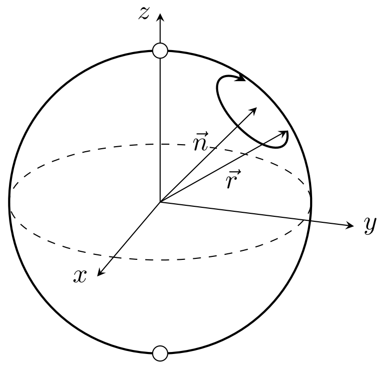
Christian B. Mendl, Irene López Gutiérrez, Keefe Huang

Tutorial 11 (Bloch sphere interpretation of rotations[1])
In this tutorial, we show that the Bloch sphere representation of a general single-qubit rotation operator

$$R_{\vec{n}}(\theta) = \mathrm{e}^{-i\theta(\vec{n}\cdot\vec{\sigma})/2} = \cos(\theta/2)I - i\sin(\theta/2)(\vec{n}\cdot\vec{\sigma})$$



is a conventional rotation (in three dimensions) by angle θ about axis $\vec{n} \in \mathbb{R}^3$. Let $\vec{r}$ denote the Bloch vector of the quantum state. It will be convenient to work with the following relation between $\vec{r}$ and the density matrix ρ of the quantum state:

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}.$$

(By exercise 11.2 below, this coincides with the hitherto definition of the Bloch vector in case $\rho = |\psi\rangle\langle\psi|$ corresponds to a pure quantum state $|\psi\rangle$.)

(a) First verify the following commutation relation of the Pauli matrices: for any $j, k \in \{1,2,3\}$,

$$[\sigma_j, \sigma_k] = 2i \sum_{\ell=1}^{3} \epsilon_{jk\ell}\, \sigma_\ell,$$

where $[A,B] = AB - BA$ is the *commutator* of A and B, and the *Levi-Civita symbol* $\epsilon_{jk\ell}$ is defined by

$$\epsilon_{jk\ell} = \begin{cases} 1 & (j,k,\ell) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & (j,k,\ell) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Conclude that, for any $\vec{a}, \vec{b} \in \mathbb{R}^3$,

$$\left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right] = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

(b) Derive the relation

$$\left\{\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right\} = 2(\vec{a}\cdot\vec{b})I$$

for any $\vec{a}, \vec{b} \in \mathbb{R}^3$, where $\{A,B\} = AB + BA$ is the *anti-commutator* of A and B.

(c) Show that the Bloch vector of the rotated quantum state is obtained by applying Rodrigues' rotation formula:

$$\vec{r}' = \cos(\theta)\vec{r} + \sin(\theta)(\vec{n}\times\vec{r}) + (1-\cos(\theta))\,(\vec{n}\cdot\vec{r})\vec{n}.$$

Remark: The interpretation as rotation applies to an arbitrary single-qubit gate U (when ignoring global phases), since it can always be represented as $U = \mathrm{e}^{i\alpha}R_{\vec{n}}(\theta)$ with $\alpha \in \mathbb{R}$ and a suitable rotation operator $R_{\vec{n}}(\theta)$.

Solution

(a) We first note that for $j = k$, the commutator is clearly zero, as is the Levi-Civita symbol.

For $j = 1$ and $k = 2$, by an explicit calculation,

$$\begin{aligned}
[\sigma_1, \sigma_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \\
&= 2i\sigma_3 = 2i\sum_{\ell=1}^{3} \epsilon_{12\ell}\,\sigma_\ell,
\end{aligned}$$

and similarly $[\sigma_2, \sigma_3] = 2i\sigma_1$ and $[\sigma_3, \sigma_1] = 2i\sigma_2$ (see also exercise 3.1). Finally, we note that an interchange $j \leftrightarrow k$ flips the sign of the commutator, $[\sigma_j, \sigma_k] = -[\sigma_k, \sigma_j]$, and likewise the sign of $\epsilon_{jk\ell}$ by definition. In summary, we have verified the relation for all possible cases of $j, k \in \{1,2,3\}$.

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 4.6

Expanding in terms of individual Pauli matrices leads to:

$$\left[\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}\right] = \sum_{j,k=1}^{3} a_j b_k \left[\sigma_j,\sigma_k\right] = 2i \sum_{j,k,\ell=1}^{3} a_j b_k\, \epsilon_{jk\ell}\, \sigma_\ell = 2i \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

(b) The statement follows from the fact that different Pauli matrices anti-commute, i.e., $\sigma_j\sigma_k = -\sigma_k\sigma_j$ for $j \neq k$ (see exercise 3.1), and that squaring a Pauli matrix gives the identity:

$$\left\{\vec{a}\cdot\vec{\sigma},\vec{b}\cdot\vec{\sigma}\right\} = \sum_{j,k=1}^{3} a_j b_k \left\{\sigma_j,\sigma_k\right\} = \sum_{j,k=1}^{3} a_j b_k\, \delta_{jk} 2I = 2(\vec{a}\cdot\vec{b})I.$$

(c) In general, applying a unitary matrix U to a quantum state $|\psi\rangle$ corresponds to a conjugation of the density matrix by U:

$$\rho \mapsto U\rho U^\dagger.$$

In our case, $U = R_{\vec{n}}(\theta)$, and $U^\dagger = R_{\vec{n}}(-\theta)$ (inverse rotation).

Inserting the Bloch representation of the density matrix leads to

$$\begin{aligned}
R_{\vec{n}}(\theta)\rho R_{\vec{n}}(-\theta) &= \frac{I}{2} + \frac{1}{2} R_{\vec{n}}(\theta)(\vec{r}\cdot\vec{\sigma})R_{\vec{n}}(-\theta) \\
&= \frac{I}{2} + \frac{1}{2}\cos(\theta/2)^2\,(\vec{r}\cdot\vec{\sigma}) + \frac{i}{2}\cos(\theta/2)\sin(\theta/2)\,(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) - \frac{i}{2}\cos(\theta/2)\sin(\theta/2)\,(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma}) \\
&\quad + \frac{1}{2}\sin(\theta/2)^2\,(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) \\
&= \frac{I}{2} + \underbrace{\frac{1}{2}\cos(\theta/2)^2\,(\vec{r}\cdot\vec{\sigma})}_{①} + \underbrace{\frac{i}{2}\cos(\theta/2)\sin(\theta/2)\,[\vec{r}\cdot\vec{\sigma},\vec{n}\cdot\vec{\sigma}]}_{②} + \underbrace{\frac{1}{2}\sin(\theta/2)^2\,(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma})}_{③}.
\end{aligned}$$

To further simplify ②, we use part (a) together with the identity $2\cos(\alpha)\sin(\alpha) = \sin(2\alpha)$ for any $\alpha \in \mathbb{R}$:

$$② = \frac{i}{2}\cos(\theta/2)\sin(\theta/2)\,2i(\vec{r}\times\vec{n})\cdot\vec{\sigma} = -\frac{1}{2}\sin(\theta)\,(\vec{r}\times\vec{n})\cdot\vec{\sigma} = \frac{1}{2}\sin(\theta)\,(\vec{n}\times\vec{r})\cdot\vec{\sigma}.$$

To simplify ③, we first note that, according to (b),

$$(\vec{n}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma}) = -(\vec{r}\cdot\vec{\sigma})(\vec{n}\cdot\vec{\sigma}) + 2(\vec{n}\cdot\vec{r})I.$$

Also, since $\vec{n}$ is a unit vector, $(\vec{n}\cdot\vec{\sigma})^2 = I$ (see lecture). Inserted into ③ leads to

$$\begin{aligned}
③ &= \sin(\theta/2)^2\,(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - \frac{1}{2}\sin(\theta/2)^2\,(\vec{r}\cdot\vec{\sigma}) \\
&= \frac{1}{2}\,(1-\cos(\theta))\,(\vec{n}\cdot\vec{r})(\vec{n}\cdot\vec{\sigma}) - \frac{1}{2}\sin(\theta/2)^2\,(\vec{r}\cdot\vec{\sigma}).
\end{aligned}$$

Combining parts ①, ②, ③, and using the identity $\cos(\alpha)^2 - \sin(\alpha)^2 = \cos(2\alpha)$, we obtain:

$$R_{\vec{n}}(\theta)\rho R_{\vec{n}}(-\theta) = \frac{I}{2} + \frac{1}{2}\underbrace{\Big(\cos(\theta)\vec{r} + \sin(\theta)(\vec{n}\times\vec{r}) + (1-\cos(\theta))\,(\vec{n}\cdot\vec{r})\vec{n}\Big)}_{\vec{r}'}\cdot\vec{\sigma}$$

The expression for the new Bloch vector $\vec{r}'$ is exactly Rodrigues' rotation formula, as required.