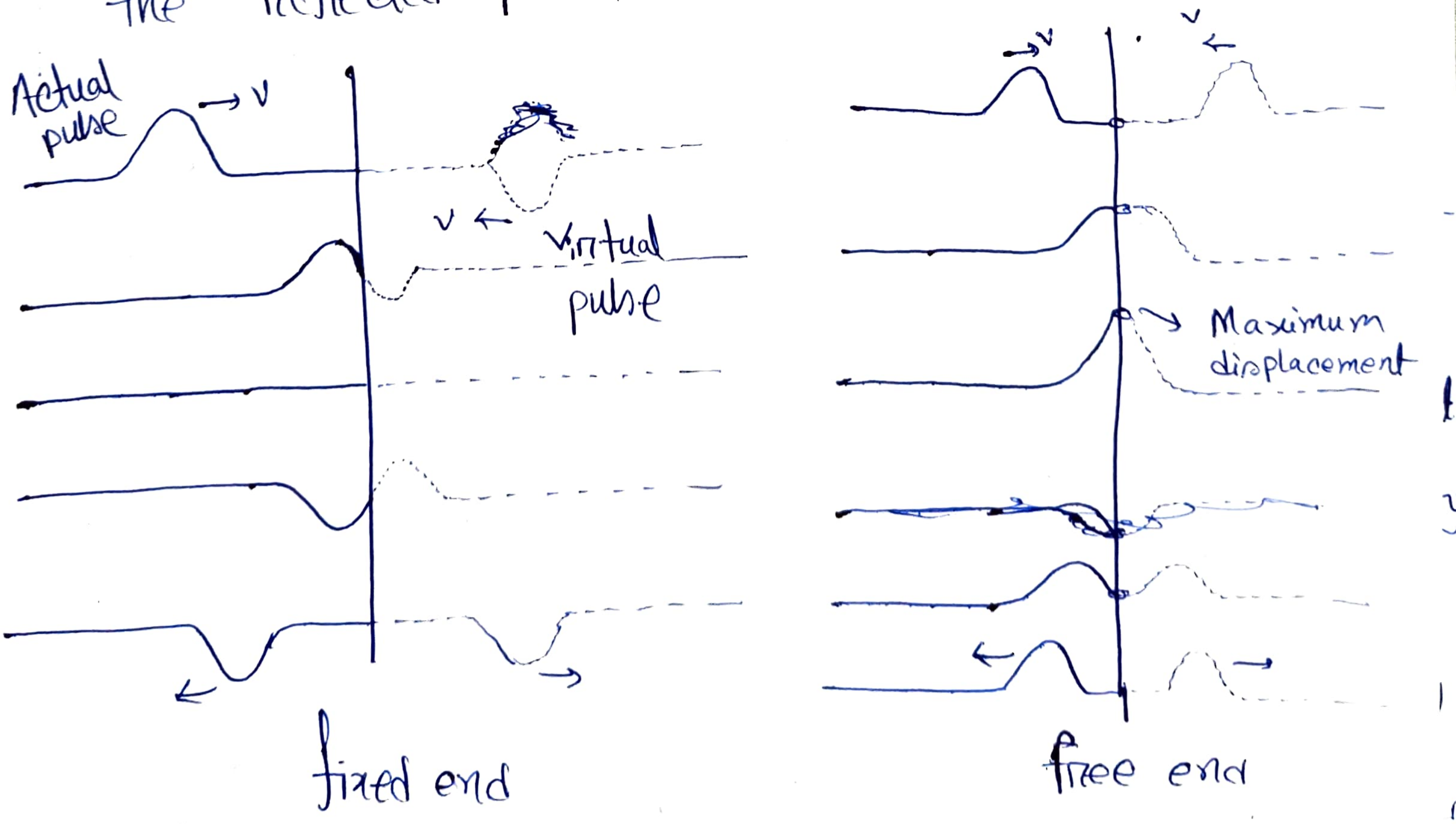


Think about a wave pulse travelling to the right, which will be incident on a rigid wall, that is the end of the string is fixed, and can only have zero displacement at all times. This is our boudary conditions. What will happen when the pulse is incident on the wall? It simply can't vanish, since the pulse carry some amount of energy, and it can't just vanish to zero.

We have to find the answer using matematics and imposing the boundary conditions. We will do it later. For now, think about the boundary condition. At all times, $\Psi(0,t)=0$, where $x=0$ is the free end.

In our previous ~~sec~~ section, we saw that the superposition of two oppositely travelling pulse, one positive and one negative, creates zero displacement at $x=0$ for all times (check by plugging $x=0$ at $\Psi(x,t) = \frac{b^3}{b^2+(x-vt)^2} - \frac{b^3}{b^2+(x+vt)^2}$).

The boundary condition at $x=0$ is then same for superposition of two pulses and wave pulse incident on the fixed end. It's not guranted the two mechanisms are equivalen, but it happens to be so. Think about an imaginary/virtual pulse of opposite polarity travelling towards left as show in the figure. Then consider the superposition. After they meet and create the zero displacement, they will re-emerge as travelling pulses again. Now, the pulse travelling to right beyond fixed point ($x=0$) will be the virtual pulse (since nothing really exists beyond $x=0$), and the pulse travelling left will be the actual pulse, which is really the reflected pulse, that was reflected from fixed point.



What really happens there is that, when the pulse reaches the fixed end, the particles will have an upward velocity ~~to~~ and ~~pull~~ try to pull the support in upward direction. By Newton's second law, the support exerts an equal and downward force on the rope. So, the rope comes down, overshoots and goes down to the equilibrium position, which creates the reflected pulse.

However, the string could have been connected to the free end, which can move up and down. We can make this happen by connecting a massless ring at the end of the string. The ring is then conneted to a rod such ~~the~~ that it can slide up and down. The surface of the rod is frictionless. The introduction of the ring is necessary to keep the tension on the string.

When the pulse reaches the ring, it reaches a maximum displacement (twice the amplitude of the pulse) and momentarily stops. (reason why this is true) The string is now stretched, so it provides increased tension. So the free end of the string is ~~pull~~ pulled back down and a reflected pulse of same polarity is formed. This can be described by a virtual pulse of same polarity since here the boundary condition is that, $\Psi(0,t) = 2 \times$ amplitude of a pulse.

Mathematics of reflection and transmission

Consider a string made up of two parts, one with mass density μ_1, from $-\infty < x < 0$ and another with mass density μ_2 from $0 < x < \infty$. Although the density is not uniform, the tension must be uniform throughout the whole string, because, without that, there will be some non-zero horizontal acceleration somewhere, which breaks our criteria of purely transverse vibration.

Assume that a pulse of the form $f(t - \frac{x}{v_1})$ is moving along from left to right and heads toward $x=0$. Look, we have changed the form of the function from $f(x - v_1 t)$ to $f(t - \frac{x}{v_1})$, because it will be convenient one. But both of them denote the same wave. We call this the incident pulse,

$$\psi_i(x,t) = f_i\left(t - \frac{x}{v_1}\right) \quad \text{with } v_1 = \sqrt{\frac{T}{\mu_1}}$$

When it hits the junction $x=0$, we expect some part of the pulse will be transmitted and some will be reflected. So, we will have,

$$\psi_r(x,t) = f_r\left(t + \frac{x}{v_1}\right) \quad \text{and} \quad \psi_t(x,t) = f_t\left(t - \frac{x}{v_2}\right)$$

with $v_2 = \sqrt{\frac{T}{\mu_2}}$.

The total wavefunction in the left $(-\infty < x < 0)$ and right $(0 < x < \infty)$ part of the string is given by,

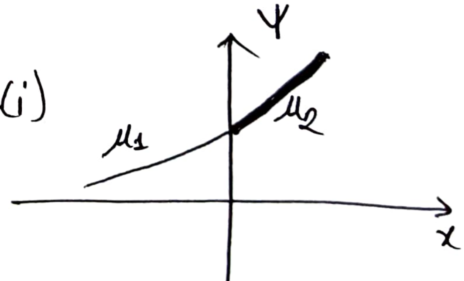
$$\Psi_L(x,t) = \Psi_i(x,t) + \Psi_r(x,t) = f_i\left(t - \frac{x}{v_1}\right) + f_r\left(t + \frac{x}{v_1}\right)$$

$$\Psi_R(x,t) = \Psi_t(x,t) = f_t\left(t - \frac{x}{v_2}\right)$$

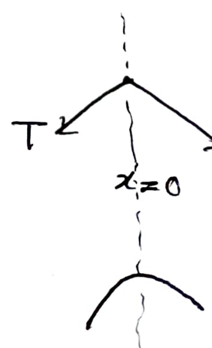
If we can express the reflected and transmitted wave in terms of the incident wave, then we will know the complete wave everywhere. Let's look at the boundary conditions here.

(i) The displacement on the left and right must be equal at $x=0$, since the string is continuous.

$$\therefore \Psi_R(0,t) = \Psi_L(0,t) \quad \text{—(i)}$$



(ii) The ~~at~~ slope must also be continuous at $x=0$, and hence the must be equal. Because if that is not true, then at the junction, there will be a 'kink', like shown in the figure. The left limit of slope and right limit of the slope won't match. Then the nearly



massless particle at $x=0$ would experience infinite force, which is not physical. So, the slopes must be equal at the junction for the left and right side, meaning it will be differentiable at $x=0$.

$$\therefore \left.\frac{\partial \Psi_L(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_R(x,t)}{\partial x}\right|_{x=0} \quad \text{— (ii)}$$

From (i), we get, $f_i(t) + f_r(t) = f_t(t)$ — (iii)

From (ii), " ", $\left.\frac{\partial f_i}{\partial x}\right|_{x=0} + \left.\frac{\partial f_r}{\partial x}\right|_{x=0} = \left.\frac{\partial f_t}{\partial x}\right|_{x=0}$ — (iv)

Now, $\frac{\partial f_i(t-\frac{x}{V_1})}{\partial x} \cancel{= \frac{\partial}{\partial(t-x}}$

$$= \frac{\partial(t-\frac{x}{V_1})}{\partial x} \frac{\partial f_i(t-\frac{x}{V_1})}{\partial(t-\frac{x}{V_1})} = -\frac{1}{V_1}\frac{\partial}{\partial(t-\frac{x}{V_1})} f_i(t-\frac{x}{V_1})$$

But, $$\frac{\partial f_i(t-\frac{x}{V_1})}{\partial t} = \frac{\partial(t-\frac{x}{V_1})}{\partial t} \cdot \frac{\partial f_i(t-\frac{x}{V_1})}{\partial(t-\frac{x}{V_1})} \cancel{= \partial f_i(t-\frac{x}{V_1}}$$

$$= \frac{\partial f_i(t-\frac{x}{V_1})}{\partial(t-\frac{x}{V_1})}$$

$$\therefore \frac{\partial f_i(t-\frac{x}{V_1})}{\partial x} = -\frac{1}{V_1}\frac{\partial f_i(t-\frac{x}{V_1})}{\partial t}$$

From (iv) —

$$-\frac{1}{V_1}\dot{f}_i(t) + \frac{1}{V_1}\dot{f}_r(t) = -\frac{1}{V_2}\dot{f}_t(t)$$

Integrating both sides w.r.t. time we get,

$$-\frac{1}{v_1} f_i(t) + \frac{1}{v_1} f_r(t) = -\frac{1}{v_2} f_t(t)$$

$$\therefore \; v_2 f_i(t) - v_2 f_r(t) = v_1 f_t(t) \quad \text{——} (v)$$

where we have set the arbitrary constant to be zero since we are assuming that the string has no displacement before the wave passes by.

Solving (iii) and (v) we get,

$$f_r(t) = \frac{v_2 - v_1}{v_2 + v_1} f_i(t) \quad \text{and} \quad f_t(t) = \frac{2v_2}{v_2 + v_1} f_i(t),$$

Now, these two equations are valid as long as the arguments of f_r and f_i denotes the same value. Say, the argument is τ. Then,

$$f_r(\tau) = \frac{v_2 - v_1}{v_2 + v_1} f_i(\tau) \quad \text{and} \quad f_t(t) = \frac{2v_2}{v_2 + v_1} f_i(\tau),$$

This τ can have any value, and can be related by any value of x and t that fits the description of travelling wave. If $\tau = t - \frac{x}{v_1}$, then,

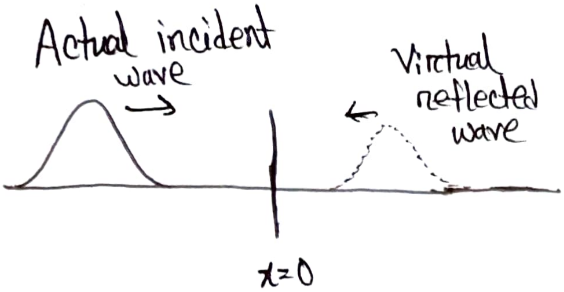
$$f_r\left(t - \frac{x}{v_1}\right) = \frac{v_2 - v_1}{v_2 + v_1} f_i\left(t - \frac{x}{v_1}\right)$$

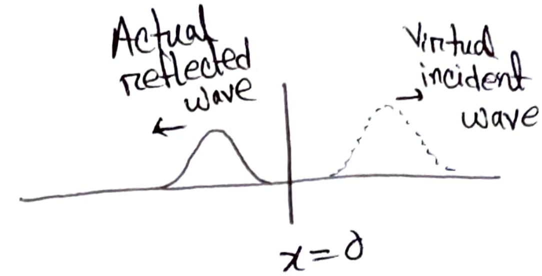
$$f_r\left(t-\frac{x}{v_1}\right) = \frac{v_2-v_1}{v_2+v_1} f_i\left(t-\frac{-x}{v_1}\right)$$

$$\Rightarrow \boxed{\psi_r(x,t) = \frac{v_2-v_1}{v_2+v_1}\psi_i(-x,t)}$$

This implies that, at a given instant of time, the value of ψ_r ~~is~~ at some position x is exactly equal to $\frac{v_2-v_1}{v_2+v_1}$ times the value of ψ_i at position $-x$. Both ψ_r and ψ_i travels with the same speed and has the ~~sha~~ same shape. Its just that the amplitude of ψ_r is scaled by a factor of $\frac{v_2-v_1}{v_2+v_1}$ from ψ_i. Additionally, if $v_2<v_1$, then ψ_r is inverted.

In our case, only negative value of x is what matters, since ψ_r exists here. For that, we need value of ψ_i at positive x. Although ψ_i doesn't exist at positive values of x, but you can imagine it as travelling to $x>0$ as if there is no boundary. Now you see, the virtual pulse picture is a consequence of the underlying mathematics.





For transmission of the wave, we had,

$$f_t(t) = \frac{2V_2}{V_1+V_2} f_i(t)$$

In a similar manner, we can write,

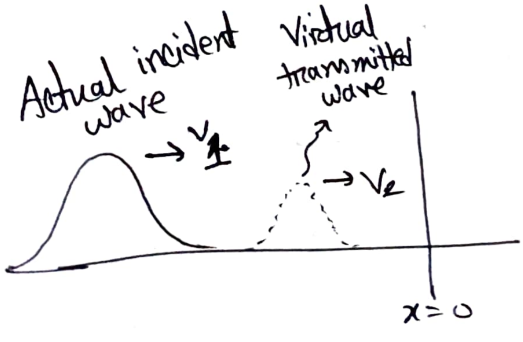
$$f_t\left(t-\frac{x}{V_2}\right) = \frac{2V_2}{V_1+V_2} f_i\left(t-\frac{x}{V_2}\right)$$

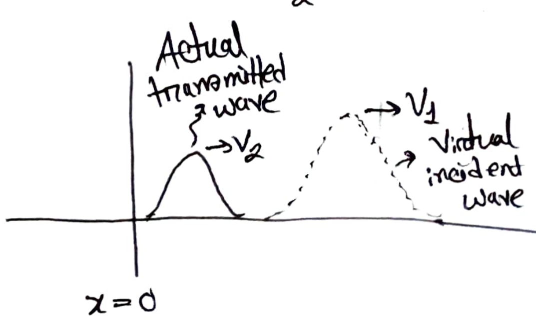
$$= \frac{2V_2}{V_1+V_2} f_i\left(t-\frac{V_1}{V_1}\cdot\frac{x}{V_2}\right)$$

$$= \frac{2V_2}{V_1+V_2} f_i\left[t-\left(\frac{V_1}{V_2}\right)\frac{x}{V_1}\right]$$

$$\boxed{\therefore\ \Psi_t(x,t) = \frac{2V_2}{V_1+V_2} \Psi_i\left(\frac{V_1}{V_2}x, t\right)}$$

So, at any instant of time t, the value of Ψ_t at position x is equal to $\frac{2V_2}{V_2+V_1}$ times the value of Ψ_i at position $\frac{V_1}{V_2}x$. The speed of Ψ_t is V_2, and the shape of the transmitted wave is either contracted or broadened depending on $\frac{V_1}{V_2}$.





Different scenerios

1. **Fixed wall/end at $x=0$:** Consider the $x>0$ part of the string to have infinite mass density, so $\mu_2 \to \infty$, and so $v_2 \to 0$. This is the same if the incident wave hits a fixed end like a rigid wall that can't move.

$$\therefore \Psi_r(x,t) = \frac{v_2 - v_1}{v_2 + v_1}\Psi_i(-x,t)$$

$$\boxed{\therefore \Psi_r(x,t) = -\Psi_i(-x,t)}$$

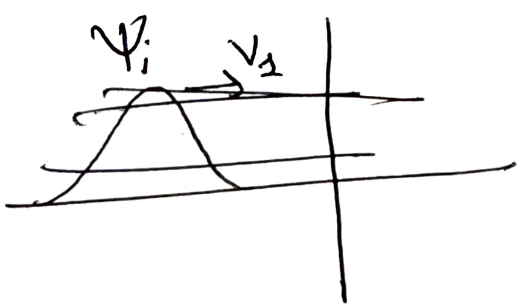
You can define,

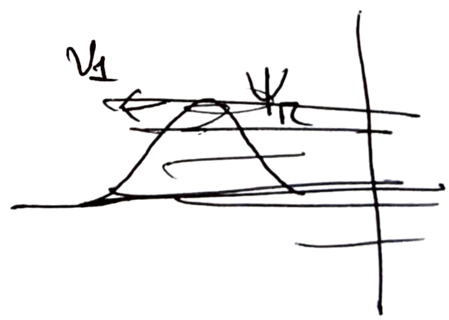
Reflectance, $R = \dfrac{v_2 - v_1}{v_2 + v_1}$

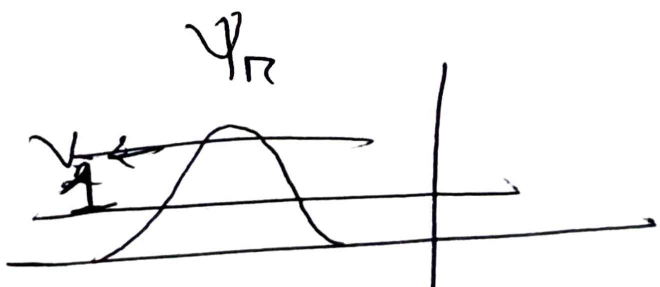
Transmittence, $T = \dfrac{2v_2}{v_2 + v_1}$

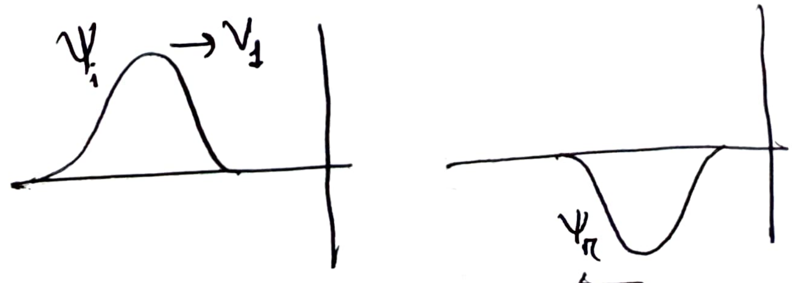
Here, $R = -1$ and $T = 0$

So, the incident wave is totally reflected having the same amplitude and shape, but is inverted [due to the (–) sign].









On the other hand, $\psi_t(x,t) = 0$ and nothing transmits.

2. Uniform string: If both parts have the same mass density so that $\mu_1 = \mu_2 = \mu$ and $v_1 = v_2 = v$, then,

$$\psi_r(x,t) = 0 \quad \text{and} \quad \psi_t(x,t) = \psi_i(x,t)$$

$$\therefore R = 0 \qquad \therefore T = 1$$

So, the wave just passes by.

3. String with zero mass on $x>0$: In this case, $\mu_2 \to 0$ and so $v_2 \to \infty$.

$$\therefore \psi_r(x,t) = \psi_i(-x,t) \quad \text{and} \quad \psi_t(x,t) = 2\psi_i(0,t)$$

~~Th~~ Technically this is a totally reflected wave with the same shape, velocity and polarity. Since the right string is massless, it can't carry any energy. All the $x>0$ string does is what the point at $x=0$ is doing. So, the whole string remains horizontal and just rises and falls as the incident wave pulse hits $x=0$, as the point at $x=0$ just raises and falls as the wave reaches here. This is the same thing as connecting a massless ring at the $x=0$ point. We saw the behaviour earlier in the virtual pulse segment.