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Exercise 8.2 (CHSH game)
The CHSH game demonstrates the power of quantum correlations and gives rise to a Bell inequality through a simple game in a real-world setting.

Alice and Bob play a game whose inputs and objective are described in the following. Alice and Bob each receive a completely random bit denoted by x resp. y (independently distributed, with equal probability for 0 or 1). After receiving their input they are not allowed to communicate until the end of the game. Alice and Bob each have to produce an output $a \in \{0, 1\}$ resp. $b \in \{0, 1\}$. They win the game whenever their inputs and outputs fulfill the condition

$$x \cdot y = a + b \mod 2.$$

As a resource Alice and Bob receive an EPR pair

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

shared between them.

(a) Draw a sketch of the setting described above.

(b) Before starting the game, Alice and Bob are allowed to agree on a strategy determining which outputs to generate for a given input. Note that Alice only has access to her input x and Bob only to his input y and they are not allowed to communicate once they received their input. In case Alice and Bob do not make use of their shared entanglement, the maximum winning probability they can achieve with a deterministic classical strategy is 75%. Give a deterministic classical strategy which Alice and Bob should follow in order to achieve the best possible classical winning probability.

We now give a strategy for Alice and Bob making use of their quantum resources. Define the unitary operator

$$U_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}$$

and parameters

$$\theta_0 = 0, \quad \theta_1 = \frac{\pi}{4},$$
$$\tau_0 = \frac{\pi}{8}, \quad \tau_1 = -\frac{\pi}{8}.$$

Based on their inputs x, y Alice and Bob will apply operators U_{θ_x} and U_{τ_y} on their local system of the shared EPR pair. Afterwards, they measure their local register in the computational basis with outcomes a, b. The probability of receiving outcomes a, b dependent on inputs x, y is given by

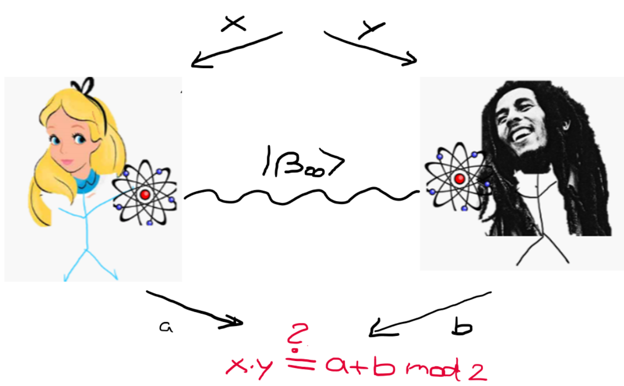
$$P(a, b \mid x, y) = \langle (U_{\theta_x} \otimes U_{\tau_y})\beta_{00} \mid P_a \otimes P_b \mid (U_{\theta_x} \otimes U_{\tau_y})\beta_{00}\rangle,$$

where $P_a = |a\rangle\langle a|$ and $P_b = |b\rangle\langle b|$ are projections onto the computational basis states for $a, b \in \{0, 1\}$. Alice and Bob use the outcome of their measurement a, b as output for the game.

(c) Show that Alice and Bob can win with probability roughly 85% when following the above quantum strategy.
Hint: The relation $(V \otimes W)|\beta_{00}\rangle = (V \cdot W^T \otimes \mathbb{1})|\beta_{00}\rangle$ for any $V, W \in \mathbb{C}^{2\times 2}$ might be helpful.

Solution

(a)



(b) A deterministic classical strategy Alice and Bob can agree on in order to reach a winning probability of 75% is that both always output 0. For any pair of input bits except for $(x, y) = (1, 1)$ Alice and Bob will win, as inserting the different combinations of x and y with $a = b = 0$ into $x \cdot y = a + b \mod 2$ shows.

More formally, we can calculate the winning probability for this strategy by defining the set $A = \{(x, y, a, b) \in \{0,1\}^4 \mid x \cdot y = a + b \mod 2\}$ such that

$$\begin{aligned} P(\text{win}) &= \sum_{(x,y,a,b)\in A} P(x, y) P(a, b \mid x, y) \\ &= \frac{1}{4} \left(P(0, 0 \mid 0, 1) + P(0, 0 \mid 1, 0) + P(0, 0 \mid 0, 0) \right) \\ &= \frac{3}{4}, \end{aligned}$$

since for the chosen strategy $P(a, b \mid x, y) = 1$ for $a = b = 0$ and $P(a, b \mid x, y) = 0$ otherwise.

(c) Let again $A = \{(x, y, a, b) \in \{0,1\}^4 \mid x \cdot y = a + b \mod 2\}$. Then the winning probability for the quantum strategy is

$$\begin{aligned} P(\text{win}) &= \sum_{(x,y,a,b)\in A} P(x, y) P(a, b \mid x, y) \\ &= \frac{1}{4} \sum_{(x,y,a,b)\in A} P(a, b \mid x, y) \\ &= \frac{1}{4} \left(P(\{(0, 1), (1, 0)\} \mid 1, 1) + \sum_{(x,y)\neq(1,1)} P(\{(0, 0), (1, 1)\} \mid x, y) \right) \\ &= \frac{1}{4} \left(P(0, 1 \mid 1, 1) + P(1, 0 \mid 1, 1) + \sum_{(x,y)\neq(1,1)} \left(P(0, 0 \mid x, y) + P(1, 1 \mid x, y) \right) \right) \\ &= \frac{1 + \frac{1}{\sqrt{2}}}{2} \approx 85\%. \end{aligned}$$

The last equation is derived by computing the value of $P(a, b \mid x, y)$ for all $(x, y, a, b) \in A$ with the formula

$$P(a, b \mid x, y) = \langle (U_{\theta_x} \otimes U_{\tau_y}) \beta_{00} \mid P_a \otimes P_b \mid (U_{\theta_x} \otimes U_{\tau_y}) \beta_{00} \rangle. \tag{1}$$

Now this can be done by inserting the matrix representation of U_{θ_x}, U_{τ_y}, P_a, P_b and $|\beta_{00}\rangle$ and explicitly calculating the matrix-vector multiplications. Alternatively, we can first simplify equation (1):

$$\begin{aligned} P(a, b \mid x, y) &= \langle (U_{\theta_x} \otimes U_{\tau_y}) \beta_{00} \mid P_a \otimes P_b \mid (U_{\theta_x} \otimes U_{\tau_y}) \beta_{00} \rangle \\ &= \langle (U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \beta_{00} \mid P_a \otimes P_b \mid (U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \beta_{00} \rangle \\ &= \langle (U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \beta_{00} \mid (P_a \otimes \mathbb{1})(\mathbb{1} \otimes P_b)(U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \mid \beta_{00} \rangle \\ &= \langle (U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \beta_{00} \mid (P_a \otimes \mathbb{1})(U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1})(\mathbb{1} \otimes P_b) \mid \beta_{00} \rangle \\ &= \langle (U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \beta_{00} \mid (P_a \cdot U_{\theta_x} \cdot U_{-\tau_y} \otimes \mathbb{1}) \frac{1}{\sqrt{2}} |bb\rangle \\ &= \frac{1}{2} \left(\langle 0| (U_{\theta_x} \cdot U_{-\tau_y})^\dagger \otimes \langle 0| + \langle 1| (U_{\theta_x} \cdot U_{-\tau_y})^\dagger \otimes \langle 1| \right) \left(P_a \cdot U_{\theta_x} \cdot U_{-\tau_y} |b\rangle \otimes |b\rangle \right) \\ &= \frac{1}{2} \langle (U_{\theta_x} \cdot U_{-\tau_y}) b | (P_a \cdot U_{\theta_x} \cdot U_{-\tau_y}) b \rangle \\ &= \frac{1}{2} \langle b| (U_{\theta_x} \cdot U_{-\tau_y})^\dagger (P_a \cdot U_{\theta_x} \cdot U_{-\tau_y}) b \rangle \\ &= \frac{1}{2} \langle b| (U_{\theta_x} \cdot U_{-\tau_y})^\dagger |a\rangle \langle a| (U_{\theta_x} \cdot U_{-\tau_y}) b \rangle \\ &= \frac{1}{2} \left| (U_{\theta_x} \cdot U_{-\tau_y})_{a,b} \right|^2. \end{aligned}$$

Here we have used the hint, and that $U_\theta^T = U_\theta^{-1} = U_{-\theta}$ since U_θ is an orthogonal rotation matrix.

Now computing (1) for a given (x, y, a, b) reduces to computing $\frac{1}{2}|(U_{\theta_x} \cdot U_{-\tau_y})_{a,b}|^2$. For example, we have

$$
\begin{aligned}
P(0,0 \mid 0,1) &= \frac{1}{2}|(U_{\theta_0} \cdot U_{-\tau_1})_{0,0}|^2 \\
&= \frac{1}{2}\left|\left(\begin{pmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{pmatrix} \cdot \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) & \sin\left(\frac{\pi}{8}\right) \\ -\sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{pmatrix}\right)_{0,0}\right|^2 \\
&= \frac{1}{2}\cos\left(\frac{\pi}{8}\right)^2.
\end{aligned}
$$