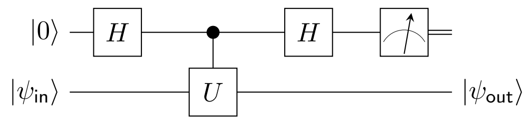
Tutorial 5 (Measuring an operator[1])

Suppose U is a single qubit operator with eigenvalues ± 1, so that U is both Hermitian and unitary, i.e., it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. Show that this is implemented by the following quantum circuit:



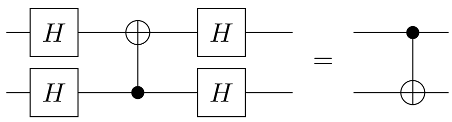
This tutorial requires the concept of an orthogonal projection (see also the linear algebra cheatsheet): a square matrix $P \in \mathbb{C}^{n \times n}$ is called an *orthogonal projection matrix* if P is Hermitian ($P^\dagger = P$) and $P^2 = P$, i.e., applying P a second time does not change the result any more. Note that a geometric projection is a special case of this abstract definition.

Exercise 5.1 (Basis transformation and measurement)

(a) Compute the probabilities when measuring $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with respect to the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ given by $|u_1\rangle = \frac{3}{5}|0\rangle + i\frac{4}{5}|1\rangle$ and $|u_2\rangle = \frac{4}{5}|0\rangle - i\frac{3}{5}|1\rangle$.

Hint: You can obtain the coefficients of $|\psi\rangle$ with respect to these basis states by computing the inner products $\langle u_j|\psi\rangle$ for $j = 1, 2$.

(b) The role of the control and target qubit of a CNOT gate can be reversed by switching to a different basis! First show that



with H the Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Use this identity to derive the following relations:

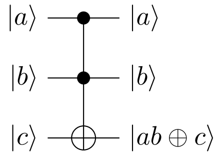
$$
\begin{aligned}
|+\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|+\rangle \\
|-\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|+\rangle \\
|+\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|-\rangle \\
|-\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|-\rangle
\end{aligned}
$$

with $|\pm\rangle$ defined as $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. In other words, with respect to the $|\pm\rangle$ basis, the second qubit assumes the role of the control and the first qubit the role of the target.

Hint: Use that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, and conversely $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$.

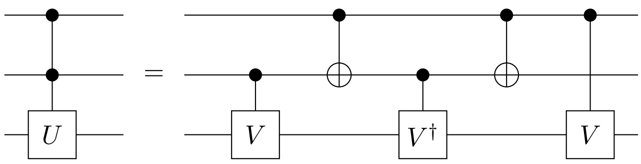
[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 4.34

Exercise 5.2 (Toffoli gate and quantum logic)
The Toffoli gate is also known as CCNOT or "controlled-controlled-NOT" operation. In classical circuits, this gate inverts the target wire when the input of its two control wires is 1. Thus, defining the Toffoli gate in terms of computational basis states leads to (for any $a, b, c \in \{0, 1\}$):



The bottom (target) qubit gets flipped precisely if both control qubits are in the $|1\rangle$ state; equivalently, the flip occurs if the product ab equals 1, which leads to the expression $ab \oplus c$.

Physical quantum computers only implement a finite gate-set as hardware operations. Thus it is typically necessary to decompose such multi-control operations in terms of gates with at most one control wire. One such decomposition of a controlled-controlled-U operation, known as the Sleator and Weinfurter construction[2], is:

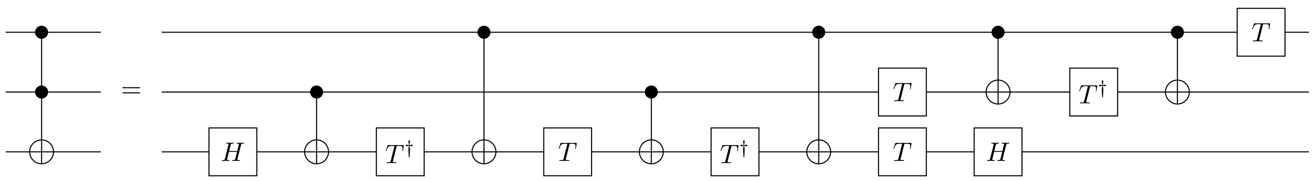


where V is a certain single-qubit gate depending on U. (The Toffoli gate corresponds to the special case $U = X$.)

(a) Which condition must V satisfy such that the equality holds? Verify your answer by inserting all four possible computational basis states for the control qubits.

(b) Find the V gate corresponding to $U = X$.
Hint: You can obtain a matrix power A^κ (with $\kappa \in \mathbb{R}$) of a normal matrix $A \in \mathbb{C}^{n \times n}$ by first computing its spectral decomposition: $A = U \operatorname{diag}(\lambda_1, \dots, \lambda_n) U^\dagger$ and U unitary; then exponentiate the eigenvalues, i.e., $A^\kappa = U \operatorname{diag}(\lambda_1^\kappa, \dots, \lambda_n^\kappa) U^\dagger$.

The so-called Clifford gates play an important role for quantum error correction and the Gottesman-Knill theorem.[3] Hence, most quantum computers support the "Clifford+T" gate set, which includes the Hadamard gate, the Pauli gates, the CNOT gate as well as the T gate, defined as $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$. It is possible to decompose the Toffoli gate in terms of these gates:



(c) Verify that the above circuit indeed implements the Toffoli gate.

[2]T. Sleator, H. Weinfurter: *Realizable Universal Quantum Logic Gates*. Phys. Rev. Lett. 74, 4087 (1995)
[3]Quantum error correction will be part of the follow-up course "Advanced Concepts of Quantum Computing" next semester.