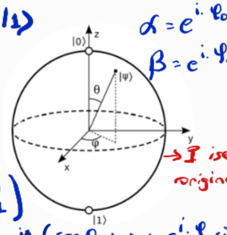
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

where $\alpha^2 + \beta^2 = 1$

$\alpha = e^{i\varphi_0} \cos\frac{\theta}{2}$

$\beta = e^{i\varphi_0} \cos\frac{\theta}{2}$



$\vec{I}$ ise orijinde

$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$

$|\psi\rangle = e^{i\alpha}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$

Hadamard Gate

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{H} \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$

Matrix exponential

$e^A = \sum_{k=0}^{\infty}\frac{1}{k!}A^k$ where $A^2 = I$ and $x \in \mathbb{R}$

$e^{iAx} = \sum_{k=0}^{\infty}\frac{1}{k!}(i.A.x)^k$

$= \cos(x).I + i.\sin(x).A$

$H Z H = X$

$H X H = Z$

$H Y H = -Y$

$|\psi\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$ ise $\langle\psi| = \frac{\langle 0| - i\langle 1|}{2}$

$H R_x(\theta) H = R_z(\theta)$

$H R_y(\theta) H = R_y(-\theta)$

$H R_z(\theta) H = R_x(\theta)$

Phase Gate

$S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

T-Gate

$T = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}$

$T^2 = S$

Rotation Operators

$A = \cos\frac{\theta}{2}.I \quad B = i.\sin\frac{\theta}{2}$

$R_x(\theta) = e^{i\theta\frac{X}{2}} = A - B.X = \begin{pmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$

$R_y(\theta) = e^{i\theta\frac{Y}{2}} = A - B.Y = \begin{pmatrix}\cos\theta/2 & -\sin\theta/2\\ \sin\theta/2 & \cos\theta/2\end{pmatrix}$

$R_z(\theta) = e^{i\theta\frac{Z}{2}} = A - B.Z = \begin{pmatrix}e^{-i\frac{\theta}{2}} & 0\\ 0 & e^{i\frac{\theta}{2}}\end{pmatrix}$

X, Y, Z : rotation around resp. axis by 180 °

H : rotation around y by 90 ° and x by 180 °

Z-Y Decomposition

$U = e^{i\alpha}\begin{pmatrix}e^{-i\frac{\beta}{2}} & 1\\ 1 & e^{i\frac{\beta}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2}\\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2}\end{pmatrix}\begin{pmatrix}e^{-i\frac{\delta}{2}} & 0\\ 0 & e^{i\frac{\delta}{2}}\end{pmatrix} = e^{i\alpha}R_z(\beta).R_y(\gamma).R_z(\delta)$

Kronecker Product

$|a,b\rangle \mapsto (A|a\rangle)\otimes(B|b\rangle)$ Example: $A = I$, $B = Y$

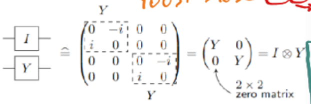
$|00\rangle \mapsto |0\rangle\otimes Y|0\rangle = i|01\rangle$

$|01\rangle \mapsto -i|00\rangle$

$|10\rangle \mapsto i|11\rangle$

$|11\rangle \mapsto -i|10\rangle$

Matrix representation:



$\begin{pmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{pmatrix} = \begin{pmatrix}Y & 0\\ 0 & Y\end{pmatrix} = I \otimes Y$

General formula: **Kronecker product** (see also exercise 2.2): matrix representation of operators:

$$A \otimes B = \begin{pmatrix}a_{11}B & a_{12}B & \cdots & a_{1n}B\\ a_{21}B & a_{22}B & \cdots & a_{2n}B\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B\end{pmatrix} \in \mathbb{C}^{mp \times nq}$$

Single-bit Gates

Principle of time evolution

$|\psi\rangle$ at current time t transition to state $|\psi'\rangle$ by complex unitary matrix U: $(U^\dagger U = UU^\dagger = I)$

$|\psi'\rangle = U.|\psi\rangle$

Pauli Vector $\sigma = (\sigma_x, \sigma_y, \sigma_z)$

Pauli-X: $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ → Makes $|0\rangle$ to $|1\rangle$, $|1\rangle$ to $|0\rangle$ } Flip

Pauli-Y: $\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

Pauli-Z: $\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ → $|0\rangle$ → Unchanged, $|1\rangle$ → Rotate by π (Change sign)

New angles

$\theta' = \theta \quad \varphi' = \varphi + \pi$

Multiple Qubits

$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$|1\rangle \otimes |0\rangle = |10\rangle$

$\alpha(|v\rangle \otimes |w\rangle) = \alpha|v\rangle \otimes |w\rangle$

$\begin{pmatrix}v_1\\v_2\end{pmatrix}\otimes\begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}v_1w_1\\ v_1w_2\\ v_2w_1\\ v_2w_2\end{pmatrix}$

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

if we measured only 1 qubit of two qubit: 0 with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$, 1 with prob $|\alpha_{10}|^2 + |\alpha_{11}|^2$

Wave function after measurement

if we measured 0: $|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$

if we measured 1: $|\psi'\rangle = \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}$

Why Quantum Gates U must be unitary

- It has to be unitary to preserve normalization
- Physical gates are discrete steps of the time evolution operator, which is unitary.

$|\psi'\rangle = U.|\psi\rangle$

Probability: measure $0 \to |\alpha|^2$, measure $1 \to |\beta|^2 \to |i|^2 = 1$

Point on the surface of Bloch sphere

$\vec{r} = \begin{pmatrix}\cos\varphi\sin\theta\\ \cos\varphi\sin\theta\\ \cos\theta\end{pmatrix}$

Circuit → Left to Right

Matrix → right to left

$A^\dagger = (A^*)^T$ → Adjoint or conjugate transpose

Hermitian if $A^\dagger = A$ and A is square

Unitary if $U^\dagger U = I$

Controlled Z

$\begin{pmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1\end{pmatrix}$

You can flip control and target qubit.

$e^{i\pi} = -1$, $e^{i\frac{\pi}{2}} = i$ $\quad e^{i\frac{\pi}{2}} = -1$

$e^{ix} = \cos(x) + i.\sin(x)$ $\quad e^{i\frac{\pi}{2}} = i$

$e^{-ix} = \cos(x) - i.\sin(x)$

CNOT → if control is 1

$|10\rangle \mapsto |01\rangle$, target flipped

control, target: $|a,b\rangle \mapsto |a, a\oplus b\rangle$

$U_{CNOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \begin{pmatrix}I & 0\\ 0 & X\end{pmatrix}$

Generalization of CNOT

$|10\rangle \to |1\rangle\otimes(X|0\rangle)$, $|11\rangle \to |1\rangle\otimes(X|1\rangle)$

$U_{CU}\begin{pmatrix}I & 0\\ 0 & U\end{pmatrix}$

Quantum Measurement

measuring Z corresponds to projecting onto z axis

$\langle Z\rangle = r_z$

Standart Deviation of observable M

$\Delta M = \sqrt{\langle M^2\rangle - \langle M\rangle^2}$, with

$\langle A\rangle = \langle\psi|A|\psi\rangle$

Note, $\Delta Z = \sqrt{1 - r_z^2}$ since $Z^2 = I$, Also $X^2 = I$, Also $Y^2 = I$

$[Z,X] = ZX - XZ = 2iY$ (two)

$\langle\psi|[Z,X]|\psi\rangle = 2ir_y$

Heisenberg uncertainty Principle

$\Delta C.\Delta D \geq \frac{|\langle\psi|[C,D]|\psi\rangle|}{2}$

Note: $r_x^2 + r_y^2 + r_z^2 = 1$

Entanglement

a n qubit state $|\psi\rangle$ $(n \geq 2)$ is called entangled if it cannot be written as tensor products of single-qubit states.

$|\psi\rangle \neq |\varphi_{n-1}\rangle \otimes \ldots \otimes |\varphi_0\rangle$

$\frac{|00\rangle + |11\rangle}{2}$ { $|0\rangle$ $|0\rangle$ H

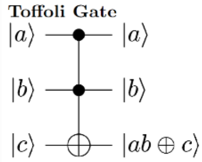
Bell States

$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\quad |B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ $\quad |B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$|x\rangle$, $|y\rangle$ verilirsen B_{xy} sonucunu verir.

Toffoli Gate



eigen values: +1, -1

Quantum Measurement

$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$|-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\alpha|0\rangle + \beta|1\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$

$+$ with probability $\frac{|\alpha+\beta|^2}{2}$

$-$ with probability $\frac{|\alpha-\beta|^2}{2}$

Controlled-U matrix

$\begin{pmatrix}1 & \\ & 1 & \\ & & U\end{pmatrix}$

$|+-\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$

$|-+\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}$

$|++\rangle = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

$|--\rangle = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

- $\sigma_j^2 = I$
- $\sigma_j \cdot \sigma_k = -\sigma_k \cdot \sigma_j$
- $[\vec{a}\cdot\sigma, \vec{b}\cdot\sigma] = 2i(\vec{a}\times\vec{b})\cdot\sigma$
- $[\sigma_j, \sigma_k] = 2i\sigma_l$
- $\{\sigma_k, \sigma_j\} = 0$
- $\{\vec{a}\sigma, \vec{b}\sigma\} = 2(\vec{a}\,\vec{b})I$

Eivec	X	Y	Z
	$\lvert+\rangle, \lvert-\rangle$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$	$\lvert 0\rangle, \lvert 1\rangle$

Linear Algebra

$||v|| = \sqrt{\langle v|v\rangle} = \sqrt{\sum|v_i|^2}$

$\langle w|v\rangle = \langle v|w\rangle^* = \sum_{i=1}^n w_j^* v_j$

$\langle v|Aw\rangle = \langle A^\dagger v|w\rangle$

$A^\dagger = (A^*)^T$

Hermitian: $A^\dagger = A$

Eigenvalues of hermitian are real

Unitary: $U^\dagger U = UU^\dagger = I$

Normal: $A^\dagger A = AA^\dagger$

Orthogonal project matrix: Hermitian $P^2 = P$

$e^{i\pi}$: Rotation by 180°

$e^{i\pi/2} = i$

$e^{ix} = \cos(x) + i\sin(x)$

$e^{iAx} = \cos(x)I + i\sin(x)A$

$e^{U^\dagger AU} = U^\dagger e^A U$ for U unitary

$(e^A)^\dagger = e^{(A^\dagger)}$

$e^A = \sum_0^\infty \frac{A^k}{k!}$

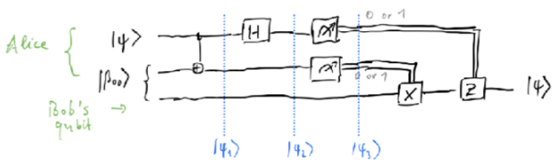
Other trivials

$tr[|\psi\rangle\langle\varphi|] = \langle\varphi|\psi\rangle$

$tr[AB] = tr[BA]$

Cauchy-Schwarz: $|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle = ||u||\cdot||v||$

Quantum Teleportation



Schrödinger

$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$

$|\psi(t)\rangle = U_t|\psi(0)\rangle$ with $U_t = e^{-iHt/\hbar}$

von Neumann

Given density matrix ϱ

$i\hbar\frac{d}{dt}|\varrho(t)\rangle = [H, \varrho(t)]$

$\varrho(t) = U(t)\varrho_0 U(-t)$ with $U(t) = e^{-iHt/\hbar}$

Quantum measurement

M_m measurement operator

$M_x = |x\rangle\langle x|$ with $x \in \mathbb{R}^2$

$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle = ||\,M_m|\psi\rangle\,||^2$

Afterward: $|\psi'\rangle = \frac{M_m|\psi\rangle}{||M_m|\psi\rangle||}$

Kronecker

$(A\otimes B) = \begin{pmatrix}a_{11}B & a_{12}B\\ a_{21}B & a_{22}B\end{pmatrix}$

$(A\otimes B)^* = A^*\otimes B^*$

$(A\otimes B)^T = A^T\otimes B^T$

$(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger$

$(A\otimes B)\otimes C = A\otimes(B\otimes C)$

$(A\otimes B)\cdot(C\otimes D) = (A\cdot C)\otimes(B\cdot D)$

Projective measurement

Projector $P = \sum_{j=1}^m |u_j\rangle\langle u_j|$

Observab. $M = \sum_m \lambda_m P_m$ with λ_m eigval of P_m

$P_m = |u_m\rangle\langle u_m|$

$p(m) = \langle\psi|P_m|\psi\rangle$

State after meas.: $|\psi'\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}}$

- $\mathbb{E}[M] = \sum_m \lambda_m p(m) = \langle\psi|M|\psi\rangle = \langle M\rangle$
- $\Delta(M) = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{\langle(M - \langle M\rangle)^2\rangle}$

Tensor Product

$\vec{v}\otimes w = |v\rangle\otimes|w\rangle = \begin{pmatrix}v_1w_1\\ v_1w_2\\ v_2w_1\\ v_2w_2\end{pmatrix}$

$(|v_1\rangle + |v_2\rangle)\otimes|w\rangle = |v_1\rangle\otimes|w\rangle + |v_2\rangle\otimes|w\rangle$

Grover Search

Complexity $\to O\left(\sqrt{\frac{N}{M}}\right)$ given certain predictions

$N \to$ Search space $M \to$ number of solutions

$1 \leq M \leq N$

$f: \{0, \dots, N-1\} \to \{0,1\}$

$f(x) = 0 \to$ Not a solution

$f(x) = 1 \to$ Solution

$\alpha \to$ Consist of non-solutions

$|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_{\substack{x=0 \\ f(x)=0}}^{N} |x\rangle$

$\beta \to$ Consist of solutions

$|\beta\rangle = \frac{1}{\sqrt{M}} \sum_{\substack{x=0 \\ f(x)=1}}^{N} |x\rangle$

Rotation Angle θ defined by

$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{M}{N}}$ such that $|\psi\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle$

$U_f|\alpha\rangle = |\alpha\rangle$

$U_f|\beta\rangle = -|\beta\rangle$

Grover Operator

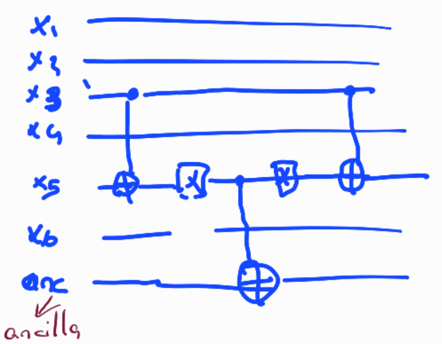
Phase Gate $\to$ $|0\rangle \to |0\rangle$

$|x\rangle \to -|x\rangle$ for $x > 0$

FULL GROVER

$(2|\psi\rangle\langle\psi| - I)U_f$

$O|x\rangle|anc\rangle = |x\rangle|anc \oplus f(x)\rangle$



Reduced density operator

Partial Trace: $tr_1[M_1 \otimes M_2]) = tr[M_1] \cdot M_2$

$tr_1[\alpha M_1 \otimes M_2 + \beta N_1 \otimes N_2] = \alpha tr_1[M_1 \otimes M_2] + \beta tr_1[N_1 \otimes N_2]$

Reduced density operator for A: $\varrho^A = tr_B[\varrho^{AB}]$

Calculate elements of Kraus Operator

$$U_{AD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E_0 = \begin{pmatrix} \langle 00|U_{AD}|00\rangle & \langle 00|U_{AD}|10\rangle \\ \langle 10|U_{AD}|00\rangle & \langle 10|U_{AD}|10\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} \langle 01|U_{AD}|00\rangle & \langle 01|U_{AD}|10\rangle \\ \langle 11|U_{AD}|00\rangle & \langle 11|U_{AD}|10\rangle \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Schmidt Decomposition

$|\psi\rangle = \sum_{i=1}^{k} \sigma_i |i_A\rangle |i_B\rangle$

Density operator

$\varrho = |\psi\rangle\langle\psi| = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

- $tr[\varrho] = 1$
- $U\varrho U^\dagger$
- Hermitian
- eigenvalues ≥ 0 ($\Rightarrow \langle\varphi|\varrho|\varphi\rangle \geq 0$)
- Blochsphere repr.: $\varrho = \frac{1 + \vec{r}\cdot\vec{\sigma}}{2}$ $\vec{r} \in \mathbb{R}^3, ||\vec{r}|| \leq 1$

$p(m|i) = \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = tr[M_m^\dagger M|\psi_i\rangle\langle\psi_i|]$

Overall prob. $p(m) = tr[M_m^\dagger M_m \varrho] = ||M_m|\psi_i\rangle||^2$

Density operator after collapse: $\varrho_m = \frac{M_m \varrho M_m^\dagger}{tr[M_m^\dagger M_m \varrho]}$

Pure State: $tr[\varrho^2] = 1$

Mixed State: $tr[\varrho^2] < 1$

Purification

Schmidt Decomposition: $\rho_A = \sum_x \lambda_x |a_x\rangle\langle a_x|$

Extend: $\rho_A \otimes B = \sum_x \sqrt{\lambda_x} |a_x\rangle \otimes |b_x\rangle$

Check that: $\rho_A = tr_B[\rho_A \otimes B]$

Entanglement Entropy

Given eigenvalues of subsystem ϱ_1:

$S_{ent} = S_{\varrho 1} = S_{\varrho 2} = -\sum_{i=1}^{k} \sigma_i^2 log(\sigma_i^2)$

Valid Density matrix

- $tr[\rho] = tr[\hat{\rho}] = 1$
- ρ is Hermition $\rho = \rho^\dagger$ $\lambda \geq 0$
- Eigen Values

$\rho' = U\rho U^\dagger$

$\rho' = R_y(\theta)\rho R_y(\theta)^\dagger$

$\vec{r} \cdot \vec{\sigma} = c_1 \cdot x + c_2 \cdot y + c_3 \cdot z$

$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$

Density

Density operator ρ of ensemble $\{p_i, |\psi_i\rangle\}_i$ defined as $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$

- Unitary Operation $|\psi_i\rangle \to U|\psi_i\rangle$

$\rho \xrightarrow{U} \sum p_i U|\psi_i\rangle\langle\psi_i|U^\dagger = U\rho U^\dagger$

- Measurement

$p(m|i) = tr[M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|]$

Chrecterizition of density operators

1. $tr[\rho] = 1$
2. ρ is positive operator (if it is Hermition and all eigenvalues are ≥ 0 yani if $\langle\varphi|\rho|\varphi\rangle \geq 0$ for all vectors $|\varphi\rangle$)

Density Operators

Pure state $\to \rho = |\psi\rangle\langle\psi|$ such that $tr[\rho^2] = \langle\psi|\psi\rangle^2 = 1$

Mixed State $\to$ " " " $tr[\rho^2] = \sum p_i^2 < 1$

Example

$\rho_A = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$tr[\rho_A^2] = \frac{1}{16}(1 + 9) < 1$ mixed

Pirincipal Quantum State

$\mathcal{E}(\rho) = tr_{env}[U(\rho \otimes \rho_{env})U^\dagger]$

Example

$\mathcal{E}(\rho) = tr_{env}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$

Alternative $\to \mathcal{E}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger$ with $(E_k)_{\ell m} = \langle \ell, k|U|m, 0\rangle$

E_0 E_1 gördüğün sorguya yoz.

reconcile

$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$

with complex matrices (Kraus Operators)

E_k obeying $\sum_k E_k^\dagger E_k \leq I$

Open Quantum System can be regarded as interaction between

- principal system (State ρ)
- Enviroment (State ρ_{env}

Quantum operations/channels

Unitary: $\varrho' = \mathcal{E}(\varrho) = U\varrho U^\dagger$

Measurement $= \mathcal{E}_m(\varrho) = M_m \varrho M_m^\dagger$

$\mathcal{E} = \sum_m p(m)\varrho_m = \sum_m M_m \varrho M_m^\dagger$

Unital: $\mathcal{E}(I) = I$ Examples:

- Bit flip: $E_0 = \sqrt{p}I$, $E_1 = \sqrt{1-p}X$
- Phase flip: $E_0 = \sqrt{(p)}I$, $E_1 = \sqrt{1-p}Z$
- Depolarizing (replace with mixed state): $\mathcal{E}(\varrho) = p\frac{I}{2} + (1-p)\varrho$
- Amplitude damping: $E_0 = diag(1, \sqrt{1-\sigma})$, $E_1 = \begin{pmatrix} 0 & \sqrt{(\gamma)} \\ 0 & 0 \end{pmatrix}$

Example: Purification

Give purpification for $|\Theta\rangle$ of ϱ_A:

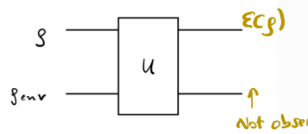
For any pure state $|\psi\rangle$ we write Schmidt decomp. $|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle$ with λ_i^2 eigenvalues of red. density matrix.

Set $|\psi\rangle = \sum_{i=0}^{1} \sqrt{(p_i)}|\Theta\rangle|i\rangle$. ϱ is diag. matrix, we know that $(\varrho_0, |\Theta_0\rangle) = (2/3, |0\rangle)$ and $(\varrho_1, |\Theta_1\rangle) = (1/3, |1\rangle)$. Thus $|\psi\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle\langle 0| + \frac{1}{\sqrt{3}}|1\rangle\langle 1|$ is purification for ϱ_A.

Example: Find kraus oeprators

Given $\mathcal{E}(\varrho) = tr(|\psi\rangle\langle\psi|\varrho)|\theta\rangle\langle\theta|$. So $\mathcal{E}(\varrho) = tr(\langle\psi|\varrho|\psi\rangle)|\theta\rangle\langle\theta| = \langle\psi|\varrho|\psi\rangle|\theta\rangle\langle\theta| = |\theta\rangle\langle\psi|\varrho|\psi\rangle\langle\theta|$. Thus kraus operator is $|\theta\rangle\langle\psi|$.

Circuit diagram:



$\mathcal{E}(\rho)$

Not observed

Math Representation

$U(\rho \otimes \rho_{env})U^\dagger$

$\mathcal{E}(\rho) = tr_{env}[U(\rho \otimes \rho_{env})U^\dagger]$

trace out enviroment

Exp

$\rho_{env} = |0\rangle\langle 0|$ and $U = CNOT$

$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$

$tr_{env}[U_{CNOT}(\rho \otimes |0\rangle\langle 0|)U_{CNOT}^\dagger]$

$tr_{env}[\rho_{00}|00\rangle\langle 00| + \rho|00\rangle\langle 10| \dots]$

burada enviroment'i calculate için her ifadenin 2. bitlerini inner product yapıyoruz.

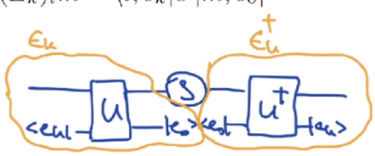
$\rho_{00}|0\rangle\langle 0| \underbrace{\langle 0|0\rangle}_{1} + \rho_{01}|0\rangle\langle 1| \underbrace{\langle 0|1\rangle}_{0} + \dots + \rho_{11}|1\rangle\langle 1| \underbrace{\langle 1|1\rangle}_{1}$

$= \rho_{00}|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1| = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix} = P_0 \rho P_0^\dagger + P_1 \rho P_1$ with $P_0 = |0\rangle\langle 0|$, $P_1 = |1\rangle\langle 1|$

Operator-sum representation

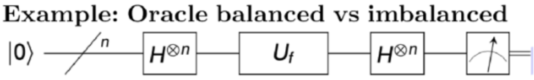
$\mathcal{E}(\varrho) = \sum_k E_k \varrho E_k^\dagger$

$(E_k)_{lm} = \langle l, e_k|U|m, e_0|$



Tracepreserving: $1 = tr[\mathcal{E}(\varrho)] = tr[\sum_k E_k \varrho E_k^\dagger] = \sum_k tr[E_k \varrho E_k^\dagger] = \sum_k tr[E_k^\dagger E_k \varrho] = tr[(\sum_k E_k^{\dagger E_k})\varrho] = tr[\varrho]$

Example: Oracle balanced vs imbalanced



$|\psi_1\rangle = \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}}$

$|\psi_2\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{\sqrt{2^n}}|x\rangle$

$|\psi_3\rangle = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x \cdot z + f(x)}}{2^n}|z\rangle$