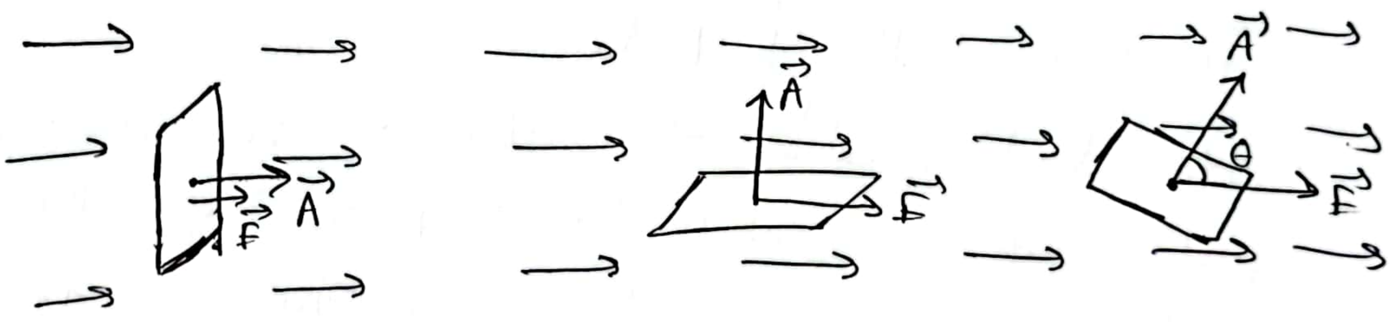
Lecture 4

The flux of electric field $\vec{E}$

We consider some arbitrary electric field in space, where the field is shown by a few field lines. For an open surface, we can define the area vector perpendicular to the plane of the surface, both ways. Let's pick a particular direction for the area vector, and say is is denoted by $\vec{A}$. ~~B~~ The electric flux through this surface is given by,

$$\phi_E = \vec{E}\cdot\vec{A}$$



$\phi = EA\cos 0°$ $\phi = EA\cos 90°$ $\phi = EA\cos\theta$

Now, let's say we have a closed surface, meaning that the surface encloses a volume. Now we take a very ~~sar~~ small, infinitesimal patch of area on the surface, and say it is denoted by

dA_j. The direction of the area vector is fixed now, its always outwards, and now since the surface is closed, you can tell the difference between the inside and the outside. The flux through that infinitesimal surface is,

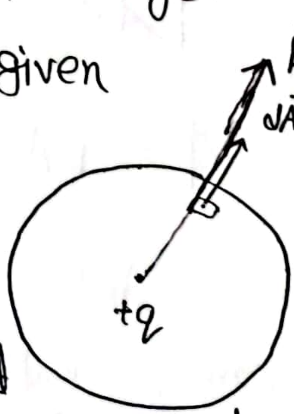
$$d\phi_E = \vec{E}_j \cdot d\vec{A}_j$$

where $\vec{E}_j$ is the electric field passing through that surface. Now, if we compute the flux over the whole surface, the in the infinitesimal limit we can write the total flux as —

$$\phi_E = \oiint_S \vec{E} . d\vec{A} = \oiint_S \vec{E} . \hat{n}\, dA$$

Gauss's law: The first Maxwell's equation

Let's first calculate the flux of the electric field that is created by a point charge. The electric field of a point charge is given by,



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}\hat{r}$$

Let's take a sphere around that charge of radius R centered at the point charge. The flux through the surface of the sphere is is given by,

$$\phi_E = \oiint_S \vec{E} \cdot d\vec{A} = \oiint_S \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R^2} \hat{r} \cdot dA\, \hat{r}$$

At each and every infinitesimal area, the area vector will point in the $\hat{r}$ direction.

$$\therefore \phi_E = \oiint_S \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R^2} dA = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R^2} \oiint_S dA$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{R^2} \times 4\pi R^2 = \frac{q}{\epsilon_0}$$

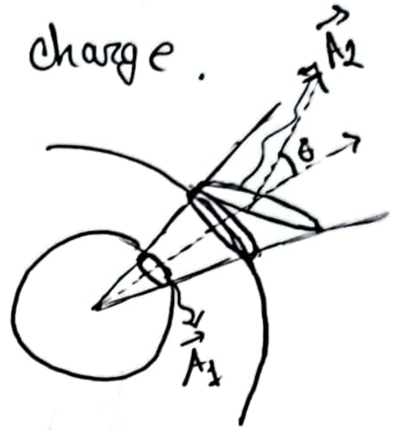
$$\boxed{\therefore \oiint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}}$$

Area $= 4\pi (2R)^2$

The result is true for a sphere with any radius. If the radius was twice, the flux would have been the same since $E \propto \frac{1}{(2R)^2}$, but the total area of the sphere will increase as $A \propto (2R)^2$.

But what about any arbitrary surface around the charge? Does this law still holds? Let's see.

We imagine first a portion of two concentric spheres around a charge. A very small area element on the smaller sphere is taken. Now, the electric field spreads out as we move away from the charge in terms of the lines, that is, the field lines have a higher density close to the charge and smaller density further away from the charge. They spread out in space according to a solid angle, that is making a cone starting at the charge.



If the areas are very small, the electric field over the surfaces can be considered constant. Now, say, $\vec{A}_1$ is the area covered in the cone in the first sphere. The flux is given by,

$$\phi_{E,1} = \vec{E}_1 \cdot \vec{A}_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_1^2} \times A_1$$

Now, let's consider the second surface area A_2, which lies along the same cone, but is in a different orintation, any arbitrary orientation. Say, now that area vector makes an angle θ with the previous one. The electric flux through this surface is given by,

$$\phi_2 = \vec{E}_2 \cdot \vec{A}_2 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_2^2} A_2 \cos\theta$$

Now, $A_2 \cos\theta$ is nothing but the projection of A_2 on the second sphere which lie along the axis of the cone. The surface area along a cone increases as a function of r^2. So, if we define, $A_3 = A_2 \cos\theta$, then,

$$A_3 = \frac{r_2^2}{r_1^2} \times A_1$$

$$\therefore \Phi_{E2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_2^2} \cdot \frac{r_2^2}{r_1^2} \times A_1 = \Phi_{E1}$$

So, the flux is the same through bothe the surfaces as long as they lie in the same cone. So, the flux is independent of shape and size of the surface we take.

$$\therefore \oiint_S \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

if our surface encloses the charge. But what if our charge is outside the surface?

First consider two small patches cut from spheres of radius R_1 and R_2, which subtends the same cone. Make a closed surface from then as shown in the figure.

Flux through radial surfaces are zero for sure since the area vectors are perpendicular to the electric field.

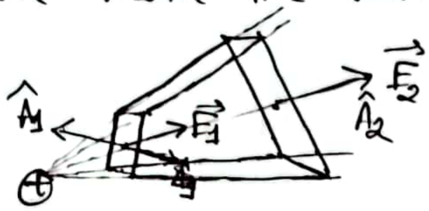
$$\phi_1 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_1^2} \cdot \hat{r} \cdot A_1(-\hat{r}) = -\frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_1^2} A_1$$

$$\phi_2 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_2^2} \hat{r} \cdot A_2 \hat{r} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_2^2} \cdot \frac{r_2^2}{r_1^2} A_2$$

$$= -\phi_1$$

$\therefore$ Total flux $= 0$

$\therefore \oiint_S \vec{E} \cdot d\vec{A} = 0$; if the charge is outside. It should be also ~~trow~~ true for any random surface. It ~~will~~ can be shown from the following figure.



$$\phi_1 = -\frac{1}{4\pi\epsilon_0} \frac{q}{r_1^2} \cdot A_1 \cos\theta \qquad \phi_2 = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r_2^2} A_2$$

$$\phi = \frac{1}{4\pi\epsilon_0} q \left[\frac{A_2}{r_2^2} - \frac{A_1}{r_1^2} \cos\theta \right] \qquad \Big|\ A_3 = A_1 \cos\theta$$

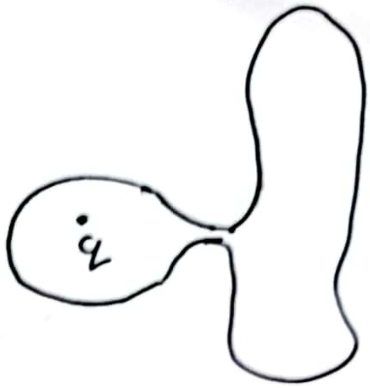
$$= \frac{1}{4\pi\epsilon_0} q \left[\frac{r_2^2}{r_1^2} \times A_3 \times \frac{1}{r_2^2} - \frac{A_3}{\cos\theta\, r_1^2} \cos\theta \right]$$

$$= 0$$

Logical example

Consider a charge q enclosed by a surface as shown in the figure, where two surfaces are joined by a neck. Now, total flux through the surface is,

$$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$$

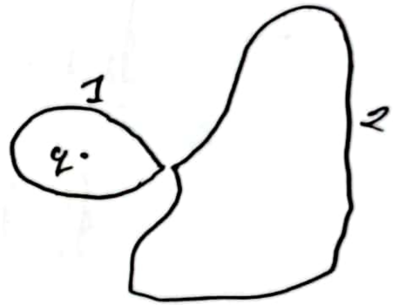


Now ~~in the~~ say the neck has a radius of ϵ. In the limit $\epsilon \to 0$, the flux should still be the same.

$$\therefore \lim_{\epsilon\to 0} \oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0}$$

In that limit you can think of the enclosing surface as two of them.

The flux through surface 1 must be $\frac{q}{\epsilon_0}$. Now, total flux,



$$\phi = \oint\!\!\oint_{S_1} \vec{E}\cdot d\vec{A} + \oint\!\!\oint_{S_2} \vec{E}\cdot d\vec{A}$$

$$\Rightarrow \frac{q}{\epsilon_0} = \frac{q}{\epsilon_0} + \oint\!\!\oint_{S_2} \vec{E}\cdot d\vec{A}$$

$$\boxed{\therefore \oint\!\!\oint_{S_2} \vec{E}\cdot d\vec{A} = 0}$$

CamScanner

Finally, our conclusions are,

$$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \begin{cases} \frac{q}{\epsilon_0}; & \text{if } q \text{ is inside } S \\ 0; & \text{if } q \text{ is outside } S \end{cases}$$

Now, say, there are not one, but two different charges. The total electric field is given by,

$$\vec{E} = \vec{E_1} + \vec{E_2}$$

Now, the flux through any closed surface enclosing those two charges will be given by,

$$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \oint\!\!\oint_S (\vec{E_1} + \vec{E_2}) d\vec{A} = \oint\!\!\oint_S \vec{E_1}\cdot d\vec{A} + \oint\!\!\oint_S \vec{E_2}\cdot d\vec{A}$$

$$\Rightarrow \oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_1}{\epsilon_0} + \frac{q_2}{\epsilon_0}$$

$$\therefore \oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_1 + q_2}{\epsilon_0}$$

This is also true for more than two charges. In general, we write,

$$\boxed{\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}}$$

where q_{enc} is the enclosed charge by the surface S. This is called Gauss's law, and the ~~imaginary~~ imaginary surface around the charges is called the Gaussian.

surface. In general,

$$q_{enc} = \sum_i q_i$$

Now, $$\oint\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

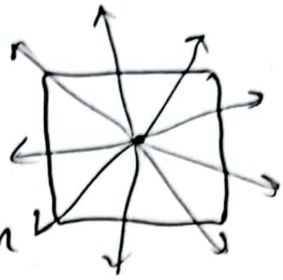
$$\Rightarrow \iiint_V (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{1}{\epsilon_0} \iiint_V \rho \, d\tau$$

$$\boxed{\therefore \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$

This is Gauss's law in differential form ~~ane~~ and, this is the first Maxwell's equation.

Using Gauss's law to find the electric field

Now, although Gauss's law is valid for any random closed surface, we can't always make good use of Gauss's law unless we have symmetries in electric field configuration. On the left hand side we have $\oint_S \vec{E}\cdot d\vec{A}$. Now, if the electric field has random values and/or directions on a particular surface, we have to calculate many different $\vec{E}\cdot d\vec{A}$ s to finally compute the integration, which is next to impossible. On the other hand, if we do not choose our surface carefully, we will overcomplicate our problem. Say, for example, we want to calculate the electric field of a point charge. But we are choosing a cube as our surface rather than a sphere. Now, the simple problem became very complicated, and we can't possibly solve this. The imaginary closed surface that we are choosing is called a Gaussian surface.



So, we need some symmetry and corresponding Gaussian surfaces to simplify our problems. Here ~~to~~ are some examples.

System	Symmetry	Gaussian surface
Infinite rod/Infinite cylinder	Cyllindrical	Coaxial cyllinder
Infinite plane	Planar	Gaussian pillbox
Sphere/spherical shell	Spherical	Concentric sphere

→ Identify the symmetry associated with charge distribution.

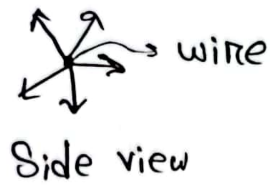
→ Determine the direction of the electric field and a Gaussian surface on which the magnitude of the electric field is constant.

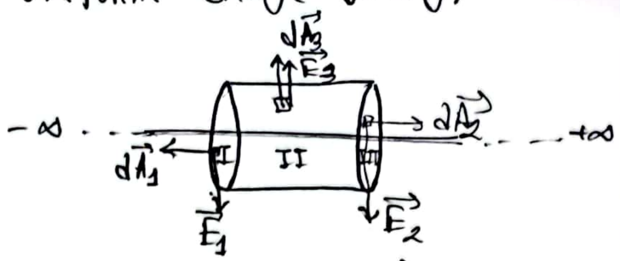
→ Divide the closed surface in different regions and calculate flux through them individually if needed.

→ Equate ϕ_E with $\frac{q_{enc}}{\epsilon_0}$ and find the magnitude of $\vec{E}$ field.

CS CamScanner

1. Infinitely long rod with uniform charge density:





So, the symmetry is cyllindrical and electric field magnitude on the ~~sur~~ curved surface of the cyllinder is the same. So, our Gaussian surface is a cyllinder. We have three regions of the cyllinder.

→ Two circular disks
→ Cyllindrical curved surface

$$\phi_I = \oint_{S_1} \vec{E_1} \cdot d\vec{A_1} = 0 \qquad \phi_{III} = \iint_{S_2} \vec{E_2} \cdot d\vec{A_2} = 0$$

$$\phi_{II} = \iint_{S_2} \vec{E_3} \cdot d\vec{A_3} = \iint_{S_2} E_3 \, dA_3 \cos 0^\circ = E_3 \iint_{S_2} dA_3$$

$$= E_3 \times 2\pi r h$$

$$\therefore \phi_E = \frac{q_{enc}}{\epsilon_0}$$

$$\boxed{q_{enc} = \lambda h}$$

$$\lambda = \frac{q_{enc}}{h}$$

$$\Rightarrow E_3 \times 2\pi r h = \frac{\lambda h}{\epsilon_0}$$

$$\boxed{\therefore E_3 = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r}}$$

CS CamScanner