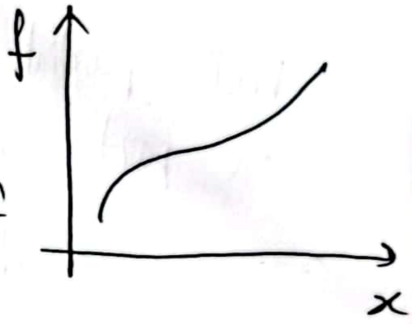
# Lecture 1

# Review of vector calculus

## Ordinary derivatives

Let's say, we have a function of single variable $x$, which is given by $f(x)$. At this point, we know the derivative of the function with respect to $x$, given by, $\frac{df}{dx}$.




Geometrically, $\frac{df}{dx}$ gives you the slope of the graph of $f(x)$ vs $x$. Now, we can write,

$$df = \left(\frac{df}{dx}\right) dx \quad \text{---(i)}$$

Equation (i) implies that, if we change $x$ by a very small amount (infinitesimal) $dx$, then the value of the function changes accordingly as stated in equation (i) and hence $\frac{df}{dx}$ is the proportionality factor.

## Gradient

Now, let's say, we have a function of three variables, say temperature in a room,

$$T(x, y, z)$$

Now, derivatives tell you how fast a function changes as you move by little distances. But, now we have three variables, and we have to specify in which direction we are moving to exactly calculate the rate of change. However, if we move in some particular direction, we technically move along x, y and z. How can we associate the change then? Well, partial derivatives comes into the rescue. A theorem on partial derivatives tells that,

$$dT = \left(\frac{\partial T}{\partial x}\right)dx + \left(\frac{\partial T}{\partial y}\right)dy + \left(\frac{\partial T}{\partial z}\right)dz$$

$$\Rightarrow dT = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\boxed{\therefore dT = (\vec{\nabla} T) \cdot d\vec{s}} \text{ ——}$$

where $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is called the gradient operator and $d\vec{s}$ is the infinitesimal displacement vector.

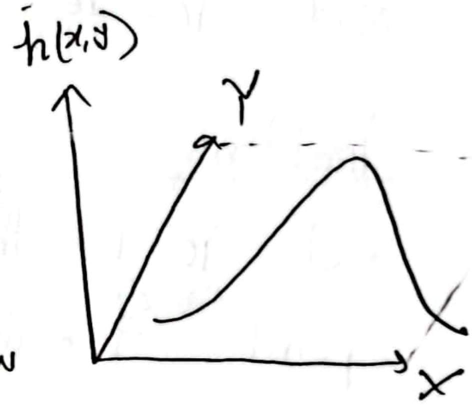
## Geometric interpretation

Gradient of T, $\vec{\nabla} T$ is a vector and hence it will have both magnitude and direction. We can now write dT as,

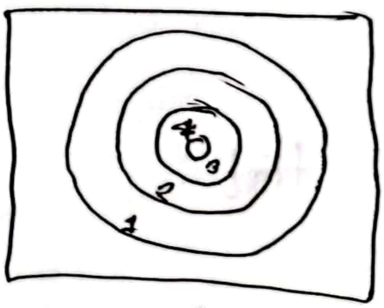
$$dT = \vec{\nabla} T \cdot d\vec{s} = |\vec{\nabla} T|\,|d\vec{s}|\cos\theta$$

where $\theta$ is angle between $\vec{\nabla}T$ and $d\vec{s}$. Now, if we want to find the direction which corresponds to maximum change in T, it will surely occur for $\theta = 0^\circ$. So, dT is maximum when we move along the direction of $\vec{\nabla}T$. So, gradient of some function points in the direction of maximum increase and $|\vec{\nabla}T|$ gives the slope (rate of change) along this maximal direction.

Let's consider the gradient for a two variable function. Consider you want to climb a hill. The height of the hill is given as a function of $x$ and $y$.




So, for each values of x and y we get the height of the hill at that point. We can draw some contour lines, where the height essentially remains the same. If you do not want to climb the hill, rather just roam around, you will move along a particular contour, say 1.




In my drawing, $h_4 > h_3 > h_2 > h_1$. If you want to climb the hill very fast, then you should move perpendicular to the contour lines, and that's where the gradient should direct.

## Problem 1

Find the gradient of magnitude of position vector $r = \sqrt{x^2+y^2+z^2}$. Does the gradient direct in the direction of maximum increase in the magnitude of position vector?

~~Problem 2~~

~~Let $\vec{r}$ be the separation vector between f~~

## Problem 2

Let $\vec{r}$ be the separation vector from a fixed point $(x', y', z')$ to the point $(x, y, z)$ and say $r$ is its magnitude. Show that,

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$$

Hint:

$$\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$

## Divergence

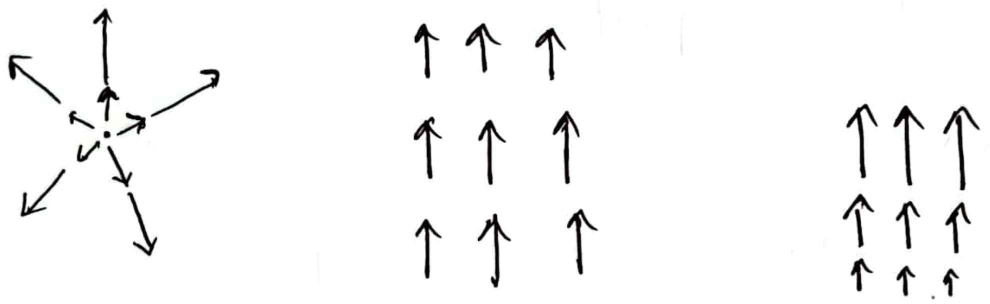
If ~~f~~ $\vec{V}$ is a vector function, meaning $\vec{V} = \vec{V}(x,y,z)$, then the divergence is defined as,

$$\vec{\nabla} \cdot \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

## Geometrical interpretation

The divergence is a measure of how much the vector $\vec{V}$ spreads out (diverges) from a particular ~~questio~~ point.



(a) (b) (c)

So, (a) has a positive divergence, (b) has a zero divergence and (c) has a positive divergence.

## Problem

Calculate the divergence of the ~~f~~ vector function, $\vec{v} = \frac{\hat{r}}{r^2}$. ~~Sketch~~

## Curl

$$\vec{V} = V_x(x,y,z)\hat{i} + V_y(x,y,z)\hat{j} + V_z(x,y,z)$$

For a vector function $\vec{V}$ we define the curl of $\vec{V}$ as,

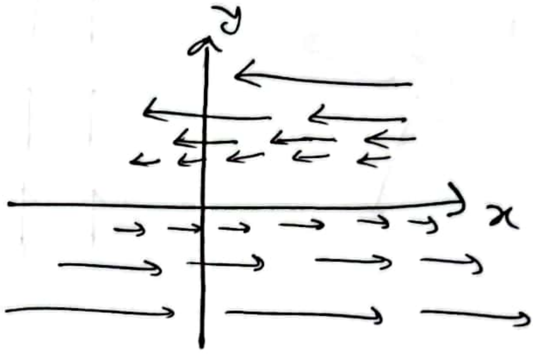
$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

## Geometric interpretation

Curl is a measure of how much the vector $\vec{V}$ swirls about a point.


ⓐ




ⓑ

The vector functions in figure ⓐ and ⓑ has curls, basically pointing in the $z$- direction. However, the figures in divergence has zero curl.

## Second derivatives

1. Divergence of a gradient: $\vec{\nabla}\cdot(\vec{\nabla}T)$

$$\vec{\nabla}\cdot(\vec{\nabla}T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$\vec{\nabla}\cdot(\vec{\nabla}T)$ is often written as $\nabla^2 T$, which is called the Laplacian operator.

2. Curl of a gradient: $\vec{\nabla}\times(\vec{\nabla}T)$

$$\vec{\nabla}\times(\vec{\nabla}T) = \vec{0}$$

3. Divergence of a curl: $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v})$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$$

4. Curl of curl: $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}$$

where,

$$\nabla^2\vec{v} = (\nabla^2 v_x)\hat{i} + (\nabla^2 v_y)\hat{j} + (\nabla^2 v_z)\hat{k}$$

# Integral calculus

## Line integral / path integral

A line integral is defined as –

$$\int_a^b \vec{v} \cdot d\vec{s}$$

where $\vec{v}$ is a vector function, $d\vec{s}$ is infinitesimal displacement vector and the integral is carried away along a prescribed path $p$ from a point 'a' to a point 'b'.

Line integral in a closed path is something that we often encounter is shown as –

$$\oint \vec{v} \cdot d\vec{s}$$

which means the integration is carried out along a closed path/loop.

Problem: Calculate the line integral of the function $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - z^2y\,\hat{k}$ along a path characterized by $x = 4t$, $y = 3t^2$ and $z = 2$, from ~~(2,-3,2) to (4,0,0)~~. $t = 5$ to $t = 7$.

# Surface integral

Surface integral is defined as —

$$\iint_S \vec{v} \cdot d\vec{A}$$

where the integral is specified over a surface S. $d\vec{A}$ denotes infinitesimal patch of area, with a direction perpendicular to the surface. The area vector is always taken in the perpendicular direction to the surface. Obviously there are two perpendicular directions, and hence the sign of the surface integral is kind of ambiguous. If the surf-ace is closed, then the surface integral is,

$$\oiint \vec{v} \cdot d\vec{A}$$

In cases of closed surface, its a general notation that we take the outward direction to be positive area vector.

For arbitrary surfaces this might get tricky. However, for surfaces with certain symmetries, we will be able to calculate the surface integral pretty much easily.

## Volume integrals

For a scalar function T, the volume integral is defined as –

$$\iiint_V T \, d\tau$$

where $d\tau$ is the infinitesimal volume element. In Cartesian coordinates, $d\tau = dx\,dy\,dz$.

## Fundamental theorem of calculus

If $f(x)$ is a single ~~vat~~ variable function, then fundamental theorem of calculus says –

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(b) - f(a)$$

So, the integral of the derivative over some region is given by the value of the function at the boundaries. This might not be surprising as we can write,

$$df = \left(\frac{df}{dx}\right) dx$$

and hence,

$$\int_a^b df = f(b) - f(a)$$

## For gradient

The fundamental theorem of calculus for gradient says,

$$\int_a^b (\vec{\nabla} T) . d\vec{s} = T(\vec{b}) - T(\vec{a})$$

There ~~is~~ are two interesting properties of the gradient.

i) The line ~~inte~~ integral of the gradient is independent of the path taken (since the line integral only depends on the endpoints).

(ii) $\oint (\vec{\nabla} T) . d\vec{s} = 0$ since $T(\vec{L}) - T(\vec{a}) = T(\vec{b}) - T(\vec{L})$
$= 0$

## For divergence

The fundamental theorem of divergence tells us ~~the~~ that,

$$\iiint_V (\vec{\nabla} . \vec{v}) d\tau = \oint_{\partial V} \vec{v} . d\vec{A}$$

where $V$ is the volume over which the volume integral is carried out and $\partial V$ is the ~~bounda~~ boundary surface that encloses the volume. This theorem is called Gauss's theorem, Green's theorem and famously Gauss-divergence theorem.

The theorem states that the integral of a derivative (here it is divergence) over a volume is equal to the value of the function at the boundary (enclosing closed surface). Boundary of a curve/line are just two points, boundary of a volume is a closed surface.

## For curls

The fundamental theorem of calculus for curls is denoted by -

$$\iint_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \oint_{\partial S} \vec{V} \cdot d\vec{s}$$

(small s)

(capital S)

The theorem is also called Stoke's theorem.

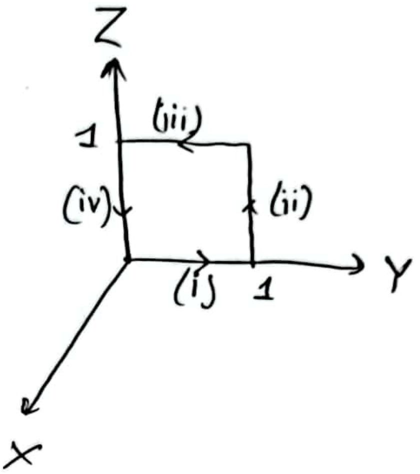
Again, the integral of the derivative of the curl over a open surface is equal to the value of the function at the boundary (here, the perimeter path that encloses the surface). It has again two interesting properties.

(i) $\iint_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{A}$ depends only on the boundary path, not on the particular surface.

(ii) $\oiint_S (\vec{\nabla}\times\vec{V})\cdot d\vec{A} = 0$ for any closed surface, since the boundary of the surface then shrinks to a point, making the right ~~a~~ hand side vanishing.

## Problem

If $\vec{V} = (2xz+3y^2)\hat{j} + (4yz^2)\hat{k}$, then ~~prove~~ show that Stoke's theorem holds for the square surface shown here.



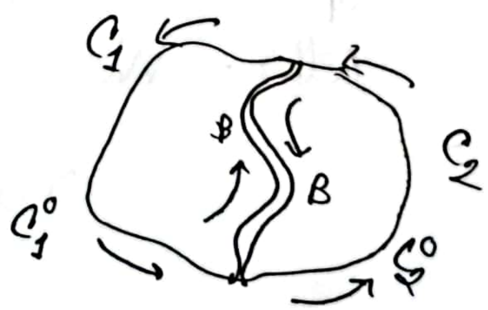

~~Curvl~~

## An attempt to visualize Stoke's theorem

~~A~~ We calculate the line integral along the curve $C$ in closed loop.


$$\therefore \Gamma = \oint_C \vec{V}\cdot d\vec{s}$$

Now, let's divide the surface into two new ~~closed~~ surfaces — $C_1$ and $C_2$.



Both the surfaces have one common path – B. ~~Surface~~ Contour $C_1$ consists of $C_1^0$ and B. ~~Surface~~ Contour $C_2$ consists of $C_2^0$ and B.

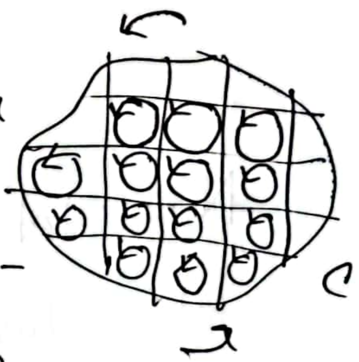
Now,
$$\Gamma = \int_{C_1} \vec{V}\cdot d\vec{s} + \int_{C_2} \vec{V}\cdot d\vec{s}$$

$$= \int_{C_1^0} \vec{V}\cdot d\vec{s} + \int_{B} \vec{V}\cdot d\vec{s} + \int_{C_2^0} \vec{V}\cdot d\vec{s} + \int_{B} \vec{V}\cdot (-d\vec{s})$$

$$\cancel{= \left( \int_{C_1^0} \vec{V}\cdot d\vec{s} + \int_{B} \vec{V}\cdot d\vec{s} \right.}$$

$$= \int_{C_1^0} \vec{V}\cdot d\vec{s} + \int_{C_2^0} \vec{V}\cdot d\vec{s} = \int_{C} \vec{V}\cdot d\vec{s}$$

So, even if we break a big contour into parts, the internal contours will cancel out and we will only be left with the original contour if we choose the internal ~~contours~~ bridging contours (B here) to go in opposite direction. We can then keep breaking the contours and cover up the whole area with circulation cells. We can then define curl as –




$$\vec{\nabla}\times\vec{V}\cdot\hat{n} = \lim_{A\to 0} \frac{\oint_C \vec{V}\cdot d\vec{s}}{A}$$

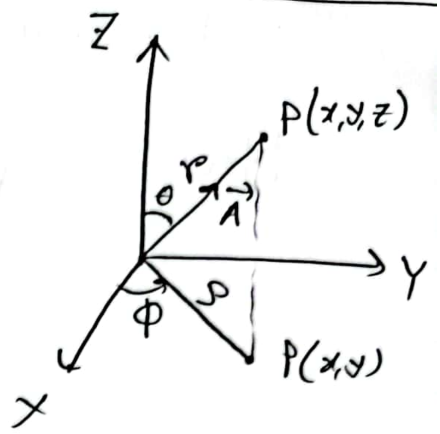
So, curl is defined as circulation of $\vec{v}$ per unit area.

Now,
$$\Gamma = \oint_C \vec{v}\cdot d\vec{s} = \sum_{i=1}^{N} \oint_{C_i} \vec{v}\cdot d\vec{s}$$
$$= \sum_{i=1}^{N} \oint_{C_i} A_i \frac{\vec{v}\cdot d\vec{s}}{A_i}$$
$$= \sum_{i=1}^{N} \cancel{\oint_{C_i}} A_i \oint_{C_i} \frac{\vec{v}\cdot d\vec{s}}{A_i}$$
$$= \sum_{i=1}^{N} A_i \, (\vec{\nabla}\times\vec{v})\cdot\hat{n}$$
$$= \sum_{i=1}^{N} (\vec{\nabla}\times\vec{v})\cdot\vec{A}_i$$

$$\boxed{\therefore \oint_C \vec{v}\cdot d\vec{s} = \iint_S (\vec{\nabla}\times\vec{v})\cdot d\vec{A}}$$

So, the circulation of a vector over a closed contour is equal to the flux of the curl of the vector through the surface bounded by the contour.

# Curvilinear coordinates and infinitesimal displacements




$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Any vector in spherical polar coordinate can be written as –

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

with

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\vec{r} = r\hat{r}$$

Infinitesimal displacements:

$\cancel{dr = dr}$

$$dS_r = dr$$

$$dS_\theta = r\,d\theta$$

$$dS_\phi = r\sin\theta\,d\phi$$

$$\therefore\ d\vec{S} = \underbrace{dr\,\hat{r} + r\,d\theta\,\hat{\theta}}_{\text{In 2D}} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$d\tau = dS_r\,dS_\theta\,dS_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$