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Tutorial 2 (Dirac notation and inner products)
The Dirac notation (also called bra-ket notation), which you have seen being used in the lecture, uses "kets", such as $|\psi\rangle$, to represent a quantum state. For our purposes, a ket is always a complex (column) vector.[1] ψ is usually the actual vector itself, or can be an identifier or index for the quantum state, as for $|0\rangle$ and $|1\rangle$.

The corresponding "bra" $\langle\psi|$ is then the conjugate-transposed $|\psi\rangle$, i.e., a row vector with complex-conjugated entries of $|\psi\rangle$. A motivation for this notation is that "bras" are linear maps from quantum states to complex numbers via the inner product. Namely, given $\phi \in \mathbb{C}^n$:

$$\langle\phi| : \mathbb{C}^n \to \mathbb{C}, \quad |\psi\rangle \mapsto \langle\phi|\psi\rangle = \sum_{j=1}^{n} \phi_j^* \, \psi_j.$$

(a) Write down the matrix representation of the following expressions:

- $|0\rangle\langle 1|$
- $|0\rangle\langle 0| + |1\rangle\langle 1|$
- $|+\rangle\langle 0|$, with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

(b) Express the Hadamard gate H using Dirac notation in the computational basis (i.e. $\{|0\rangle, |1\rangle\}$).

(c) Given the qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, compute $H|\psi\rangle$ using only the bra-ket notation.

(d) For any $\psi, \phi \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$, verify that

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle,$$

with $A^\dagger = (A^*)^T$ denoting the conjugate transpose (adjoint) of A.

(e) Prove that unitary matrices are norm-preserving, i.e., $\|U\psi\| = \|\psi\|$ for all unitary $U \in \mathbb{C}^{n\times n}$ and $\psi \in \mathbb{C}^n$.
Hint: Use that $\|\psi\|^2 = \langle\psi|\psi\rangle$ and part (d).

Solution

(a) Using the column and row vector forms leads to

$$\begin{aligned}
|0\rangle\langle 1| &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix},\\
|0\rangle\langle 0| + |1\rangle\langle 1| &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I_2 \quad \text{(identity matrix)},\\
|+\rangle\langle 0| &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}.
\end{aligned}$$

(b) Recall that the Hadamard gate is defined as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}.$$

We can re-write this in bra-ket notation as

$$H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|).$$

(c) Here we can use the fact that $|0\rangle$ and $|1\rangle$ form an orthonormal basis, and therefore: $\langle a|b\rangle = \delta_{ab}$ for $a, b \in \{0, 1\}$. Thus

$$\begin{aligned}
H|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\\
&= \frac{1}{2}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) = |0\rangle.
\end{aligned}$$

[1] In general, quantum states can also be complex-valued functions (e.g., electronic orbitals of atoms), but these will not play a role in this course.

For comparison, the equivalent vector notation for the same operation reads

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

(d) Using the definition of the inner product and the index notation for a matrix-vector product, we can write

$$\langle\phi|A\psi\rangle = \sum_{j=1}^{n}\phi_j^*\sum_{k=1}^{n}A_{jk}\psi_k = \sum_{j,k=1}^{n}\phi_j^*A_{jk}\psi_k.$$

Now, note that $A_{jk} = (A^\dagger)^*_{kj}$ which we can use to rewrite the above expression as

$$\langle\phi|A\psi\rangle = \sum_{j,k=1}^{n}(A^\dagger)^*_{kj}\phi_j^*\psi_k = \langle A^\dagger\phi|\psi\rangle.$$

(e) Using our result from part (d) and that $U^\dagger U = I$ by definition of a unitary matrix,

$$\|U\psi\|^2 = \langle U\psi|U\psi\rangle = \langle U^\dagger U\psi|\psi\rangle = \langle\psi|\psi\rangle = \|\psi\|^2.$$

This means that $\|U\psi\| = \|\psi\|$.