Solvable Structures for Hamiltonian Systems

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1 Solvable Structures for Ordinary Differential Equations

Let $U \subseteq \mathbb{R}^n$ be a open, convex connected subset of $\mathbb{R}^n$. In particular, U is a simply connected set, thus we can use Poincare Lemma whenever we need. Define

$$\mathfrak{X}(U) \to \text{Smooth Vector Fields on } U$$

$$\Omega^k(U) \to \text{Smooth } k\text{-forms on } U$$

This two set has a module structure on $C^\infty(U)$.

For $x \in U$, we have the tangent space T_xU to U at the point x. Suppose we have chosen a r dimensional subspace $\mathcal{A}_x \leq T_xU$ at every point x in U. Such a choice is called a rank r distribution over U. By choosing a basis, we can write

$$\mathcal{A} = \langle A_1, \dots, A_r \rangle$$

We say $\mathcal{A}$ is involutive if it is closed under bracket operation, i.e.

$$[\mathcal{A}, \mathcal{A}] \subseteq \mathcal{A}$$

The importance of this concept comes from the theorem of Fröbenius:

Theorem 1.1. *Let $\mathcal{A}$ be a rank r distribution on U. There exists an integral manifold of $\mathcal{A}$ at arbitrary points of U iff $\mathcal{A}$ is involutive.*

We can see from this theorem that we can always obtain a solution from our distribution.

Now, we will define a symmetry of a rank r involutive distribution $\mathcal{A} = \langle A_1, \dots, A_r \rangle$ as a vector field $Y \in \mathfrak{X}(U)$ satisfying

- $A_1, \dots, A_r, Y$ are pointwise linearly independent on U
- $[Y, A_i] \in \mathcal{A}$ for $i = 1, \dots, r$

This definition simply states that a vector field outside of an involutive distribution is a symmetry if it remains in the distribution after applying it to a basis of the distribution. Since we're still in our structure after we apply an independent vector field, we call it a symmetry of our distribution.

This notion of a symmetry is a generalization of a notion of solvable Lie algebra.

Now, suppose we have a rank r involutive distribution $\mathcal{A} = \langle A_1, \ldots, A_r \rangle$ with a symmetry Y_1. Then, we can form the rank $(r+1)$ involutive distribution

$$\mathcal{A}^{(1)} = \langle A_1, \ldots, A_r, Y_1 \rangle$$

Now suppose Y_2 is a symmetry of $\mathcal{A}^{(1)}$. Then, we can form the rank $(r+2)$ involutive distribution

$$\mathcal{A}^{(2)} = \langle A_1, \ldots, A_r, Y_1, Y_2 \rangle$$

Suppose we are able to complete this process and find $n-r$ vector fields

$$\{Y_1, \ldots, Y_{n-r}\}$$

We will call this as a solvable structure on $\mathcal{A}$.

Let's give a formal definition:

Let $\mathcal{A} = \langle A_1, \ldots, A_r \rangle$ be a rank r involutive distribution on $U \subseteq \mathbb{R}^n$. The ordered system of vector fields $\{A, Y_1, \ldots, Y_{n-r}\}$ is called a solvable structure for $\mathcal{A}$ if

- Y_1 is a symmetry of $\mathcal{A}$,
- Y_k is a symmetry of the involutive rank $(r-k+1)$ distribution $\langle A_1, \ldots, A_r, Y_1, \ldots, Y_{k-1} \rangle$ for $k = 1, 2, \ldots, n-r$.

Now we can consider a system of ODEs which is determined by the vector field A. The solutions to this system is given by the integral curves of the A, which are the integral manifolds of the distribution

$$\mathcal{A} = \langle A \rangle$$

Assume $\{Y_1, \ldots, Y_{n-1}\}$ is a solvable structure for $\mathcal{A}$.

Define $\lambda \in C^\infty(U)$ by

$$\lambda = Y_{n-1} \lrcorner \cdots \lrcorner Y_1 \lrcorner A \lrcorner \tau$$

Define a sequence of 1-forms $\omega_i \in \Omega^1(U)$ by

$$\omega_i = \frac{1}{\lambda} Y_{n-1} \lrcorner \cdots \lrcorner \hat{Y}_i \lrcorner \cdots \lrcorner Y_1 \lrcorner A \lrcorner \tau$$

where τ is the volume form on U and $\hat{Y}_i$ denotes the omission of the vector field Y_i.

Clearly, one have

$$\omega_i(Y_j) = \pm\delta_{ij}$$

$$\omega_i(A) = 0$$

In particular, ω_i's are linearly independent and

$$\mathcal{A} = \ker\omega_1 \cap \ker\omega_2 \cap \cdots \cap \ker\omega_{n-1}$$

Thus, in order to solve the system of differential equations we are considering, we should solve the system of Pfaffian equations

$$\omega_1 = \omega_2 = \cdots = \omega_n = 0$$

Let's look at the ω_i's more closely. First, recall the formula

$$d\omega_i(X,Y) = X d\omega_i(Y) - Y d\omega_i(X) - \omega_i([X,Y]) = -\omega_i([X,Y])$$

The last equality holds in the case $X, Y \in \{A, Y_1, \ldots, Y_{n-1}\}$, which is a basis for the tangent spaces of U.

Now, we have

$$d\omega_{n-1}(X,Y) = -\omega_{n-1}([X,Y]) = 0$$

since necessarily one have $[X,Y] \in \langle A, Y_1, \ldots, Y_{n-2}\rangle$.

$$d\omega_{n-2}(X,Y) = -\omega_{n-2}([X,Y]) \neq 0 \Rightarrow Y_{n-2} \in [X,Y] \Rightarrow Y_{n-1} \in X \quad \text{or} \quad Y_{n-1} \in Y$$

These implies that

$$\begin{aligned} d\omega_{n-1} &= 0 \\ d\omega_{n-2} &= \theta \wedge \omega_{n-1} \end{aligned}$$

and in general

$$d\omega_i = \sum_{j=i+1}^{n-1} \theta_{ij} \wedge \omega_j$$

where $\theta_{ij} \in \Omega^1(U)$. Then, by Poincare lemma, we have

$$\begin{aligned} d\omega_{n-1} = 0 &\Rightarrow \omega_{n-1} = dI_{n-1}, \qquad I_{n-1} \in C^\infty(U) \\ \omega_{n-1} = 0 &\Rightarrow M_{n-1} = \{x \in U \mid I_{n-1} = c_{n-1}\} \end{aligned}$$

On the submanifold M_{n-1}, one have

$$\begin{aligned} d\omega_{n-2}|_{M_{n-1}} = 0 &\Rightarrow \omega_{n-2}|_{M_{n-1}} = dI_{n-2}, \qquad I_{n-2} \in C^\infty(U) \\ \omega_{n-2}|_{M_{n-1}} = 0 &\Rightarrow M_{n-2} = \{x \in U \mid I_{n-1} = c_{n-1}, \quad I_{n-2} = c_{n-2}\} \end{aligned}$$

In that way, we find a sequence of submanifolds

$$M_{n-1} \supset M_{n-2} \supset \cdots \supset M_1, \qquad \dim M_k = k$$

Where,

$$M_1 = \{x \in U \mid I_{n-1} = c_{c-1} \ , I_{n-2} = c_{n-2} \ , ..., \ I_1 = c_1\}$$

corresponds to the solution of the system $\omega_1 = \cdots = \omega_n = 0$.

2 Review of Classical Mechanics

In classical mechanics, the state of a system is specified by a point $x = (q, p) \in \mathbb{R}^{2n}$ in phase space where $q = (q_1, \dots, q_n)$ are generalized coordinates and $p = (p_1, \dots, p_n)$ are conjugate momenta, satisfying the Hamiltonian system of equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, \dots, n$$

Where the the dot represents the time derivative and $H = H(q, p) \in C^\infty(U)$ is the total energy of the system. Solutions to the Hamilton's equations are the integral curves of the vector field

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$$

which is called the Hamiltonian vector field associated with H.

Now, one can introduce the Poisson bracket $\{\ ,\ \} : C^\infty(U) \times C^\infty(U) \to C^\infty(U)$ as

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}$$

The time evolution of a smooth function $F \in C^\infty(U)$ along the flow of X_H is given by

$$\dot{F} = \mathcal{L}_{X_H} F = \{F, H\}$$

The Hamiltonian vector fields have the property that

$$[X_F, X_H] = -X_{\{F,H\}}$$

Thus, we can say F is an integral of motion if X_F is a symmetry of X_H

Let X_H be a Hamiltonian vector field on $U \subseteq \mathbb{R}^{2n}$. We say X_H is completely integrable if there exists n integrals $F_i \in C^\infty(U)$ such that

- $\{F_i, H\} = \{F_i, F_j\} = 0$
- $dF_1, dF_2, \dots, dF_n$ are linearly independent on U

Notice that the last condition is equivalent to the functional independency of the functions $F_1, \dots, F_n$. Integral curves of X_H lies in the submanifolds

$$M_c = \{(q,p) \in \mathbb{R}^{2n} \mid F_i(q,p) = c_i\}$$

If these submanifolds are compact and connected, according to Arnold-Liouville theorem, $M_c \simeq T^n$ and there exists a symplectic transformation $(q,p) \mapsto (Q,P)$ such that

$$H = H(P_1, P_2, \dots, P_n)$$

Then, one can solve the Hamilton's equations of motion as

$$Q_i(t) = \frac{\partial H}{\partial P_i} t + Q_i(0) \qquad P_i(t) = P_i(0)$$

In the new coordinates, the functions F_i are also functions of $P_1, \dots, P_n$.

Now, we are able to integrate the completely integrable Hamiltonian systems by quadratures. However, in most cases, finding action-angle variables explicitly is difficult. This is why we introduced the solvable structures.

3 Solvable Structures for Hamiltonian Systems

Let X_H be a completely integrable Hamiltonian vector field on $U \subseteq \mathbb{R}^{2n}$ with corresponding integral of motions $F_1, \dots, F_n$. Our objective is to construct a solvable structure for

$$\mathcal{A} = \langle A \rangle = \langle \partial_t + X_H \rangle$$

on the extended phase space $\mathbb{R}^{2n+1}$. We will try to find a solvable structure

$$\{X_{F_1}, \dots, X_{F_n}, X_{G_1}, \dots, X_{G_n}\} \qquad (*)$$

for some unknown functions $G_i = G_i(q,p)$. We have

- $[X_{F_i}, A] = [X_{F_i}, X_{F_j}] = 0$

and we show that we can choose G_i such that

- $[X_{G_i}, X_{G_j}] = 0 \quad (1)$
- $[X_{G_i}, X_{F_j}] = \sum_{l=1}^{n} f_{ij}^l X_{F_l}$
- $[X_{G_i}, A] = \sum_{l=1}^{n} h_{il} X_{F_l}$

for some $f_{ij}^l, h_{il} \in C^\infty(U)$. Then, $(*)$ becomes a solvable structure.

Notice that the commutation relations are invariant under $(q,p) \mapsto (Q,P)$ since

$$\varphi_*[X_F, X_G] = [X_{\varphi_* F}, X_{\varphi_* G}]$$

Hence, we may assume all our functions are expressed in action-angle variables. Then, we have

$$X_H = \sum_{k=1}^{n} \frac{\partial H}{\partial P_k} \frac{\partial}{\partial Q_k} \qquad X_{F_i} = \sum_{k=1}^{n} \frac{\partial F_i}{\partial P_k} \frac{\partial}{\partial Q_k} \qquad X_{G_i} = \sum_{k=1}^{n} \frac{\partial G_i}{\partial P_k} \frac{\partial}{\partial Q_k} - \frac{\partial G_i}{\partial Q_k} \frac{\partial}{\partial P_k}$$

Now, we have

$$(1) \iff \{G_i, G_j\} = \sum_{k=1}^{n} \frac{\partial G_i}{\partial Q_k} \frac{\partial G_j}{\partial P_k} - \frac{\partial G_j}{\partial Q_k} \frac{\partial G_i}{\partial P_k} = c_{ij} \in \mathbb{R} \quad (1')$$

Now, if we define

$$H_{ij} = \{F_j, G_i\}$$

Then (2) reads as

$$(2) \iff \sum_{k=1}^{n} \frac{\partial H_{ij}}{\partial P_k} \frac{\partial}{\partial Q_k} - \frac{\partial H_{ij}}{\partial Q_k} \frac{\partial}{\partial P_k} = \sum_{k=1}^{n} \left(\sum_{l=1}^{n} f_{ij}^{l} \frac{\partial F_l}{\partial P_k} \right) \frac{\partial}{\partial Q_k}$$

$$\iff \frac{\partial H_{ij}}{\partial Q_k} = 0 \qquad \frac{\partial H_{ij}}{\partial P_k} = \sum_{l=1}^{n} f_{ij}^{l} \frac{\partial F_l}{\partial P_k}$$

$$\iff \sum_{l=1}^{n} \frac{\partial^2 G_i}{\partial Q_k \partial Q_l} \frac{\partial F_j}{\partial F_l} = 0 \quad (2') \qquad -\sum_{l=1}^{n} \frac{\partial}{\partial P_k} \left(\frac{\partial G_i}{\partial Q_l} \frac{\partial F_j}{\partial F_l} \right) = \sum_{l=1}^{n} f_{ij}^{l} \frac{\partial F_l}{\partial P_k} \quad (2'')$$

Now, we can solve (2′) as

$$(2') \iff G_i = \sum_{j=1}^{n} g_{ij}(P) Q_j + \alpha_i$$

due to the nonsingularity of the Jacobian matrix. Then, we have

$$(1') \iff \sum_{l=1}^{n} \sum_{k=1}^{n} \left(g_{ik} \frac{\partial g_{jl}}{\partial P_k} - g_{jk} \frac{\partial g_{il}}{\partial P_k} \right) Q_l = c_{ij}$$

$$\iff \sum_{k=1}^{n} \left(g_{ik} \frac{\partial g_{jl}}{\partial P_k} - g_{jk} \frac{\partial g_{il}}{\partial P_k} \right) = 0$$

For example, one can choose $g_{ij}(P) = \alpha_{ij} \in \mathbb{R}$ to solve this system.
Now, (2″) reduces to

$$(2'') \iff -\sum_{l=1}^{n} \frac{\partial}{\partial P_k} \left(g_{il} \frac{\partial F_j}{\partial P_l} \right) = \sum_{l=1}^{n} f_{ij}^{l} \frac{\partial F_l}{\partial P_k}$$

We can solve f_{ij}^l from there since the Jacobian matrix is nonsingular.

Now, we must deal with (3), which turns into

$$(3) \iff -\sum_{k=1}^{n} \frac{\partial}{\partial P_j}\Big(g_{ik}\,\frac{\partial H}{\partial P_k}\Big) = \sum_{k=1}^{n} h_{ik}\,\frac{\partial F_k}{\partial P_j}$$

which can be solved for h_{ij} since the Jacobian matrix is nonsingular.

Finally, we must show $\{A, X_{F_1}, \ldots, X_{F_n}, X_{G_1}, \ldots, X_{G_n}\}$ is linearly independent set. Since A involves ∂_t, $A \notin \langle X_{F_1}, \ldots, X_{F_n}, X_{G_1}, \ldots, X_{G_n}\rangle$. Now, assume

$$a_1 X_{F_1} + \cdots + a_n X_{F_n} + b_1 X_{G_1} + \cdots + b_n X_{G_n} = 0$$

Expanding this gives

$$\sum_{i=1}^{n} a_i \frac{\partial F_i}{\partial P_k} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_i \frac{\partial g_{ij}}{\partial P_k} Q_j = 0 \quad (*)$$

$$\sum_{i=1}^{n} b_i g_{ik} = 0 \quad (**)$$

Assuming $\det[g_{ij}] \neq 0$ on U, we get

$$(**) \Rightarrow b_i = 0$$

and since the Jacobian matrix is nonsingular

$$(*) \Rightarrow a_i = 0$$

Thus, we have proved that completely integrable Hamiltonian systems admits a solvable structure.

Thus, we may apply the machinery of solvable structures to completely integrable systems.

First, we can choose $g_{ij}(P) = \delta_{ij}$, giving

$$X_H = \sum_{k=1}^{n} \frac{\partial H}{\partial P_k}\frac{\partial}{\partial Q_k} \qquad X_{F_i} = \sum_{k=1}^{n} \frac{\partial F_i}{\partial P_k}\frac{\partial}{\partial Q_k} \qquad X_{G_i} = -\frac{\partial}{\partial P_i}$$

Let τ be the volume form on $\mathbb{R}^{2n+1}$:

$$\tau = dt \wedge dQ_1 \wedge \cdots \wedge dQ_n \wedge dP_1 \wedge \cdots \wedge dP_n$$

and define $\lambda \in C^\infty(U)$ as

$$\lambda = X_{G_n} \lrcorner \cdots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \cdots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau$$

$$= dt(A) dP_1\left(-\frac{\partial}{\partial P_1}\right) \ldots dP_n\left(-\frac{\partial}{\partial P_n}\right) \times dQ_1 \wedge \cdots \wedge dQ_n(X_{F_1}, \ldots, X_{F_n})$$

$$= (-1)^n \left| \frac{\partial(F_1,\dots,F_n)}{\partial(P_1,\dots,P_n)} \right|$$

Since $F_1,\dots,F_n$ are functionally independent over U, we have $\lambda \neq 0$.

The Pfaffian forms are given by

$$\omega_k = \frac{1}{\lambda} X_{G_n} \lrcorner \cdots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \cdots \lrcorner \hat{X}_{F_k} \lrcorner \cdots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau$$

$$\omega_{n+k} = \frac{1}{\lambda} X_{G_n} \lrcorner \cdots \lrcorner \hat{X}_{G_k} \lrcorner \cdots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \cdots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau$$

for $k = 1, 2, \dots, n$.

Now, ω_{n+k} can be evaluated as

$$\omega_{n+k} = \frac{1}{\lambda} X_{G_n} \lrcorner \cdots \lrcorner \hat{X}_{G_k} \lrcorner \cdots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \cdots \lrcorner X_{F_1} \lrcorner A \lrcorner (-1)^{n-k} dt \wedge dQ_1 \wedge \cdots \wedge dP_k$$

$$= \frac{1}{\lambda} (-1)^{n-1} (-1)^{n-k} dQ_1 \wedge \cdots \wedge dQ_n (X_{F_1},\dots,X_{F_n}) = (-1)^{n+k+1} dP_k$$

while ω_k is given by

$$\omega_k = \frac{1}{\lambda} X_{G_n} \lrcorner \cdots \lrcorner X_{G_1} \lrcorner X_{F_n} \lrcorner \cdots \lrcorner \hat{X}_{F_k} \lrcorner \cdots \lrcorner X_{F_1} \lrcorner A \lrcorner \tau$$

$$= \frac{(-1)^n}{\lambda} \left(\left| \frac{\partial(H,F_1,\dots,\hat{F}_k,\dots,F_n)}{\partial(P_1,\dots,P_n)} \right| dt + \sum_{j=1}^{n} (-1)^j \left| \frac{\partial(F_1,\dots,\hat{F}_k,\dots,F_n)}{\partial(P_1,\dots,\hat{P}_k,\dots,P_n)} \right| dQ_j \right)$$

The integral curves of $A = \partial_t + X_H$ are solutions to $\omega_1 = \cdots = \omega_n = \omega_{n+1} = \cdots = \omega_{2n} = 0$. Now, we have

$$\omega_{n+1} = \cdots = \omega_{2n} = 0 \to N = \{(t,Q,P) \in \mathbb{R}^{2n+1} \mid P_1 = c_1, P_2 = c_2, \dots, P_n = c_n\}$$

Then, we have $\omega_k|_N = dI_k$ where

$$I_k = \frac{(-1)^n}{\lambda} \left(\left| \frac{\partial(H,F_1,\dots,\hat{F}_k,\dots,F_n)}{\partial(P_1,\dots,P_n)} \right| t + \sum_{j=1}^{n} (-1)^j \left| \frac{\partial(F_1,\dots,\hat{F}_k,\dots,F_n)}{\partial(P_1,\dots,\hat{P}_k,\dots,P_n)} \right| Q_j \right)$$

Now, defining

$$\Delta_{ij} = (-1)^{j+1} \left| \frac{\partial(F_1,\dots,\hat{F}_i,\dots,F_n)}{\partial(P_1,\dots,\hat{P}_j,\dots,P_n)} \right| \qquad B_i = \left| \frac{\partial(H,F_1,\dots,\hat{F}_i,\dots,F_n)}{\partial(P_1,\dots,P_n)} \right| t + (-1)^{n+1} \lambda I_i$$

gives

$$\sum_{j=1}^{n} \Delta_{ij} Q_j = B_i$$

Now, if (M_{ij}) is the cofactor matrix of $DF := C$, we have

$$\Delta_{ij} = (-1)^{i+1} M_{ij} \to \det \Delta = \begin{cases} (-1)^{n/2} \det M & n \text{ even} \\ (-1)^{\frac{n-1}{2}} \det M & n \text{ odd} \end{cases}$$

Now, we can solve this system as

$$Q_j = \frac{\det(\Delta_j)}{\det(\Delta)}$$

where

$$\Delta_j = \begin{bmatrix} \Delta_{11} & \dots & \Delta_{1,j-1} & B_1 & \Delta_{1,j+1} & \dots & \Delta_{1n} \\ \Delta_{21} & \dots & \Delta_{2,j-1} & B_2 & \Delta_{2,j+1} & \dots & \Delta_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{n1} & \dots & \Delta_{n,j-1} & B_n & \Delta_{n,j+1} & \dots & \Delta_{nn} \end{bmatrix}$$

The determinant $\det(\Delta_j)$ can be expressed as $\det(\Delta_j) = \det(\Delta_j^{(1)}) + \det(\Delta_j^{(2)})$ where

$$\det(\Delta_j^{(1)}) = \begin{bmatrix} \Delta_{11} & \dots & \Delta_{1,j-1} & \left|\frac{\partial(H,\hat{F}_1,F_2,\ldots,F_n)}{\partial(P_1,\ldots,P_n)}\right| & \Delta_{1,j+1} & \dots & \Delta_{1n} \\ \Delta_{21} & \dots & \Delta_{2,j-1} & \left|\frac{\partial(H,F_1,\hat{F}_2,\ldots,F_n)}{\partial(P_1,\ldots,P_n)}\right| & \Delta_{2,j+1} & \dots & \Delta_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{n1} & \dots & \Delta_{n,j-1} & \left|\frac{\partial(H,F_1,F_2,\ldots,\hat{F}_n)}{\partial(P_1,\ldots,P_n)}\right| & \Delta_{n,j+1} & \dots & \Delta_{nn} \end{bmatrix} t$$

and

$$\det(\Delta_j^{(2)}) = \begin{bmatrix} \Delta_{11} & \dots & \Delta_{1,j-1} & K_1 & \Delta_{1,j+1} & \dots & \Delta_{1n} \\ \Delta_{21} & \dots & \Delta_{2,j-1} & K_2 & \Delta_{2,j+1} & \dots & \Delta_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta_{n1} & \dots & \Delta_{n,j-1} & K_n & \Delta_{n,j+1} & \dots & \Delta_{nn} \end{bmatrix}$$

where $K_i = (-1)^{n+1} \lambda I_i$.

Using the Laplace expansion along the first row gives

$$\left|\frac{\partial(H, F_1, \ldots, \hat{F}_i, \ldots, F_n)}{\partial(P_1, P_2, \ldots, P_n)}\right| = \sum_{j=1}^{n} \frac{\partial H}{\partial P_j} (-1)^{j+1} \left|\frac{\partial(F_1, \ldots, \hat{F}_i, \ldots, F_n)}{\partial(P_1, \ldots, \hat{P}_j, \ldots, P_n)}\right| = \sum_{j=1}^{n} \Delta_{ij} \frac{\partial H}{\partial P_j}$$

Substituting this gives

$$\det(\Delta_j) = \det(\Delta)\frac{\partial H}{\partial P_j}t + \det(\Delta_j^{(2)})$$

$$\to Q_j(t) = \frac{\partial H}{\partial P_j}t + \frac{\det(\Delta_j^{(2)})}{\det(\Delta)}$$

In there, the last term is a constant, which can be determined from the initial conditions. Thus, we have

$$Q_j(t) = \frac{\partial H}{\partial P_j}t + Q_{j0}$$

Thus, we get an interpretation of Arnold-Liouville theorem in terms of solvable structures.

4 Calogero-Moser System

The rational Calogero-Moser system describes the motion of N identical particles on a line interacting with a $\sim \frac{1}{r^2}$ potential. The Hamiltonian of this system is given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + g^2 \sum_{i<j} \frac{1}{(q_i - q_j)^2}$$

For simplicity, we will consider the case $N = 2$. Then, two integral of motions can be chosen as

$$F_1 = p_1 + p_2 \qquad F_2 = p_1^2 + p_2^2 + \frac{2g^2}{(q_1 - q_2)^2}$$

and the Hamiltonian vector field is given by

$$X_H = p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} + \frac{2g^2}{(q_1 - q_2)^3}\left(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}\right)$$

We want to construct a solvable structure of the form

- $[X_{G_1}, X_{G_2}] = 0$
- $[X_{G_i}, X_{F_j}] = f_{ij}^1 X_{F_1} + f_{ij}^2 X_{F_2}$
- $[X_{G_i}, A] = h_{i1}X_{F_1} + h_{i2}X_{F_2}$

Now, we have

$$[X_{G_i}, X_{F_j}] = X_{H_{ij}} \qquad H_{ij} = \{F_j, G_i\}$$

Thus, if we have $H_{ij} = H_{ij}(F_1, F_2)$, we get

$$X_{H_{ij}} \in \langle X_{F_1}, X_{F_2} \rangle$$

Thus, we have

$$\frac{\partial G_i}{\partial q_1} + \frac{\partial G_i}{\partial q_2} = -H_{i1}(F_1, F_2)$$

$$2p_1 \frac{\partial G_i}{\partial q_1} + 2p_2 \frac{\partial G_i}{\partial q_2} + \frac{4g^2}{(q_1 - q_2)^3} \left(\frac{\partial G_i}{\partial p_1} - \frac{\partial G_i}{\partial p_2} \right) = -H_{i2}(F_1, F_2)$$

To simplify the equations, assume

$$G_1 = G_1(q_1, q_2) = q_1 + q_2$$

Then we get

$$H_{11} = -2 \qquad H_{12} = -2F_1$$

Also assuming

$$(G_2)_{q_1} + (G_2)_{q_2} = 0$$

implies

$$H_{21} = 0$$

Furthermore, the first condition holds if

$$\{G_1, G_2\} = 0 \Rightarrow (G_2)_{p_1} + (G_2)_{p_2} = 0$$

The simplest choice for G_2 to satisfy both conditions is

$$G_2 = (q_1 - q_2)(p_1 - p_2)$$

The vector fields X_{F_i}, X_{G_i} are pointwise linearly independent since

$$\left| \frac{\partial(F_1, F_2, G_1, G_2)}{\partial(q_1, q_2, p_1, p_2)} \right| = 4(p_1 - p_2)^2 + \frac{16g^2}{(q_1 - q_2)^2} \neq 0$$

Since this Jacobian is nonsingular, we can replace the coordinates (t, q_1, q_2, p_1, p_2) with (t, F_1, F_2, G_1, G_2) and calculate the Pfaffian forms in this coordinates. This gives

$$\omega_4 = \frac{F_1 dF_1 - F_2 dF_2}{2(2F_2 - F_1^2)}$$

$$\omega_3 = \frac{1}{2} dF_1$$

$$\omega_2 = -\frac{1}{2} dt + \frac{dG_2}{2(2F_2 - F_1^2)}$$

$$\omega_1 = -\frac{1}{2}dG_1 + \frac{F_1 dG_2}{2(2F_2 - F_1^2)}$$

Now, we have

$$\omega_3 = dI_3 = d\left(\frac{1}{2}F_1\right) \qquad \omega_4 = dI_4 = d\left(-\frac{1}{4}\ln\left(-F_1^2 + 2F_2\right)\right)$$

giving

$$F_1 = 2C_3 \qquad F_2 = \frac{1}{2}e^{-4C_4} + 2C_3^2$$

Restricting ω_2 to this manifold gives

$$\omega_2 = dI_2 = d\left(-\frac{1}{2}t + \frac{1}{2}e^{4C_4 G_2}\right)$$

giving

$$G_2 = e^{-4C_4}(t + 2C_2)$$

Restricting ω_1 to that manifold gives

$$\omega_1 = dI_1 = d\left(C_3 t - \frac{1}{2}G_1\right)$$

giving

$$G_1 = 2(C_3 t - C_1)$$

Now, we have a 4 equation, which can be solved for q_1, q_2, p_1, p_2.
Now, let us determine the action-angle variables. We have

$$\omega_3 = dP_1 \quad \omega_4 = -dP_2 \Rightarrow P_1 = \frac{1}{2}(p_1+p_2) \quad P_2 = \frac{1}{4}\ln\left((p_1 - p_2)^2 + \frac{4g^2}{(q_1 - q_2)^2}\right)$$

In terms of these, the Hamiltonian can be expressed as

$$H = P_1^2 + \frac{1}{4}e^{4P_2}$$

Similarly, by comparing ω_1 and ω_2 by our previous findings, we get

$$Q_1 = q_1 + q_2 \qquad Q_2 = (q_1 - q_2)(p_1 - p_2)$$

One can easily verify that $(q, p) \mapsto (Q, P)$ is a canonical transformation.