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Exercise 6.1 (Heisenberg uncertainty principle for a single qubit)
Imagine we prepare multiple copies of an arbitrary single-qubit state $|\psi\rangle$, described by the Bloch vector (r_x, r_y, r_z). Some of these copies are measured using the observable Z, and the remaining copies using the observable X.

(a) Compute the expectation values of both measurements.
Hint: These two measurements are projective measurements and have a geometric interpretation on the Bloch sphere. You can use without proof the identities $\cos(\alpha/2)^2 - \sin(\alpha/2)^2 = \cos(\alpha)$ and $2\cos(\alpha/2)\sin(\alpha/2) = \sin(\alpha)$ for any $\alpha \in \mathbb{R}$.

(b) What will be their corresponding standard deviations? Recall from the lecture that the standard deviation of an observable M is defined as $\Delta M = \sqrt{\langle M^2\rangle - \langle M\rangle^2}$, with $\langle A\rangle \equiv \langle\psi| A |\psi\rangle$ for any observable A.

(c) Evaluate the commutator $[Z, X]$ and its expectation value $\langle\psi| [Z, X] |\psi\rangle$.

(d) Insert your results and explicitly verify that the Heisenberg uncertainty principle is satisfied.

Solution

(a) In the Bloch sphere, measuring Z corresponds to projecting onto the z-axis, and measuring X means projecting onto the x-axis. Therefore,

$$\langle Z\rangle = r_z \quad \text{and} \quad \langle X\rangle = r_x.$$

Alternatively, one can compute these expectation values starting from the Bloch angles θ and φ, which define the quantum state $|\psi\rangle$ via

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + \mathrm{e}^{i\varphi}\sin(\theta/2)\,|1\rangle\,,$$

and the Bloch vector via

$$\vec{r} = \begin{pmatrix} \cos(\varphi)\sin(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

(b) Note that $X^2 = Z^2 = I$. Thus, together with the normalization of the quantum state $|\psi\rangle$,

$$\Delta Z = \sqrt{1 - r_z^2},$$
$$\Delta X = \sqrt{1 - r_x^2}.$$

(c) Inserting the definitions of Z and X directly leads to

$$[Z, X] = ZX - XZ = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2iY.$$

Analogous to the expectation values of X and Z, one obtains $\langle Y\rangle = r_y$, and thus $\langle\psi| [Z, X] |\psi\rangle = 2ir_y$.

(d) The Heisenberg uncertainty principle states that

$$\Delta Z \Delta X \geq \frac{|\langle\psi| [Z, X] |\psi\rangle|}{2}.$$

In the present setting,

$$\sqrt{1 - r_z^2}\sqrt{1 - r_x^2} \overset{!}{\geq} |r_y|$$
$$(1 - r_z^2)(1 - r_x^2) \overset{!}{\geq} r_y^2$$
$$1 - r_z^2 - r_x^2 + r_z^2 r_x^2 \overset{!}{\geq} r_y^2$$

The Bloch vector must be normalized, i.e., $r_x^2 + r_y^2 + r_z^2 = 1$. Using this, the above inequality can be simplified to

$$r_z^2 r_x^2 \geq 0,$$

which is always true. The Heisenberg uncertainty principle is satisfied.