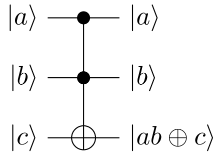
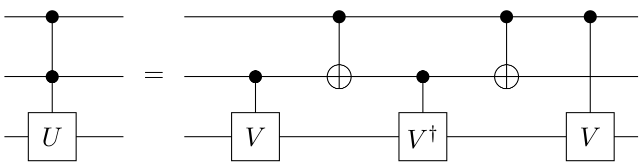
Christian B. Mendl, Irene López Gutiérrez, Keefe Huang

Exercise 5.2 (Toffoli gate and quantum logic)
The Toffoli gate is also known as CCNOT or "controlled-controlled-NOT" operation. In classical circuits, this gate inverts the target wire when the input of its two control wires is 1. Thus, defining the Toffoli gate in terms of computational basis states leads to (for any $a, b, c \in \{0, 1\}$):



The bottom (target) qubit gets flipped precisely if both control qubits are in the $|1\rangle$ state; equivalently, the flip occurs if the product ab equals 1, which leads to the expression $ab \oplus c$.

Physical quantum computers only implement a finite gate-set as hardware operations. Thus it is typically necessary to decompose such multi-control operations in terms of gates with at most one control wire. One such decomposition of a controlled-controlled-U operation, known as the Sleator and Weinfurter construction[1], is:

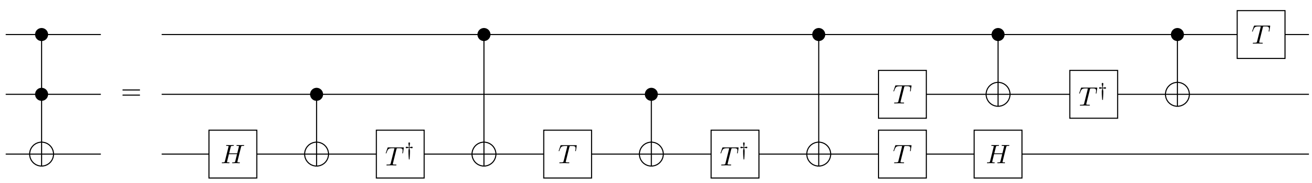


where V is a certain single-qubit gate depending on U. (The Toffoli gate corresponds to the special case $U = X$.)

(a) Which condition must V satisfy such that the equality holds? Verify your answer by inserting all four possible computational basis states for the control qubits.

(b) Find the V gate corresponding to $U = X$.
Hint: You can obtain a matrix power A^κ (with $\kappa \in \mathbb{R}$) of a normal matrix $A \in \mathbb{C}^{n\times n}$ by first computing its spectral decomposition: $A = U \operatorname{diag}(\lambda_1, \dots, \lambda_n)U^\dagger$ and U unitary; then exponentiate the eigenvalues, i.e., $A^\kappa = U \operatorname{diag}(\lambda_1^\kappa, \dots, \lambda_n^\kappa)U^\dagger$.

The so-called Clifford gates play an important role for quantum error correction and the Gottesman-Knill theorem.[2] Hence, most quantum computers support the "Clifford+T" gate set, which includes the Hadamard gate, the Pauli gates, the CNOT gate as well as the T gate, defined as $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{smallmatrix}\right)$. It is possible to decompose the Toffoli gate in terms of these gates:



(c) Verify that the above circuit indeed implements the Toffoli gate.

Solution hints

(a) The condition is $V^2 = U$.

(b) We need to find $V = \sqrt{X}$, i.e., $\kappa = \frac{1}{2}$ in the hint. The final result is

$$\sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} = \frac{1+i}{2} I + \frac{1-i}{2} X.$$

(c) Check the action of the circuit for the four possible inputs in the computational basis on the control qubits.

[1] T. Sleator, H. Weinfurter: *Realizable Universal Quantum Logic Gates*. Phys. Rev. Lett. 74, 4087 (1995)
[2] Quantum error correction will be part of the follow-up course "Advanced Concepts of Quantum Computing" next semester.