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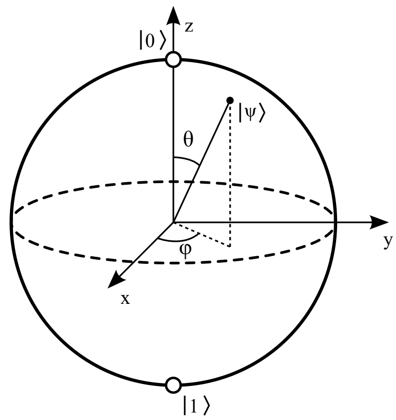
Exercise 2.1 (Bloch sphere and single qubit rotation gates)
Recall from the lecture that an arbitrary single qubit quantum state can be parametrized as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, φ and γ are real numbers, which can be chosen such that $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. The angles θ and φ define the Bloch sphere representation of $|\psi\rangle$, as shown on the right.



https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg

(a) Find the Bloch angles θ and φ of $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$, and the corresponding Bloch vector

$$\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)).$$

For a real unit vector $\vec{v} \in \mathbb{R}^3$, the rotation by an angle ω about the $\vec{v}$ axis is defined as

$$R_{\vec{v}}(\omega) = \exp(-i\omega\,\vec{v}\cdot\vec{\sigma}/2) = \cos(\omega/2)I - i\sin(\omega/2)(\vec{v}\cdot\vec{\sigma}),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector. The rotations R_x, R_y, R_z about the standard axes correspond to the special cases $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$ and $\vec{v} = (0,0,1)$, respectively.

(b) Compute $R_x(\frac{2\pi}{3})|\psi\rangle$ for the state $|\psi\rangle$ defined in (a), and visualize this operation on the Bloch sphere.

Hint: $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

(c) The *Z-Y decomposition* theorem states the following: given any unitary 2×2 matrix U, there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta).$$

Find the Z-Y decomposition of the Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Hint: There exists a solution with $\beta = 0$.

Solution hints

(a) Absorb the prefactor i of $|\psi\rangle$ into $e^{i\gamma}$ by setting $\gamma = \frac{\pi}{2}$. The last entry of the Bloch vector $\vec{r}$ is $-\frac{1}{2}$.

(b) You should obtain $R_x(\frac{2\pi}{3})|\psi\rangle = i|0\rangle$. R_x rotates $|\psi\rangle$ within the y-z-plane to the north pole. (The prefactor i in $i|0\rangle$ does not affect the Bloch vector representation.)

(c) Set $\gamma = \frac{\pi}{2}$ and $\delta = \pi$, and absorb an overall prefactor in $e^{i\alpha}$.