
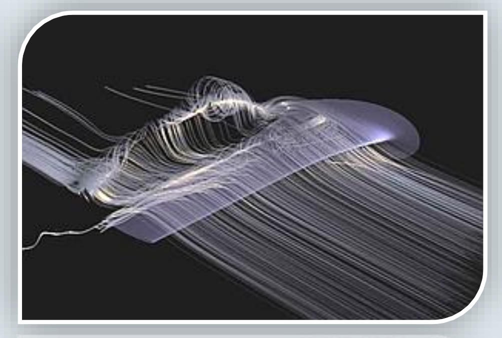
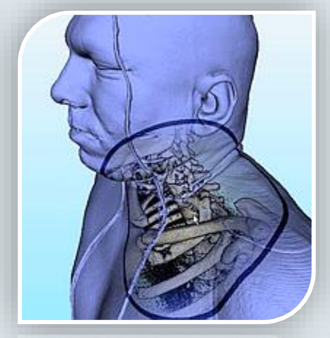


Visual Data Analytics

Data Reconstruction and Interpolation

Dr. Johannes Kehrer

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Sampling & Reconstruction

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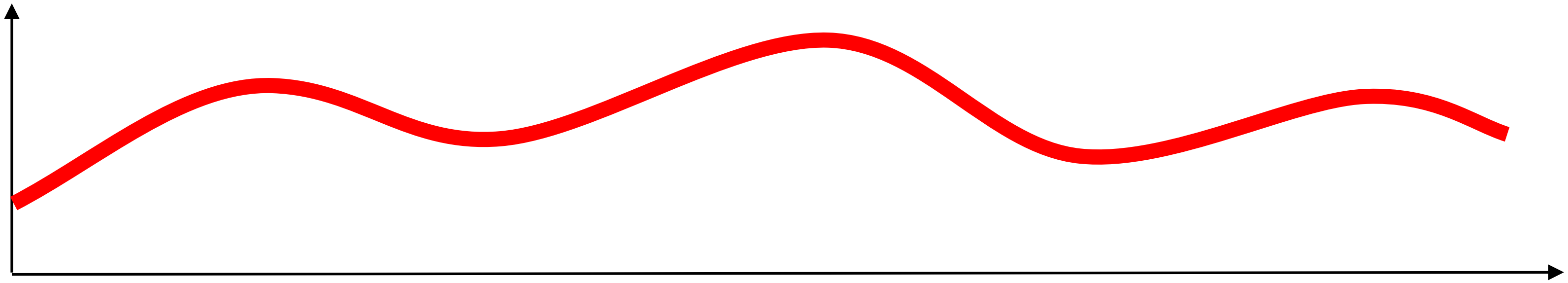
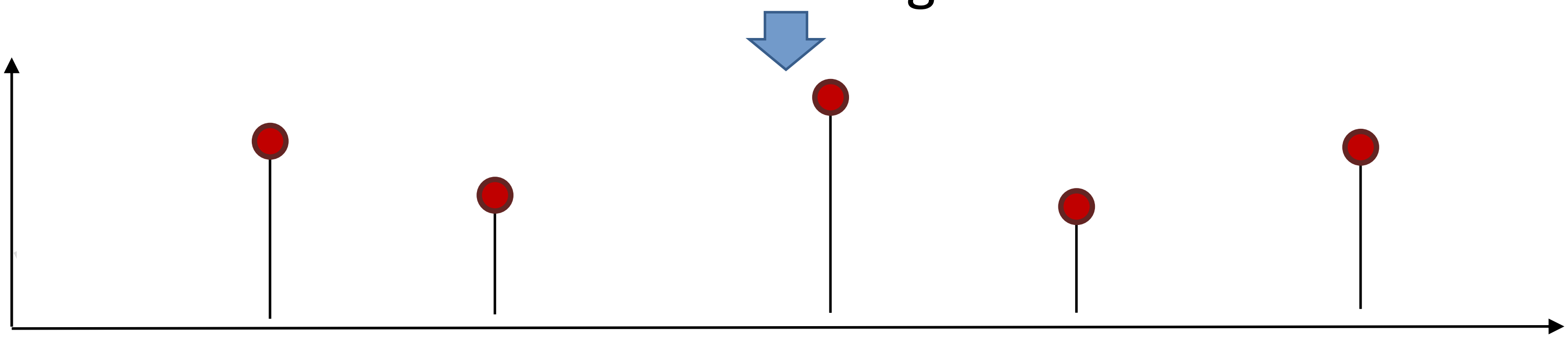
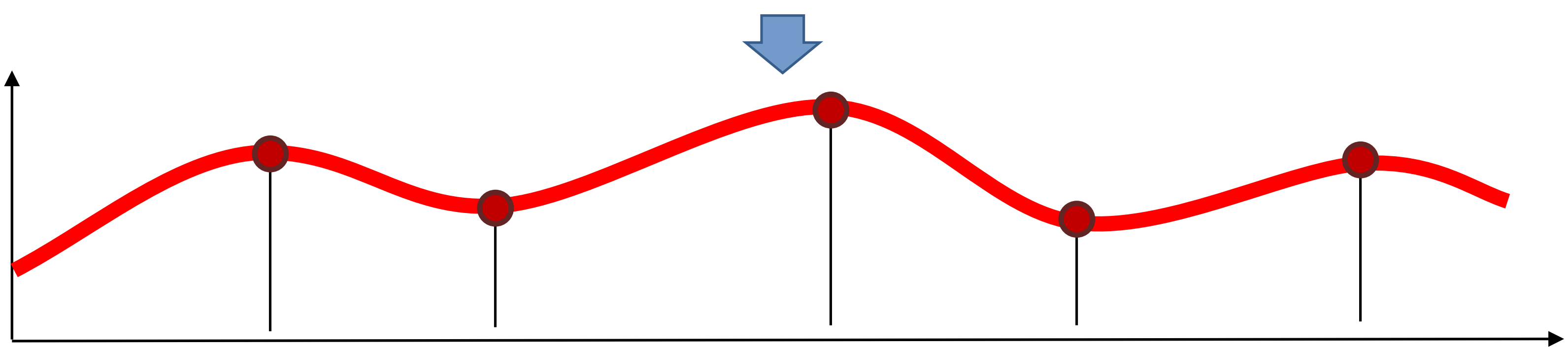


Continuous signal



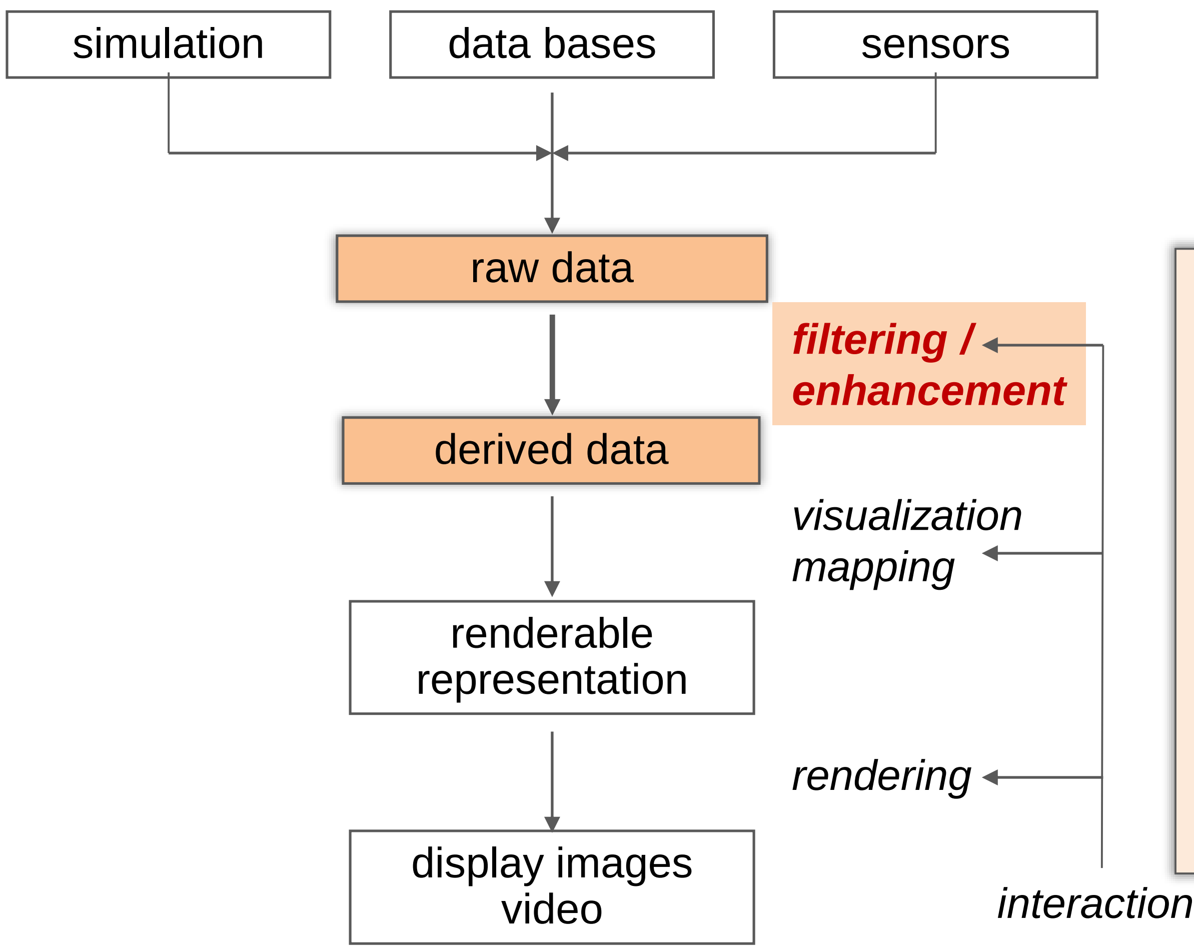
Sampled signal



Reconstructed signal

Data reconstruction

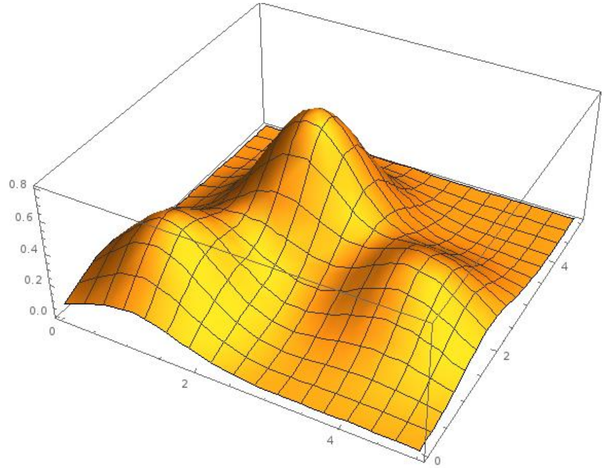
- From discrete samples to a continuous representation

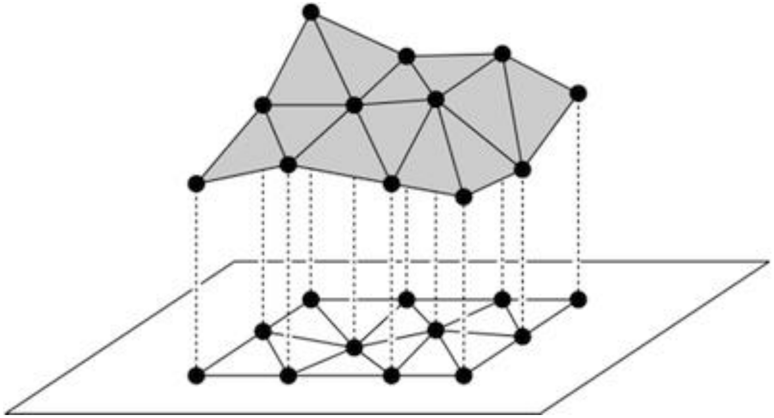


Process to obtain useful data (e.g., 3D volume)
- data format conversion
- co-registration of data sets
- resampling to grid
- interpolation/approximation of missing values
- data reduction (remove nonrelevant data)
- clipping/cleaning/denoising
- characteristic properties (gradients, curvature, extrema, segmentation)

Overview

- Scattered data interpolation
  - Continuous interpolation functions
  - Piecewise interpolation via triangulation





- Interpolation on grids (next lecture)

Data reconstruction

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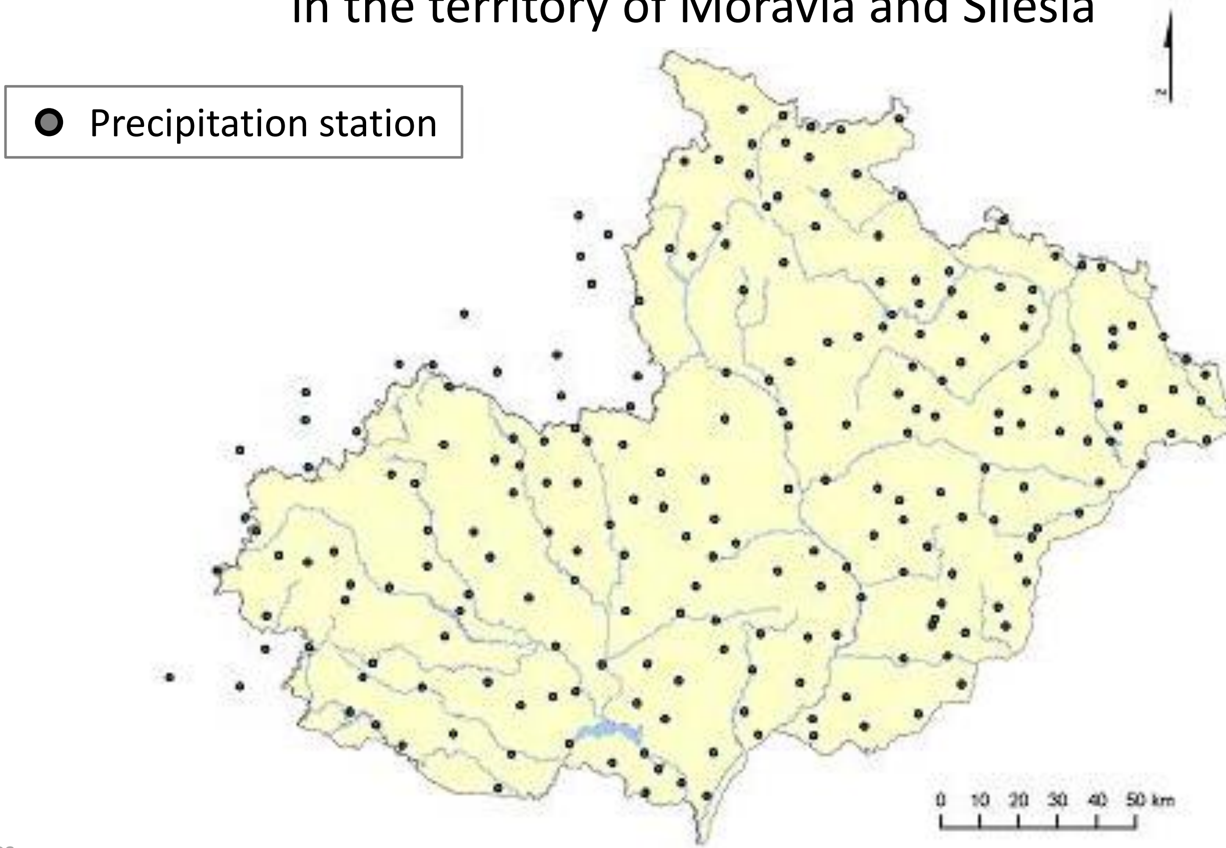
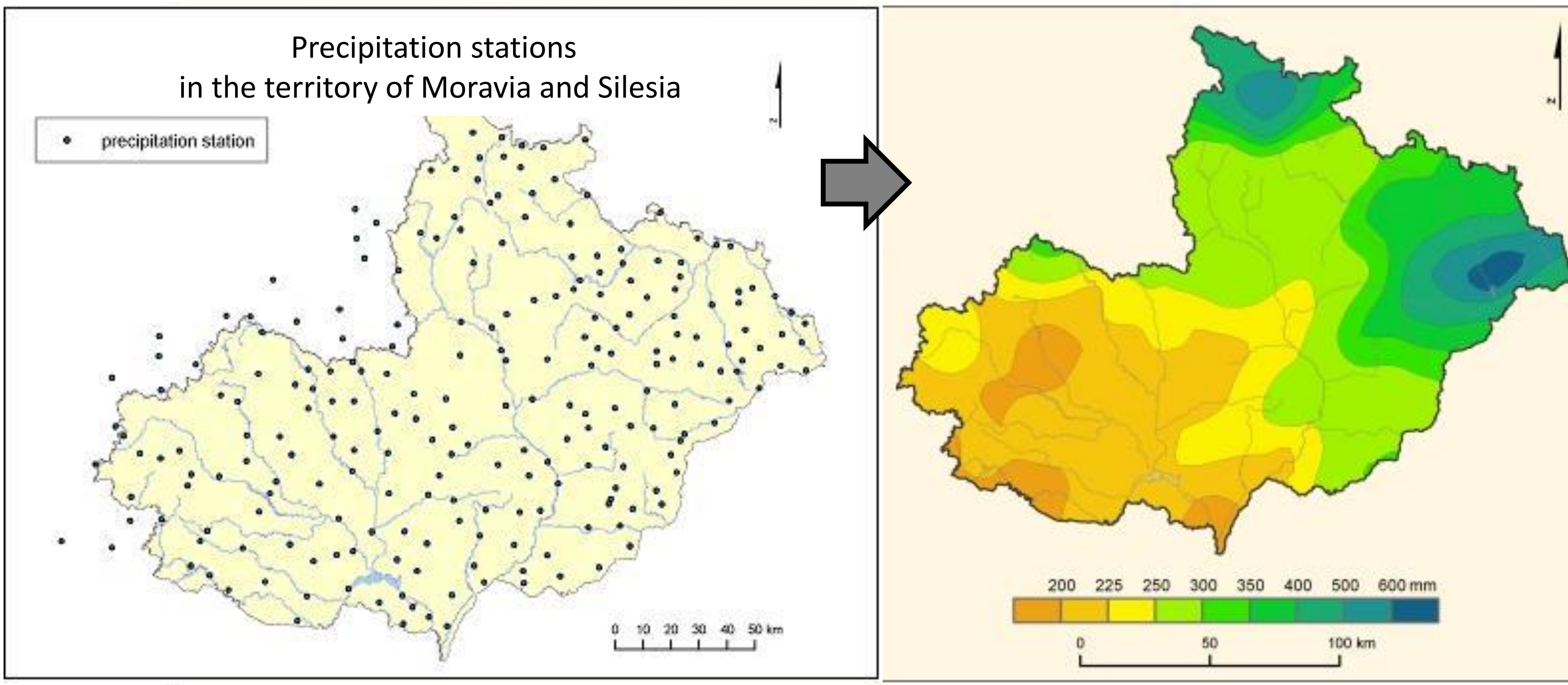
- Initial data often given at a discrete set of scattered points (samples) in the domain

Precipitation stations
in the territory of Moravia and Silesia



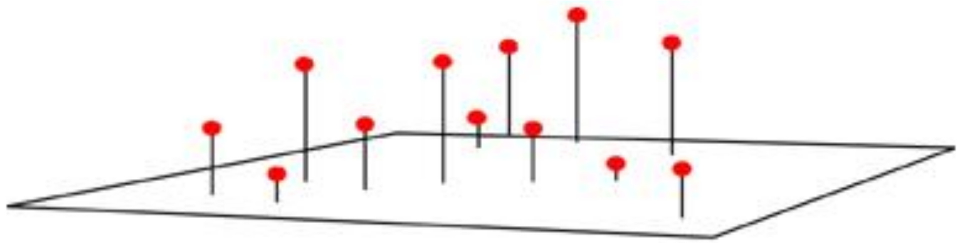
Data reconstruction

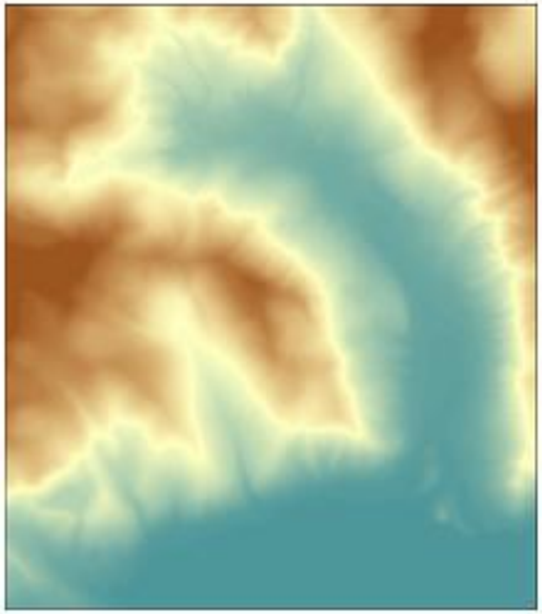
- We want to derive a continuous representation from data given at scattered points
  - Better communication of spatial data distribution



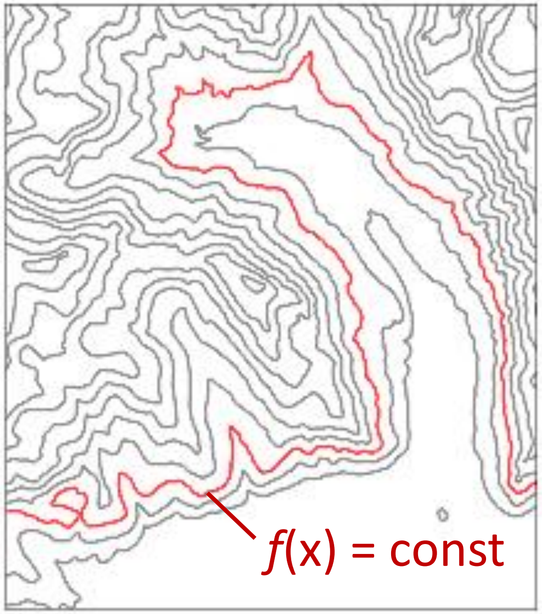
Data reconstruction

- Some analysis techniques require a continuous representation





Data distribution

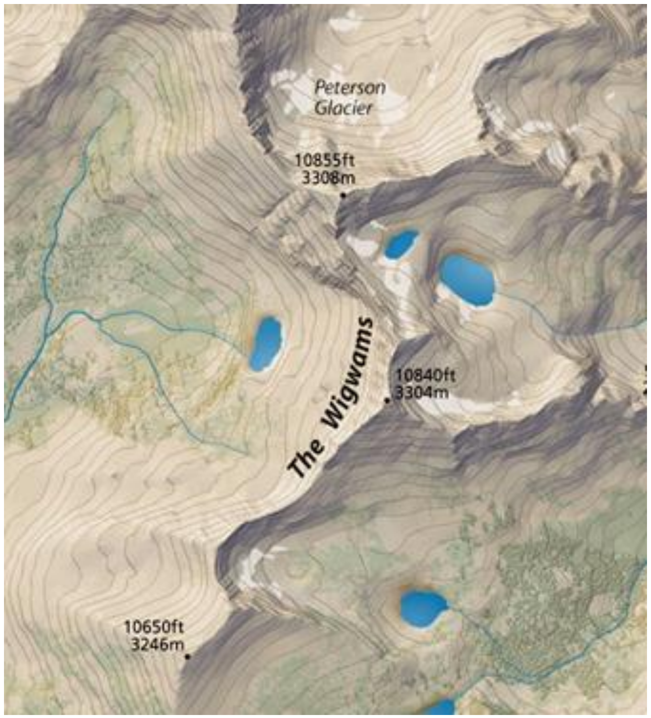


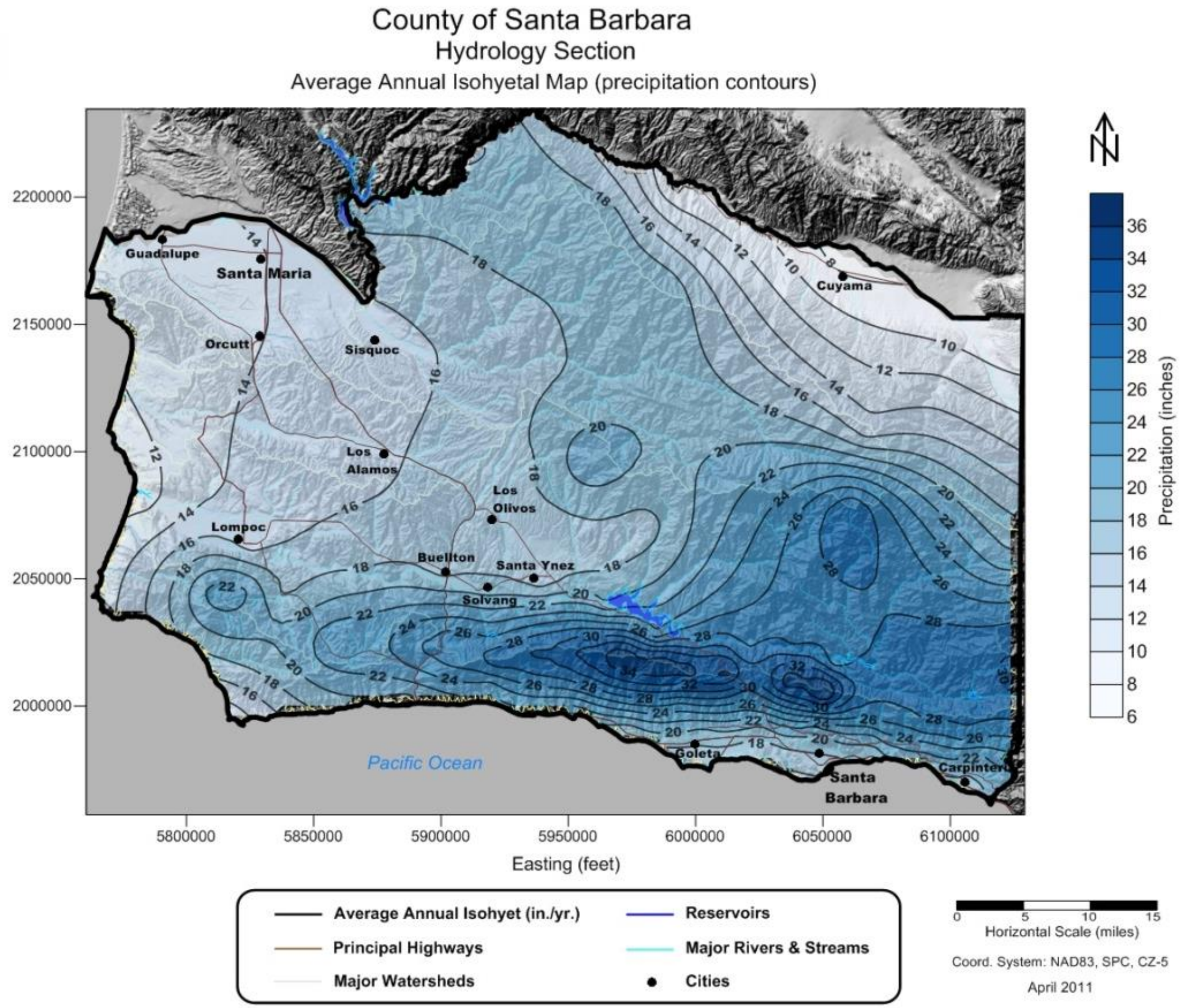
Isocontours – curves on which all points have a certain value

Data reconstruction

- Isocontours/lines in continuous data fields

Geographic map with isolines





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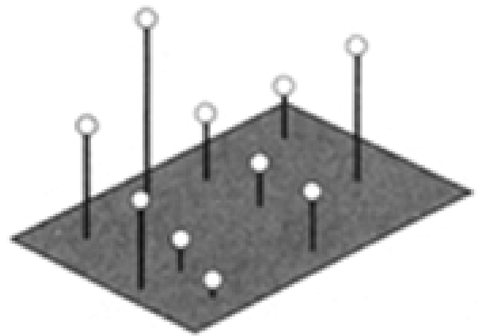
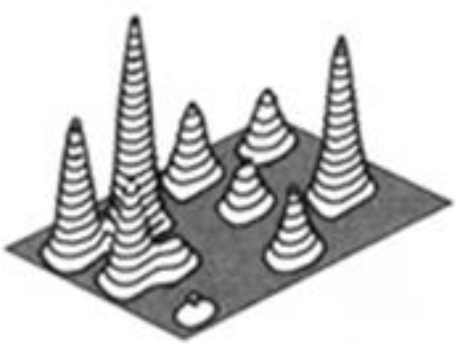
Data reconstruction

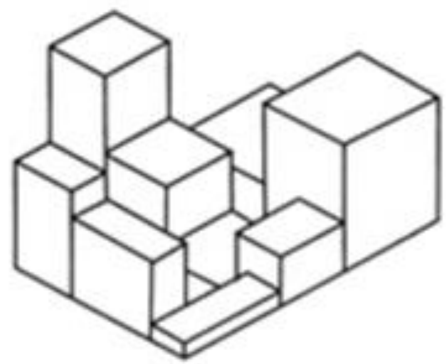
- Data reconstruction from scattered points



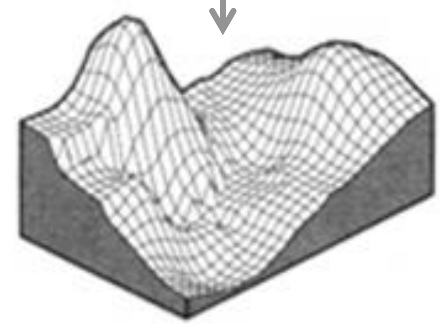
Assumes some similarity between data values inversely proportional to distance



data interpolation



piece-wise constant interpolation



continuous interpolation

Data reconstruction

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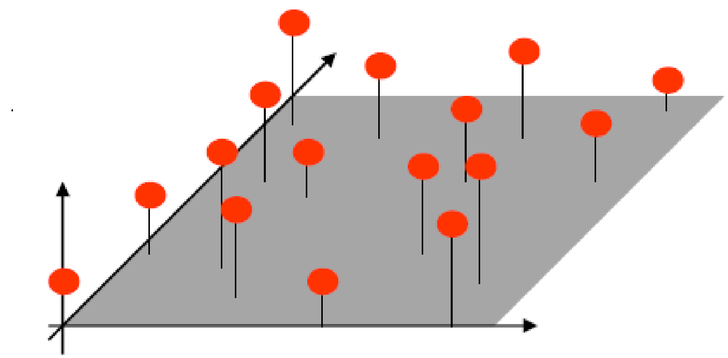
- Data reconstruction from scattered points



- Strategies to obtain values in-between data points…
  1. from a continuous function which interpolates the given values and varies smoothly in-between
  2. from a grid which is constructed from the given points, i.e., a triangulation

Continuous representation

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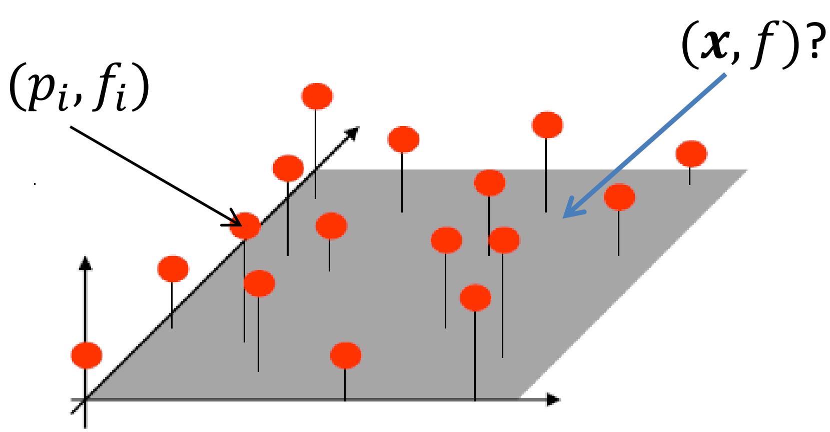
Ingenuity for life

- Given a set of scattered points p_i in a 2D parameter domain with scalar values f_i
  - The principles are applicable to arbitrary parameter domain dimensions (1D/2D/3D)

- **Goal:** Construct a continuous function f from given set of p_i, f_i which approximates ("follows") the given values



Continuous representation

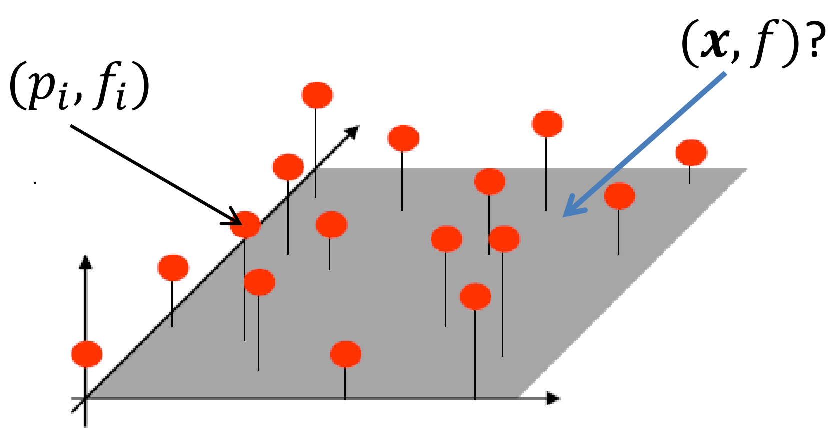
- Radial Basis Functions
  - Independent of dimension of parameter domain (1D/2D/3D)
  - Function f represented as weighted sum of N radial functions φ

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} f_i \, \varphi(\|p_i - \boldsymbol{x}\|)$$

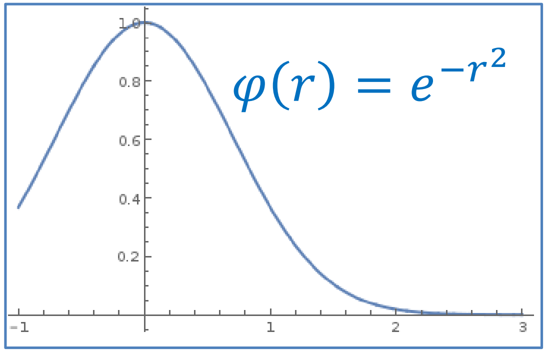
  - Each (p_i, f_i) influences $f(\boldsymbol{x})$ based on Euclidean distance

$$r = \|p_i - \boldsymbol{x}\|$$

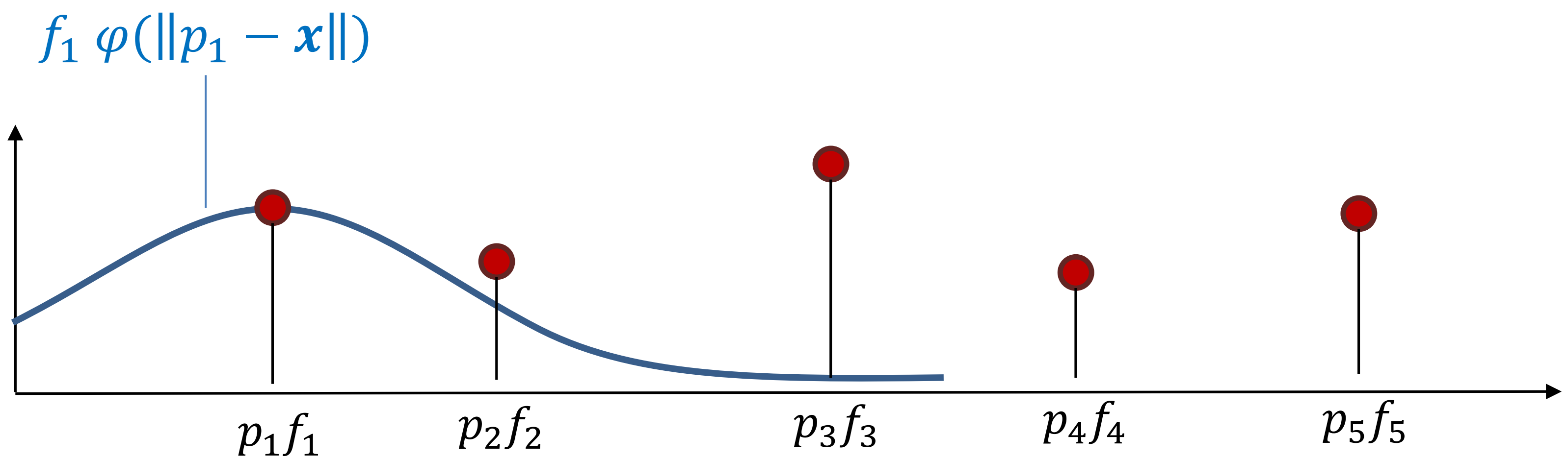
  - Nearby points have higher influence than far-away points



Continuous representation



- Radial Basis Functions
  - Each radial function $\varphi(r)$ is centered around a data point p_i
  - Value of $\varphi(r)$ decreases quickly with increasing distance $r = \|p_i - \boldsymbol{x}\|$ to the function's center p_i



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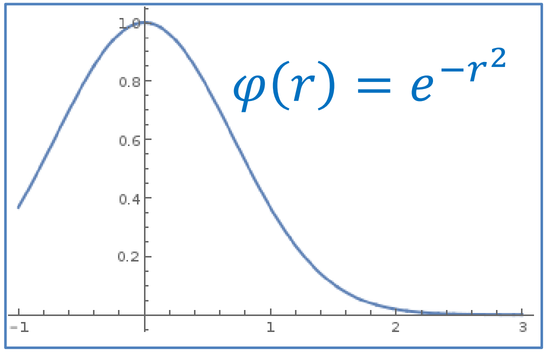
Continuous representation



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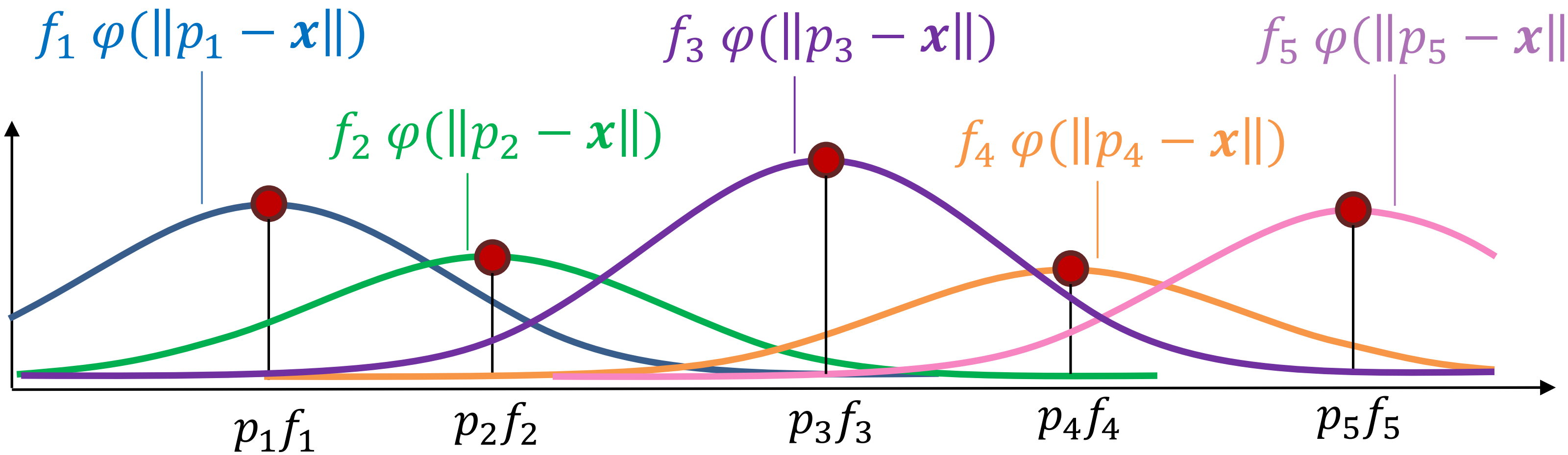

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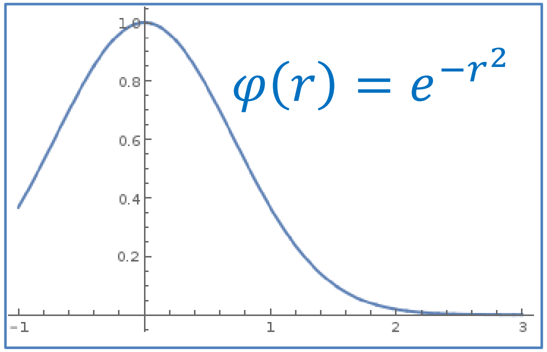
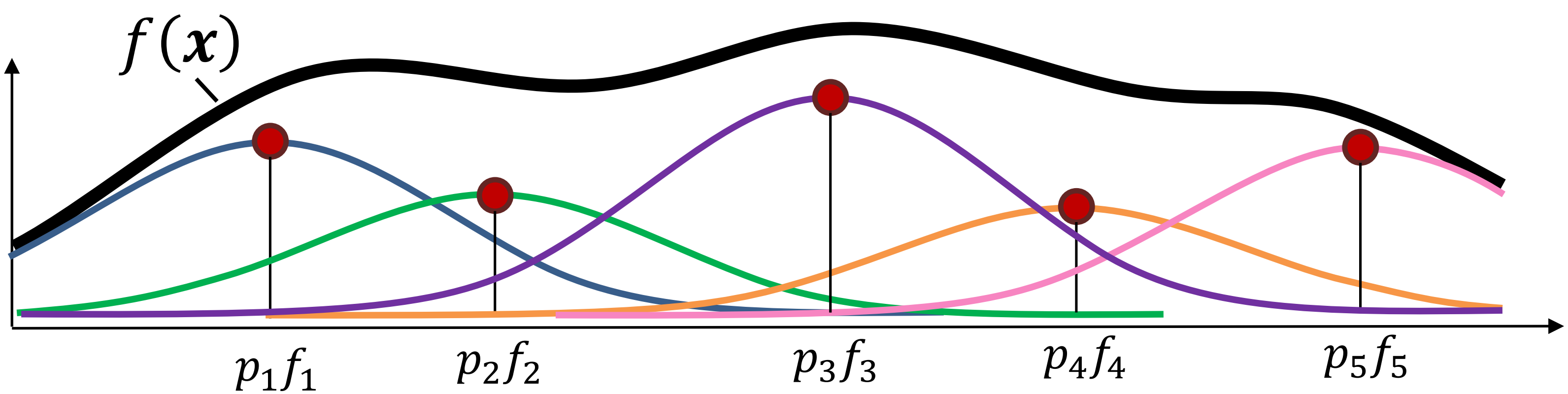
Continuous representation



- Radial Basis Functions
  - Each radial function $\varphi(r)$ is centered around a data point p_i
  - Value of $\varphi(r)$ decreases quickly with increasing distance $r = ||p_i - \boldsymbol{x}||$ to the function's center p_i
  - Function f represented as weighted sum of N radial functions φ

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} f_i \, \varphi(||p_i - \boldsymbol{x}||)$$

where $||\cdot||$ is the Euclidean distance (length of a vector)



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Continuous representation

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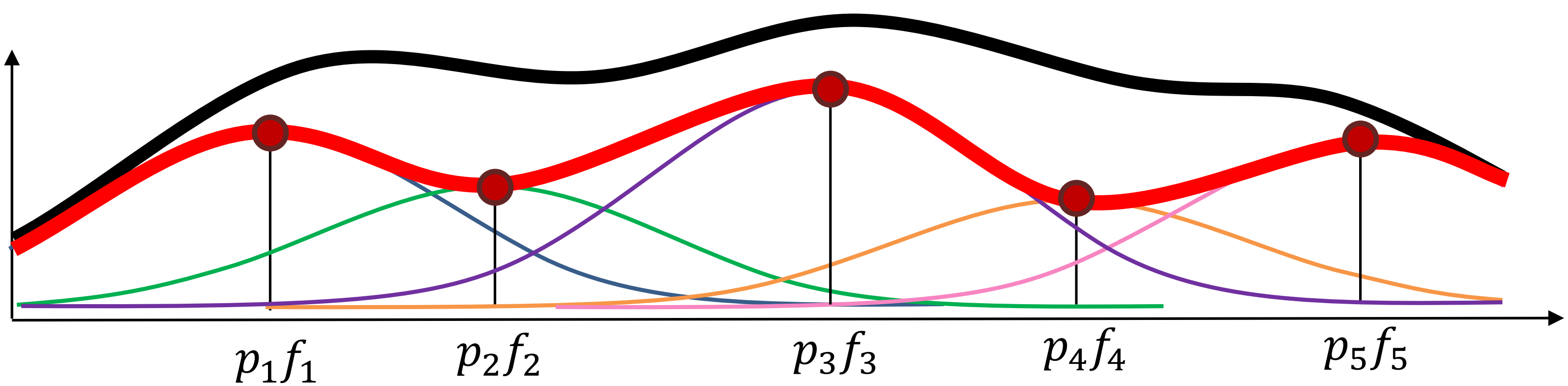
Ingenuity for life

- Radial Basis Functions
  - Instead of the **black** curve we want the **red** one, i.e., a curve which is going through the initial data points
  - This is called an interpolation
  - **Question:** How do we have to select the weights w_i so that the red curve is obtained?

$$f(\boldsymbol{x}) = \sum_i w_i \, \varphi(\|p_i - \boldsymbol{x}\|)$$



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Continuous representation

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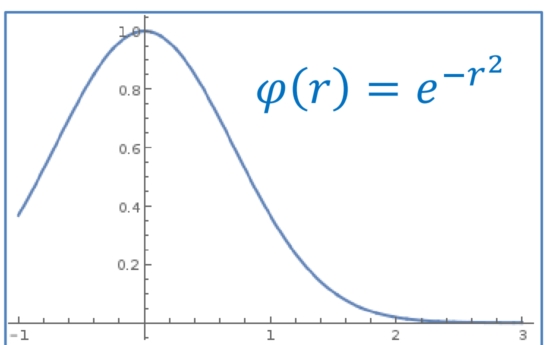
Ingenuity for life

- Example:
  - Data points: $p_1 = 1$, $p_2 = 3$, $p_3 = 4$
  - Data values: $f_1 = 1$, $f_2 = {}^{3}/_{5}$, $f_3 = 0$
- Find the weights w_i such that f interpolates all points

$$f(\boldsymbol{x}) = \sum_{i=1}^{N} w_i \, \varphi(\|p_i - \boldsymbol{x}\|) \quad \text{where} \quad \varphi(r) = e^{-r^2}$$



r	0	0.5	1	1.5	2	2.5	3
$\varphi(r)$	1	$^4/_5$	$^2/_5$	$^1/_{10}$	0	0	0

Continuous representation

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Ingenuity for life

- Radial Basis Functions – finding the weights w_i
  - For $j = 1, \dots, N$,
    specify w_i such that $f(p_j)$ interpolates the value f_j

$$f(p_j) = \sum_{i=1}^{N} w_i \varphi(\|p_i - p_j\|) = f_j$$

Continuous representation

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Ingenuity for life

- Radial Basis Functions – finding the weights w_i
  - For $j = 1, \ldots, N$,
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$$f(p_j) = \sum_{i=1}^{N} w_i \varphi(\|p_i - p_j\|) = f_j$$

  - Yields a system of linear equations (per point) to be solved for w_i

$$\begin{bmatrix} \varphi(\|p_1 - p_1\|) & \varphi(\|p_2 - p_1\|) & \cdots & \varphi(\|p_N - p_1\|) \\ \varphi(\|p_1 - p_2\|) & \varphi(\|p_2 - p_2\|) & \cdots & \varphi(\|p_N - p_2\|) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(\|p_1 - p_N\|) & \varphi(\|p_2 - p_N\|) & \cdots & \varphi(\|p_N - p_N\|) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

N equations in N unkowns

Continuous representation

- Radial Basis Functions – finding the weights w_i
  - For $j = 1, \dots, N$,
    specify w_i such that $f(p_j)$ interpolates the value f_j

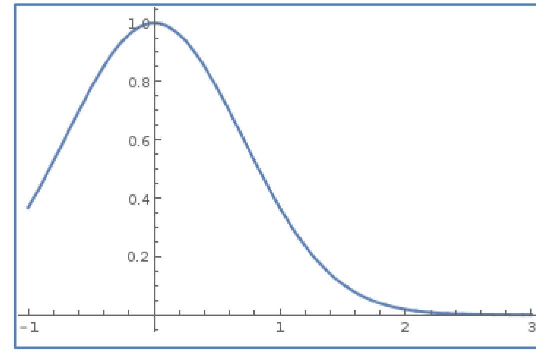
$$f(p_j) = \sum_{i=1}^{N} w_i \varphi(\|p_i - p_j\|) = f_j$$

  - Yields a system of linear equations (per point) to be solved for w_i

$$\begin{bmatrix} \varphi(0) & \varphi(\|p_2 - p_1\|) & \cdots & \varphi(\|p_N - p_1\|) \\ \varphi(\|p_1 - p_2\|) & \varphi(0) & \cdots & \varphi(\|p_N - p_2\|) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(\|p_1 - p_N\|) & r(\|p_2 - p_N\|) & \cdots & \varphi(0) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

Continuous representation

- Example (cont.)
  - Data points: $p_1 = 1,\ p_2 = 3,\ p_3 = 4$
  - Data values: $f_1 = 1,\ f_2 = {}^3/_5\,,\ f_3 = 0$
  - Radial function: $\varphi(r) = e^{-r^2}$



$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & {}^2/_5 \\ 0 & {}^2/_5 & 1 \end{bmatrix}}_{\boldsymbol{R}} \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\boldsymbol{w}} = \underbrace{\begin{bmatrix} 1 \\ {}^3/_5 \\ 0 \end{bmatrix}}_{\boldsymbol{f}}$$

$1 = w_1$

$3/5 = w_2 + \frac{2}{5} w_3$

$0 = 2/5\, w_2 + w_3$

If matrix $\boldsymbol{R}$ is invertible,

$\boldsymbol{w} = \boldsymbol{R}^{-1}\boldsymbol{f}$ with solution: $w_1 = 1, w_2 = {}^5/_7\,, w_3 = -\,{}^2/_7$

Continuous representation

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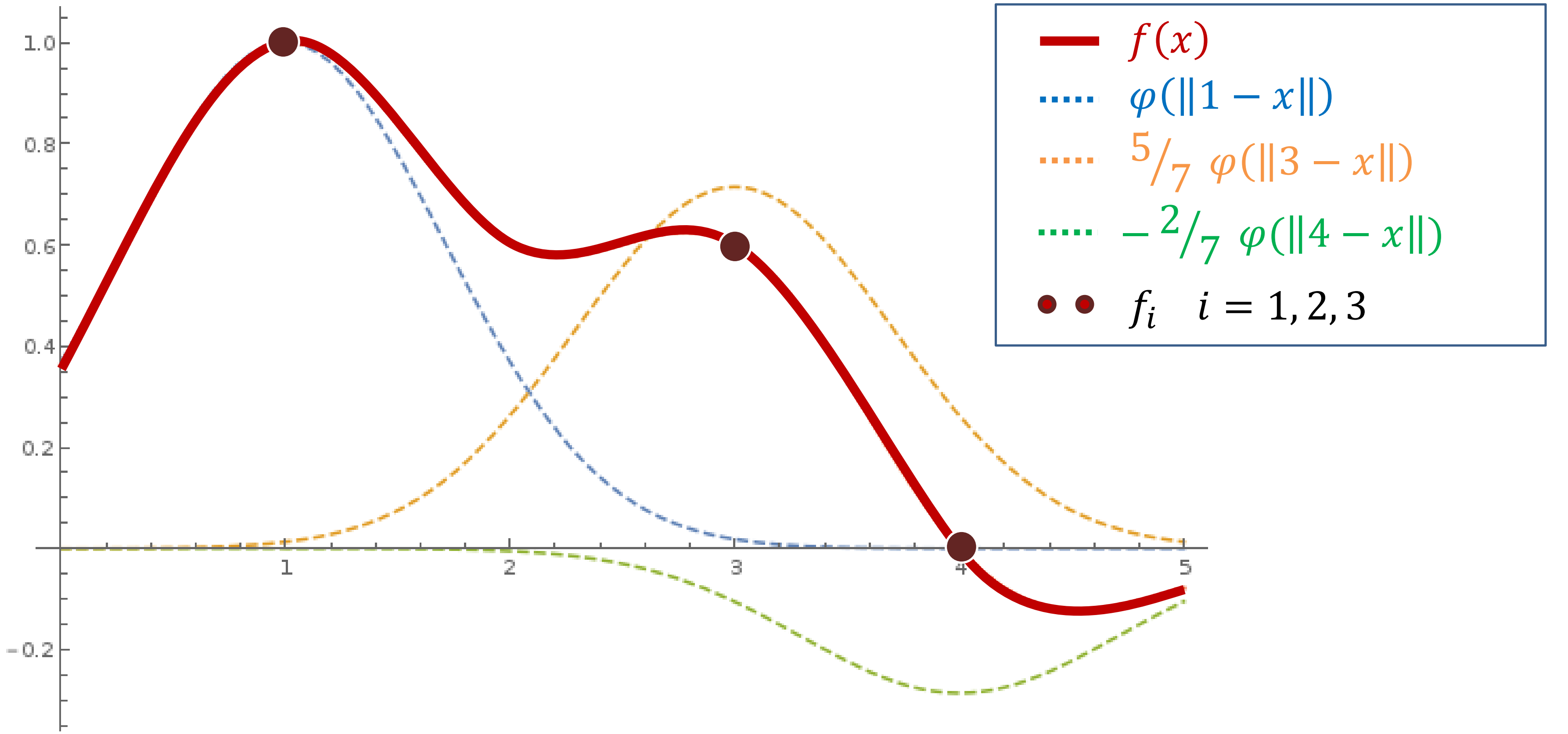
- Example

$$f(x) = \varphi(\|1 - x\|) + {}^{5}/_{7}\, \varphi(\|3 - x\|) - {}^{2}/_{7}\, \varphi(\|4 - x\|)$$



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Continuous representation

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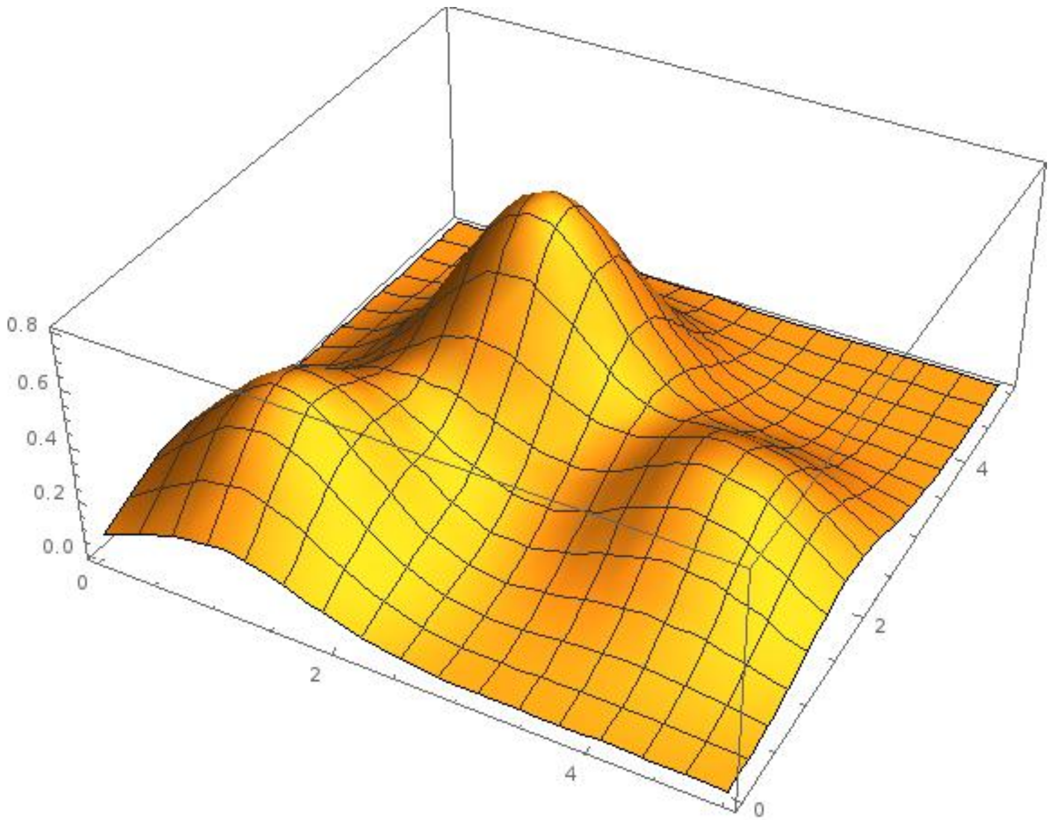
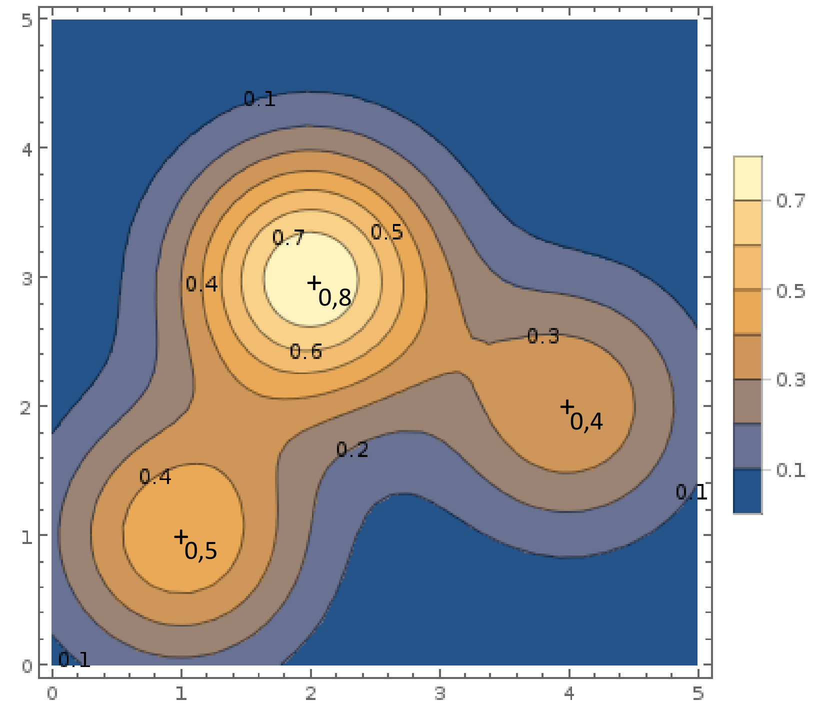
- 2D Example
  - Data points: $p_1 = (1,1)^T$, $p_2 = (2,3)^T$, $p_3 = (4,2)^T$
  - Data values: $f_1 = 0.5$, $f_2 = 0.8$, $f_3 = 0.4$

$$f(\boldsymbol{x}) = 0.49\,\varphi(\|(1,1)^T - \boldsymbol{x}\|) + 0.79\,\varphi(\|(2,3)^T - \boldsymbol{x}\|) + 0.39\,\varphi(\|(4,2)^T - \boldsymbol{x}\|)$$





Continuous representation

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- Drawbacks of radial basis functions
  - Every sample point has influence on whole domain
  - Adding a new sample requires re-solving the equation system
  - Computationally expensive (solving a system of linear equations)

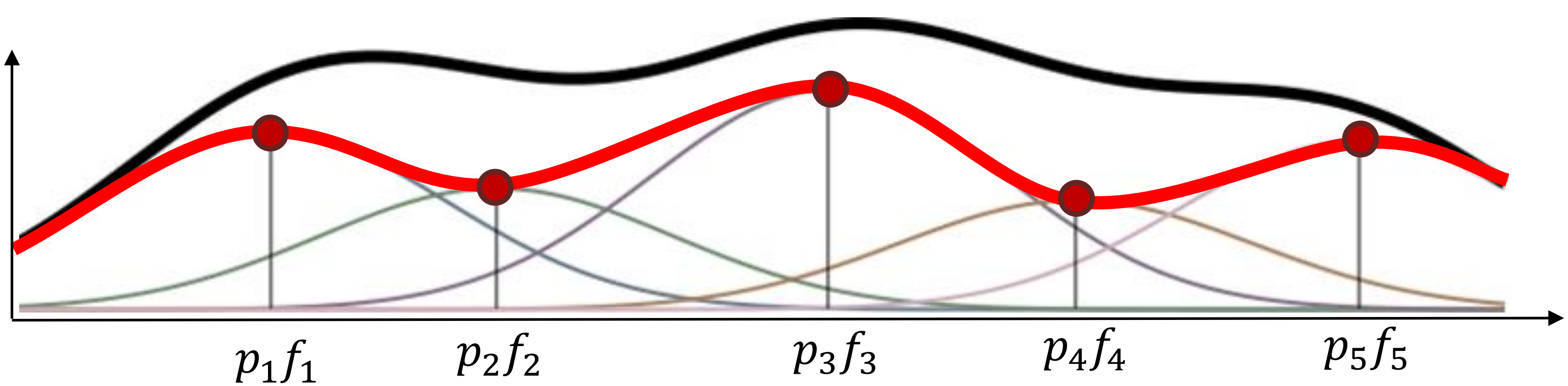
- What can we do?
  - Find a different radial function
  - Give up finding a smooth reconstruction
    - Try finding a piecewise (local) reconstruction function

Continuous representation

SIEMENS
Ingenuity for life

- Radial Basis Functions
  - Instead of the **black** curve we want the **red** one, i.e., the curve which is going through the initial data points
  - This is called an interpolation
  - **Question:** How do we have to select the radial function φ so that the red curve is obtained?

$$f(\boldsymbol{x}) = \sum_i f_i \varphi(\|p_i - \boldsymbol{x}\|)$$



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Continuous representation

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Ingenuity for life

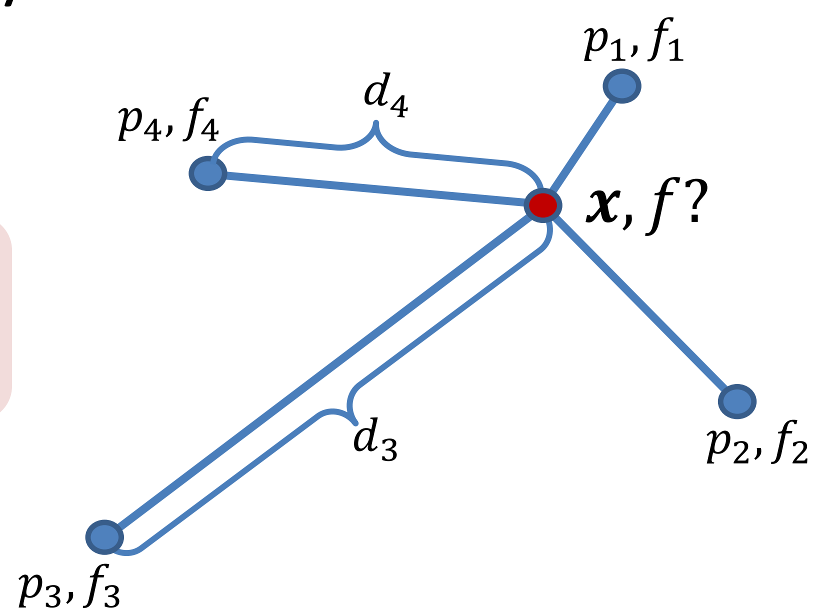
- Inverse distance weighting
  - Sample positions p_i and values f_i
  - Assumption: Nearby points are more similar than those further away → they have more influence

$$f(\boldsymbol{x}) = \sum_i f_i \, \varphi(\|p_i - \boldsymbol{x}\|)$$

$d_i = \|p_i - \boldsymbol{x}\|,$

$$\boxed{\varphi(r) = \frac{1}{r^2} \Big/ \sum_{i=1}^{N} \frac{1}{d_i^2}}$$

Attention for $d_i < \epsilon$



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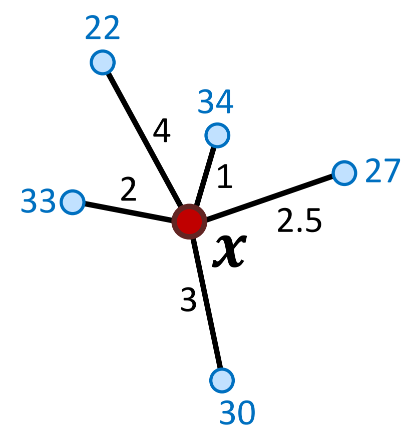
Continuous representation

SIEMENS
Ingenuity for life

- Inverse distance weighting
  - Sample positions p_i and values f_i
  - Assumption: Nearby points are more similar than those further away → they have more influence

$$f(\boldsymbol{x}) = \sum_i f_i \, \varphi(\|p_i - \boldsymbol{x}\|) =$$

$$= \boxed{\sum_{i=1}^{N} \frac{f_i}{\|p_i - \boldsymbol{x}\|^2} \Big/ \sum_{i=1}^{N} \frac{1}{\|p_i - \boldsymbol{x}\|^2}}$$

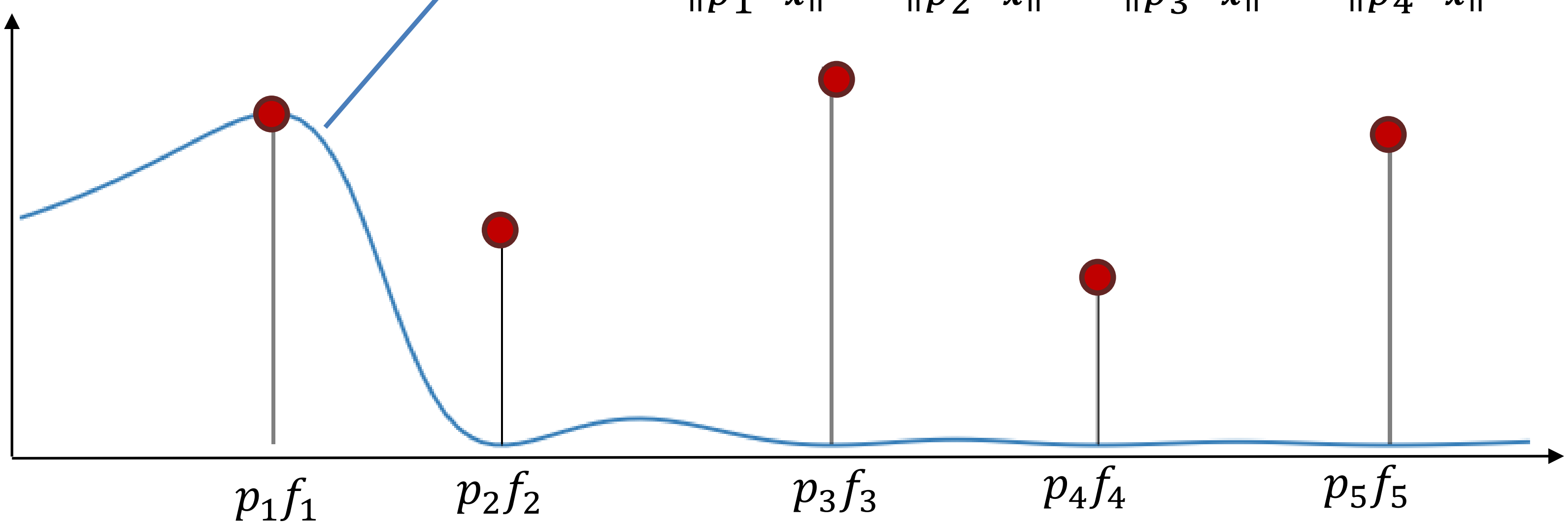


$$f(\boldsymbol{x}) = \frac{\frac{22}{4^2} + \frac{34}{1^2} + \frac{27}{2.5^2} + \frac{30}{3^2} + \frac{33}{2^2}}{\frac{1}{4^2} + \frac{1}{1^2} + \frac{1}{2.5^2} + \frac{1}{3^2} + \frac{1}{2^2}} = 32.38$$

Continuous representation

- Inverse distance weighting
  - Sample positions p_i and values f_i
  - Assumption: Nearby points are more similar than those further away

$$f_1 \varphi(\|p_1 - \boldsymbol{x}\|) = \frac{\frac{f_1}{\|p_1-\boldsymbol{x}\|^2}}{\frac{1}{\|p_1-\boldsymbol{x}\|^2}+\frac{1}{\|p_2-\boldsymbol{x}\|^2}+\frac{1}{\|p_3-\boldsymbol{x}\|^2}+\frac{1}{\|p_4-\boldsymbol{x}\|^2}+\frac{1}{\|p_5-\boldsymbol{x}\|^2}}$$

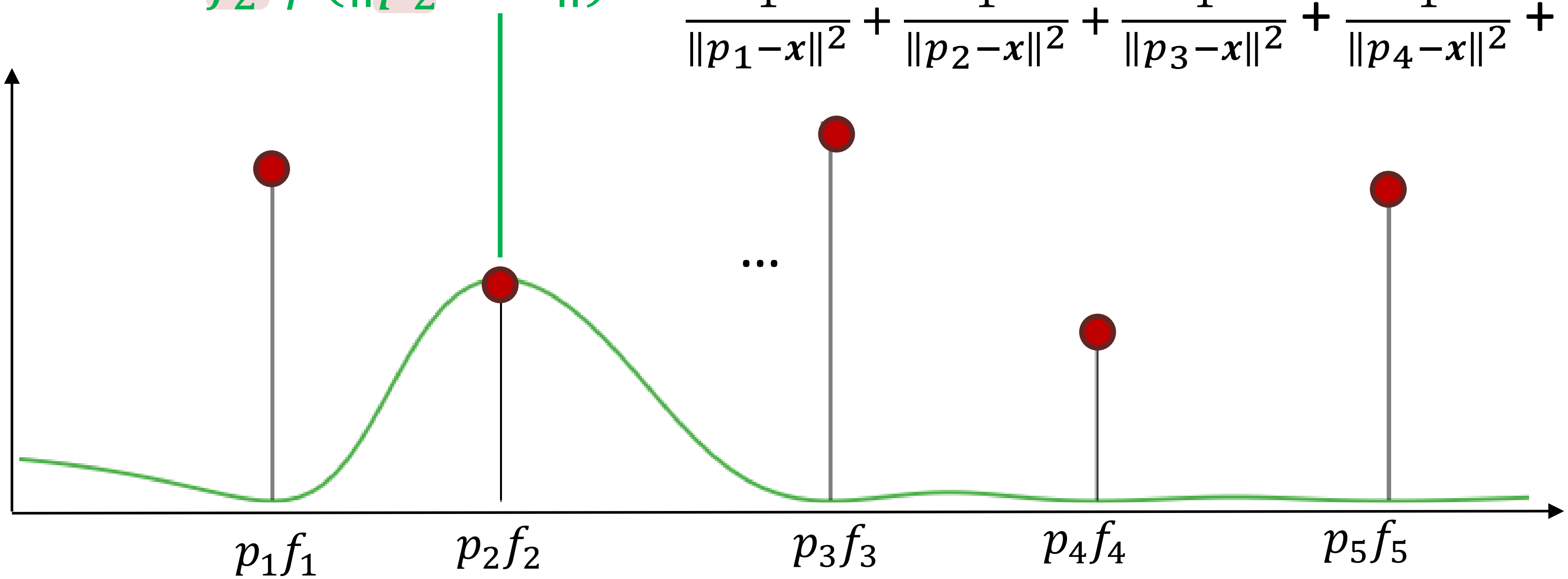


function is zero at $p_i, \quad i \neq 1$

Continuous representation

- Inverse distance weighting
  - Sample positions p_i and values f_i
  - Assumption: Nearby points are more similar than those further away

$$f_2\,\varphi(\|p_2 - \boldsymbol{x}\|) = \frac{\frac{f_2}{\|p_2-x\|^2}}{\frac{1}{\|p_1-x\|^2}+\frac{1}{\|p_2-x\|^2}+\frac{1}{\|p_3-x\|^2}+\frac{1}{\|p_4-x\|^2}+\frac{1}{\|p_5-x\|^2}}$$

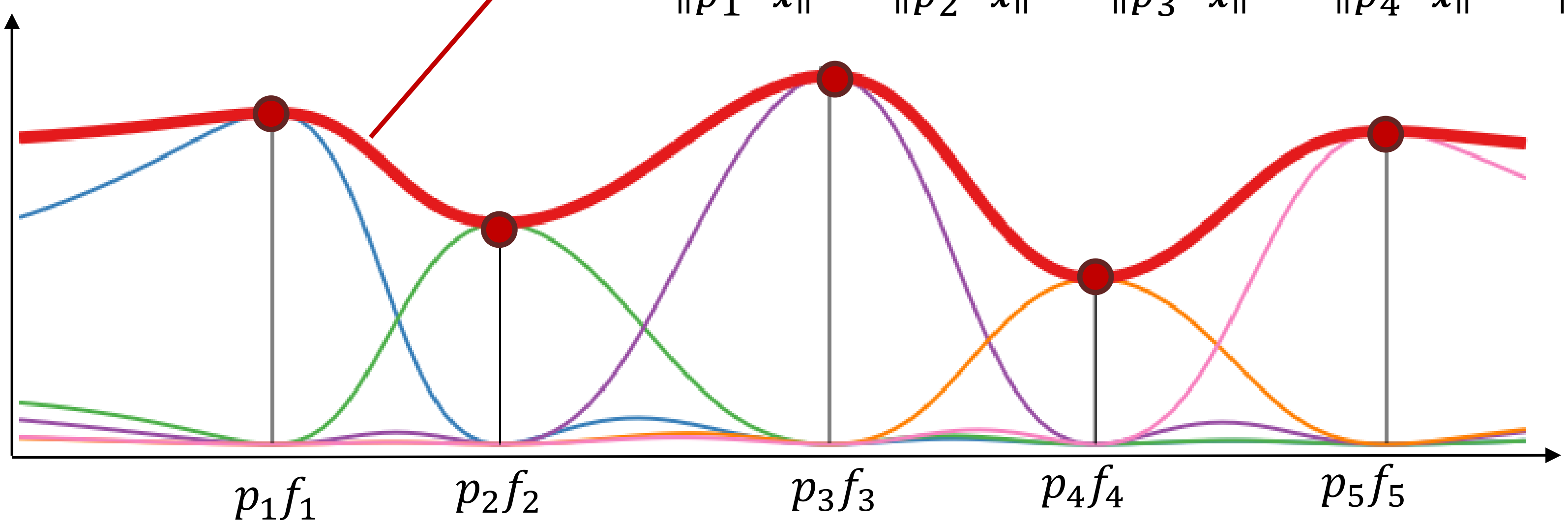


function is zero at p_i, $i \neq 2$

Continuous representation

- Inverse distance weighting
  - Sample positions p_i and values f_i
  - Assumption: Nearby points are more similar than those further away

$$f(\boldsymbol{x}) = \frac{\frac{f_1}{\|p_1-x\|^2} + \frac{f_2}{\|p_2-x\|^2} + \frac{f_3}{\|p_3-x\|^2} + \frac{f_4}{\|p_4-x\|^2} + \frac{f_5}{\|p_5-x\|^2}}{\frac{1}{\|p_1-x\|^2} + \frac{1}{\|p_2-x\|^2} + \frac{1}{\|p_3-x\|^2} + \frac{1}{\|p_4-x\|^2} + \frac{1}{\|p_5-x\|^2}}$$



multivis.net/lecture/radial-basis.html

Continuous representation

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- Inverse distance weighting
  - We no longer have to solve a system of linear equations to find the weights
  - However, every sample point still has global influence

- What can we do?
  - Give up smooth reconstruction by constructing a grid from the given points, i.e., a triangulation

Generating a triangulation

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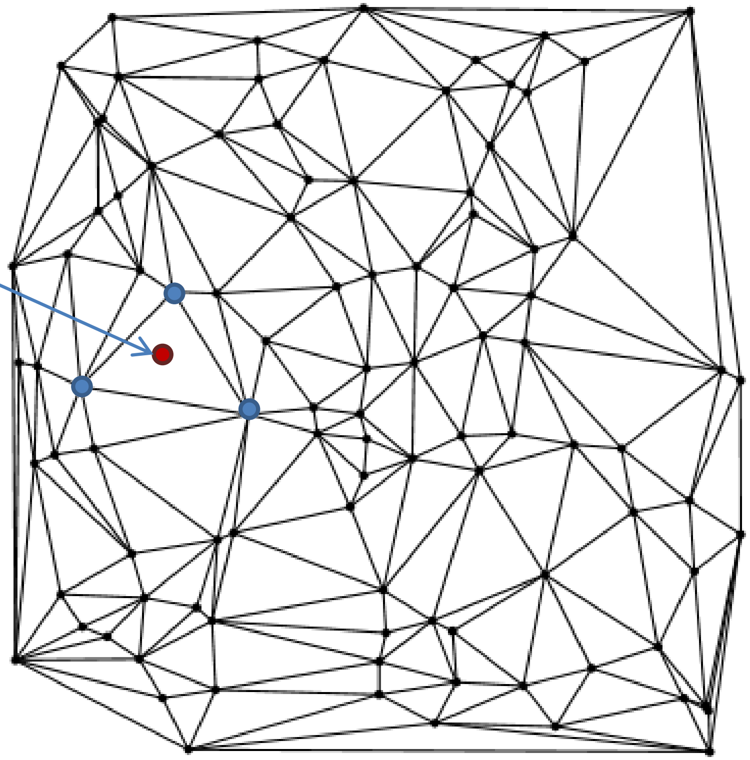
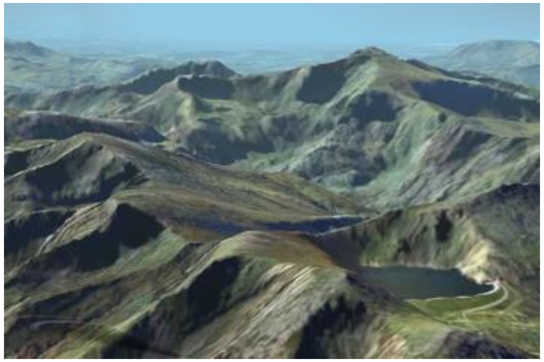
- Try finding a piecewise (local) reconstruction function
  - Connect the points so that a triangulation is obtained
  - Interpolate locally within the triangles

Value obtained by only considering values at triangle corners





Generating a triangulation

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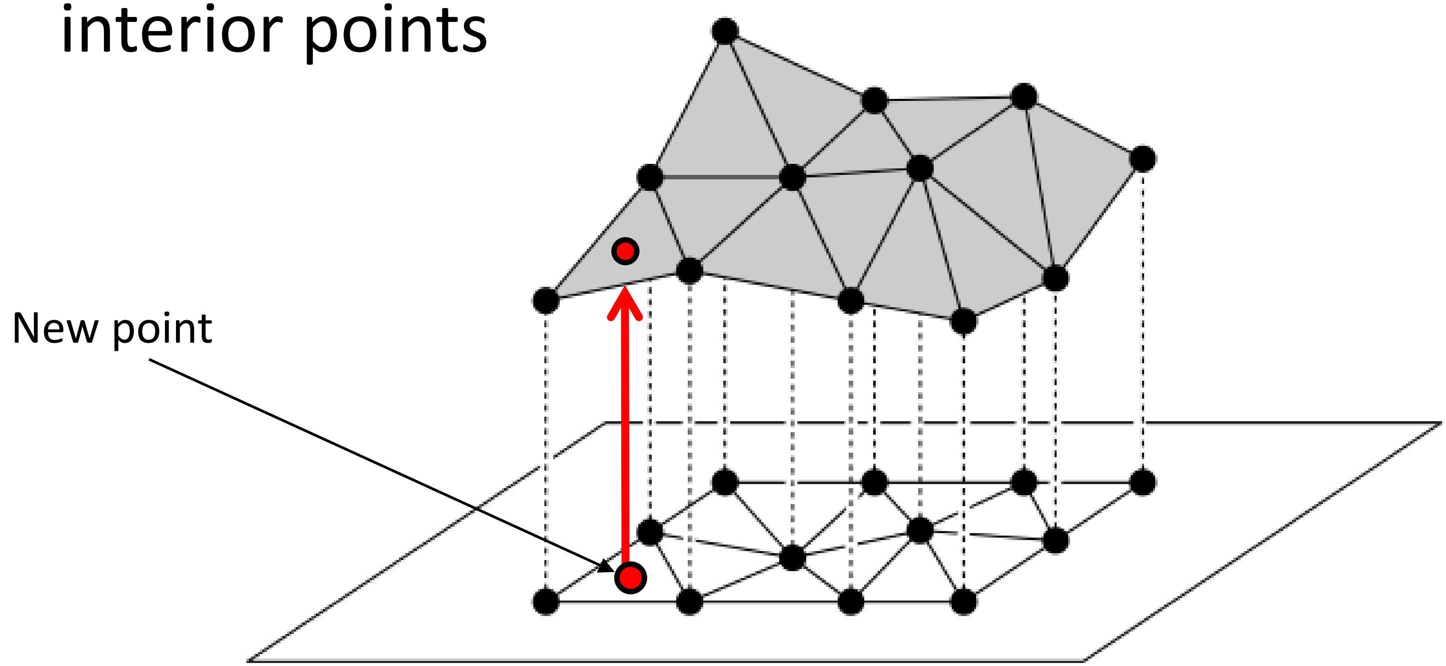
- Once a triangulation is given
  - Let the scalar values at vertices be interpolated across the triangles
  - I.e., piecewise linear interpolation of values at interior points



Generating a triangulation

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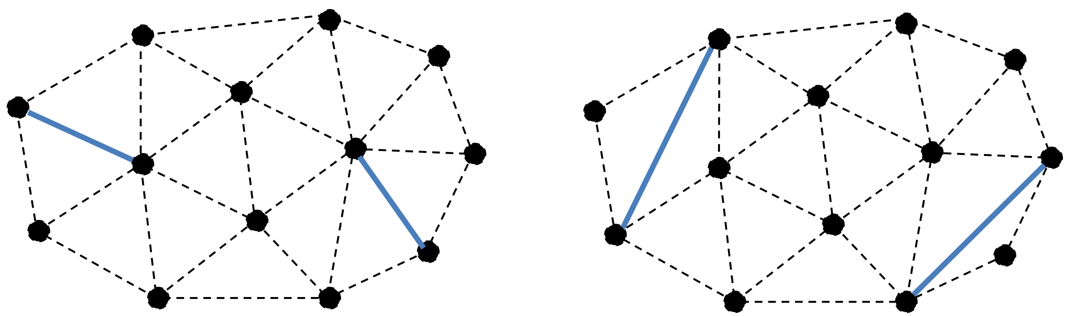
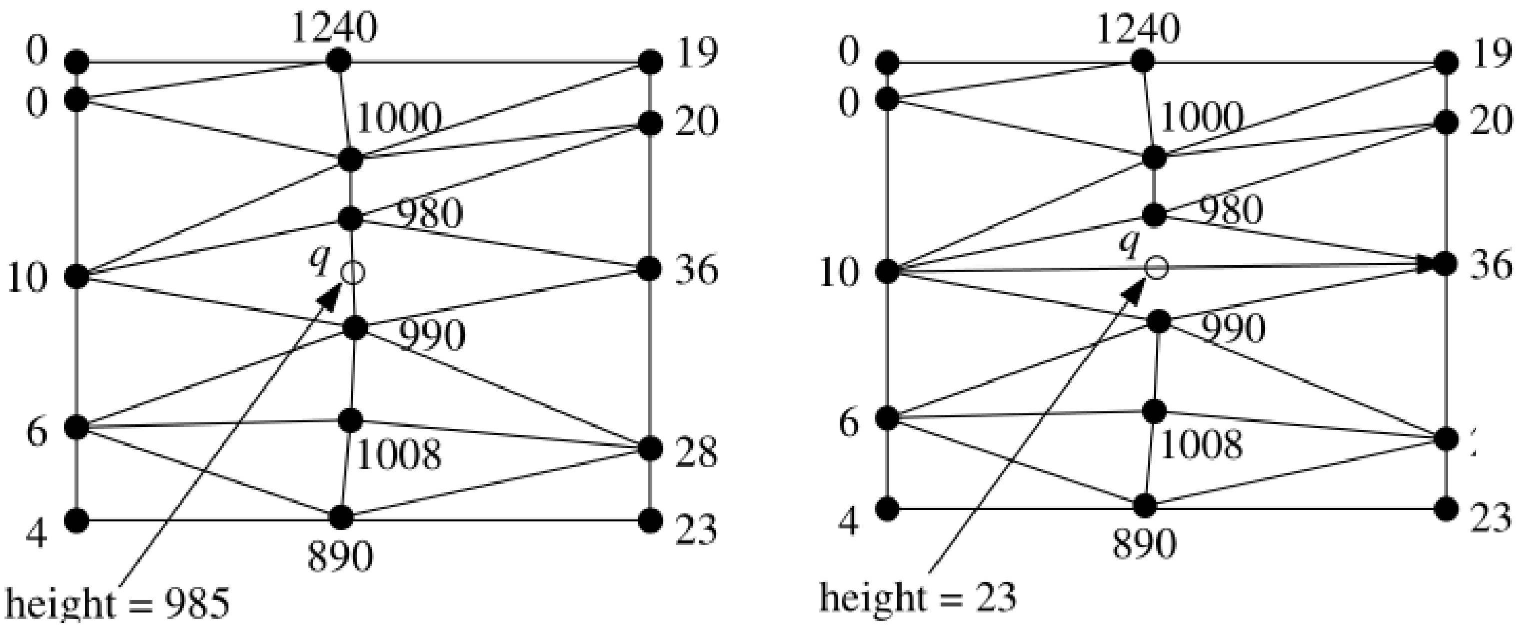
- Given irregularly distributed positions without connectivity information
- For a set of points many triangulations exist
- The challenge is to find the connectivity so that a "good" triangulation is generated



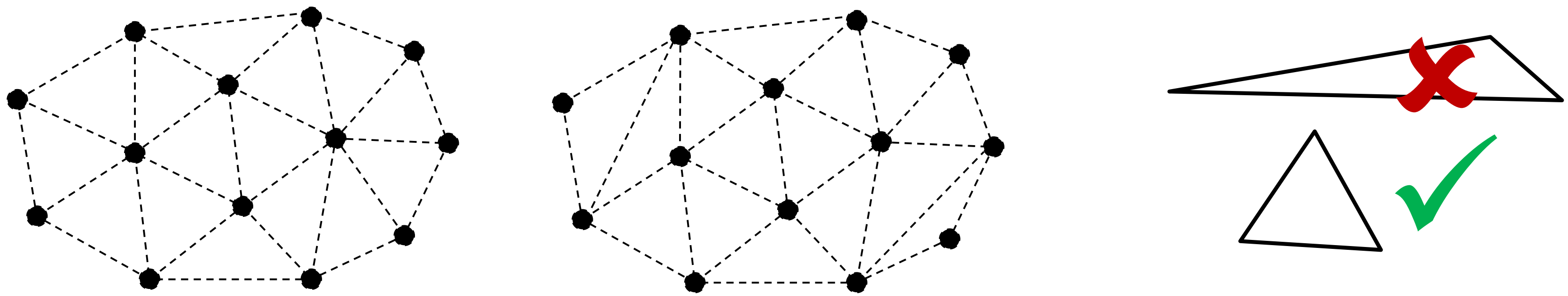
Generating a triangulation

- What is a good triangulation?



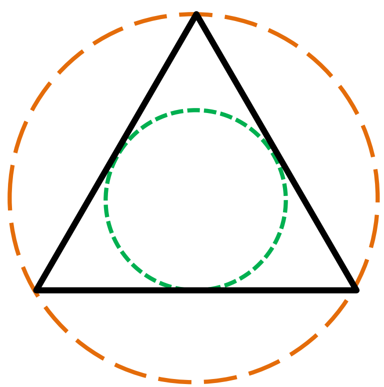
Generating a triangulation

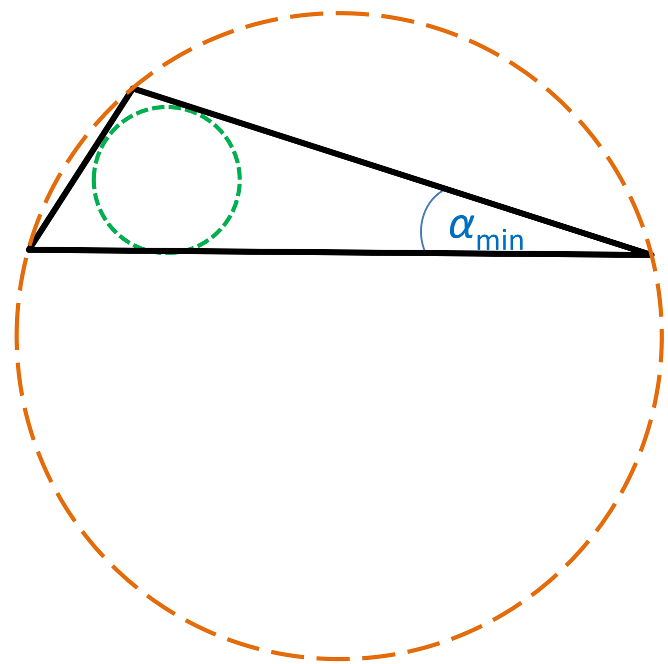
- What is a good triangulation?
  - A measure for the quality of a triangulation is the aspect ratio of the so-defined triangles
  - Avoid long, thin triangles
  - Make triangles as “round” as possible



Generating a triangulation

- An “optimal” triangulation
  - Makes triangles as “round” as possible
  - Maximizes the minimum angle in the triangulation
  - Maximizes $\dfrac{\text{radius of in-circle}}{\text{radius of circumcircle}}$

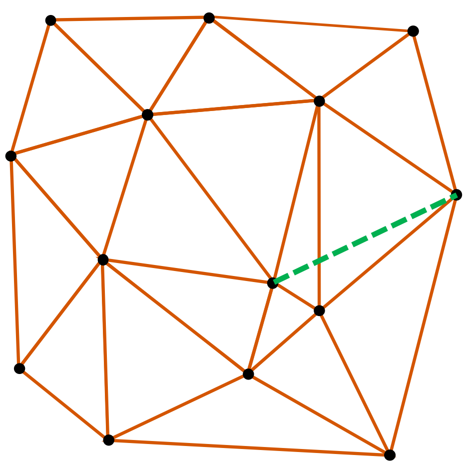


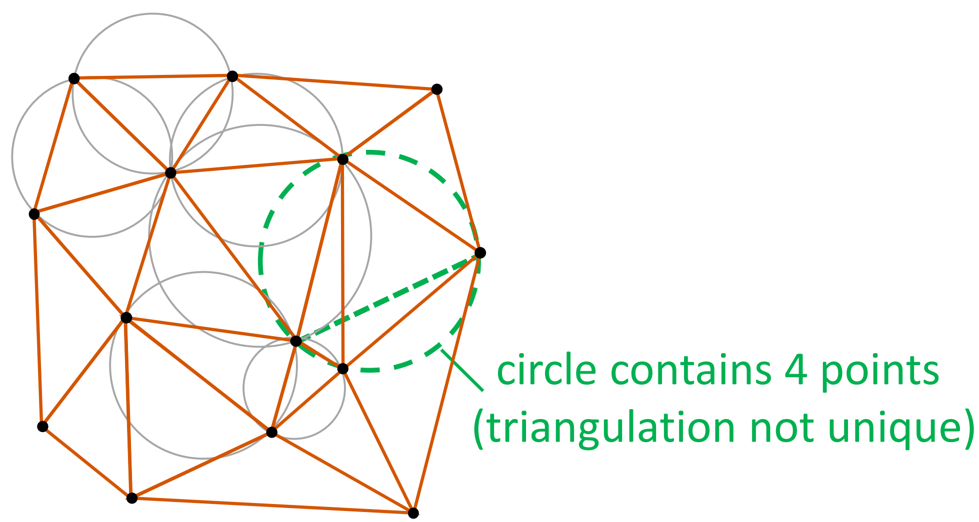


- A Delaunay triangulation is an optimal triangulation

Delaunay triangulation

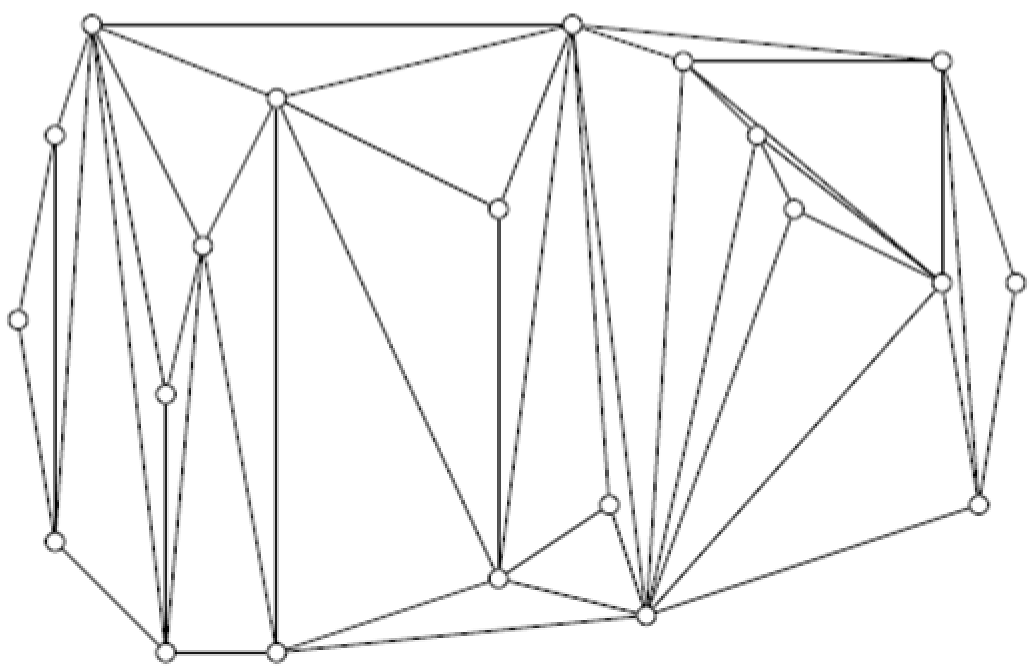
- Delaunay triangulation
  - The circumcircle of any triangle does not contain another point of the set
    - Maximizes the minimum angle in the triangulation
    - Such a triangulation is unique (independent of the order of samples) for all but trivial cases





Delaunay triangulation

- How to build a Delaunay triangulation from an initial, non-optimal triangulation?
  – Can be performed by successively improving the initial triangulation via local operations



Delaunay triangulation

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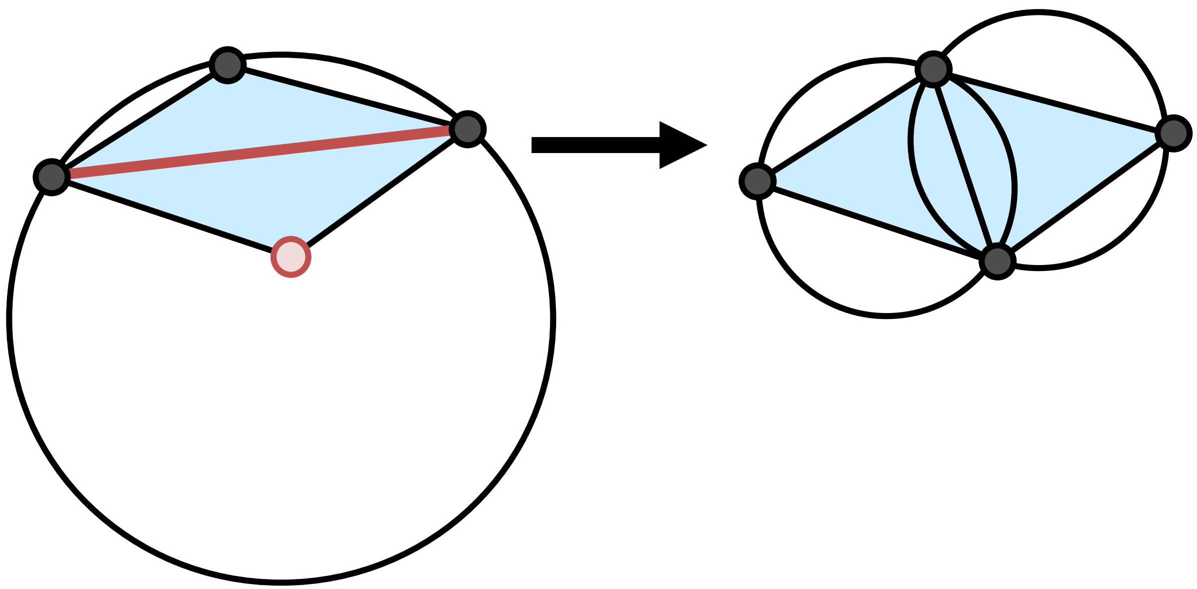
- Edge flip operation



  - An edge is local Delaunay if there exists an empty circumcircle
  - If an edge shared by two triangles is illegal, a flip operation generates a new edge that is legal!

Delaunay triangulation

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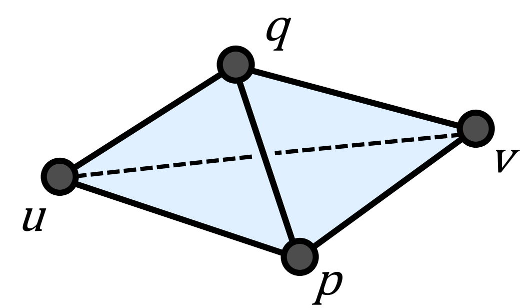
Edge flip algorithm:

```
put all edges in stack and mark them
while stack is non-empty do
  pop edge uv from stack and unmark it
  if uv is illegal then
    substitute pq for uv    //edge flip
    for ab ∈ {up, pv, vq, qu} do
      if ab is unmarked then
        push ab on the stack and mark it
      endif
    endfor
  endif
endwhile
```



Edge flip - example

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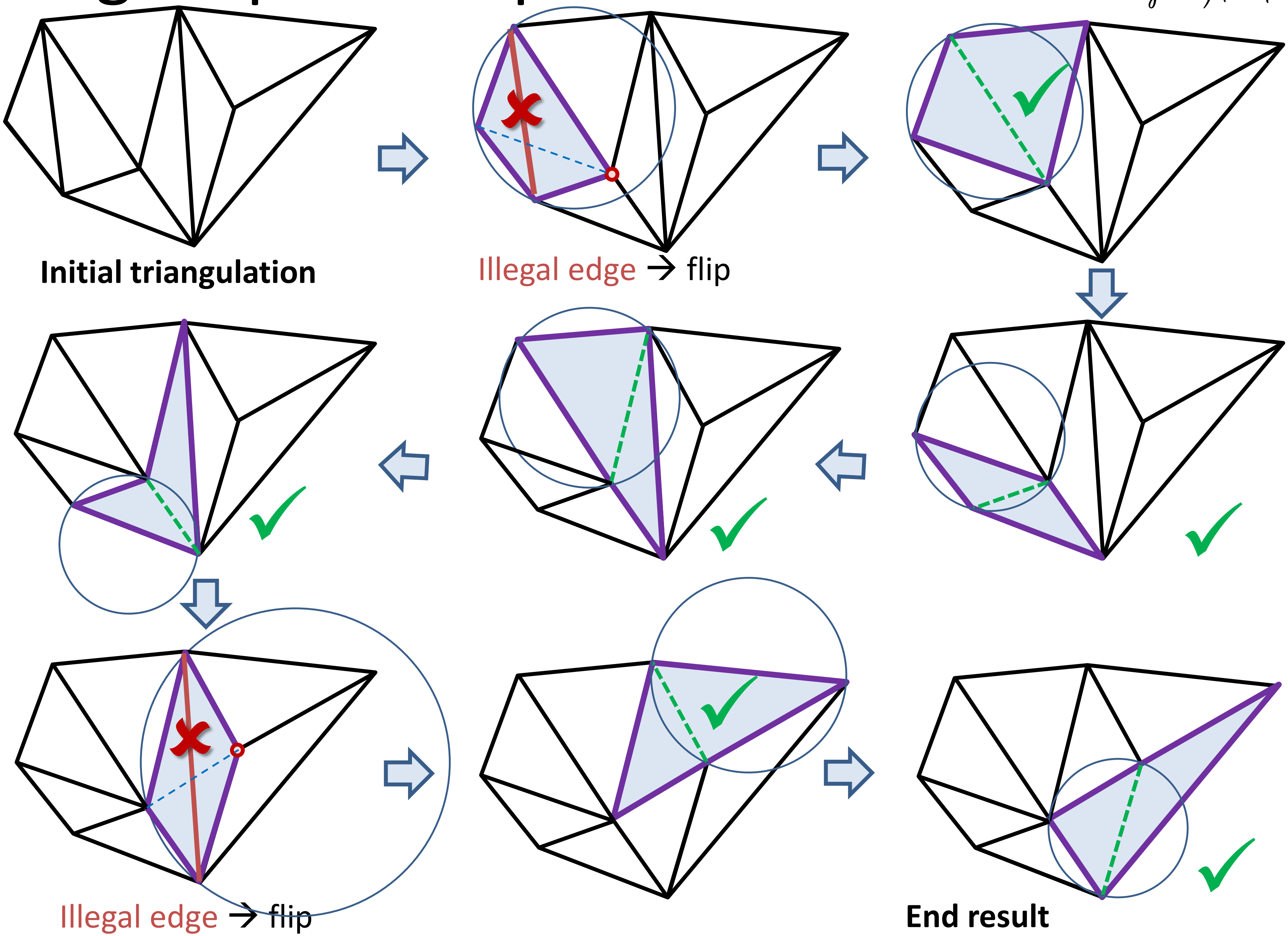
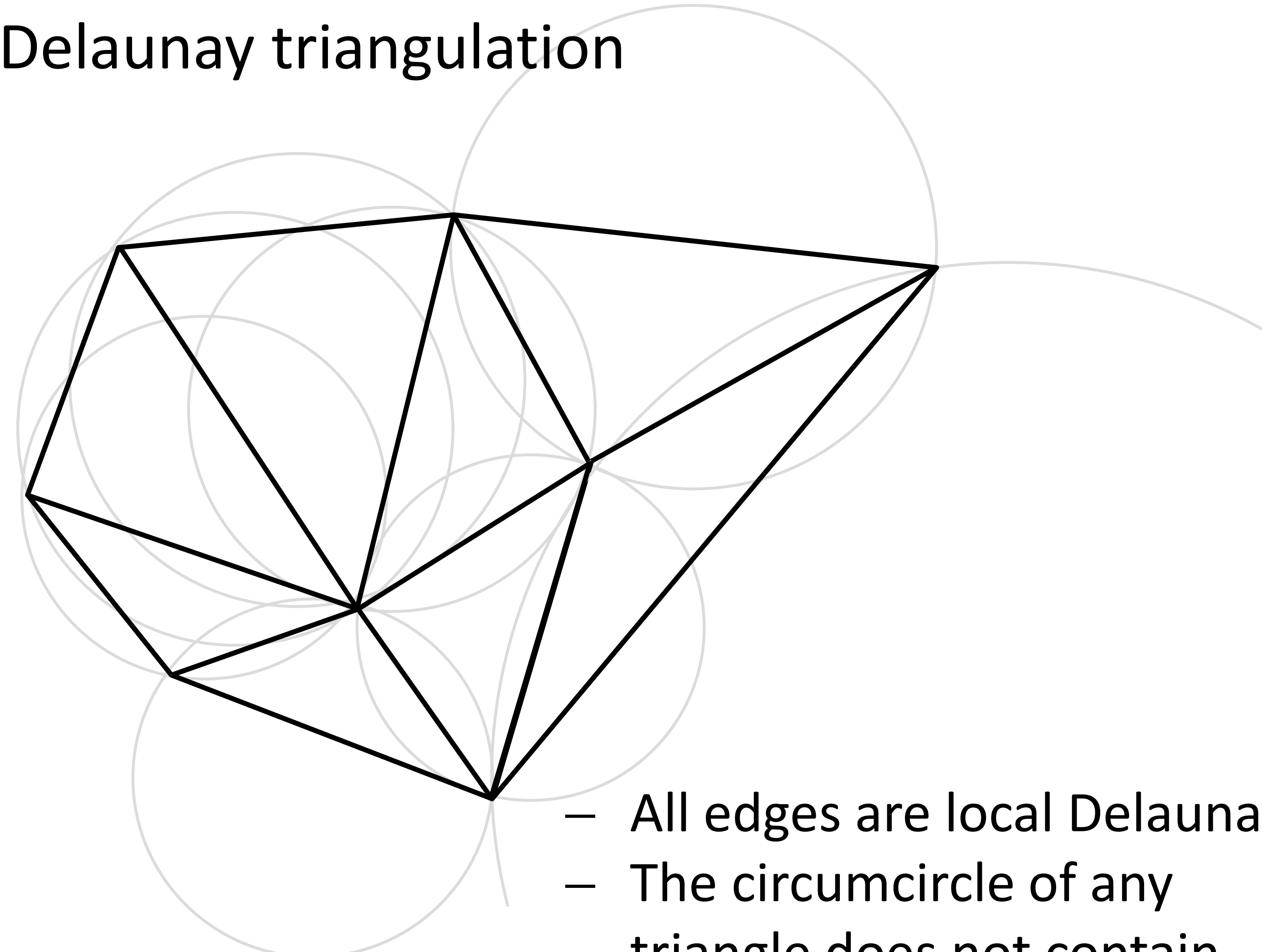


Edge flip - example

- **Result:** Delaunay triangulation

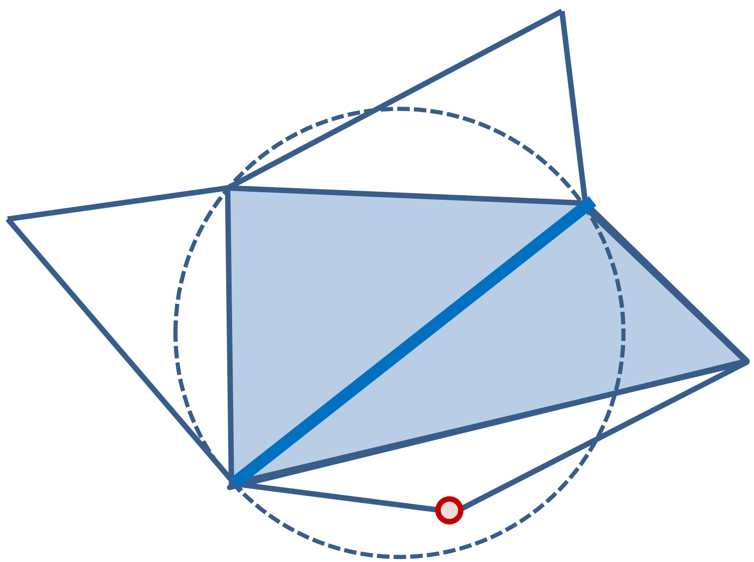


  - All edges are local Delaunay
  - The circumcircle of any triangle does not contain another point of the set

http://multivis.net/lecture/delaunay.html

Delaunay triangulation

- Local vs. global optimality
  - Edge is locally Delaunay … but not globally



Delaunay triangulation

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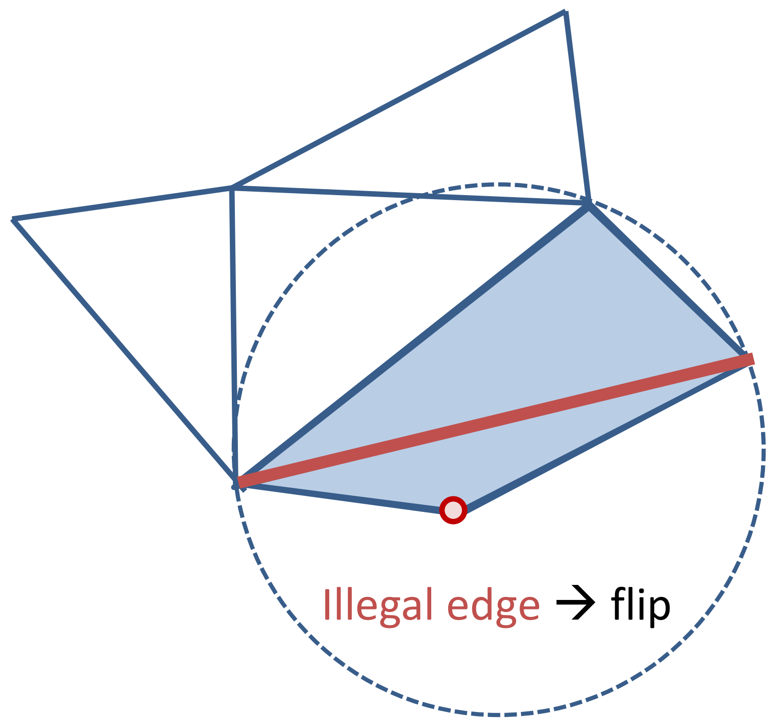
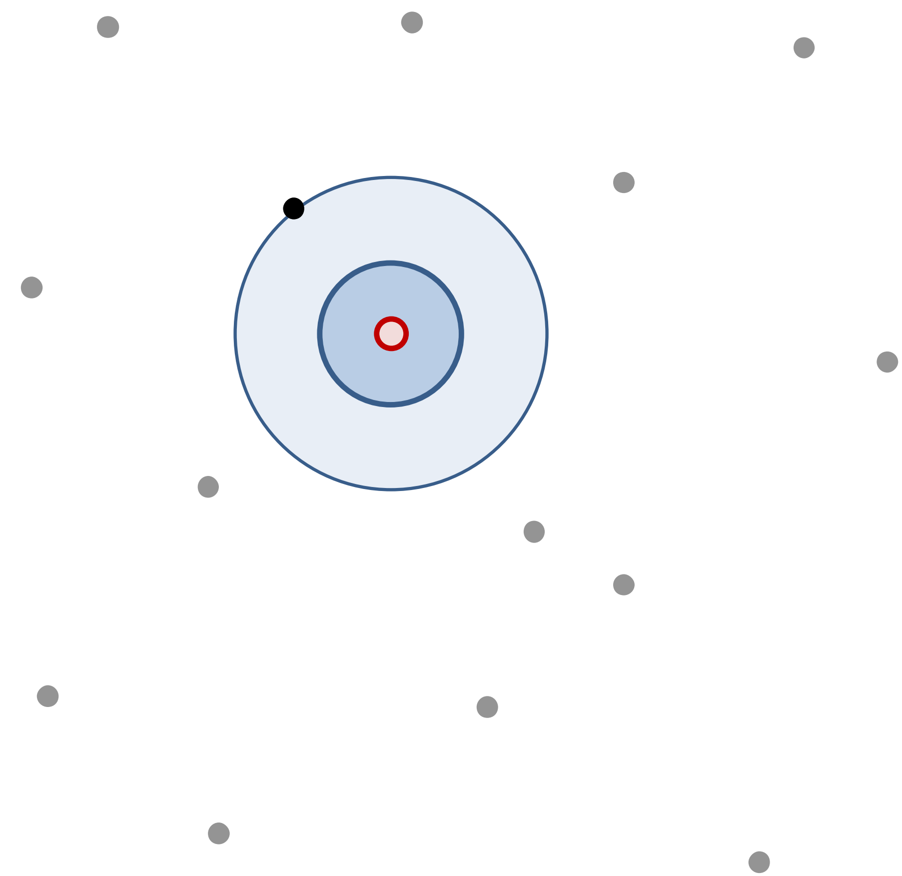
- Local vs. global optimality
  - If a triangulation is locally Delaunay everywhere → globally Delaunay



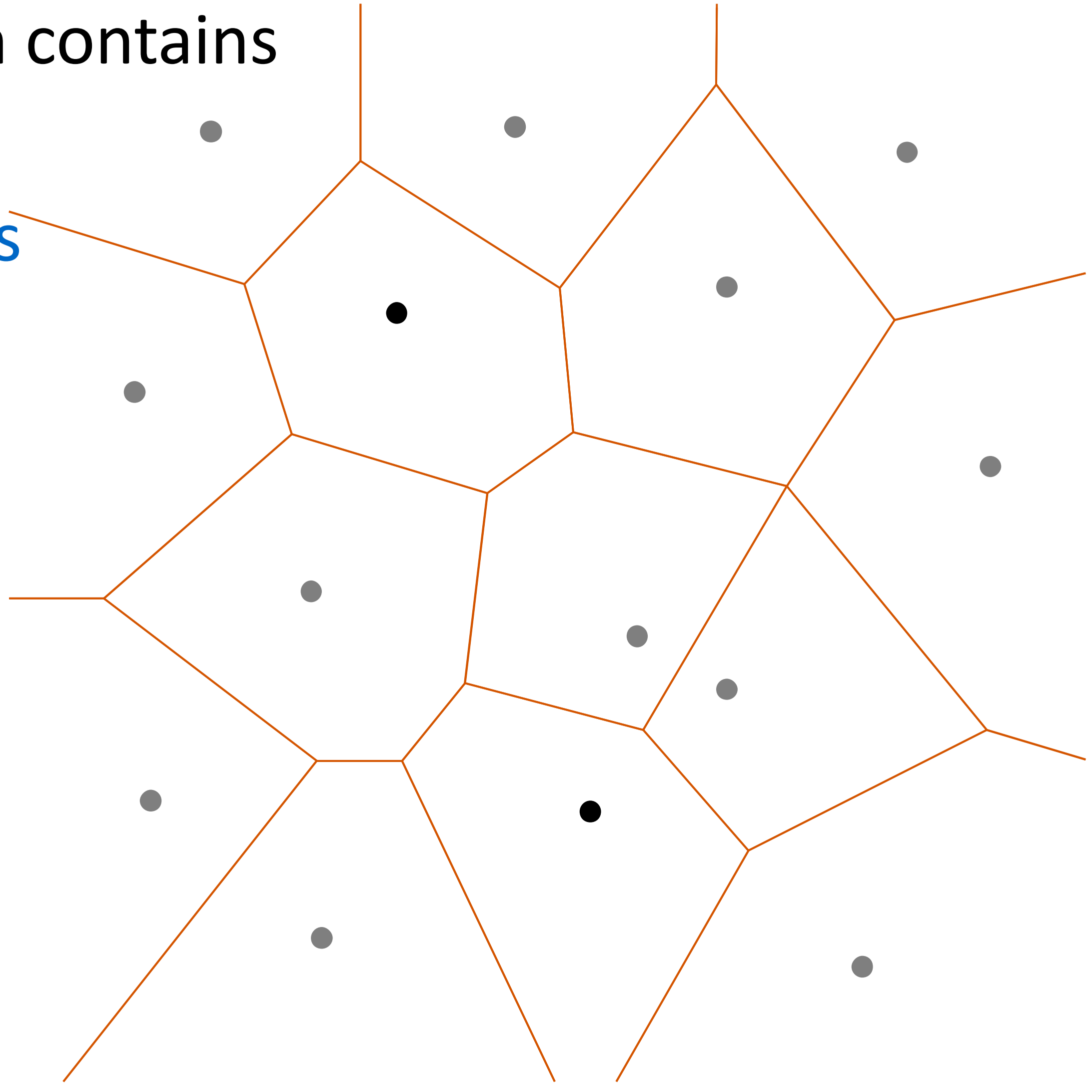
Voronoi diagram

- **Problem:** Looking for nearest neighbor



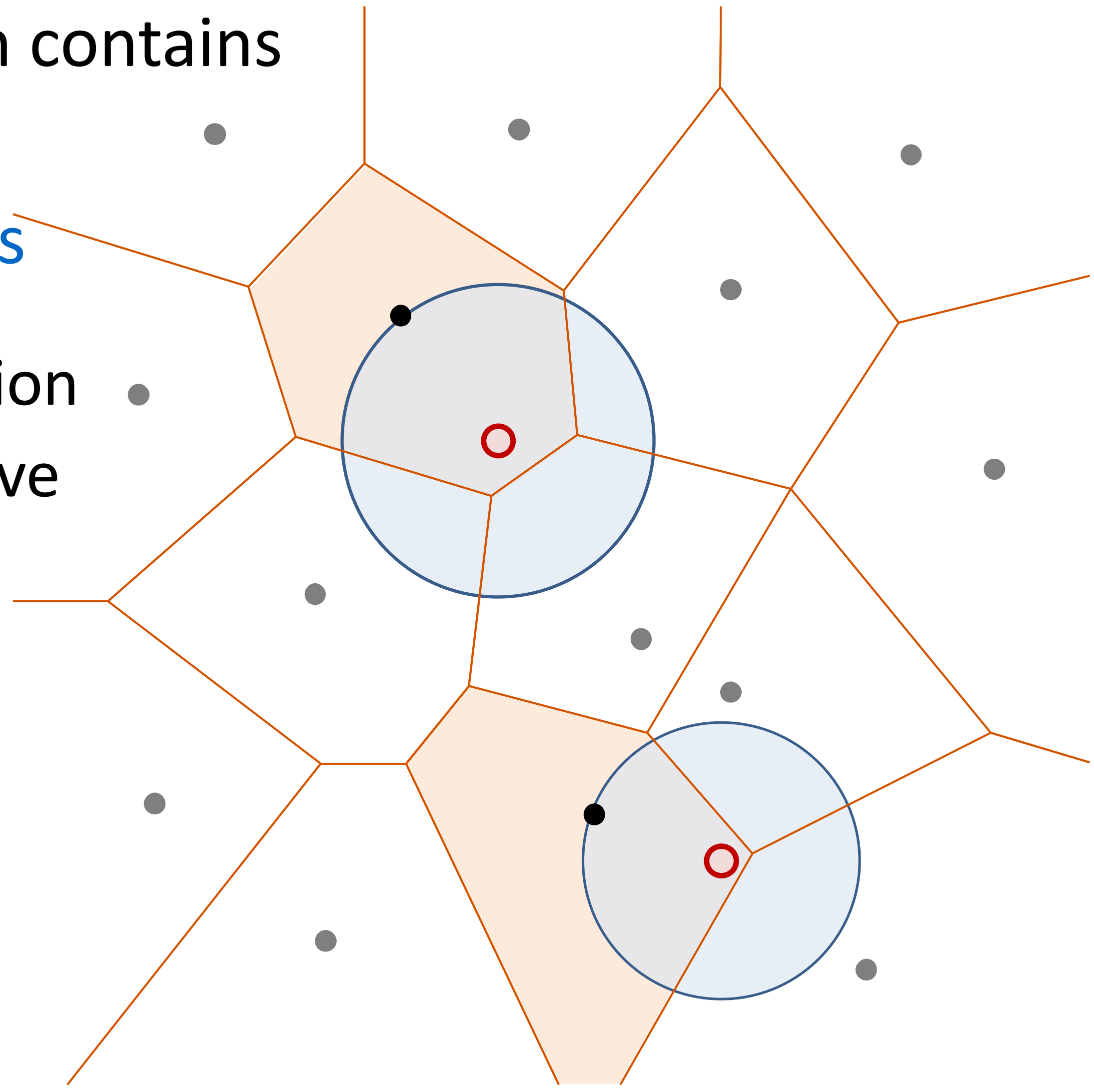
Voronoi diagram

- Partitions domain into Voronoi regions
  - Each Voronoi region contains one initial sample – the Voronoi samples



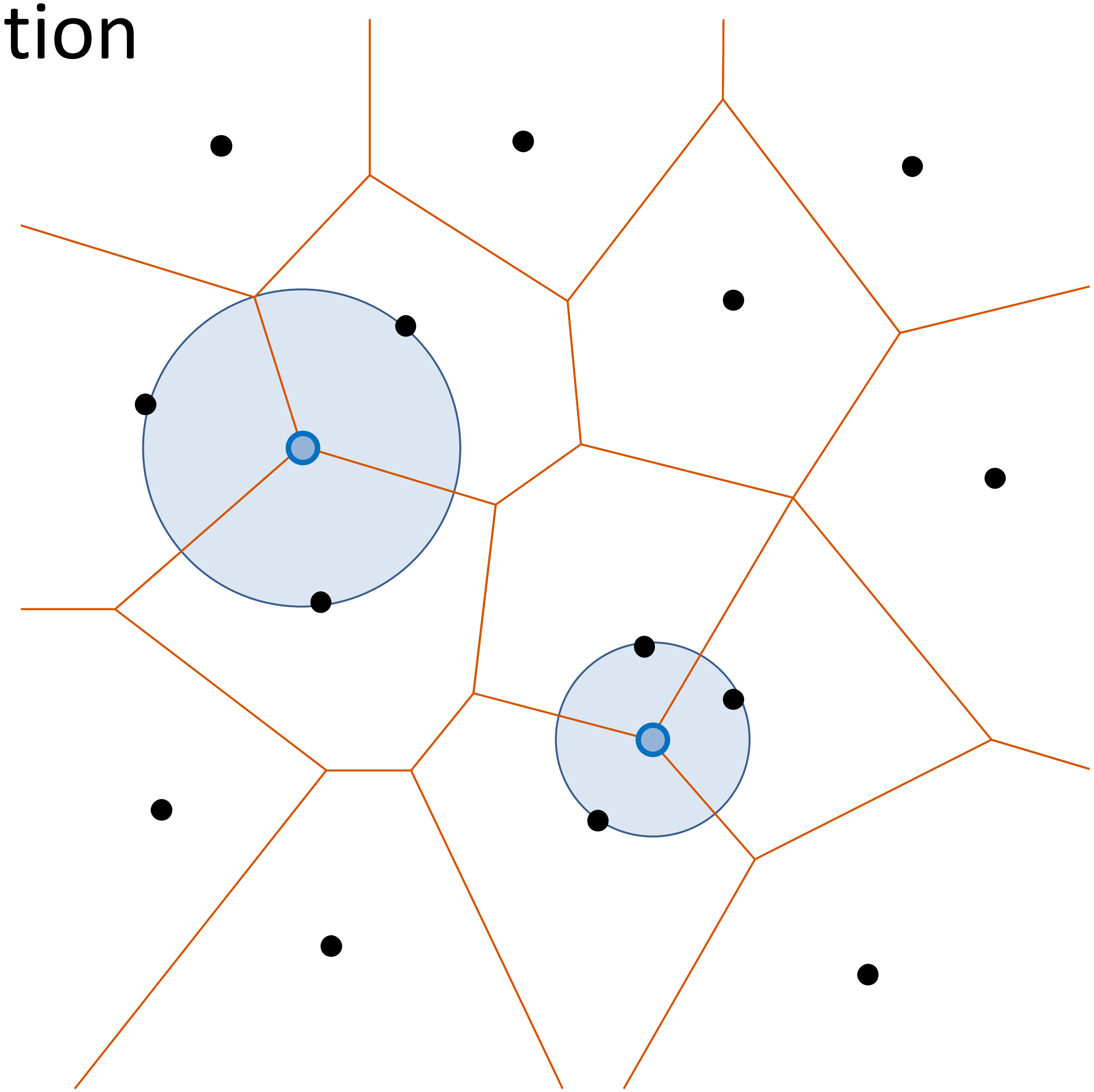
Voronoi diagram

- Partitions domain into Voronoi regions
  - Each Voronoi region contains one initial sample – the Voronoi samples
  - Points in Voronoi region are closer to respective sample than to any other sample



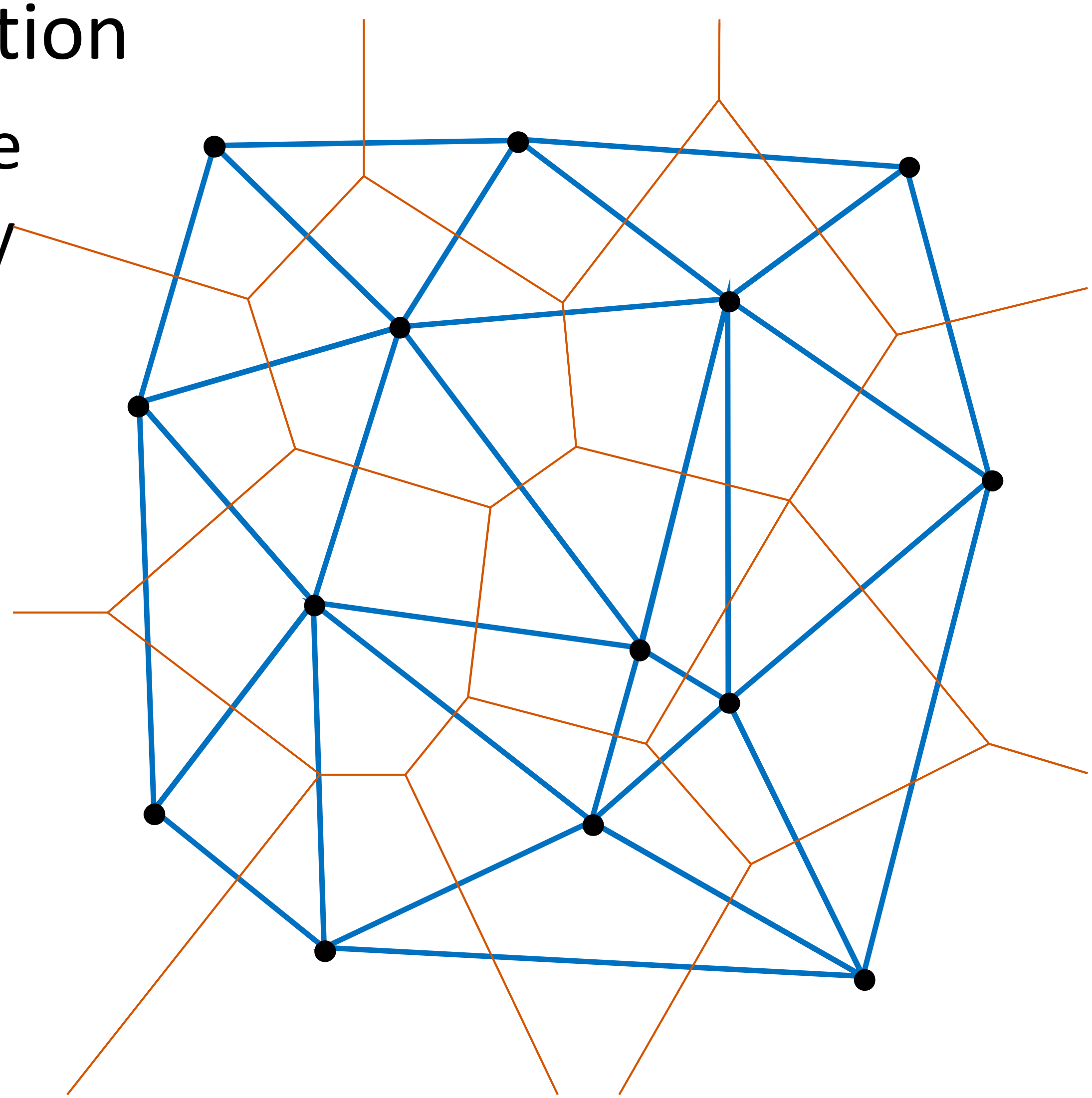
Voronoi diagram

- Centers of circumcircles of Delauney triangulation



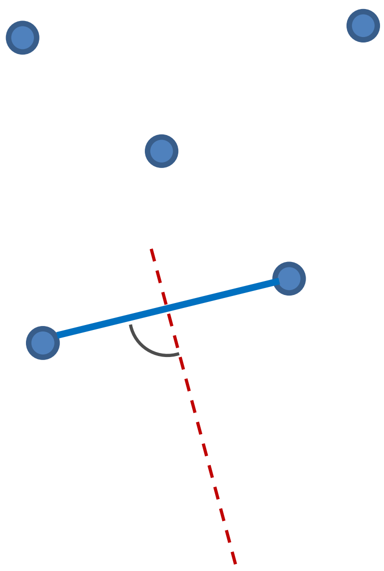
Voronoi diagram

- The geometric dual (topologically equal) of Delaunay triangulation
  – Voronoi samples are vertices in Delauney triangulation



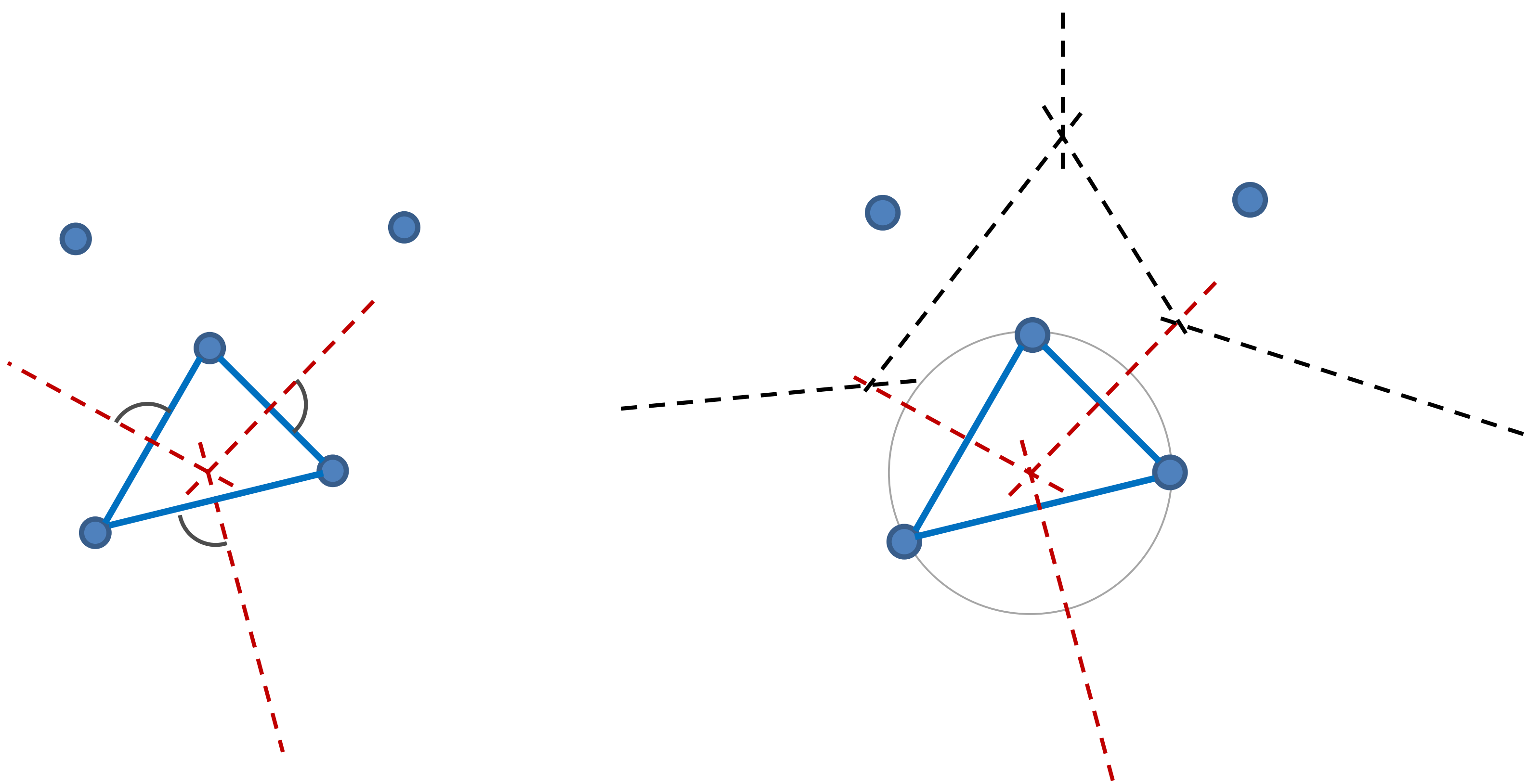
Voronoi diagram

- Construction
  - Points in a Voronoi region are closer to the respective sample than to any other sample



Voronoi diagram

- Construction
  - Points in a Voronoi region are closer to the respective sample than to any other sample



Contact information

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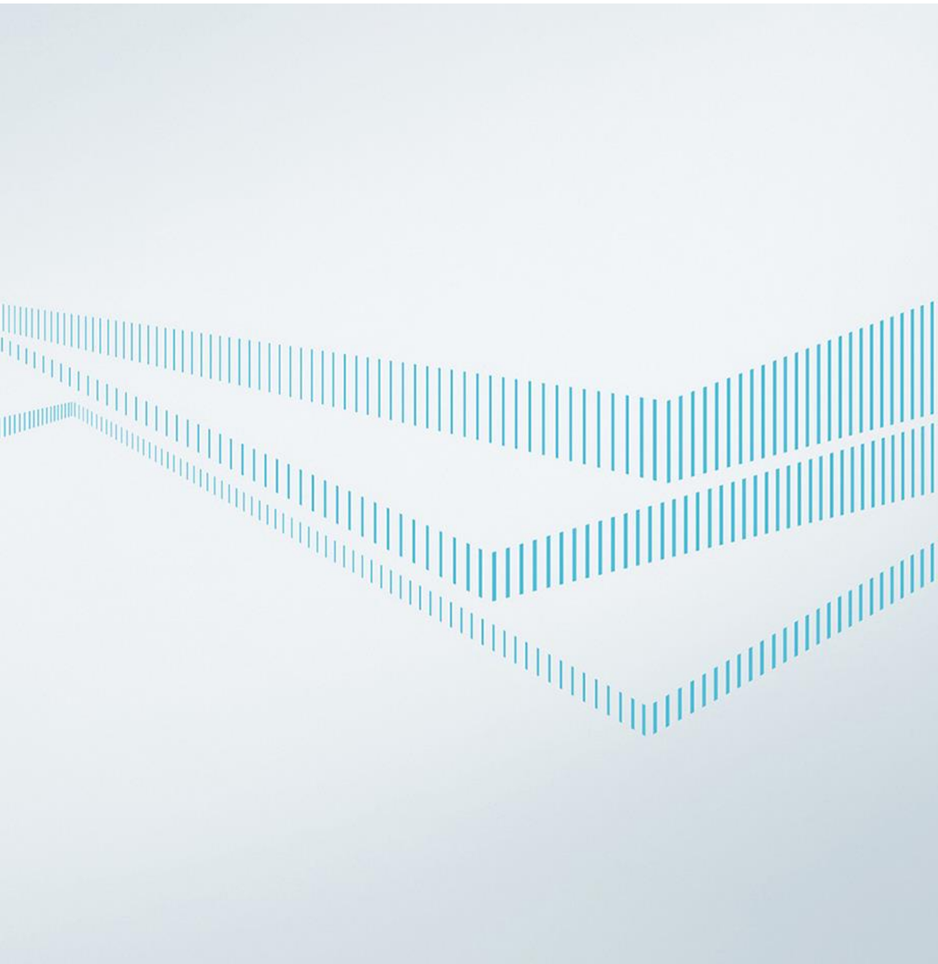

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