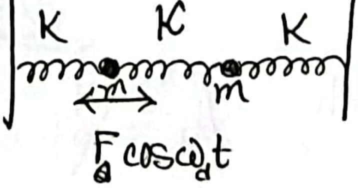
Lecture 8

Driven and damped coupled oscillator

Let's say, we now introduce damping in our coupled oscillator and apply a driving force $F_d = F_0 \cos\omega_d t$ on the left mass. The equation of motion will now be—



$$m\ddot{x}_1 = -Kx_1 - \kappa(x_1 - x_2) - b\dot{x}_1 + F_0 \cos\omega_d t$$

$$m\ddot{x}_2 = -Kx_2 - \kappa(x_2 - x_1) - b\dot{x}_2$$

$(+)$

$$m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1+x_2) - \kappa(x_1 - x_2 + x_2 - x_1) - b(\dot{x}_1 + \dot{x}_2) + F_0 \cos\omega_d t$$

$$\therefore\ m(\ddot{x}_1 + \ddot{x}_2) = -K(x_1+x_2) - b(\dot{x}_1+\dot{x}_2) + F_0 \cos\omega_d t$$

Similarly, $m(\ddot{x}_1 - \ddot{x}_2) = -K(x_1 - x_2) - 2\kappa(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + F_0 \cos\omega_d t$

Defining the normal coordinates as, $q_1 = x_1 + x_2$ and $q_2 = x_1 - x_2$,

$$m\ddot{q}_1 = -Kq_1 - b\dot{q}_1 + F_0 \cos\omega_d t \Rightarrow m\ddot{q}_1 + b\dot{q}_1 + Kq_1 = F_0 \cos\omega_d t \quad \text{—(I)}$$

$$m\ddot{q}_2 = -(K+2\kappa)q_2 - b\dot{q}_2 + F_0 \cos\omega_d t \Rightarrow m\ddot{q}_2 + b\dot{q}_2 + (K+2\kappa)q_2 = F_0 \cos\omega_d t \quad \text{—(II)}$$

But we are pretty much familiar with equation (I) and (II). They look exactly same as the single mass-spring driven damped oscillator. The solutions in steady state

$$q_1 = A_1 \cos(\omega_d t + \phi_1) \quad \text{and} \quad q_2 = A_2 \cos(\omega_d t + \phi_2)$$

with, $\cancel{A_1 = \sqrt{\quad}}$

$$A_1 = \frac{F_0/m}{\sqrt{(\omega_1^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}} \qquad A_2 = \frac{F_0/m}{\sqrt{(\omega_2^2 - \omega_d^2)^2 + \gamma^2 \omega_d^2}}$$

$$\tan \phi_1 = \frac{-\gamma \omega_d}{\omega_1^2 - \omega_d^2} \qquad \tan \phi_2 = \frac{-\gamma \omega_d}{\omega_2^2 - \omega_d^2}$$

where $\omega_1 = \sqrt{\frac{K}{m}}$ and $\omega_2 = \sqrt{\frac{K + 2\kappa}{m}}$

Then again, $x_1 = \frac{q_1 + q_2}{2}$ and $x_2 = \frac{q_1 - q_2}{2}$

$$\therefore x_1 = \frac{A_1}{2} \cos(\omega_d t + \phi_1) + \frac{A_2}{2} \cos(\omega_d t + \phi_2)$$

$$x_2 = \frac{A_1}{2} \cos(\omega_d t + \phi_1) - \frac{A_2}{2} \cos(\omega_d t + \phi_2)$$

If damping is small, then we have resonance frequencies to be exactly ω_1 and ω_2. So, if $\omega_d = \omega_1$, both x_1 and x_2 has very high amplitudes. They then move in phase (since A_1 will be much greater than A_2 at resonance). If $\omega_d = \omega_2$, then also x_1 and x_2 are large. But now, A_2 dominates over A_1 and so,

$$x_1 \approx A_2 \cos(\omega_d t + \phi_2) \text{ and}$$

$$x_2 \approx -A_2 \cos(\omega_d t + \phi_2)$$

So, they move out of phase, with roughly equal amplitudes. It seems like resonance frequencies cause the system to be in normal modes.

Three masses and four springs

Let's now extend the problem with three equal masses and four springs with equal spring constant. For unequal spring constants and unequal masses, calculations get real messy.

If x_1, x_2 and x_3 are the displacements of these three masses from left to right, then, equations of motions are –

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3)$$

$$m\ddot{x}_3 = -k(x_3 - x_2) - kx_3$$

$$\Rightarrow \quad m\ddot{x}_1 = -2kx_1 + kx_2 + 0$$

$$m\ddot{x}_2 = +kx_1 - 2kx_2 + kx_3$$

$$m\ddot{x}_3 = 0 + kx_2 - 2kx_3$$

We form the matrices like before –

$$M=\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix},\quad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\quad K=\begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}$$

$$\therefore\ M\ddot{X} = -KX$$

Like before, we assume the solution,

$$X=\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i\omega t}$$

$$\therefore\ (i\omega)^2 MX = -KX$$

$$\Rightarrow (K-\omega^2 M)X = 0 \quad \Rightarrow (K-\omega^2 M)\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i\omega t} = 0$$

~~As long~~ You will again then be left with same equation before,

$$(K-\omega^2 M)\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1)$$

Now, $\det\left[K-\omega^2 M\right] = 0$

$$\Rightarrow \det\begin{pmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 m & -k \\ 0 & -k & 2k-\omega^2 m \end{pmatrix} = 0$$

$$\Rightarrow (2k-\omega^2 m)\left[(2k-\omega^2 m)^2 - k^2\right] - k\left[0-(-k)(2k-\omega^2 m)\right] = 0$$

$$\Rightarrow (2k-\omega^2 m)\left[(2k-\omega^2 m)^2 - k^2 - k^2\right] = 0$$

$$\therefore\ (2k-\omega^2 m)\left[(2k-\omega^2 m)^2 - 2k^2\right] = 0$$

$$\therefore\ 2k - \omega^2 m = 0 \qquad \text{and} \qquad (2k-\omega^2 m)^2 - 2k^2 = 0$$

$$\therefore\ \omega^2 = \frac{2k}{m} \qquad \Rightarrow \quad (2k - \omega^2 m \pm \sqrt{2}k)(2k - \omega^2 m \mp \sqrt{2}k) = 0$$

$$\therefore\ \omega^2 = 2\omega_0^2 \qquad \therefore\ \omega^2 = \frac{2k}{m} \pm \frac{\sqrt{2}k}{m}$$

with $\omega_0 = \sqrt{k/m}$ $\qquad \therefore\ \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2$

Let's now plug these ω^2 in (1) to find the amplitudes

$\omega^2 = 2\omega_0^2 = \frac{2k}{m}$:

$$\begin{pmatrix} 2k - \frac{2k}{m}\times m & -k & 0 \\ -k & 2k - \frac{2k}{m}\times m & -k \\ 0 & -k & 2k - \frac{2k}{m}\times m \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -k & 0 \\ -k & 0 & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow -k \begin{pmatrix} A_2 \\ A_1 + A_3 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \therefore\ A_2 = 0$$

$$A_1 + A_3 = 0$$

$$\therefore\ A_1 = -A_3$$

$$\therefore \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\omega = (2 \pm \sqrt{2})\frac{k}{m}$:

$$\begin{pmatrix} 2k - (2+\sqrt{2})k & -k & 0 \\ -k & \ddots & -k \\ 0 & -k & \ddots \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow k \begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad -\sqrt{2}A_1 - A_2 = 0 \qquad \Rightarrow -A_2 = \sqrt{2}A_1$$

$$-A_1 + \sqrt{2}A_2 - A_3 = 0 \qquad \Rightarrow -A_2 = \frac{1}{\sqrt{2}}(A_1 + A_3)$$

$$-A_2 + -\sqrt{2}A_3 = 0 \qquad \Rightarrow -A_2 = \sqrt{2}A_3$$

$$\therefore \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Similarly, for $\omega^2 = (2-\sqrt{2})\frac{K}{m}$, $\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \propto \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

Now, the most general solution is obviously found by taking the linear combination of six solutions (three $+\omega$ and three $-\omega$).

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{i\omega_1 t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-i\omega_1 t} + c_3\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} e^{i\omega_2 t}$$

$$+ c_4\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} e^{-i\omega_2 t} + c_5\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} e^{i\omega_3 t} + c_6\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} e^{-i\omega_3 t}$$

with $\omega_1 = \sqrt{\frac{2K}{m}}$, ~~and~~ $\omega_2 = \sqrt{(2+\sqrt{2})\frac{K}{m}}$ and

$$\omega_3 = \sqrt{(2-\sqrt{2})\frac{K}{m}}$$

As like our previous argument, the displacements must be real, and hence, the solutions are –

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A_m\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{2}\omega_0 t + \phi_m) + A_f\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right)$$

$$+ A_s\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

where A_m corresponds to middle oscillation, A_f corresponds to fast oscillation and A_s corresponds to slow oscillation, corresponding to the frequencies. There are total six undetermined constants, which can be found by six initial conditions — three position and three velocities. Now,

$$x_1 = A_m \cos(\sqrt{2}\,\omega_0 t + \phi_m) + A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

$$x_2 = -\sqrt{2} A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + \sqrt{2} A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

$$x_3 = -A_m \cos(\sqrt{2}\,\omega_0 t + \phi_m) + A_f \cos\left(\sqrt{2+\sqrt{2}}\,\omega_0 t + \phi_f\right) + A_s \cos\left(\sqrt{2-\sqrt{2}}\,\omega_0 t + \phi_s\right)$$

N masses

Now, we are ready to derive the result for N masses. We will take equal masses and equal spring constants. So, all our masses are m and spring constants are K, and they are connected with each other, with the two remote springs connected to two fixed wall.