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Tutorial 2 (Dirac notation and inner products)
The Dirac notation (also called bra-ket notation), which you have seen being used in the lecture, uses "kets", such as $|\psi\rangle$, to represent a quantum state. For our purposes, a ket is always a complex (column) vector.[1] ψ is usually the actual vector itself, or can be an identifier or index for the quantum state, as for $|0\rangle$ and $|1\rangle$.

The corresponding "bra" $\langle\psi|$ is then the conjugate-transposed $|\psi\rangle$, i.e., a row vector with complex-conjugated entries of $|\psi\rangle$. A motivation for this notation is that "bras" are linear maps from quantum states to complex numbers via the inner product. Namely, given $\phi \in \mathbb{C}^n$:

$$\langle\phi| : \mathbb{C}^n \to \mathbb{C}, \quad |\psi\rangle \mapsto \langle\phi|\psi\rangle = \sum_{j=1}^{n} \phi_j^* \, \psi_j.$$

(a) Write down the matrix representation of the following expressions:

- $|0\rangle\langle 1|$
- $|0\rangle\langle 0| + |1\rangle\langle 1|$
- $|+\rangle\langle 0|$, with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

(b) Express the Hadamard gate H using Dirac notation in the computational basis (i.e. $\{|0\rangle, |1\rangle\}$).

(c) Given the qubit state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, compute $H|\psi\rangle$ using only the bra-ket notation.

(d) For any $\psi, \phi \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$, verify that

$$\langle\phi|A\psi\rangle = \langle A^\dagger\phi|\psi\rangle,$$

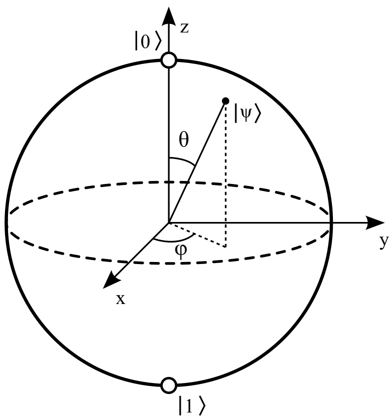
with $A^\dagger = (A^*)^T$ denoting the conjugate transpose (adjoint) of A.

(e) Prove that unitary matrices are norm-preserving, i.e., $\|U\psi\| = \|\psi\|$ for all unitary $U \in \mathbb{C}^{n\times n}$ and $\psi \in \mathbb{C}^n$.
Hint: Use that $\|\psi\|^2 = \langle\psi|\psi\rangle$ and part (d).

Exercise 2.1 (Bloch sphere and single qubit rotation gates)
Recall from the lecture that an arbitrary single qubit quantum state can be parametrized as

$$|\psi\rangle = \mathrm{e}^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

where θ, φ and γ are real numbers, which can be chosen such that $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. The angles θ and φ define the Bloch sphere representation of $|\psi\rangle$, as shown on the right.



https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg

(a) Find the Bloch angles θ and φ of $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$, and the corresponding Bloch vector

$$\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta)).$$

For a real unit vector $\vec{v} \in \mathbb{R}^3$, the rotation by an angle ω about the $\vec{v}$ axis is defined as

$$R_{\vec{v}}(\omega) = \exp(-i\omega\,\vec{v}\cdot\vec{\sigma}/2) = \cos(\omega/2)I - i\sin(\omega/2)(\vec{v}\cdot\vec{\sigma}),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector. The rotations R_x, R_y, R_z about the standard axes correspond to the special cases $\vec{v} = (1,0,0)$, $\vec{v} = (0,1,0)$ and $\vec{v} = (0,0,1)$, respectively.

(b) Compute $R_x(\frac{2\pi}{3})|\psi\rangle$ for the state $|\psi\rangle$ defined in (a), and visualize this operation on the Bloch sphere.
Hint: $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$.

[1] In general, quantum states can also be complex-valued functions (e.g., electronic orbitals of atoms), but these will not play a role in this course.

(c) The *Z-Y decomposition* theorem states the following: given any unitary 2×2 matrix U, there exist real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$U = \mathrm{e}^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

Find the Z-Y decomposition of the Hadamard gate $H = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

Hint: There exists a solution with $\beta = 0$.

Exercise 2.2 (Basic single qubit gates)
Imagine you are playing a game against an adversary. The game consists of multiple trials through which the adversary performs one of the following with equal probability:

1. They flip a coin and send you $|0\rangle$ or $|1\rangle$ depending on the outcome.

OR

2. They send you the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Your goal is to decide which of the two they performed, and you win if you can decide correctly for $\frac{3}{4}$ of the trials on average.

(a) Before you make your guess (based on a quantum measurement on the qubit), you are allowed to perform **one** of the gates X, Y, Z or H. Compute the outputs you would obtain in each situation with each of these gates.

(b) Which of the gates would allow you to win the game? Explain your strategy.