Consider the N-mass oscillator that we solved. Our solution was –

$$x_n(t) = \sum_{p=1}^{N} a_p \sin\left(p\frac{n\pi}{N+1}\right)\cos(\omega_p t - \phi_p)$$

with $\omega_p = 2\omega_0 \sin\left(\frac{p\pi}{2(N+1)}\right)$

Here we have took the linear combination for all possible modes. This is the most general solution. But this equation requires a total of $2N$ number of undetermined constants, N number of a_p and N number of ϕ_p, that has to be ~~bed~~ determined from the initial conditions. But how do we find this huge number of undetermined constants, where N might be very large?

Now, if you remember how we built the idea of the normal modes, you know that, we first found the normal frequencies, then calculated the amplitudes (normal modes) matrix A, which happend to be eigenvector of ~~$M^{-1}K - \omega^2 I$~~ $M^{-1}K$ matrix. Now, the normal modes were the building blocks of any type of motion of the blocks. We were basically taking the linear combination to find the motion of the blocks at any time.

Think about the symmetric case of equal mass and equal spring constants. Start with $N=2$ masses.

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} k+k_c & -k_c \\ -k_c & k+k_c \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & 1/m \end{pmatrix}$$

$$\therefore M^{-1}K = \begin{pmatrix} \frac{k+k_c}{m} & -\frac{k_c}{m} \\ -\frac{k_c}{m} & \frac{k+k_c}{m} \end{pmatrix}$$ which is a symmetric matrix. There is a theorem in Linear Algebra, that eigenvectors (corresponding to different eigenvalues) of a symmetric matrix are orthogonal. Here orthogonality of eigenvector A goes like,

$$A_i^T A_j = L\,\delta_{ij}$$

You can find the proof anywhere

So, $A_i^T A_j = 0$ if $i \neq j$.

Let's check for $N=2$ case. There, $A_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$A_1^T A_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

If you write, $A_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_{2,1} \\ a_{2,2} \end{pmatrix}$

with $a_{n,p}$ = value of p^{th} normal mode of n^{th} mass, then

$$A_1^T A_2 = a_{1,1}\, a_{2,1} + a_{2,1}\, a_{2,2} = \sum_{p=1}^{2} a_{n,p}\, a_{m,p}$$

with $n=1$ and $m=2$.

For N-coupled oscillator, $A_{n,p} = \sin\left(n \frac{p\pi}{N+1}\right)$

Since $M^{-1}K$ matrix is symmetric, the eigenvalues are orthogonal.

$$\therefore \sum_{p=1}^{N} A_{n,p}\, A_{m,p} = 0 \quad \text{if } n \neq m$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$\Rightarrow \sum_{p=1}^{N} \sin\left(n \frac{p\pi}{N+1}\right) \sin\left(m \frac{p\pi}{N+1}\right) = 0 \quad \text{if } n \neq m$$

If $n=m$, then $\displaystyle\sum_{p=1}^{N} \sin^2\left(n \frac{p\pi}{N+1}\right) = \sum_{p=1}^{N} \frac{1}{2}\left[1 - \cos \frac{2np\pi}{N+1}\right]$

But $\displaystyle\sum_{p=1}^{N} \cos\left(\frac{2np\pi}{N+1}\right)$ is zero for all $N > 1$.

$$\therefore \sum_{p=1}^{N} \sin^2\left(n \frac{p\pi}{N+1}\right) = \sum_{p=1}^{N} \frac{1}{2} = \frac{N}{2}$$

$$\therefore \sum_{p=1}^{N} \sin\left(n \frac{p\pi}{N+1}\right) \sin\left(m \frac{p\pi}{N+1}\right) = \frac{N}{2}\,\delta_{n,m}$$

which is called the discrete sine transform.

Now, let's see how this will help us to solve

2N initial

$$x_n(t) = \sum_{p=1}^{N} a_p \sin\left(n\frac{p\pi}{N+1}\right) \cos(\omega_p t - \phi_p)$$

At $t=0$,

$$x_n(0) = \sum_{p=1}^{N} a_p \sin\left(n\frac{p\pi}{N+1}\right) \cos\phi_p$$

Multiplying both sides with $\sin\left(n\frac{p'\pi}{N+1}\right)$ and summing over $n=1$ to N we get,

$$\sum_{n=1}^{N} x_n(0) \sin\left(p'\frac{n\pi}{N+1}\right) = \sum_{p=1}^{N}\sum_{n=1}^{N} a_p \sin\left(p\frac{n\pi}{N+1}\right) \sin\left(p'\frac{n\pi}{N+1}\right) \cos\phi_p$$

$$\Rightarrow \sum_{n=1}^{N} x_n(0) \sin\left(p'\frac{n\pi}{N+1}\right) = \sum_{p=1}^{N} a_p \delta_{pp'} \cos\phi_p \frac{N}{2}$$

$$\Rightarrow \sum_{n=1}^{N} x_n(0) \sin\left(p'\frac{n\pi}{N+1}\right) = \frac{N}{2} a_{p'} \cos\phi_{p'}$$

$$\therefore \quad a_p \cos\phi_p = \frac{2}{N} \sum_{n=1}^{N} x_n(0) \sin\left(p\frac{n\pi}{N+1}\right) \quad \text{— (1)}$$

Similarly, differentiating the expression of $x_n(t)$ we get,

$$\dot{x}_n(t) = -\sum_{p=1}^{N} a_p \omega_p \sin\left(n\frac{p\pi}{N+1}\right) \sin(\omega_p t - \phi_p)$$

$$\Rightarrow \dot{x}_n(0) = \sum_{p=1}^{N} a_p \omega_p \sin\left(n\frac{p\pi}{N+1}\right) \sin\phi_p$$

Using the same procedure above, we get,

$$a_p \sin\phi_p = \frac{2}{N\omega_p} \sum_{n=1}^{N} \dot{x}_n(0) \sin\left(p\frac{n\pi}{N+1}\right)$$

$$\therefore\ a_p \cos\phi_p = \frac{2}{N} \sum_{n=1}^{N} x_n(0) \sin\left(n \frac{p\pi}{N+1}\right) \quad \text{---} \ (*)$$

$$a_p \sin\phi_p = \frac{2}{N\omega_p} \sum_{n=1}^{N} \dot{x}_n(0) \sin\left(n \frac{p\pi}{N+1}\right) \quad \text{---} \ (**)$$

And, this two equation gives you the solution for all a_p and ϕ_p!

Let's solve a problem. Say, the first mass only was displaced by A, with all other remaining in equilibrium. They all were released from zero initial velocities.

$$\therefore\ x_1(0) = A,\quad x_2(0) = 0, \ \ldots\ldots, \ x_N(0) = 0$$

$$\dot{x}_1(0) = 0,\quad \dot{x}_2(0) = 0, \ \ldots\ldots, \ \dot{x}_N(0) = 0$$

From $(**)$ we get,

$$a_p \sin\phi_p = 0 \qquad \therefore\ \phi_p = 0 \text{ for all } p = 1, 2, \ldots N$$

Plugging this in $(*)$ we get,

$$a_p = \frac{2}{N} \sum_{n=1}^{N} x_n(0) \sin\left(n \frac{p\pi}{N+1}\right)$$

$$= \frac{2}{N} A \sin\left(\frac{p\pi}{N+1}\right)$$

$$\boxed{\therefore\ a_p = \frac{2A}{N} \sin\left(\frac{p\pi}{N+1}\right)}$$

and, you exactly know how the motion will look like. You can now solve for any number of mass!

Now, let's get back to where we left off before Fourier analysis. We had the wave equation at hand, which was given by–

For transverse oscillation:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y(x,t)}{\partial x^2}$$

For longitudinal motion:

$$\frac{\partial^2 \xi(x,t)}{\partial t^2} = \frac{E}{\mu}\frac{\partial^2 \xi(x,t)}{\partial x^2}$$

For finding the solution to the wave equation, let's think about our general procedure. We had our solution of the form.

$$y(x,t) = a(x)e^{i\omega t}$$

$$x_n(t) = a_n e^{i\omega t}$$

This was our starting point

Plugging this in the wave equation we get,

$$\cancel{\partial^2 y(x,t)} \quad -\omega^2 a(x)\, e^{i\omega t} = \frac{T}{\mu}\frac{d^2}{dx^2} a(x)\, e^{i\omega t}$$

$$\rightarrow \frac{d^2 a(x)}{dx^2} = -\frac{\omega^2 \mu}{T} a(x)$$

This is nothing but our good old second order linear, homogenous equation with the solution -

$$a(x) = A e^{\pm ikx} \quad \text{with } k = \omega\sqrt{\frac{\mu}{T}}$$

where k is called the wave number, bearing a dimension of $\left[\frac{1}{L}\right]$. The physical meaning of k is - if λ is the wavelength (which is the distance between the points with same phase), then a ~~whole~~ whole cycle must be complete, and so kx will be equal to 2π.

$$\therefore k\lambda = 2\pi \qquad \boxed{\therefore k = \frac{2\pi}{\lambda}}$$

Now, the general solution of $y(x,t)$ for a given mode with frequency ω is given by,

$$y(x,t) = A_1 e^{i(kx+\omega t)} + A_2 e^{-i(kx+\omega t)} + B_1 e^{i(kx-\omega t)} + B_2 e^{-i(kx-\omega t)}$$

Since $y(x,t)$ is real, we must have $A_1^* = A_2$ and $B_1^* = B_2$ and,

$$y(x,t) = A\cos(kx+\omega t+\phi_A) + B\cos(kx-\omega t+\phi_B) \qquad (1)$$

This equation represents two travelling waves, one towards right and another towards left. We will talk about travelling waves later. Now, expanding we get

$$y(x,t) = A[\cos(kx+\omega t)\cos\phi_A - \sin(kx+\omega t)\sin\phi_A] + B[\cos(kx-\omega t)\cos\phi_B - \sin(kx-\omega t)\sin\phi_B]$$

$\therefore y(x,t) = C_1 \cos(kx+\omega t) + C_2 \sin(kx+\omega t) + C_3 \cos(kx-\omega t) + C_4 \sin(kx-\omega t)$ —— (II)

with $C_1 = A\cos\phi_A$, $C_2 = -A\sin\phi_A$ and so on.

This equation also represents travelling waves, each of the four terms. Expanding more, we get,

$$y(x,t) = (C_1+C_3)\cos\omega t\cos kx + (C_3 - C_1)\sin kx \sin\omega t + (C_2 - C_4)\cos\omega t \sin kx$$
$$+ (C_2+C_4)\cos kx \sin\omega t$$

$$\boxed{\therefore y(x,t) = D_1 \cos kx \cos\omega t + D_2 \sin kx \sin\omega t + D_3 \sin kx \cos\omega t + D_4 \cos kx \sin\omega t}$$ —— (III)

These four terms represents four different standing waves. All the points on the string reaches maximum displacement at the same time and crosses zero at the same time. Take just one term – $D_1 \cos kx \cos\omega t$. $D_1 \cos kx$ gives the amplitudes as a function of x, and $\cos\omega t$ govern the time evolution. So, all of them move in phase. They do not travel anywhere. They just oscillate up and down remaining at the same horizontal position. This is ~~wat~~ what we call a standing wave.

Equation (II) and (III) implies that, any travelling wave can be written as a sum of standing waves and vice versa! Look, we have four undetermined constants for a mode, where we previously had only two (one amplitude and one phase). Why four now? The reason is that, we haven't yet imposed the wall boundary conditions. If we do use that, then,

$$y(0,t) = 0 \quad \text{and} \quad y(L,t) = 0$$

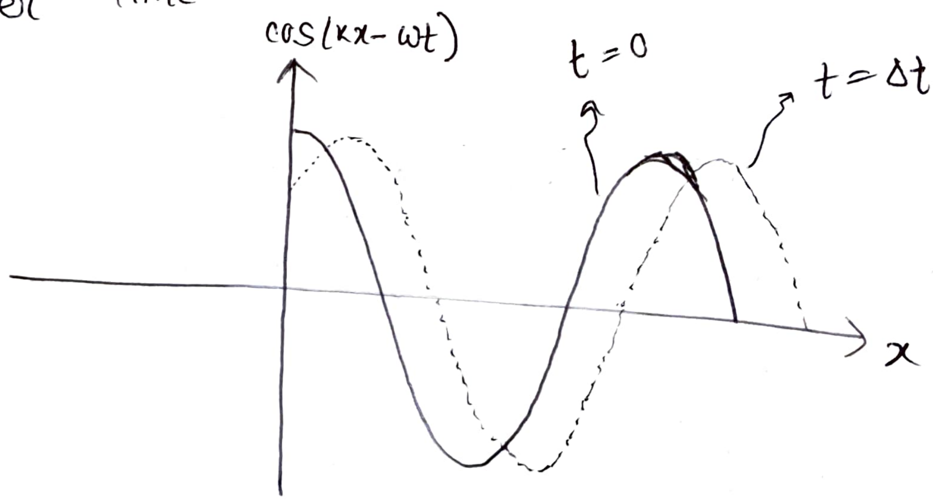
(ii) $\Rightarrow$

$$\Rightarrow \quad D_1 \cos \omega t + D_4 \sin \omega t = 0 \quad ; \text{ for all } t.$$

This is only possible if D_1 and D_4 is zero, and we are again back to two undetermined constants.

Travelling waves

Consider the third term of equation (2) — ~~$C_3 \cos(kx-\omega$~~ $C_3 \cos(kx - \omega t)$. Let's plot $\cos(kx-\omega t)$ as a function of x for two different times separated by Δt. If we take the former time to be $t=0$, then the later time is Δt.



The solid ~~curv~~ curve is a plot of just $\cos kx$, and the dashed curve is a plot of $\cos(kx-\phi)$, with $\phi = \omega \Delta t$. It is shifted to the right because it now takes larger value of x to obtain the same phase.

Now, what is the horizontal shift between the two ~~curves~~ curves? Finding the distance between the maxima will suffice to calculate the horizontal shift.

Maxima of solid curve occurs when, $\cos(kx - \omega . 0) = 1$

$$\Rightarrow \cos kx = \cos 0 \quad \begin{pmatrix} \text{or} \\ 2n\pi \end{pmatrix}$$

$$\therefore x = 0$$

" " dashed " " ", $\cos(kx - \omega\Delta t) = \cos 0$

$$\Rightarrow kx - \omega \Delta t = 0$$

$$\therefore x = \frac{\omega \Delta t}{k}$$

$\therefore$ Horizontal shift $= \frac{\omega \Delta t}{k} - 0 = \frac{\omega \Delta t}{k}$

So, the velocity of the wave will be, $v = \frac{\frac{\omega \Delta t}{k}}{\Delta t}$

$$\boxed{\therefore v = \frac{\omega}{k}}$$

But again, ~~th~~ $k = \omega \sqrt{\frac{\mu}{T}}$ and so, $v = \sqrt{\frac{T}{\mu}}$

So, the wave equation now can be written as-

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2} \quad \Bigg| \quad \frac{\partial^2 \xi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \xi(x,t)}{\partial x^2}$$

So, $\cos(kx-\omega t)$ represents a wave that travels with a velocity v towards right, so is for $\sin(kx-\omega t)$. Similarly, $\cos(kx+\omega t)$ and $\sin(kx+\omega t)$ represents waves travelling towards left with a velocity of $v=\frac{\omega}{k}$. But remember, none of the masses are actually travelling with this velocity. With the small amplitude approximations, the actual velocities of the masses are very small.

Using the trigonometric identities you can clearly see how standing waves combine to form travelling waves and vice-versa.

$$\overset{T.W.}{\cos(kx-\omega t)} = \overset{S.W.}{\cos kx \cos\omega t} + \overset{S.W.}{\sin kx \sin\omega t}$$

$$\underbrace{\cos kx \cos\omega t}_{S.W.} = \frac{1}{2}\left[\underbrace{\cos(kx-\omega t)}_{T.W.}\right] + \frac{1}{2}\left[\underbrace{\cos(kx+\omega t)}_{T.W.}\right]$$

A more general solution

Our wave equation is now,

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

We claim, $f(x-vt)$ will be a solution to the wave equation. This is a very strong claim. We are saying, any function of the form, not necessarily the sinusoidals, any arbitrary function $f(x-vt)$ will be a solution to the wave equation. Let's plug in and verify!

$$\frac{\partial^2 f(x-vt)}{\partial t^2} = \cancel{(-v)^2 \frac{\partial^2 f(x-vt)}{\partial t^2} = v^2 \frac{\partial^2 (x-vt)}{\partial t^2} \frac{\partial^2 f(x)}{\partial^2 (x-vt)}}$$

$$\cancel{\frac{\partial^2 f(x-vt)}{\partial x^2} = \frac{\partial^2 f(x-vt)}{\partial x^2}}$$

But now, $$\frac{\partial^2 f(x-vt)}{\partial t^2} = \left(\frac{\partial z}{\partial t}\right)^2 \cdot \frac{\partial^2}{\partial^2 z} f(z) \text{ with } z = x-vt$$

$$= (-v)^2 \frac{\partial^2 f(z)}{\partial^2 z} = v^2 \frac{d^2 f(z)}{d^2 f(z)}$$

$$v^2 \frac{\partial^2 f(x-vt)}{\partial x^2} = v^2 \left(\frac{\partial z}{\partial x}\right)^2 \cdot \frac{\partial}{\partial^2 z} f(z) = (1)^2 v^2 \frac{\partial^2 f(z)}{\partial^2 (z)}$$

$$= v^2 \frac{d^2 f(z)}{d^2 z}$$

$$\therefore \frac{\partial^2 f(x-vt)}{\partial t^2} = v^2 \frac{\partial^2 f(x-vt)}{\partial x^2}$$

But why is it true? The answer lies in Fourier analysis. You can write the Fourier transform as-

$$f(z) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dx$$

If you plug $z = x-vt$, then, $f(x-vt) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{i(kx-\omega t)} \, dx$

And we see something beautiful. $f(x-vt)$ is the linear combination of (integral) $e^{i(kx-\omega t)}$ with multiplicative constants $\tilde{f}(k)$ [interpret this in integral term as we did in Fourier transform. But that's what we expect. Our solution to the wave equation says $y(x,t)$ must be a linear combination of $e^{i(kx-\omega t)}$, which represents travelling waves towards right. So, $f(x-vt)$ will satisfy the wave equation (so will $f(x+vt)$). So, any arbitrary function $f(x-vt)$ is a solution to the wave equation!