Dunkl-Pauli Oscillator

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1 The Pauli Equation

Consider a two dimensional oscillator with a time dependent mass $m(t)$ and time dependent frequency $\omega(t)$. The Pauli equation for such a system is given by

$$\left[\frac{1}{2m(t)}(\vec{\sigma}\cdot\vec{\pi})^2 + \frac{1}{2}m(t)\omega(t)r^2\right]\psi(\vec{r},t) = i\frac{\partial}{\partial t}\psi(\vec{r},t)$$

In there, ψ is a doublet

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$\vec{\sigma}$ stands for the Pauli matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and $\vec{\pi}$ represents the kinetic momentum

$$\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$$

Let's consider a vector potential of the form

$$\vec{A} = \frac{B(t)}{2}(-y\hat{i} + x\hat{j})$$

This leads to

$$(\vec{\sigma}\cdot\vec{\pi})^2 = (\sigma_1\vec{\pi}_1+\sigma_2\vec{\pi}_2)^2 = \begin{pmatrix}0 & \pi_1 - i\pi_2\\ \pi_1 + i\pi_2 & 0\end{pmatrix}^2 = \begin{pmatrix}\pi_1^2+\pi_2^2+i[\pi_1,\pi_2] & 0\\ 0 & \pi_1^2+\pi_2^2-i[\pi_1,\pi_2]\end{pmatrix}$$

$$= \pi_1^2 + \pi_2^2 + i\sigma_3[\pi_1,\pi_2]$$

2 Dunkl Operators and Modified Heisenberg Algebra

In Dunkl quantum mechanics, one makes the substitution

$$p_j \to \frac{1}{i} D_j$$

where D_j is the Dunkl operator defined by

$$D_j = \frac{\partial}{\partial x_j} + \frac{v_j}{x_j}\left(1 - R_j\right)$$

In there, R_j is a reflection operator defined by

$$R_j f(x_j) = -f(x_j)$$

First, let's analyze some commutation relations. We have

$$R_j \frac{\partial}{\partial x_j} f(x_j) = \frac{\partial}{\partial(-x_j)} f(-x_j) = -\frac{\partial}{\partial x_j} f(-x_j) = -\frac{\partial}{\partial x_j} R_j f(x_j)$$

$$\Rightarrow R_j \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j} R_j$$

By this relation, we get

$$D_j^2 = \left(\frac{\partial}{\partial x_j} + \frac{v_j}{x_j} - \frac{v_j}{x_j} R_j\right)\left(\frac{\partial}{\partial x_j} + \frac{v_j}{x_j} - \frac{v_j}{x_j} R_j\right) = \frac{\partial^2}{\partial x_j^2} + \frac{2v_j}{x_j} - \frac{v_j}{x_j^2}(1 - R_j)$$

Let's analyze what happen to Heisenberg algebra. Clearly, one have

$$[x_i, x_j] = 0 \qquad [D_i, D_j] = 0 \ \ (i \neq j)$$

For $i = j$ case, we have trivially $[D_i, D_i] = 0$. Thus

$$[x_i, x_j] = 0 \qquad [D_i, D_j] = 0$$

The only nontrivial commutator is

$$[x_i, D_j] = \left[x_i, \frac{\partial}{\partial x_j} + \frac{v_j}{x_j} - \frac{v_j}{x_j} R_j\right] = -\delta_{ij} - \frac{v_j}{x_j}[x_i, R_j]$$

We have

$$[x_i, R_j] = \begin{cases} x_i R_i - R_i x_i = 2x_i R_i & i = j \\ x_i R_j - R_j x_i = 0 & i \neq j \end{cases} = 2x_j R_j \delta_{ij}$$

$$\Rightarrow [x_i, D_j] = -\delta_{ij}(1 + 2v_j R_j)$$

3 The Hamiltonian

Let's use our results to explicitly calculate the Hamiltonian. First, we have

$$\vec{\pi} = \begin{pmatrix} \frac{D_1}{i} + \frac{eB(t)}{2c} y \\ \frac{D_2}{i} - \frac{eB(t)}{2c} x \end{pmatrix}$$

leading to

$$\pi_1^2 = -D_1^2 + \frac{eB(t)y}{ic} D_1 + \frac{e^2 B^2(t)}{4c^2} y^2$$

$$\pi_2^2 = -D_2^2 - \frac{eB(t)x}{ic} D_2 + \frac{e^2 B^2(t)}{4c^2} x^2$$

Moreover, we have

$$[\pi_1, \pi_2] = -\frac{eB(t)}{ic}(1 + v_1 R_1 + v_2 R_2)$$

Hence, as a final result we have

$$H = -\frac{1}{2m(t)} \Delta_D + \frac{m(t)\Omega^2(t)}{2}(x^2+y^2) + i\frac{\omega_c(t)}{2}(xD_2 - yD_1) - \frac{e}{2m(t)c}(1+v_1R_1+v_2R_2)\sigma_z B(t)$$

where

$$\Delta_D = D_1^2 + D_2^2 \qquad \omega_c(t) = \frac{eB(t)}{m(t)c} \qquad \Omega(t) = \omega^2(t) + \frac{\omega_c^2(t)}{4}$$

In polar coordinates, the Hamiltonian becomes

$$H = -\frac{1}{2m(t)} \frac{\partial^2}{\partial r^2} - \frac{1 + 2v_1 + 2v_2}{2m(t)r} \frac{\partial}{\partial r} + \frac{m(t)}{2} \Omega^2(t) r^2 + \frac{\mathcal{J}_\theta^2 - 2v_1 v_2 (1 - R_1 R_2)}{2m(t) r^2}$$
$$+ \frac{\omega_c(t)}{2} \left[\mathcal{J}_\theta - 2(1 + v_1 R_1 + v_2 R_2) \cdot S_z\right]$$

where the Dunkl angular operator is defined to be

$$\mathcal{J}_\theta = i \left[\partial_\theta + v_2 \cot\theta (1 - R_2) - v_1 \tan\theta (1 - R_1)\right]$$

4 Transforming the Equation

Consider the solutions of the form

$$\psi_{m_s}(r,\theta,t) = \phi(r,\theta,t)\chi_{m_s}$$

where χ_{m_s} is the eigenvectors of S_z with eigenvalues $m_s/2$.
Moreover, to kill the last term in the Hamiltonian, make the substitution

$$\phi(r,\theta,t) = \exp\left[-\frac{i}{2}\left(\mathcal{J}_\theta - m_s\left(1 + v_1 R_1 + v_2 R_2\right)\right)\int^t \omega_c(t')dt'\right] F(r,\theta,t)$$

This leads to a new Hamiltonian

$$\widetilde{H} = -\frac{1}{2m(t)}\frac{\partial^2}{\partial r^2} - \frac{1+2v_1+2v_2}{2m(t)r}\frac{\partial}{\partial r} + \frac{m(t)}{2}\Omega^2(t)r^2 + \frac{\mathcal{J}_\theta^2 - 2v_1v_2(1-R_1R_2)}{2m(t)r^2}$$

If we define

$$\delta = v_1 + v_2 + \frac{1}{2} \qquad \mathcal{P} = -i\left(\frac{\partial}{\partial r} + \frac{\delta}{r}\right) \qquad P^2 = \mathcal{P}^2 + \frac{\mathcal{J}_\theta^2}{r^2}$$

we can express the new Hamiltonian as

$$\widetilde{H} = \frac{P^2}{2m(t)} + \frac{\delta(\delta-1) - 2v_1v_2(1-R_1R_2)}{2m(t)r^2} + \frac{m(t)\Omega^2(t)}{2}r^2$$

5 Lewis-Riesenfeld Invariant

Let's recall the Lewis-Riesenfeld invariant approach. Suppose we are given an operator $I(t)$ satisfying

$$\frac{dI(t)}{dt} = \frac{\partial I(t)}{\partial t} + \frac{1}{i}\left[I(t), \widetilde{H}(t)\right] = 0$$

Suppose we solve the eigenvalue problem of $I(t)$:

$$I(t)\mathcal{F}(r,\theta,t) = \epsilon\mathcal{F}(r,\theta,t)$$

Then, we look for the solutions of the form

$$F(r,\theta,t) = \mathcal{F}(r,\theta,t)e^{i\eta(t)}$$

Substituting this equation gives

$$\frac{d}{dt}\eta(t) = \left\langle \mathcal{F}(r,\theta,t)\middle| i\frac{\partial}{\partial t} - \widetilde{H}\middle|\mathcal{F}(r,\theta,t)\right\rangle$$

Thus, with this method we are able to solve the eigenvalue problem for $\widetilde{H}$.

To find a Lewis-Riesenfeld invariant, consider the operators

$$T_1 = P^2 + \frac{\delta(\delta-1) - 2v_1v_2(1-R_1R_2)}{2m(t)r^2}$$

$$T_2 = r^2$$

$$T_3 = r\mathcal{P} + \mathcal{P}r$$

satisfying the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[T_1, T_2] = -2iT_3 \qquad [T_2, T_3] = 4iT_2 \qquad [T_1, T_3] = -4iT_1$$

In terms of this algebra, we can express the Hamiltonian as

$$\widetilde{H} = \frac{T_1}{2m} + \frac{m\Omega^2(t)}{2}T_2$$

Let's look for a Lewis-Riesenfeld invariant in the form

$$I(t) = \frac{1}{2}\left[\alpha(t)T_1 + \beta(t)T_2 + \gamma(t)T_3\right]$$

Then, the condition $dI/dt = 0$ gives

$$\dot{\alpha} + \frac{2\gamma}{m} = 0 \qquad \dot{\beta} - 2\gamma m\Omega^2(t) = 0 \qquad m\dot{\gamma} - \alpha m^2\Omega^2(t) + \beta = 0$$

A solution to this system can be given as

$$\alpha = \rho^2 \qquad \beta = \frac{1}{\rho^2} + m^2\dot{\rho}^2 \qquad \gamma = -m\rho\dot{\rho}$$

where ρ satisfies the Ermakov-Pinney equation

$$\ddot{\rho} + \frac{\dot{m}}{m}\dot{\rho} + \Omega^2(t)\rho = \frac{1}{m^2\rho^3}$$

Then, we have the invariant

$$I(t) = \frac{1}{2}\left[\rho^2 T_1 + \left(\frac{1}{\rho^2} + m^2\dot{\rho}^2\right)T_2 - m\rho\dot{\rho}T_3\right]$$

6 One More Transformation

Let's focus on finding $\mathcal{F}(r,\theta,t)$, i.e. solving the equation

$$I(t)\mathcal{F}(r,\theta,t) = \epsilon\mathcal{F}(r,\theta,t)$$

To simplify the equation, consider the unitary transformation

$$\mathcal{F}(r,\theta,t) = \exp\left(\frac{im\dot{\rho}}{2\rho}r^2\right)G(r,\theta)$$

Then, the Lewis-Riesenfeld invariant transforms as

$$I(t) \to \tilde{I}(t) = \exp\left(-\frac{im\dot{\rho}}{2\rho}r^2\right)I(t)\exp\left(\frac{im\dot{\rho}}{2\rho}r^2\right)$$

With the aid of the Campbell identity $e^X Y e^{-X} = Y + [X,Y] + 1/2[X,[X,Y]] + \ldots$, we get

$$\tilde{I}(t) = \frac{1}{2}\left(\rho^2 T_1 + \frac{1}{\rho^2}T_2\right)$$

So, our equation becomes

$$\tilde{I}(t)G(r,\theta) = \psi G(r,\theta)$$

If we make the ansatz

$$G(r,\theta) = r^{-\delta}Q(r)\Theta_\epsilon(\theta)$$

where

$$\mathcal{J}_\theta\Theta_\epsilon(\theta) = \lambda_\epsilon\Theta_\epsilon(\theta)$$

Then we get the equation

$$\rho^2 Q'' - \frac{r^2}{\rho^2}Q - \frac{(\sigma_l^\epsilon)^2 - 1/4}{r^2/\rho^2} + 2\psi_{n,l}^\epsilon = 0$$

Thus, we get a hypergeometric equation in terms of r/ρ.
By explicit calculation, one can also shows that

$$\frac{d}{dt}\eta(t) = -\frac{\psi_{n,l}^\epsilon}{m\rho^2} \Rightarrow \eta(t) = -\int^t \frac{\psi_{n,l}^\epsilon}{m(t')\rho^2(t')}dt'$$