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Exercise 1

Considering the following model

$$v''' = v(t)v'(t) - s(t)\left(v''\right)^2 \quad \forall t > 0$$
$$v(0) = 1, \quad v'(0) = 0, \quad v''(0) = 2$$

1. We let

$$\begin{aligned} y(t) &= v'(t) \\ y'(t) &= x(t) = v''(t) \\ \text{and } x'(t) &= v''' \end{aligned}$$

we therefore get the following system

$$\begin{aligned} v'(t) &= y(t) \\ y'(t) &= x(t) \\ x'(t) &= v(t)y(t) - s(t)(x(t))^2 \end{aligned}$$

with the following initial conditions

$$v(0) = 1, \quad y(0) = 0, \quad x(0) = 2$$

thus

$$Y'(t) = \begin{pmatrix} v'(t) \\ y'(t) \\ x'(t) \end{pmatrix}$$

and

$$F(t, Y) = \begin{pmatrix} y(t) \\ x(t) \\ v(t)y(t) - s(t)\left(x(t)\right)^2 \end{pmatrix} \qquad (P)$$

and

$$Y(0) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

2. The explicit Mid-point scheme solution to (P) using $\delta t_n = \delta t$ is as follows

$$
\begin{aligned}
\begin{pmatrix} v'(t) \\ y'(t) \\ x'(t) \end{pmatrix} &= \begin{pmatrix} w(t) \\ z(t) \\ v(t)w(t) - s(t)\left(z(t)\right)^2 \end{pmatrix} \\
Y'(t) &= \begin{pmatrix} v'(t) \\ y'(t) \\ x'(t) \end{pmatrix} \\
Y &= \begin{pmatrix} v \\ w \\ z \end{pmatrix} \\
F(t, v, w, z) &= \begin{pmatrix} w \\ z \\ vw - s(t)z^2 \end{pmatrix} \\
Y(0) &= Y_0 \\
t_{n+\frac{1}{2}} &= t_n + \frac{1}{2}\delta t \\
Y_{n+\frac{1}{2}} &= \begin{pmatrix} v_{n+\frac{1}{2}} \\ w_{n+\frac{1}{2}} \\ z_{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} v_n \\ w_n \\ z_n \end{pmatrix} + \frac{\delta t}{2} \begin{pmatrix} w_n \\ z_n \\ v_n w_n - s(t_n) z_n^2 \end{pmatrix} \\
y_{n+1} &= \begin{pmatrix} v_{n+1} \\ w_{n+1} \\ z_{n+1} \end{pmatrix} + \begin{pmatrix} w_{n+\frac{1}{2}} \\ z_{n+\frac{1}{2}} \\ v_{n+\frac{1}{2}} w_{n+\frac{1}{2}} - s_{t_{n+\frac{1}{2}}} z_{n+\frac{1}{2}} \end{pmatrix}
\end{aligned}
$$

3. Taking $t_0 = 0, \delta t = \frac{1}{2}, s(t) = 1 + t^2$ to solve for $v_1 \approx v(t_1)$ and $v_2 \approx v(t_2)$ with $n = 0$.

$$
\begin{aligned}
Y(0) = Y_0 &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \\
t_{\frac{1}{2}} = t_0 + \frac{\delta t}{2} &= \frac{0.5}{2} = \frac{1}{4} \\
\begin{pmatrix} v_{\frac{1}{2}} \\ w_{\frac{1}{2}} \; z_{\frac{1}{2}} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \\
\begin{pmatrix} v_1 \\ w_1 \\ z_1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{2} + \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} - (1 + (\frac{1}{4})^2) \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.72 \end{pmatrix}
\end{aligned}
$$

For $n = 1 v_2 \approx v(t_2)$

$$t_{\frac{3}{2}} = t_1 + \frac{\delta t}{2} = 0.5 + \frac{0.5}{2} = \frac{3}{4}$$

$$\begin{pmatrix} v_{\frac{3}{2}} \\ w_{\frac{3}{2}} \\ z_{\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.72 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0.5 \\ 1.72 \\ (1.25 \times 0.5) - (1 + 0.5^2 \times 1.72^2) \end{pmatrix} = \begin{pmatrix} 1.375 \\ 0.93 \\ 0.95 \end{pmatrix}$$

$$\begin{pmatrix} v_2 \\ w_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.72 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1.375 \\ 0.93 \\ (1.375 \times 0.93) - (1 + 0.75^2) \times 0.95^2) \end{pmatrix} = \begin{pmatrix} 1.9375 \\ 0.965 \\ 1.6543 \end{pmatrix}$$

Exercise 2

Given the conservation of the energy of a thin spherical air layer (as a model below)

$$C\frac{dT}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T^4 \quad \text{where } \forall t \in [0, 310] \text{ and } T(0) = 270, \quad T = T(t) \tag{1}$$

where the following quantities are used

$$C = 85, \alpha = 0.3, S_0 = 1367, \epsilon = 0.6, \sigma = 5.67 \times 10^{-8}$$
$$T = T(t) \quad \text{globally averaged sufrace temperature}$$

1. To find the equilibrium temperature T_{eq}.

$$\frac{dT}{dt} = 0$$
$$\implies \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T^4 = 0$$
$$\implies T_{eq} = \sqrt[4]{\frac{(1-\alpha)S_0}{4\epsilon\sigma}}$$

2. To write $T(t) = Teq + \tilde{T}(t)$ near the equilibrium, where $\tilde{T}(t)$ is a small time-dependent temperature perturbation($|\tilde{T}| << T_{eq}$). And to prove that

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma\left(T_{eq} + \tilde{T}\right)^4 \tag{2}$$

Proof.

$$T(t) = T_{eq} + \tilde{T} \quad |\tilde{T} << T_{eq}|$$
$$\frac{dT}{dt} = \frac{dT_{eq}}{dt} + \frac{d\tilde{T}}{dt}$$
$$\implies \frac{dT}{dt} = \frac{d\tilde{T}}{dt}$$
$$C\frac{dT}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T^4$$
$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma\left(T_{eq} + \tilde{T}\right)^4$$

$\square$

3. Assuming that

$$\left(1+\frac{\tilde{T}}{T_{eq}}\right)^4 = \left(1+\frac{4\tilde{T}}{T_{eq}}\right) \tag{3}$$

Proof.

$$\begin{aligned}
\frac{d\tilde{T}}{dt} &= -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T} \\
C\frac{dT}{dt} &= \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4\left(1+\frac{\tilde{T}}{T_{eq}}\right)^4 \\
&= \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4\left(1+\frac{4\tilde{T}}{T_{eq}}\right) \\
&= \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4\left(4\epsilon\sigma T_{eq}^3\right)\tilde{T} \\
\implies C\frac{d\tilde{T}}{dt} &= -\left(4\epsilon\sigma T_{eq}^3\right)\tilde{T} \\
\implies \frac{d\tilde{T}}{dt} &= -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T}
\end{aligned}$$

$\square$

4. Given $\tilde{T}(0) = 10, A = 10, t \in [0, 310]$, we find the exact solution of (1) as follows.

$$\begin{aligned}
\frac{d\tilde{T}}{dt} &= -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right) \\
\int \frac{d\tilde{T}}{dt} &= -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\int dt \\
\ln(\tilde{T}) &= -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)t + A_1 \\
\tilde{T} &= A\exp^{-\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)t} \\
\tilde{T} &= 10\exp^{-\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)t}
\end{aligned}$$

5. To solve (1) numerically, one considering the following scheme

$$(P)\begin{cases} y_0 = y(t_0) \\ K_1 = f(t_n, y_n) \\ t_{n+\frac{3}{4}} = y_n + \frac{3}{4}\delta t K_1 \\ K_2 = f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \\ y_{n+1} = y_n + \frac{1}{3}\delta t(K_1 + 2K_2) \end{cases}$$

   (a) Prove that (P) is the one-step method.

Proof. Which means

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{3}\delta t(K_1 + 2K_2) \equiv y_{n+1} = y_n + \delta t\phi(t_n, y_n, \delta t) \\
y_{n+1} &= y_n + \frac{1}{3}\delta t \left(f(t_n, y_n) + 2f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \right) \\
&= y_n + \frac{1}{3}\delta t \left[f(t_n, y_n) + 2f\left(t_n + \frac{3}{4}\delta t, y_n + \frac{3}{4}\delta t f(t_n, y_n) \right) \right] \\
&= y_n + \frac{1}{3}\delta t\phi(t_n, y_n, \delta t) \\
&\text{where } \phi(t_n, y_n, \delta t) = \left[f(t_n, y_n) + 2f\left(t_n + \frac{3}{4}\delta t, y_n + \frac{3}{4}\delta t f(t_n, y_n) \right) \right]
\end{aligned}
$$

□

(b)