Lecture 5

Transient phenomena

All the previous description of forced oscillation were based on the steady state. That is, we have only concentrated on the particular solution. But as we argued, this is not the complete solution. The complete solution to the forced oscillation is given by,

$$x(t) = x_h(t) + x_p(t)$$

$$\therefore \quad x(t) = A_h\, e^{-\frac{\gamma}{2}t} \cos(\omega t + \theta) + A_p \cos(\omega_d t - \phi) \quad \text{——(i)}$$

This equation correctly dictates the motion of the driven damped oscillator, with two undetermined constan[ts] A_h and θ, which depends on the initial conditions. But a[s] we have seen, the homogenous solution $x_h(t)$ eventually die away with time due to damping. So, if we wait lo[ng] enough (depending on the strength of the damping) $t \gg \frac{1}{\gamma}$, the oscillator will eventually reach the steady state and the oscillator will then be totally described by,

$$x(t) = A_p \cos(\omega_d t - \phi) \quad \text{——(ii)}$$

Equation (i) is called the transient solution and (ii) is called the steady state solution to the driven damped oscillator.

Now, $A = \dfrac{\quad}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}}$

Since for large t, we are essentially left with the steady state solution, then, if two different oscillator starts with wildly different initial conditions, but ~~even~~ are subjected to the same driving force, the motions will essentially be same for both the oscillators. All the memories of the past will be lost!

Driven oscillation for undamped oscillator

Although this is an unrealistic case, let's consider briefly what happens if we try to drive an undamped oscillator. This can be achieved from our previous calculations by setting $\gamma = 0$.

$$x(t) = A_h e^{-\frac{\gamma}{2}t}\left[\cos(\omega t+\theta) + A_p \cos(\omega_d t - \phi)\right.$$

$$A_p = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}} \qquad \phi = \tan^{-1}\frac{\gamma\omega_d}{\omega_0^2-\omega_d^2} = \tan^{-1}\left(\frac{0}{\cdots}\right.$$

$$\therefore x(t) = A_h \cos(\omega t+\theta) + \frac{F_0/m}{\omega_0^2-\omega_d^2}\cos(\omega_d t - \phi)$$

Here, $\phi = 0$; if $\omega_d < \omega_0$ ✓

and $\phi = \pi$; if $\omega_d > \omega_0$.

For $\omega_d < \omega_0$, we write,

$$x(t) = A_h \cos(\omega t+\theta) + B\cos(\omega_d t) \quad \text{——— (iii)}$$

A_h and θ can be found from initial conditions. One possible initial condition, that we might impose is –

$$\text{At } t=0, \quad x=0, \quad v=\frac{dx}{dt}=0$$

$$\therefore \quad 0 = A_h \cos\theta + B$$

and,
$$\frac{dx}{dt} = -\omega A_h \sin(\omega t+\theta) - \omega_d B \sin(\omega_d t)$$

$$\therefore \quad \left.\frac{dx}{dt}\right|_{t=0} = -\omega A_h \sin\theta$$

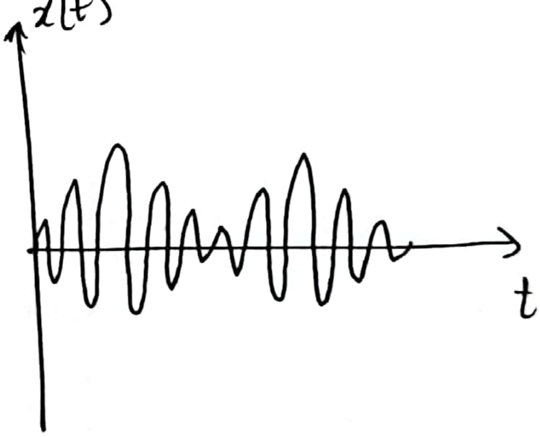
$$\therefore \quad 0 = -\omega A_h \sin\theta$$

$$\therefore \quad \theta = 0 \text{ or } \pi .$$

Taking $\theta = 0$, we get, $B = -A_h$

$$\boxed{\therefore \quad x(t) = B\left(\cos\omega_d t - \cos\omega_0 t\right)}$$ from equation (iii) — (iv)

The oscillatory motion continues ~~to~~ forever in the zero damping case. Let's check what happens ~~to~~ just after $t=0$. Since $\omega_d t, \omega_0 t \ll 1$, we can write –



A typical graph of (iv)

$$\cos\omega_d t = 1 - \frac{\omega_d^2 t^2}{2}$$

$$\cos\omega_0 t = 1 - \frac{\omega_0^2 t^2}{2}$$

$$\therefore \quad x(t) = \frac{F_0/m}{\omega_0^2-\omega_d^2}\left[1 - \frac{\omega_d^2 t^2}{2} - 1 + \frac{\omega_0^2 t^2}{2}\right]$$

Now, $A = \frac{\quad}{\sqrt{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}}$

$\therefore$ ~~x(t)~~ $\boxed{\therefore x(t) = \frac{1}{2}\frac{F_0}{m}t^2}$

This is what we should expect. Before the restoring force is called into play, the mass starts out in the direction of the applied force with acceleration F_0/m.

Power in driven damped oscillator

In a driven and damped oscillator, the driving force feeds energy into the system during some parts of the motion and takes energy out during other parts (except at resonance, where it always feeds energy in). The damping force always takes energy out of the system. But, in the steady state, the motion is periodic, and hence the energy should stay the same on average, since the amplitude is not changing. So, the average net power from the driving force must equal to the average net power loss due to damping force.

Power is the rate at which work is done.

$$P = \frac{dW}{dt} = F\frac{dx}{dt} \quad \Big| \; dW = F\,dx$$

$$= Fv$$

Energy of a driven damped oscillator

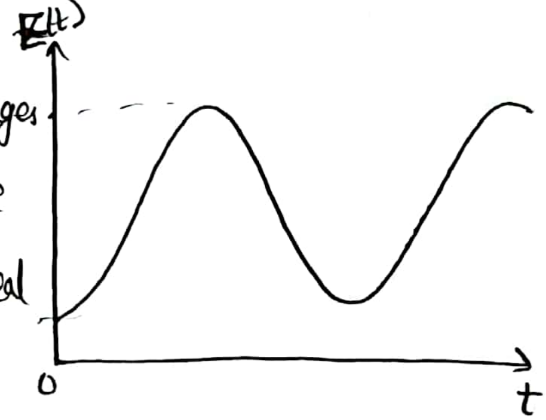
Kinetic energy, $K(t) = \frac{1}{2} m \dot{x}(t)^2 = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t - \phi)$

Potential energy, $U(t) = \frac{1}{2} k x^2 = \frac{1}{2} k A^2 \cos^2(\omega t - \phi)$

$$\therefore E(t) = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t - \phi) + \frac{1}{2} kA^2 \cos^2(\omega t - \phi)$$

If $\omega_d \neq \omega_0$, ~~en~~ total mechanical energy changes with time, over a period of oscillation. A typical energy curve is shown in the following figure for $\phi = 0$.

Since total mechanical energy changes with time, let's calculate ~~as~~ a the time average of the mechanical energy.



Now, $\langle \sin^2(\omega t - \phi) \rangle = 1/2$ and $\langle \cos^2(\omega t - \phi) \rangle = \frac{1}{2}$

$$\therefore \langle U(t) \rangle = \frac{1}{4} kA^2 \qquad \langle K(t) \rangle = \frac{1}{4} m\omega^2 A^2$$

$$\therefore \langle E(t) \rangle = \frac{1}{4} (k + m\omega^2) A^2 = \frac{1}{4} (\cancel{k}\, m\omega_0^2 + m\omega^2) A^2$$

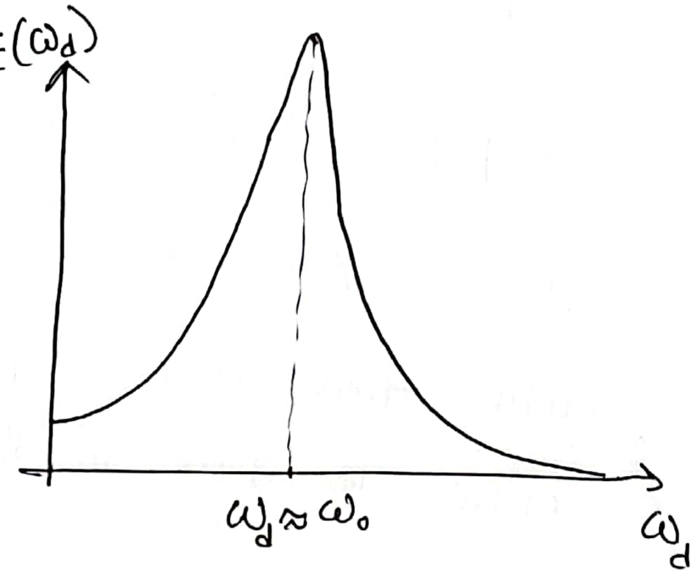
$$= \frac{1}{4} m (\omega_0^2 + \omega^2) A^2$$

Now, $A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}}$

$$\therefore \quad \langle E(\omega_d)\rangle = \frac{1}{4}\,\frac{F_0^2/m^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2} \times m\,(\omega_0^2+\omega_d^2)$$

$$\therefore \langle E(\omega_d)\rangle = \frac{F_0^2}{4m}\,\frac{\omega_0^2+\omega_d^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}$$

The average energy is maximum when the oscillator is operating at $\omega_d \approx \omega_0$, that is in resonance.



Now, power dissipated by damping force:

$$P_{damping} = F_{damping} V = -b\dot{x}\times\dot{x}$$

$$= -b\dot{x}^2$$

$$= -b\left[-\omega_d A \sin(\omega t-\phi)\right]^2$$

$$= -b\omega^2 A^2 \sin^2(\omega_d t-\phi)$$

$$x(t) = A\cos(\omega_d t-\phi)$$

$$\frac{dx}{dt} = -\omega_d A\sin(\omega_d t-\phi)$$

Since $\sin^2(\omega t-\phi)$ is always positive, the power due to damping is always negative.

Power supplied by the driving force:

$$P_{driving} = F_{driving} V = \left(F_0 \cos\omega_d t\right)\dot{x}$$

$$= F_0 \cos\omega t\left[-\omega_d A\sin(\omega_d t-\phi)\right]$$

$$= -\omega A F_0 \cos\omega_d t\left[\sin\omega_d t\cos\phi - \cos\omega_d t\sin\phi\right]$$

$$= -\omega A F_0\left[\cos\omega t\sin\omega_d t\cdot\cos\phi - \cos^2\omega_d t\sin\phi\right]$$

$$= -\omega A F_0\left[\frac{1}{2}\sin 2\omega_d t\cos\phi - \cos^2\omega_d t\sin\phi\right]$$

Let's calculate the average powers. To calculate average power dissipation or power supplied in one complete cycle for $T=\frac{2\pi}{\omega_d}$. Let's first calculate the followings.

$$\langle\sin^2(\omega_d t-\phi)\rangle = \frac{\int_0^T \sin^2(\omega_d t-\phi)\,dt}{T}$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left[1 - \cos 2(\omega_d t - \phi)\right] dt = \frac{1}{2T}\left[(T-0) - \frac{\sin 2(\omega_d t - \phi)}{2\omega_d}\right]_0^T$$

$$= \frac{1}{2T}\left[T - \frac{\sin(2\omega_d T - 2\phi)}{2\omega_d} + \frac{\sin(-2\phi)}{2\omega_d}\right]$$

$$= \frac{1}{2} - \frac{\sin(4\pi - 2\phi)}{4\omega_d T} + \frac{\sin(-2\phi)}{4\omega_d T}$$

$$= \frac{1}{2} - \frac{\sin(-2\phi)}{4\omega_d T} + \frac{\sin(-2\phi)}{4\omega_d T} = \frac{1}{2}$$

Similarly,

$$\langle \cos^2(\omega_d t + \phi) \rangle = 1/2$$

$$\langle \sin(2\omega_d t) \rangle = 0$$

$$\therefore \langle P_{damping} \rangle = -b\omega_d^2 A^2 \times \frac{1}{2} = -\frac{1}{2} b\, \omega_d^2 A^2$$

$$\langle P_{driving} \rangle = +\omega_d A F_0 \cdot \frac{1}{2} \sin\phi = \frac{1}{2}\omega_d A F_0 \sin\phi$$

Now, $\tan\phi = \dfrac{\gamma\omega_d}{\omega_0^2 - \omega_d^2}$ $\quad \therefore \sin\phi = \dfrac{\gamma\omega_d}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \gamma^2\omega_d^2}}$

$$\therefore \sin\phi = \frac{\gamma\omega_d}{F_0/mA} = \frac{mA\gamma\omega_d}{F_0}$$

$\left| \dfrac{b}{m} = \gamma \quad \therefore b = \gamma m \right.$

$$\therefore \langle P_{driving} \rangle = \frac{1}{2}\omega_d A F_0 \times \frac{mA\gamma\omega_d}{F_0} = \frac{1}{2} b\, \omega_d^2 A^2$$

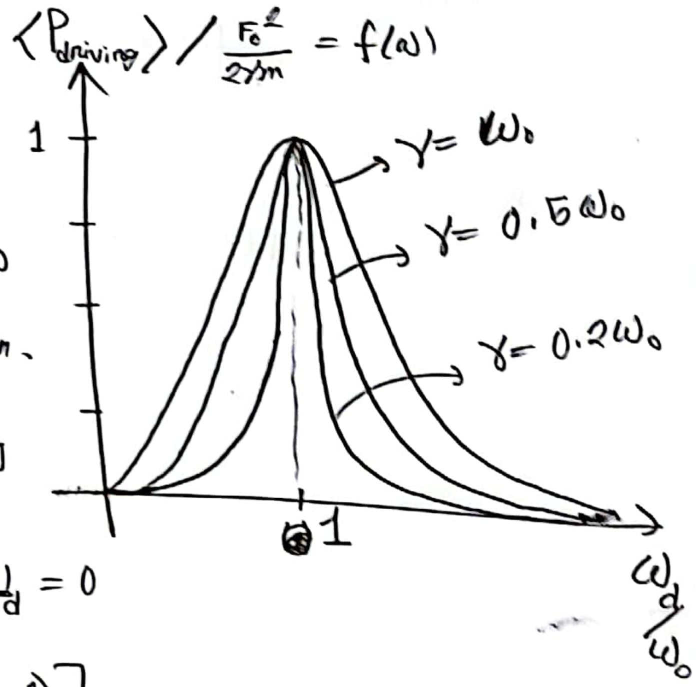
$$\boxed{\therefore \langle P_{damping} \rangle + \langle P_{driving} \rangle = 0}$$

Now, $$\langle P_{driving} \rangle = \frac{1}{2} \gamma m \omega_d^2 \frac{(F_0/m)^2}{(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}$$

$$= \frac{F_0^2 \gamma m}{2m^2(\gamma^2)} \frac{\gamma^2 \omega_d^2}{(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}$$

$$\therefore \langle P_{driving} \rangle = \frac{F_0^2}{2\gamma m} f(\omega_d) \quad \text{with } f(\omega_d) = \frac{\gamma^2 \omega_d^2}{(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}$$

The reason of writing like this is that $f(\omega_d)$ is now dimensionless. The maximum of $\langle P_{driving} \rangle$ occurs when $f(\omega_d)$ is maximum. ~~This happens when the denominator is minimum.~~



$$\cancel{\therefore \frac{d}{d\omega_d}\left[(\omega_0^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2\right] = 0}$$

$$\cancel{\rightarrow -2(\omega_0^2 - \omega_d^2) \cdot 2\omega_d + 2\gamma^2 \omega_d = 0}$$

$$\cancel{\Rightarrow 2\omega_d \left[\gamma^2 - 2(\omega_0^2 - \omega_d^2)\right] = 0}$$

$$\therefore \frac{d}{d\omega_d}\left[f(\omega_d)\right] = 0 \quad \text{and} \quad \frac{d^2}{d\omega_d^2} f(\omega_d) \Big|_{\omega_d = \omega_0} < 0$$

→ You will be able to show that, this happens for $\omega_d = \omega_0$.

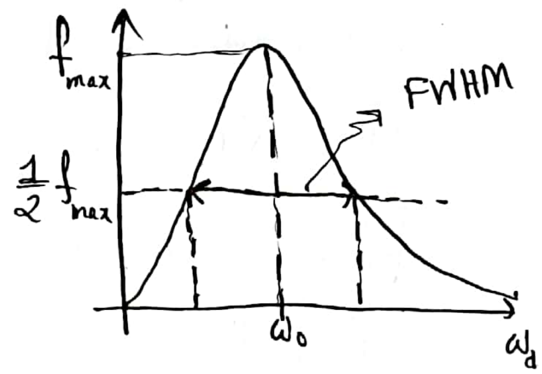
At $\omega_d = \omega_0$, $f(\omega_d) = 1$.

$$\therefore \langle P_{driving} \rangle_{max} = \frac{F_0^2}{2\gamma m}$$

As we can see, the widths of the graphs gets narrower

with decreasing γ. Let's define the full width at half maximum (FWHM) to find the dependence of width with γ.
So, we want to calculate the width of the curve (in terms of ω_d) at the value when $\langle P_{driving}\rangle = \frac{1}{2}\langle P_{driving}\rangle_{max}$.
For this, concentrating on $f(\omega_d)$ will be enough since $\langle P_{driving}\rangle$ is $f(\omega_d)$ multiplied by constant $\langle P_{driving}\rangle_{max}$.

$$f_{max}(\omega_d) = 1$$



Now,
$$f(\omega_d) = \frac{\gamma^2\omega_d^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}$$

$$\Rightarrow \frac{1}{2} = \frac{\gamma^2\omega_d^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}$$

$$\Rightarrow 2\gamma^2\omega_d^2 = (\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2 \Rightarrow \gamma^2\omega_d^2 = (\omega_0^2-\omega_d^2)^2$$

$$\therefore \omega_0^2-\omega_d^2 = \pm\gamma\omega_d$$

$$\therefore \omega_d^2+\gamma\omega_d-\omega_0^2=0 \qquad \omega_d^2-\gamma\omega_d-\omega_0^2=0$$

$$\therefore \omega_d = \frac{-\gamma\pm\sqrt{\gamma^2+4\omega_0^2}}{2} \qquad \omega_d = \frac{\gamma\pm\sqrt{\gamma^2+4\omega_0^2}}{2}$$

We are only interested in positive roots.

$$\therefore \omega_{d_1} = \frac{\gamma+\sqrt{\gamma^2+4\omega_0^2}}{2} \qquad \omega_{d_2} = \frac{-\gamma+\sqrt{\gamma^2+4\omega_0^2}}{2}$$

Then, $\text{FWHM} = \omega_{d_1} - \omega_{d_2} = \gamma$

$\therefore$ The full width at half maximum is exactly $=\gamma$.

Reason behind $\langle P_{driving} \rangle$ curve and phase lags

~~Fast driving $\omega \gg \omega_0$~~

Slow driving ($\omega_d \ll \omega_0$)

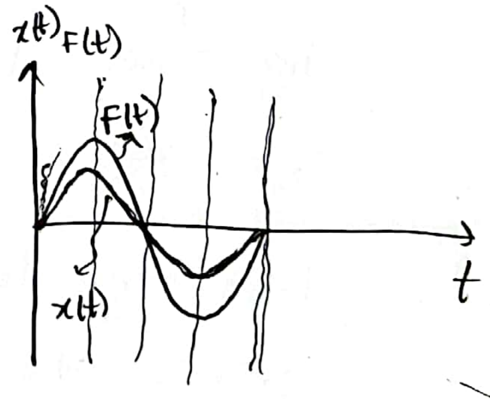
When $\omega_d \ll \omega_0$, $\phi \approx 0$ and $A = \frac{F_0/m}{\omega_0^2} = \frac{F_0}{k}$

$$\therefore x(t) = \frac{F_0}{k} \cos \omega_d t$$

So, the mass just follows the driving force. They are in phase. So, when the force is upward, the block moves upward and vice versa.

Here, spring force, $F_s = -kx = -F_0 \cos \omega_d t = -F_d$. So, the driving force just balances the spring force. The damping is kind of irrelevant since the velocity is very small. The block will be hardly moving (hardly accelerating), meaning the net force will be zero.

To see why $\langle P_{driving} \rangle$ is very small in this limit, let's consider the graphs. In the first quarter cycle, $\dot{x}(t)$ is positive, $F(t)$ is positive, so the power is positive. You are doing positive work because the force is in the direction of motion.



In second quarter cycle, $\dot{x}(t)$ is negative, but $F(t)$ is still positive, so power is negative. You are doing negative work since the block is coming back.

to the origin, but the force is away from the origin. Similarly in third quarter cycle, you are doing positive work and in fourth negative. So, overall effect is that, they cancel each other out.

Fast driving ($\omega_d \to \infty$ or $\omega_d \gg \omega_0$)

~~When $\omega_d \gg \omega_0$, $\phi \approx \pi$ and $A \approx \frac{F}{\omega^2}$~~

When $\omega_d \gg \omega_0$,

$$\phi \approx \pi \quad \text{and} \quad A \approx \frac{F_0/m}{\omega_d^2}$$

$$\therefore x(t) = \frac{F_0}{m\omega_d^2} \cos(\omega_d t - \pi) = -\frac{F_0}{m\omega_d^2} \cos \omega_d t$$

The amplitude is very small (since ω_d is very large). Note that,

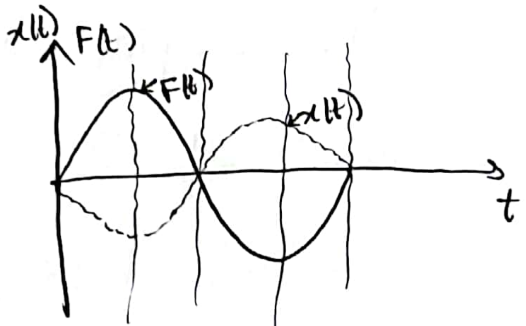
$$m\ddot{x}(t) = +\frac{F_0}{m\omega_d^2}\omega_d^2 \cos \omega_d t = \overset{F_0/m}{F_d} \cos \omega_d t.$$

So, the driving force seems entirely responsible for the acceleration. Since the mass hardly moves, spring and damping force plays no role here. The velocity and position are very small, all we are gonna count is acceleration.

Now, why the phase lag of π? Because the driving force provides all the force essentially. So, it must be in phase with acceleration. But acceleration is always 180° out of phase with position. So, the driving force is 180° out of phase with position.

Now, about the power due to driving force.

In first quarter cycle, the block is going in the opposite direction of the direction in which the force is being applied. So, the work done



is negative (you are slowing the mass down). In the second quarter cycle, the direction of motion and force are same, giving positive work. Same happens for other two quarter cycles. So, eventually net work done is zero and so is power.

Resonance ($\omega = \omega_0$)

Now, $\phi = \frac{\pi}{2}$ and $A = A_{max} = \frac{F_0/m}{\gamma\omega} = \frac{F_0/m}{\gamma\omega_0} = \frac{F_0}{m\gamma\omega_0}$

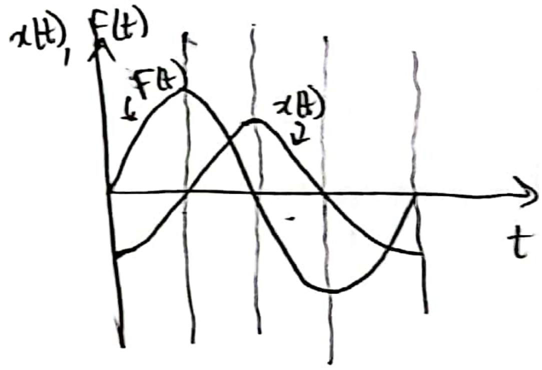
$\therefore x(t) = \frac{F_0}{m\gamma\omega_0}\cos\left(\omega_0 t - \frac{\pi}{2}\right) = \frac{F_0}{m\gamma\omega_0}\sin\omega_0 t$

Here, $F_{damping} = -b\dot{x} = -m\gamma \cdot \frac{F_0}{m\gamma\omega_0} \times \omega_0 \cos\omega_0 t$

$= -F_0\cos\omega_0 t$

So, the damping force is exactly opposite to the driving force and they cancel each ~~th~~ other out. So, the effect of driving force is to cancel the damping force. It makes sense since the oscillator will oscillate with the natural frequency ω_0, the spring and mass are doing exactly what they were supposed to do without the damping and driving force.

Now, $\phi = \frac{\pi}{2}$, here, because, the amplitude is very large. So, we have to provide a lot of energy in the system. To do that, the driving force has to ~~work with~~ act over largest possible distance. For greater power, we need driving force to be large, when velocity is large. So, we want driving force and velocity to be in phase. But since the phase difference between velocity and position is $\frac{\pi}{2}$, so should be with driving force and position.



In this case, for first half cycle, the mass is moving from negative maximum to positive maximum, and the force is always in ~~the~~ the direction of motion. This is also same in the next half cycle (just in opposite sense). So, the work done is positive throughout the whole process, and we ~~we get the~~ get the maximum power.