Exercise 1.2

(a)

```
{{2, -ⅈ, 5}, {3, 0, 1}}.{4, ⅈ, -3} // MatrixForm
```

$$\begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

```
{{-2, 7}, {3, 1 + 2 ⅈ}}.{{5, -4}, {6 ⅈ, 0}} // MatrixForm
```

$$\begin{pmatrix} -10 + 42\, i & 8 \\ 3 + 6\, i & -12 \end{pmatrix}$$

(b)

```
(* this matrix is not normal *)
Amat = {{0, 0}, {1, 0}};
% // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

```
(* not the same *)
Amat.Transpose[Amat] // MatrixForm
Transpose[Amat].Amat // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(c)

```
Amat = {{0, 3/5, 4/5}, {-3/5, 0, 0}, {-4/5, 0, 0}};
% // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ -\frac{3}{5} & 0 & 0 \\ -\frac{4}{5} & 0 & 0 \end{pmatrix}$$

The matrix is normal since it is anti-symmetric, i.e., its adjoint is the same as -A, and A commutes with -A.

```
(* for the homework submission, you should compute this with "pen and paper" *)
(* minus sign to adhere to convention from the lecture *)
-CharacteristicPolynomial[Amat, λ]
Solve[% == 0, λ]
```

$\lambda + \lambda^3$

$\{\{\lambda \to 0\}, \{\lambda \to -i\}, \{\lambda \to i\}\}$

Thus the eigenvalues are 0 and $\pm i$.

```
(* eigenspace corresponding to eigenvalue 0 *)
NullSpace[A_mat]
(* normalized eigenvector *)
v_0 = FullSimplify[%[[1]] / Norm[%[[1]]]]
```

$\left\{\left\{0, -\frac{4}{3}, 1\right\}\right\}$

$\left\{0, -\frac{4}{5}, \frac{3}{5}\right\}$

```
(* eigenspace corresponding to eigenvalue i *)
NullSpace[i IdentityMatrix[3] - A_mat]
(* normalized eigenvector *)
v_i = FullSimplify[%[[1]] / Norm[%[[1]]]]
```

$\left\{\left\{-\frac{5\,i}{4}, \frac{3}{4}, 1\right\}\right\}$

$\left\{-\frac{i}{\sqrt{2}}, \frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}\right\}$

```
(* eigenspace corresponding to eigenvalue -i *)
NullSpace[(-i) IdentityMatrix[3] - A_mat]
(* normalized eigenvector *)
v_-i = FullSimplify[%[[1]] / Norm[%[[1]]]]
```

$\left\{\left\{\frac{5\,i}{4}, \frac{3}{4}, 1\right\}\right\}$

$\left\{\frac{i}{\sqrt{2}}, \frac{3}{5\sqrt{2}}, \frac{2\sqrt{2}}{5}\right\}$

(d)

```
U[θ_] := {{Cos[θ], i Sin[θ]}, {i Sin[θ], Cos[θ]}}

FullSimplify[U[θ].ConjugateTranspose[U[θ]], Assumptions → {θ ∈ Reals}] // MatrixForm
```

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(e)

$1 = \mathrm{Det}[\mathrm{id}] = \mathrm{Det}[U^\dagger.U] = \mathrm{Det}[U^\dagger]\,\mathrm{Det}[U] = \mathrm{Det}[U]^*\,\mathrm{Det}[U] = |\mathrm{Det}[U]|^2$