Centre of Mass

For a discrete system of N particles of masses and positions given by m_i & $\vec{r}_i$, $i=1,\dots,N$ the centre of mass of the system is defined as:

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{M} \quad \text{where} \quad M = \sum_{i=1}^{N} m_i$$

If the system of interest cannot be expressed in terms of discrete particles but some kind of continuous mass distribution then the definition of the centre of mass needs to be generalized accordingly.

Suppose we have a body whose mass density is given by $\rho(\vec{r})$. The units of $\rho(\vec{r})$ is Kg/m^3.

Then in a small volume dV located at $\vec{r}$ the total mass contained is

$$d\mu = \rho(\vec{r})\, dV$$

Then the total mass of the body is given by

$$M = \iiint \rho(\vec{r})\, dV$$

Then the centre of mass of such a system is defined by:

$$\boxed{\vec{R} = \frac{1}{M} \iiint \vec{r}\, \rho(\vec{r})\, dV}$$

It should clear that this integral should be $N \to \infty$ limit of the Riemann sum:

$$\vec{R} = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} \vec{r}_i\, \rho(\vec{r}_i)\, \Delta V}{\sum_{j=1}^{N} \rho(r_j)\, \Delta V}$$

where $\Delta V = \frac{V}{N}$ with V being the total volume of the body & N being the number of small pieces it has been divided into.

Comments:

1. The integral expression for the centre of mass is not always possible to compute analytically except for objects with some high level of symmetry. However, it is still a very useful and insightful expression.

Examples:

1. A rod with non-uniform mass density.

Consider a rod of length L whose linear mass density is given by $\lambda = \lambda_0 (x/L)$ where λ_0 is constant. $[\lambda] = [M][L^{-1}]$

Then the mass $d\mu$ of a small element between x and $x + dx$ is

$$d\mu = \lambda(x)\, dx \quad \Rightarrow \quad M = \int_0^L \lambda(x)\, dx = \frac{\lambda_0}{L} \frac{x^2}{2}\Big|_0^L = \frac{\lambda_0 L^2}{2L} = \frac{\lambda_0 L}{2}$$

Then the centre of mass is at:

$$X = \frac{1}{M}\int x\, d\mu = \frac{1}{M}\int x\, \lambda(x)\, dx = \frac{2}{\lambda_0 L} \frac{\lambda_0}{L} \int_0^L x^2\, dx = \frac{2}{L^2} \frac{L^3}{3}$$

$$X = \frac{2}{3} L.$$

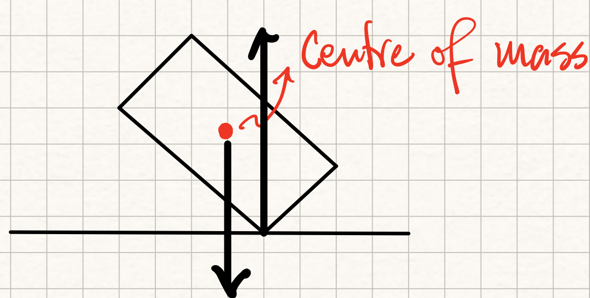
A practical method for finding the centre of mass of an object:

The motion of an object if it is not a point particle has two components:

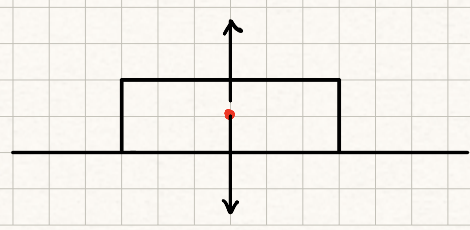
1. The motion of its centre of mass
2. The motion of the body relative to the centre of mass.

To find the centre-of-mass of an object we can take advantage of the fact that if the force is along a line that goes through the centre of mass then there is no motion of the second type.

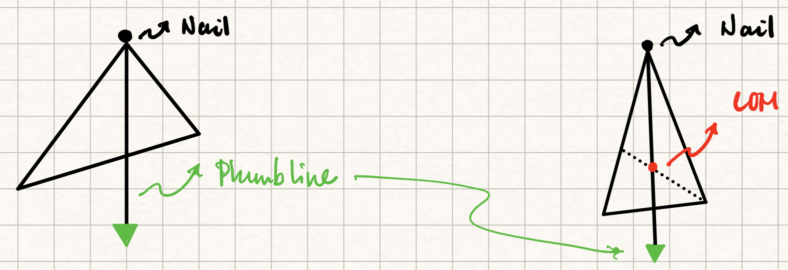
Relative motion



No relative motion



Suppose we have a thin object and we have to find its centre of mass. We can hang it from a nail such that there is no relative or translational motion. We can hang a plumb line from the nail and we know that the COM is going to be on the line. If we do this from two independent points the point at



Comment:

1. Here we are using the fact that in a uniform gravitational field the centre of mass coincides its centre of gravity (the point through which gravity acts). If the field were not uniform then we couldn't use this trick.

Impulse & change of Momentum:

We have seen that when the external force on a system is zero the total momentum of the system does not change. Next we explore the circumstances when there is an external force on the system which for the moment we take to be a particle:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

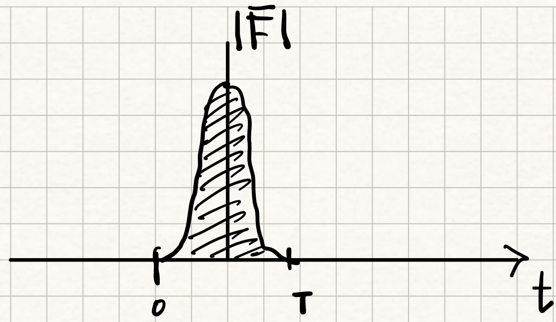
which implies

$$\Delta\vec{p} = \vec{P}(t_2) - \vec{P}(t_1) = \int_{t_1}^{t_2} \vec{F}(t)\, dt$$

If the force is constant during the time interval from t_1 to t_2 then

$$\Delta\vec{p} = F(t_2 - t_1)$$

Often in collision problems the force on a particle acts only for a short time. The time profile of such a force is



In such cases the integrand in the integral has very compact support and it is called the impulse:

$$\text{Impulse} = \int_0^T \vec{F}(t)\,dt$$

Then we see that

$$\boxed{\Delta\vec{P} = \text{Impulse} = \int_0^T \vec{F}(t)\,dt}$$

The Impulse-Momentum Theorem.

If $\langle \vec{F} \rangle$ is the average force during the time interval $\Delta T = T - 0$ then we can write

$$\Delta\vec{P} = \langle \vec{F} \rangle \Delta T$$

Work & Kinetic Energy

Suppose a particle moves through some distance during the time a force is applied on it. Then the impulse momentum theorem gives us how the momentum changes. But during such a process if the only process that the particle/system undergoes is mechanical then there is another set of useful concepts which we develop now.

Consider the Work $W \equiv \int_0^{\vec{x}} \vec{F} \cdot d\vec{x}'$

Then Newton's Law gives us:

$$\int_0^{\vec{x}} \vec{F} \cdot d\vec{x}' = \int_0^{\vec{x}} m \frac{d\vec{v}}{dt} \cdot d\vec{x}'$$

$$= \int_0^T m \frac{d\vec{v}}{dt} \cdot \frac{d\vec{x}'}{dt}\,dt = \int_0^T m \frac{d\vec{v}}{dt} \cdot \vec{v}\,dt = \frac{1}{2} m \int_0^T \frac{d}{dt} v^2\,dt = \frac{1}{2} m v^2(T) - \frac{1}{2} m v^2(0)$$

$$\boxed{W = \frac{1}{2} m v^2(T) - \frac{1}{2} m v^2(0)}$$

The quantity $\frac{1}{2} m v^2(t)$ is called the kinetic energy of the system and the above result is known as the Work-Energy theorem. The work-energy Theorem states that the change in kinetic energy of a system is equal to the work done on it.