Shock Tubes using Finite Volume Methods By MATLAB

The conservative form of 1D Euler equations [1]:

$$\frac{\partial W}{\partial t}+\frac{\partial F}{\partial x}=0 \tag{1}$$

$$W=\begin{pmatrix}\rho\\ \rho u\\ \rho E\end{pmatrix},\qquad F=\begin{pmatrix}u\rho\\ p+\rho u^2\\ u\rho E\end{pmatrix},\qquad p=\rho RT\ or\ \ p=\rho(\gamma-1)\left(E-\frac{1}{2}u^2\right),\qquad H=E+\frac{p}{\rho}$$

$$\rho\mathrm{E}=\frac{p}{(\gamma-1)}+\frac{\rho}{2}u^2\ or\ E=\frac{p}{(\gamma-1)\rho}+\frac{1}{2}u^2$$

where ρ is the density, p is the pressure, u is the velocity (horizontal), E is the internal energy, H is the static enthalpy, and γ is the ratio of specific heats. Equation (1) can be rewritten as follows:

$$\frac{\partial W}{\partial t}+A\frac{\partial W}{\partial x}=0,\qquad A=\frac{\partial F}{\partial x} \tag{2}$$

A is the convective flux Jacobian matrix which is given as:

$$A=\begin{pmatrix}0 & 1 & 0\\ \frac{1}{2}u^2(-1+\gamma)-u^2 & u(3-\gamma) & \gamma-1\\ u\left(\frac{1}{2}u^2(-1+\gamma)-H\right) & H-u^2(-1+\gamma) & u\gamma\end{pmatrix} \tag{3}$$

A can be expressed as $A=\mathrm{R\Lambda L}$ where Λ is the diagonal matrix of real eigenvalues of A, L and R are the left and right matrices of eigenvectors of A; $L=R^{-1}$.

$$L=\begin{pmatrix}1-\frac{u^2(\gamma-1)}{2\alpha^2} & \frac{u(\gamma-1)}{\alpha^2} & \frac{1-\gamma}{\alpha^2}\\ \frac{1}{2}u^2(\gamma-1)-u\alpha & \alpha-u(\gamma-1) & \gamma-1\\ u\alpha+\frac{1}{2}u^2(\gamma-1) & -\alpha-u(\gamma-1) & \gamma-1\end{pmatrix}, \tag{4}$$

$$R=\begin{pmatrix}1 & \beta & \beta\\ u & (u+\alpha)\beta & (u-\alpha)\beta\\ \frac{u^2}{2} & (H+u\alpha)\beta & (H-u\alpha)\beta\end{pmatrix},\alpha^2=\frac{\gamma\,\mathrm{p}}{\rho},\beta=\frac{1}{2\alpha^2}$$

where a is the speed of sound.
With $W(x,0)=W_0(x)$, equation (1) can be written as:

$$\frac{\partial W_i}{\partial t}+\frac{F_{i+1/2}-F_{i-1/2}}{\Delta\mathrm{x}}=0 \tag{5}$$

W_i is the cell-centered values stored at nods and $F_{i\pm1/2}$ are the fluxes at left and right cell interface.

$$F_{i+1/2}=\frac{1}{2}\left({F^R}_{i+1/2}+{F^L}_{i+1/2}\right)-\frac{1}{2}\mathrm{R\Lambda L}\left({W^R}_{i+1/2}+{W^L}_{i+1/2}\right) \tag{6}$$

Λ is the diagonal matrix of absolute eigenvalues of A and L and R. The Roe solver is an approximate Riemann solver and it uses below formulation to find the numerical fluxes at the interface:

$$F_{i+1/2}=\frac{1}{2}({F^R}_{i+1/2}+{F^L}_{i+1/2})-\frac{1}{2}\bar{R}\bar{\Lambda}\bar{L}({W^R}_{i+1/2}+{W^L}_{i+1/2}) \tag{7}$$

The bars denote the Roe's averages between the left and right interfaces. To get $\bar{A}$, $\bar{u}$, $\bar{H}$, and $\bar{\alpha}$ should be calculated first using Roe average based on expansion of Taylor series for F about W^L, W^R points.

$$F(W)=F(W^L)+A(W^L)\big((W-W^L)\big),F(W)=F(W^R)+A(W^R)\big((W-W^R)\big) \tag{8}$$

$$F(W^L) - F(W^R) = A(W^L - W^R) + (A(W^R) - A(W^L))W$$
$$Roe's\ \bar{A}\ ,\ \bar{A}(W^L - W^R)\ \ goes\ to\ A(W)\ as\ \bar{W}\ goes\ to\ W.$$
$$Therefore\text{: } F(W^L) - F(W^R) = \bar{A}(W^L - W^R)$$

The Roe averaging formulas to compute approximate values for constructing $\bar{R}$ and $\bar{L}$ are given below

$$\bar{u} = \frac{u_R\sqrt{\rho_R} + u_L\sqrt{\rho_L}}{\sqrt{\rho_R} + \sqrt{\rho_L}}, \qquad \bar{H} = \frac{H_R\sqrt{\rho_R} + H_L\sqrt{\rho_L}}{\sqrt{\rho_R} + \sqrt{\rho_L}}, \qquad \bar{\alpha} = \sqrt{(\gamma - 1)\left[\bar{H} - \frac{1}{2}\bar{u}^2\right]} \tag{9}$$

The eigenvalues of the Jacobian matrix are

$$\lambda_1 = \bar{u}, \qquad \lambda_2 = \bar{u} + \alpha, \qquad \lambda_3 = \bar{u} - \alpha$$

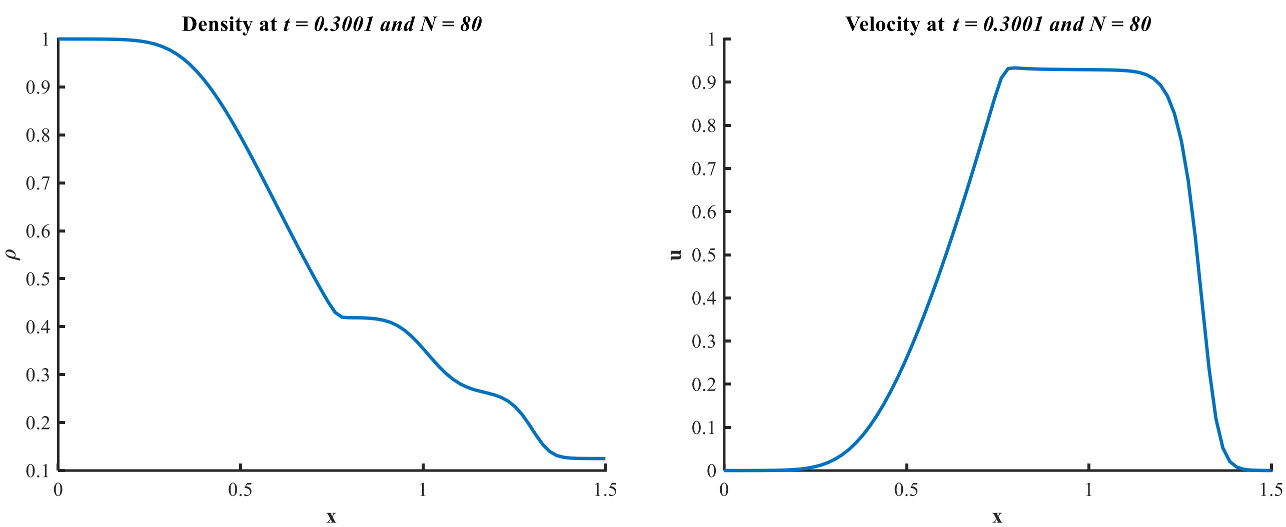
After obtaining $A_{i+1/2}$, a first-order upwind Roe's scheme is used. The conservative form of flux for Roe's solver can be written as:

$$F({W^t}_L, {W^t}_R) = \frac{1}{2}(F({W^t}_L) + F({W^t}_R) - \frac{1}{2}\bar{R}\bar{\Lambda}\bar{L}({W^t}_L + {W^t}_R) \tag{10}$$

The first-order upwind Roe's scheme can be written as:

$${W^{t+1}}_i = {W^t}_i - \frac{\Delta t}{\Delta x}\left(F({W^t}_L, {W^t}_R) - F({W^t}_L, {W^t}_R)\right) \tag{11}$$

Figure 1 – Figure 5 show the density, velocity, and pressure distributions at N=80, 160, 320,640, and 2000 respectively. The resolution of the expansion fan, contact discontinuity, and shock increases as the level of mesh refinement increases. Contact discontinuities (surfaces that separate zones of different density and temperature) becomes sharper as N increases. The expansion fan becomes sharper too. The shock become sharp and its distance decreases as N increases.



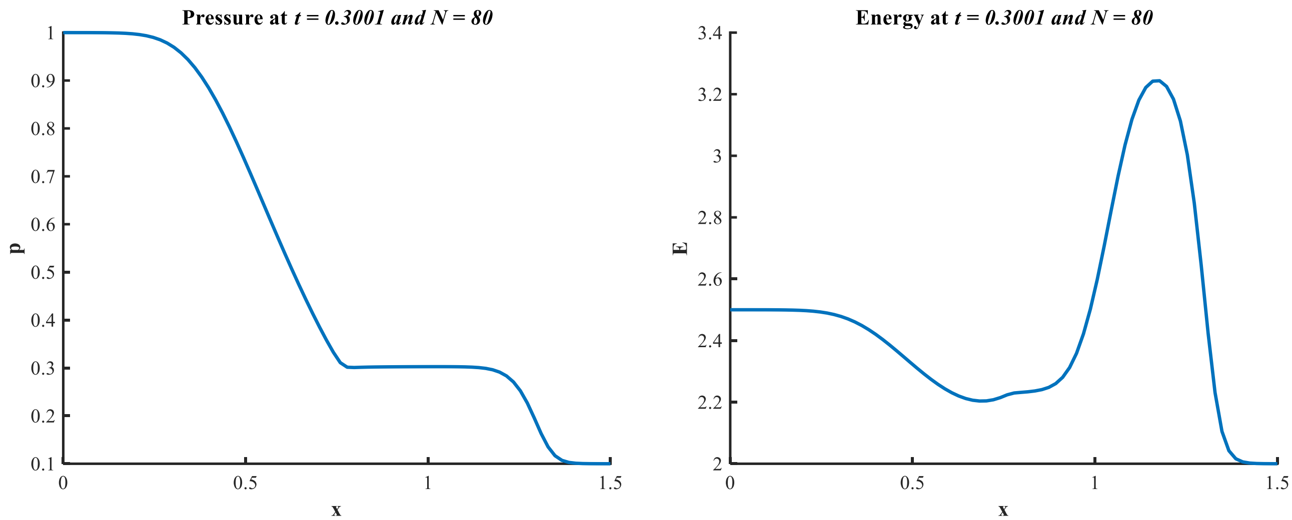


Figure 1: The density, velocity, and pressure distributions at N=80.

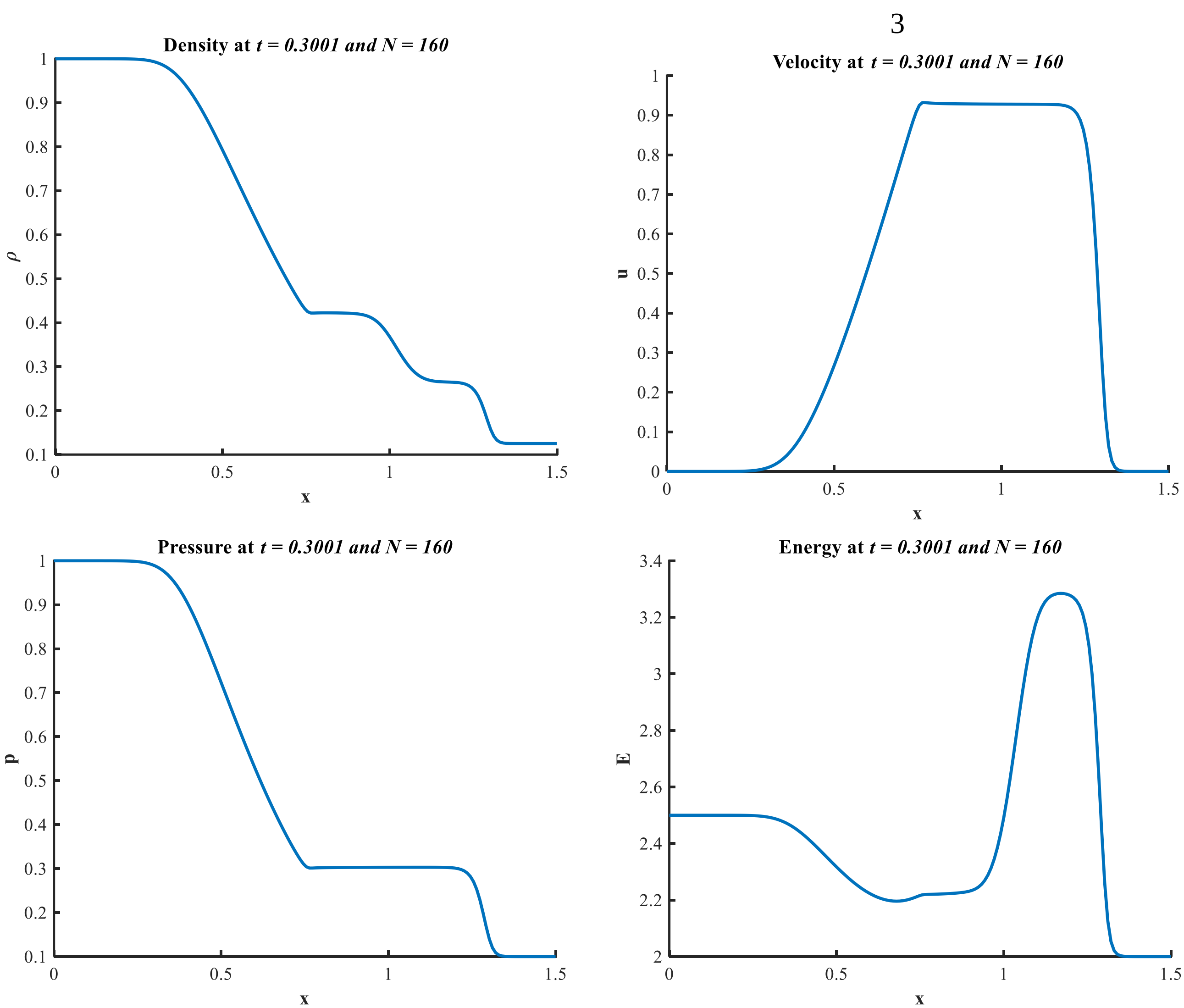


Figure 2: The density, velocity, and pressure distributions at N=160.

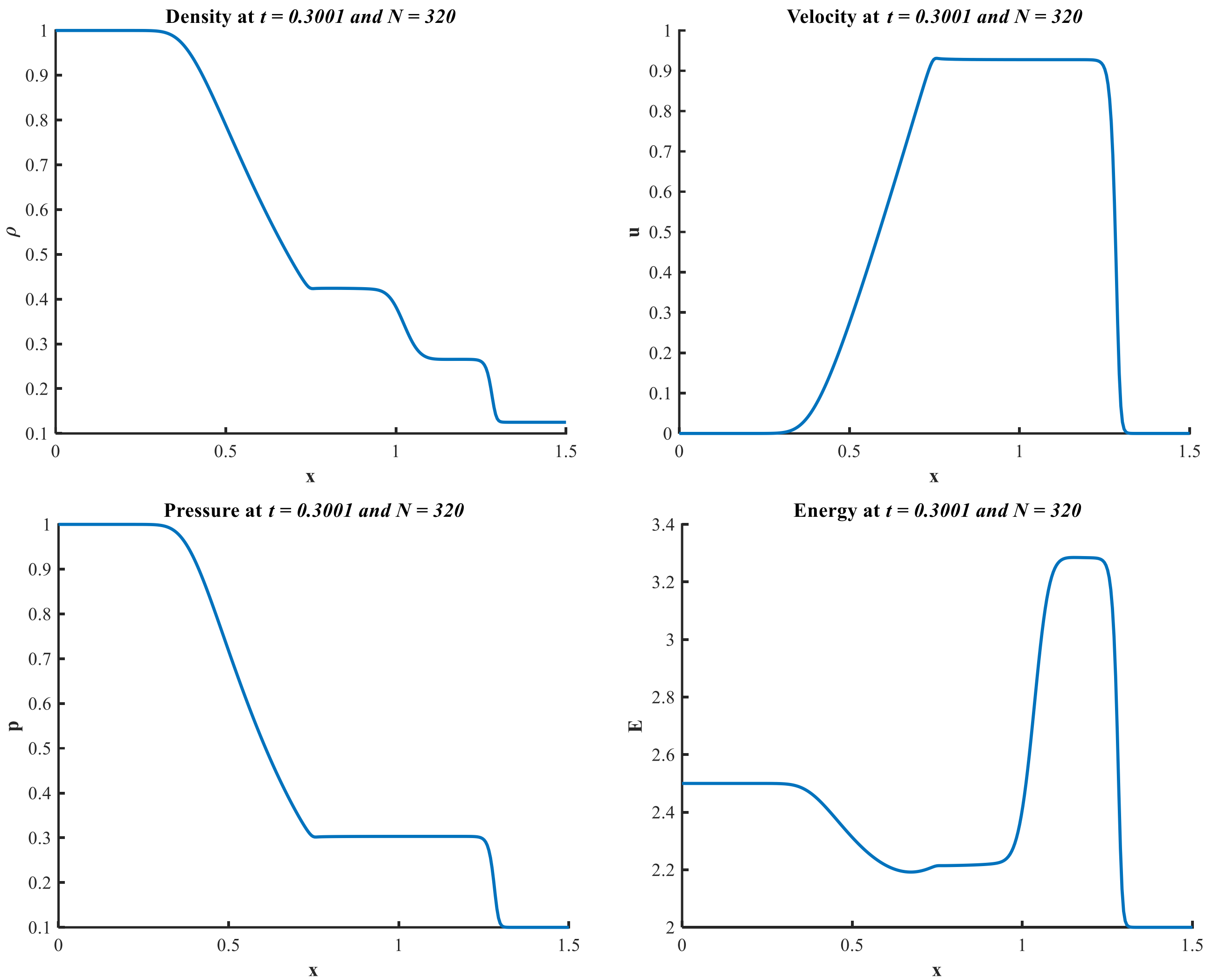
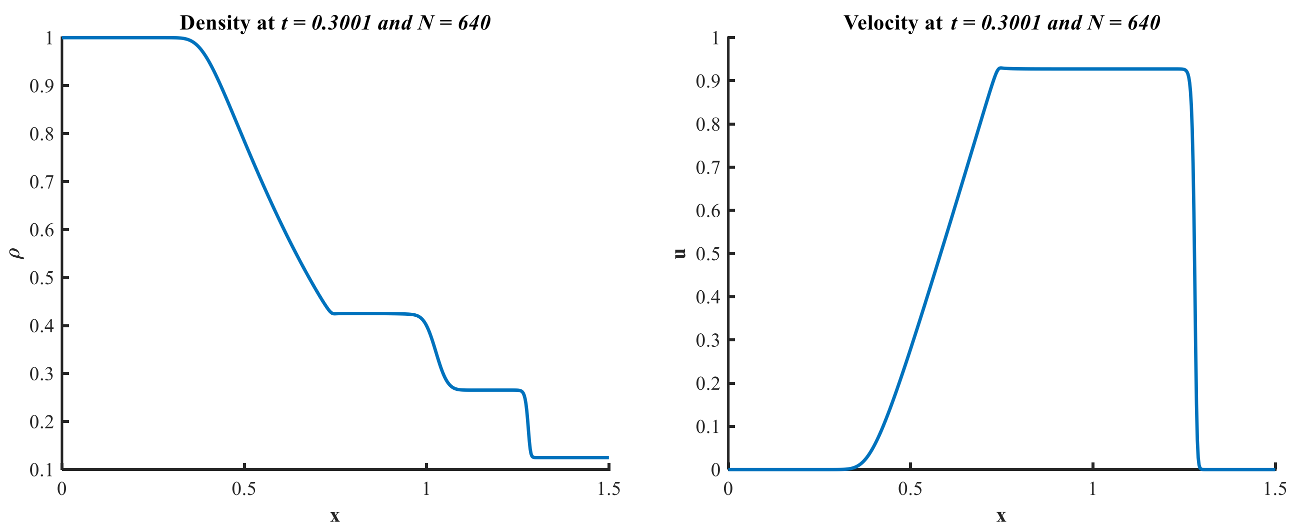


Figure 3: The density, velocity, and pressure distributions at N=320.



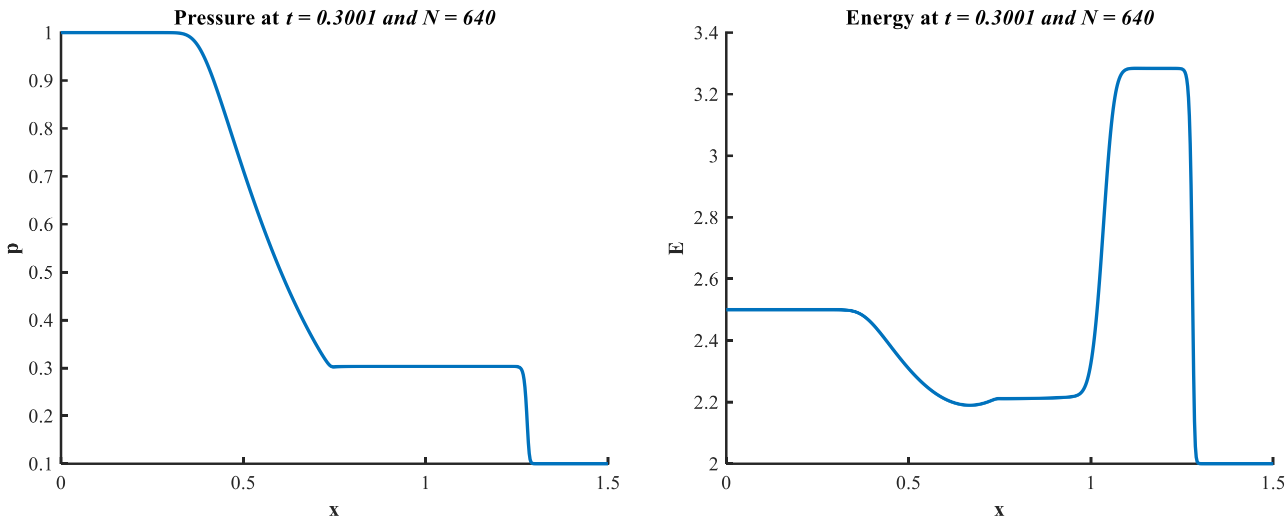


Figure 4: The density, velocity, and pressure distributions at N=640.

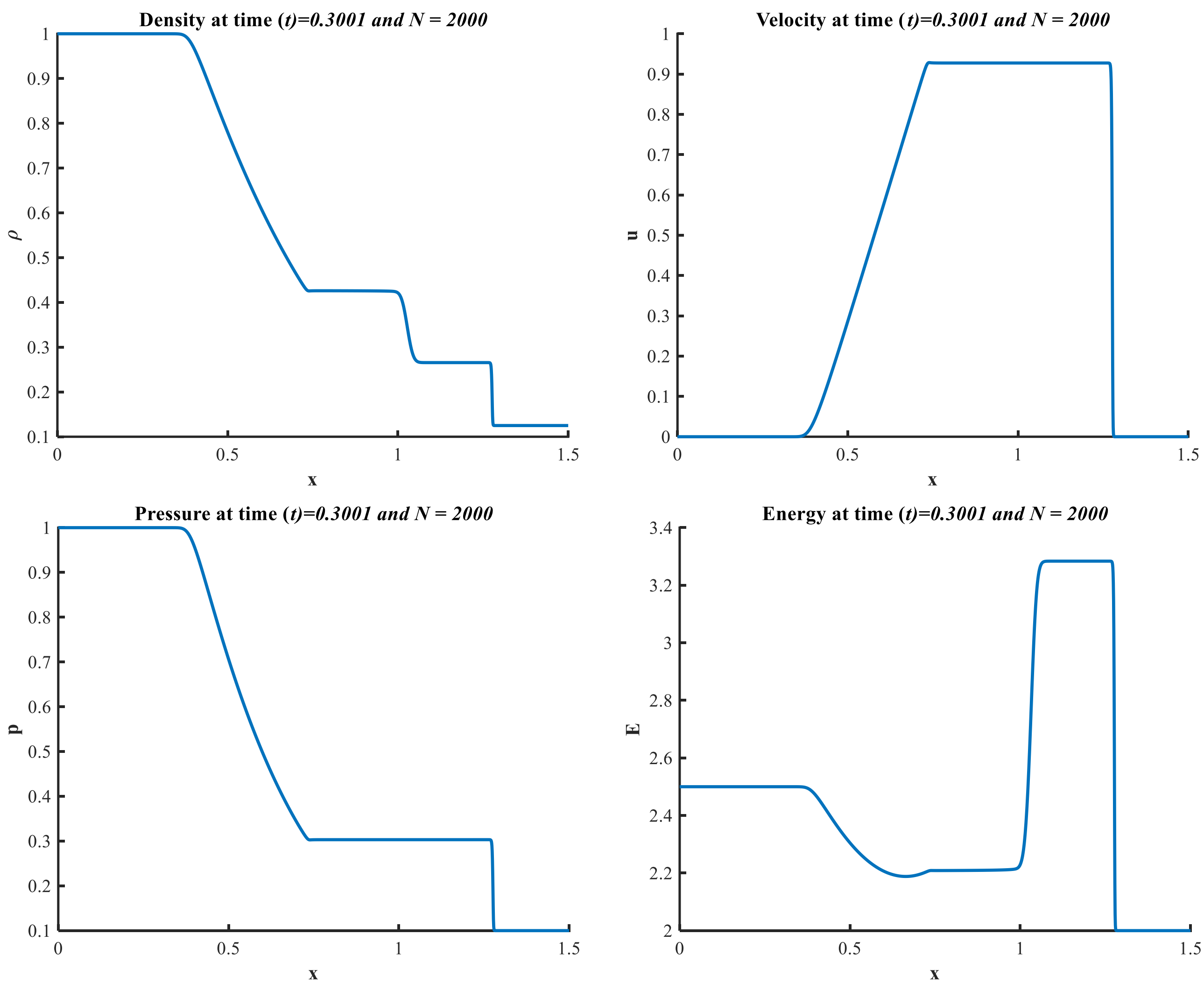


Figure 5: The density, velocity, and pressure distributions at N=2000.