**Tutorial 11** (Bloch sphere interpretation of rotations[1])

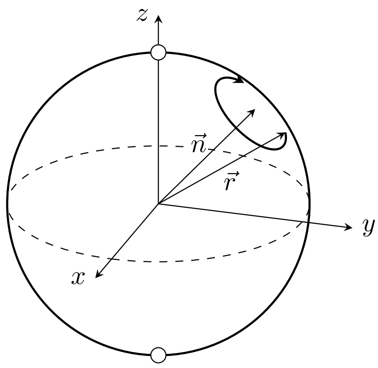
In this tutorial, we show that the Bloch sphere representation of a general single-qubit rotation operator

$$R_{\vec{n}}(\theta) = \mathrm{e}^{-i\theta(\vec{n}\cdot\vec{\sigma})/2} = \cos(\theta/2)I - i\sin(\theta/2)(\vec{n}\cdot\vec{\sigma})$$

is a conventional rotation (in three dimensions) by angle $\theta$ about axis $\vec{n} \in \mathbb{R}^3$. Let $\vec{r}$ denote the Bloch vector of the quantum state. It will be convenient to work with the following relation between $\vec{r}$ and the density matrix $\rho$ of the quantum state:

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}.$$

(By exercise 11.2 below, this coincides with the hitherto definition of the Bloch vector in case $\rho = |\psi\rangle\langle\psi|$ corresponds to a pure quantum state $|\psi\rangle$.)




(a) First verify the following commutation relation of the Pauli matrices: for any $j, k \in \{1, 2, 3\}$,

$$[\sigma_j, \sigma_k] = 2i \sum_{\ell=1}^{3} \epsilon_{jk\ell}\, \sigma_\ell,$$

where $[A, B] = AB - BA$ is the *commutator* of $A$ and $B$, and the *Levi-Civita symbol* $\epsilon_{jk\ell}$ is defined by

$$\epsilon_{jk\ell} = \begin{cases} 1 & (j,k,\ell) \text{ is an even (cyclic) permutation of } (1,2,3) \\ -1 & (j,k,\ell) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Conclude that, for any $\vec{a}, \vec{b} \in \mathbb{R}^3$,

$$\left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right] = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}.$$

(b) Derive the relation

$$\left\{\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right\} = 2(\vec{a}\cdot\vec{b})I$$

for any $\vec{a}, \vec{b} \in \mathbb{R}^3$, where $\{A, B\} = AB + BA$ is the *anti-commutator* of $A$ and $B$.

(c) Show that the Bloch vector of the rotated quantum state is obtained by applying Rodrigues' rotation formula:

$$\vec{r}' = \cos(\theta)\vec{r} + \sin(\theta)(\vec{n}\times\vec{r}) + (1-\cos(\theta))\,(\vec{n}\cdot\vec{r})\vec{n}.$$

Remark: The interpretation as rotation applies to an arbitrary single-qubit gate $U$ (when ignoring global phases), since it can always be represented as $U = \mathrm{e}^{i\alpha}R_{\vec{n}}(\theta)$ with $\alpha \in \mathbb{R}$ and a suitable rotation operator $R_{\vec{n}}(\theta)$.

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 4.6

**Exercise 11.1** (von Neumann equation and time evolution with density operators)

(a) Based on the Schrödinger equation (cf. tutorial 3), derive the following *von Neumann equation* for a density matrix $\rho(t) = \sum_j p_j \, |\psi_j(t)\rangle \langle\psi_j(t)|$:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = [H, \rho(t)].$$

Here $[\cdot, \cdot]$ is the matrix commutator.

Hint: Use the product rule for computing the time derivative of each term $|\psi_j(t)\rangle \langle\psi_j(t)|$.

(b) What is the formal solution for $\rho(t)$ expressed in terms of the time evolution operator $U(t) = \mathrm{e}^{-iHt/\hbar}$?

(c) We consider the specific single-qubit Hamiltonian operator

$$H = JX$$

with parameter $J \in \mathbb{R}$. Compute the time-dependent density matrix $\rho(t)$ starting from the initial state $\rho_0 = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}$ at $t = 0$. For simplicity, you can set $\hbar = 1$.

(d) Since the map $\rho \mapsto [H, \rho]$ is linear, we can represent it as matrix-vector multiplication after "vectorizing" $\rho$, i.e., collecting its entries in a vector, denoted $\vec{\rho}$ in the following. For the commutator, this leads to

$$\vec{\rho} \mapsto \mathrm{vec}([H, \rho]) = \left(H \otimes I - I \otimes H^T\right) \vec{\rho},$$

where the identity matrix has the same dimension as $H$. Thus we can represent the von Neumann equation equivalently in the "superoperator" form

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\vec{\rho}(t) = \mathcal{H}\vec{\rho}(t), \quad \mathcal{H} = H \otimes I - I \otimes H^T.$$

Write down the formal solution of this differential equation, and determine $\mathcal{H}$ for the Hamiltonian from (c).

**Exercise 11.2** (Bloch sphere for mixed state qubits[2])

(a) Show that an arbitrary density operator $\rho$ for a qubit may be written as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2},$$

where $\vec{r} \in \mathbb{R}^3$ is a real vector such that $\|\vec{r}\| \leq 1$. ($\vec{r}$ is called the *Bloch vector* of $\rho$.)

Hint: Note that $\{I, \sigma_1, \sigma_2, \sigma_3\}$ forms a basis of $2 \times 2$ matrices. Argue that the corresponding coefficients to represent a density matrix are real. Why is the coefficient of $I$ equal to $\frac{1}{2}$? Finally, compute the eigenvalues of the above representation and use the positivity of $\rho$ to derive the condition $\|\vec{r}\| \leq 1$.

(b) Show that a state $\rho$ is pure if and only if $\|\vec{r}\| = 1$.

(c) Verify that for pure states $\rho = |\psi\rangle \langle\psi|$, the above definition of the Bloch vector $\vec{r}$ coincides with the Bloch vector of $|\psi\rangle$ (cf. exercise 2.1).

Hint: Insert $|\psi\rangle = \mathrm{e}^{i\gamma}(\cos\frac{\theta}{2}\,|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}\,|1\rangle)$ into $|\psi\rangle \langle\psi|$, read off the entries of $\vec{r}$ based on $|\psi\rangle \langle\psi| = (I + \vec{r} \cdot \vec{\sigma})/2$, and verify that $\vec{r} = (\cos(\varphi)\sin(\theta), \sin(\varphi)\sin(\theta), \cos(\theta))$.

[2] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press (2010), Exercise 2.72