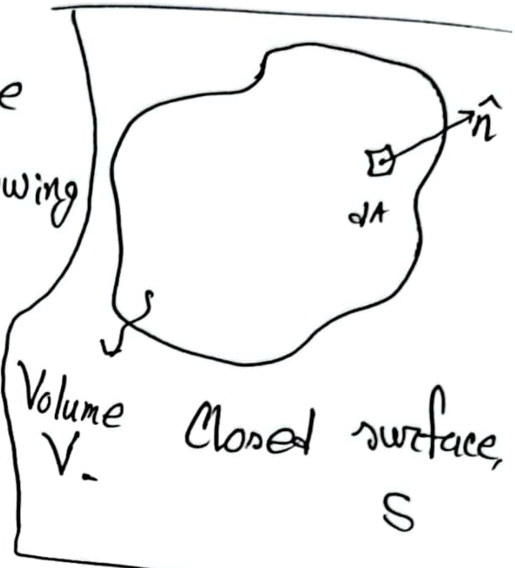
Lecture 1 Q and A

1. Proof of Gauss-divergence theorem:

Flux of a vector field

Let's say we want to find the flow of say heat that is flowing out of the volume V. Let $\vec{h}$ denotes the heat that flows through a unit area per unit time.



Now, we can calculate the heat flowing out of the volume by just calculating the total heat flow out of the surface S. Let's consider infinitesimal area element dA, which is Cartesian coordinates can be considered as $dA = dx\,dy$ in XY plane.

Now, the heat flow out of the surface element dA is the area times the component of $\vec{h}$ perpendicular to dA. So, the flow outward is -

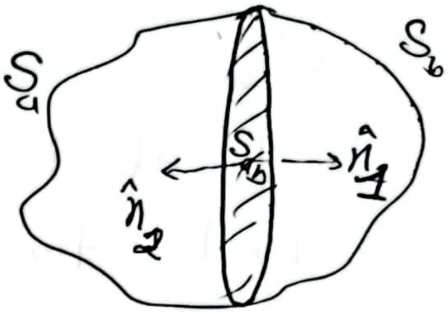
$$\vec{h} \cdot \hat{n}\, dA = \vec{h} \cdot d\vec{A}$$

The total flow is then given by,

$$\Phi = \iint_S \vec{h} \cdot d\vec{A}$$

This surface integral S is called the flux of $\vec{h}$ through the surface.

Now, let's see an interesting property. We divide the volume into to — V_1 and V_2. Accordingly the surfaces enclosing V_1 and V_2 are S_1 and S_2.



Flux through S_1 = Flux through S_a + Flux through S_{ab}

$$= \iint_{S_a} \vec{h}\cdot d\vec{A} + \iint_{S_{ab}} \vec{h}\cdot \hat{n}_1 dA$$

$\hat{n}_1$ is unit normal to S_{ab} for S_1

Flux through $S_2 = \iint_{S_b} \vec{h}\cdot d\vec{A} + \iint_{S_{ab}} \vec{h}\cdot\hat{n}_2\, d\vec{A}$

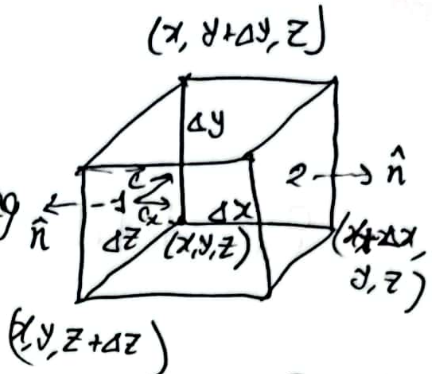
Since its obvious that $\hat{n}_1 = -\hat{n}_2$, then total flux through S = Flux through S_1 + Flux through S_2

= Flux through S_a + Flux through S_b

We may now subdivide V_1 and V_2 and continue the same procedure. Ultimately, the flux through the outer boundary surface is the sum of all the ~~little~~ flux through the little surfaces.

If the little volumes are very small, we ~~cat at~~ can always approximate them as ~~squares~~ cubes.

Let's calculate the flux through the cube. We can compute this by finding the flux through six faces.



Consider face 1. The $\hat{n}$ is in $-x$ direction. So the flux outward on this face is the negative x component of $\vec{C}$, integrated over the area of the face.

$\therefore$ The flux is $= -\int C_x \, dy \, dz$

Since we are considering a small cube, we can approximate the integral by the value of C_x at the center of face 1, that we call point (1), multiplied by area of the face, given by $\Delta y \Delta z$.

$$\therefore \text{Flux out of } 1 = -C_x(1) \Delta y \Delta z$$

$$\text{Flux out of } 2 = -C_x(2) \Delta y \Delta z$$

Because at a distance δx apart

Now, $C_x(1)$ and $C_x(2)$ are slightly different. If Δx is small enough, then –

$$C_x(2) = C_x(1) + \frac{\partial C_x}{\partial x} \Delta x$$; excluding higher order terms

$$\therefore \text{Flux out of } 2 = \left[C_x(1) + \frac{\partial C_x}{\partial x} \Delta x \right] \Delta y \Delta z$$

Total

$\therefore$ Flux out of 1 and 2 $= \frac{\partial C_x}{\partial x} \Delta x \Delta y \Delta z$

" " " " 3 and 4 $= \frac{\partial C_y}{\partial y} \Delta x \Delta y \Delta z$

" " " " 5 and 6 $= \frac{\partial C_z}{\partial z} \Delta x \Delta y \Delta z$

So, total flux $= \left(\frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} \right) \Delta x \Delta y \Delta z$

$= (\vec{\nabla} \cdot \vec{C}) \Delta V$

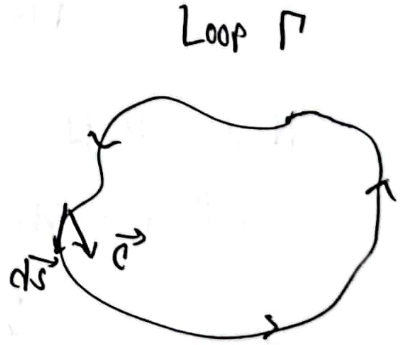
$$\boxed{\therefore \iint_S \vec{C} \cdot \hat{n}\, d\vec{A} = \iiint_V (\vec{\nabla} \cdot \vec{C})\, dV} \rightarrow \text{Gauss's theorem}$$

where S is any closed surface and V is the volume inside it.

2. Proof of Stoke's theorem:

Circulation of a vector field

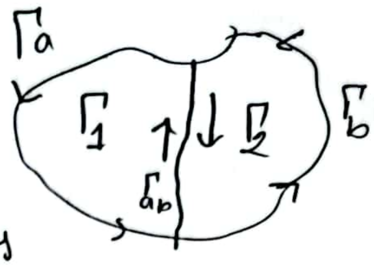
If Γ is a closed loop in space, then the line integral along this loop of a vector $\vec{C}$ is called the circulation and is given by,



$$\oint_\Gamma \vec{C} \cdot d\vec{s}$$

Now, we can divide the surface that is enclosed by the loop Γ into two parts – S_1 and S_2. Corresponding new closed loops are Γ_1 and Γ_2.

$$\Gamma_1 = \Gamma_a + \Gamma_{ab} \quad \text{and} \quad \Gamma_2 = \Gamma_b + \Gamma_{ab}$$



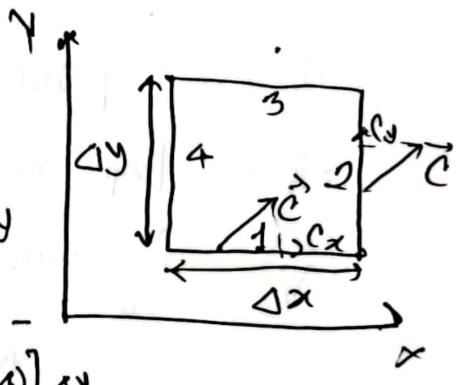
Now, the circulation due to Γ_{ab} for Γ_1 and Γ_{ab} for Γ_2 will always cancel out, and so,

$$\Gamma = \Gamma_a + \Gamma_b$$

So, the circulation around one and another loop sums up to the circulation around the whole loop.

We can now break this in many more parts and this still holds.

We can consider the little parts as squares if they are small eneough. What is the circulation around a square? Let's see.



$$\oint \vec{C}\cdot d\vec{s} = C_x(1)\,\Delta x + C_y(2)\,\Delta y - C_x(3)\,\Delta x - C_y(4)\,\Delta y$$

$$= \left[C_x(1) - C_x(3)\right]\Delta x + \left[C_y(2) - C_y(4)\right]\Delta y$$

Now,

$$C_x(3) = C_x(1) + \frac{\partial C_x}{\partial y}\Delta y$$

$$C_y(4) = C_y(2) + \frac{\partial C_y}{\partial x}\Delta x$$

$$\therefore \oint \vec{C}\cdot d\vec{s} = \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right)\Delta x\,\Delta y$$

The term in the paranthesis is just the z component of curl of $\vec{C}$.

$$\therefore \oint \vec{C} \cdot d\vec{s} = (\vec{\nabla} \times \vec{C})_z \, \Delta a$$

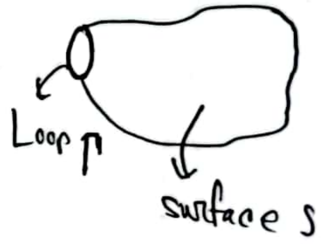
In general, z is basically the unit normal $\hat{n}$ ~~to~~ for the area of the square.

$$\therefore \oint \vec{C} \cdot d\vec{s} = (\vec{\nabla} \times \vec{C})_n \, \Delta a = (\vec{\nabla} \times \vec{C}) \cdot \hat{n} \, \Delta a$$

In the infinitesimal limit, then –

$$\boxed{\oint_{\Gamma} \vec{C} \cdot d\vec{s} = \iint_{S} (\vec{\nabla} \times \vec{C}) \cdot \hat{n} \, dA}$$ → Stoke's theorem

For a closed surface, things are a bit different. Consider the loop Γ that encloses this big surface. As we shrink the loop down to a point, the line integral around that loop will vanish. Consequently, $(\vec{\nabla} \times \vec{C})_n$ must also vanish. So, for closed surface,



$$\boxed{\oiint (\vec{\nabla} \times \vec{C}) \cdot \hat{n} \, dA = 0}$$

Example of divergences

Say, $\vec{V} = x\hat{i}$ $\qquad$ $\vec{V} = x\hat{i} - y\hat{j}$

$\vec{\nabla}\cdot\vec{V} = 1$ $\qquad$ $\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 1 - 1 = 0$

Problem

$$\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \vec{\nabla}\cdot\frac{\vec{r}}{r^3} \quad \text{with} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad r = \sqrt{x^2+y^2+z^2}$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right] + \frac{\partial}{\partial y}\left[\frac{y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right] + \frac{\partial}{\partial z}\left[\frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right]$$

$$= \frac{\sqrt{x^2+y^2+z^2} - x\,3\,\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x}{x^2+y^2+z^2} + \frac{\sqrt{x^2+y^2+z^2} - y\,3\,\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y}{x^2+y^2+z^2} + \cdots$$

$$= \frac{3}{\sqrt{x^2+y^2+z^2}} - 3\frac{x^2}{(x^2+y^2+z^2)^{3/2}} - 3\frac{y^2}{(x^2+y^2+z^2)^{3/2}} - 3\frac{z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{3}{\sqrt{x^2+y^2+z^2}} - 3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= 0$$

$$\therefore \vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \begin{cases} 0 \;; & r \neq 0 \\ \text{undefined} \;; & r = 0 \end{cases}$$

Del operator in curvilinear coordinates

Any vector in ~~Cart~~ spherical polar coordinate has components given by,

$$\vec{A} = A_r\,\hat{r} + A_\theta\,\hat{\theta} + A_\phi\,\hat{\phi}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Infinitesimal volume $= dr\; r\,d\theta\; r\sin\theta\, d\phi$

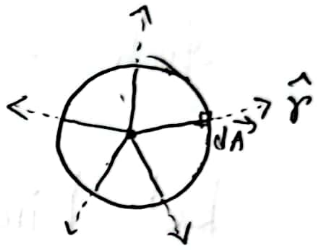
$= r^2 \sin\theta\; dr\, d\theta\, d\phi$

$$\therefore\quad \vec{\nabla}\cdot\frac{\hat{r}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$

The Dirac-delta function

The divergence of $\vec{V} = \frac{\hat{r}}{r^2}$ is zero everywhere except at the origin, where it is undefined. Let's calculate the surface integral, that is flux that is passing through a sphere of radius R centered at the origin.

$$\text{Flux} = \iint_S \vec{V} \cdot d\vec{A}$$



$$= \iint_S \frac{\hat{r}}{R^2} \cdot \hat{r}\, dA$$

$$= \frac{1}{R^2} \iint_S dA = \frac{1}{R^2} \times 4\pi R^2 = 4\pi$$

So, the value of the surface integral is 4π, ~~everywh~~ which is independent of the R. So, however small or big the sphere is, the flux will be ~~the~~ the same. Now, divergence theorem tells you

$$\iiint_V \vec{\nabla} \cdot \vec{V}\, d\tau = \iint_{\partial V} \vec{V} \cdot d\vec{A}$$

$$\therefore \iiint_V \vec{\nabla} \cdot \vec{V}\, d\tau = 4\pi$$

Now, $\vec{\nabla} \cdot \vec{V} = 0$ everywhere except at the origin.

So, all the contribution must come from the origin. This kind of peculiarities actually motivated physicists to introduce the Dirac delta function, which is actually a distribution. Paul Dirac introduced this function at first to use it for normalization of state vectors in Quantum Mechanics. Dirac delta function is defined as –

$$\delta(x) = \begin{cases} +\infty ; & x = 0 \\ 0 ; & x \neq 0 \end{cases}$$

which satisfies the identity, $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

Dirac delta function has the property that is called the sifting property, given by,

$$\int_{-\infty}^{\infty} f(x)\, \delta(x - x')\, dx = f(x')$$

Now, ~~we~~ in three dimension, $\int \delta^3(\vec{r})\, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\, dx\, dy\, dz = 1$

and $\iiint_{\text{All space}} f(\vec{r})\, \delta^3(\vec{r} - \vec{r}')\, d\tau = f(\vec{r}')$

We then write, $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^3(\vec{r})$.

Then, $\iiint_V \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) d\tau = 4\pi \iiint_V \delta^3(\vec{r})\, d\tau = 4\pi$

and everything is fine now!