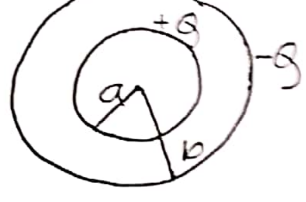
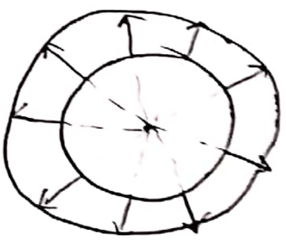
Spherical capacitor

Consider two conducting spherical shells that are concentric. Let, there are charges $+Q$ and $-Q$ on the inner and outer shells respectively. Due to the properties of the conductors, all the charges will reside on the outer and inner surface of the inner and outer shell respectively.



Now, the field due to the outer shell is basically zero, as we already know that there is no electric field inside a spherical shell with uniform charge density. So, the field is entirely due to the inner shell and the field lines will be spherically symmetric.



$$\phi_+ - \phi_- = -\int_{r=b}^{r=a} E\hat{r}\cdot dr\,\hat{r} = -\int_b^a E\,dr$$

$$= -\int_b^a K\frac{Q}{r^2}\,dr = -KQ\left[-\frac{1}{r}\right]_b^a$$

$$\therefore \Delta\phi = KQ\left[\frac{1}{a} - \frac{1}{b}\right]$$

So, the capacitance, $C = \dfrac{Q}{\Delta\phi} = \dfrac{Q}{\frac{1}{4\pi\epsilon_0} Q\left[\frac{1}{a} - \frac{1}{b}\right]}$

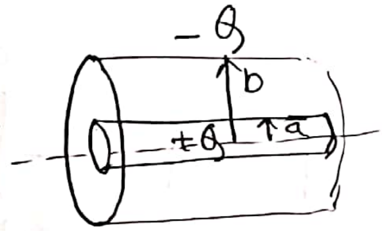
$$\therefore \boxed{C = \frac{4\pi\epsilon_0}{\frac{1}{a}-\frac{1}{b}}} = \frac{4\pi\epsilon_0 ab}{b-a}$$

If the outer shell extend to infinity, meaning $b \to \infty$, then $C = 4\pi\epsilon_0 a$, and we regain the expression for the capacitance of a spherical conductor.

Also, if the distance between the two shells is $d = b-a$ is much smaller than b, then essentially if we define $r \approx a \approx b$ and the area to be $4\pi r^2$, $C = \frac{4\pi r^2 \epsilon_0}{d} = \frac{\epsilon_0 A}{d}$.

Cyllindrical capacitor

Consider a very long (techically infinite cylinder (solid or hollow, doesn't matter), and another cylinder enclosing it throughout all length. If the radii of the cylinders are a and b respectively, calculate the capacitance of the cylindrical capacitor.



$$= \frac{1}{2}\epsilon_0 E^2 (A \times d) = \frac{1}{2}\epsilon_0 E^2 \times \text{volume}$$

$$\therefore \; U = \frac{1}{2}\epsilon_0 E^2 \times \text{Volume}$$

So, energy density, $U' = \frac{1}{2}\epsilon_0 E^2$, which agrees with our result from previous calculations.

Electric fields in matter

Dielectrics

Matter can be found in various different forms. However, they can be divided into two large classes — conductors and insulators. We already talked about the conductors, where there is an unlimited supply of free electrons. In dielectrics/insulators, all charges are attached to particular atoms or molecules. All they can do is move a bit within the atom or molecule, if some external electric field is applied. We want to study the behaviour of such materials under the influence of external electric fields.

Energy stored in a capacitor

Say, at some particular time of charging a capacitor of capacitance C, there is a charge of $+q$ and $-q$ on the plates. Suppose, we increase the charge from q to $q+dq$ by transporting positive charge of ~~ma~~ amount dq from the negative to positive plate, working against the potential difference $\Delta\phi$. The work done is given by,

$$dW = \Delta\phi \, dq = \frac{q}{C} dq$$

$$\Delta\phi = -\int_i^f \vec{E}\cdot d\vec{s}$$

$$\Rightarrow q\Delta\phi = -\int_i^f \vec{F}\cdot d\vec{s}$$

$$\therefore q\Delta\phi = -W$$

The total work done in the whole process if the final charge stored is Q is given by,

$$W = \int_0^Q \frac{q}{C} dq = \frac{1}{C} \frac{q^2}{2} \Big|_0^Q$$

$$\therefore W = \frac{Q^2}{2C}$$

~~Theis~~ This is the amount of potential energy stored in the capacitor.

$$Q = C\Delta\phi$$

$$\therefore U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C(\Delta\phi)^2 = \frac{1}{2}Q\Delta\phi$$

For a parallel plate capacitor, $U = \frac{1}{2}C(\Delta\phi)^2$

$$= \frac{1}{2}\frac{\epsilon_0 A}{d}\cdot(Ed)^2$$

Induced dipoles

If we apply electric field $\vec{E}$, what happens to a neutral atom? An atom is consisted of a positively charged core and an ~~ele~~ electron cloud surrounding it. Due to the electric field, the nucleus (core) is pushed in the direction of the electric field and the electron cloud is slightly displaced to the opposite direction of the electric field. If the field is large enough, the atom can be ionized (meaning electron might start moving freely). However, for small enough electric fields, an equilibrium takes place, since the nucleus and electron cloud will attract them after the displacement because the uniformity is broken. The net effect is that, there is a slight displacement of the charges and the atom is left polarized. The atom now acts like a tiny dipole with dipole moment $\vec{p}$, which points in the same direction of the electric field. The induced dipole moment is roughly proportional to the applied electric field and we can write,

$$\vec{p} = \alpha \vec{E}$$

where α is called the atomic polarizability, differs from atom to atom.

With siplest approximation, you can ~~thing a~~ think an atom as a point charge and a spherical electron cloud. If the separation between the center of the sphere and point charge becomes d under the influence of ~~electric~~ field, then the electric field will be given by,

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{qd}{a^3}$$ if a is the radius of the sphere.

Because, d is the displacement of the nucleus relative to the spherical electron cloud, when the equilibrium is achieved. Since equilibrium is achieved, it means that the electric field on the nucleus due to the negative electron sphere exactly balances the external electric field on the nucleus. So,

$$E_{eletron} = E_{ext}$$

Now, the electric field due to a charged sphere of radius 'a' at a distance of 'd' inside the sphere is given by,

$$E_{electron} = \frac{q}{4\pi\epsilon_0 a^3} \cdot d$$

$$\therefore \quad E_{ext} = \frac{1}{4\pi\epsilon_0} \cdot \frac{qd}{a^3} = \frac{p}{4\pi\epsilon_0 a^3}$$

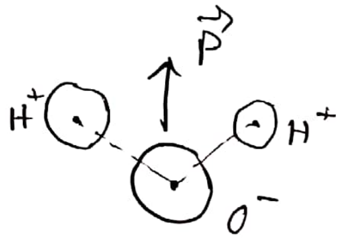
$$\Rightarrow \quad p = (4\pi\epsilon_0 a^3)\, E_{ext}$$

$$\boxed{\therefore \; p \propto E_{ext}} \quad \text{with} \quad \alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 V$$

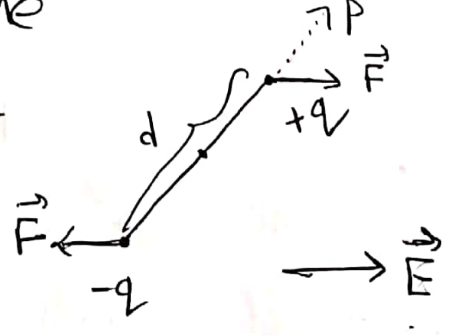
where V is the volume of the atom.

Torque on a dipole and allignment of polar molecules

The dipole moment $\vec{p}$ was induced in insulators/dielectri due to the introduction of external electric field. But there are molecules with built in, permanent dipole moments. Water molecule is a ~~very~~ very good example of such molecule. These molecules are called polar molecules. The dipole moment vector is shown in the figure for H_2O. Now, what happens if we place them in extenal electric field? To answer this question,



lets see what happen for a simple dipole with one positive and one negative charge at a distance 'd' in an uniform electric field. The force on $+q$ is $q\vec{E}$ and $-q$ is $-q\vec{E}$. This will introduce a torque on the dipole system



given by,

$$\vec{\tau} = \vec{r} \times \vec{F}_+ + \vec{r} \times \vec{F}_-$$

where we are taking our origin at the middle of the dipole.

$$\therefore \vec{\tau} = \frac{\vec{d}}{2} \times q\vec{E} + \left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E}) \cancel{= q\vec{E} \times \vec{d}}.$$

$$= \vec{d} \times q\vec{E} = q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$

$$\boxed{\therefore \vec{\tau} = \vec{P} \times \vec{E}} \quad \text{——} (1)$$

This torque will be independent of the choice of our origin since the total force here is zero.
Now, equation (1) is true regardless of the fact that ~~is~~ it is a simple dipole or dipole like water molecule. The proof is left to the reader as an exercise! :P :V

If the electric field was not uniform, then the forces will also not be uniform. ~~So~~ Now, force on a dipole is,

$$\vec{F} = \vec{F}_+ - \vec{F}_- = q\left(\vec{E}_+ - \vec{E}_-\right) = q\Delta\vec{E}$$

where $\Delta\vec{E}$ is the difference between the electric fields between + charge and - charge.

If the dipole is very short, then

$$\left.\begin{aligned} \Delta E_x &= \frac{\partial E_x}{\partial x}\Delta x \\ \Delta E_y &= \frac{\partial E_y}{\partial y}\Delta y \\ \Delta E_z &= \frac{\partial E_z}{\partial z}\Delta z \end{aligned}\right\}$$

$$\therefore \cancel{\Delta E_x\hat{i} + \Delta E_y\hat{j} + \Delta E_z\hat{k}}$$
$$\cancel{= (\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}).\vec{\nabla}\vec{E}}$$
$$\cancel{= \vec{d}.\nabla E}$$
$$\cancel{= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z}$$

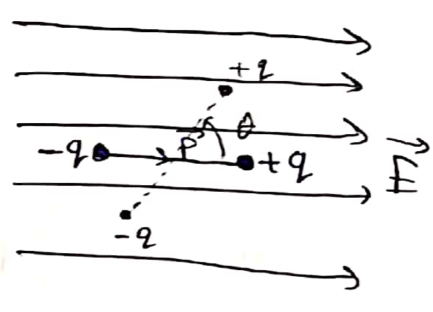
$$\therefore \Delta E_x\hat{i} + \Delta E_y\hat{j} + \Delta E_z\hat{k} = \left[\left(\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}\right).\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right](E_x\hat{i} + E_y\hat{j} + E_z\hat{k})$$

$$\therefore \Delta\vec{E} = (\vec{d}.\vec{\nabla})\vec{E}$$

$$\therefore \vec{F} = q(\vec{d}.\vec{\nabla})\vec{E} = (q\vec{d}.\vec{\nabla})\vec{E} = (\vec{P}.\vec{\nabla})\vec{E}$$

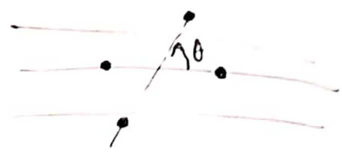
However, equation (i) works pretty well even in non-uniform electric field as ~~th~~ long as the $\vec{E}$-field doesn't change violently at short distances, since atoms are very short in dimension.

The solid line shown in the figure is the stable equilibrium for the dipole in this electric field, meaning it has the minimum potential energy orientation. If we want to rotate it by an angle θ from its original position, then, the work done against the electric field is given by,



$$\int_0^\theta \tau\, d\theta = \int_0^\theta PE\sin\theta\, d\theta = -PE\cos\theta\Big|_0^\theta = PE(1-\cos\theta)$$

$$W = \int q\vec{E} \cdot r\,d\theta\,\hat{\theta} = \int qE\hat{i} \cdot r\,d\theta\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$



$$= -\int q\left(\frac{d}{2}\right)\sin\theta\,d\theta$$

$$= -\frac{1}{2}\int p\sin\theta\,d\theta$$

For both, $W = -\int p\sin\theta\,d\theta$

Polarization

Either insulator or polar molecules, if they are placed in an external electric field, there are ~~tiny~~ a net dipole moment associated inside the material. For insulator, each atom acts like a small dipole with dipole moment in the direction of the electric field. For polar molecules, permanent dipoles, the dipoles allign themselves in the direction of electric field. So, in any way, the material is polarized. A convenient measure of this is the polarization vector, defined as –

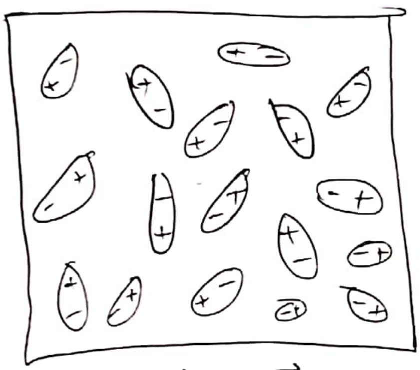
$$\vec{P} = \text{dipole moment per unit volume}.$$

If we define N to be the number of dipoles per unit volume, then,

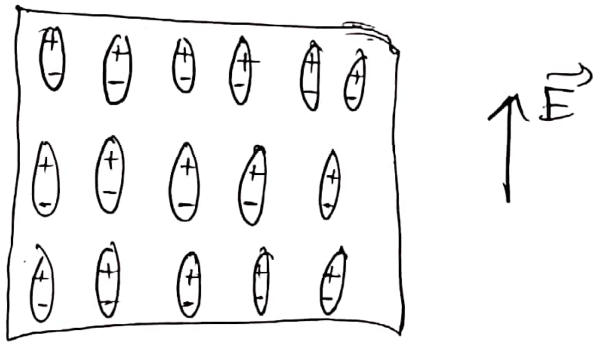
$$\vec{P} = \vec{p}N$$

since all the dipole moments $\vec{p}$ are alligned in the same direction.

Now, since all the dipoles are alligned in the same direction, the figure might look something like shown in the following.



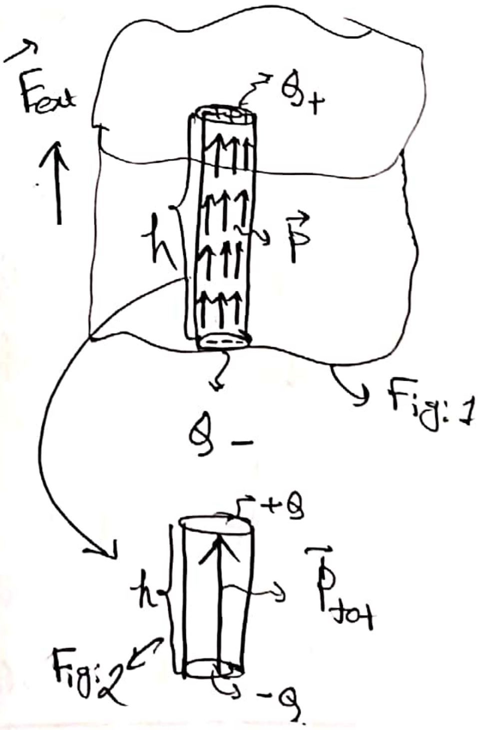
Before $\vec{E}$ was applied



After applying $\vec{E}$

Now, if we consider the whole object, then the internal charges can be considered to cancel out in pairs, and all we are left with a net positive and negative charge on the surface of the object. If we consider a small cylinder, then

$$\vec{P} = N\vec{p}$$



According to Fig. 2, the total dipole moment can be written as. Qh.

$$\therefore Qh = N\vec{p} \times Ah$$

$$\therefore Q = N\vec{p} \times A$$

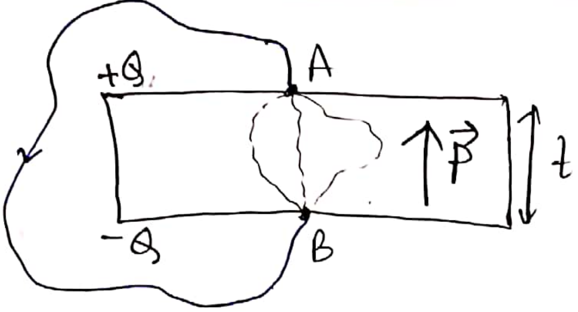
Now, since, $Q = \sigma A$,

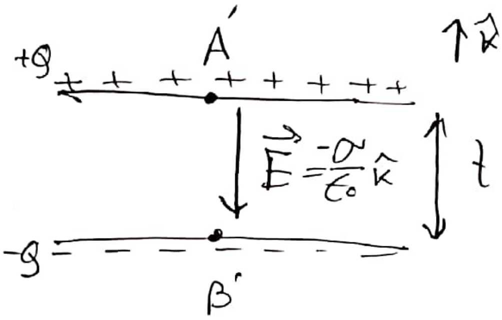
$$\boxed{\therefore\ \sigma = P} \rightarrow \text{Polarization}$$

In general, $\sigma = \vec{P}\cdot\hat{n} = P\cos\theta$

$$\sigma = \frac{Q}{A'} = \frac{Q}{A/\cos\theta}$$

The electric field inside matter





Consider the dielectric slab on the left after being polarized and the two sheet configuration on the right. The electric field inside the slab is not really uniform. Close to the atoms, the electric fields vary violently, both in terms of magnitude and direction. However, one thing is the same. The ~~potential~~ potential difference between A and B, say $\phi_B - \phi_A$, must be the same as between A' and B', say $\phi'_B - \phi'_A$. This is due to the fact that, electric field is conservative, meanig its independ of the path taken what the value of $\int \vec{E}\cdot d\vec{s}$ will be.

$$\phi'_B - \phi'_A = \int_{z_A}^{z_B} \frac{\sigma}{\epsilon_0}\, dz = \frac{\sigma t}{\epsilon_0} = \frac{Pt}{\epsilon_0}$$

This is true because outside the slab and the two sheet configuration is practically the same. So, potential difference between A and B should also be same, as we could take the path $A \to B$ outside the slab also.

Since, the path integral yields the same value, we can conclude that the average electric field inside the slab must also be, $\vec{E}_{in} = -\frac{\vec{P}}{\epsilon_0}$. By average we mean, $\langle \vec{E} \rangle = \frac{\int \vec{E}\, dV}{V}$

Potential and bound charges

Consider the polarized object. For a single dipole,

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}'}{r'^2}$$

where $\vec{r}'$ is the vector from dipole to a point

For a polarized object, $\vec{p} = P\, d\tau$

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P} \cdot \hat{r}'}{r^2}\, d\tau$$

Now, $\nabla'\left(\frac{1}{r'}\right) = \frac{\hat{r}'}{r'^2}$

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0} \int_V \vec{P} \cdot \nabla\left(\frac{1}{r'}\right) d\tau$$

$$= \frac{1}{4\pi\epsilon_0} \left[\int_V \vec{\nabla} \cdot \left(\frac{\vec{P}}{r'}\right) d\tau - \int \frac{1}{r'} (\vec{\nabla} \cdot \vec{P})\, d\tau \right]$$

$$\therefore\ V = \frac{1}{4\pi\epsilon_0} \oint_{\partial V} \frac{1}{r'} \vec{P} \cdot d\vec{A} - \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{r'} (\vec{\nabla} \cdot \vec{P})\, d\tau$$

The first term looks like a potential due to a surface charge density $\sigma = \vec{P} \cdot \hat{n}$ and the second term due to a volume charge density $\rho_b = -\vec{\nabla} \cdot \vec{P}$. This ρ_b arises from non-uniform polarization $\vec{P}$.

Now, in the presence of dielectric media, there will both be free and bound charges (inside the material)

$$\therefore\ \rho = \rho_{free} + \rho_{bound}$$

Using Gauss's law, $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$

$$\Rightarrow \epsilon_0 (\vec{\nabla} \cdot \vec{E}) = \rho_{free} + \rho_{bound}$$

$$\Rightarrow \epsilon_0 (\vec{\nabla} \cdot \vec{E}) = \rho_f + (-\vec{\nabla} \cdot \vec{P})$$

$$\Rightarrow \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\boxed{\therefore\ \vec{\nabla} \cdot \vec{D} = \rho_f}$$

where we defined electric displacement vector as $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.

This is very important, since bound charges can have many different orientation and we do not know that. Working with free charges are easier.

Now, as we discussed earlier, in many sustances, polarization is directly proportional to ~~external~~ electric field.

$$\therefore \vec{P} = \epsilon_0 \chi_e \vec{E} \quad \text{—(11)} \quad \text{where } \boxed{\vec{E} \text{ is the total field.}}$$

where χ_e is called electric susceptibility of the medium. ϵ_0 just makes χ_e dimensionless. The materials that follows equation (11) are called linear dielectrics.

Now, $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1+\chi) \vec{E}$

So, $\vec{D}$ is also proportional to $\vec{E}$ and we write,

$$\vec{D} = \epsilon \vec{E}$$

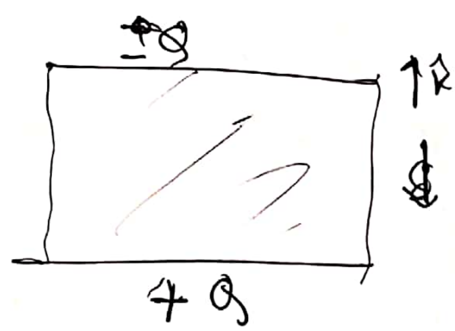
with $\epsilon \equiv \epsilon_0 (1+\chi_e)$, where ϵ is called the permittivity of the material. The relative permittivity is defined as,

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

It is also called dielectric constant.

Dielectric inside capacitor

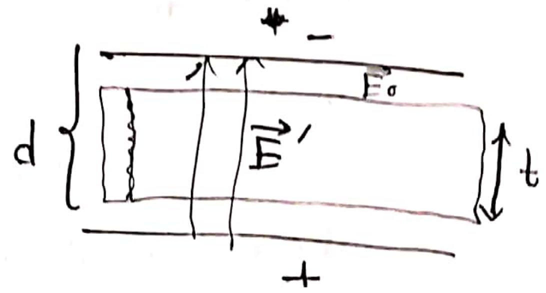
The electric field inside will now be,



$$\vec{E} = \vec{E} + \vec{E}_{pol} = \left(\frac{\sigma}{\epsilon} - \frac{P}{\epsilon_0}\right) \hat{k}$$

$$= -\frac{\cancel{P}}{\cancel{\epsilon}\epsilon_n}$$

Dielectric inside capacitor



$$V_+ - V_- = -\int_{z_-}^{z_+} \vec{E}\cdot d\vec{s}$$

$$= -\int \vec{E}_0 \cdot d\vec{s} - \int \frac{\vec{E}_0}{\epsilon_r}\cdot d\vec{s}$$

$$= -\frac{\sigma}{\epsilon_0}(d-t) - \frac{\sigma}{\epsilon_0\epsilon_r}(t)$$

$$= -\frac{\sigma}{\epsilon_0}\left[(d-t) + \frac{t}{\epsilon_r}\right]$$

$$\vec{E}' = \vec{E}_0 - \frac{\vec{P}}{\epsilon_0}$$

$$= \vec{E}_0 - \frac{\epsilon_0 \chi_e \vec{E}'}{\epsilon_0}$$

$$\Rightarrow \vec{E}_0 = (1+\chi_e)\vec{E}'$$

$$\therefore \vec{E}' = \frac{\vec{E}_0}{1+\chi_e} = \frac{\vec{E}_0}{\epsilon_r}$$

$$\therefore C = \frac{Q}{|\Delta V|} = \frac{\sigma A}{\frac{\sigma}{\epsilon_0}\left[(d-t)+\frac{t}{\epsilon_r}\right]}$$

$$\boxed{\therefore C = \frac{\epsilon_0 A}{(d-t) + \frac{t}{\epsilon_r}}}$$