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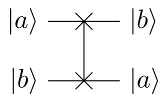
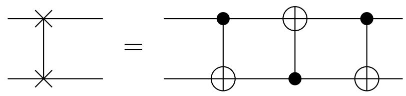
Exercise 4.1 (Basic quantum circuits)

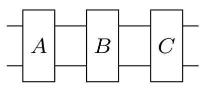
(a) Find the matrix representation (with respect to the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$) of the *swap-gate* $|a,b\rangle \mapsto |b,a\rangle$, which is written in circuit form as



Also show that the swap operation is equivalent to the following sequence of three CNOT gates:

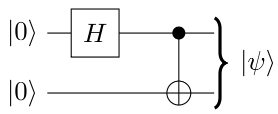


Hint: You can either work directly with basis states, e.g. $|a,b\rangle \overset{\text{CNOT}}{\mapsto} |a, a\oplus b\rangle$, or use matrix representations. In the latter case, note that a sequence of gates like



(with A, B, C unitary 4×4 matrices) corresponds to the matrix product CBA since the circuit is read from left to right, but the input vector in the matrix representation is multiplied from the right.

(b) Compute the output $|\psi\rangle$ of the following "entanglement circuit" applied to the input $|00\rangle$:



with $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ denoting the Hadamard gate.

(c) Build the CNOT gate from the controlled-Z gate and two Hadamard gates, and verify your construction.

Solution hints

(a) The swap gate interchanges $|01\rangle \leftrightarrow |10\rangle$. Think about which elements are these in the two-qubit state vector. The swap matrix consists only of 0s and 1s. To prove the equivalency with the three CNOT gates using matrix representations, first convince yourself that the flipped CNOT gate reads

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(b) The output is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

(c) Recall that H is self-inverse and note that $HZH = X$.