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Exercise 2.2 (Basic single qubit gates)

Imagine you are playing a game against an adversary. The game consists of multiple trials through which the adversary performs one of the following with equal probability:

1. They flip a coin and send you $|0\rangle$ or $|1\rangle$ depending on the outcome.

OR

2. They send you the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Your goal is to decide which of the two they performed, and you win if you can decide correctly for $\frac{3}{4}$ of the trials on average.

- (a) Before you make your guess (based on a quantum measurement on the qubit), you are allowed to perform **one** of the gates X , Y , Z or H . Compute the outputs you would obtain in each situation with each of these gates.
- (b) Which of the gates would allow you to win the game? Explain your strategy.

Solution

- (a) If we apply X then the outcome for each scenario will be:

1. $|1\rangle$ or $|0\rangle$.
2. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

If we apply Y :

1. $i|1\rangle$ or $-i|0\rangle$.
2. $\frac{i}{\sqrt{2}}(-|0\rangle + |1\rangle)$.

If we apply Z :

1. $|0\rangle$ or $-|1\rangle$.
2. $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

If we apply H :

1. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
2. $|0\rangle$.

- (b) Applying a Hadamard gate allows us to win the game. In the second scenario we would always measure 0, so if we measure 1 we know with certainty that we are in the first scenario. Therefore, our strategy would work as follows:

- Apply a Hadamard gate.
- Measure in the standard basis to obtain either 0 or 1.
- If we obtain 1 say we are in scenario 1.
- If we obtain 0 say we are in scenario 2.

Note that we will obtain 1 in $\frac{1}{4}$ of the trials, for which we will always be correct. For the remaining $\frac{3}{4}$ we will be correct in $\frac{2}{3}$ of the trials. Therefore, we will be correct $1 \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{3}{4} = \frac{3}{4}$ of the times, as required.