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Exercise 6.1 (Heisenberg uncertainty principle for a single qubit)
Imagine we prepare multiple copies of an arbitrary single-qubit state $|\psi\rangle$, described by the Bloch vector (r_x, r_y, r_z). Some of these copies are measured using the observable Z, and the remaining copies using the observable X.

(a) Compute the expectation values of both measurements.
Hint: These two measurements are projective measurements and have a geometric interpretation on the Bloch sphere. You can use without proof the identities $\cos(\alpha/2)^2 - \sin(\alpha/2)^2 = \cos(\alpha)$ and $2\cos(\alpha/2)\sin(\alpha/2) = \sin(\alpha)$.

(b) What will be their corresponding standard deviations? Recall from the lecture that the standard deviation of an observable M is defined as $\Delta M = \sqrt{\langle M^2\rangle - \langle M\rangle^2}$, with $\langle A\rangle \equiv \langle\psi| A |\psi\rangle$ for any observable A.

(c) Evaluate the commutator $[Z, X]$ and its expectation value $\langle\psi| [Z, X] |\psi\rangle$.

(d) Insert your results and explicitly verify that the Heisenberg uncertainty principle is satisfied.

Solution hints

(a) You should arrive at $\langle Z\rangle = r_z$ and $\langle X\rangle = r_x$.

(b) Note that $X^2 = Z^2 = I$.

(c) $[Z, X] = 2iY$.

(d) We already know the terms on the left-hand side of Heisenberg inequality, so we can focus on the right:

$$\langle\psi| [Z, X] |\psi\rangle = \langle\psi| 2iY |\psi\rangle = 2i\langle Y\rangle = 2ir_y$$

Plug this into the equation and check that the inequality is satisfied.