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Tutorial 6 (The no-cloning theorem¹)

The *no-cloning theorem* states that, surprisingly, one cannot make a copy of an unknown quantum state. In more detail, we consider a system of two qubits (source and target): the first qubit is the source state $|\psi\rangle$, and the second qubit starts out in some standard state $|s\rangle$, for example $|s\rangle = |0\rangle$. Thus the initial state is $|\psi\rangle \otimes |s\rangle \equiv |\psi\rangle |s\rangle$. One would like to copy $|\psi\rangle$ into $|s\rangle$, that is, find some unitary transformation $U \in \mathbb{C}^{4 \times 4}$ such that

$$|\psi\rangle \otimes |s\rangle \mapsto U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle.$$

Show that such a copying procedure is impossible: the equation cannot hold for arbitrary source qubits $|\psi\rangle$.

Solution Suppose the copying procedure works for two particular states $|\psi\rangle$ and $|\varphi\rangle$. Then

$$\begin{aligned} U(|\psi\rangle \otimes |s\rangle) &= |\psi\rangle \otimes |\psi\rangle, \\ U(|\varphi\rangle \otimes |s\rangle) &= |\varphi\rangle \otimes |\varphi\rangle. \end{aligned}$$

Taking inner products gives for the left sides

$$\langle U(|\psi\rangle \otimes |s\rangle) | U(|\varphi\rangle \otimes |s\rangle) \rangle = (\langle \psi | \otimes \langle s |) U^\dagger U (|\varphi\rangle \otimes |s\rangle) = (\langle \psi | \otimes \langle s |) (|\varphi\rangle \otimes |s\rangle) = \langle \psi | \varphi \rangle \langle s | s \rangle = \langle \psi | \varphi \rangle$$

and for the right sides

$$\langle |\psi\rangle \otimes |\psi\rangle | |\varphi\rangle \otimes |\varphi\rangle \rangle = \langle \psi | \varphi \rangle \langle \psi | \varphi \rangle = (\langle \psi | \varphi \rangle)^2,$$

leading to $\langle \psi | \varphi \rangle = (\langle \psi | \varphi \rangle)^2$. But this can only hold for $\langle \psi | \varphi \rangle = 1$ or $\langle \psi | \varphi \rangle = 0$, so either $|\psi\rangle = |\varphi\rangle$ or $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal. In other words, general quantum cloning is impossible; for example, a potential quantum cloner cannot copy the qubit states $|\psi\rangle = |0\rangle$ and $|\varphi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

¹M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), page 532