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Tutorial 13 (Experimentally resolving the quantum measurement process[1])

Recall from the lecture that a projective measurement is described by a Hermitian operator M. Writing its spectral decomposition as $M = \sum_m \lambda_m P_m$, where P_m is the projector onto the eigenspace of eigenvalue λ_m, the P_m's take the role of the measurement operators. If the measurement outcome is not recorded, then the overall process is represented by the quantum channel

$$\mathcal{E}_{\mathsf{proj}}(\rho) = \sum_m P_m \rho P_m.$$

In the history of quantum mechanics, the interpretation as mathematical projection onto subspaces traces back to an article by G. Lüders[2]. He concluded that the quantum superposition within an eigenspace of dimension 2 or larger "survives" the measurement process, and that two commuting observables M, $\tilde{M}$, $[M, \tilde{M}] = 0$, are "compatible" with each other, i.e., measuring M does not affect the outcome statistics of $\tilde{M}$. In this tutorial, we discuss an experimental realization [1] of such a "Lüders process" retaining the superposition. (The term "Lüders process" and "ideal measurement" refer to a projective measurement here.)

The principal quantum system is formed by three electronic states $|0\rangle$, $|1\rangle$ and $|2\rangle$ of a $^{88}\mathrm{Sr}^+$ ion, as indicated in Fig. 1(a). Such a "qutrit" is a generalization of qubits to statevectors from $\mathbb{C}^3$. The ion has an additional short-lived excited state $|e\rangle$. In the experiment, a laser with variable power drives

$$|0\rangle \to g_0 |0\rangle + g_1 |e\rangle\,.$$

$|e\rangle$ quickly decays to $|0\rangle$, emitting a photon in the process: $|e\rangle |n = 0\rangle \to |0\rangle |n = 1\rangle$, where $|n\rangle$ is the quantum state of the photon environment. The (indirect) measurement process consists of the detection of the emitted photon, which indicates the occupancy of $|0\rangle$, but leaves a superposition between $|1\rangle$ and $|2\rangle$ intact. The coefficients g_0 and g_1 satisfy $|g_0|^2 + |g_1|^2 = 1$ and are used to demonstrate a transition from "no measurement" ($g_0 = 1$) to an ideal measurement ($g_0 = 0$). (In the experiment, fluorescence detection is actually only employed at the end for state tomography, but not during the measurement, i.e., one ignores the outcome.)

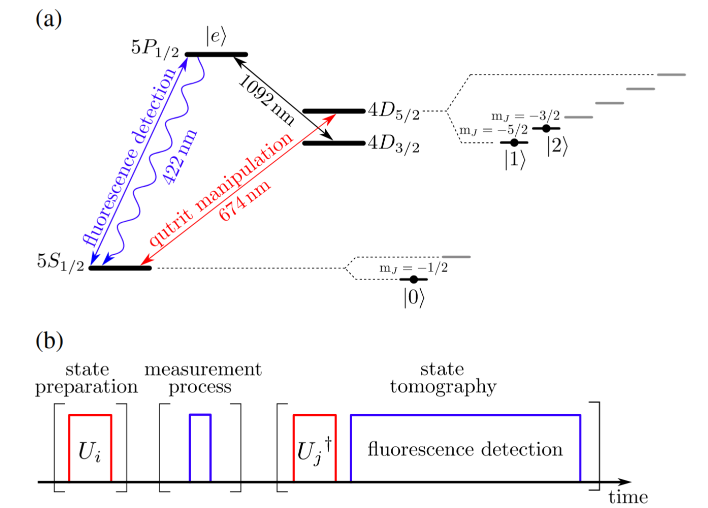


Figure 1: (a) Electronic states of the ion $^{88}\mathrm{Sr}^+$. (b) Experimental sequence to characterize the process.

(a) The system is initialized to

$$|\psi\rangle = (\alpha_0 |0\rangle + \alpha_1 |1\rangle + \alpha_2 |2\rangle) |n = 0\rangle\,.$$

What is the state of the system, $|\psi'\rangle$, after the excitation by the laser and $|e\rangle$ has decayed back into $|0\rangle$?

The Kraus operators which model this whole operation (i.e., the drive into $|e\rangle$ and the subsequent decay) are

$$E_0 = g_1 |0\rangle \langle 0| \quad \text{and} \quad E_1 = g_0 |0\rangle \langle 0| + |1\rangle \langle 1| + |2\rangle \langle 2|\,.$$

(b) Compute the reduced density matrix of the ion at the beginning of the experiment, $\rho_{\mathsf{ion}} = \mathrm{tr}_{\mathsf{env}}[|\psi\rangle \langle\psi|]$, and apply the quantum operation to ρ_{ion}.

(c) Verify that your result for (b) matches

$$\rho'_{\mathsf{ion}} = \mathrm{tr}_{\mathsf{env}}[|\psi'\rangle \langle\psi'|].$$

(d) Which measurement process corresponds to the case when $g_1 = 1$?

The experiment uses *process tomography* for characterization. A detailed explanation is beyond the scope of this tutorial; as brief summary: an additional laser (shown in red in Fig. 1) performs the initial state preparation, formally by applying a unitary matrix to $|0\rangle$. In the experiment, nine specific initial states $|\psi_i\rangle = U_i |0\rangle$ are used in different runs, corresponding to the unitaries $\{U_i\}$. Before the final fluorescence detection, the red laser realizes the action of one of the adjoint unitaries $U_j^\dagger$.

[1] F. Pokorny et al.: *Tracking the dynamics of an ideal quantum measurement*. Phys. Rev. Lett. 124, 080401 (2020)

[2] G. Lüders: *Über die Zustandsänderung durch den Meßprozeß*. Ann. Phys. 443, 322–328 (1950)

(e) Show that, in general, for a projective measurement with operators P_m, applying a unitary $U^\dagger$ beforehand changes the outcome probabilities as if using the operators $UP_mU^\dagger$.

The experiment represents the process in terms of the so-called Choi matrix, as shown in Fig. 2. As $g_0 \to 0$, the process becomes an ideal (projective) measurement.

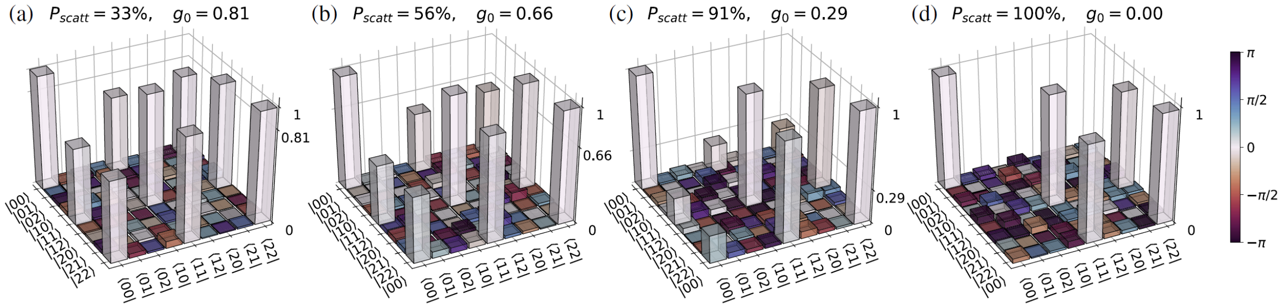
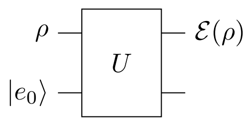


Figure 2: Choi matrices reconstructed from experimental data for different values of g_0, from [1].

(f) Which feature of the Choi matrix indicates that the superposition between $|1\rangle$ and $|2\rangle$ is preserved?

Exercise 13.1 (Quantum operations as coupling to an environment, and amplitude damping[3])
Any quantum operation can be represented by embedding the principal system into an environment, which we can assume (without loss of generality) to start in some state $|e_0\rangle$, and then applying a unitary transformation to the combined system, as illustrated in the following diagram:



From that, one obtains $\mathcal{E}(\rho)$ by "tracing out" the environment; for this purpose we first extend $|e_0\rangle$ to a basis $\{|e_k\rangle\}$ of the environment, and then compute the partial trace:

$$\mathcal{E}(\rho) = \mathrm{tr}_{\mathsf{env}}\left[U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger\right] = \sum_k \langle e_k| U(\rho \otimes |e_0\rangle\langle e_0|)U^\dagger |e_k\rangle = \sum_k E_k \rho E_k^\dagger$$

with the matrix entries of E_k given by $(E_k)_{\ell,m} = \langle \ell, e_k| U |m, e_0\rangle$. The last term is the operator-sum representation of the quantum operation.

Amplitude damping models effects due to the loss of energy from a quantum system, for example by loosing a photon (elementary particle of light) from a cavity. In this case one can think of $|0\rangle$ and $|1\rangle$ as the physical system with zero or one photon, respectively. Specifically, the operator-sum representation of amplitude damping is given by

$$\mathcal{E}_{\mathsf{AD}}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

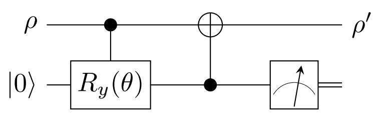
with

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{1}$$

and a real parameter $\gamma \in [0, 1]$, which one can interpret as the probability. Note that E_1 maps $|1\rangle \mapsto \sqrt{\gamma}\,|0\rangle$.

(a) Show that the operation elements $\{E_k\}$ in Eq. (1) satisfy $\sum_{k\in\{0,1\}} E_k^\dagger E_k = I$.

We now want to verify that the following circuit describes the amplitude damping operation, with $\gamma = \sin(\theta/2)^2$:



Recall that R_y is the rotation operator

$$R_y(\theta) = \mathrm{e}^{-i\theta Y/2} = \cos(\theta/2)I - i\sin(\theta/2)Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

[3] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 8.20

(b) Find the 4×4 matrix representation U_{AD} of the controlled-$R_y(\theta)$ gate followed by the flipped CNOT gate in the above circuit.

(c) Finally, read off the corresponding operation elements with entries $(E_0)_{\ell,m} = \langle \ell, 0| U_{\mathsf{AD}} |m, 0\rangle$ and $(E_1)_{\ell,m} = \langle \ell, 1| U_{\mathsf{AD}} |m, 0\rangle$, and confirm that they agree with Eq. (1).

Exercise 13.2 (Bloch sphere representation of the phase damping channel)
Phase damping models decoherence in realistic physical situations and is described by the quantum channel

$$\mathcal{E}_{\mathsf{PD}}(\rho) = \sum_{k=0}^{1} E_k \rho E_k^\dagger,$$

with operation elements

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$$

and "scattering" probability $\lambda \in [0, 1]$. We assume $0 < \lambda < 1$ in the following.

(a) A quantum channel $\mathcal{E}$ is called *unital* if $\mathcal{E}(I) = I$. Show that the phase damping channel is unital.

(b) Recall that an arbitrary density operator ρ for a mixed state qubit can be represented as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2},$$

with $\vec{r} \in \mathbb{R}^3$ the *Bloch vector* of ρ and $\vec{\sigma}$ the vector of Pauli matrices. Compute the Bloch vector $\vec{r}'$ of the output state $\rho' = \mathcal{E}_{\mathsf{PD}}(\rho)$ of the phase damping channel in dependence of $\vec{r}$. Also provide a short geometric interpretation.

(c) In which case(s) does $\mathcal{E}_{\mathsf{PD}}(\rho)$ describe a pure quantum system?

(d) Compute the density matrix after n repeated applications of the phase damping operation, $\mathcal{E}_{\mathsf{PD}}(\dots \mathcal{E}_{\mathsf{PD}}(\mathcal{E}_{\mathsf{PD}}(\rho)))$, and take the limit $n \to \infty$. You may work with a symbolic 2×2 matrix representation of ρ, or its Bloch representation and part (b).