Exercise 3

If we have a linear model

$$\mathbf{y} = \mathbf{w} \cdot \mathbf{x} \tag{1}$$

to show that

$$\mathbf{w} = \langle \mathbf{xx} \rangle^{-1} \cdot \langle \mathbf{xy} \rangle \tag{2}$$

we define the mean squared loss as follow

$$\mathcal{L}\left(\mathbf{x}^{\mu}\mathbf{y}^{\mu}\right) = \frac{1}{2}\sum_{\mu=1}^{n}\left(\mathbf{y}^{\mu} - \mathbf{w} \cdot \mathbf{x}^{\mu}\right)^2 \tag{3}$$

where

$$\langle \mathbf{xx} \rangle := \frac{1}{n}\sum_{\mu=1}^{n} \mathbf{x}^{\mu}\mathbf{x}^{\mu} \tag{4}$$

and

$$\langle \mathbf{xy} \rangle := \frac{1}{n}\sum_{\mu=1}^{n} \mathbf{x}\mathbf{y}^{\mu}\mathbf{y}^{\mu} \tag{5}$$

Solution

Proof.

$$\begin{aligned}
\mathcal{L}\left(\mathbf{x}^{\mu}\mathbf{y}^{\mu}\right) &= \frac{1}{2}\sum_{\mu=1}^{n}\left(\mathbf{y}^{\mu} - \mathbf{w} \cdot \mathbf{x}^{\mu}\right)^2 \\
\frac{\partial \mathcal{L}}{\partial \mathbf{w}} &= \frac{\partial}{\partial \mathbf{w}}\left(\left(\mathbf{y}^{\mu} - \mathbf{w} \cdot \mathbf{x}^{\mu}\right)^2\right) \\
&= \frac{1}{2}\left[\left(\mathbf{y} - \mathbf{w} \cdot \mathbf{x}\right)^2 + \left(\mathbf{y}^2 - \mathbf{w} \cdot \mathbf{x}^2\right)^2 + \cdots + \left(\mathbf{y}^n - \mathbf{w} \cdot \mathbf{x}^n\right)^2\right] \\
&= -\mathbf{xy} + \mathbf{wx^2} - \mathbf{x^2y^2} + \cdots + \mathbf{x^ny^n} + \mathbf{wx^{2n}} = 0 \\
\mathbf{w}\left(\mathbf{x^2} + \mathbf{x^4} + \cdots + \mathbf{x^{2n}}\right) &= \mathbf{xy} + \mathbf{x^2y^2} + \cdots + \mathbf{x^ny^n} \\
\mathbf{w} &= \frac{\mathbf{xy} + \mathbf{x^2y^2} + \cdots + \mathbf{x^ny^n}}{\left(\mathbf{x^2} + \mathbf{x^4} + \cdots + \mathbf{x^{2n}}\right)} \\
\mathbf{w} &= \frac{\langle \mathbf{xy} \rangle}{\langle \mathbf{xx} \rangle} \\
\therefore \mathbf{w} &= \langle \mathbf{xx} \rangle^{-1}\langle \mathbf{xy} \rangle
\end{aligned}$$

□