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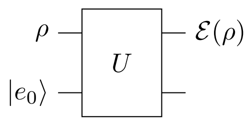
**Exercise 13.1** (Quantum operations as coupling to an environment, and amplitude damping[1])
Any quantum operation can be represented by embedding the principal system into an environment, which we can assume (without loss of generality) to start in some state $|e_0\rangle$, and then applying a unitary transformation to the combined system, as illustrated in the following diagram:




From that, one obtains $\mathcal{E}(\rho)$ by "tracing out" the environment; for this purpose we first extend $|e_0\rangle$ to a basis $\{|e_k\rangle\}$ of the environment, and then compute the partial trace:

$$\mathcal{E}(\rho) = \mathrm{tr}_{\mathsf{env}}\left[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right] = \sum_k \langle e_k| U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle = \sum_k E_k \rho E_k^\dagger$$

with the matrix entries of $E_k$ given by $(E_k)_{\ell,m} = \langle \ell, e_k| U |m, e_0\rangle$. The last term is the operator-sum representation of the quantum operation.

*Amplitude damping* models effects due to the loss of energy from a quantum system, for example by loosing a photon (elementary particle of light) from a cavity. In this case one can think of $|0\rangle$ and $|1\rangle$ as the physical system with zero or one photon, respectively. Specifically, the operator-sum representation of amplitude damping is given by

$$\mathcal{E}_{\mathsf{AD}}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

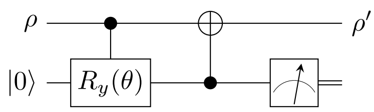
with

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{1}$$

and a real parameter $\gamma \in [0,1]$, which one can interpret as the probability. Note that $E_1$ maps $|1\rangle \mapsto \sqrt{\gamma}\,|0\rangle$.

(a) Show that the operation elements $\{E_k\}$ in Eq. (1) satisfy $\sum_{k\in\{0,1\}} E_k^\dagger E_k = I$.

We now want to verify that the following circuit describes the amplitude damping operation, with $\gamma = \sin(\theta/2)^2$:




Recall that $R_y$ is the rotation operator

$$R_y(\theta) = \mathrm{e}^{-i\theta Y/2} = \cos(\theta/2)I - i\sin(\theta/2)Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

(b) Find the $4\times 4$ matrix representation $U_{\mathsf{AD}}$ of the controlled-$R_y(\theta)$ gate followed by the flipped CNOT gate in the above circuit.

(c) Finally, read off the corresponding operation elements with entries $(E_0)_{\ell,m} = \langle \ell, 0| U_{\mathsf{AD}} |m, 0\rangle$ and $(E_1)_{\ell,m} = \langle \ell, 1| U_{\mathsf{AD}} |m, 0\rangle$, and confirm that they agree with Eq. (1).

## Solution

(a) Since $\gamma$ is real, complex conjugation does not change $\gamma$, and thus

$$\sum_{k\in\{0,1\}} E_k^\dagger E_k = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\gamma \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} = I.$$

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 8.20

(b) The controlled-$R_y(\theta)$ gate has the following matrix representation with respect to the standard computational basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$,

$$U_{\text{controlled-}R_y(\theta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix},$$

and the flipped-CNOT gate

$$U_{\text{flipped-CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Multiplying these two matrices (in the correct order!) gives $U_{\text{AD}}$:

$$U_{\text{AD}} = U_{\text{flipped-CNOT}} \cdot U_{\text{controlled-}R_y(\theta)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ 0 & 0 & \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(c) $E_0$ and $E_1$ are submatrices of $U_{\text{AD}}$:

$$E_0 = \begin{pmatrix} \langle 00| U_{\text{AD}} |00\rangle & \langle 00| U_{\text{AD}} |10\rangle \\ \langle 10| U_{\text{AD}} |00\rangle & \langle 10| U_{\text{AD}} |10\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix},$$
$$E_1 = \begin{pmatrix} \langle 01| U_{\text{AD}} |00\rangle & \langle 01| U_{\text{AD}} |10\rangle \\ \langle 11| U_{\text{AD}} |00\rangle & \langle 11| U_{\text{AD}} |10\rangle \end{pmatrix} = \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix},$$

in agreement with Eq. (1).