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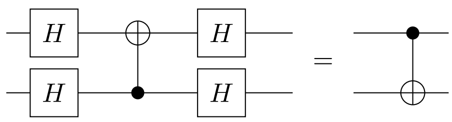
Exercise 5.1 (Basis transformation and measurement)

(a) Compute the probabilities when measuring $|\psi\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ with respect to the orthonormal basis $\{|u_1\rangle, |u_2\rangle\}$ given by $|u_1\rangle = \frac{3}{5}|0\rangle + i\frac{4}{5}|1\rangle$ and $|u_2\rangle = \frac{4}{5}|0\rangle - i\frac{3}{5}|1\rangle$.

Hint: You can obtain the coefficients of $|\psi\rangle$ with respect to these basis states by computing the inner products $\langle u_j|\psi\rangle$ for $j = 1, 2$.

(b) The role of the control and target qubit of a CNOT gate can be reversed by switching to a different basis! First show that



with H the Hadamard gate: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Use this identity to derive the following relations:

$$
\begin{aligned}
|+\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|+\rangle \\
|-\rangle|+\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|+\rangle \\
|+\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |-\rangle|-\rangle \\
|-\rangle|-\rangle &\overset{\text{CNOT}}{\mapsto} |+\rangle|-\rangle
\end{aligned}
$$

with $|\pm\rangle$ defined as $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. In other words, with respect to the $|\pm\rangle$ basis, the second qubit assumes the role of the control and the first qubit the role of the target.

Hint: Use that $H|+\rangle = |0\rangle$ and $H|-\rangle = |1\rangle$, and conversely $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$.

Solution hints

(a) You should arrive at the following measurement probabilities:

$$
\begin{aligned}
p(u_1) &= |\alpha_1|^2 = \frac{1}{50}, \\
p(u_2) &= |\alpha_2|^2 = \frac{49}{50}.
\end{aligned}
$$

(b) To prove the first identity, you can use the relations $HXH = Z$ and $HZH = X$, see Exercise 3.1(d), and the fact that the controlled-Z gate is invariant when interchanging the roles of the control and target qubits.