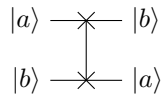


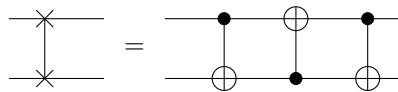
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Exercise 4.1 (Basic quantum circuits)

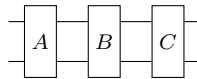
- (a) Find the matrix representation (with respect to the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$) of the *swap-gate* $|a, b\rangle \mapsto |b, a\rangle$, which is written in circuit form as



Also show that the swap operation is equivalent to the following sequence of three CNOT gates:

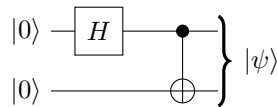


Hint: You can either work directly with basis states, e.g. $|a, b\rangle \xrightarrow{\text{CNOT}} |a, a \oplus b\rangle$, or use matrix representations. In the latter case, note that a sequence of gates like



(with A, B, C unitary 4×4 matrices) corresponds to the matrix product CBA since the circuit is read from left to right, but the input vector in the matrix representation is multiplied from the right.

- (b) Compute the output $|\psi\rangle$ of the following “entanglement circuit” applied to the input $|00\rangle$:



with $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ denoting the Hadamard gate.

- (c) Build the CNOT gate from the controlled- Z gate and two Hadamard gates, and verify your construction.

Solution

- (a) The swap gate interchanges $|01\rangle \leftrightarrow |10\rangle$, thus

$$U_{\text{swap}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Concerning the sequence of three CNOT-gates:

$$|a, b\rangle \xrightarrow{\text{CNOT}} |a, a \oplus b\rangle \xrightarrow{\text{flipped CNOT}} |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle \xrightarrow{\text{CNOT}} |b, (a \oplus b) \oplus b\rangle = |b, a\rangle.$$

Here we have used that $a \oplus a = 0$ for $a \in \{0, 1\}$.

Alternative solution using matrix representations:

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_{\text{flipped CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

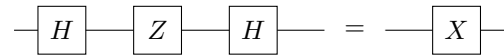
which gives as matrix representation of the three CNOT-gates:

$$U_{\text{CNOT}} \cdot U_{\text{flipped CNOT}} \cdot U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = U_{\text{swap}}.$$

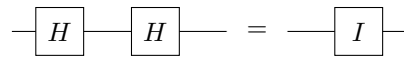
(b)

$$|00\rangle \xrightarrow{H \otimes I} \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi\rangle$$

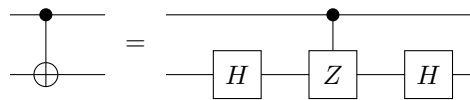
(c) Based on the matrix representation of the Hadamard gate and Pauli matrices, one directly verifies that $HZH = X$ and $H^2 = I$ (identity operation). Expressed in circuit form:



and



These properties lead to the following identity:



The circuit on the right is the requested construction.