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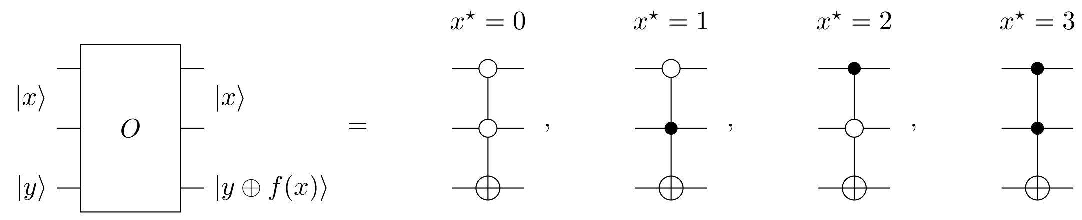
Exercise 9.1 (Two-bit quantum search)
We consider the quantum search (Grover's) algorithm for the special case $n = 2$, i.e., a search space with $N = 4$ elements, and $M = 1$ (exactly one solution). The solution is denoted $x^\star$, and correspondingly $f(x^\star) = 1$, $f(x) = 0$ for all $x \neq x^\star$.

The oracle, which is able to recognize the solution, can be realized as follows (depending on $x^\star$):

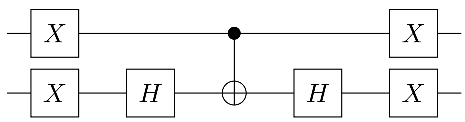


Note that the rightmost gate for $x^\star = 3$ is the Toffoli gate (cf. Exercise 5.2): the first and second qubits act as controls, and the third qubit as target, which is flipped precisely if both controls are set to 1. The empty circles in the gates for $x^\star = 0, 1, 2$ mean that the control is activated by 0 (instead of 1).

As derived in the lecture, the Grover operator G performs a rotation by angle θ in the plane spanned by the orthonormal states $|\alpha\rangle$ and $|\beta\rangle$; thus k applications to the initial equal superposition state $|\psi\rangle = \cos(\frac{\theta}{2})|\alpha\rangle + \sin(\frac{\theta}{2})|\beta\rangle$ results in

$$G^k |\psi\rangle = \cos\left((\tfrac{1}{2} + k)\theta\right)|\alpha\rangle + \sin\left((\tfrac{1}{2} + k)\theta\right)|\beta\rangle .$$

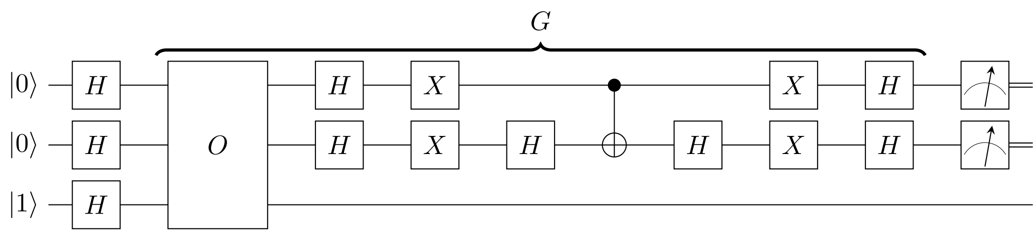
(a) Show that the following circuit implements the negated phase gate appearing in the Grover operator, that is, $-(2|00\rangle\langle 00| - I)$:



The global factor (-1) does not influence the final quantum measurement results and will be ignored from now on.

(b) Compute the angle θ defined via $\sin(\frac{\theta}{2}) = \sqrt{M/N}$. Why is a single application of G sufficient to reach the desired solution state $|\beta\rangle$ exactly, that is, $G|\psi\rangle = |\beta\rangle$?

In summary, the quantum search circuit with one use of G and the above realization of the phase gate is:

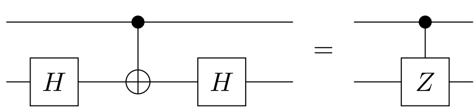


(c) Assemble this circuit in the IBM Q Circuit Composer for one of the four possible oracles of your choice, and verify that the final measurement indeed yields the solution $x^\star$.

Hint: You can use the Pauli-X gate to initialize the oracle qubit to $|1\rangle$. The Toffoli gate is available in the Circuit Composer.

Solution

(a) We have identified the controlled-NOT gate with the controlled-X gate in the lecture. Furthermore, since $HXH = Z$ and $H^2 = I$, the following identity holds:



The controlled-Z gate acts on computational basis states as

$$|00\rangle \mapsto |00\rangle\,, \quad |01\rangle \mapsto |01\rangle\,, \quad |10\rangle \mapsto |10\rangle\,, \quad |11\rangle \mapsto -|11\rangle\,,$$

which we can compactly represent as $I - 2\,|11\rangle\,\langle 11|$, or written in matrix form:

$$\text{controlled-}Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

Now the overall circuit operation (including the leading and trailing $X \otimes X$ gates) expressed in bra-ket notation reads

$$\begin{aligned} &(X \otimes X)(I - 2\,|11\rangle\,\langle 11|)(X \otimes X) \\ &= I - 2(X \otimes X)\,|11\rangle\,\langle 11|\,(X \otimes X) \\ &= I - 2(X\,|1\rangle \otimes X\,|1\rangle)(\langle 1|\,X \otimes \langle 1|\,X) \\ &= I - 2(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|) \\ &= I - 2\,|00\rangle\,\langle 00|\,. \end{aligned}$$

For the first equal sign we have used that $(X \otimes X)^2 = X^2 \otimes X^2 = I_2 \otimes I_2 = I_4$, where I_2 is the 2×2 identity matrix and likewise I_4 the 4×4 identity matrix.

Alternatively, we can use the matrix representation of $X \otimes X$:

$$X \otimes X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

Combined with the above matrix representation of controlled-Z leads to

$$(X \otimes X)(\text{controlled-}Z)(X \otimes X) = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

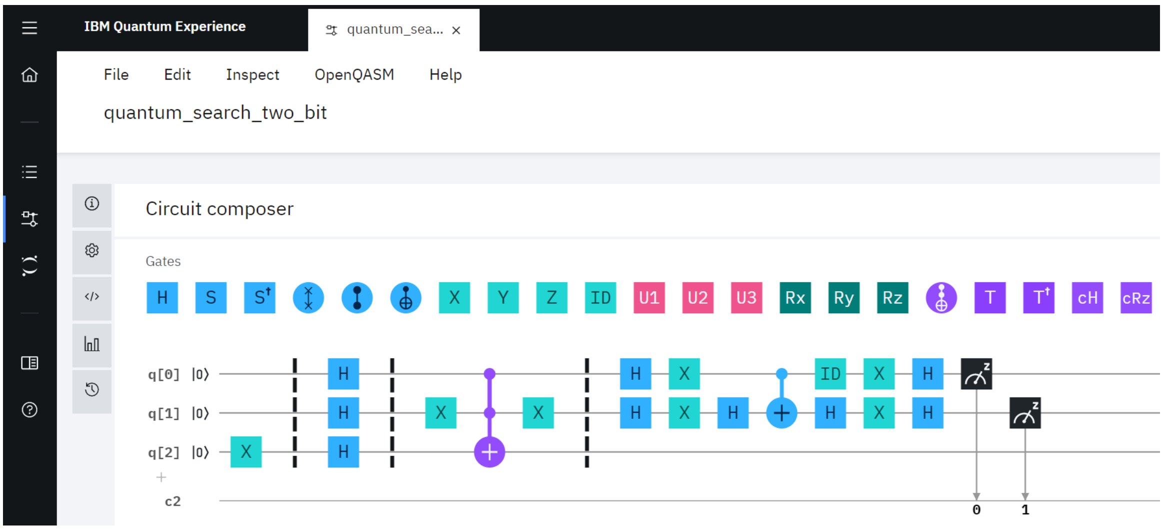
which is the matrix form of $I - 2\,|00\rangle\,\langle 00|$, as required.

(b) Here $M = 1$, $N = 4$, and $\sin(\frac{\theta}{2}) \stackrel{!}{=} \sqrt{M/N} = \frac{1}{2}$ for $\theta = \frac{\pi}{3}$. A single application of G yields

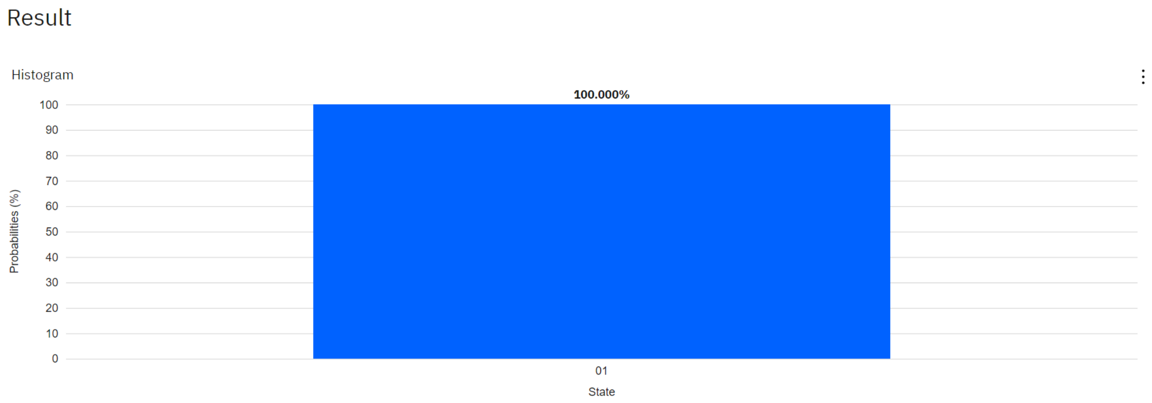
$$G\,|\psi\rangle = \cos\left(\tfrac{3}{2}\theta\right)|\alpha\rangle + \sin\left(\tfrac{3}{2}\theta\right)|\beta\rangle\,,$$

and inserting $\theta = \frac{\pi}{3}$ confirms that $G\,|\psi\rangle = |\beta\rangle$.

(c) A realization of the circuit (for $x^\star = 2$) in IBM Q is:



Simulating this circuit (using the `ibmq_qasm_simulator`) leads to the following result:



IBM Q uses the convention that the "first" (topmost) qubit corresponds to the least significant bit in the basis state enumeration, such that the reported state "01" actually corresponds to $|10\rangle$ following the convention of this exercise sheet. Thus we have indeed found the solution $x^\star = 2$, as expected.