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Exercise 3.1 (Properties of Pauli matrices and matrix exponential)
As usual, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ denotes the Pauli vector.

(a) Verify that the Pauli matrices anti-commute with each other, i.e.,

$$\{\sigma_1, \sigma_2\} = 0, \quad \{\sigma_2, \sigma_3\} = 0, \quad \{\sigma_3, \sigma_1\} = 0,$$

where $\{A, B\} = A\,B + B\,A$ denotes the *anti-commutator* of two matrices.

(b) Verify the following commutation relations (here $[A, B] = A\,B - B\,A$ denotes the *commutator* of two matrices):

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2.$$

(c) Use the series expansion of the matrix exponential to derive that, for any $A \in \mathbb{C}^{n\times n}$ and unitary matrix $U \in \mathbb{C}^{n\times n}$,

$$\mathrm{e}^{U^\dagger A U} = U^\dagger\, \mathrm{e}^A\, U.$$

Remark: In case A is normal, one can combine this relation with the spectral decomposition to evaluate e^A, since the matrix exponential of a diagonal matrix is the pointwise exponential of the diagonal entries.

(d) Show that

$$HXH = Z \quad \text{and} \quad HZH = X,$$

where H denotes the Hadamard gate. (Since H is Hermitian and self-inverse, i.e., $H^2 = I$, H can thus be interpreted as base change matrix between the eigenvectors of X and Z.)

(e) Combine parts (c) and (d) to argue that

$$HR_x(\theta)H = R_z(\theta) \quad \text{for all } \theta \in \mathbb{R}.$$

Solution

(a) The anti-commutation relations can be verified by an explicit calculation:

$$\begin{aligned}\{\sigma_1, \sigma_2\} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0,\end{aligned}$$

and similarly for the other two.

(b) The commutation relations can likewise be verified by an explicit calculation:

$$\begin{aligned}[\sigma_1, \sigma_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \\ &= 2i\sigma_3,\end{aligned}$$

and similarly for the other two.

(c) We exploit that $UU^\dagger = I$:

$$\mathrm{e}^{U^\dagger A U} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{U^\dagger A U U^\dagger A U \cdots U^\dagger A U}_{k \text{ times}} = \sum_{k=0}^{\infty} \frac{1}{k!} U^\dagger A^k U = U^\dagger\, \mathrm{e}^A\, U.$$

(d) We can calculate explicitly

$$\begin{aligned} HZH &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} = X. \end{aligned}$$

Multiplying this relation from left and right by H and using that $H^2 = I$ leads to

$$HXH = H(HZH)H = H^2ZH^2 = Z.$$

(e) We insert the definition of $R_x(\theta)$ from the lecture, and use that $H^\dagger = H$:

$$HR_x(\theta)H = H\mathrm{e}^{-i\theta X/2}H \overset{\text{(c)}}{=} \mathrm{e}^{-i\theta HXH/2} \overset{\text{(d)}}{=} \mathrm{e}^{-i\theta Z/2} = R_z(\theta).$$