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Exercise 13.2 (Bloch sphere representation of the phase damping channel)
Phase damping models decoherence in realistic physical situations and is described by the quantum channel

$$\mathcal{E}_{\mathsf{PD}}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger,$$

with operation elements

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

and "scattering" probability $\lambda \in [0,1]$. We assume $0 < \lambda < 1$ in the following.

(a) A quantum channel $\mathcal{E}$ is called *unital* if $\mathcal{E}(I) = I$. Show that the phase damping channel is unital.

(b) Recall that an arbitrary density operator ρ for a mixed state qubit can be represented as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2},$$

with $\vec{r} \in \mathbb{R}^3$ the *Bloch vector* of ρ and $\vec{\sigma}$ the vector of Pauli matrices. Compute the Bloch vector $\vec{r}'$ of the output state $\rho' = \mathcal{E}_{\mathsf{PD}}(\rho)$ of the phase damping channel in dependence of $\vec{r}$. Also provide a short geometric interpretation.

(c) In which case(s) does $\mathcal{E}_{\mathsf{PD}}(\rho)$ describe a pure quantum system?

(d) Compute the density matrix after n repeated applications of the phase damping operation, $\mathcal{E}_{\mathsf{PD}}(\dots\mathcal{E}_{\mathsf{PD}}(\mathcal{E}_{\mathsf{PD}}(\rho)))$, and take the limit $n \to \infty$. You may work with a symbolic 2×2 matrix representation of ρ, or its Bloch representation and part (b).

Solution

(a) Since $\lambda \in (0,1)$, $\sqrt{1-\lambda}$ and $\sqrt{\lambda}$ are both real numbers. Inserting the identity matrix leads directly to

$$\mathcal{E}_{\mathsf{PD}}(I) = E_0 E_0^\dagger + E_1 E_1^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1-\lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

(b) We first note that

$$E_0 X E_0^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} = \sqrt{1-\lambda}\, X,$$
$$E_1 X E_1^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} = 0 \quad \text{(zero matrix)},$$

and similarly $E_0 Y E_0^\dagger = \sqrt{1-\lambda}\, Y$ and $E_1 Y E_1^\dagger = 0$. Analogous to (a), one computes that $\mathcal{E}(Z) = Z$. Now inserting the Bloch representation of ρ with $\vec{r} = (r_1, r_2, r_3)$ leads to

$$\mathcal{E}_{\mathsf{PD}}(\rho) = \frac{1}{2}\left(\mathcal{E}_{\mathsf{PD}}(I) + \sum_{\alpha=1}^{3} r_\alpha\, \mathcal{E}_{\mathsf{PD}}(\sigma_\alpha)\right) = \frac{1}{2}\left(I + \sqrt{1-\lambda}\, r_1\, X + \sqrt{1-\lambda}\, r_2\, Y + r_3\, Z\right) = \frac{I + \vec{r}' \cdot \vec{\sigma}}{2}$$

with $\vec{r}' = (\sqrt{1-\lambda}\, r_1, \sqrt{1-\lambda}\, r_2, r_3)$.

Thus the x- and y-components of $\vec{r}$ shrink by the factor $\sqrt{1-\lambda}$, while the z-component does not change. In other words, the Bloch vector approaches its projection onto the z-axis.

(c) The Bloch vector $\vec{r}$ of any density matrix ρ satisfies $\|\vec{r}\| \le 1$, with equality if and only if ρ describes a pure state (see exercise 11.2). According to (b), the quantum channel $\mathcal{E}_{\mathsf{PD}}$ scales the first and second entry of the input Bloch vector $\vec{r}$ by the factor $\sqrt{1-\lambda} < 1$. Thus the Bloch vector $\vec{r}'$ of $\mathcal{E}_{\mathsf{PD}}(\rho)$ has unit length, $\|\vec{r}'\| = 1$, precisely if $\vec{r}' = \vec{r} = (0, 0, \pm 1)$ (north or south pole of the Bloch sphere). The corresponding density matrices for these cases are $\rho = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\rho = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, describing a quantum system in state $|0\rangle$ and $|1\rangle$, respectively.

(d) According to (b), the Bloch vector after n repeated applications of the channel is equal to

$$\left(\sqrt{1-\lambda}^n r_1, \sqrt{1-\lambda}^n r_2, r_3\right) \overset{n\to\infty}{\to} (0,0,r_3)$$

since $\sqrt{1-\lambda} < 1$ by assumption. The matrix representation of the limiting density matrix is thus

$$\rho^{(\infty)} = \frac{1}{2}\begin{pmatrix} 1+r_3 & 0 \\ 0 & 1-r_3 \end{pmatrix},$$

that is, the phase damping channel "damps" the off-diagonal entries to zero.

Note: physically, $\rho^{(\infty)}$ represents a (classical) ensemble of the basis states $|0\rangle$ and $|1\rangle$, without any quantum superposition of these states.