Ex 1.1

$a = 3+4i \quad , \quad b = 2-i$

a) • $a+b = \boxed{5+3i}$

• $ab = 6 - 3i + 8i - 4\underbrace{i^2}_{(-1)} = \boxed{10+5i}$

• $1/a = \dfrac{3-4i}{(3+4i)(3-4i)} = \dfrac{3-4i}{25} = \boxed{\dfrac{3}{25} - \dfrac{4i}{25}}$

• $a^* = \boxed{3-4i}$

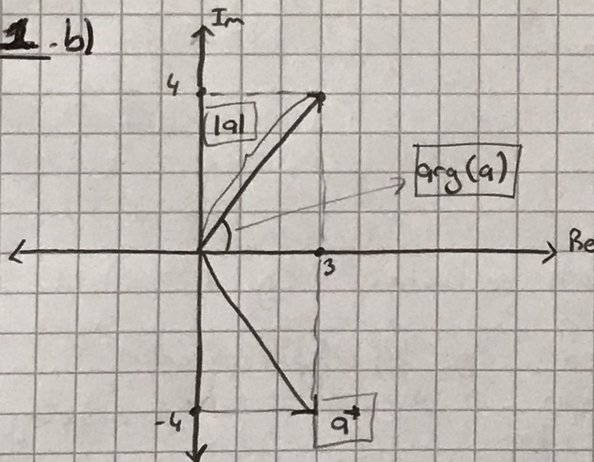
• $|a| = \sqrt{3^2+4^2} = 5$ $\qquad \dfrac{3-4i}{5} = e^{i\arg(a)}$ $\qquad \arg(3+4i) = \arctan\left(\dfrac{4}{3}\right) = 1.326\pi$

$\boxed{a = |5|\, e^{i\, 1.326\pi}}$

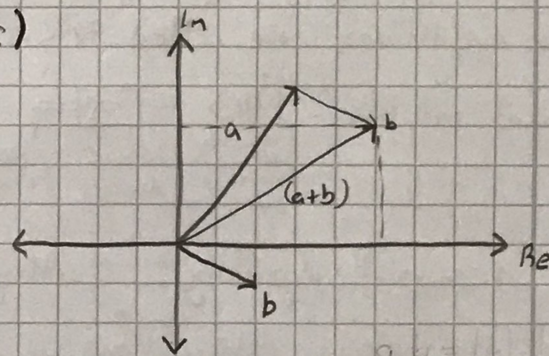
• $\|\psi\| = \sqrt{\sum_{i=1}^{n} \psi \cdot \psi^*}$

$= \sqrt{(3+4i).(3-4i) + (2-i).(2+i)} = \sqrt{25+5} = \boxed{\sqrt{30}}$

1.1.b)



c)



$a.b = |a|.|b|.e^{i(\arg(a)+\arg(b))}$

after we calculate each of this elements, we can construct the final vector easily.

Ex 1.2

a)

$$\begin{pmatrix} 2 & -i & 5 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ i \\ -3 \end{pmatrix} = \begin{pmatrix} 8+1-15 \\ 12+0-3 \end{pmatrix} = \boxed{\begin{pmatrix} -6 \\ 9 \end{pmatrix}}$$

$$\begin{pmatrix} -2 & 7 \\ 3 & 1+2i \end{pmatrix} \cdot \begin{pmatrix} 5 & -4 \\ 6i & 0 \end{pmatrix} = \begin{pmatrix} -10+42i & 8+0 \\ 15+6i-12 & -12+0 \end{pmatrix} = \boxed{\begin{pmatrix} -10+42i & 8 \\ 3+6i & -12 \end{pmatrix}}$$

b) a matrix is normal if

$A^\dagger A = A A^\dagger$ since we care about real-valued matrices

we can omit the conjugate part $(A^*)^T$ and left wit A^T

so we need to find a matrix A that satisfies $A^T A \neq A A^T$

which meas we can try a non-symmetrical ~~matrix~~ $\boxed{\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}}$

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 10 \\ 10 & 20 \end{pmatrix} \neq \begin{pmatrix} 17 & 11 \\ 11 & 13 \end{pmatrix}$$ which satisfies our condition.

1.2.c)

to show it's normal, $A^\dagger A = A A^\dagger$

$$\begin{pmatrix} 0 & -3/5 & -4/5 \\ 3/5 & 0 & 0 \\ 4/5 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -3/5 & -4/5 \\ 3/5 & 0 & 0 \\ 4/5 & 0 & 0 \end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{pmatrix}$$

since they're equal, the matrix is normal

- $\chi_A(\lambda) = \det(\lambda I - A)$

$$\begin{vmatrix} \lambda & -3/5 & -4/5 \\ 3/5 & \lambda & 0 \\ 4/5 & 0 & \lambda \end{vmatrix} = \lambda \cdot \det\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + 3/5 \cdot \det\begin{pmatrix} 3/5 & 0 \\ 4/5 & \lambda \end{pmatrix} - 4/5 \cdot \det\begin{pmatrix} 3/5 & \lambda \\ 4/5 & 0 \end{pmatrix}$$

$$\lambda^3 + \underbrace{\frac{9\lambda}{25} + \frac{16\lambda}{25}}$$ $\boxed{\text{c.p.}}$

$$\lambda^3 + \frac{25\lambda}{25} = \boxed{\lambda^3 + \lambda}$$

Eigenvalues

$\lambda^3 + \lambda = \lambda(\lambda^2 + 1)$

$\boxed{\lambda_1 = 0 \quad \lambda_2 = i \quad \lambda_3 = -i}$

$$\begin{bmatrix} 0 & 3/5 & 4/5 \\ -3/5 & 0 & 0 \\ -4/5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$3/5\, y + 4/5\, z = 0 \Rightarrow \frac{3y}{5} = \frac{-4z}{5} = 15y = -20z$

$\qquad\qquad\qquad\qquad\qquad -4 \quad +3$

$-3/5\, x = 0 \qquad x = 0$

$-4/5\, x = 0$

for $\lambda_1 = 0$ eigen vector is $\boxed{(0, -4, 3)}$

d) $A^\dagger A = I$ has to be satisfied

$A^\dagger = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \qquad A^\dagger A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ so it's unitary

e) If U is unitary then $U^\dagger U = I$ and $\det(U^\dagger U) = 1$

we also know $\det(U^T) = \det(U)$ so

$\det(U^\dagger) = \det(U^*)^T = \det(U^*) \Rightarrow \det(U^*) \cdot \det(U) = 1$

$\sqrt{U^* \cdot U} = |U| \qquad\qquad |\det(U)^2| = 1$ hence

$|\det(U)| = 1$