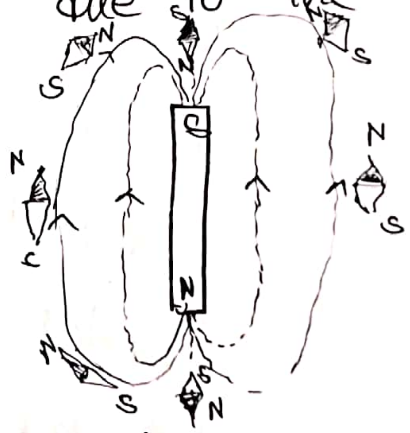
Lecture 13

Magnetism

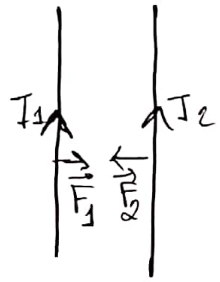
In the winter of 1819-1820, Hans Christian Oersted was lecturing on electricity and magnetism at the University of Copenhagen. Magnetism back then referred to the magnets found in nature, which attracts or repulse each other, ~~earth's~~ compass needles and earth's magnetic behaviours, with ~~wit~~ which compasses worked. It was known at that time that a bar magnet creates some sort of "magnetic field" around it, with having two poles — called north and south. Just like electric field in space $\vec{E}$, the magnetic field, denoted by $\vec{B}$, extends in all places due to the bar magnet. If we had a compass needle, we could place at different points near the magnet. The directions of the compass will position themselves in the directions of the magnetic field in those places.

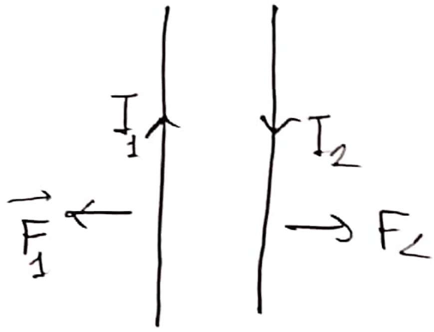


The similar poles repulse each other and opposite poles attract each other. It was understood by people then that the earth should act as a giant bar magnet, for which the north pole of the compass would point in the direction of earth's magnetic south and vice-versa. However, the question always bothered people was — why magnetism works, or are there any other form of magnetism. People then knew there should be some relation between currents and electric charge. But magnetism and electricity appeared to have nothing to do with one another.

Back to Oersted teaching his students. Oersted had a vague notion that magnetism might have something to do with electricity. For the manifestation he tried before the class passing a current through a wire where the compass was placed in a particular orientation. His compass was directly below the wire, and the needle was perpendicular to the wire. Oersted found no noteworthy deflection. However, after the class, Oersted changed the needle direction parallel to the wire, and now he observed a large deflection. He reversed the direction of the current, and the deflection was now in opposite direction. And hence, Oersted discovered, electric currents produce magnetic fields around the wire.

After Oersted, people soon found that two parallel current carrying wires applies force on each other.





The forces on the wires are like action at a distance force, and static electric charge has nothing to do with it. It's the motion of the charges that creates the force, and we call the force magnetic. Observing the motion of a free charged particle, instead of a wire carrying current, results in the same thing. In a cathode ray tube, electrons that follows straight path are deflected if there is a current carrying wire near it. This interactions between moving charged particles can be described by introduction of a magnetic field. Remember, electric field was simply a way of describing action at a distance between stationary charges, expressed by Coulomb's law. We say, the electric current has associated with it a magnetic field on its surrounding space. Any moving charged particle that finds ~~itself~~ itself in this field, experiences a field proportional to the field. But the force also depends on the velocity. So, we have a velocity dependent force, depending along with the charge and field $\vec{B}$.

Some experimental observations

(i) The magnitude of magnetic force exerted on a charged particle,

$$F_B \propto q \quad \text{and} \quad F_B \propto v$$

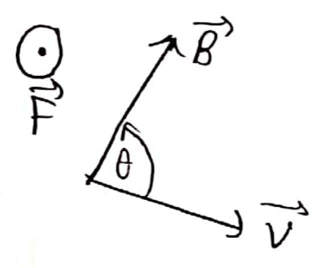
(ii) The magnitude and direction of F_B depends on $\vec{v}$ and $\vec{B}$.

(iii) The ~~magnit~~ magnetic force $\vec{F_B}$ is perpendicular to the plane created by $\vec{v}$ and $\vec{B}$.

(iv) When the sign of the charged particle switches, the direction of the force also switch.

All these results can be summarized into the following equation-

$$\vec{F}_B = q\,\vec{v} \times \vec{B}$$



The magnitude is given by, $F_B = |q| v B \sin\theta$

$$\therefore B = \frac{F_B}{|q| v \sin\theta}$$

We can use this equation to define the magnetic field $\vec{B}$. The SI unit is Tesla (T).

$$1T = \frac{1N}{1\,C\,ms^{-1}} = \frac{1N}{1\,Am}$$

Another commonly used unit is Gauss. $\boxed{1T = 10^4 \text{ Gauss}}$

In the presence of both electric and magnetic field,

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right]$$

This is known as the Lorentz force.

Work done by the magnetic force

The work done by the magnetic force for an infinitesimal displacement $d\vec{s}$ is given by,

$$dW_{mag} = \vec{F}_{mag}\cdot d\vec{s} = q(\vec{v}\times\vec{B})\cdot\vec{v}dt$$

Now, $\vec{v}\times\vec{B}$ is always perpendicular to $\vec{v}$, and hence the dot product is obviously zero.

$$\therefore dW_{mag} = 0 \qquad \boxed{\therefore W_{mag} = 0}$$

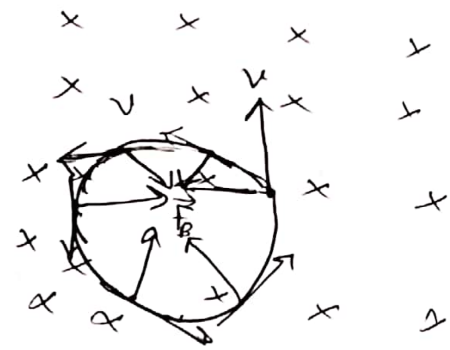
Since no work is done by the magnetic force, it can't change the kinetic energy of a charged particle, since we know from work-energy theorem that,

$$W = \Delta K$$

$$\boxed{\therefore \Delta K = 0}$$

So, magnetic force can't change the magnitude of velocity of the charged particle. But it can, of course change the direction of velocity, and it does.

Consider a charged particle is moving with some velocity. There is a constant magnetic field in the perpendicular direction of velocity, such that there is no component of velocity in the direction of magnetic field. Since the ~~fo~~ magnetic force is perpendicular to both $\vec{v}$ and $\vec{B}$, it must act like a centripetal force in this situation. So, the particle will start rotating in a circular path. If the particle is moving in a circular path of radius r, then –



$$F_B = \frac{mv^2}{r}$$

$$\Rightarrow qvB = \frac{mv^2}{r}$$

$$\boxed{\therefore\ r = \left(\frac{m}{q}\right)\frac{v}{B}}$$

One can plot ~~a~~ a graph of r vs B by adjusting the B values and measuring the radii. The plot should be a straight line passing through the origin. This very experiment was done by Sir J. J. Thomson to find the $\frac{e}{m}$ ratio some

of electrons, where he used a stream of electrons.

Now, lets see whats happening mathematically.

$\vec{V} = V_x\hat{i} + V_y\hat{j}$ $\qquad \vec{B} = B\hat{k}$

$$\therefore \vec{F}_B = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & 0 \\ 0 & 0 & B \end{vmatrix} = q\left[\hat{i}\,(BV_y) + \hat{j}\,(-BV_x)\right]$$

Now, $F_x = ma_x$ $\qquad \frac{dV_x}{dt} = \frac{qB}{m}V_y$

$\Rightarrow qBV_y = m\frac{dV_x}{dt}$

$\Rightarrow qB\frac{dV_y}{dt} = m\frac{d^2V_x}{dt^2}$

$\Rightarrow qB\left(-\frac{qB}{m}V_x\right) = m\frac{d^2V_x}{dt^2}$

$\therefore \frac{d^2V_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 V_x$

$\therefore \frac{d^2V_x}{dt^2} + \omega^2 V_x = 0$

$F_y = ma_y$ $\qquad \frac{dV_y}{dt} = -\frac{qB}{m}V_x$

$\Rightarrow -qBV_x = m\frac{dV_y}{dt}$

$\Rightarrow -qB\frac{dV_x}{dt} = m\frac{d^2V_y}{dt^2}$

$\Rightarrow \cancel{\frac{d^2V_y}{dt^2}} \; -qB\left(\frac{qB}{m}V_y\right) = m\frac{d^2V_y}{dt^2}$

$\Rightarrow \frac{d^2V_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 V_y$

$\Rightarrow \frac{d^2V_y}{dt^2} + \omega^2 V_y$

The general result is

$V_x = A_x\cos\omega t + C_x\sin\omega t$

At $t=0$, $V_x = V_{x_0}$

$\therefore V_{x_0} = A_x$

Now, $\frac{dV_x}{dt} = -\omega A\sin\omega t + \omega C_x\cos\omega t$

~~At t=0:~~

$\Rightarrow \frac{qB}{m}V_y = ----$

$V_y = A_y\cos\omega t + C_y\sin\omega t$

At $t=0$, $V_y = V_{y_0}$.

$\therefore V_{y_0} = A_y$

$\Rightarrow \frac{dV_y}{dt} = -\omega A_y\sin\omega t + \omega C_y\cos\omega t$

$\Rightarrow -\frac{qB}{m}V_x = ----$

At $t=0$,

$$\frac{qB}{m} V_{y_0} = \frac{qB}{m} C_x$$

$$\therefore C_x = V_{y_0}$$

$$-\frac{qB}{m} V_{x_0} = \frac{qB}{m} C_y$$

$$C_y = -V_{x_0}$$

$$\therefore V_x = V_{x_0} \cos\omega t + V_{y_0} \sin\omega t$$

$$V_y = V_{y_0} \cos\omega t - V_{x_0} \sin\omega t$$

You can now check that, $v^2 = \sqrt{V_{x_0}^2 + V_{y_0}^2} = \sqrt{V_x^2 + V_y^2}$

You are always free to choose the ~~ini~~ initial velocity direction as your, say X axis.

$$\therefore V_{x_0} = V_0 \quad \text{and} \quad V_{y_0} = 0$$

$$\therefore V_x = V_0 \cos\omega t$$

$$V_y = V_0 \sin\omega t$$

Now, $\frac{dx}{dt} = V_0 \cos\omega t$

$$\Rightarrow \int dx = \int V_0 \cos\omega t \, dt$$

$$\Rightarrow x = \frac{V_0}{\omega} \sin\omega t + c$$

$$\frac{dy}{dt} = V_0 \sin\omega t$$

$$\int dy = \int V_0 \sin\omega t \, dt$$

$$\Rightarrow y = -\frac{V_0}{\omega} \cos\omega t + d$$

$$\therefore (x-c)^2 + (y-d)^2 = \frac{V_0^2}{\omega^2} (\sin^2\omega t + \cos^2\omega t)$$

$$\Rightarrow (x-c)^2 + (y-d)^2 = \frac{V_0^2}{\left(\frac{qB}{m}\right)^2} = \left(\frac{mV_0}{qB}\right)^2 = r^2$$

$$\boxed{\therefore (x-c)^2 + (y-d)^2 = r^2}$$

However, if we had a component of velocity parallel to B, then say,

$$\vec{B} = B\hat{j} \quad \text{and} \quad \vec{v} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\vec{F}_B = q\left(\vec{v}\times\vec{B}\right) = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ 0 & B & 0 \end{vmatrix} = q\left[\hat{i}(-V_zB) + \hat{j}\times 0 + \hat{k}\,(BV_x)\right]$$

$$\therefore \vec{F}_B = q(BV_x\hat{k} - BV_z\hat{i})$$

The calculations will be pretty much same as before, along x and z axis. Along Y-axis, there is no force. So, V_y = constant.

$$\therefore y(t) = V_y t$$

So, the motion will be a combination of circular motion, that moves in the y direction with a constant velocity. So, the motion will be hellical.

$$\boxed{\therefore \vec{r}(t) = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j} + V_y t\,\hat{k}}$$

Let's now say a line charge of density λ is travelling through a wire at speed v. The current can be defined as the amount of charge passing a particular point per unit time. Within time Δt, a total amount of charge $\lambda(v\Delta t)$ will pass through point P.

$$\therefore \text{Current, } I = \frac{\lambda v \Delta t}{\Delta t} = \lambda v$$

We can then define current as a vector, with,

$$\vec{I} = \lambda \vec{v}. \qquad \Big| \; \lambda = \frac{dq}{dl}$$

Now, the force that the wire will be experiencing is given by,

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B})\, \lambda\, dl$$

$$\Rightarrow \vec{F}_{mag} = \int (\vec{I} \times \vec{B})\, dl$$

Essentially $\vec{I}$ and $\vec{dl}$ are in the same direction. So, we can also write,

$$\vec{F}_{mag} = \int I (\vec{dl} \times \vec{B})$$

If I is a constant throughout the wire, then,

$$\boxed{\vec{F}_{mag} = I \int \vec{dl} \times \vec{B}}$$

If the current is passing through a volume of wire, then,

$$\vec{F}_{mag} = \int (\vec{V} \times \vec{B}) \rho \, d\tau = \int (\rho \vec{V} \times \vec{B}) \, d\tau$$

$$\boxed{\therefore \; F_{mag} = \int (\vec{J} \times \vec{B}) \, d\tau}$$

with $\vec{J}$ being the current density defined as

$$\vec{J} = \rho \vec{v} \quad \text{and} \quad I = \iint \vec{J} \cdot d\vec{A}$$

Steady currents and Biot-Savart law

Stationary charges produce electric fields that are constant in time. In magnetostatics, a constant magnetic field in time is created by a steady current. By steady current we mean a current that is going on for forever, without any change or any charge piling up. So, in the magnetostatics regime, we will have,

$$\frac{\partial \rho}{\partial t} = 0$$ and consequently due to the continuity equation, $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$, we also have,

$$\vec{\nabla} \cdot \vec{J} = 0 .$$

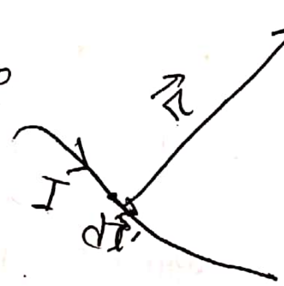
Magnetic field of steady current

The magnetic field due to a steady line current is given by Biot-Savart law,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\imath}}{\imath^2} \, dl'$$

$$= \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl'} \times \hat{\imath}}{\imath^2}$$

with $\vec{dl'}$ is an element of length along the wire, and $\vec{\imath}$ is the vector from the source to the point $\vec{r}$. The constant μ_0 is called the permeability of free space, given by,



$$\mu_0 = 4\pi \times 10^{-7} \; N/A^2$$