Christian B. Mendl, Irene López Gutiérrez, Keefe Huang

Exercise 3.1 (Properties of Pauli matrices and matrix exponential)
As usual, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (X, Y, Z)$ denotes the Pauli vector.

(a) Verify that the Pauli matrices anti-commute with each other, i.e.,

$$\{\sigma_1, \sigma_2\} = 0, \quad \{\sigma_2, \sigma_3\} = 0, \quad \{\sigma_3, \sigma_1\} = 0,$$

where $\{A, B\} = A\,B + B\,A$ denotes the *anti-commutator* of two matrices.

(b) Verify the following commutation relations (here $[A, B] = A\,B - B\,A$ denotes the *commutator* of two matrices):

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2.$$

(c) Use the series expansion of the matrix exponential to derive that, for any $A \in \mathbb{C}^{n\times n}$ and unitary matrix $U \in \mathbb{C}^{n\times n}$,

$$\mathrm{e}^{U^\dagger A U} = U^\dagger\, \mathrm{e}^A\, U.$$

Remark: In case A is normal, one can combine this relation with the spectral decomposition to evaluate e^A, since the matrix exponential of a diagonal matrix is the pointwise exponential of the diagonal entries.

(d) Show that

$$HXH = Z \quad \text{and} \quad HZH = X,$$

where H denotes the Hadamard gate. (Since H is Hermitian and self-inverse, i.e., $H^2 = I$, H can thus be interpreted as base change matrix between the eigenvectors of X and Z.)

(e) Combine parts (c) and (d) to argue that

$$HR_x(\theta)H = R_z(\theta) \quad \text{for all } \theta \in \mathbb{R}.$$

Solution hints

(a) Explicit calculation to evaluate matrix products.

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(c) In the series expansion of $\mathrm{e}^{U^\dagger A U}$, the inner unitary terms cancel since $UU^\dagger = I$.

(d) Can show one of the relations by an explicit calculation, and the other by multiplying with H from both sides and using that $H^2 = I$.

(e) Insert the definition of $R_x(\theta)$ from the lecture, and use that $H^\dagger = H$.