

**Tutorial 2** (Dirac notation and inner products)

The Dirac notation (also called bra-ket notation), which you have seen being used in the lecture, uses “kets”, such as  $|\psi\rangle$ , to represent a quantum state. For our purposes, a ket is always a complex (column) vector.<sup>1</sup>  $\psi$  is usually the actual vector itself, or can be an identifier or index for the quantum state, as for  $|0\rangle$  and  $|1\rangle$ .

The corresponding “bra”  $\langle\psi|$  is then the conjugate-transposed  $|\psi\rangle$ , i.e., a row vector with complex-conjugated entries of  $|\psi\rangle$ . A motivation for this notation is that “bras” are linear maps from quantum states to complex numbers via the inner product. Namely, given  $\phi \in \mathbb{C}^n$ :

$$\langle\phi| : \mathbb{C}^n \rightarrow \mathbb{C}, \quad |\psi\rangle \mapsto \langle\phi|\psi\rangle = \sum_{j=1}^n \phi_j^* \psi_j.$$

(a) Write down the matrix representation of the following expressions:

- $|0\rangle\langle 1|$
- $|0\rangle\langle 0| + |1\rangle\langle 1|$
- $|+\rangle\langle 0|$ , with  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

(b) Express the Hadamard gate  $H$  using Dirac notation in the computational basis (i.e.  $\{|0\rangle, |1\rangle\}$ ).

(c) Given the qubit state  $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , compute  $H|\psi\rangle$  using only the bra-ket notation.

(d) For any  $\psi, \phi \in \mathbb{C}^n$  and  $A \in \mathbb{C}^{n \times n}$ , verify that

$$\langle\phi|A\psi\rangle = \langle A^\dagger \phi|\psi\rangle,$$

with  $A^\dagger = (A^*)^T$  denoting the conjugate transpose (adjoint) of  $A$ .

(e) Prove that unitary matrices are norm-preserving, i.e.,  $\|U\psi\| = \|\psi\|$  for all unitary  $U \in \mathbb{C}^{n \times n}$  and  $\psi \in \mathbb{C}^n$ .

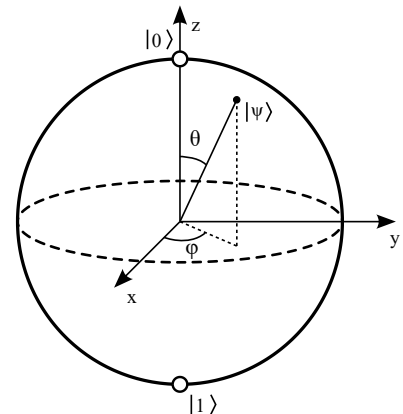
Hint: Use that  $\|\psi\|^2 = \langle\psi|\psi\rangle$  and part (d).

**Exercise 2.1** (Bloch sphere and single qubit rotation gates)

Recall from the lecture that an arbitrary single qubit quantum state can be parametrized as

$$|\psi\rangle = e^{i\gamma} \left( \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle \right)$$

where  $\theta$ ,  $\varphi$  and  $\gamma$  are real numbers, which can be chosen such that  $0 \leq \theta \leq \pi$  and  $0 \leq \varphi \leq 2\pi$ . The angles  $\theta$  and  $\varphi$  define the Bloch sphere representation of  $|\psi\rangle$ , as shown on the right.



[https://commons.wikimedia.org/wiki/File:Bloch\\_sphere.svg](https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg)

(a) Find the Bloch angles  $\theta$  and  $\varphi$  of  $|\psi\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ , and the corresponding Bloch vector

$$\vec{r} = (\cos(\varphi) \sin(\theta), \sin(\varphi) \sin(\theta), \cos(\theta)).$$

For a real unit vector  $\vec{v} \in \mathbb{R}^3$ , the rotation by an angle  $\omega$  about the  $\vec{v}$  axis is defined as

$$R_{\vec{v}}(\omega) = \exp(-i\omega \vec{v} \cdot \vec{\sigma}/2) = \cos(\omega/2)I - i \sin(\omega/2)(\vec{v} \cdot \vec{\sigma}),$$

where  $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  is the Pauli vector. The rotations  $R_x$ ,  $R_y$ ,  $R_z$  about the standard axes correspond to the special cases  $\vec{v} = (1, 0, 0)$ ,  $\vec{v} = (0, 1, 0)$  and  $\vec{v} = (0, 0, 1)$ , respectively.

(b) Compute  $R_x(\frac{2\pi}{3})|\psi\rangle$  for the state  $|\psi\rangle$  defined in (a), and visualize this operation on the Bloch sphere.

Hint:  $\cos(\frac{\pi}{3}) = \frac{1}{2}$  and  $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ .

<sup>1</sup>In general, quantum states can also be complex-valued functions (e.g., electronic orbitals of atoms), but these will not play a role in this course.

- (c) The *Z-Y decomposition* theorem states the following: given any unitary  $2 \times 2$  matrix  $U$ , there exist real numbers  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  such that

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

Find the Z-Y decomposition of the Hadamard gate  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ .

Hint: There exists a solution with  $\beta = 0$ .

### Exercise 2.2 (Basic single qubit gates)

Imagine you are playing a game against an adversary. The game consists of multiple trials through which the adversary performs one of the following with equal probability:

1. They flip a coin and send you  $|0\rangle$  or  $|1\rangle$  depending on the outcome.

OR

2. They send you the state  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ .

Your goal is to decide which of the two they performed, and you win if you can decide correctly for  $\frac{3}{4}$  of the trials on average.

- (a) Before you make your guess (based on a quantum measurement on the qubit), you are allowed to perform **one** of the gates  $X$ ,  $Y$ ,  $Z$  or  $H$ . Compute the outputs you would obtain in each situation with each of these gates.
- (b) Which of the gates would allow you to win the game? Explain your strategy.