Christian B. Mendl, Pedro Hack, Irene López Gutiérrez, Keefe Huang due: 12 Dec 2022, 08:00 on Moodle

Tutorial 7 (Experimental quantum teleportation)
In this tutorial, we will discuss one of the first lab experiments[1] for quantum teleportation, which uses the polarization and "spatial path information" of photons as logical qubits. Quantum teleportation requires the following ingredients, which are challenging to realize experimentally:

1. the creation and distribution of entangled qubits,
2. measurement in the Bell basis.

The experimental setup is illustrated in the following Figure 1. It uses non-linear crystal excitations and beam splitters to generate entanglement.

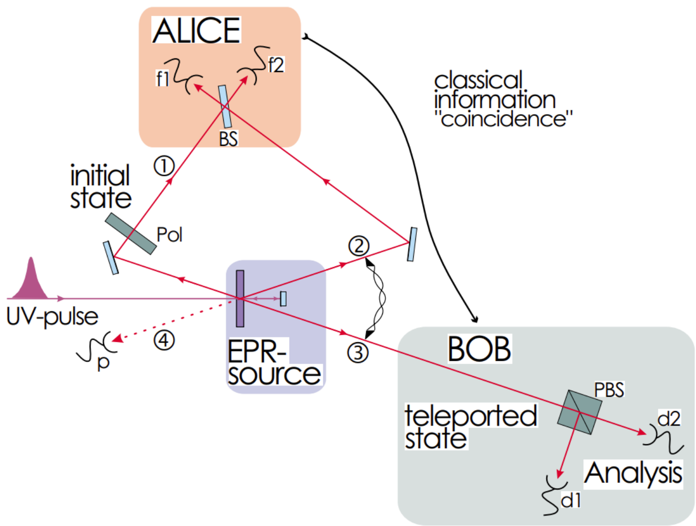


Figure 1: Quantum teleportation setup [1]

Discuss the various steps of the quantum teleportation process.

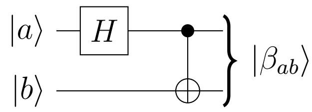
Exercise 7.1 (Bell states and superdense coding)
Recall that the *Bell states* are defined as

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

which can be summarized as $|\beta_{ab}\rangle = \frac{1}{\sqrt{2}}(|0,b\rangle + (-1)^a|1,1-b\rangle)$ for $a,b \in \{0,1\}$.

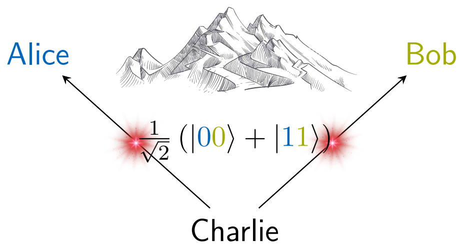
(a) Verify that the following quantum circuit creates the Bell states for inputs $|a,b\rangle$:

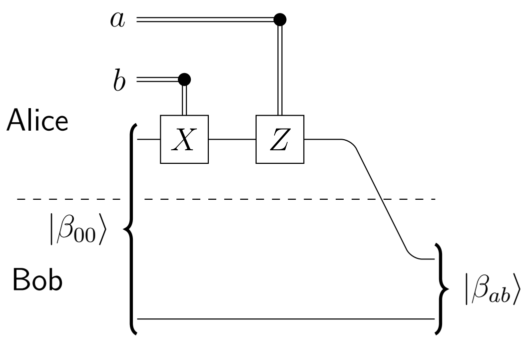
[1] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, A. Zeilinger: *Experimental quantum teleportation.* Nature 390, 575 (1997); see also: T. Jennewein, G. Weihs, A. Zeilinger: *Photon statistics and quantum teleportation experiments.* J. Phys. Soc. Jpn. 72, 168–173 (2003)



Note: this generalizes exercise 4.1(b). Since the circuit implements a unitary transformation of the standard basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Bell states form an orthonormal basis of the two qubit state space as well.

Superdense coding is a surprising use of entanglement to transmit two bits of classical information by sending just a single qubit! The setup agrees with quantum teleportation: two parties, usually referred to as Alice and Bob, live far from each other but share a pair of qubits in the entangled Bell state $|\beta_{00}\rangle$. They could have generated the pair during a visit in the past, or a common friend Charlie prepared it and sent one qubit to Alice and the other to Bob, as shown on the right.





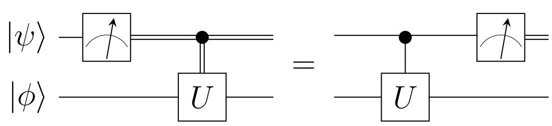
Now Alice's task is to communicate two bits 'ab' of classical information to Bob. Alice can achieve that by applying X and/or Z gates to her qubit before sending it to Bob, depending on the information she wants to transmit: for '00', she does nothing to her qubit, for '01' she applies X, for '10' she applies Z, and for '11' she applies first X and then Z, i.e., $ZX = iY$. It turns out that the resulting states are precisely the Bell states, which Bob can distinguish by performing a measurement with respect to this basis. The diagram on the left summarizes the protocol.

(b) Verify for all combinations of $a, b \in \{0, 1\}$ that the output of the circuit is indeed the Bell state $|\beta_{ab}\rangle$.

(c) Suppose E is an operator on Alice's qubit (e.g., $E = M_m^\dagger M_m$ in the general measurement framework, with M_m a measurement operator). Show that $\langle\beta_{ab}|E \otimes I\,|\beta_{ab}\rangle$ takes the same value for all four Bell states. Assuming an adversarial "Eve" intercepts Alice's qubit on the way to Bob, can Eve infer anything about the classical information which Alice tries to send?

Exercise 7.2 (Quantum teleportation circuit using IBM Q and Qiskit)

(a) By the *principle of deferred measurement*, measurement operations can always be moved to the end of the circuit, and classically controlled operations by conditional quantum operations. Verify this statement for the following controlled-U circuit, by inserting $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and computing the intermediate states:



7.1.a-) Lets apply Hadamard gate

$$|a,b\rangle \xrightarrow{H\otimes I} \frac{|0\rangle + (-1)^a|1\rangle}{\sqrt{2}}|b\rangle = \frac{1}{\sqrt{2}}(|0,b\rangle + (-1)^a|1,b\rangle)$$

then apply CNOT

$$\xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|0,b\rangle + (-1)^a|1,1-b\rangle) = |\beta_{ab}\rangle$$

b-) Lets try all four states:

for $a=0$ $b=0$, it will do nothing $\boxed{|\beta_{00}\rangle}$

for $a=0, b=1$ she will apply X

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{X\otimes I} \frac{1}{\sqrt{2}}((X|0\rangle)|0\rangle + (X|1\rangle)|1\rangle) = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \boxed{|\beta_{01}\rangle}$$

for $a=1, b=0$ Z gate changes sign of coefficien $|1\rangle$

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{Z\otimes I} \frac{1}{\sqrt{2}}((Z|0\rangle)|0\rangle + (Z|1\rangle)|1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \boxed{|\beta_{10}\rangle}$$

for $a=1, b=1$ she will apply Z and X

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{ZX\otimes I} \frac{1}{\sqrt{2}}((Z(X|0\rangle))|0\rangle + (Z(X|1\rangle))|1\rangle)$$

$$= \frac{1}{\sqrt{2}}((Z|1\rangle)|0\rangle + (Z|0\rangle)|1\rangle) = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle) \boxed{= |\beta_{11}\rangle}$$

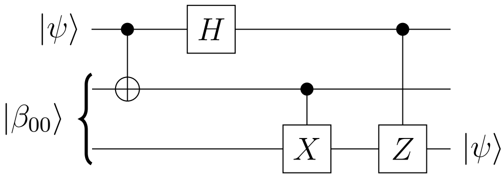
c-) $\langle \beta_{ab}|E\otimes I|\beta_{ab}\rangle =$

$$= \frac{1}{2}\Big(\underbrace{\langle 0,b|E\otimes I|0,b\rangle}_{\langle 0|E|0\rangle\underbrace{\langle b|b\rangle}_{1}} + (-1)^a\underbrace{\langle 0,b|E\otimes I|1,1-b\rangle}_{\langle 0|E|1\rangle\underbrace{\langle b|1-b\rangle}_{0}} + (-1)^a\underbrace{\langle 1,1-b|E\otimes I|0,b\rangle}_{\langle 1|E|0\rangle\underbrace{\langle 1-b|b\rangle}_{0}} + \underbrace{\langle 1,1-b|E\otimes I|1,1-b\rangle}_{\langle 1|E|1\rangle\underbrace{\langle 1-b|1-b\rangle}_{1}}$$

$$= \boxed{\frac{1}{2}(\langle 0|E|0\rangle + \langle 1|E|1\rangle)}$$

Eve can not infer anything about a,b because result does not contain a,b

(b) For the purpose of simulating the quantum teleportation circuit, we can exploit the principle of deferred measurement to omit the measurements altogether and rewrite the circuit in the following modified form:[2]



Construct this circuit in the IBM Q Circuit Composer. Insert a rotation operation at the beginning to prepare the initial qubit as $|\psi\rangle = R_y(\frac{\pi}{3})\,|0\rangle$. To check that the circuit works as intended, first compute the amplitudes α and β of the representation $R_y(\frac{\pi}{3})\,|0\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Now insert a measurement operation at the end of the bottom qubit line, run 1024 "shots" your of circuit and compare the resulting measurement histogram with the expected probabilities $|\alpha|^2$ and $|\beta|^2$.

Also print a picture of your circuit and the corresponding `OPENQASM` code (shown in the Circuit Editor).

Hint: You can use the gates from exercise 3.1(b) to prepare the initial entangled pair $|\beta_{00}\rangle$.

(c) Construct the circuit in (b) using Qiskit, and execute the circuit via Aer's statevector_simulator. Print your code together with the final state vector.

Hint: See also https://github.com/Qiskit/qiskit-tutorials/blob/master/tutorials/simulators/1_aer_provider.ipynb.

[2] Note that the usual quantum teleportation protocol assumes that Alice and Bob are far from each other, such that conditional quantum operations between their qubits would be impractical.